GENERATING "LARGE" SUBGROUPS AND SUBSEMIGROUPS

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Generating "large" subgroups and subsemigroups

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I was admitted as a research student in September, 2012 and as a candidate for the degree of PhD Mathematics in September, 2012; the higher study for which this is a record was carried out in the University of St Andrews between 2012 and 2016.

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Abstract

In this thesis we will be exclusively considering uncountable groups and semigroups. Roughly speaking the underlying problem is to find “large” subgroups (or subsemigroups) of the object in question, where we consider three different notions of “largeness”:

- we classify all the subsemigroups of the set of all mapping from a countable set back to itself which contains a specific uncountable subsemigroup;
- we investigate topological “largeness”, in particular subgroups which are finitely generated and dense;
- we investigate if it is possible to find an integer r such that any countable collection of elements belongs to some r-generated subsemigroup, and more precisely can these elements be obtain by multiplying the generators in a prescribed fashion.

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Contents

Chapter 1

Introduction and preliminaries

This chapter is dedicated to introducing the notions and definitions which will be used in the rest of the thesis. Concepts which are relevant to a particular chapter only are introduced in the corresponding chapter for ease of referencing.

1.1 Set theory

The axiom system used in the thesis is the standard Zermelo-Fraenkel axioms together with the Axiom of Choice. In this section, assuming the knowledge of ordinals, we define cardinal numbers and introduce some basic notation used throughout. The section is based on [10], the proofs of the results in this section are not included, as they do not offer any insight to the problems we will be dealing with in the later chapters.

Let X and Y be arbitrary sets. Then $f \subseteq X \times Y$ is a FUNCTION, usually written as $f : X \longrightarrow Y$, if for every $x \in X$ there is a unique $y \in Y$ such that $(x, y) \in f$. It is more customary to denote $(x, y) \in f$ as $(x)f = y$, and we are going to adopt this notation. A function $f : X \longrightarrow Y$ is said to be INJECTIVE if $(x)f = (x')f$ implies $x = x'$ for all $x, x' \in X$, f is said to be SURJECTIVE if for all $y \in Y$ there is $x \in X$ such that $(x)f = y$, and f is BIJECTIVE if it is both injective and surjective.

Let X be an arbitrary set, and let $n \in \mathbb{N}$. Then a subset of X^n is an n-ARY RELATION on X. In the particular case where $n = 2$, n-ary relations are referred to as BINARY RELATIONS. A binary relation $\leq$ on X is a PARTIAL ORDER if the following are satisfied:

- (REFLEXIVITY) $x \leq x$ for all $x \in X$;
- (ANTISYMMETRY) if $x \leq y$ and $y \leq x$ for some $x, y \in X$, then $x = y$;

- (TRANSITIVITY) if $x \le y$ and $y \le z$ for any $x, y, z \in X$, then $x \le z$.

The pair $(X, \le)$ is called PARTIALLY ORDERED SET. If in addition for every pair of elements $x, y \in X$ either $x \le y$ or $y \le x$, then $\le$ is a TOTAL ORDER. We use the convention that $x < y$ if and only if $x \le y$ and $x \ne y$.

Let $(X, \le)$ be a partially ordered set, and let A be a subset of P. Then A is a CHAIN if $\le$ is a total order on A. An element $u \in P$ is an UPPER BOUND OF A if $a \le u$ for every $a \in A$, and $m \in P$ is a MAXIMAL ELEMENT OF P if there are no $p \in P$ such that $m < p$. Note that a maximal element of a partially ordered set does not have to be unique. A WELL ORDER on a set X is a total order such that every non-empty subset Y of X has an element which is smaller than every other element in Y. The Zermelo's Theorem, also known as Well-ordering Theorem, states that every set can be well-ordered, see Theorem 4.3.3 in [10]. The following classical result is equivalent to Zermelo's Theorem, and also equivalent to the Axiom of Choice.

Theorem 1.1.1 (Kuratowski-Zorn Lemma, see Theorem 4.3.3 in [10])**.** *If $(P, \le)$ is a partially ordered set such that every chain in P has an upper bound, then P has a maximal element.*

We say that two sets X and Y have the SAME CARDINALITY if there is a bijection $f : X \longrightarrow Y$. Recall that for every well-ordered $(X, \le)$ set there is a unique ordinal α such that there is a bijection $f : X \longrightarrow \alpha$ so that $x \le y$ if and only if $(x)f \subseteq (y)f$ for all $x, y \in X$. It then follows from Zermelo's Theorem that for every set X there exists an ordinal α such that X has the same cardinality as α. Also recall that ordinal numbers are defined in such a way, that for every ordinal α, $\subseteq$ is a well-order on α. Define CARDINALITY OF X to be the smallest ordinal which has the same cardinality as X, denoted by $|X|$. Note that the definitions of cardinality and having the same cardinality are consistent, in other words there is a bijection between X and Y if and only if $|X|$ and $|Y|$ are equal. An ordinal κ is a CARDINAL NUMBER if $\kappa = |X|$ for some set X.

For ordinals α and β, we say that $\alpha \le \beta$ if $\alpha \subseteq \beta$. Since cardinality of a set is an ordinal, the same definition of $\le$ applies to cardinal numbers.

Theorem 1.1.2 (Theorem 5.1.2 in [10])**.** *Let A and B be arbitrary sets. Then $|A| \le |B|$ if and only if there is an injective function $f : A \longrightarrow B$.*

We denote by ω cardinality of the natural numbers. It is also worth mentioning, that for us natural numbers start at 0. If λ is a cardinal, then 2^λ denotes cardinality of the set of all subsets of a set λ.

Theorem 1.1.3 (Cantor's Theorem, Theorem 5.1.6 in [10])**.** *Let λ be a cardinal number. Then $\lambda < 2^\lambda$.*

For a given cardinal number κ, consider the set

$$S_\kappa = \{\lambda \in 2^{2^\kappa} : \lambda \text{ is a cardinal and } \kappa < \lambda\}.$$

Hence $2^\kappa \in S_\kappa$ by Cantor's Theorem. Since 2^{2^κ} is well-ordered by $\leq$, the set S_κ has an element which is smaller than any other element in the set. Denote this unique cardinal number by κ^+, the CARDINAL SUCCESSOR OF κ.

Proposition 1.1.4 (Proposition 5.1.7 in [10])**.** *Let $\mathcal{F}$ be a family of cardinal numbers. Then $\bigcup\{\kappa : \kappa \in \mathcal{F}\}$ is a cardinal number.*

Using Proposition 1.1.4, we can now for every ordinal α define a cardinal number $\aleph_\alpha$ by induction on $\beta \leq \alpha$. Let $\aleph_0 = \omega$. For every $\beta < \alpha$, let

$$\aleph_{\beta+1} = \aleph_\beta^+,$$

and for every limit $\lambda \leq \alpha$, let

$$\aleph_\lambda = \bigcup_{\beta<\lambda} \aleph_\beta.$$

Then $\aleph_\alpha$ is the α-th cardinal number.

For any two cardinal numbers κ and λ we can define the operations

$$\kappa \oplus \lambda = |(\kappa \times \{0\}) \cup (\lambda \times \{1\})|$$
$$\kappa \otimes \lambda = |\kappa \times \lambda|.$$

Proposition 1.1.5 (Corollary 5.2.5 in [10])**.** *Let κ and λ be infinite cardinal numbers. Then $\kappa \oplus \lambda = \kappa \otimes \lambda = \max(\kappa, \lambda)$.*

If X and Y are sets, then we denote by Y^X, the set of all functions from X to Y.

Proposition 1.1.6 (Proposition 5.2.12 in [10])**.** *If λ and κ are cardinal numbers such that $\omega \leq \lambda$ and $2 \leq \kappa \leq \lambda$. Then $|\kappa^\lambda| = 2^\lambda$.*

It follows immediately from Proposition 1.1.6 that if X is an infinite set, then $|X^X| = 2^{|X|}$.

A subset X of an ordinal α is UNBOUNDED IN α if there is no $\beta \in \alpha$ such that $\gamma < \beta$ for all $\gamma \in X$. For an ordinal number α, let $\mathrm{cf}(\alpha)$ be the smallest ordinal number β such that there is a function $f : \beta \longrightarrow \alpha$ so that $(\beta)f = \{(\gamma)f : \gamma \in \beta\}$ is unbounded in α. The ordinal $\mathrm{cf}(\alpha)$ is known as COFINALITY OF α. It can be shown that $\mathrm{cf}(\alpha) = 1$ if and only if $\mathrm{cf}(\alpha) < \omega$, which is equivalent to α being a successor ordinal. An ordinal number α is REGULAR if α is a limit ordinal and $\mathrm{cf}(\alpha) = \alpha$. An ordinal is IRREGULAR if it is not regular.

1.2 Groups and semigroups

Let S be an arbitrary set, and let $\cdot : S \times S \longrightarrow S$ be a function. Then S is a SEMIGROUP if it satisfies the ASSOCIATIVITY LAW, that is

$$(x \cdot y) \cdot z = x \cdot (y \cdot z)$$

for all x, y, and $z \in S$. An element $e \in S$ is an IDENTITY if $x \cdot e = e \cdot x = x$ for all $x \in S$. A semigroup S with an identity is called a MONOID. A monoid G is a GROUP if for all $x \in G$ there is $x^{-1} \in G$ such that $x \cdot x^{-1} = x^{-1} \cdot x = e$. A semigroup S is COMMUTATIVE, also called ABELIAN, if $x \cdot y = y \cdot x$ for all $x, y \in S$.

Let G and H be groups with the operations denoted by $\cdot_G$ and $\cdot_H$ respectively. Define an operation on $G \times H$ as follows:

$$(g, h) \cdot (g', h') = (g \cdot_G g', h \cdot_H h') \in G \times H$$

for every $(g, h), (g', h') \in G \times H$. Then it is routine to verify that $G \times H$ is a group with respect to this operation. If G is a group and $n \in \mathbb{N}$, we write G^n to mean

$$\underbrace{G \times \cdots \times G}_{n}.$$

Another group theory notion, we will occasionally make use of, is the one of conjugation. Let G be a group, and let $g, h \in G$. Then g and h are CONJUGATE if there is $k \in G$ such that $h = k^{-1}gk$. The set of all elements of G which are conjugate to g is called the CONJUGACY CLASS OF G CONTAINING g.

Throughout the thesis we, without mention, assume that the operation of a semigroup S is denoted by $\cdot$, unless specified otherwise. Also whenever possible without causing ambiguity, we omit the mention of $\cdot$ and write xy instead of $x \cdot y$.

1.2.1 Generation

Let S be a semigroup, and let T be a subset of a semigroup S. Then T is SUBSEMIGROUP of S, written $T \leq S$, if T is a semigroup under the operation of S. Let I be a non-empty set, and let $\{S_i : i \in I\}$ be a set of subsemigroups of S. It follows easily from the definition that $\bigcap_{i=I} S_i$ is a semigroup with respect to $\cdot$, and so a subsemigroup of S. Suppose that X is a subset of S, and let

$$\langle X \rangle = \bigcap \{T \leq S : X \subseteq T\}.$$

Note that $S \leq S$ and $X \subseteq S$, and so the set in the definition of $\langle X \rangle$ is non-empty. It follows from the definition of $\langle X \rangle$, that if T is any subsemigroup of S such that X is a subset of T, then $\langle X \rangle \leq T$. Hence $\langle X \rangle$ is the smallest subsemigroup of S

containing X, called the SUBSEMIGROUP GENERATED BY X. We say that X is a GENERATING SET FOR S, or that S is a SEMIGROUP GENERATED BY X, if $\langle X\rangle = S$.

There is also an alternative, more constructive, way of defining the subsemigroup of S generated by X. Let

$$T = \bigcup_{n\geq 1} \{x_1 \cdot x_2 \cdots x_n : x_i \in X \text{ for all } i \in \{1,\dots,n\}\}.$$

Then $X \subseteq T$ and if $x_1 \cdot x_2 \cdots x_n, x'_1 \cdot x'_2 \cdots x'_k \in T$, then $x_1 \cdot x_2 \cdots x_n \cdot x'_1 \cdot x'_2 \cdots x'_k$ is in T. Hence T is a subsemigroup of S containing X. Suppose that Q is a subsemigroup of S such that $X \subseteq Q \leq T$. Since Q is a semigroup, $x_1 \cdot x_2 \cdots x_n \in Q$ for every $n \geq 1$ and $x_i \in X$ where $i \in \{1,\dots,n\}$. Hence $Q = T$, and so T is the smallest subsemigroup of S containing X, in other words

$$\langle X\rangle = \{x_1 \cdot x_2 \cdots x_n : n \geq 1 \text{ and } x_i \in X \text{ for all } i \in \{1,\dots,n\}\}.$$

Same concepts can be applied to groups as well. Let G be a group. Then a subset H of G is a SUBGROUP OF G, written $H \leq G$, if H is a group on its own right with respect to the operation of G. Let I be a non-empty set, and let $\{G_i : i \in I\}$ be a set of subgroups of G. It follows from the definition that $\bigcap_{i=I} G_i$ is a subgroup of G. Suppose that X is a subset of G, and let

$$\langle X\rangle = \bigcap\{H \leq G : X \subseteq H\}.$$

By the same argument as used for semigroups $\langle X\rangle$ is non-empty, and so a subgroup of G. It follows from the definition of $\langle X\rangle$ that if H is any subgroup of G containing X, then $\langle X\rangle \leq H$. Hence $\langle X\rangle$ is the smallest subgroup containing X, called the SUBGROUP GENERATED BY X. In the same way as for semigroups, we say that X is a GENERATING SET FOR G, or that G is a GROUP GENERATED BY X, if $\langle X\rangle = G$.

Let X^{-1} be the set of inverses of the elements in X. Define

$$H = \bigcup_{n\in\mathbb{N}} \{x_1 \cdot x_2 \cdots x_n : x_i \in X \cup X^{-1} \text{ for all } i \in \{1,\dots,n\}\}, \tag{1.1}$$

where the product of length 0 is assumed to be the identity of G. It follows that if $x_1 \cdot x_2 \cdots x_n$, $x'_1 \cdot x'_2 \cdots x'_k \in H$, then $x_1 \cdot x_2 \cdots x_n \cdot x'_1 \cdot x'_2 \cdots x'_k \in H$, and also that the identity of G is in H. Consider $g = x_1^{\varepsilon_1} \cdot x_2^{\varepsilon_2} \cdots x_n^{\varepsilon_n} \in H$ where $n \geq 1$, $x_i \in X$ and $\varepsilon_i \in \{-1, 1\}$ for all $i \in \{1,\dots,n\}$. Then $x_n^{-\varepsilon_n} \cdots x_2^{-\varepsilon_2} \cdot x_1^{-\varepsilon_1} \in H$ is the inverse of g, and so H is a subgroup of G. Moreover, X is a subset of H. Suppose that K is a subgroup of G such that $X \subseteq K \leq H$. Since K is a group, $x_1 \cdot x_2 \cdots x_n \in K$ for all $n \in \mathbb{N}$, and $x_i \in X \cup X^{-1}$ where $i \in \{1,\dots,n\}$. Hence $K = H$, and so H is the smallest subgroup of G containing X, in other words

$\langle X \rangle = H$.

1.2.2 Free groups and free semigroups

An ALPHABET is a non-empty set A, and the elements of A are referred to as LETTERS. Then a WORD over the alphabet A is a finite sequence of letters in A, we will simply denote it by $a_1 \dots a_n$ where $a_i \in A$ for all $i \in \{1, \dots, n\}$. Let A^+ be the set of all words over a given alphabet A. Concatenation of words defines a binary operation on A^+, namely for $a_1 \dots a_n$, $b_1 \dots b_m \in A^+$

$$(a_1 \dots a_n)(b_1 \dots b_m) = a_1 \dots a_n b_1 \dots b_m.$$

Then this binary operation is associative, and so A^+ is a semigroup, called the FREE SEMIGROUP ON A. It can be shown that A^+ is the free object in the category of semigroups, in other words for every semigroup S, and $f : A \longrightarrow S$, there is a unique semigroup homomorphism $F : A^+ \longrightarrow S$ such that $(a)f = (a)F$ for all $a \in A$.

Choose a set disjoint from A^+, with exactly one element. We denote this element by 1. Define $A^* = A^+ \cup \{1\}$ with the binary operation being concatenation for any elements of A^+ and $w1 = 1w = w$ for every word $w \in A^+$. We may think of 1 as the EMPTY WORD, or the word containing no letters. Then A^* is a monoid, called the FREE MONOID on A. Similarly to A^+, the free monoid is the free object in the category of monoids. For more details see [30].

Let $w = a_1 \dots a_n \in A^+$ such that $a_i \in A$ for all $i \in \{1, \dots, n\}$. A word $u \in A^*$ is a SUBWORD OF w if $u = 1$ or $u = a_i \dots a_j$ for some $i, j \in \{1, \dots, n\}$ such that $i \leq j$. A word $p \in A^*$ is a PREFIX of w if either $p = a_1 \dots a_i$ for some $i \in \{1, \dots, n\}$, or $p = 1$. Similarly, a word $s \in A^*$ is a SUFFIX of w if either $s = a_i \dots a_n$ for some $i \in \{1, \dots, n\}$, or $s = 1$. A subword, prefix, or suffix of w is a PROPER SUBWORD, PREFIX, or SUFFIX OF w respectively, if it is strictly shorter than w and not equal to 1. We say that the LENGTH of w is the number of letters in the word, denoted by $|w|$, in other words $|a_1 \dots a_n| = n$ and $|1| = 0$.

Let A^{-1} be a set such that A and A^{-1} are disjoint such that $|A| = |A^{-1}|$. Denote by a^{-1} image of $a \in A$ under a fixed bijection between A and A^{-1}. Choose a set disjoint from $A \cup A^{-1}$, which contains exactly one element, and denote that element by 1, as in the definition of free monoids we refer to 1 as the empty word. Define $w = x_1 \dots x_n$ to be a REDUCED WORD if either $w = 1$, or w is a word over $A \cup A^{-1}$ such that if $x_i = a \in A$, then neither x_{i-1} or x_{i+1}, if defined, can be equal to a^{-1} for every $i \in \{1, \dots, n\}$. Let $F(A)$ be the set of all reduced words over $A \cup A^{-1}$, together with the element the empty word 1.

Next we want to define a binary operation on $F(A)$. The empty word 1 acts as the identity, namely $w1 = 1w = w$ for all $w \in F(A)$. Let $x_1 \dots x_n, y_1 \dots, y_m \in$

$F(A)$ and suppose that k is the largest natural number such that $x_{n-i} = y_{i+1}^{-1}$ or $x_{n-i}^{-1} = y_{i+1}$ for all $i \in \{0, \ldots, k-1\}$. Then $k \leq \min(n, m)$, and so we define

$$(x_1 \ldots x_n)(y_1 \ldots y_m) = \begin{cases} x_1 \ldots x_{n-k} y_{k+1} \ldots y_m & \text{if } k < n \text{ and } k < m; \\ y_{k+1} \ldots y_m & \text{if } n = k < m; \\ x_1 \ldots x_{n-k} & \text{if } m = k < n; \\ 1 & \text{if } n = m = k. \end{cases}$$

The definition ensures that the product of reduced words is still a reduced word.

Theorem 1.2.1 (Theorem 9.1 in [33]). *If A is a non-empty set, then $F(A)$ is a group under a binary operation defined above.*

Note that free groups are the free objects in the category of groups, in the same way as free semigroups or free monoids, namely for every group G, and $f : A \longrightarrow G$, there is a unique group homomorphism $F : F(A) \longrightarrow G$ such that $(a)f = (a)F$ for all $a \in A$, we refer to F as the CANONICAL HOMOMORPHISM INDUCED BY f. More formal treatment of free groups can be found in [33, Chapter 1, Section 9].

1.3 Functions and partial permutations

1.3.1 Definitions and notation

Let X and Y be sets, and let $f : X \longrightarrow Y$. The set X is then the DOMAIN OF f, denoted by $\text{dom}(f)$, and the RANGE OF f, denoted by $\text{ran}(f)$, is the set $(X)f = \{(x)f : x \in X\} \subseteq Y$. Let A be a subset of Y. Then the PREIMAGE OF A UNDER f is the set $(A)f^{-1} = \{x \in X : (x)f \in A\}$. If $g : Y \longrightarrow Z$, then COMPOSITION OF FUNCTIONS f AND g is defined to be the function $f \circ g$ such that for all $x \in X$

$$(x)f \circ g = ((x)f)\, g.$$

Example 1.3.1. Let Ω be a set, and let Ω^Ω denote the set of all functions $f : \Omega \longrightarrow \Omega$. By the definition composition $\circ$ is a function from $\Omega^\Omega \times \Omega^\Omega$ to Ω^Ω. Let $f, g, h \in \Omega^\Omega$. Then

$$(x)\,(f \circ g) \circ h = ((x)f \circ g)\, h = (((x)f)\, g)\, h = ((x)f)\, g \circ h = (x)f \circ (g \circ h)$$

for all $x \in \Omega$. Therefore $\circ$ is associative, and so Ω^Ω is a semigroup with respect to the composition of functions.

An alternative definition of being bijective for a function can be provided using composition of functions.

Proposition 1.3.2. *Let $f : X \longrightarrow Y$. Then f is bijective if and only if there is $g : Y \longrightarrow X$ such that*

$$(x)f \circ g = x \quad \text{and} \quad (y)g \circ f = y$$

for all $x \in X$ and $y \in Y$.

If $f : X \longrightarrow Y$ is a bijection, then $g : Y \longrightarrow X$, as defined in Proposition 1.3.2, is an INVERSE OF f, usually denoted by f^{-1}. Let $A \subseteq X$. Then the IMAGE of A under f is the set

$$(A)f = \{(x)f : x \in A\},$$

and the RESTRICTION of f to A is a function $f|_A : A \longrightarrow Y$ given by

$$f|_A = f \cap (A \times Y).$$

A function $f : X \longrightarrow Y$ is an EXTENSION OF g, if there is some subset A of X such that $f|_A = g$. We say that $f : X \longrightarrow X$ is the IDENTITY FUNCTION if $(x)f = x$ for all $x \in X$. Define $\text{fix}(f) = \{x \in X : (x)f = x\}$ and $\text{supp}(f) = \{x \in X : (x)f \neq x\}$, called the FIX and the SUPPORT OF f respectively. If $f : X \longrightarrow X$ is a bijection, and $x \in X$, then the ORBIT OF x UNDER f is the set $\{(x)f^n : n \in \mathbb{N}\}$.

Let X be an arbitrary set. Then p is a PARTIAL PERMUTATION of X if there is a subset X' of X such that $p : X' \longrightarrow X$ and p is injective. Since a partial permutation is also a function, domain and range of a partial permutation are defined to be domain and range of the function. If f and g are two partial permutations of X, then we define their composition

$$f \circ g : \text{dom}(f) \cap (\text{dom}(g) \cap \text{ran}(f))\, f^{-1} \longrightarrow \text{ran}(g) \cap (\text{dom}(g) \cap \text{ran}(f))\, g$$

to be $(x)f \circ g = ((x)f)g$. Denote the composite $f \circ f^{-1}$ by f^0, being the identity function on $\text{dom}(f)$. Let $\mathcal{A}$ be any collection of functions of the form $f : X \longrightarrow X$. Then we denote $\{f|_A : f \in \mathcal{A} \text{ and } A \subseteq X \text{ is finite}\}$ by $\mathcal{A}^{<\omega}$.

In Chapter 3, the sets $\{(x)f^n : n \in \mathbb{Z} \text{ and } x \in \text{dom}(f^n)\}$ for a partial permutation f, will play an important role. Even though, the aforementioned sets agree with the definition of an orbit for a bijective function, in this case an "orbit" is a non-standard term, so we opt to use different terminology. If f is a partial permutation and $x \in \text{dom}(f) \cup \text{ran}(f)$, we define the COMPONENT of x under f to be the set

$$\{(x)f^k : k \in \mathbb{Z} \text{ and } x \in \text{dom}(f^k)\}.$$

A component of f is COMPLETE if $x \in \text{dom}(f^k)$ for every $k \in \mathbb{Z}$. A component that is not complete is INCOMPLETE.

1.3.2 Parameters associated with functions

Let $f \in \Omega^\Omega$ and let $\Sigma \subseteq \Omega$. If $f|_\Sigma$ is injective and $(\Sigma)f = (\Omega)f$, then Σ is a TRANSVERSAL OF f. For $f \in \Omega^\Omega$, define the following parameters

$$\begin{aligned} c(f) &= |\Omega \setminus \Sigma|, \text{ where } \Sigma \text{ is any transversal of } f, \\ d(f) &= |\Omega \setminus (\Omega)f|, \\ k(f) &= |\{x \in \Omega : |\{y \in \Omega : (y)f = x\}| = |\Omega|\}. \end{aligned}$$

The parameters $d(f), c(f)$, and $k(f)$ were termed the DEFECT, COLLAPSE, and INFINITE CONTRACTION INDEX, respectively, of f in [31]. Define the KERNEL OF $f \in \Omega^\Omega$ to be

$$\ker(f) = \{(x, y) \in \Omega \times \Omega : (x)f = (y)f\}.$$

Proposition 1.3.3 (Section 2 in [31]). *The parameter c is well-defined.*

Proof. Let $f \in \Omega^\Omega$, and let Σ be a transversal of f. Then

$$\Omega \setminus \Sigma = \bigcup_{y \in (\Omega)f} \left(\{x \in \Omega : (x)f = y\} \setminus \Sigma\right).$$

For each each $y \in (\Omega)f$ there is exactly one element x in Σ such that $(x)f = y$. Hence $|\left(\{x \in \Omega : (x)f = y\} \setminus \Sigma\right)| = |\{x \in \Omega : (x)f = y\}| - 1$, and so

$$c(f) = |\Omega \setminus \Sigma| = \sum_{y \in (\Omega)f} \left(|\{x \in \Omega : (x)f = y\}| - 1\right).$$

The right hand side of the equation is independent of the transversal, and so $c(f)$ is independent of the choice of transversal. □

The parameters c and d quantify how far away a function $f \in \Omega^\Omega$ is from being injective and surjective respectively. That is, f is injective if and only if $c(f) = 0$, and f is surjective if and only if $d(f) = 0$. In the next proposition, we will show that all combinations of values of c and d are possible.

Proposition 1.3.4. *Let Ω be infinite and let λ and μ be cardinals such that $\lambda, \mu \leq |\Omega|$. Then there is $f \in \Omega^\Omega$ such that $c(f) = \lambda$ and $d(f) = \mu$.*

Proof. Since Ω is infinite there are sets $\Sigma, \Gamma \subseteq \Omega$ such that $|\Sigma| = |\Gamma| = |\Omega|$, $|\Omega \setminus \Sigma| = \lambda$, and $|\Omega \setminus \Gamma| = \mu$. Let $y \in \Gamma$, and let $f \in \Omega^\Omega$ be any function such that f bijectively maps Σ to Γ and $(x)f = y$ for all $x \in \Omega \setminus \Sigma$. Then Σ is a transversal of f and the image of f is Γ. From the choice of sets Σ and Γ, it follows that $c(f) = \lambda$ and $d(f) = \mu$. □

In the next three lemmas we show how the parameters c and d interact with composition of functions. The following result is proved in [17] and part (i) was earlier proved in [31]. We include the proofs for the sake of completeness.

Lemma 1.3.5 (Lemma 5.4 in [17])**.** *Let $f, g \in \Omega^\Omega$, and let μ be an infinite cardinal such that $\mu \leq |\Omega|$. Then the following are true*

(i) $d(g) \leq d(f \circ g) \leq d(f) + d(g)$;

(ii) *if* $c(g) = 0$, *namely* g *is an injection, then* $d(f \circ g) = d(f) + d(g)$;

(iii) $c(f) \leq c(f \circ g) \leq c(f) + c(g)$;

(iv) *if* $d(f) = 0$, *namely* f *is a surjection, then* $c(f \circ g) = c(f) + c(g)$;

(v) *if* $c(g) < \mu \leq d(f)$, *then* $d(f \circ g) \geq \mu$;

(vi) *if* $d(f) < \mu \leq c(g)$, *then* $c(f \circ g) \geq \mu$.

Proof. **(i).** Since $(\Omega)f \subseteq \Omega$, it follows that $(\Omega)g \supseteq (\Omega)f \circ g$, and so $\Omega \setminus (\Omega)g \subseteq \Omega \setminus (\Omega)f \circ g$. Hence $d(g) \leq d(f \circ g)$.

If $x \in (\Omega)g \setminus (\Omega)f \circ g$, then there is $y \in \Omega \setminus (\Omega)f$ such that $(y)g = x$, and so $x \in (\Omega \setminus (\Omega)f)g$. Hence

$$(\Omega)g \setminus (\Omega)f \circ g \subseteq (\Omega \setminus (\Omega)f)\, g. \tag{1.2}$$

It then follows from

$$\Omega \setminus (\Omega)f \circ g = (\Omega \setminus (\Omega)g) \cup ((\Omega)g \setminus (\Omega)f \circ g) \tag{1.3}$$

that $\Omega \setminus (\Omega)f \circ g \subseteq (\Omega \setminus (\Omega)g) \cup (\Omega \setminus (\Omega)f)\, g$. By the definition of a function $|(X)g| \leq |X|$ for any subset X of the domain of f, and so $d(f \circ g) \leq d(f) + d(g)$.

(ii). If $x \in (\Omega \setminus (\Omega)f)g$, then there is $y \in \Omega \setminus (\Omega)f$ such that $x = (y)g$. Since g is injective, $x \neq (z)g$ for every $z \in (\Omega)f$, and so $x \in (\Omega)g \setminus (\Omega)f \circ g$. It then follows from (1.2) that

$$(\Omega)g \setminus (\Omega)f \circ g = (\Omega \setminus (\Omega)f)\, g.$$

Together with (1.3) it implies that

$$\Omega \setminus (\Omega)f \circ g = (\Omega \setminus (\Omega)f)\, g \cup (\Omega \setminus (\Omega)g)\, .$$

Moreover, the sets in the above displayed equation are disjoint. Finally, since g is injective $|\,(\Omega \setminus (\Omega)f)\, g| = |\Omega \setminus (\Omega)f|$, and so $d(f \circ g) = d(f) + d(g)$.

(iii). Let $T_f \subseteq \Omega$ be any transversal for f. Then by the definition of a transversal $(T_f)f \circ g = (\Omega)f \circ g$, and so there is a transversal $T_{f\circ g}$ of $f \circ g$ such that $T_{f\circ g} \subseteq T_f$. Hence $c(f) \leq c(f \circ g)$, and also $\Omega \setminus T_{f\circ g}$ is a disjoint union of

$\Omega \setminus T_f$ and $T_f \setminus T_{f\circ g}$. Since f is injective on T_f, $|T_f \setminus T_{f\circ g}| = |(T_f \setminus T_{f\circ g})f|$, and so

$$|\Omega \setminus T_{f\circ g}| = |\Omega \setminus T_f| + |\,(T_f \setminus T_{f\circ g})\, f|. \tag{1.4}$$

Since $f\circ g$ is injective on $T_{f\circ g}$, it follows that g is injective on $(T_{f\circ g})f$, and so there is $T_g \subseteq \Omega$ a transversal of g such that $(T_{f\circ g})f \subseteq T_g$. If $x \in T_f \setminus T_{f\circ g}$, then there is $y \in T_{f\circ g} \subseteq T_f$ such that $(x)f\circ g = (y)f\circ g$. Since x and y are both elements of a transversal of f, it follows that $(x)f \neq (y)f$. It also follows from $(T_{f\circ g})f \subseteq T_g$ that $(y)f \in T_g$, and so $(x)f \in \Omega \setminus T_g$. Hence

$$(T_f \setminus T_{f\circ g})\, f \subseteq \Omega \setminus T_g. \tag{1.5}$$

It follows from (1.4) that

$$|\Omega \setminus T_{f\circ g}| \leq |\Omega \setminus T_f| + |\Omega \setminus T_g|,$$

which is the same as $c(f\circ g) \leq c(f) + c(g)$.

(iv). Let T_f, T_g, and $T_{f\circ g}$ be transversals of f, g, and $f\circ g$ respectively, as in part (iii). To be more precise, $T_{f\circ g} \subseteq T_f$ and $(T_{f\circ g})g \subseteq T_g$. Let $x \in \Omega \setminus T_g$. Since f is surjective there is $y \in T_f$ so that $(y)f = x$. Recall that $(T_{f\circ g})f \subseteq T_g$, and so if $y \in T_{f\circ g}$, then $x = (y)f \in T_g$, contradicting the choice of x. Hence $y \in T_f \setminus T_{f\circ g}$, which together with (1.5) implies that

$$(T_f \setminus T_{f\circ g})\, f = \Omega \setminus T_g.$$

Then $c(f\circ g) = c(f) + c(g)$ by (1.4).

(v). Let T_g be a transversal of g. Then by the hypothesis $|\Omega \setminus T_g| = c(g) < \mu$ and $|\Omega \setminus (\Omega)\, f| = d(f) \geq \mu$. Hence

$$|T_g \cap (\Omega \setminus (\Omega)\, f)\,| = |\,(\Omega \setminus (\Omega)\, f) \setminus (\Omega \setminus T_g)\,| \geq \mu. \tag{1.6}$$

If $x \in T_g \cap (\Omega \setminus (\Omega)\, f)$ is such that $(x)g \in (\Omega)\, f\circ g$, then there exists $y_x \in (\Omega)\, f$ such that $(x)g = (y_x)g$, and so $y_x \in \Omega \setminus T_g$. Let ϕ be a mapping such that $(x)\phi = y_x$ for every $x \in T_g \cap (\Omega \setminus (\Omega)\, f)$ such that $(x)g \in (\Omega)\, f\circ g$. Then the image of ϕ is contained in $\Omega \setminus T_g$. If $y_x = (x)\phi = (x')\phi = y_{x'}$ for some x and x', then $(x)g = (y_x)g = (y_{x'})g = (x')g$. Since $x, x' \in T_g$, it follows that $x = x'$ and so ϕ is injective. Hence

$$|\{x \in T_g \cap (\Omega \setminus (\Omega)\, f) : (x)g \in (\Omega)\, f\circ g\}| \leq |\Omega \setminus T_g| < \mu.$$

Then $|\{x \in T_g \cap (\Omega \setminus (\Omega)\, f) : (x)g \notin (\Omega)\, f\circ g\}| \geq \mu$, by (1.6). Since g acts injectively on T_g

$$|\,(\{x \in T_g \cap (\Omega \setminus (\Omega)\, f) : (x)g \notin (\Omega)\, f\circ g\})\, g| \geq \mu,$$

and thus

$$|\Omega \setminus (\Omega)\, f \circ g| \geq |\, (\{x \in T_g \cap (\Omega \setminus (\Omega)\, f) : (x)g \notin (\Omega)\, f \circ g\})\, g| \geq \mu,$$

which is the same as $d(f \circ g) \geq \mu$.

(vi). Let T_f, T_g, and $T_{f\circ g}$ be transversals of f, g, and $f \circ g$ respectively, as in part (iii). That is $T_{f\circ g} \subseteq T_f$ and $(T_{f\circ g})g \subseteq T_g$. By the hypothesis $|\Omega \setminus (T_f)\, f| < \mu$ and $|\Omega \setminus T_g| \geq \mu$. Hence

$$|\, (\Omega \setminus T_g) \cap (T_f)f| = |\, (\Omega \setminus T_g) \setminus (\Omega \setminus (T_f)f)\, | \geq \mu.$$

Since transversal T_g was chosen so that $(T_{f\circ g})f \subseteq T_g$, it follows from $(T_f \setminus T_{f\circ g})f \supseteq (T_f)f \setminus (T_{f\circ g})f$, that $|(T_f \setminus T_{f\circ g})f| \geq |(T_f)f \setminus T_g|$. Then $(T_f)f \setminus T_g = (\Omega \setminus T_g) \cap (T_f)f$ implies that

$$|(T_f \setminus T_{f\circ g})f| \geq \mu.$$

Finally, it follows from the fact that f act injectively on T_f

$$|\Omega \setminus T_{f\circ g}| \geq |\, (T_f \setminus T_{f\circ g})\, f| \geq \mu$$

as required. □

The following technical lemma can be used to show that parameters c and d of a function are finite.

Lemma 1.3.6. *Let $f, g, h \in \Omega^\Omega$ be such that $f \in \langle g, h\rangle$ and $c(f)$, $d(f)$, $c(g)$, and $d(g)$ are all finite. Then $c(h)$ and $d(h)$ are both finite or $f = g^n$ for some $n \in \mathbb{N}$.*

Proof. Suppose that $f \notin \langle g\rangle$. We will show by induction on the length of the product that $c(h)$ and $d(h)$ are finite.

First of all, suppose that $f = g \circ h$. If $d(h)$ is infinite then so is $d(f)$ by Lemma 1.3.5(i), and if $c(h)$ is infinite, then $c(f)$ is also infinite by Lemma 1.3.5(vi). Hence both $c(h)$ and $d(h)$ must be finite. Similarly, if $f = h \circ g$ parts (iii) and (v) of Lemma 1.3.5 shows that $c(h)$ and $d(h)$ are finite, and if $f = h^2$ parts (i) and (iii) of Lemma 1.3.5 can be used to show the same result.

For $n \geq 2$, suppose that if $f \in \langle g, h\rangle$ is a product of length at most n with $c(f)$, $d(f)$, $c(g)$, $d(g)$ finite, and $f \neq g^m$ for any $m \in \mathbb{N}$, then $c(h)$ and $d(h)$ are both finite. Suppose that $f \in \langle g, h\rangle$ is a product of length $n+1$, $f \neq g^m$ for any $m \in \mathbb{N}$, and as before the parameters c and d of f and g are all finite. If $f = h \circ f' \circ h$ for some $f' \in \langle g, h\rangle$, then both $c(h)$ and $d(h)$ must be finite by parts (i) and (iii) of Lemma 1.3.5. Otherwise, there is $f' \in \langle g, h\rangle$ a product of length n such that $f = f' \circ g$ or $f = g \circ f'$. Since the parameters c and d of f

and g are finite, it follows that $c(f')$ and $d(f')$ are finite, by the discussion in the previous paragraph. It follows from $f \neq g^m$ for all $m \in \mathbb{N}$, that $f' \neq g^m$ for any $m \in \mathbb{N}$, and so the inductive hypothesis implies that $c(h)$ and $d(h)$ are both finite. Therefore, by induction, the conclusion holds for all $f \in \langle g, h \rangle$. □

The final of the technical lemmas gives us a more precise relation between parameters c and d of the composition of functions and its constituent parts.

Lemma 1.3.7. *Let $f, g \in \Omega^\Omega$ be such that the parameters c and d of f and g are all finite. Then $c(f \circ g) + d(f) + d(g) = d(f \circ g) + c(f) + c(g)$.*

Proof. Let T_f, T_g, and $T_{f\circ g}$ be transversals of f, g, and $f \circ g$ respectively. It was shown in the proof of Lemma 1.3.5(iii) that these transversals can be chosen so that $T_{f\circ g} \subseteq T_f$ and $(T_{f\circ g})f \subseteq T_g$.

First note that $(T_{f\circ g})f \subseteq T_g \cap (\Omega)f$. Let $x \in T_g \cap (\Omega)f$. Then there is $y \in T_f$ such that $x = (y)f$. Choose $z \in T_{f\circ g}$ such that $(z)f \circ g = (y)f \circ g$. Since both $(z)f$ and $x = (y)f$ are elements of T_g, $(z)f = (y)f$. It then follows from the fact that y and z are in T_f that $y = z$. Therefore $y \in T_{f\circ g}$, and so $(T_{f\circ g})f = T_g \cap (\Omega)f$. Also $(T_f \setminus T_{f\circ g})f = (\Omega)f \setminus (T_{f\circ g})f$ as f is injective on T_f, and so

$$|T_f \setminus T_{f\circ g}| = |(\Omega)f \setminus T_g|.$$

It follows from the fact that $\Omega \setminus T_{f\circ g}$ is a disjoint union of $\Omega \setminus T_f$ and $T_f \setminus T_{f\circ g}$ that

$$|\Omega \setminus T_{f\circ g}| = |\Omega \setminus T_f| + |(\Omega)f \setminus T_g|. \tag{1.7}$$

Since g act injectively on T_g, it follows that $(T_g \setminus (\Omega)f)g = (\Omega)g \setminus (\Omega)f \circ g$, and so

$$|T_g \setminus (\Omega)f| = |(\Omega)g \setminus (\Omega)f \circ g|.$$

Note that $(\Omega)f \circ g \subseteq (\Omega)g$ implying that $\Omega \setminus (\Omega)f \circ g$ is a disjoint union of $\Omega \setminus (\Omega)g$ and $(\Omega)g \setminus (\Omega)f \circ g$. Then

$$|\Omega \setminus (\Omega)\, f \circ g| = |\Omega \setminus (\Omega)g| + |T_g \setminus (\Omega)\, f|. \tag{1.8}$$

Since Ω is a disjoint union of T_g and $\Omega \setminus T_g$ and $(\Omega \setminus (\Omega)f) \cap (\Omega \setminus T_g) = \Omega \setminus (T_g \cup (\Omega)f)$, it follows that $|T_g \setminus (\Omega)\, f| + |\Omega \setminus (T_g \cup (\Omega)\, f)\,| = |\Omega \setminus (\Omega)\, f|$. Then (1.8) implies that

$$|\Omega \setminus (\Omega)\, f \circ g| + |\Omega \setminus (T_g \cup (\Omega)\, f)\,| = |\Omega \setminus (\Omega)g| + |\Omega \setminus (\Omega)\, f|. \tag{1.9}$$

In the same way, since Ω is a disjoint union of $(\Omega)f$ and $\Omega \setminus (\Omega)f$, it follows that $|\,(\Omega)\, f \setminus T_g| + |\Omega \setminus (T_g \cup (\Omega)\, f)\,| = |\Omega \setminus T_g|$, and so by (1.7).

$$|\Omega \setminus T_{f\circ g}| + |\Omega \setminus (T_g \cup (\Omega)\, f)\,| = |\Omega \setminus T_f| + |\Omega \setminus T_g|. \tag{1.10}$$

By parts (i) and (iii) of Lemma 1.3.5, $d(f \circ g)$ and $c(f \circ g)$ are finite implying that $\Omega \setminus (T_g \cup (\Omega)f)$ is also finite. Hence by combining (1.9) and (1.10) we obtain $c(f \circ g) + d(f) + d(g) = d(f \circ g) + c(f) + c(g)$, as required. □

Lemma 1.3.7 can be used to define an interesting subsemigroup of Ω^Ω.

Example 1.3.8. Let $n \in \mathbb{N}$ be arbitrary, and let

$$S = \{f \in \Omega^\Omega : c(f),\ d(f) \text{ are finite and } n \text{ divides } c(f) - d(f)\}.$$

Suppose that $f, g \in S$. Then $c(f \circ g)$ and $d(f \circ g)$ are finite by parts (i) and (iii) of Lemma 1.3.5. It follows from Lemma 1.3.7 that

$$c(f \circ g) - d(f \circ g) = c(f) - d(f) + c(g) - d(g).$$

Hence n divides $c(f \circ g) - d(f \circ g)$, and so $f \circ g \in S$. Therefore, S is a subsemigroup of Ω^Ω.

Given a set Ω, of particular interest to us is the set of all bijective functions from Ω back to Ω. Recall that $f \in \Omega^\Omega$ is bijective if and only if $c(f) = 0$ and $d(f) = 0$. Then by parts (i) and (iii) of Lemma 1.3.5 the composition of two bijective functions is bijective. It is then easy to show that the set of all bijective functions forms a group, which is known as the SYMMETRIC GROUP ON Ω and we denote it by $\operatorname{Sym}(\Omega)$.

Proposition 1.3.9. *If Ω is infinite, then the cardinality of* $\operatorname{Sym}(\Omega)$ *is* $2^{|\Omega|}$.

Proof. Since $\operatorname{Sym}(\Omega) \subseteq \Omega^\Omega$ and the cardinality of Ω^Ω is $2^{|\Omega|}$ by Proposition 1.1.6, it follows that $|\operatorname{Sym}(\Omega)| \leq 2^{|\Omega|}$. Since Ω is infinite, $\Omega \times \Omega$ has the same cardinality as Ω, and so $\operatorname{Sym}(\Omega)$ and $\operatorname{Sym}(\Omega \times \Omega)$ also have the same cardinality. Then by Theorem 1.1.2 it is sufficient to find an injection from Ω^Ω to $\operatorname{Sym}(\Omega \times \Omega)$.

Suppose that $f \in \Omega^\Omega$. Then define a function g_f to be such that

$$(x, x)g_f = ((x)f, x), \quad ((x)f, x)g_f = (x, x),$$

and it acts as identity everywhere else. Then $g_f \in \operatorname{Sym}(\Omega \times \Omega)$. If $f \neq f'$, then there is $x \in \Omega$ such that $(x)f \neq (x)f'$, and so $(x, x)g_f = ((x)f, x) \neq ((x)f', x) = (x, x)g_{f'}$. Hence a function mapping f to g_f is injective, as required. □

1.4 Topology and Baire category theory

In this section we will introduce the notions related to topology and metric spaces which will be of use in the later chapters.

1.4.1 Topological spaces

Let X be an arbitrary set, and let τ be a set of subsets of X. Then the pair (X, τ) is a TOPOLOGICAL SPACE if the following conditions are satisfied:

1) $\varnothing, X \in \tau$;

2) $\bigcup_{i \in I} X_i \in \tau$ for any non-empty set I such that $X_i \in \tau$ for all $i \in I$;

3) $\bigcap_{i=1}^{n} X_i$ for any $n \in \mathbb{N}$ and $X_i \in \tau$ for all $i \in \{1, \ldots, n\}$.

Then the set τ is called a TOPOLOGY on X, and its members are referred to as OPEN SETS. A set $F \subseteq X$ is CLOSED if it is a complement of an open set. Also a subset Y of X is a G_δ set if it is an intersection of countably many open subsets of X.

If $Y \subseteq X$, let $\sigma = \{U \cap Y : U \in \tau\}$. Then $\varnothing = \varnothing \cap Y$, and $Y = X \cap Y \in \sigma$. Suppose that $Y_i \in \sigma$ for some set I and all $i \in I$. Then by definition of σ there are $X_i \in \tau$ such that $Y_i = X_i \cap Y$ for all $i \in I$, and since $\bigcup_{i \in I} X_i \in \tau$

$$\bigcup_{i \in I} Y_i = \bigcup_{i \in I} X_i \cap Y = Y \cap \bigcup_{i \in I} X_i \in \sigma.$$

Finally, let $n \in \mathbb{N}$ and for $i \in \{1, \ldots, n\}$, and let $Y_i \in \sigma$. Then as before $Y_i = X_i \cap Y$ for some $X_i \in \tau$ and all $i \in \{1, \ldots, n\}$, and since $\bigcap_{i=1}^{n} X_i \in \tau$

$$\bigcap_{i=1}^{n} Y_i = \bigcap_{i=1}^{n} X_i \cap Y = Y \cap \bigcup_{i=1}^{n} X_i \in \sigma.$$

Hence (Y, σ) is a topological space, called a SUBSPACE OF (X, τ), and σ is the SUBSPACE TOPOLOGY ON Y.

A BASIS $\mathcal{B}$ for a topological space (X, τ) is a subset of τ such that every element of τ can be expressed as a union of elements of $\mathcal{B}$.

Example 1.4.1. Let X be a set, and let τ be the set of all subsets of X. Then $\varnothing, X \in \tau$, and if $\{X_i : i \in I\}$ is any collection of subsets of X, then

$$\bigcup_{i \in I} X_i, \bigcap_{i \in I} X_i \in \tau.$$

Hence (X, τ) is a topological space. The topology τ is known as DISCRETE TOPOLOGY. It follows from the definition that $\{\{x\} : x \in X\}$ is a basis for (X, τ).

The following result provides a necessary and sufficient condition under which a collection of subsets is a basis for some topology.

Proposition 1.4.2 (see Section 13 in [56])**.** *Let X be a set, and let $\mathcal{B}$ be a set of subsets of X. Then $\mathcal{B}$ is a basis if only if for every $x \in X$ there is $B \in \mathcal{B}$*

with $x \in B$, and for any $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$ there is $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

Proof. ($\Rightarrow$) Since $\mathcal{B}$ is a basis there is a non-empty set I and open sets $B_i \in \mathcal{B}$ for all $i \in I$ such that $X = \bigcup_{i \in I} B_i$. Hence for each $x \in X$ there is some $i \in I$ such that $x \in B_i$. Let $B_1, B_2 \in \mathcal{B}$ be arbitrary. Since B_1 and B_2 are both open sets, $B_1 \cap B_2$ is also open, and so $B_1 \cap B_2 = \bigcup_{i \in I} C_i$ for some set I where $C_i \in \mathcal{B}$ for all $i \in I$. Hence for all $x \in B_1 \cap B_2$ there is $i \in I$ such that $x \in C_i \subseteq B_1 \cap B_2$.

($\Leftarrow$) Let τ be the set of all arbitrary unions of elements of $\mathcal{B}$, including the empty set. We will show that (X, τ) is a topological space. By definition of $\varnothing \in \tau$, and since for all $x \in X$ there is $B \in \mathcal{B}$ such that $x \in B$, it follows that

$$X = \bigcup_{B \in \mathcal{B}} B \in \tau.$$

Let I be a non-empty set, and let $U_i \in \tau$ for all $i \in I$. For each $i \in I$ there is J_i and $B_{i,j} \in \mathcal{B}$ for each $j \in J_i$ such that $U_i = \bigcup_{j \in J_i} B_{i,j}$. Then

$$\bigcup_{i \in I} U_i = \bigcup_{i \in I} \bigcup_{j \in J_i} B_{i,j} \in \tau.$$

Note that if $B_1, B_2 \in \mathcal{B}$, then for each $x \in B_1 \cap B_2$ there is $B_x \in \mathcal{B}$ such that $x \in B_x \subseteq B_1 \cap B_2$. Hence $B_1 \cap B_2 = \bigcup_{x \in B_1 \cap B_2} B_x \in \tau$. Suppose that $U = \bigcup_{i \in I} B_i \in \tau$. Let $B \in \mathcal{B}$. Since $B \cap B_i \in \tau$ for each $i \in I$, it follows from the fact that τ is closed with respect to arbitrary unions that

$$B \cap U = B \cap \bigcup_{i \in I} B_i = \bigcup_{i \in I} B \cap B_i \in \tau$$

If $U' \in \tau$, then $U' \cap B_i \in \tau$ by the above for all $i \in I$. Hence

$$U' \cap U = U' \cap \bigcup_{i \in I} B_i = \bigcup_{i \in I} U' \cap B_i \in \tau. \tag{1.11}$$

Finally, let $U_1, \ldots, U_n \in \tau$. Suppose that $\bigcap_{i=1}^{k} U_i \in \tau$ for some $i \in \{1, \ldots, n-1\}$. Then $\bigcap_{i=1}^{k+1} U_i \in \tau$ by (1.11). Hence by induction, $\bigcap_{i=1}^{n} U_i \in \tau$, and so τ is a topology for X. □

A subset A of a topological space (X, τ) is DENSE if for every open set $U \in \tau$ the intersection $A \cap U$ is non-empty.

Let $\{(X_i, \tau_i) : i \in I\}$ be a collection of topological spaces. Define

$$X = \prod_{i \in I} X_i = \{f : I \longrightarrow \bigcup_{i \in I} X_i : (i)f \in X_i\},$$

and let $\mathcal{B}$ be the collection of all sets of the form $\prod_{i \in I} U_i$ where $U_i \in \tau_i$ for all

$i \in I$ and $U_i = X_i$ for all but finitely many $i \in I$. Then $X \in \mathcal{B}$ and for any $B_1, B_2 \in \mathcal{B}$ it follows that $B_1 \cap B_2 \in \mathcal{B}$. Hence $\mathcal{B}$ is a basis for some topology τ on X by Proposition 1.4.2, we refer to (X, τ) as the PRODUCT TOPOLOGICAL SPACE. Note that every element of $\mathcal{B}$ be expressed as a union of elements of the form $\prod_{i \in I} B_i$ where B_i is a basic open set of X_i and $B_i = X_i$ for all but finitely many $i \in I$. Hence in the definition of $\mathcal{B}$ it is sufficient to only consider products of basic open sets.

Example 1.4.3. If Ω is an infinite set, then TRANSFORMATION MONOID Ω^Ω is the set of all functions from Ω back to Ω. For each $i \in \Omega$, let $X_i = \Omega$ and equip it with the discrete topology τ_i, as in Example 1.4.1. Consider the product topology of the collection $\{(X_i, \tau_i) : i \in \Omega\}$. The underlying set is

$$\prod_{i \in \Omega} X_i = \{f : \Omega \longrightarrow \Omega\} = \Omega^\Omega$$

and since $\{\{x\} : x \in \Omega\}$ is a basis for (X_i, τ_i), the basic open sets for the product topology are of the form $\prod_{i \in \Omega} U_i$ where there is a finite set $A \subseteq \Omega$ such that $U_i = \{x_i\}$ for some $x_i \in X_i = \Omega$ and all $i \in A$, and $U_i = X_i$ for all $i \in \Omega \setminus A$. Note that if we define $\phi : A \longrightarrow \Omega$ to be $(i)\phi = x_i$, then

$$[\phi] = \{f \in \Omega^\Omega : f|_A = \phi\} = \prod_{i \in \Omega} U_i.$$

Hence $\{[\phi] : \phi : A \longrightarrow \Omega \text{ for some finite } A \subseteq \Omega\}$ forms a basis of Ω^Ω.

Suppose that (X, τ) and (Y, σ) are two topological spaces, and let $f : X \longrightarrow Y$. Then f is a CONTINUOUS FUNCTION if for every open set $U \in \sigma$, the set $(U)f^{-1} = \{x \in X : (x)f \in U\}$ is an open subset of X. If $\mathcal{B}$ is a basis for (Y, σ), then every $U \in \sigma$ can be expressed as $U = \bigcup_{i \in I} B_i$ for some I and $B_i \in \mathcal{B}$ for all $i \in I$. Then

$$(U)f^{-1} = \bigcup_{i \in I} (B_i)f^{-1},$$

and so for f to be continuous it is sufficient that $(B)f^{-1}$ is open for every open basic set of (Y, σ). Since $\mathcal{B} \subseteq \sigma$ it is also a necessary condition.

A function $f : X \longrightarrow Y$ is a HOMEOMORPHISM if it is a continuous bijection such that the inverse $f^{-1} : Y \longrightarrow X$ is also continuous. It follows from the definition of continuity that f^{-1} is continuous if and only if for every open subset U of X, the set $(U)f$ is open in Y.

1.4.2 Metric spaces

Let X be a set, and let $d : X \times X \longrightarrow [0, \infty)$. Then (X, d) is a METRIC SPACE, with a METRIC d, if the following are satisfied:

1) $d(x, y) = 0$ if and only if $x = y$;

2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

We say that a topological space (X, τ) is METRIZABLE if there is a metric d such that the sets

$$B(x, r) = \{y \in X : d(x, y) < r\}$$

where $x \in X$ and $r \geq 0$, form a basis for the topology τ. The sets $B(x, r)$ are called OPEN BALLS.

Consider the converse problem, that is start with a metric space (X, d). Then it can be verified, using Proposition 1.4.2, that the set of all open balls on X forms a basis for some topology on X. Hence X is also a topological space with open sets being arbitrary unions of open balls, which we will refer to as TOPOLOGY INDUCED BY THE METRIC.

A sequence $(x_n)_{n\in\mathbb{N}}$ of elements of a metric space (X, d) is CONVERGENT if there is $x \in X$ so that for every $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that $n \geq N$ implies that $d(x_n, x) < \varepsilon$. A more general definition of convergence can be given for topological spaces. A sequence $(x_n)_{n\in\mathbb{N}}$ of elements of a topological space converge to an element x, if for every open set U containing x there is $N \in \mathbb{N}$ such that $x_n \in U$ for all $n \geq N$. Since $y \in B(x, \varepsilon)$ is equivalent to $d(x, y) < \varepsilon$, it follows that if we consider a metric space as a topological space with the topology induced by the metric, the two definitions of convergence are equivalent.

A sequence $(x_n)_{n\in\mathbb{N}}$ is a CAUCHY SEQUENCE if for every $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that $d(x_n, x_m) \leq \varepsilon$ for all $n, m \geq N$. We say that a metric space is COMPLETE if every Cauchy sequence is convergent. A topological space is a POLISH SPACE if it is metrizable with a complete metric and it has a countable dense subset.

The following well known result shows that if a sequence is a subset of a closed set, then it can only converge to a point within the closed set.

Proposition 1.4.4. *Let (X, d) be a metric space, let F be a closed subset of X, and let $(x_n)_{n\in\mathbb{N}} \subseteq F$ be a sequence converging to a point $x \in X$. Then $x \in F$.*

The following proposition allows us to obtain Polish spaces from other Polish spaces.

Proposition 1.4.5 (Proposition 3.3 in [41])**.**

(i) *A closed subspace of a Polish space is a Polish space.*

(ii) *The product topological space of a countable sequence of Polish spaces is a Polish space.*

Proof. **(i)** Let X be a Polish space, let Y be a closed subset of X, and let d be a complete metric on X which induces the topology on X. Define d' to be the restriction of d onto $Y \times Y$. Then d' is a metric on Y which induced the subspace topology on Y. Let $(y_n)_{n\in\mathbb{N}}$ be a Cauchy sequence of elements in Y. Since $Y \subseteq X$, it follows that the sequence $(y_n)_{n\in\mathbb{N}}$ converges to some $y \in X$. However, $y \in Y$ by Proposition 1.4.4 and the fact that Y is closed subset of X. Hence Y is completely metrizable.

Finally, if D is a countable dense subset of X, then every open subset U of X can be written as a union of subsets $B(d, q)$ where $d \in D$ and $q \in \mathbb{Q}$. Note that the collection of such $B(d, q)$ is countable. Let D' be a countable set such that for any $d \in D$ and $q \in \mathbb{Q}$ if the set $B(d, q) \cap Y$ is non-empty, then D' intersects $B(d, q) \cap Y$ non-trivially. Let U be an open subset of Y. Then by the definition of the subspace topology there is an open subset V of X such that $U = V \cap Y$. Since U is a union of elements of the form $B(d, q) \cap Y$, the set U intersect D' non-trivially. Hence Y has a countable dense subset, and so is Polish.

(ii) Let $\{X_n : n \in \mathbb{N}\}$ be a sequence of Polish spaces, and suppose that d_n is a complete metric on X_n inducing the topology on X_n for all $n \in \mathbb{N}$. Recall that elements of $X = \prod_{n\in\mathbb{N}} X_n$ are of the form $(x_0, x_1, \ldots)$ where $x_n \in X_n$ for all $n \in \mathbb{N}$. Define $d' : X \times X \longrightarrow [0, \infty)$ by

$$d'((x_0, x_1, \ldots), (y_0, y_1, \ldots)) = \sum_{n\in\mathbb{N}} 2^{-n} \frac{d_n(x_n, y_n)}{1 + d_n(x_n, y_n)}.$$

Since d_n is a metric on X_n for all $n \in \mathbb{N}$, it follows that $d(\mathbf{x}, \mathbf{y}) = 0$ if and only if $\mathbf{x} = \mathbf{y}$ and $d(\mathbf{x}, \mathbf{y}) = d(\mathbf{y}, \mathbf{x})$ for all $\mathbf{x}, \mathbf{y} \in X$. If $x, y, z \geq 0$ and $x + y \geq z$, by simply multiplying it out, we see that the following inequality holds

$$\frac{x}{1 + x} + \frac{y}{1 + y} \geq \frac{z}{1 + z}.$$

It follows from the inequality that $d(\mathbf{x}, \mathbf{z}) \leq d(\mathbf{x}, \mathbf{y}) + d(\mathbf{y}, \mathbf{z})$ for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in X$, and so d' is a metric on X.

Suppose that $(\mathbf{x}_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in X, and let $\mathbf{x}_n = (x_{n,0}, x_{n,1}, \ldots)$ where $x_{n,m} \in X_m$ for all $n, m \in \mathbb{N}$. It follows that for each $n \in \mathbb{N}$ the sequence $(x_{n,m})_{m\in\mathbb{N}}$ is Cauchy in X_n, and so it converges to some $x_n \in X_n$. Let $\mathbf{x} = (x_0, x_1, \ldots)$.

Let $\varepsilon > 0$. Then there is some $K \in \mathbb{N}$ such that $\sum_{n\geq K} 2^{-n} \leq \varepsilon/2$. Since $(x_{n,m})_{m\in\mathbb{N}}$ converges to x_n for all $n \in \mathbb{N}$, there is $N \in \mathbb{N}$ such that for all $n < K$ and all $m \geq N$, $d_n(x_{n,m}, x_n) < \varepsilon/(2K)$. Then for $m \geq N$

$$d'(\mathbf{x}_n, \mathbf{x}) \leq \sum_{n=0}^{K-1} 2^{-n} \frac{d_n(x_n, y_n)}{1 + d_n(x_n, y_n)} + \frac{\varepsilon}{2} \leq \sum_{n=0}^{K-1} d_n(x_n, y_n) + \frac{\varepsilon}{2} < \varepsilon.$$

Hence d' is a complete metric.

Finally, let D_n be a countable dense subset of X_n, and let $\mathcal{A}$ be the collection of all sets of the form $\prod_{n\in\mathbb{N}} U_n$ such that there is a finite $A \subseteq \mathbb{N}$ so that $U_n = B(x, r)$ where $x \in D_n$ and $r \in \mathbb{Q}$ for all $n \in A$, and $U_n = X_n$ for all $n \in \mathbb{N} \setminus A$. Then $\mathcal{A}$ is countable. Let D be a countable set intersecting each element of $\mathcal{A}$ non-trivially. If $\prod_{n\in\mathbb{N}} U_n$ is a basic open set in the product topology, it follows from the definition that U_n is an open subset of X_n for all $n \in \mathbb{N}$ and is equal to X_n for all $n \in \mathbb{N} \setminus A$ for some finite A. For each $n \in A$, let $r_n \in \mathbb{Q}$ and $x_n \in D_n$ be such that $B(x_n, r_n) \subseteq U_n$, which is possible since D_n is dense in X_n. If $V_n = B(x_n, r_n)$ for all $n \in A$ and $V_n = X_n$ for all $n \in \mathbb{N} \setminus A$, then D intersects $\prod_{n\in\mathbb{N}} V_n \subseteq \prod_{n\in\mathbb{N}} U_n$ non-trivially. Therefore D is a countable dense subset, and so X is a Polish space. □

Example 1.4.6. Let Ω be a countable set, and let τ be the discrete topology on Ω. Then Ω itself is a countable dense subset of Ω. Let $d : \Omega \times \Omega \longrightarrow [0, \infty)$ be defined by

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}.$$

It can be verified that d is indeed a metric. Then the open ball $B(x, 1/2)$ is just the singleton $\{x\}$ for any $x \in \Omega$, and so the topology induced by d is discrete. Finally, note that a sequence $\{x_n : n \in \mathbb{N}\}$ with the metric d is Cauchy only if there is $K \in \mathbb{N}$ such that $x_n = x_m$ for all $n, m \geq K$, in which case the sequence converges to x_K. Therefore d is a complete metric, and so Ω is a Polish space with the discrete topology.

Since Ω is countable, it follows from Example 1.4.3 and Proposition 1.4.5(ii) that Ω^Ω is Polish space with the product topology.

It is interesting to note that every closed subset of a metrizable topological space must be G_δ.

Proposition 1.4.7 (Theorem 3.7 in [41])**.** *Let (X, τ) be a metrizable topological space. Then every closed subset of X can be expressed as a countable intersection of open sets.*

Proof. Let d be a metric on X inducing the topology τ. For $x \in X$, and a non-empty subset Y of X, define

$$d(x, Y) = \inf\{d(x, y) : y \in Y\}.$$

Let $x, y \in X$, let Y be non-empty subset of X, and let $\varepsilon > 0$ be arbitrary. Then $d(x, z) < d(x, Y) + \varepsilon$ for some $z \in Y$, and so

$$d(y, Y) \leq d(y, z) \leq d(x, y) + d(x, z) < d(x, y) + d(x, Y) + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, it then follows that $d(y, Y) - d(x, Y) \leq d(x, y)$. The same argument but with x and y swapped shows that

$$|d(x, Y) - d(y, Y)| \leq d(x, y).$$

If $d(x, Y) < \varepsilon$ and $y \in X$ is such that $d(x, y) < \varepsilon - d(x, Y)$, it then follows that $d(y, Y) < \varepsilon$, and so $x \in B(x, \varepsilon - d(x, Y)) \subseteq \{y \in X : d(y, Y) < \varepsilon\}$. Hence $\{x \in X : d(x, Y) < \varepsilon\}$ is open. Since for every non-empty closed set F

$$F = \bigcap_{n \in \mathbb{N}} \{x \in X : d(x, F) < 1/(n + 1)\},$$

it follows that F is a G_δ set. □

The next result together with Proposition 1.4.7 generalises Proposition 1.4.5(i). The proof can be found in [41].

Theorem 1.4.8 (Theorem 3.11 in [41])**.** *A subspace of a Polish space is Polish if and only if it is a G_δ subset of X.*

Example 1.4.9. Let Ω be a countable set. Let $\text{Inj}(\Omega)$ be the set of all injective functions from Ω to Ω, and let $\text{Surj}(\Omega)$ be the set of all surjective functions. It follows from $\text{Sym}(\Omega) = \text{Inj}(\Omega) \cap \text{Surj}(\Omega)$ that if both $\text{Inj}(\Omega)$ and $\text{Surj}(\Omega)$ are G_δ subsets of Ω^Ω, then so is $\text{Sym}(\Omega)$.

Let $x, y \in \Omega$ such that $x \neq y$. Suppose that $f \in \Omega^\Omega$ such that $(x)f \neq (y)f$. Then $f \in [f|_{\{x,y\}}] \subseteq \{g \in \Omega^\Omega : (x)g \neq (y)g\}$. Hence the set $\{g \in \Omega^\Omega : (x)g \neq (y)g\}$ is a union of basic open sets, and so an open set. Since $\mathcal{A} = \Omega^2 \setminus \{(x, x) : x \in \Omega\}$ is countable,

$$\text{Inj}(\Omega) = \bigcap_{(x,y) \in \mathcal{A}} \{g \in \Omega^\Omega : (x)g \neq (y)g\}$$

implies that $\text{Inj}(\Omega)$ is G_δ.

Let $x \in \Omega$ and suppose that $f \in \Omega^\Omega$ is such that there is $y \in \Omega$ with $(y)f = x$. Then

$$f \in [f|_{\{y\}}] \subseteq \{g \in \Omega^\Omega : \text{there exists } t \in \Omega \text{ such that } (t)g = x\},$$

and so the set $\{g \in \Omega^\Omega : \text{there exists } y \in \Omega \text{ such that } (y)f = x\}$ is a union of basic open sets, and hence open. Hence

$$\text{Surj}(\Omega) = \bigcap_{x \in \Omega} \{g \in \Omega^\Omega : \text{there exists } t \in \Omega \text{ such that } (t)g = x\}$$

implies that $\text{Surj}(\Omega)$ is a G_δ set.

Therefore, Sym(Ω) is a Polish space with the subspace topology by Theorem 1.4.8 and Example 1.4.6. Hence by Proposition 1.4.5(i) any closed subspace of Sym(Ω) is a Polish space.

1.4.3 Topological groups

Let G be a group. Then G is a TOPOLOGICAL GROUP if there is a topology on G so that multiplication, thought of as a function $\cdot : G \times G \longrightarrow G$, and inversion $^{-1} : G \longrightarrow G$ are both continuous. A topological group G is a POLISH GROUP if the topology makes G into a Polish space.

Proposition 1.4.10. *Let G be a topological group, and let $H \leq G$. Then H is a topological group with the subspace topology.*

Proof. Let $\cdot_G : G \times G \longrightarrow G$ denote the operation of G, and $\cdot_H : H \times H \longrightarrow H$ the operation of H. Since H is a subgroup of G, it follows that $\cdot_H$ is a restriction of $\cdot_G$ to $H \times H$. Let U be an open subset of H. Then there is V an open subset of G such that $U = V \cap H$. If P is the preimage of U under $\cdot_H$ and R is the preimage of V under $\cdot_G$, then $P = R \cap (H \times H)$. Since $\cdot_G$ is continuous, R is an open subset of $G \times G$, and so P is an open subset of $H \times H$ with subspace topology. Therefore, $\cdot_H$ is continuous.

Similarly, let $_G^{-1} : G \longrightarrow G$ and $_H^{-1} : H \longrightarrow H$ be inversion functions of G and H respectively. Then $_H^{-1}$ is a restriction of $_G^{-1}$ to $H \times H$. Let U be an open subset of H and V an open subset of G such that $U = V \cap H$. If P is a preimage of U under $_H^{-1}$ and R is a preimage of V under $_G^{-1}$, then $P = R \cap H$. Hence as in the previous paragraph, continuity of $_G^{-1}$ implies continuity of $_H^{-1}$. Therefore, H is a topological group. □

Theorem 1.4.8 and Proposition 1.4.10 can be used to show that subgroup of a Polish group is a Polish group if and only if it is a G_δ subset.

Example 1.4.11. Consider Sym(Ω) with subspace topology of Ω^Ω as in Example 1.4.9. Denote by $\circ : \mathrm{Sym}(\Omega) \times \mathrm{Sym}(\Omega) \longrightarrow \mathrm{Sym}(\Omega)$ the operation of Sym(Ω), and by $^{-1} : \mathrm{Sym}(\Omega) \longrightarrow \mathrm{Sym}(\Omega)$ the inversion of Sym(Ω). Let $\phi \in \mathrm{Sym}(\Omega)^{<\omega}$ be arbitrary. Then the preimage of $[\phi]$ under $\circ$ is the set $A = \{(f, g) \in \mathrm{Sym}(\Omega)^2 : f \circ g \in [\phi]\}$. Let $\mathcal{A}$ be the set of all $\psi \in \mathrm{Sym}(\Omega)^{<\omega}$ such that $\mathrm{dom}(\psi) = \mathrm{dom}(\phi)$. It follows that if $\psi \in \mathcal{A}$, then $\mathrm{dom}(\psi^{-1}) = \mathrm{ran}(\psi)$ and $\mathrm{ran}(\psi^{-1}) = \mathrm{dom}(\phi)$. Hence $\psi \circ \psi^{-1} \circ \phi = \phi$, and so

$$A = \{(f, g) \in \mathrm{Sym}(\Omega)^2 : f \circ g \in [\phi]\} = \bigcup_{\psi \in \mathcal{A}} [\psi] \times [\psi^{-1} \circ \phi]$$

is open. Similarly if B is a preimage of $[\phi]$ under $^{-1}$, then

$$B = \{f^{-1} : f \in [\phi]\} = [\phi^{-1}]$$

is also open. Therefore both $\circ$ and $^{-1}$ are continuous functions, and so $\mathrm{Sym}(\Omega)$ is a topological group. Then Example 1.4.9 implies that $\mathrm{Sym}(\Omega)$ is in fact a Polish group, and so every closed subgroup is also a Polish group.

Next we will show that if G is a topological group, and $g \in G$, then multiplying on the right by g is a homeomorphism, that is $\phi : G \longrightarrow G$ given by $(h)\phi = hg$ is a homeomorphism. An analogous result holds for multiplication on the left. First, we will prove a result for general topological spaces.

Proposition 1.4.12. *Let $f : X \times Y \longrightarrow Z$ be a continuous function, where $X \times Y$ is equipped with product topology of X and Y. Then*

(i) *the function $L_x : Y \longrightarrow Z$ given by $(y)L_x = (x, y)f$ for all $x \in X$ and all $y \in Y$ is continuous;*

(ii) *the function $R_y : X \longrightarrow Z$ given by $(x)R_y = (x, y)f$ for all $x \in X$ and all $y \in Y$ is continuous.*

Proof. **(i)** Let U be an open subset of Z, and let $y \in (U)L_x^{-1}$, or in other words $(x, y)f \in U$. Since f is continuous, $(U)f^{-1}$ is open in $X \times Y$, and by the definition of product topology there is a non-empty set I and for each $i \in I$ there is a set A_i open in X and a set B_i open in Y, so that

$$(U)f^{-1} = \bigcup_{i \in I} A_i \times B_i.$$

It follows from the fact that $(x, y)f \in U$ that there is some $i \in I$ so that $(x, y) \in A_i \times B_i \subseteq (U)f^{-1}$, and so $(\{x\} \times B_i)f \subseteq U$. Hence $y \in B_i \subseteq (U)L_x^{-1}$, which implies that $(U)L_x^{-1}$ is a union of open sets, and thus an open set. Therefore L_x is continuous.

(ii) The proof is almost identical. □

Since the operation of a topological group G is a continuous function from $G \times G$ to G, Proposition 1.4.12 immediately implies that multiplication on the left or on the right by an element of G is also a continuous function. The next result is a corollary of Proposition 1.4.12.

Corollary 1.4.13. *Let G be a topological group, let $g \in G$, and let $L_g, R_g : G \longrightarrow G$ be defined as follows:*

$$(h)L_g = gh \quad \text{and} \quad (h)R_g = hg.$$

Then L_g and R_g are both homeomorphisms.

Proof. It follows from Proposition 1.4.12 that both L_g and R_g are continuous functions. Also for each $h \in G$

$$(h)L_g \circ L_{g^{-1}} = g^{-1}gh = h \quad \text{and} \quad (h)L_{g^{-1}} \circ L_g = gg^{-1}h = h.$$

Hence L_g is a bijection by Proposition 1.3.2, with a continuous inverse $L_{g^{-1}}$, and so a homeomorphism. The proof for R_g is almost identical. □

1.4.4 Meagre and comeagre sets

Let (X, τ) be a topological space. Then a subset A of X is COMEAGRE if it contains an intersection of countably many open and dense subsets of X. The complement of a comeagre set is called MEAGRE. We say that a set N is NOWHERE DENSE in X if for every non-empty open set U there is a non-empty open subset $V \subseteq U$ such that $V \cap N$ is empty. Let A be a meagre set. Hence $X \setminus A$ is comeagre, and so $X \setminus A \supseteq \bigcap_{i\in\mathbb{N}} U_i$ for some open dense subsets U_i of X. Then

$$A \subseteq \bigcup_{i\in\mathbb{N}} X \setminus U_i.$$

Let U be an open set in X. Since U_i is dense, it follows that $U \cap U_i$ is a non-empty open subset. However, the intersection of $U \cap U_i$ and $X \setminus U_i$ is empty, and hence $X \setminus U_i$ is nowhere dense. Therefore, a subset of X is meagre only if it is contained in a countable union of nowhere dense sets. It can similarly be shown that if a subset is contained in a countable union of nowhere dense sets, then it is meagre, giving an alternative definition of meagre sets.

Next we show that comeagre sets are invariant under homeomorphisms.

Proposition 1.4.14. *Let X and Y be a topological spaces, and let $f : X \longrightarrow Y$ be a homeomorphism. Then $(C)f$ is comeagre in Y for every comeagre subset C of X.*

Proof. Let C be a comeagre subset of X, and for all $n \in \mathbb{N}$, let A_n be an open and dense subset of X such that $\bigcap_{n\in\mathbb{N}} A_n \subseteq C$. Since f is injective

$$\bigcap_{n\in\mathbb{N}} (A_n) f = \left(\bigcap_{n\in\mathbb{N}} A_n \right) f \subseteq (C)f.$$

Hence it is sufficient to show that $(A_n)f$ is open and dense for every $n \in \mathbb{N}$. Since A_n is an open set and f^{-1} is continuous, $(A_n)f$ is open in Y for every $n \in \mathbb{N}$.

Let $n \in \mathbb{N}$, and let U be an open subset of Y. Since f is continuous, it follows that $(U)f^{-1}$ is an open subset of X. Hence there is $x \in (U)f^{-1} \cap A_n$, as A_n is dense. Therefore $(x)f \in (A_n)f \cap U$, proving that $(A_n)f$ is dense, and so $(C)f$ is comeagre. □

The following example demonstrates that in Proposition 1.4.14 we can not drop the condition that the inverse of f is continuous. The example is by Emil Jeřábek, which was published on MATHOVERFLOW website [38]. Let (X, τ) be a topological space, and let Y be a subset of X. We will require the following observations:

- there is the smallest closed subset of X containing Y, called CLOSURE OF Y;
- Y is dense in the closure of Y;
- the set of all open intervals on $\mathbb{R}$ forms a basis for a topology which makes $\mathbb{R}$ into a Polish space.

Since these observations are only relevant to Example 1.4.15, the proofs will be omitted.

Example 1.4.15 ([38])**.** Let $\mathbb{N}^{\mathbb{N}}$ be the set of sequences over $\mathbb{N}$ (can also be thought as the set of all functions from $\mathbb{N}$ to $\mathbb{N}$) equipped with the topology described in Example 1.4.3, and let $[0,1)$ be equipped with the subspace topology of $\mathbb{R}$. Since $\mathcal{R}$ is a Polish space and $[0,1) = \bigcap_{n\in\mathbb{N}}(-1/n, 1)$, Theorem 1.4.8 implies that $[0,1)$ is also a Polish space. Define $f : \mathbb{N}^{\mathbb{N}} \longrightarrow [0,1)$ by

$$(x_0, x_1, \ldots)f = 0.\underbrace{1\ldots1}_{x_0}0\underbrace{1\ldots1}_{x_1}0\ldots$$

where the right hand side of the displayed equation is written in the binary representation, which, for the sake of brevity, we will write as $0.1^{x_0}01^{x_1}\ldots$ from now on. It follows from the definition that f is injective, and since every number in $[0,1)$ can be written without an infinite string of 1s, f is bijective.

Since the set of all open intervals is a basis for the topology on $\mathbb{R}$, the set of all intervals (a,b) and $[0,a)$ such that $0 < a < b < 1$ is a basis for the topology on $[0,1)$. Let $a, b \in \mathbb{R}$ be such that $0 < a < b < 1$, and let $x \in \mathbb{N}^{\mathbb{N}}$ be such that $y = (x)f \in (a,b)$. For any $r \in \mathbb{R}$ such that $r = 0.x_1x_2\ldots$ is the binary representation of r and any $k \in \mathbb{N}$, define $r_k = (x_1, \ldots, x_k) \in \{0,1\}^k$. Then there is $k \in \mathbb{N}$ such that a_k, b_k, and y_k are all different. Since $a < x < b$, it follows that $a_k < y_k < b_k$ with the lexicographic order. Recall that $[y_k] = \{w \in \mathbb{N}^{\mathbb{N}} : w \text{ starts with } y_k\}$ is a basic open set. Then $x \in [y_k] \subseteq ((a,b))\, f^{-1}$, and so $((a,b))\, f^{-1}$ is an open set of $\mathbb{N}^{\mathbb{N}}$. Similar argument shows that $([0,a))\, f^{-1}$ is open for all $a \in \mathbb{R}$ such that $0 < a < 1$. Therefore, f is a continuous bijection.

Consider the following set

$$U = \{(x_0, x_1, \ldots) \in \mathbb{N}^{\mathbb{N}} : \text{ there exists } n \in \mathbb{N} \text{ so that } x_n \text{ is even}\}.$$

If follows easily from the definition of basic open sets for $\mathbb{N}^{\mathbb{N}}$ that U is an open dense set, and thus a comeagre set. We will show that $(U)f$ is not a comeagre subset of $[0,1)$.

Suppose that $(U)f$ is comeagre. Then $(U)f \supseteq \bigcap_{n\in\mathbb{N}} G_n$ where G_n is open and dense for all $n \in \mathbb{N}$. Let A be the set of all real numbers of the form $a2^{-k}$ such that $a \in \mathbb{N}$ with the binary representation $a = 1^{x_0}0\ldots1^{x_r}0$ where x_i is odd for $i \in \{0, \ldots, r-1\}$ and x_r is even, and $k = \sum_{i=0}^{r}(x_i + 1)$. Let P be the closure of A in $[0, 1)$. Then A is dense in P. Since $[0, 1)$ is a Polish space, and P is a closed subset Proposition 1.4.5(i) implies that P is a Polish space. Note that $A \subseteq P \cap (U)f$. Suppose $x \in (U)f \setminus A$. If the binary representation of x is $0.1^{x_0}01^{x_1}0\ldots$ and $r \in \mathbb{N}$ is smallest such that x_r is even, then

$$(0.1^{x_0}0\ldots 1^{x_r}010, 0.1^{x_0}0\ldots 1^{x_r}011) \subseteq (U)f \setminus A,$$

and so $(U)f \setminus A$ is open in $[0, 1)$. Since $A \subseteq (U)f$,

$$[0, 1) \setminus ((U)f \setminus A) = ([0, 1) \setminus (U)f) \cup A$$

is a closed subset of $[0, 1)$ containing A. Thus $P \subseteq ([0, 1) \setminus (U)f) \cup A$, and so $P \cap (U)f \subseteq A$, implying that $A = P \cap (U)f$. Then $A \supseteq \bigcap_{n\in\mathbb{N}} G_n \cap P$, and $G_n \cap P$ are open in P, by the definition of subspace topology. Moreover, the fact that A is dense in P, implies that $G_n \cap P \supseteq A$ is also dense in P for all $n \in \mathbb{N}$. Therefore A is a comeagre subset of P. On the other hand, one element sets are nowhere dense in P, and thus the countable set A is meagre. Since both A and $X \setminus A$ are comeagre, it follows that $\varnothing = A \cap (X \setminus A)$ is a comeagre set. However, by Theorem 1.4.17 (which we will prove in the next section), comeagre subsets of Polish spaces are dense, which is a contradiction as P is Polish. Therefore, $(U)f$ is not comeagre in $[0, 1)$.

It follows from the alternative definition, that meagre sets are closed under subsets. Moreover, under certain reasonable conditions (for example, there is no minimal non-empty open set under inclusion) every set of size 1 is nowhere dense, and so all countable or finite set are meagre sets. These are the properties we expect any notion of "smallness" to satisfy. Roughly speaking we want to consider meagre sets to be topologically "small", and comeagre sets to be topologically "big". However, in a general topological space a set can be both meagre and comeagre, see Example 1.4.16.

Example 1.4.16. Let X be countable, and let τ be a collection of subsets of X such that $U \in \tau$ if and only if the set $X \setminus U$ is finite. Then (X, τ) is a topological space. Let $x \in X$ and let $U \in \tau$. If $x \in U$, then $U \setminus \{x\} \in \tau$, and so $\{x\}$ is nowhere dense. For any subset Y of X, both $X \setminus Y$ and Y are a countable union of nowhere dense sets. Therefore Y is both meagre and comeagre.

In the next section, we introduce the notion which guarantees that no set is both meagre and comeagre.

1.4.5 Baire category theory

A topological space (X, τ) is a Baire space if every comeagre set in X is dense. The following classical result, see Theorem 8.4 in [41], shows that every

Polish space (topological groups considered in this thesis are all Polish) is a Baire space. We will include the proof for the sake of completeness.

Theorem 1.4.17 (The Baire Category Theorem)**.** *Every complete metric space is Baire with the induced topology.*

Proof. Let (X, d) be a complete metric space. Recall that the topology on X is given by the basis consisting of open balls $B(x, r) = \{y \in X : d(x, y) < r\}$ where $x \in X$ and $r > 0$. Define a closed ball to be $\overline{B(x, r)} = \{y \in X : d(x, y) \leq r\}$.

Let $\{U_n : i \in \mathbb{N}\}$ be a sequence of open dense subsets of X, and let U be an arbitrary open set. Since $U \cap U_0$ is a non-empty open set there is $r_0 < 1/2$ and $x_0 \in U \cap U_0$ such that $\overline{B(x_0, r_0)} \subseteq U \cap U_0$. Since $B(x_0, r_0)$ is open, $B(x_0, r_0) \cap U_1$ is non-empty, and so there is $x_1 \in B(x_0, r_0) \cap U_1$ and $r_1 \leq 1/3$ such that $\overline{B(x_1, r_1)} \subseteq B(x_0, r_0) \cap U_1$. Continuing this we obtain $x_k \in B(x_{k-1}, r_{k-1}) \cap U_k$ and $r_k \leq 1/(k+2)$ for each $k \geq 1$ so that $\overline{B(x_k, r_k)} \subseteq B(x_{k-1}, r_{k-1}) \cap U_k$. Then $\{x_k : k \in \mathbb{N}\}$ is a Cauchy sequence, and so it converges to some $x \in X$. The sequence $\{x_k : x \geq N\}$ also converges to x for every $N \in \mathbb{N}$. Since $x_k \in \overline{B(x_N, r_N)}$ for all $k \geq N$ and all $N \in \mathbb{N}$, it follows from Proposition 1.4.4 that $x \in \overline{B(x_N, r_N)}$ for all $N \in \mathbb{N}$. Therefore,

$$x \in \bigcap_{k \in \mathbb{N}} \overline{B(x_k, r_k)} \subseteq \bigcap_{k \in \mathbb{N}} U_k.$$

Moreover $x \in B(x_0, r_0) \subseteq U$, and so the intersection of U and $\bigcap_{n \in \mathbb{N}} U_n$ is non-empty. Since U is an arbitrary open set, $\bigcap_{n \in \mathbb{N}} U_n$ is dense. Finally, since every comeagre set contains an intersection of countably many open dense subsets, it also is dense. $\square$

As discussed in the previous section, we want to think of comeagre sets as topologically "big". If (X, τ) is a non-empty Baire space and C is a comeagre subset of X, then C is dense and so non-empty. Hence we refer to elements of C as TYPICAL elements of X.

Example 1.4.18. It follows from Example 1.4.9 that $\mathrm{Sym}(\Omega)$ and every closed subgroup of $\mathrm{Sym}(\Omega)$ are Baire spaces.

1.5 Graphs

A GRAPH is a pair (V, E), where V is a set, known as THE SET OF VERTICES, and E is a set of subsets of V each of size 2. A set $\{x, y\} \in E$ is the EDGE BETWEEN x AND y. Two vertices of a graph are ADJACENT if there is an edge between them. For any cardinal n, the COMPLETE GRAPH K_n is the graph with n vertices and an edge between every pair of distinct vertices. The complete graph with a countable infinite set of vertices is denoted by K_ω.

If Γ and Δ are graphs with disjoint sets of vertices (and hence edges), then the DISJOINT UNION of Γ and Δ is the graph with vertices and edges being the unions of the vertices and edges, respectively, of Γ and Δ. The DUAL OF A GRAPH Γ is a graph Δ with the vertex set being the same as the vertex set of Γ and Δ has an edge between a pair of two distinct vertices if and only if the two vertices are not adjacent in Γ. If U is a set of vertices of a graph Γ, then the SUBGRAPH INDUCED BY U is the graph with vertices U and edges between $u \in U$ and $v \in U$ if and only if u and v are adjacent in Γ. For a fixed $n \in \mathbb{N}$ a graph Γ is K_n-FREE GRAPH if no induced subgraph of Γ is a complete graph on n points. It follows immediately from the definition that every K_1-free graph is empty and every K_2-free graph does not have any edges, thus we will be only interested in K_n-free graphs if $n \geq 3$.

If $x \in \Gamma$, the CONNECTED COMPONENT OF Γ CONTAINING x is the subgraph induced by the set of vertices $y \in \Gamma$ such that there are $n \in \mathbb{N}$ and vertices $x_0 = x, x_1, \ldots, x_n = y$ such that x_i is adjacent to x_{i+1} for all $i \in \{0, \ldots, n-1\}$.

Another very important tool we require is the Alice's restaurant property. We will have three versions of the property — for graphs; for K_n-free graphs; and partial orders. The first two will be defined in this section.

Let Γ be a graph. Then Γ has ALICE'S RESTAURANT PROPERTY FOR GRAPHS if for every pair of finite disjoint induced subgraphs U and V of Γ, there is $w \in \Gamma \setminus (U \cup V)$ such that w is adjacent to every vertex in U and to no vertex in V.

Let $n \geq 3$, and let Γ be a K_n-free graph. We say that Γ has the ALICE'S RESTAURANT PROPERTY FOR K_n-FREE GRAPHS if for every pair of finite disjoint induced subgraphs U and V of Γ where U is K_{n-1}-free, there exists a vertex $w \in \Gamma \setminus (U \cup V)$ such that w is adjacent to every vertex in U and to no vertex in V.

1.6 Fraïssé limits

In this section we will describe some model theory ideas, most importantly those concerning Fraïssé limits. For a more complete treatment see Hodges' "A shorter model theory" [28], which we use as the reference for this section.

A STRUCTURE A is an object consisting of following four parts

- a set, called DOMAIN of A, and denoted by $\mathrm{dom}(A)$;
- a collection of CONSTANT SYMBOLS and a subset of A, called CONSTANT ELEMENTS, which are named by the constant symbols. If c is the constant symbol, then we denote constant element corresponding to c by c^A;

- for every $n > 0$, a collection of n-ary RELATION SYMBOLS and a set of n-ary relations on $\mathrm{dom}(A)$, each named by one of the n-ary relation symbols. If R is a relation symbol, then we denote the relation corresponding to the symbol by R^A;

- for every $n > 0$, a collection of n-ary FUNCTION SYMBOLS and a set of functions from $\mathrm{dom}(A)^n$ to $\mathrm{dom}(A)$, each named by one of the n-ary function symbols. If F is a function symbol, then we denote the function corresponding to the symbol by F^A.

A SIGNATURE OF A STRUCTURE A consists of the set of constant symbols, and for every $n > 0$ the sets of n-ary relation symbols and n-ary function symbols. If L is a signature of A, we can refer to A as an L-STRUCTURE. A structure with a signature which contains no constant or function symbols is a RELATIONAL STRUCTURE.

Let A and B be L-structures. Then a HOMOMORPHISM for A to B, written as $f : A \longrightarrow B$, is a function $f : \mathrm{dom}(A) \longrightarrow \mathrm{dom}(B)$ so that the following are true: $(c^A)f = c^B$ for each constant symbol c of L; for $n > 0$, an n-ary relation symbol R, and a $x \in \mathrm{dom}(A)^n$, $x \in R^A$ implies $(x)f \in R^B$; for $n > 0$, an n-ary function symbol F, and a $x \in \mathrm{dom}(A)^n$, $((x)F^A)f = ((x)f)F^B$. An EMBEDDING OF A TO B is a homomorphism $f : A \longrightarrow B$ which is injective and for $n > 0$, an n-ary relation symbol R, and a $x \in \mathrm{dom}(A)^n$, $x \in R^A$ if and only if $(x)f \in R^B$. An ISOMORPHISM is a surjective embedding, and an AUTOMORPHISM OF A is an isomorphism $f : A \longrightarrow A$. By $\mathrm{Aut}(A)$ we denote the set of all automorphisms of A, which can be shown to be a group. Similarly $\mathrm{End}(A)$ denotes the set of all homomorphisms from A to A, also known as ENDOMORPHISMS.

Let A and B be L-structures such that $\mathrm{dom}(A) \subseteq \mathrm{dom}(B)$ and the inclusion map $i : A \longrightarrow B$, given by $(a)i = a$ for all $a \in A$, is an embedding. Then A is a SUBSTRUCTURE OF B, and B is an EXTENSION OF A.

Lemma 1.6.1 (see Lemma 1.2.2. in [28])**.** *Let B be an L-structure and X a subset of* $\mathrm{dom}(B)$*. Then the following are equivalent*

a. $X = \mathrm{dom}(A)$ *for some substructure A of B.*

b. For every constant c in L, $c^B \in X$; and for all $n > 0$, every n-ary function symbol F in L and every $x \in X^n$, $(x)F^B \in X$.

Moreover, it the above conditions hold then A is uniquely determined.

Let B be an L-structure and let Y be a subset of the domain of B. Let $X_0 = Y \cup \{c^B : c \text{ is a constant symbol of } L\}$ and for $k \geq 1$ define

$$X_k = X_{k-1} \cup \bigcup_{n>0} \{(x)F^B : F \text{ is an } n\text{-ary function symbol and } x \in X_{k-1}^n\}.$$

Let $X = \bigcup_{k\in\mathbb{N}} X_k$. By the definition $c^B \in X$ for all constant symbols c of L. Let $n > 0$ and let $x \in X^n$. Since $X_0 \subseteq X_1 \subseteq \ldots$, it follows that $x \in X_k^n$ for some $k \in \mathbb{N}$, and by the definition $(x)F^B \in X$. Hence by Lemma 1.6.1 there is a substructure A of B such that $\text{dom}(A) = X$. It can also be shown that A is the smallest substructure of B containing Y. We call the unique smallest substructure of B containing Y the SUBSTRUCTURE GENERATED BY Y, denoted by $\langle Y\rangle_B$. The structure B is FINITELY GENERATED if there is a finite $Y \subseteq \text{dom}(B)$ such that $B = \langle Y\rangle_B$. The AGE OF B is the collection of all finitely generated structures which can be embedded into B, up to isomorphism. That is, when considering the age of a structure we only care about its elements up to isomorphism. It is worth mentioning that if A is a finitely generated relational structure, then A is finite.

An L-structure A is ULTRAHOMOGENEOUS if for any finitely generated substructures B and C of A and an isomorphism $f : B \longrightarrow C$ there is an automorphism g of A such that $(x)f = (x)g$ for all $x \in \text{dom}(B)$.

The main theorem of this section can now be stated.

Theorem 1.6.2 (Fraïssé Theorem, see Theorem 6.1.2 in [31])**.** *Let L be a countable signature and let $\mathbf{K}$ be a non-empty countable set of finitely generated L-structures satisfying the following conditions:*

- (HEREDITARY PROPERTY) *If $A \in \mathbf{K}$ and B is a finitely generated substructure of A, then $B \in \mathbf{K}$;*
- (JOINT EMBEDDING PROPERTY) *If $A, B \in \mathbf{K}$, then there is $C \in \mathbf{K}$ such that both A and B can be embedded in C;*
- (AMALGAMATION PROPERTY) *If $A, B, C \in \mathbf{K}$ and $f : A \longrightarrow B$, $g : A \longrightarrow C$ are embeddings, then there is $D \in \mathbf{K}$ and embeddings $h : B \longrightarrow D$, $k : C \longrightarrow D$ such that $f \circ h = g \circ k$.*

Then there exists a countable L-structure $\mathcal{K}$, unique up to isomorphism, such that $\mathbf{K}$ is the age of $\mathcal{K}$ and $\mathcal{K}$ is ultrahomogeneous.

We refer to the structure $\mathcal{K}$ in Theorem 1.6.2 as the FRAÏSSÉ LIMIT OF THE FAMILY $\mathbf{K}$. The following theorem relates Fraïssé limits to the permutation groups.

Theorem 1.6.3 (Theorem 5.8 in [9])**.** *Let Ω be a countable set. Then G is a closed subgroup of $\text{Sym}(\Omega)$ if and only if $G = \text{Aut}(A)$ for some ultrahomogeneous relational structure A with domain Ω.*

Before giving the proof we need another notion. Let G be a subgroup of $\text{Sym}(\Omega)$. Then a subset O of Ω^n is an ORBIT OF G ON Ω^n if there are $x_1, \ldots, x_n \in \Omega$ such that

$$O = \{((x_1)g, \ldots, (x_n)g) : g \in G\}.$$

Proof. (⇒) Let G be a closed subgroup of $\mathrm{Sym}(\Omega)$. We will define a relational structure A with domain Ω. For every $n \in \mathbb{N}$, and every orbit O_i of G on Ω^n, let R_i be an n-ary relation symbol in the language, and let O_i be the relation corresponding to R_i. If $f \in G$, and $(x_1, \ldots, x_n) \in R_i^A$ for some i, then $((x_1)f, \ldots (x_n)f) \in R_i^A$, by the definition of $O_i = R_i^A$. Hence f is an automorphism of A, in other words $G \leq \mathrm{Aut}(A)$.

Let f be an automorphism of A, and let $\{x_1, x_2, \ldots\}$ be an enumeration of Ω. Then for any $(x_1, \ldots, x_n) \in \Omega^n$ there is a symbol R_i such that $(x_1, \ldots, x_n) \in R_i^A$. Since f is an automorphism $((x_1)f, \ldots, (x_n)f) \in R_i^A$. Hence $(x_1, \ldots, x_n)$ and $((x_1)f, \ldots, (x_n)f)$ are both elements of the same orbit O_i of G on Ω^n, and so there is $g_n \in G$ such that $(x_m)g_n = (x_m)f$ for all $m \in \{1, \ldots, n\}$. Then the sequence $(g_n)_{n\in\mathbb{N}}$ converges to f, and since G is closed Proposition 1.4.4 implies that $f \in G$. Therefore G is the automorphism group of A.

Finally, we show that A is ultrahomogeneous. Let B and C be finite substructures of A, and let $q : B \longrightarrow C$ be an isomorphism. Then for each n-ary relation R_i, by the definition of substructure, $R_i^B = O_i \cap \mathrm{dom}(B)^n$ and $R_i^C = O_i \cap \mathrm{dom}(C)^n$. Let $B = \{y_1, \ldots, y_n\}$ for some $n \in \mathbb{N}$. Then there is a symbol R_i so that $(y_1, \ldots, y_n) \in R_i^B$. Since q is an isomorphism $((y_1)q, \ldots (y_n)q) \in R_i^C$, and thus both $(y_1, \ldots, y_n)$ and $((y_1)q, \ldots, (y_n)q)$ are elements of O_i. Hence there is $g \in G$ such that $(y_m)q = (y_m)g$ for all $m \in \{1, \ldots, n\}$. Therefore A is ultrahomogeneous.

(⇐) Let $g \in \mathrm{Sym}(\Omega) \setminus \mathrm{Aut}(A)$. Then g is not an automorphism of A and so there are $n \in \mathbb{N}$, an n-ary relation R, and $(x_1, \ldots, x_n) \in R^A$ such that $((x_1)g, \ldots, (x_n)g) \notin R^A$. Then the basic open set $[g|_{\{x_1,\ldots,x_n\}}]$ contains g and is contained in $\mathrm{Sym}(\Omega) \setminus \mathrm{Aut}(A)$, thus $\mathrm{Sym}(\Omega) \setminus \mathrm{Aut}(A)$ is an open set. Hence $\mathrm{Aut}(A)$ is a closed subset of $\mathrm{Sym}(\Omega)$. □

An analogous result holds for Ω^Ω and endomorphism monoids of ultrahomogeneous relational structures. Even though the concept of an orbit does not apply to monoids, we can still use the sets of the form

$$\{((x_1)g, \ldots, (x_n)g) : g \in M\}$$

where M is a submonoid of Ω^Ω and $x_1, \ldots, x_n \in \Omega$. Otherwise, the same proof can be used to prove the following result.

Theorem 1.6.4. *Let Ω be a countable set. Then M is a closed submonoid of Ω^Ω if and only if $M = End(A)$ for some ultrahomogeneous relational structure A with domain Ω.*

Chapter 2

A finite interval in a subsemigroup lattice

This chapter is based on the paper by the J. D. Mitchell and the author [40]. It is included in this thesis with the permission of the coauthor.

2.1 Description of the problem

Let $(X, \leq)$ be a partially ordered set, and let $Y \subseteq X$. Recall that an UPPER BOUND of Y is an element $u \in X$ such that $y \leq u$ for all $y \in Y$, and similarly $l \in X$ is a LOWER BOUND of Y if $l \leq y$ for all $y \in Y$. An upper bound u of Y is the SUPREMUM of Y if it is the least upper bound, assuming it exists, in other words if u' is an upper bound of Y, then $u \leq u'$. In the same way we may define the INFIMUM to be the greatest lower bound. We denote the supremum of Y by $\sup(Y)$, and the infimum of Y by $\inf(Y)$. A partially ordered set $(X, \leq)$ is a LATTICE if every pair of elements of X has the supremum and the infimum. Similarly, $(X, \leq)$ is a COMPLETE LATTICE if every subset of X has a supremum and an infimum.

Let S be a semigroup. Then define a binary relation $\leq$ on the set of all subsemigroups of S by $T \leq U$ if and only if T is a subsemigroup of U. It is easy to show that $\leq$ is a partial order on the set of all subsemigroups of S. Suppose that $\{T_i : i \in I\}$ is a family of semigroups such that $T_i \leq S$ for all $i \in I$. It was shown in Section 1.2 that $\bigcap_{i\in I} T_i$ is a subsemigroup of S. If V is a subsemigroup of T_i for all $i \in I$, then $V \subseteq \bigcap_{i\in I} T_i$, and so V is a subsemigroup of $\bigcap_{i\in I} T_i$. Hence $\bigcap_{i\in I} T_i$ is the infimum of the family $\{T_i : i \in I\}$. On the other hand, consider the semigroup generated by the subset $\bigcup_{i\in I} T_i$ of S, as usually denoted by $\langle \bigcup_{i\in I} T_i\rangle$. Then by the definition, it is the smallest subsemigroup of S which contains T_i for all $i \in I$. Therefore $\langle \bigcup_{i\in I} T_i\rangle$ is the supremum of $\{T_i : i \in I\}$, and thus the set of all subsemigroups of S forms a complete lattice, we call it the SUBSEMIGROUP LATTICE OF S.

Let $(X, \leq)$ be a complete lattice. An element x of X is a COMPACT ELEMENT if for every subset A of X such that $x \leq \sup(A)$ there is a finite subset B of A with $x \leq \sup(B)$. A complete lattice $(X, \leq)$ is an ALGEBRAIC LATTICE if every element $x \in X$ is equal to the supremum of the set of all compact elements smaller than x.

Proposition 2.1.1 ([]). *Let S be a semigroup, and let T be a subsemigroup of S. Then T is a compact element in the subsemigroup lattice of S if and only if T is finitely generated.*

Proof. ($\Rightarrow$) Let T be a compact element in the subsemigroup lattice of S, and for $x \in T$, let $\langle x \rangle$ be a subsemigroup of S generated by the element x. Then $\sup_{x \in T} \langle x \rangle \geq \langle y \rangle$, and so $y \in \sup_{x \in T} \langle x \rangle$ for all $y \in T$. Hence $T \leq \sup_{x \in T} \langle x \rangle$, and by the definition of compact elements there is a finite subset F such that $T \leq \sup_{x \in F} \langle x \rangle$. Also note that since all $\langle x \rangle \subseteq T$, the subsemigroup T is an upper bound for $\{\langle x \rangle : x \in F\}$, and so $T = \sup_{x \in F} \langle x \rangle$. Hence

$$T = \sup_{x \in F} \langle x \rangle = \langle \bigcup_{x \in F} \langle x \rangle \rangle = \langle F \rangle,$$

and so T is finitely generated.

($\Leftarrow$) Let T be a finitely generated subsemigroup of S, and let $\{U_i : i \in I\}$ be a family of subsemigroups of S such that $T \leq \sup_{i \in I} U_i = \langle \bigcup_{i \in I} U_i \rangle$. If $x \in T$, then it can be written as a finite product of the elements in $\bigcup_{i \in I} U_i$. Hence there is a finite subset J_x of I such that $x \in \langle \bigcup_{i \in J_x} U_i \rangle$. Let A be a finite generating set of T, and let $J = \bigcup_{x \in A} J_x$. Then J is finite and $A \subseteq \langle \bigcup_{i \in J} U_i \rangle$. Hence

$$T \leq \sup_{i \in J} U_i,$$

and so T is a compact element. □

Proposition 2.1.1 can now be used to show that subsemigroup lattice is an algebraic lattice. Let T be a subsemigroup of S, and let $\mathbf{F} = \{F \leq T : F \text{ is a compact element}\}$. Since T is an upper bound of $\mathbf{F}$, $\sup \mathbf{F} \leq T$. Let A be a generating set for T. Then

$$T = \langle A \rangle = \langle \bigcup_{x \in A} \langle x \rangle \rangle = \sup_{x \in A} \langle x \rangle \leq \sup \mathbf{F},$$

as $\langle x \rangle \in \mathbf{F}$ for all $x \in A$ by Proposition 2.1.1. It then follows that $T = \sup \mathbf{F}$, and so the subsemigroup lattice of S is an algebraic lattice. This result comes as no surprise and is consistent with the terminology. Actually, the fact that the subsemigroup lattice is an algebraic lattice is a consequence of a much stronger result. In order to state the result we need to define an algebraic closure operator.

Let $(X, \leq)$ be a partially ordered set. Then $c : X \longrightarrow X$ is an ALGEBRAIC CLOSURE OPERATOR if the following are satisfied for all $x, y \in X$: $x \leq (x)c$; $x \leq y$ implies that $(x)c \leq (y)c$; $((x)c)c = (x)c$; and $(A)c = \bigcup\{(B)c : B \subseteq A \text{ and } B \text{ is finite}\}$ for every $A \subseteq X$. Since in this chapter we are only interested in the subsemigroup lattice, the result is only included for the sake of completeness and we will omit the proof and the further discussion of the concepts within.

Theorem 2.1.2 (Lemma 7.19 in [14])**.** *Let $(X, \leq)$ be a partial order, let c be an algebraic closure operator on X, and let $\mathcal{L}_c = \{A \subseteq X : (A)c = A\}$. Then $\mathcal{L}_c$ is an algebraic lattice such that an element A is compact if and only if $A = (B)c$ for some finite subset B of X.*

Let Ω be an arbitrary infinite set. Then as discussed before, the set of all subsemigroups of Ω^Ω forms a semigroup lattice. In the next proposition we show that Ω^Ω has $2^{|\Omega|}$ many finitely generated subsemigroups, and so $2^{|\Omega|}$ compact elements in the semigroup lattice of Ω^Ω, by Proposition 2.1.1.

Proposition 2.1.3. *Let Ω be an infinite set. Then there are $2^{|\Omega|}$ many distinct finitely generated subsemigroups of Ω^Ω.*

Proof. Let $\mathbf{F}$ be the set all finitely generated subsemigroups of Ω^Ω. Recall that $|\Omega^\Omega| = 2^{|\Omega|}$, and so there are only $2^{|\Omega|}$ finite subsets of Ω^Ω. Hence $|\mathbf{F}| \leq 2^{|\Omega|}$.

Suppose that $\mathbf{F} < 2^{|\Omega|}$. Recall that every finitely generated semigroup is either finite or countable, and so if $\mathbf{F}$ is finite, then $\langle \bigcup_{F \in \mathbf{F}} F \rangle$ is either finite or countable. If $\mathbf{F}$ is infinite, then it follows that $|\bigcup_{F \in \mathbf{F}} F| \leq |\mathbf{F} \times \aleph_0| = |\mathbf{F}|$, and also $\bigcup_{F \in \mathbf{F}} F$ is infinite. Hence

$$|\langle \bigcup_{F \in \mathbf{F}} F \rangle| = |\bigcup_{F \in \mathbf{F}} F| \leq |\mathbf{F}| < 2^{|\Omega|}.$$

In both cases, $\langle \bigcup_{F \in \mathbf{F}} F \rangle$ is a proper subset of Ω^Ω, and so there is $f \in \Omega^\Omega \setminus \langle \bigcup_{F \in \mathbf{F}} F \rangle$. However, $\langle f \rangle$ is a finitely generated subsemigroup of Ω^Ω, and thus $\langle f \rangle \in \mathbf{F}$, which is a contradiction. Therefore $\mathbf{F} = 2^{|\Omega|}$, as required. □

It turns out that the subsemigroup lattice of Ω^Ω is in some way a "universal" algebraic lattice, as described in the following result by Pinsker and Shelah.

Theorem 2.1.4 (Theorem 1.1 in [62])**.** *Every complete algebraic lattice with at most $2^{|\Omega|}$ compact elements can be embedded into the subsemigroup lattice of Ω^Ω.*

Since Ω^Ω has $2^{|\Omega|}$ many elements, it follows that it has $2^{2^{|\Omega|}}$ many subsets. Moreover, the number of distinct subsemigroups of Ω^Ω is also $2^{2^{|\Omega|}}$.

Proposition 2.1.5. *Let Ω be infinite. Then Ω^Ω has $2^{2^{|\Omega|}}$ many distinct subsemigroups.*

Proof. Let a and b be two distinct elements of Ω, and let $\Sigma = \Omega \setminus \{a, b\}$. Then $|\Sigma| = |\Omega|$. For a subset A of Σ, define $f_A \in \Omega^\Omega$ by

$$(x)f_A = \begin{cases} a & \text{if } x \in A \cup \{a\} \\ b & \text{otherwise} \end{cases}.$$

Let $\mathcal{F}$ be the set of all such f_A. Since there $2^{|\Omega|}$ many subsets of Σ and f_A and f_B are distinct for distinct A and B, it follows that $|\mathcal{F}| = 2^{|\Omega|}$. Moreover, if A and B are subsets of Σ, then $f_A \circ f_B = f_A$, as $b \notin A$. Hence every subset of $\mathcal{F}$ is a subsemigroup of Ω^Ω, and so there are at least $2^{|\mathcal{F}|} = 2^{2^{|\Omega|}}$ many subsemigroups of Ω^Ω. However, Ω^Ω has $2^{2^{|\Omega|}}$ many subsets, thus there are $2^{2^{|\Omega|}}$ many subsemigroups of Ω^Ω. □

A subsemigroup M of S is MAXIMAL if for every subsemigroup T of S such that $M \lneq T$, it follows that $T = S$. We already know that Ω^Ω has $2^{2^{|\Omega|}}$ distinct subsemigroups by Proposition 2.1.5, but more surprisingly the number of maximal subsemigroups is also $2^{2^{|\Omega|}}$, see [17, Theorem C]. The maximal subsemigroups of other semigroups were also studied in the literature. For example, the maximal subsemigroups of a finite semigroup are, in a sense, determined by their maximal subgroups, see [25]. The question for infinite semigroups was investigated by Levi and Wood, in [45], and Hotzel, in [29], in the case of Baer-Levi semigroups, and by Shneperman, in [68], in the case of the endomorphism monoid of a finite dimension complex vector space, where maximal compact subsemigroups were considered.

The analogous problem of finding maximal subgroups is studied to even a higher degree. For example the maximal subgroups of the symmetric group on Ω for an infinite Ω were studied in [3, 4, 6, 7, 8, 11, 49, 53, 55, 64], and the case of finite Ω has been investigated by Aschbacher and Scott, in [2], O'Nan and Scott, in [66], and also Liebeck, Praeger and Saxl in [46]. Of particular interest, perhaps, are the results by H. D. Macpherson and Peter M. Neumann, see [52], and Fred Richman, see [64]. It follows from their results that the number of maximal subgroups of $\mathrm{Sym}(\Omega)$ is $2^{2^{|\Omega|}}$, if Ω is infinite. The result is obtained by showing that the stabiliser of an ultrafilter is a maximal subgroup of $\mathrm{Sym}(\Omega)$.

A set $\mathcal{F}$ of subsets of Ω is a FILTER if the following are satisfied:

1. if $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$;
2. if $A \subseteq B \subseteq \Omega$ and $A \in \mathcal{F}$, then $B \in \mathcal{F}$.

A filter $\mathcal{F}$ is a PROPER FILTER if $\mathcal{F}$ is not the set of all subsets of Ω, and a proper filter is an ULTRAFILTER if it is maximal with respect to containment among all proper filters. Let $a \in \Omega$ and let $\mathcal{K} = \{A \subseteq \Omega : a \in A\}$. Then $\mathcal{K}$ is a proper filter. If $\mathcal{K}'$ is a filter properly containing $\mathcal{K}$, then there is $B \in \mathcal{K}'$ such that $a \notin B$, and so $\varnothing = \{a\} \cap B \in \mathcal{K}'$, implying that $\mathcal{K}'$ is the set of all subsets of Ω. Hence $\mathcal{K}$ is an ultrafilter. An ultrafilter $\mathcal{F}$ a PRINCIPAL ULTRAFILTER if $\mathcal{F} = \{A \subseteq \Omega : a \in A\}$ for some $a \in \Omega$. For a filter $\mathcal{F}$, the stabiliser of $\mathcal{F}$ is

$$\mathrm{Stab}(\mathcal{F}) = \{f \in \mathrm{Sym}(\Omega) : (A)f \in \mathcal{F} \text{ for all } A \in \mathcal{F}\}.$$

The following theorem shows that there $2^{2^{|\Omega|}}$ ultrafilters on an infinite set Ω.

Theorem 2.1.6 (Theorem 7.6 in [37])**.** *Let Ω be an infinite set. Then there are $2^{2^{|\Omega|}}$ ultrafilters on Ω.*

Since there are precisely $|\Omega|$ many distinct principal ultrafilters, it follows from Theorem 2.1.6 that there are $2^{2^{|\Omega|}}$ non-principal ultrafilters.

In [17], East, Mitchell, and Péresse use ultrafilters to show that there are $2^{2^{|\Omega|}}$ many maximal subsemigroups of Ω^Ω. For every non-principal ultrafilter $\mathcal{F}$ they show that there are at least two maximal subsemigroups of Ω^Ω containing $\mathrm{Stab}(\mathcal{F})$, all of them distinct, and so the result follows from the above. In contrast to this result, Pinsker, in [60], showed that if $|\Omega| = \aleph_\alpha$, then there are $2|\alpha| + 5$ many maximal subsemigroups of Ω^Ω containing $\mathrm{Sym}(\Omega)$. The result was initially proved by Gavrilov, in [21], for a countable Ω, and then expanded by Pinsker to arbitrary cardinalities.

Theorem 2.1.7 (Theorem 4 in [60])**.** *Let Ω be an infinite set. If Ω has regular cardinality κ, then the maximal subsemigroups of Ω^Ω containing $\mathrm{Sym}(\Omega)$ are as follows:*

- $A = \{f \in \Omega^\Omega : k(f) < \kappa\}$;
- $G_\lambda = \{f \in \Omega^\Omega : |\Omega \setminus (\Omega \setminus A)f| \geq \lambda \text{ for all } A \subseteq X \text{ of cardinality } \lambda\}$;
- $M_\lambda = \{f \in \Omega^\Omega : c(f) \geq \lambda \text{ or } d(f) < \lambda\}$;

where $\lambda = 1$ or $\aleph_0 \leq \lambda \leq \kappa$.

If Ω has singular cardinality, then the same is true, but with subsemigroup A replaced by

$$A' = \{f \in \Omega^\Omega : \text{there is } \lambda < \kappa \text{ and } A \subset \Omega \text{ of cardinality } \kappa \text{ such that } |\{x \in \Omega : (x)f = y\}| \leq \lambda \text{ for all } y \in A\}.$$

In this chapter we are exclusively interested in the case where $|\Omega| = \aleph_0$. Then the five maximal subsemigroups containing $\text{Sym}(\Omega)$ are:

$$\begin{aligned} S_1 &= \{f \in \Omega^\Omega : c(f) = 0 \text{ or } d(f) > 0\} = G_1 \\ S_2 &= \{f \in \Omega^\Omega : c(f) > 0 \text{ or } d(f) = 0\} = M_1 \\ S_3 &= \{f \in \Omega^\Omega : c(f) < \omega \text{ or } d(f) = \omega\} = G_{\aleph_0} \\ S_4 &= \{f \in \Omega^\Omega : c(f) = \omega \text{ or } d(f) < \omega\} = M_{\aleph_0} \\ S_5 &= \{f \in \Omega^\Omega : k(f) < \omega\} = A. \end{aligned} \tag{2.1}$$

If S and T are subsemigroups of Ω^Ω, then the INTERVAL FROM T TO S, denoted (T, S), is defined to be the set of proper subsemigroups of S properly containing T, in particular

$$(T, S) = \{U \leq \Omega^\Omega : T \lneq U \lneq S\}.$$

Pinsker showed in [61] that the interval $(\text{Sym}(\Omega), \Omega^\Omega)$ has cardinality 2^{2^κ}, where $|\Omega| = \aleph_\alpha$ and $\kappa = \max\{\alpha, \aleph_0\}$. Moreover, if $\aleph_\alpha$ is a regular cardinal and S is the intersection of the maximal subsemigroups of Ω^Ω containing $\text{Sym}(\Omega)$, as described in Theorem 2.1.7, then the interval $(\text{Sym}(\Omega), S)$ also has cardinality 2^{2^κ}. As mentioned before, of particular interest to us is the case where $|\Omega| = \aleph_0$. Then $\kappa = \aleph_0$, and so the interval $(\text{Sym}(\Omega), S)$ has the same cardinality as the number of subsemigroups in Ω^Ω. However, the question remains, whether the interval (S, Ω^Ω) is also of the cardinality $2^{2^{\aleph_0}}$. It turns out that this is not the case. More precisely, we will prove in Theorem 2.1.8 that there are only 38 subsemigroups in the interval from the intersection of the maximal subsemigroups described in (2.1) to Ω^Ω when Ω is countably infinite.

Throughout the rest of this chapter, we will assume that Ω is countably infinite. For the sake of brevity, if $I \subseteq \{1, 2, 3, 4, 5\}$, then we will denote the intersection $\bigcap_{i \in I} S_i$ by S_I, so that $S_{1,2}$ denotes $S_1 \cap S_2$ and so on. Define

$$\begin{aligned} U &= \{f \in \Omega^\Omega : d(f) = \omega \text{ or } (0 < c(f) < \omega)\} \cup \text{Sym}(\Omega) \\ V &= \{f \in \Omega^\Omega : c(f) = \omega \text{ or } (0 < d(f) < \omega)\} \cup \text{Sym}(\Omega). \end{aligned}$$

The following theorem is the main result of this chapter.

Theorem 2.1.8. *Let Ω be a countable set. Then the interval $(S_{1,2,3,4,5}, \Omega^\Omega)$ consists of 38 semigroups, 30 of which are all possible intersections of any non-empty proper subset of $\{S_1, S_2, S_4, S_3, S_5\}$ as well as the following 8 semigroups:*

$$U,\ V,\ S_1 \cap U,\ S_2 \cap V,\ S_5 \cap U,\ S_5 \cap V,\ S_{1,5} \cap U,\ \text{and } S_{2,5} \cap V.$$

The subsemigroup lattice of the interval $(S_{1,2,3,4,5}, \Omega^\Omega)$ is depicted in Figure 2.1. We will describe the relevant semigroups given in the Theorem 2.1.8, in terms of the parameters d, c, and k (in a similar way to (2.1)) in Section 2.2. In Section 2.3 a number of technical lemmas are proved which are extensively used in the proof of the Theorem 2.1.8 in Section 2.4.

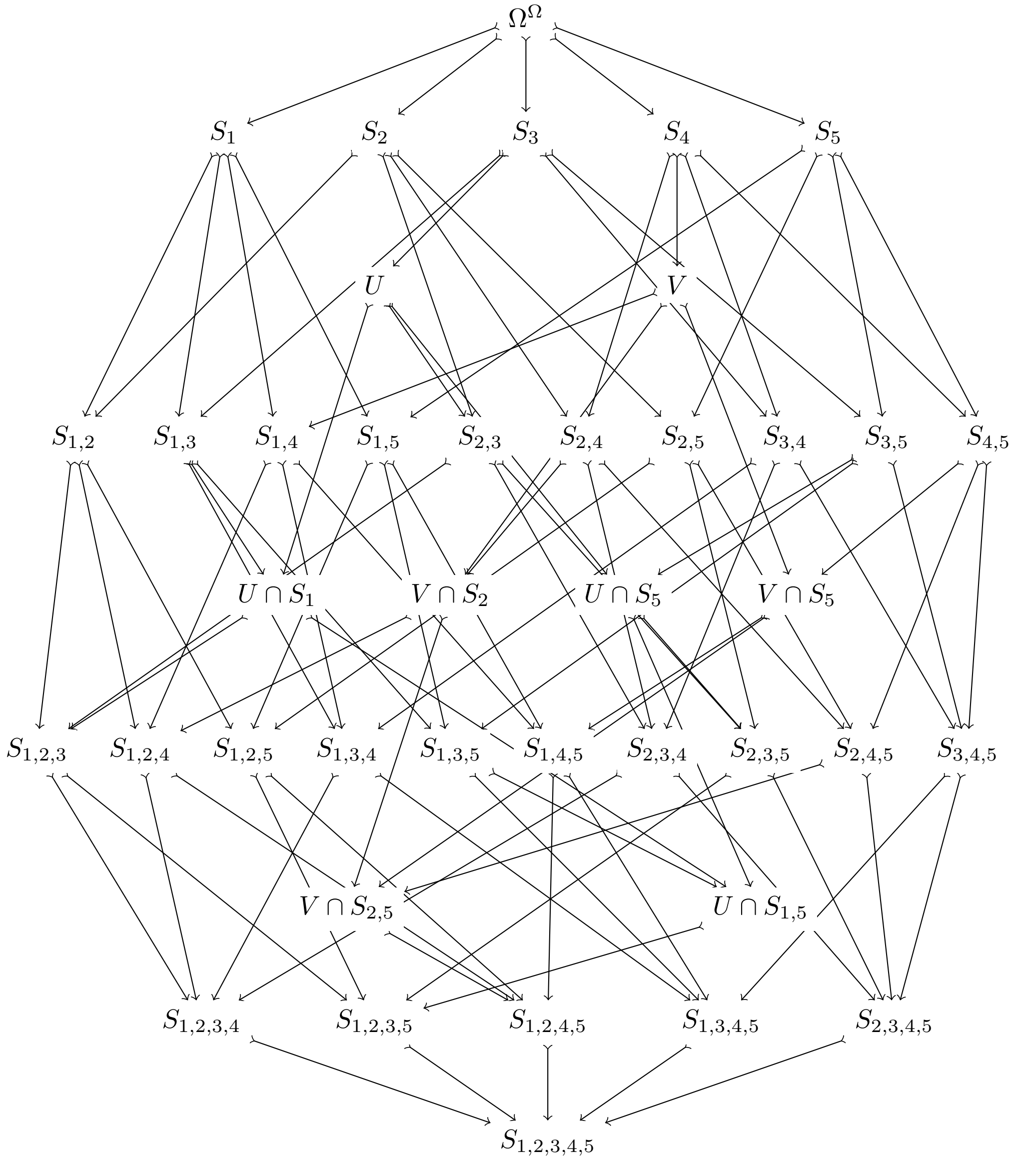

Figure 2.1: Subsemigroup lattice of $(S_{1,2,3,4,5}, \Omega^\Omega)$

2.2 Definitions of the semigroups in Theorem 2.1.8

In this section we give the definitions of all the semigroups mentioned in Theorem 2.1.8, for ease of reference. Since $\mathrm{Sym}(\Omega)$ is a subgroup of every semigroup in question, in the following table we will only describe non bijective transformations.

S	If $f \in S$, then is either in $\mathrm{Sym}(\Omega)$ or satisfies the following condition		
S_1	$c(f) = 0$	or	$d(f) > 0$
S_2	$c(f) > 0$	or	$d(f) = 0$
S_3	$c(f) < \omega$	or	$d(f) = \omega$
S_4	$c(f) = \omega$	or	$d(f) < \omega$
S_5	$k(f) < \omega$		
U	$0 < c(f) < \omega$	or	$d(f) = \omega$
V	$c(f) = \omega$	or	$0 < d(f) < \omega$
$S_{1,2}$	$c(f), d(f) > 0$		
$S_{1,3}$	$d(f) = \omega$ or $c(f) = 0$	or	$0 < c(f), d(f) < \omega$
$S_{1,4}$	$c(f) = d(f) = \omega$	or	$0 < d(f) < \omega$
$S_{1,5}$	$d(f) > 0$ and $k(f) < \omega$	or	$c(f) = 0$
$S_{2,3}$	$c(f) = d(f) = \omega$	or	$0 < c(f) < \omega$
$S_{2,4}$	$c(f) = \omega$ or $d(f) = 0$	or	$0 < c(f), d(f) < \omega$
$S_{2,5}$	$c(f) > 0$ and $k(f) < \omega$	or	$d(f) = 0$ and $k(f) < \omega$
$S_{3,4}$	$c(f) = d(f) = \omega$	or	$c(f), d(f) < \omega$
$S_{3,5}$	$c(f) < \omega$	or	$d(f) = \omega$ and $k(f) < \omega$
$S_{4,5}$	$c(f) = \omega$ and $k(f) < \omega$	or	$d(f) < \omega$ and $k(f) < \omega$
$V \cap S_2$	$c(f) = \omega$	or	$0 < c(f), d(f) < \omega$
$U \cap S_1$	$d(f) = \omega$	or	$0 < c(f), d(f) < \omega$
$V \cap S_5$	$k(f) < c(f) = \omega$	or	$0 < d(f) < \omega$ and $k(f) < \omega$
$U \cap S_5$	$k(f) < d(f) = \omega$	or	$0 < c(f) < \omega$
$S_{1,2,3}$	$c(f) > 0$ and $d(f) = \omega$	or	$0 < c(f), d(f) < \omega$
$S_{1,2,4}$	$c(f) = \omega$ and $d(f) > 0$	or	$0 < c(f), d(f) < \omega$
$S_{1,2,5}$	$c(f), d(f) > 0$ and $k(f) < \omega$		
$S_{1,3,4}$	$c(f) = d(f) = \omega$	or	$0 < d(f) < \omega$ and $c(f) < \omega$

$S_{1,3,5}$	$d(f) = \omega$ and $k(f) < \omega$	or	$c(f) < \omega$ and $d(f) > 0$
$S_{1,4,5}$	$c(f) = d(f) = \omega$ and $k(f) < \omega$	or	$0 < d(f) < \omega$ and $k(f) < \omega$
$S_{2,3,4}$	$c(f) = d(f) = \omega$	or	$0 < c(f) < \omega$ and $d(f) < \omega$
$S_{2,3,5}$	$c(f) = d(f) = \omega$ and $k(f) < \omega$	or	$0 < c(f) < \omega$
$S_{2,4,5}$	$c(f) = \omega$ and $k(f) < \omega$	or	$0 < c(f) < \omega$ and $d(f) < \omega$
$S_{3,4,5}$	$c(f) = d(f) = \omega$ and $k(f) < \omega$	or	$c(f), d(f) < \omega$
$V \cap S_{2,5}$	$k(f) < \omega$ and $c(f) = \omega$	or	$0 < c(f), d(f) < \omega$
$U \cap S_{1,5}$	$k(f) < \omega$ and $d(f) = \omega$	or	$0 < c(f), d(f) < \omega$
$S_{1,2,3,4}$	$c(f) = d(f) = \omega$	or	$0 < c(f), d(f) < \omega$
$S_{1,2,3,5}$	$d(f) = \omega, c(f) > 0$ and $k(f) < \omega$		
$S_{1,2,4,5}$	$c(f) = d(f) = \omega$ and $k(f) < \omega$	or	$c(f) > 0, 0 < d(f) < \omega$ and $k(f) < \omega$
$S_{1,3,4,5}$	$c(f) = d(f) = \omega$ and $k(f) < \omega$	or	$c(f) < \omega$ and $0 < d(f) < \omega$
$S_{2,3,4,5}$	$c(f) = d(f) = \omega$ and $k(f) < \omega$	or	$d(f) < \omega$ and $0 < c(f) < \omega$
$S_{1,2,3,4,5}$	$c(f) = d(f) = \omega$ and $k(f) < \omega$	or	$0 < c(f), d(f) < \omega$

In order to show that all the sets described in the table are subsemigroups, we only need to show that S_1, S_2, S_3, S_4, S_5, U, and V are subsemigroups, since all the others are just intersections of them. We already know from Theorem 2.1.7 that S_i is a semigroup for $i \in \{1, \ldots, 5\}$, and we demonstrate that so are U and V in the following proposition.

Proposition 2.2.1. *U and V are semigroups.*

Proof. Let $f, g \in U$. If $f \in \mathrm{Sym}(\Omega)$, then $d(f \circ g) = d(g)$ and $c(f \circ g) = c(g)$ and so $f \circ g \in U$. Similarly, if $g \in \mathrm{Sym}(\Omega)$, then $f \circ g \in U$. If $d(g) = \omega$, then $d(f \circ g) = \omega$ by Lemma 1.3.5(i), and so $f \circ g \in U$. If $d(f) = \omega$ and $0 < c(g) < \omega$, then by Lemma 1.3.5(v), $d(f \circ g) = \omega$, and thus $f \circ g \in U$. If $0 < c(f), c(g) < \omega$, then, by Lemma 1.3.5(iii), $0 < c(f \circ g) < \omega$ and so $f \circ g \in U$.

Let $f, g \in V$. If $f \in \mathrm{Sym}(\Omega)$ or $g \in \mathrm{Sym}(\Omega)$, then $d(f \circ g) = d(g)$ and $c(f \circ g) = c(g)$ and so $f \circ g \in V$. If $c(f) = \omega$, then $c(f \circ g) = \omega$ by Lemma 1.3.5(iii), and so $f \circ g \in V$. If $c(g) = \omega$ and $0 < d(f) < \omega$, then by Lemma 1.3.5(vi), $c(f \circ g) = \omega$, and thus $f \circ g \in V$. If $0 < d(f), d(g) < \omega$, then, by Lemma 1.3.5(i), $0 < d(f \circ g) < \omega$ and so $f \circ g \in V$. □

It is routine, using Proposition 1.3.4, to show that U and V are not equal to any of the intersections of the subsemigroups S_1, S_2, S_3, S_4, and S_5. It then follows from the descriptions of the semigroups that $S_{2,3} \lneq U \lneq S_3$ and $S_{1,4} \lneq V \lneq S_4$.

2.3 Technical lemmas

We require several technical results to prove Theorem 2.1.8, which we present in this section.

Lemma 2.3.1. *Let $f, g \in \Omega^\Omega$ be such that $\ker(f) = \ker(g)$ and $d(f) = d(g)$. Then $f \in \langle g, \mathrm{Sym}(\Omega)\rangle$.*

Proof. Let $h' : (\Omega)f \longrightarrow (\Omega)g$ be defined by $(x)h' = (y)g$ if $x = (y)f$. Then $(y)f = (y')f$ if and only if $(y, y') \in \ker(f) = \ker(g)$ which is equivalent to $(y)g = (y')g$. Hence h' is a well-defined injective function. If $x \in (\Omega)g$, then there is $y \in \Omega$ such that $(y)g = x$, and so $((y)f)h' = (y)g = x$. Therefore h' is surjective, and thus a bijection between $(\Omega)f$ and $(\Omega)g$. Let $h \in \Omega^\Omega$ be any function such that h agrees with h' on $(\Omega)f$ and bijectively maps $\Omega \setminus (\Omega)f$ to $\Omega \setminus (\Omega)g$, which is possible since $d(f) = d(g)$. Then $h \in \mathrm{Sym}(\Omega)$ and $f \circ h = g$. $\square$

The next three lemmas form the essential part of the proof of Theorem 2.1.8. Roughly speaking, all of the lemmas will consider the question, given $f, g \in \Omega^\Omega$ with certain properties, whether f can be generated by g and elements of $S_{1,2,3,4,5}$?

Lemma 2.3.2. *Let $u, v, f \in \Omega^\Omega$. Then the following hold:*

(i) *if $0 < c(u), c(f) < \omega$ and $d(u) = d(f)$, then $f \in \langle S_{1,2,3,4,5}, u\rangle$;*

(ii) *if $c(u) < \omega$, $d(u) = \omega$, $c(v) > 0$, $d(f) = d(v) = 0$, and $0 < c(f) < \omega$, then $f \in \langle S_{1,2,3,4,5}, u, v\rangle$;*

(iii) *if u is injective, $c(f) < \omega$, and $d(u) = d(f) = \omega$, then $f \in \langle S_{1,2,3,4,5}, u\rangle$;*

(iv) *if u is injective, $d(u) > 0$, $d(v) = d(f) = \omega$, and $c(v), c(f) < \omega$, then $f \in \langle S_{1,2,3,4,5}, u, v\rangle$.*

Proof. **(i).** Let $g \in \Omega^\Omega$ be such that $\ker(g) = \ker(f)$ and such that $(\Omega)g$ is a transversal of u. Then $c(g) = c(f)$ and $d(g) = c(u)$ and thus $g \in S_{1,2,3,4,5}$. Since $d(g \circ u) = d(u) = d(f)$ and $\ker(g \circ u) = \ker(g) = \ker(f)$, Lemma 2.3.1 implies that $f \in \langle g \circ u, \mathrm{Sym}(\Omega)\rangle \leq \langle u, S_{1,2,3,4,5}\rangle$.

(ii). If $c(v) < \omega$, then $f \in \langle S_{1,2,3,4,5}, v\rangle$ by Lemma 2.3.2(i). Suppose $c(v) = \omega$ and let $t \in S_{1,2,3,4,5}$ be such that $0 < c(t), d(t) < \omega$. Then $0 < c(t \circ u) < \omega$ and $d(t \circ u) = \omega$ by parts (i) and (iii) of Lemma 1.3.5. Choose $g \in \Omega^\Omega$ such that $(\Omega)t \circ u$ is a transversal of g, $(\Omega)g$ is a transversal of v and $k(g) < \omega$. Then $c(g) = d(g) = \omega$, and so $g \in S_{1,2,3,4,5}$. Also $c(t \circ u \circ g \circ v) = c(t \circ u)$, and so $0 < c(t \circ u \circ g \circ v) < \omega$

and $d(t \circ u \circ g \circ v) = 0$ and thus $f \in \langle S_{1,2,3,4,5}, t \circ u \circ g \circ v\rangle \leq \langle S_{1,2,3,4,5}, u, v\rangle$ by Lemma 2.3.2(i).

(iii). If $c(f) = 0$, then $\ker(f) = \ker(u)$ and so $f \in \langle u, S_{1,2,3,4,5}\rangle$ by Lemma 2.3.1. Suppose $0 < c(f) < \omega$. If $g \in \Omega^\Omega$ is such that $\ker(g) = \ker(f)$ and $0 < d(g) < \omega$, then $g \in S_{1,2,3,4,5}$. But $\ker(g \circ u) = \ker(g)$ since u is injective, and so $\ker(g \circ u) = \ker(f)$. Also $d(g \circ u) \geq d(u) = \omega$, and so $d(g \circ u) = d(u) = d(f)$. Thus Lemma 2.3.1 implies that $f \in \langle g \circ u, \mathrm{Sym}(\Omega)\rangle \leq \langle u, S_{1,2,3,4,5}\rangle$.

(iv). If v is injective or $d(u) = \omega$, then $f \in \langle S_{1,2,3,4,5}, v\rangle$ or $f \in \langle S_{1,2,3,4,5}, u\rangle$, respectively, by part (iii). Suppose $0 < c(v)$ and $d(u) < \omega$. Let $w \in \Omega^\Omega$ be such that $c(w) = 0$ and $(\Omega)w$ is a transversal of v. Then $c(w \circ v) = 0$ and $d(w \circ v) > d(v) = \omega$, and so $f \in \langle S_{1,2,3,4,5}, w \circ v\rangle$ by part (iii). Let $t \in \Omega^\Omega$ be any function such that $(\Omega)u$ is a transversal of t and $d(t) = d(w)$. Then $c(t) = d(u)$ and $0 < c(t) < \omega$. Since $d(w) = c(v)$, it follows that $0 < d(t) < \omega$. Hence $t \in S_{1,2,3,4,5}$. Since w and $u \circ t$ are injective, $d(u \circ t) = d(t)$ and $d(t) = d(w)$. Then Lemma 2.3.1 implies that $w \in \langle u \circ t, \mathrm{Sym}(\Omega)\rangle \leq \langle u, S_{1,2,3,4,5}\rangle$. Combining the above, $f \in \langle S_{1,2,3,4,5}, w \circ v\rangle \leq \langle S_{1,2,3,4,5}, u, v\rangle$, as required. □

We also require a result which roughly speaking is dual to Lemma 2.3.2, where the values of c and d are interchanged, and because of different nature of parameters c and d, in some cases the parameter k is introduced. It is worth noting that even though the two lemmas are not formally dual, they can be interchanged to show different parts of the proof of Theorem 2.1.8.

Lemma 2.3.3. *Let $u, v, f \in \Omega^\Omega$. Then the following hold:*

(i) *if $0 < d(u), d(f) < \omega$, $c(u) = c(f)$ and $k(f) < \omega$, then $f \in \langle S_{1,2,3,4,5}, u\rangle$;*

(ii) *if $c(u) = \omega$, $d(u) < \omega$, $c(f) = c(v) = 0$, $d(v) > 0$, and $0 < d(f) < \omega$, then $f \in \langle S_{1,2,3,4,5}, u, v\rangle$;*

(iii) *if u is surjective, $d(f), k(f) < \omega$ and $c(u) = c(f) = \omega$, then $f \in \langle S_{1,2,3,4,5}, u\rangle$;*

(iv) *if u is surjective, $c(u) > 0$, $c(v) = c(f) = \omega$, $k(f) < \omega$ and $d(v), d(f) < \omega$, then $f \in \langle S_{1,2,3,4,5}, u, v\rangle$.*

Proof. **(i).** Let $g, h \in \Omega^\Omega$ be such that $(\Omega)u$ is a transversal of g, $d(g) = d(f)$, $\ker(h) = \ker(f)$, and $(\Omega)h$ is a transversal of u. Then $\ker(f) = \ker(h \circ u \circ g)$ and $d(f) = d(h \circ u \circ g)$. Hence $f \in \langle h \circ u \circ g, \mathrm{Sym}(\Omega)\rangle$ by Lemma 2.3.1 and so it suffices to show that $g, h \in S_{1,2,3,4,5}$.

Since $c(g) = d(u)$ and $d(g) = d(f)$ by the choice of g, it follows that $0 < c(g), d(g) < \omega$ and so $g \in S_{1,2,3,4,5}$. Also h was chosen such that $c(h) = c(f)$, $d(h) = c(u)$, and $k(h) = k(f) < \omega$. So, if $c(f) = c(u) = 0$, then $h \in \mathrm{Sym}(\Omega)$. If $0 < c(f) = c(u) < \omega$, then $0 < c(h), d(h) < \omega$ and thus $h \in S_{1,2,3,4,5}$. Finally, if $c(f) = c(u) = \omega$, then $c(h) = d(h) = \omega$, $k(h) < \omega$, and so $h \in S_{1,2,3,4,5}$.

(ii). If $d(v) < \omega$, then $f \in \langle S_{1,2,3,4,5}, v\rangle$ by Lemma 2.3.3(i). Suppose $d(v) = \omega$. Let $g \in \Omega^\Omega$ be such that $(\Omega)v$ is a transversal of g, $(\Omega)g$ is contained in a transversal T of u with $0 < |T \setminus (\Omega)g| < \omega$, and $k(g) < \omega$. Then $c(g) = d(g) = \omega$, and so $g \in S_{1,2,3,4,5}$. Also $c(v \circ g \circ u) = 0$ and $0 < d(v \circ g \circ u) < \omega$. Therefore $f \in \langle S_{1,2,3,4,5}, v \circ g \circ u\rangle \leq \langle S_{1,2,3,4,5}, u, v\rangle$ by Lemma 2.3.3(i).

(iii). Let $g \in \Omega^\Omega$ be such that $\ker(g) = \ker(f)$ and $(\Omega)g$ is contained in a transversal T of u with $|T \setminus (\Omega)g| = d(f)$. Then $c(g) = d(g) = \omega$, $k(g) < \omega$ and thus $g \in S_{1,2,3,4,5}$. Also $\ker(f) = \ker(g \circ u)$ and $d(f) = d(g \circ u)$, and so $f \in \langle g \circ u, \mathrm{Sym}(\Omega)\rangle \leq \langle u, S_{1,2,3,4,5}\rangle$.

(iv). If $d(v) = 0$ or $c(u) = \omega$, then the result follows from Lemma 2.3.3(iii). Suppose that $0 < c(u), d(v) < \omega$ and let $g \in \Omega^\Omega$ be such that $(\Omega)v$ is a transversal of g and $(\Omega)g$ is a transversal of u. Then $c(g) = d(v)$ and $d(g) = c(u)$ which implies $0 < c(g), d(g) < \omega$ and $g \in S_{1,2,3,4,5}$. Therefore $f \in \langle S_{1,2,3,4,5}, v \circ g \circ u\rangle$ by Lemma 2.3.3(iii) since $c(v \circ g \circ u) = \omega$, $k(v \circ g \circ u) < \omega$ and $d(v \circ g \circ u) = 0$. □

The final technical lemma we require relates to generating transformations with parameter k being infinite, whereas the previous two lemmas are concerned with generating mappings with finite k value.

Lemma 2.3.4. *Let $u, v, t, f \in \Omega^\Omega$. Then the following hold:*

(i) *if $k(u) = k(f) = d(f) = \omega$, then $f \in \langle S_{1,2,3,4,5}, u\rangle$;*

(ii) *if $k(u) = k(f) = \omega$, $d(f), d(u) > 0$, $c(v) = \omega$, and $0 < d(v) < \omega$ then $f \in \langle S_{1,2,3,4,5}, u, v\rangle$;*

(iii) *if $k(u) = k(f) = \omega$, $c(v) = \omega$, and $d(v) = 0$, then $f \in \langle S_{1,2,3,4,5}, u, v\rangle$;*

(iv) *if $k(u) = k(f) = \omega$, $c(v) = \omega$, $d(v) < \omega$, $c(t) > 0$, and $d(t) = 0$, then $f \in \langle S_{1,2,3,4,5}, u, v, t\rangle$.*

Proof. **(i).** Let $\{K_i : i \in \mathbb{N}\}$ be the kernel classes of f, in other words the sets $\{x \in \Omega : (x)f = y\}$ for every $y \in (\Omega)f$. Since $k(f) \geq 1$, we may assume that K_0 is infinite. Let $\{L_i : i \in \mathbb{N}\}$ be the infinite kernel classes of u, and let $g \in \Omega^\Omega$ be such that $(K_0)g = \{x\}$ where $x \in L_0$, $g|_{K_i}$ is injective and $(K_i)g \subseteq L_{2i}$ for all $i \in \mathbb{N}$. Then $c(g) = d(g) = \omega$ and $k(g) = 1$ which implies that $g \in S_{1,2,3,4,5}$. Also $\ker(g \circ u) = \ker(f)$ and $d(g \circ u) = d(f) = \omega$, and so $f \in \langle g \circ u, \mathrm{Sym}(\Omega)\rangle \leq \langle u, S_{1,2,3,4,5}\rangle$ by Lemma 2.3.1.

(ii). If $d(f) = \omega$, then the result follows from Lemma 2.3.4(i). Suppose $0 < d(f) < \omega$ and let $g \in \Omega^\Omega$ be such that $\ker(f) = \ker(g)$ and $(\Omega)g$ is a transversal of v. Then $k(g) = k(f) = \omega$ and $d(g) = c(v) = \omega$, and so $g \in \langle S_{1,2,3,4,5}, u\rangle$ by Lemma 2.3.4(i). If $h \in \Omega^\Omega$ is such that $(\Omega)v$ is a transversal of h and $d(h) = d(f)$, then $0 < c(h), d(h) < \omega$ and $h \in S_{1,2,3,4,5}$. Hence $\ker(f) = \ker(g \circ v \circ h)$ and $d(f) = d(g \circ v \circ h)$. Then Lemma 2.3.1 implies that $f \in \langle g \circ v \circ h, \mathrm{Sym}(\Omega)\rangle \leq \langle u, v, S_{1,2,3,4,5}\rangle$.

(iii). Let $g \in \Omega^\Omega$ be such that $\ker(f) = \ker(g)$ and $(\Omega)g$ is contained in a transversal T of v with $|T \setminus (\Omega)g| = d(f)$. Then $k(g) = d(g) = \omega$, and so $g \in \langle S_{1,2,3,4,5}, u\rangle$ by (i). Since $\ker(f) = \ker(g \circ v)$ and $d(f) = d(g \circ v)$, Lemma 2.3.1 implies that $f \in \langle g \circ v, \mathrm{Sym}(\Omega)\rangle \leq \langle S_{1,2,3,4,5}, u, v\rangle$, as required.

(iv). If $d(v) = 0$ or $c(t) = \omega$, then $f \in \langle u, v, S_{1,2,3,4,5}\rangle$ or $f \in \langle u, t, S_{1,2,3,4,5}\rangle$, respectively, from Lemma 2.3.4(iii). Suppose $0 < c(t), d(v) < \omega$ and let $g \in \Omega^\Omega$ be such that $(\Omega)v$ is a transversal of g and $(\Omega)g$ is a transversal of t. Then $0 < c(g), d(g) < c(t)$ and so $g \in S_{1,2,3,4,5}$. Also $c(v \circ g \circ t) = \omega$ and $d(v \circ g \circ t) = 0$, and hence $f \in \langle S_{1,2,3,4,5}, u, v \circ g \circ t\rangle \leq \langle u, v, t, S_{1,2,3,4,5}\rangle$ by Lemma 2.3.4(iii). □

2.4 The proof of Theorem 2.1.8

We prove the Theorem 2.1.8 by finding the maximal subsemigroups in S_1, S_2, S_3, S_4 and S_5 containing $S_{1,2,3,4,5}$, then the maximal subsemigroups in each of those semigroups which contain $S_{1,2,3,4,5}$ and so on. We give the descriptions of the maximal subsemigroups found at each stage in a separate proposition, each of which can be proved using the following strategy.

Let S be a semigroup, let T be a subsemigroup of S, and let $\mathcal{M} = \{M_i : i \in I\}$ be a collection of subsemigroups of S, none of which are equal to S, all containing T, and such that $M_i \not\leq M_j$ for all $i, j \in I$ with $i \neq j$. Suppose that if U is a subsemigroup of S containing T and U is not contained in any $M_i \in \mathcal{M}$ then $U = S$.

Lemma 2.4.1. *If $\mathcal{M}$ is as above, then $\mathcal{M}$ is the set of maximal subsemigroups of S containing T.*

Proof. Let $M \in \mathcal{M}$, and suppose that U is subsemigroup of S such that $M \lneq U$. Then U is not a subsemigroup of any $M' \in \mathcal{M}$, and so $U = S$ by the condition on $\mathcal{M}$. Hence M is a maximal subsemigroup of S.

Suppose M is a maximal subsemigroup of S and $M \notin \mathcal{M}$. Then $M \lneq M'$ for every $M' \in \mathcal{M}$, since $M' \neq S$. It follows that $M = S$, which is a contradiction. Hence if M is maximal subsemigroup of M, then $M \in \mathcal{M}$, and so $\mathcal{M}$ is the set of all maximal subsemigroups of S. □

There are essentially 38 cases in the proof of Theorem 2.1.8. However, there are 14 pairs of cases where the proof of one case can be obtained from the proof of the other by interchanging the values of c and d, interchanging any part of Lemma 2.3.2 by the same part of Lemma 2.3.3 (or vice versa), and by interchanging part (ii), (iii), or (iv) of Lemma 2.3.4 with part (i) of the same lemma. Therefore we will not present duplicate proofs, but will only give a proof of one of the cases in each of these pairs.

Proposition 2.4.2.

(i) *The maximal subsemigroups of* S_1 *containing* $S_{1,2,3,4,5}$ *are:* $S_{1,2}$, $S_{1,3}$, $S_{1,4}$ *and* $S_{1,5}$*;*

(ii) *the maximal subsemigroups of* S_2 *containing* $S_{1,2,3,4,5}$ *are:* $S_{1,2}$, $S_{2,3}$, $S_{2,4}$ *and* $S_{2,5}$*;*

(iii) *the maximal subsemigroups of* S_3 *containing* $S_{1,2,3,4,5}$ *are:* $S_{1,3}$, U, $S_{3,4}$ *and* $S_{3,5}$*;*

(iv) *the maximal subsemigroups of* S_4 *containing* $S_{1,2,3,4,5}$ *are:* V, $S_{2,4}$, $S_{3,4}$ *and* $S_{4,5}$*;*

(v) *the maximal subsemigroups of* S_5 *containing* $S_{1,2,3,4,5}$ *are:* $S_{1,5}$, $S_{2,5}$, $S_{3,5}$ *and* $S_{4,5}$*.*

Proof. **(i).** It suffices to show that if M is any subsemigroup of S_1 containing $S_{1,2,3,4,5}$ but not contained in any of the semigroups $S_{1,2}$, $S_{1,3}$, $S_{1,4}$ and $S_{1,5}$, then $M = S_1$. Let $u_1 \in M \setminus S_{1,2}$, $u_2 \in M \setminus S_{1,3}$, $u_3 \in M \setminus S_{1,4}$ and $u_4 \in M \setminus S_{1,5}$. Then the following hold:

$$c(u_1) = 0 \text{ and } d(u_1) > 0, \qquad c(u_2) = \omega \text{ and } 0 < d(u_2) < \omega$$
$$c(u_3) < \omega \text{ and } d(u_3) = \omega, \qquad k(u_4) = \omega \text{ and } d(u_4) > 0.$$

Let $f \in S_1 \setminus S_{1,2,3,4,5}$ be arbitrary. Then one of the following holds:

1. f is injective and $0 < d(f) < \omega$ in which case by Lemma 2.3.3(ii), $f \in \langle S_{1,2,3,4,5}, u_1, u_2\rangle$;

2. $0 < d(f) < \omega$, $k(f) < \omega$ and $c(f) = \omega$ in which case by Lemma 2.3.3(i), $f \in \langle S_{1,2,3,4,5}, u_2\rangle$;

3. $c(f) < \omega$ and $d(f) = \omega$ in which case by Lemma 2.3.2(iv), $f \in \langle S_{1,2,3,4,5}, u_1, u_3\rangle$;

4. $k(f) = \omega$ and $d(f) > 0$ in which case by Lemma 2.3.4(ii), $f \in \langle S_{1,2,3,4,5}, u_2, u_4\rangle$.

Therefore
$$S_1 \leq \langle S_{1,2,3,4,5}, u_1, u_2, u_3, u_4\rangle \leq M \leq S_1,$$
giving equality throughout.

(ii). Let M be any subsemigroup of S_2 containing $S_{1,2,3,4,5}$ which is not contained in any of the given semigroups. Let $u_1 \in M \setminus S_{1,2}$, $u_2 \in M \setminus S_{2,3}$, $u_3 \in M \setminus S_{2,4}$ and $u_4 \in M \setminus S_{2,5}$. Then

$$c(u_1) > 0 \text{ and } d(u_1) = 0, \qquad c(u_2) = \omega \text{ and } d(u_2) < \omega$$
$$0 < c(u_3) < \omega \text{ and } d(u_3) = \omega, \qquad k(u_4) = \omega.$$

Let $f \in S_2 \setminus S_{1,2,3,4,5}$ be arbitrary. Then one of the following holds:

1. $0 < c(f) < \omega$ and $d(f) = 0$ in which case by Lemma 2.3.2(ii), $f \in \langle S_{1,2,3,4,5}, u_1, u_3 \rangle$;

2. $0 < c(f) < \omega$ and $d(f) = \omega$ in which case by Lemma 2.3.2(i), $f \in \langle S_{1,2,3,4,5}, u_3 \rangle$;

3. $d(f) < \omega$, $k(f) < \omega$ and $c(f) = \omega$ in which case by Lemma 2.3.3(iv), $f \in \langle S_{1,2,3,4,5}, u_1, u_2 \rangle$;

4. $k(f) = \omega$ in which case by Lemma 2.3.4(iv), $f \in \langle S_{1,2,3,4,5}, u_1, u_2, u_4 \rangle$.

Therefore

$$S_2 \leq \langle S_{1,2,3,4,5}, u_1, u_2, u_3, u_4 \rangle \leq M \leq S_2,$$

giving equality throughout.

(iii). The proof of this case follows by an argument analogous to that used in the proof of part (iv) as discussed before the proposition. It is also necessary in the case (4) of (iv) to replace the assumption that $k(f) = \omega$ by the assumption that $k(f) = \omega$ and $d(f) = \omega$, and apply Lemma 2.3.4(i) instead of Lemma 2.3.4(iv).

(iv). Let M be any subsemigroup of S_4 containing $S_{1,2,3,4,5}$ which is not contained in any of the given semigroups. Let $u_1 \in M \setminus V$, $u_2 \in M \setminus S_{2,4}$, $u_3 \in M \setminus S_{3,4}$ and $u_4 \in M \setminus S_{4,5}$. Then

$$\begin{aligned} 0 < c(u_1) < \omega \text{ and } d(u_1) = 0, \qquad & c(u_2) = 0 \text{ and } 0 < d(u_2) < \omega \\ c(u_3) = \omega \text{ and } d(u_3) < \omega, \qquad & k(u_4) = \omega. \end{aligned}$$

Let $f \in S_4 \setminus S_{1,2,3,4,5}$ be arbitrary. Then one of the following holds:

1. $d(f) < \omega$, $k(f) < \omega$ and $c(f) = \omega$ in which case by Lemma 2.3.3(iv), $f \in \langle S_{1,2,3,4,5}, u_1, u_3 \rangle$;

2. $c(f) = 0$ and $0 < d(f) < \omega$ in which case by Lemma 2.3.3(i), $f \in \langle S_{1,2,3,4,5}, u_2 \rangle$;

3. $0 < c(f) < \omega$ and $d(f) = 0$ in which case by Lemma 2.3.2(i), $f \in \langle S_{1,2,3,4,5}, u_1 \rangle$;

4. $k(f) = \omega$ in which case by Lemma 2.3.4(iv), $f \in \langle S_{1,2,3,4,5}, u_1, u_3, u_4 \rangle$.

Therefore

$$S_4 \leq \langle S_{1,2,3,4,5}, u_1, u_2, u_3, u_4 \rangle \leq M \leq S_4,$$

giving equality throughout.

(v). Let M be any subsemigroup of S_5 containing $S_{1,2,3,4,5}$ which is not contained in any of the given semigroups. Let $u_1 \in M \setminus S_{1,5}$, $u_2 \in M \setminus S_{2,5}$,

$u_3 \in M \setminus S_{3,5}$ and $u_4 \in M \setminus S_{4,5}$. Then

$$c(u_1) > 0, k(u_1) < \omega \text{ and } d(u_1) = 0, \qquad c(u_2) = 0 \text{ and } d(u_2) > 0$$
$$c(u_3) = \omega, k(u_3) < \omega \text{ and } d(u_3) < \omega, \qquad c(u_4) < \omega \text{ and } d(u_4) = \omega.$$

Let $f \in S_5 \setminus S_{1,2,3,4,5}$ be arbitrary. Then one of the following holds:

1. $d(f) < \omega$, $k(f) < \omega$ and $c(f) = \omega$ in which case by Lemma 2.3.3(iv), $f \in \langle S_{1,2,3,4,5}, u_1, u_3\rangle$;
2. $c(f) < \omega$ and $d(f) = \omega$ in which case by Lemma 2.3.2(iv), $f \in \langle S_{1,2,3,4,5}, u_2, u_4\rangle$;
3. $0 < c(f) < \omega$ and $d(f) = 0$ in which case by Lemma 2.3.2(ii), $f \in \langle S_{1,2,3,4,5}, u_1, u_4\rangle$;
4. $c(f) = 0$ and $0 < d(f) < \omega$ in which case by Lemma 2.3.3(ii), $f \in \langle S_{1,2,3,4,5}, u_2, u_3\rangle$.

Therefore

$$S_5 \leq \langle S_{1,2,3,4,5}, u_1, u_2, u_3, u_4\rangle \leq M \leq S_5,$$

giving equality throughout. □

Proposition 2.4.3.

(i) *The maximal subsemigroups of V containing $S_{1,2,3,4,5}$ are: $S_{1,4}$, $V \cap S_2$ and $V \cap S_5$;*

(ii) *the maximal subsemigroups of U containing $S_{1,2,3,4,5}$ are: $U \cap S_1$, $S_{2,3}$ and $U \cap S_5$.*

Proof. **(i).** Let $u_1 \in V \setminus S_{1,4}$, $u_2 \in V \setminus (V \cap S_2)$ and $u_3 \in V \setminus (V \cap S_5)$. Then

$$c(u_1) = \omega \text{ and } d(u_1) = 0, \quad c(u_2) = 0 \text{ and } 0 < d(u_2) < \omega, \quad k(u_3) = \omega.$$

Let $f \in V \setminus S_{1,2,3,4,5}$ be arbitrary. Then one of the following holds:

1. $0 < d(f) < \omega$ and $c(f) = 0$ in which case by Lemma 2.3.3(i), $f \in \langle S_{1,2,3,4,5}, u_2\rangle$;
2. $d(f) < \omega$, $k(f) < \omega$ and $c(f) = \omega$ in which case by Lemma 2.3.3(iii), $f \in \langle S_{1,2,3,4,5}, u_1\rangle$;
3. $k(f) = \omega$ in which case by Lemma 2.3.4(iii), $f \in \langle S_{1,2,3,4,5}, u_1, u_3\rangle$.

Hence if M is any subsemigroup of V containing $S_{1,2,3,4,5}$ which is not contained in any of the semigroups in the statement of the proposition, then $M = V$.

(ii). The proof is analogous to (i). □

Proposition 2.4.4.

(i) *The maximal subsemigroups of* $S_{1,2}$ *containing* $S_{1,2,3,4,5}$ *are:* $S_{1,2,3}$*,* $S_{1,2,4}$ *and* $S_{1,2,5}$*;*

(ii) *the maximal subsemigroups of* $S_{1,3}$ *containing* $S_{1,2,3,4,5}$ *are:* $U \cap S_1$*,* $S_{1,3,4}$ *and* $S_{1,3,5}$*;*

(iii) *the maximal subsemigroups of* $S_{1,4}$ *containing* $S_{1,2,3,4,5}$ *are:* $S_{1,2,4}$*,* $S_{1,3,4}$ *and* $S_{1,4,5}$*;*

(iv) *the maximal subsemigroups of* $S_{1,5}$ *containing* $S_{1,2,3,4,5}$ *are:* $S_{1,2,5}$*,* $S_{1,3,5}$ *and* $S_{1,4,5}$*;*

(v) *the maximal subsemigroups of* $S_{2,3}$ *containing* $S_{1,2,3,4,5}$ *are:* $S_{1,2,3}$*,* $S_{2,3,4}$ *and* $S_{2,3,5}$*;*

(vi) *the maximal subsemigroups of* $S_{2,4}$ *containing* $S_{1,2,3,4,5}$ *are:* $V \cap S_2$*,* $S_{2,3,4}$ *and* $S_{2,4,5}$*;*

(vii) *the maximal subsemigroups of* $S_{2,5}$ *containing* $S_{1,2,3,4,5}$ *are:* $S_{1,2,5}$*,* $S_{2,3,5}$ *and* $S_{2,4,5}$*;*

(viii) *the maximal subsemigroups of* $S_{3,4}$ *containing* $S_{1,2,3,4,5}$ *are:* $S_{1,3,4}$*,* $S_{2,3,4}$ *and* $S_{3,4,5}$*;*

(ix) *the maximal subsemigroups of* $S_{3,5}$ *containing* $S_{1,2,3,4,5}$ *are:* $S_{1,3,5}$*,* $U \cap S_5$ *and* $S_{3,4,5}$*;*

(x) *the maximal subsemigroups of* $S_{4,5}$ *containing* $S_{1,2,3,4,5}$ *are:* $V \cap S_5$*,* $S_{2,4,5}$ *and* $S_{3,4,5}$*.*

Proof. **(i).** Let M be any subsemigroup of $S_{1,2}$ containing $S_{1,2,3,4,5}$ which is not contained in any of the given semigroups. Then there exist $u_1, u_2, u_3 \in M$ such that:

$$c(u_1) = \omega \text{ and } 0 < d(u_1) < \omega, \quad 0 < c(u_2) < \omega \text{ and } d(u_2) = \omega,$$

and

$$k(u_3) = \omega \text{ and } d(u_3) > 0.$$

Let $f \in S_{1,2} \setminus S_{1,2,3,4,5}$ be arbitrary. Then one of the following holds:

1. $d(f) = \omega$ and $0 < c(f) < \omega$ in which case Lemma 2.3.2(i) implies that $f \in \langle S_{1,2,3,4,5}, u_2 \rangle$;
2. $c(f) = \omega$, $k(f) < \omega$ and $0 < d(f) < \omega$ in which case by Lemma 2.3.3(i), $f \in \langle S_{1,2,3,4,5}, u_1 \rangle$;

3. $k(f) = \omega$ and $d(f) > 0$ in which case Lemma 2.3.4(ii) implies that $f \in \langle S_{1,2,3,4,5}, u_1, u_3 \rangle$.

Thus $M = S_{1,2}$.

(ii). The proof is analogous to (vi).

(iii). Let M be any subsemigroup of $S_{1,4}$ containing $S_{1,2,3,4,5}$ which is not contained in any of the given semigroups. Then there exist $u_1, u_2, u_3 \in M$ such that:

$$c(u_1) = 0 \text{ and } 0 < d(u_1) < \omega, \quad c(u_2) = \omega \text{ and } 0 < d(u_2) < \omega,$$

and

$$k(u_3) = \omega \text{ and } d(u_3) > 0.$$

Let $f \in S_{1,4} \setminus S_{1,2,3,4,5}$ be arbitrary. Then one of the following holds:

1. $0 < d(f) < \omega$ and $c(f) = 0$ and so $f \in \langle S_{1,2,3,4,5}, u_1 \rangle$ by Lemma 2.3.3(i);
2. $0 < d(f) < \omega$, $k(f) < \omega$ and $c(f) = \omega$ and so $f \in \langle S_{1,2,3,4,5}, u_2 \rangle$ by Lemma 2.3.3(i);
3. $k(f) = \omega$ and $d(f) > 0$ and so $f \in \langle S_{1,2,3,4,5}, u_2, u_3 \rangle$ by Lemma 2.3.4(ii).

Thus $M = S_{1,4}$, as required.

(iv). Let M be any subsemigroup of $S_{1,5}$ containing $S_{1,2,3,4,5}$ which is not contained in any of the given semigroups. Then there exist $u_1, u_2, u_3 \in M$ such that:

$$c(u_1) = 0 \text{ and } d(u_1) > 0, \quad c(u_2) = \omega,\ k(u_2) < \omega, \text{ and } 0 < d(u_2) < \omega,$$

and

$$d(u_3) = \omega \text{ and } c(u_3) < \omega.$$

Let $f \in S_{1,5} \setminus S_{1,2,3,4,5}$ be arbitrary. Then one of the following holds:

1. $0 < d(f) < \omega$ and $c(f) = 0$ and so $f \in \langle S_{1,2,3,4,5}, u_1, u_2 \rangle$ by Lemma 2.3.3(ii);
2. $0 < d(f) < \omega$, $k(f) < \omega$ and $c(f) = \omega$ and so $f \in \langle S_{1,2,3,4,5}, u_2 \rangle$ by Lemma 2.3.3(i);
3. $d(f) = \omega$ and $c(f) < \omega$ and so $f \in \langle S_{1,2,3,4,5}, u_1, u_3 \rangle$ by Lemma 2.3.2(iv).

Thus $M = S_{1,5}$, as required.

(v). The proof is analogous to (iii).

(vi). Let M be any subsemigroup of $S_{2,4}$ containing $S_{1,2,3,4,5}$ which is not contained in any of the given semigroups. Then there exist $u_1, u_2, u_3 \in M$ such that:

$$d(u_1) = 0 \text{ and } 0 < c(u_1) < \omega, \quad d(u_2) < \omega \text{ and } c(u_2) = \omega, \quad k(u_3) = \omega.$$

Let $f \in S_{2,4} \setminus S_{1,2,3,4,5}$ be arbitrary. Then one of the following holds:

1. $d(f) = 0$ and $0 < c(f) < \omega$ in which case Lemma 2.3.2(i) implies that $f \in \langle S_{1,2,3,4,5}, u_1 \rangle$;

2. $c(f) = \omega$, $k(f) < \omega$ and $d(f) < \omega$ in which case Lemma 2.3.3(iv) implies that $f \in \langle S_{1,2,3,4,5}, u_1, u_2 \rangle$;

3. $k(f) = \omega$ in which case by Lemma 2.3.4(iv), $f \in \langle S_{1,2,3,4,5}, u_1, u_2, u_3 \rangle$.

Thus $M = S_{2,4}$.

(vii). The proof is analogous to (iv).

(viii). Let M be any subsemigroup of $S_{3,4}$ containing $S_{1,2,3,4,5}$ which is not contained in any of the given semigroups. Then there exist $u_1, u_2, u_3 \in M$ such that:

$$0 < c(u_1) < \omega \text{ and } d(u_1) = 0, \quad c(u_2) = 0 \text{ and } 0 < d(u_2) < \omega,$$

and

$$k(u_3) = \omega \text{ and } d(u_3) = \omega.$$

Let $f \in S_{3,4} \setminus S_{1,2,3,4,5}$ be arbitrary. Then one of the following holds:

1. $0 < c(f) < \omega$ and $d(f) = 0$ and so $f \in \langle S_{1,2,3,4,5}, u_1 \rangle$ by Lemma 2.3.2(i);

2. $c(f) = 0$ and $0 < d(f) < \omega$ and so $f \in \langle S_{1,2,3,4,5}, u_2 \rangle$ by Lemma 2.3.3(i);

3. $k(f) = d(f) = \omega$ and so $f \in \langle S_{1,2,3,4,5}, u_3 \rangle$ by Lemma 2.3.4(i).

Therefore $M = S_{3,4}$.

(ix). Let M be any subsemigroup of $S_{3,5}$ containing $S_{1,2,3,4,5}$ which is not contained in any of the given semigroups. Then there exist $u_1, u_2, u_3 \in M$ such that:

$$0 < c(u_1) < \omega \text{ and } d(u_1) = 0, \quad c(u_2) < \omega \text{ and } d(u_2) = \omega,$$

and

$$c(u_3) = 0 \text{ and } 0 < d(u_3) < \omega.$$

Let $f \in S_{3,5} \setminus S_{1,2,3,4,5}$ be arbitrary. Then one of the following holds:

1. $0 < c(f) < \omega$ and $d(f) = 0$ and so $f \in \langle S_{1,2,3,4,5}, u_1 \rangle$ by Lemma 2.3.2(i);

2. $c(f) < \omega$ and $d(f) = \omega$ and so $f \in \langle S_{1,2,3,4,5}, u_2, u_3 \rangle$ by Lemma 2.3.2(iv);

3. $c(f) = 0$ and $0 < d(f) < \omega$ and so $f \in \langle S_{1,2,3,4,5}, u_3 \rangle$ by Lemma 2.3.3(i).

Hence $M = S_{3,5}$.

(x). The proof is analogous to (ix). □

Proposition 2.4.5.

(i) *The maximal subsemigroups of* $V \cap S_2$ *containing* $S_{1,2,3,4,5}$ *are:* $S_{1,2,4}$ *and* $V \cap S_{2,5}$*;*

(ii) *the maximal subsemigroups of* $U \cap S_1$ *containing* $S_{1,2,3,4,5}$ *are:* $S_{1,2,3}$ *and* $U \cap S_{1,5}$*;*

(iii) *the maximal subsemigroups of* $V \cap S_5$ *containing* $S_{1,2,3,4,5}$ *are:* $S_{1,4,5}$ *and* $V \cap S_{2,5}$*;*

(iv) *the maximal subsemigroups of* $U \cap S_5$ *containing* $S_{1,2,3,4,5}$ *are:* $S_{2,3,5}$ *and* $U \cap S_{1,5}$*.*

Proof. **(i).** Let $u_1 \in (V \cap S_2) \setminus S_{1,2,4}$ and let $u_2 \in (V \cap S_2) \setminus (V \cap S_{2,5})$. Then:

$$d(u_1) = 0 \text{ and } c(u_1) = \omega, \quad k(u_2) = \omega.$$

Let $f \in V \cap S_2 \setminus S_{1,2,3,4,5}$ be arbitrary. Then one of the following holds:

1. $d(f) < \omega, k(f) < \omega$ and $c(f) = \omega$ and so $f \in \langle S_{1,2,3,4,5}, u_1\rangle$ by Lemma 2.3.3(iii);
2. $k(f) = \omega$ and so $f \in \langle S_{1,2,3,4,5}, u_1, u_2\rangle$ by Lemma 2.3.4(iii).

So, if M is any subsemigroup of $V \cap S_2$ containing $S_{1,2,3,4,5}$ which is not contained in any of the given semigroups, then $M = V \cap S_2$.

(ii). The proof is analogous to (i).

(iii). Let $u_1 \in (V \cap S_5) \setminus S_{1,4,5}$ and $u_2 \in (V \cap S_5) \setminus (V \cap S_{2,5})$. Then

$$c(u_1) = \omega,\ k(u_1) < \omega \text{ and } d(u_1) = 0, \quad c(u_2) = 0 \text{ and } 0 < d(u_2) < \omega.$$

If $f \in (V \cap S_5) \setminus S_{1,2,3,4,5}$ is arbitrary, then one of the following holds:

1. $c(f) = \omega$, $k(f) < \omega$ and $d(f) < \omega$ and so Lemma 2.3.3(iii) implies that $f \in \langle S_{1,2,3,4,5}, u_1\rangle$;
2. $c(f) = 0$, $0 < d(f) < \omega$, and $k(f) < \omega$ and so Lemma 2.3.3(i) implies that $f \in \langle S_{1,2,3,4,5}, u_2\rangle$.

So, if M is any subsemigroup of $V \cap S_5$ containing $S_{1,2,3,4,5}$ which is not contained in any of the given semigroups, then $M = V \cap S_5$.

(iv). The proof is analogous to (iii). □

Proposition 2.4.6.

(i) *The maximal subsemigroups of* $S_{1,2,3}$ *containing* $S_{1,2,3,4,5}$ *are:* $S_{1,2,3,4}$ *and* $S_{1,2,3,5}$*;*

(ii) *the maximal subsemigroups of* $S_{1,2,4}$ *containing* $S_{1,2,3,4,5}$ *are:* $S_{1,2,3,4}$ *and* $S_{1,2,4,5}$*;*

(iii) *the maximal subsemigroups of* $S_{1,2,5}$ *containing* $S_{1,2,3,4,5}$ *are:* $S_{1,2,3,5}$ *and* $S_{1,2,4,5}$*;*

(iv) *the maximal subsemigroups of* $S_{1,3,4}$ *containing* $S_{1,2,3,4,5}$ *are:* $S_{1,2,3,4}$ *and* $S_{1,3,4,5}$*;*

(v) *the maximal subsemigroups of* $S_{1,3,5}$ *containing* $S_{1,2,3,4,5}$ *are:* $S_{1,3,4,5}$ *and* $U \cap S_{1,5}$*;*

(vi) *the maximal subsemigroups of* $S_{1,4,5}$ *containing* $S_{1,2,3,4,5}$ *are:* $S_{1,2,4,5}$ *and* $S_{1,3,4,5}$*;*

(vii) *the maximal subsemigroups of* $S_{2,3,4}$ *containing* $S_{1,2,3,4,5}$ *are:* $S_{1,2,3,4}$ *and* $S_{2,3,4,5}$*;*

(viii) *the maximal subsemigroups of* $S_{2,3,5}$ *containing* $S_{1,2,3,4,5}$ *are:* $S_{1,2,3,5}$ *and* $S_{2,3,4,5}$*;*

(ix) *the maximal subsemigroups of* $S_{2,4,5}$ *containing* $S_{1,2,3,4,5}$ *are:* $S_{2,3,4,5}$ *and* $V \cap S_{2,5}$*;*

(x) *the maximal subsemigroups of* $S_{3,4,5}$ *containing* $S_{1,2,3,4,5}$ *are:* $S_{1,3,4,5}$ *and* $S_{2,3,4,5}$*.*

Proof. **(i).** Let $u_1 \in S_{1,2,3} \setminus S_{1,2,3,4}$ and let $u_2 \in S_{1,2,3} \setminus S_{1,2,3,5}$. Then

$$0 < c(u_1) < \omega \text{ and } d(u_1) = \omega, \quad k(u_2) = d(u_2) = \omega.$$

If $f \in S_{1,2,3} \setminus S_{1,2,3,4,5}$ is arbitrary, then one of the following holds:

1. $0 < c(f) < \omega$ and $d(f) = \omega$ and so $f \in \langle u_1, S_{1,2,3,4,5}\rangle$ by Lemma 2.3.2(i);
2. $k(f) = d(f) = \omega$ and so $f \in \langle u_2, S_{1,2,3,4,5}\rangle$ by Lemma 2.3.4(i).

So, if M is any subsemigroup of $S_{1,2,3}$ containing $S_{1,2,3,4,5}$ which is not contained in any of the given semigroups, then $M = S_{1,2,3}$.

(ii). Let $u_1 \in S_{1,2,4} \setminus S_{1,2,3,4}$ and $u_2 \in S_{1,2,4} \setminus S_{1,2,4,5}$. Then

$$c(u_1) = \omega,\ k(u_1) < \omega \text{ and } 0 < d(u_1) < \omega, \quad k(u_2) = \omega \text{ and } d(u_2) > 0.$$

If $f \in S_{1,2,4} \setminus S_{1,2,3,4,5}$ is arbitrary, then one of the following holds:

1. $c(f) = \omega$, $k(f) < \omega$ and $0 < d(f) < \omega$ and so $f \in \langle u_1, S_{1,2,3,4,5}\rangle$ by Lemma 2.3.3(i);
2. $k(f) = \omega$ and $d(f) > 0$ and so $f \in \langle u_1, u_2, S_{1,2,3,4,5}\rangle$ by Lemma 2.3.4(ii).

So, if M is any subsemigroup of $S_{1,2,4}$ containing $S_{1,2,3,4,5}$ which is not contained in any of the given semigroups, then $M = S_{1,2,4}$.

(iii). Let $u_1 \in S_{1,2,5} \setminus S_{1,2,3,5}$ and $u_2 \in S_{1,2,5} \setminus S_{1,2,4,5}$. Then

$$c(u_1) = \omega,\ k(u_1) < \omega \text{ and } 0 < d(u_1) < \omega, \quad 0 < c(u_2) < \omega \text{ and } d(u_2) = \omega.$$

If $f \in S_{1,2,5} \setminus S_{1,2,3,4,5}$ is arbitrary, then one of the following holds:

1. $0 < c(f) < \omega$ and $d(f) = \omega$ and so $f \in \langle u_2, S_{1,2,3,4,5}\rangle$ by Lemma 2.3.2(i);
2. $c(f) = \omega$, $k(f) < \omega$ and $0 < d(f) < \omega$ and so $f \in \langle u_1, S_{1,2,3,4,5}\rangle$ by Lemma 2.3.3(i).

So, if M is any subsemigroup of $S_{1,2,5}$ containing $S_{1,2,3,4,5}$ which is not contained in any of the given semigroups, then $M = S_{1,2,5}$.

(iv). Let $u_1 \in S_{1,3,4} \setminus S_{1,2,3,4}$ and $u_2 \in S_{1,3,4} \setminus S_{1,3,4,5}$. Then

$$c(u_1) = 0 \text{ and } 0 < d(u_1) < \omega, \quad k(u_2) = d(u_2) = \omega.$$

If $f \in S_{1,3,4} \setminus S_{1,2,3,4,5}$ is arbitrary, then one of the following holds:

1. $c(f) = 0$ and $0 < d(f) < \omega$ and so $f \in \langle u_1, S_{1,2,3,4,5}\rangle$ by Lemma 2.3.3(i);
2. $k(f) = d(f) = \omega$ and so $f \in \langle u_2, S_{1,2,3,4,5}\rangle$ by Lemma 2.3.4(i).

So, if M is any subsemigroup of $S_{1,3,4}$ containing $S_{1,2,3,4,5}$ which is not contained in any of the given semigroups, then $M = S_{1,3,4}$.

(v). Let $u_1 \in S_{1,3,5} \setminus (U \cap S_{1,5})$ and $u_2 \in S_{1,3,5} \setminus S_{1,3,4,5}$. Then

$$c(u_1) = 0 \text{ and } 0 < d(u_1) < \omega, \quad c(u_2) < \omega \text{ and } d(u_2) = \omega.$$

If $f \in S_{1,3,5} \setminus S_{1,2,3,4,5}$ is arbitrary, then one of the following holds:

1. $c(f) = 0$ and $0 < d(f) < \omega$ and so $f \in \langle u_1, S_{1,2,3,4,5}\rangle$ by Lemma 2.3.3(i);
2. $c(f) < \omega$ and $d(f) = \omega$ and so $f \in \langle u_1, u_2, S_{1,2,3,4,5}\rangle$ by Lemma 2.3.2(iv).

So, if M is any subsemigroup of $S_{1,3,5}$ containing $S_{1,2,3,4,5}$ which is not contained in any of the given semigroups, then $M = S_{1,3,5}$.

(vi). The proof is analogous to (viii).

(vii). The proof is analogous to (iv).

(viii). Let $u_1 \in S_{2,3,5} \setminus S_{1,2,3,5}$ and $u_2 \in S_{2,3,5} \setminus S_{2,3,4,5}$. Then

$$0 < c(u_1) < \omega \text{ and } d(u_1) = 0, \quad 0 < c(u_2) < \omega \text{ and } d(u_2) = \omega.$$

If $f \in S_{2,3,5} \setminus S_{1,2,3,4,5}$ is arbitrary, then one of the following holds:

1. $0 < c(f) < \omega$ and $d(f) = 0$ and so $f \in \langle u_1, S_{1,2,3,4,5}\rangle$ by Lemma 2.3.2(i);

2. $0 < c(f) < \omega$ and $d(f) = \omega$ and so $f \in \langle u_2, S_{1,2,3,4,5}\rangle$ by Lemma 2.3.2(i).

So, if M is any subsemigroup of $S_{2,3,5}$ containing $S_{1,2,3,4,5}$ which is not contained in any of the given semigroups, then $M = S_{2,3,5}$.

(ix). The proof is analogous to (v).

(x). Let $u_1 \in S_{3,4,5} \setminus S_{1,3,4,5}$ and let $u_2 \in S_{3,4,5} \setminus S_{2,3,4,5}$. Then

$$0 < c(u_1) < \omega \text{ and } d(u_1) = 0, \quad c(u_2) = 0 \text{ and } 0 < d(u_2) < \omega.$$

If $f \in S_{3,4,5} \setminus S_{1,2,3,4,5}$ is arbitrary, then one of the following holds:

1. $0 < c(f) < \omega$ and $d(f) = 0$, thus $f \in \langle u_1, S_{1,2,3,4,5}\rangle$ by Lemma 2.3.2(i);

2. $c(f) = 0$ and $0 < d(f) < \omega$, so $f \in \langle u_2, S_{1,2,3,4,5}\rangle$ by Lemma 2.3.3(i).

So, if M is any subsemigroup of $S_{3,4,5}$ containing $S_{1,2,3,4,5}$ which is not contained in any of the given semigroups, then $M = S_{3,4,5}$. □

Proposition 2.4.7.

(i) *The only maximal subsemigroup of* $V \cap S_{2,5}$ *containing* $S_{1,2,3,4,5}$ *is* $S_{1,2,4,5}$*;*

(ii) *the only maximal subsemigroup of* $U \cap S_{1,5}$ *containing* $S_{1,2,3,4,5}$ *is* $S_{1,2,3,5}$*.*

Proof. **(i)** Let $u \in (V \cap S_{2,5}) \setminus S_{1,2,4,5}$, and let $f \in \mathcal{S}_{1,2,4,5} \setminus S_{1,2,3,4,5}$. Then

$$c(u) = \omega, d(u) = 0 \text{ and } k(u) < \omega$$

and

$$c(f) = \omega, d(f) < \omega \text{ and } k(f) < \omega.$$

Hence $f \in \langle u, S_{1,2,3,4,5}\rangle$ by Lemma 2.3.3(iii). Therefore if M is any subsemigroup of $V \cap S_{2,5}$ containing $S_{1,2,3,4,5}$ which is not contained in $S_{1,2,4,5}$, then $M = V \cap S_{2,5}$.

(ii) The proof is analogous to (i). □

Proposition 2.4.8. $S_{1,2,3,4,5}$ *is maximal in* $S_{1,2,3,4}$, $S_{1,2,3,5}$, $S_{1,2,4,5}$, $S_{1,3,4,5}$, *and* $S_{2,3,4,5}$.

Proof. If $u \in S_{1,2,3,4} \setminus S_{1,2,3,4,5}$, then $k(u) = d(u) = \omega$. Hence by Lemma 2.3.4(i)

$$\langle u, S_{1,2,3,4,5}\rangle \supseteq \{f \in \Omega^\Omega : k(f) = d(f) = \omega\} \cup S_{1,2,3,4,5} = S_{1,2,3,4},$$

and so $S_{1,2,3,4,5}$ is maximal in $S_{1,2,3,4}$.

If $u \in S_{1,2,3,5} \setminus S_{1,2,3,4,5}$, then $0 < c(u) < \omega$ and $d(u) = \omega$. Therefore by Lemma 2.3.2(i)

$$\langle u, S_{1,2,3,4,5}\rangle \supseteq \{f \in \Omega^\Omega : 0 < c(f) < \omega \text{ and } d(f) = \omega\} \cup S_{1,2,3,4,5} = S_{1,2,3,5}$$

and so $S_{1,2,3,4,5}$ is maximal in $S_{1,2,3,5}$. The proof that $S_{1,2,3,4,5}$ is maximal in $S_{1,2,4,5}$ is analogous to the proof that $S_{1,2,3,4,5}$ is maximal in $S_{1,2,3,5}$, except that Lemma 2.3.3(i) is used instead.

If $u \in S_{1,3,4,5} \setminus S_{1,2,3,4,5}$, then $c(u) = 0$ and $0 < d(u) < \omega$. Thus by Lemma 2.3.3(i)

$$\langle u, S_{1,2,3,4,5} \rangle \supseteq \{f \in \Omega^{\Omega} : c(f) = 0 \text{ and } 0 < d(f) < \omega\} \cup S_{1,2,3,4,5} = S_{1,3,4,5}.$$

In particular, $S_{1,2,3,4,5}$ is maximal in $S_{1,3,4,5}$. The proof that $S_{1,2,3,4,5}$ is maximal in $S_{2,3,4,5}$ is analogous to the proof that $S_{1,2,3,4,5}$ is maximal in $S_{1,3,4,5}$, except that Lemma 2.3.2(i) is used instead. $\square$

Chapter 3

Topological generation

This chapter is based on the research conducted together with J. D. Mitchell, for the preprint see [39]. It is included in the thesis with the permission of the coauthor. In this chapter we will define and investigate the topological rank of Fraïssé limits. We use the notations defined in Section 1.3, most importantly the notion of incomplete components.

3.1 Introduction to topological generation

One of the most natural first questions, when encountering a new algebraic structure, be it a group, a semigroup, or a monoid, is to find a generating set for it. Ideally, we want to find the smallest possible such set, also known as MINIMAL GENERATING SET. The size of a minimal generating set is the RANK. Finding a minimal generating set is a classical problem in the literature of groups and semigroups.

Let G be a group, and let X be a subset of G. Then by (1.1)

$$\langle X\rangle = \bigcup_{n\in\mathbb{N}} \{x_1 \cdot x_2 \cdots x_n : x_i \in X \cup X^{-1} \text{ for all } i \in \{1, \dots, n\}\},$$

where id is the identity of G. Then

$$|X| \leq |\langle X\rangle| \leq \sum_{n\in\mathbb{N}} |X|^n.$$

Hence if X is finite, then $\langle X\rangle$ is either finite or countable, and if X is infinite then $\langle X\rangle$ has the same cardinality as X. Therefore, if G is uncountable and X is a generating set for G, it follows that $|G| = |X|$. So in this case, the rank of G is the same as the size of G, and so there is no new information captured by the notion of the rank. As a consequence a couple of alternative notions, which extend the idea of minimal generating sets and ranks, have arisen — topological rank, Sierpiński rank, and universal sequence rank. As mention previously we be will dealing with topological rank in this chapter. Universal sequence rank and, to some extent, Sierpiński rank are considered in Chapter 4.

Recall that a group G is a topological group if there is a topology on G so that multiplication and inversion, thought of as functions $\cdot : G \times G \longrightarrow G$ and ${}^{-1} : G \longrightarrow G$, are both continuous. Let G be a topological group. If the set $X \subseteq G$ generates a dense subgroup of G, then we say that X is a TOPOLOGICAL GENERATING SET for G, and $n \in \mathbb{N}$ is the TOPOLOGICAL RANK of G if it is the smallest positive integer such that $n = |X|$ for some topological generating set X for G.

Note that if the topology on G is discrete, in other words every subset of G is open, then the only dense subset of G is G itself. Hence in the case of the discrete topology the notions of generation and topological generation agree. The next result gives a sufficient condition for a topology on a finite group to be discrete.

Proposition 3.1.1. *Let X be a finite topological space such that for every $x, y \in X$ there is an open set U such that $x \in U$ and $y \notin U$. Then the topology on X is discrete.*

The topological space satisfying the condition in Proposition 3.1.1 is known as FRÉCHET SPACE.

The argument in Proposition 3.1.1, however, does not extend to infinite topological spaces. Let X be any infinite set, and define τ to be the set consisting of $\varnothing$ and every subset Y of X such that $X \setminus Y$ is finite. It is routine to verify that τ defines a topology on X. Moreover, for any two distinct elements x and y of X, $X \setminus \{y\} \in \tau$. Hence there is $U \in \tau$ such that $x \in U$, but $y \notin U$, however τ is not a discrete topology.

Throughout this chapter we will primarily be interested in Polish groups, in other words metrizable topological groups with a countable dense subset. Suppose that G is a Polish group, and let $d : G \times G \longrightarrow \mathbb{R}$ be the metric inducing the topology. For any two distinct x and y in G, let U be an open ball around x of radius $d(x, y)/2 > 0$. Then $x \in U$ and $y \notin U$. By Proposition 3.1.1 every finite Polish group has the discrete topology. Hence topological generation extends the notion of generation for finite Polish groups.

In Example 1.4.11 we have shown that for a countable Ω, the group $\mathrm{Sym}(\Omega)$ and all closed subgroups of $\mathrm{Sym}(\Omega)$ are Polish groups. Since $\mathrm{Sym}(\Omega)$ is uncountable by Proposition 1.3.9, it follows that a generating set for it must also be uncountable. In the next example we will show that if Ω is countable, then $\mathrm{Sym}(\Omega)$ has a topological generating set of size 2, which gives us an example of a group with rank different from topological rank.

Example 3.1.2. Let $\Omega = \mathbb{Z}$, let $\alpha = (0\ 1)$, and let $\beta = (\ldots -1\ 0\ 1 \ldots)$. Then for any $i, j \in \mathbb{N}$ such that $i < j$

$$(i\ j) = (j-1\ j) \circ (j-2\ j-1) \circ \ldots \circ (i+1\ i+2) \circ (i\ i+1) \circ \\ (i+1\ i+2) \circ \ldots \circ (j-2\ j-1) \circ (j-1\ j).$$

Also $(k\ k+1) = \beta^{-k} \circ \alpha \circ \beta^k$ for all $k \in \mathbb{Z}$. Hence every transposition $(i\ j)$ is an element of $\langle \alpha, \beta \rangle$.

A classical result in the finite permutation group theory states that all permutations on a finite set X can be generated by all transpositions on the set X. Therefore, $\langle \alpha, \beta \rangle$ contains all permutations with finite support. Finally, if $[\phi] \in \mathcal{B}$, then since $\mathrm{dom}(\phi)$ is finite, there exists $f \in \mathrm{Sym}(\Omega)$ with finite support extending ϕ. Hence $[\phi] \cap \langle \alpha, \beta \rangle$ is non-empty for every basic open set $[\phi]$, and so $\langle \alpha, \beta \rangle$ is dense in $\mathrm{Sym}(\Omega)$. Therefore α and β are topological generators of $\mathrm{Sym}(\Omega)$, and the topological rank is 2.

The problem of whether a given uncountable Polish group has finite topological rank is well studied in the literature. The earliest results date back to Prasad [63] and Grząślewicz [26] where they independently have shown that the group of all invertible measure preserving transformations of the unit interval has a topological rank 2. In [51] Macpherson demonstrated that the automorphism group of the random graph has topological rank 2, Sołecki in [70] proved that the group of isometries of Urysohn space has topological rank 2, and also Kechris and Rosendal in [42] proved that each of the following groups also have topological rank 2: the homeomorphism group of the Cantor space; the group of measure preserving homeomorphisms of the Cantor space; and the automorphism group of the infinitely splitting rooted tree. In fact the results by both Macpherson, and Kechris and Rosendal are stronger, namely if G is one of the groups mentioned above, then there are $f, g \in G$ such that the set $\{f^{-n} g f^n : n \in \mathbb{Z}\}$ is dense in G. Groups with this property are said to have CYCLICALLY DENSE CONJUGATION CLASS.

A further question to consider when talking about topological 2-generation is — given a topologically 2-generated group G, how easy is it to find a dense 2-generated subgroup of G? The question can be tackled in the spirit of the following classical result by Piccard in the theory of finite symmetric groups.

Theorem 3.1.3 (see [59])**.** *Let $n \in \mathbb{N}$, and let $a \in \mathrm{Sym}(n)$ be a non-identity permutation. In addition, suppose that if $n = 4$, then a is not one of the following permutations: $(1\ 2)(3\ 4)$; $(1\ 3)(2\ 4)$; or $(1\ 4)(2\ 3)$. Then there exists $b \in \mathrm{Sym}(n)$ such that $\langle a, b \rangle = \mathrm{Sym}(n)$.*

Suppose that $n = 4$, and let $H = \{\mathrm{id}, (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\}$. Then H is a subgroup of G, and since conjugation does not change the cycle structure of a permutation, it follows that H is a normal subgroup of $\mathrm{Sym}(4)$. Suppose that $a \in H$ is such that $a \neq \mathrm{id}$, and let $b \in \mathrm{Sym}(4)$. It follows from H being normal that for each $x \in H$ there is $y \in H$ such that

$$b \circ x = y \circ b,$$

and so every element of $\langle a, b\rangle$ be be written as $x \circ b^n$ for some $x \in H$ and $n \in \mathbb{N}$. However, the order of b is at most 4 and the size of H is 4, and thus $|\langle a, b\rangle| \leq 16 < 24 = |\,\mathrm{Sym}(4)|$. Hence the condition in Theorem 3.1.3 is sharp.

In [12] Darji and Mitchell demonstrated that for a countable Ω, given $f \in \mathrm{Sym}(\Omega)$ it is always possible to find $g \in \mathrm{Sym}(\Omega)$ such that $\langle f, g\rangle$ is dense in $\mathrm{Sym}(\Omega)$, and specified what type of permutation g has to be.

Let f be a permutation on Ω. Then f is a SHIFT if it has exactly one cycle (fixed points are considered to be cycles of length 1). Let $\mathcal{K}$ be a Fraïssé limit. If $f \in \mathrm{Aut}(\mathcal{K})$ is arbitrary, then we define the following subsets of $\mathrm{Aut}(\mathcal{K})$:

$$\begin{aligned} D_f(\mathcal{K}) &= \{g \in \mathrm{Aut}(\mathcal{K}) : \langle f, g\rangle \text{ is dense in } \mathrm{Aut}(\mathcal{K})\}, \\ \mathcal{I}(\mathcal{K}) &= \{g \in \mathrm{Aut}(\mathcal{K}) : g \text{ has no finite orbits}\}, \\ \mathcal{I}_\Sigma(\mathcal{K}) &= \{g \in \mathcal{I}(\mathcal{K}) : \Sigma \subset \mathcal{K} \text{ is a set of orbit representatives for } g\}, \end{aligned} \tag{3.1}$$

where the SET OF ORBIT REPRESENTATIVES of an automorphism g consists of exactly one element in every orbit of g. Note that if Ω is a set with no relations, then $\mathrm{Aut}(\Omega) = \mathrm{Sym}(\Omega)$.

Theorem 3.1.4 (Darji and Mitchell [12])**.** *Let Ω be a countably infinite set, and let $f \in \mathrm{Sym}(\Omega)$ be a non-identity permutation. Then*

(i) *if f has finite support, then there exists a shift g such that $\langle f, g\rangle$ is dense in $\mathrm{Sym}(\Omega)$, that is $g \in D_f(\Omega)$;*

(ii) *if f has infinite support, then $D_f(\Omega) \cap \mathcal{I}(\Omega)$ is comeagre in $\mathcal{I}(\Omega)$;*

(iii) *if $f \in \mathcal{I}(\Omega)$, then $D_f(\Omega)$ is comeagre in $\mathrm{Sym}(\Omega)$.*

Similar results were also obtained for the automorphism groups of the random graph $\mathrm{Aut}(R)$, and the group of order preserving automorphisms of $\mathbb{Q}$, denoted by $\mathrm{Aut}(\mathbb{Q}, \leq)$.

Theorem 3.1.5 (Darji and Mitchell [13])**.** *Let $f \in \mathrm{Aut}(\mathbb{Q}, \leq)$ be a non-identity element. Then there exists $g \in \mathrm{Aut}(\mathbb{Q}, \leq)$ such that $\langle f, g\rangle$ is dense.*

Theorem 3.1.6 (Darji and Mitchell [13])**.** *Let $f \in \mathrm{Aut}(R)$ be a non-identity element. Then $D_f(R) \cap \mathcal{I}(R)$ is comeagre in $\mathcal{I}(R)$.*

In this chapter we will investigate the topological generation of the automorphism group of the universal partial order, and the automorphism groups of ultrahomogeneous graphs. Lachlan and Woodrow showed that there are essentially four types of such countably infinite graphs, described in the following theorem.

Theorem 3.1.7 (Lachlan and Woodrow [44])**.** *The countable ultrahomogeneous graphs up to isomorphism are:*

(i) *the random graph R;*

(ii) *the universal K_n-free graph H_n, for every $n \in \mathbb{N}$, $n \geq 3$;*

(iii) *the graph ωK_n consisting of the disjoint union of countably many copies of K_n, for every $n \in \mathbb{N}$;*

(iv) *the graph nK_ω consisting of the disjoint union of $n \in \mathbb{N}$ copies of K_ω, for $n \geq 2$;*

and the duals of these graphs.

The RANDOM GRAPH is the Fraïssé limit of the class of all finite graphs, and UNIVERSAL K_n-FREE GRAPH is the Fraïssé limit of the class of all finite K_n-free graphs. Since the automorphism group of the dual of a graph is the same as the automorphism group of the original graph, it is sufficient to consider the automorphism groups of the four graphs listed in Theorem 3.1.7.

Suppose that Γ is a graph consisting of the disjoint union of countably many copies of K_n or finitely many copies of K_ω. We denote by $L_1, L_2, \ldots$ the connected components of Γ. Every $f \in \text{Aut}(\Gamma)$ or $f \in \text{Aut}(\Gamma)^{<\omega}$ induces a partial permutation $\overline{f}$ of the indices of the connected components of Γ, $\mathbb{N}$ or $\{1, 2, \ldots, n\}$, which is defined by

$$(i)\overline{f} = j \quad \text{if} \quad (L_i)f = L_j.$$

If $f \in \text{Aut}(nK_\omega)$ is a non-identity element and $\Sigma \subseteq nK_\omega$, then we define:

$$\begin{aligned} \mathcal{A}_f &= \{g \in \text{Aut}(nK_\omega) : \langle \overline{f}, \overline{g} \rangle = S_n\} \\ \mathcal{A}_{f,\Sigma} &= \{g \in \mathcal{A}_f : \Sigma \text{ is a set of orbit representatives for } g\}. \end{aligned} \tag{3.2}$$

If $n \neq 4$, then, by Theorem 3.1.3, $\mathcal{A}_f \neq \varnothing$ for all f such that $\overline{f}$ is non-identity.

The main results of the chapter are stated in the following two theorems.

Theorem 3.1.8.

(i) *$D_f(H_n) \cap \mathcal{I}(H_n)$ is comeagre in $\mathcal{I}(H_n)$ for all $f \in \text{Aut}(H_n)$ such that f is not identity.*

(ii) *$D_f(H_n) \cap \mathcal{I}_\Sigma(\omega K_n)$ is comeagre in $\mathcal{I}_\Sigma(\omega K_n)$, for all $f \in \text{Aut}(\omega K_n)$ such that support of $\overline{f}$ is infinite, and Σ is a finite subset of ωK_n.*

(iii) *Suppose that* $f \in \mathrm{Aut}(nK_\omega)$ *is such that for every finite subset* Γ *of* nK_ω *which is setwise stabilised by* f *there are components* L *and* L' *of* nK_ω *such that* $|L \cap \Gamma| \neq |L' \cap \Gamma|$*. Then* $D_f(nK_\omega) \cap \mathcal{A}_{f,\Sigma}$ *is comeagre in* $\mathcal{A}_{f,\Sigma}$ *for every finite subset* Σ *of* nK_ω*.*

It can be shown that the class of all finite partially ordered sets satisfies the conditions of Theorem 1.6.2 (Fraïssé Theorem). Hence there is a countable unique, up to isomorphisms, ultrahomogeneous partially ordered set in which every finite partially ordered set can be embedded. We call this partial order THE UNIVERSAL PARTIALLY ORDERED SET and denote it by $\mathcal{P}$.

Theorem 3.1.9. $\mathrm{Aut}(\mathcal{P})$ *has a cyclically dense conjugacy class.*

Theorem 3.1.9 is much weaker than analogous results for the structures considered in Theorem 3.1.8. The main obstacle proving stronger results for the universal partially ordered set is the transitivity property of the partial order, which prevents easily combining isomorphisms on the finite substructures. A property which is key in the proof of Theorem 3.1.8 presented here. It is also worth mentioning that it was shown in [43] that $\mathrm{Aut}(\mathcal{P})$ has a comeagre conjugacy class.

The chapter is organised as follows: some preliminaries and technical results are provided in Section 3.2; Theorem 3.1.9 is proved in Section 3.3; and the three parts of Theorem 3.1.8 will be proved in the remaining three sections.

3.2 Preliminaries

Throughout of the rest of this chapter we will assume that Ω is always countable.

First we will show in the following two propositions that R and H_n satisfy appropriate Alice restaurant properties, as defined in Section 1.5.

Proposition 3.2.1 (Lemma 3 in [19])**.** *The random graph* R *has Alice's restaurant property for graphs.*

Proof. Let U and V be two finite disjoint subsets of vertices of R. Define Γ to be a graph with the vertex set $U \cup V \cup \{u\}$ where $u \notin U \cup V$ such that for $x, y \in \Gamma$, x is adjacent to y if and only if one of the following is true:

- $x, y \in U \cup V$ and x is adjacent to y in R;
- one of x and y is in U and the other is u.

Then Γ is a finite graph, and so by definition of R, there is a subgraph Γ' of R isomorphic to Γ. Let $\Sigma \subseteq \Gamma'$ be the isomorphic copy of $U \cup V$, and let u' be the image of u under the same isomorphism. Since R is ultrahomogeneous, there is an automorphism f of R such that $(\Sigma)f = U \cup V$. Then $w = (u')f$ is adjacent to every vertex in U and not adjacent to any vertex in V. □

An analogous result holds for the universal K_n-free graph.

Proposition 3.2.2. *Let $n \geq 3$. Then the universal K_n-free graph H_n has Alice's restaurant property for K_n-free graphs.*

Proof. Let U and V be two finite disjoint subsets of vertices of H_n such that U is K_{n-1}-free. Define Γ to be a graph with the vertex set $U \cup V \cup \{u\}$ where $u \notin U \cup V$ such that for $x, y \in \Gamma$, x is adjacent to y if and only if one of the following is true:

- $x, y \in U \cup V$ and x is adjacent to y in H_n;
- one of x and y is in U and the other is u.

Since U is K_{n-1}-free, it follows that Γ is a finite K_n-free graph, and so by definition of H_n, there is a subgraph Γ' of H_n isomorphic to Γ. Let $\Sigma \subseteq \Gamma'$ be the isomorphic copy of $U \cup V$, and let u' be the image of u under the same isomorphism. The graph H_n is ultrahomogeneous, and thus there is an automorphism f of H_n such that $(\Sigma)f = U \cup V$. Then $w = (u')f$ is adjacent to every vertex in U and not adjacent to any vertex in V. □

A similar notion to the Alice's restaurant property for graphs exists for partially ordered sets as well. Let $(\mathcal{A}, \leq)$ be a partially ordered set. We say that $\mathcal{A}$ has the ALICE'S RESTAURANT PROPERTY FOR PARTIALLY ORDERED SETS if for every triple A, B, C of finite subsets of $\mathcal{A}$ such that $b \leq a$, $a \nleq c$, and $c \nleq b$ for all $a \in A$, $b \in B$, and $c \in C$, there is an element $w \in \mathcal{A} \setminus (A \cup B \cup C)$ such that $b \leq w \leq a$ and w is incomparable to c for all $a \in A$, $b \in B$, and $c \in C$. We require a stronger version of the Propositions 3.2.1 and 3.2.2 in the case of partially ordered sets.

Proposition 3.2.3. *Any countable partially ordered set which has Alice's restaurant property is isomorphic to $\mathcal{P}$.*

Proof. Let $\mathcal{A}$ be a countable partially ordered set which has Alice's restaurant property. By Theorem 1.6.2, it is sufficient to show that $\mathcal{A}$ is ultrahomogeneous and every finite partially ordered set can be embedded into $\mathcal{A}$.

Let $F = \{x_1, \ldots, x_n\}$ be a finite partially ordered set, and let $y_1 \in \mathcal{A}$ be arbitrary. Suppose that for some $k \in \{1, \ldots, n-1\}$ we have found $y_1, \ldots, y_k \in \mathcal{A}$ such that the partially ordered subsets $\{x_1, \ldots, x_k\} \subseteq F$ and $\{y_1, \ldots, y_k\} \subseteq \mathcal{A}$ are isomorphic. Define $A, B, C \subseteq \{y_1, \ldots, y_k\}$ as follows:

$$A = \{y_i : 1 \leq i \leq k \text{ and } x_i > x_{k+1}\}, \quad B = \{y_i : 1 \leq i \leq k \text{ and } x_i < x_{k+1}\},$$

and

$$C = \{y_i : 1 \leq i \leq k \text{ and } x_i \text{ is incomparable to } x_{k+1}\}.$$

Then $A \cup B \cup C = \{y_1, \ldots, y_k\}$ and the triple (A, B, C) satisfies the hypothesis of Alice's restaurant property. So there is $y_{k+1} \in \mathcal{A} \setminus \{y_1, \ldots, y_k\}$ such that for all $i \in \{1, \ldots, k\}$ $y_i > y_{k+1}$ if and only if $x_i > x_{k+1}$; $y_i < y_{k+1}$ if and only if $x_i < x_{k+1}$; and y_i is incomparable to y_{k+1} if and only if x_i is incomparable to x_{k+1}. Hence $\{x_1, \ldots, x_{k+1}\}$ is isomorphic to $\{y_1, \ldots, y_{k+1}\}$. Therefore, by induction we can construct a finite sub-structure of $\mathcal{A}$ which is isomorphic to F.

Suppose that A and B are finite substructures of $\mathcal{A}$ and $q_0 : A \longrightarrow B$ is an isomorphism. Enumerate $\mathcal{A} = \{x_i : i \in \mathbb{N}\}$ and suppose that for some $k \in \mathbb{N}$ there is an isomorphism q_k between finite substructures of $\mathcal{A}$ such that q_k extends q_{k-1} if $k > 0$, and $x_k \in \operatorname{dom}(q_k) \cap \operatorname{ran}(q_k)$.

Suppose that $x_{k+1} \in \operatorname{dom}(q_k)$. Then let $q'_k = q_k$. If $x_{k+1} \notin \operatorname{dom}(q_k)$, define $A, B, C \subseteq \operatorname{dom}(q_k)$ as follows:

$$A' = \{x \in \mathcal{A} : x > x_{k+1}\}, \quad B' = \{x \in \mathcal{A} : x < x_{k+1}\},$$

and

$$C' = \{x \in \mathcal{A} : x \text{ is incomparable to } x_{k+1}\}.$$

Then the triple (A', B', C') satisfies the hypothesis of Alice's restaurant property, except for not being finite. Since q_k is an isomorphism and $\operatorname{ran}(q_k)$ is finite, the triple $(A = (A')q_k, B = (B')q_k, C = (C')q_k)$ satisfies the hypothesis of Alice's restaurant property. Then there is $w \in \mathcal{A} \setminus \operatorname{ran}(q_k)$ such that $w < a$ for all $a \in A$, $w > b$ for all $b \in B$, and w is incomparable to all $c \in C$. Hence $q'_k = q_k \cup \{(x_{k+1}, w)\}$ is an isomorphism between finite substructures of $\mathcal{A}$ and $x_{k+1} \in \operatorname{dom}(q'_k)$.

If $x_{k+1} \in \operatorname{ran}(q'_k)$, then $q_{k+1} = q'_k$ is as required. If $x_{k+1} \notin \operatorname{ran}(q'_k)$, then $x_{k+1} \notin \operatorname{dom}(q'^{-1}_k)$, and so the argument of the last paragraph can be applied to extend q'^{-1}_k to q^{-1}_{k+1} such that q_{k+1} is an isomorphism between finite substructures of $\mathcal{A}$ and $x_{k+1} \in \operatorname{dom}(q^{-1}_{k+1}) = \operatorname{ran}(q_{k+1})$.

Hence, for all $k \in \mathbb{N}$, we define q_k, as described above. Let $f = \bigcup_{k \in \mathbb{N}} q_k$. Since q_k is an isomorphism and q_k extends q_{k-1} for all $k > 0$, it follows that f is a well-defined isomorphism. Also $x_k \in \operatorname{dom}(q_k) \cap \operatorname{ran}(q_k) \subseteq \operatorname{dom}(f) \cap \operatorname{ran}(f)$ for all $k \in \mathbb{N}$, and so $\operatorname{dom}(f) \cap \operatorname{ran}(f) = \mathcal{A}$. Therefore, f is an automorphism extending q, and so $\mathcal{A}$ is ultrahomogeneous, as required. □

Note that Proposition 3.2.3 is much stronger than Propositions 3.2.1 and 3.2.2. In truth, analogous stronger results hold for both R and H_n. However, we choose not to include these because the proofs follow the same idea as the proof of Proposition 3.2.3, they are somewhat lengthy, and the results are not used anywhere in the thesis. In fact, Alice's restaurant properties mentioned in this section are all special cases of EXISTENTIAL CLOSURE, which is beyond the scope of this thesis. By [28, Example 2 in Section 7.1] every Fraïssé limit $\mathcal{K}$ with signature L is existentially closed in the class of all L-structures which have their age contained in the age of $\mathcal{K}$. This result can be used to prove the last three propositions.

We now will show that $\mathcal{I}(\mathcal{K})$ and $\mathcal{I}_\Sigma(\mathcal{K})$ (as defined in (3.1)) are Baire spaces with the subspace topology of $\mathrm{Aut}(\mathcal{K})$ for any countably infinite structure $\mathcal{K}$, and that $\mathcal{A}_{f,\Sigma}$ and $\mathcal{A}_f$ (defined in (3.2)) are Baire subspaces of $\mathrm{Aut}(nK_\omega)$.

Lemma 3.2.4. *Let $\mathcal{K}$ be a countably infinite structure. Then $\mathcal{I}(\mathcal{K})$ is a closed subspace of* $\mathrm{Aut}(\mathcal{K})$*.*

Proof. Let $f \in \mathrm{Aut}(\mathcal{K})\setminus\mathcal{I}(\mathcal{K})$, then f has a finite orbit O and hence $[f|_O]\cap \mathrm{Aut}(\mathcal{K})$ is a subset of $\mathrm{Aut}(\mathcal{K}) \setminus \mathcal{I}(\mathcal{K})$. Hence $\mathrm{Aut}(\mathcal{K}) \setminus \mathcal{I}(\mathcal{K})$ is open, and so $\mathcal{I}(\mathcal{K})$ is closed. □

Lemma 3.2.5. *Let $f \in nK_\omega$. Then $\mathcal{A}_f$ is a closed subspace of* $\mathrm{Aut}(nK_\omega)$*.*

Proof. Let $g \in \mathrm{Aut}(nK_\omega) \setminus \mathcal{A}_f$. Then $\langle \overline{f}, \overline{g}\rangle \neq S_n$. Let $\Gamma \subseteq nK_\omega$ be a finite set containing at least one vertex in every connected component of nK_ω. Then for all $h \in [g|_\Gamma]$ we have that $\overline{h} = \overline{g}$ and thus $h \notin \mathcal{A}_f$. Therefore, the open set $[g|_\Gamma]$ is a subset of $\mathrm{Aut}(nK_\omega) \setminus \mathcal{A}_f$ and thus $\mathcal{A}_f$ is closed. □

Recall that it was shown in Example 1.4.11 that $\mathrm{Sym}(\Omega)$ is a Polish group, and so is every closed subgroup of $\mathrm{Sym}(\Omega)$. If $\mathcal{K}$ is a Fraïssé limit, then $\mathrm{Aut}(\mathcal{K})$ is a Polish group by Theorem 1.6.3. Then since $\mathcal{I}(\mathcal{K})$ and $\mathcal{A}_f$ for all $f \in \mathrm{Aut}(nK_\omega)$ are closed in $\mathrm{Aut}(\mathcal{K})$ and $\mathrm{Aut}(nK_\omega)$ respectively, it follows from Proposition 1.4.5(i) that both $\mathcal{I}(\mathcal{K})$ and $\mathcal{A}_f$ are Polish spaces. Hence $\mathrm{Aut}(\mathcal{K})$, $\mathcal{I}(\mathcal{K})$, and $\mathcal{A}_f$ are all Baire space by Theorem 1.4.17. That $\mathcal{I}_\Sigma(\mathcal{K})$ and $\mathcal{A}_{f,\Sigma}$ are Baire follows immediately from the next lemma, and the preceding discussion.

Let $\mathcal{K}$ be a Fraïssé limit, let S be a subset of $\mathrm{Aut}(\mathcal{K})$, and let $\Sigma \subseteq \mathcal{K}$. Define $S_\Sigma = \{g \in S : \Sigma \text{ is a set of orbit representatives of } g\}$. The definition agrees with the definitions of $\mathcal{I}_\Sigma(\mathcal{K})$ and $\mathcal{A}_{f,\Sigma}$.

Lemma 3.2.6. *Let Ω be countable, let T be a Polish subspace of* $\mathrm{Sym}(\Omega)$ *and let $\Sigma \subseteq \Omega$ be finite. Then T_Σ is a Polish space.*

Proof. Let K be the set of those $g \in T$ such that distinct elements of Σ belong to different orbits of g. We will show that K is a closed subset of T. If $T = K$, then K is closed in T. Otherwise, let $g \in T \setminus K$. Then there exist $x, y \in \Sigma$ and $m \in \mathbb{N}$ such that $(x)g^m = y$. If $\Gamma = \{(x)g^i : 0 \leq i \leq m\}$, then $[g|_\Gamma] \cap T$ is a subset of $T \setminus K$. Hence $T \setminus K$ is open, and so K is closed in T, implying that K is a Polish space by Proposition 1.4.5(i).

For an arbitrary $x \in \Omega$, we denote by A_x the set of all those $g \in K$ such that the orbit of x under g has non-trivial intersection with Σ. Then $T_\Sigma = \bigcap_{x\in\Omega} A_x \subseteq K$. Suppose that $g \in A_x$. Then there is $n \in \mathbb{Z}$ and $y \in \Sigma$ such that $(y)g^n = x$. If $\Gamma' = \{(y)g^i : -|n| \leq i \leq |n|\}$. Then $[g|_{\Gamma'}] \cap K$ is a subset of A_x, and so A_x is open in K for all $x \in \Omega$. Therefore T_Σ is a G_δ subset of K, and by Theorem 1.4.8 it is a Polish space. □

We end this section by proving two lemmas that will be used repeatedly later in the chapter.

Lemma 3.2.7. *Let $\mathcal{K}$ be a Fraïssé limit. Then for every $f \in \operatorname{Aut}(\mathcal{K})$ and any $p \in \operatorname{Aut}(\mathcal{K})^{<\omega}$*

$$\{g \in \operatorname{Aut}(\mathcal{K}) : \langle f, g\rangle \cap [p] \neq \varnothing\}$$

is an open set in $\operatorname{Aut}(\mathcal{K})$.

Proof. Let $f \in \operatorname{Aut}(\mathcal{K})$, and let $p \in \operatorname{Aut}(\mathcal{K})^{<\omega}$. Suppose $g \in \operatorname{Aut}(\mathcal{K})$ is such that $\langle f, g\rangle \cap [p]$ is non-empty. Then there are $n_1, \ldots, n_{2k} \in \mathbb{Z}$ such that

$$f^{n_1} g^{n_2} \cdots g^{n_{2k}} \in [p].$$

For $i \in \{0, \ldots, k\}$, let $\Gamma_i = (\operatorname{dom}(p))\, f^{n_1} g^{n_2} \cdots g^{n_{2i}}$ and define

$$\Gamma = \bigcup_{i=1}^{k} \Gamma_i.$$

Since $\operatorname{dom}(p)$ is finite, it follows that each Γ_i is finite, and therefore Γ is finite.

Let $h \in [g|_\Gamma]$. By the definition of Γ, it follows that for all $x \in \operatorname{dom}(h)$ and all $i \in \{0, \ldots, k\}$

$$(x) f^{n_1} g^{n_2} \cdots g^{n_{2i+1}} = (x) f^{n_1} h^{n_2} \cdots h^{n_{2i+1}}.$$

Therefore, $\langle f, h\rangle \cap [p]$ is non-empty for all $h \in [g|_\Gamma]$, and so the set $\{g \in \operatorname{Aut}(\mathcal{K}) : \langle f, g\rangle \cap [p] \neq \varnothing\}$ is open. □

The last lemma of this section relates comeagreness of $D_f(\mathcal{K}) \cap S_\Sigma$ in S_Σ to comeagreness of $D_f(\mathcal{K}) \cap S$ in S, for any Polish subspace S of $\operatorname{Aut}(\mathcal{K})$.

Lemma 3.2.8. *Let $\mathcal{K}$ be any Fraïssé limit, let $f \in \operatorname{Aut}(\mathcal{K})$, and let $S \subseteq \operatorname{Aut}(\mathcal{K})$ be a Baire space such that every $q \in S^{<\omega}$ has an extension in S with only finitely many orbits. If S_Σ is Baire, and $D_f(\mathcal{K}) \cap S_\Sigma$ is dense in S_Σ for every finite $\Sigma \subseteq \mathcal{K}$, then $D_f(\mathcal{K}) \cap S$ is comeagre in S.*

Proof. Recall that $D_f(\mathcal{K}) = \{g \in \operatorname{Aut}(\mathcal{K}) : \langle f, g\rangle \text{ is dense in } \operatorname{Aut}(\mathcal{K})\}$. Then since $\{[q] : q \in \operatorname{Aut}(\mathcal{K})^{<\omega}\}$ is a basis for the topology on $\operatorname{Aut}(\mathcal{K})$, it follows that

$$D_f(\mathcal{K}) \cap S = \bigcap_{p \in \operatorname{Aut}(\mathcal{K})^{<\omega}} \{g \in S : \langle f, g\rangle \cap [p] \neq \varnothing\}.$$

The set $\{g \in S : \langle f, g\rangle \cap [p] \neq \varnothing\}$ is open in S by Lemma 3.2.7, and so it suffices to show that $\{g \in S : \langle f, g\rangle \cap [p] \neq \varnothing\}$ is dense in S for all $p \in \operatorname{Aut}(\mathcal{K})^{<\omega}$.

Let $q \in S^{<\omega}$. By the hypothesis there is $g \in S$ which extends q and has a finite number of orbits. Let Σ be any set of orbit representatives of g. Then

$q \in S_\Sigma^{<\omega}$. Since $D_f(\mathcal{K}) \cap S_\Sigma$ is comeagre in S_Σ and S_Σ is Baire, it follows that $D_f(\mathcal{K}) \cap S_\Sigma$ is dense in S_Σ. Hence there is $h \in [q]$ such that $h \in D_f(\mathcal{K}) \cap S_\Sigma$. In other words, for each $q \in \mathrm{Aut}(\mathcal{K})^{<\omega}$ there exists $h \in [q]$ such that $\langle f, h\rangle$ is dense in $\mathrm{Aut}(\mathcal{K})$ and so $\{g \in S : \langle f, g\rangle \cap [p] \neq \varnothing\}$ is dense in S. □

3.3 Universal partial order

We will show in this section, that the topological rank of $\mathrm{Aut}(\mathcal{P})$ is 2.

The following lemma provides a sufficient condition to extend an isomorphism of a finite substructure of $\mathcal{P}$. In ordered to state the lemma, we need some more notation. Let $x \in \mathcal{P}$. Then we define

$$\begin{aligned} A(x) &= \{u \in \mathcal{P} : u \geq x\}, \\ B(x) &= \{u \in \mathcal{P} : u \leq x\}, \\ C(x) &= \{u \in \mathcal{P} : u \text{ and } x \text{ are incomparable}\}. \end{aligned}$$

Lemma 3.3.1. *Let $q \in \mathrm{Aut}(\mathcal{P})^{<\omega}$, and let $x, y \in \mathcal{P}$. Suppose that $x \notin \mathrm{dom}(q)$, $y \notin \mathrm{ran}(q)$, $A(y) \cap \mathrm{ran}(q) = (A(x))\,q$, and $B(y) \cap \mathrm{ran}(q) = (B(x))\,q$. Then $q \cup \{(x, y)\} \in \mathrm{Aut}(\mathcal{P})^{<\omega}$.*

Proof. Since $\mathcal{P}$ is ultrahomogeneous, it is sufficient to show that $q \cup \{(x, y)\}$ is an isomorphism between two finite partially ordered subsets of $\mathcal{P}$. By the hypothesis, q is a partial isomorphism, and so it suffices to show that $x \leq z$ if and only if $y \leq (z)q$, and $z \leq x$ if and only if $(z)q \leq y$ for every $z \in \mathrm{dom}(q)$.

Let $z \in \mathrm{dom}(q)$. Then $x \leq z$ if and only if $z \in A(x)$ which is equivalent to $(z)q \in A(y) \cap \mathrm{ran}(q)$, in other words $y \leq (z)q$.

Similarly to the previous case, $z \leq x$ if and only if $z \in B(x)$ which is equivalent to $(z)q \in B(y) \cap \mathrm{ran}(q)$, or alternatively $(z)q \leq y$. □

Next we present a corollary of Lemma 3.3.1.

Corollary 3.3.2. *Let $q \in \mathrm{Aut}(\mathcal{P})^{<\omega}$, and let $x \notin \mathrm{dom}(q)$. Then there is $y \notin \mathrm{dom}(q) \cup \mathrm{ran}(q)$ such that $q \cup \{(x, y)\} \in \mathrm{Aut}(\mathcal{P})^{<\omega}$.*

Proof. Let $A = (A(x))\,q$, let $B = (B(x))\,q$, and let

$$C = (C(x))\,q \cup \{z \in \mathrm{dom}(q) : z \nleq b \text{ and } a \nleq z \text{ for all } a \in A \text{ and } b \in B\}.$$

We will show that the triple A, B, C satisfy the condition of the Alice's restaurant property.

Let $a \in A$, and let $b \in B$. First consider the case where $c \in (C(x))\,q$. Then there are $a', b', c' \in \mathcal{P}$ such that $b' \leq x \leq a'$, x and c' are incomparable, and also $a = (a')q$, $b = (b')q$, and $c = (c')q$. Note that if $c' \leq b'$, then $c' \leq x$, which is impossible, and so $c' \not\leq b'$. A similar argument shows that $a' \not\leq c'$. Since q is an isomorphism $b = (b')q \leq (a')q = a$, and similarly $c = (c')q \not\leq (b')q = b$, $a = (a')q \not\leq (c')q = c$. In the case where $c \in C \setminus (C(x))\,q$, $a \not\leq c$ and $c \not\leq b$. Hence A, B, C satisfy the condition of the Alice's restaurant property.

By Alice's restaurant property there is $y \in \mathcal{P} \setminus (A \cup B \cup C)$ such that $A(y) \cap \mathrm{ran}(q) = A = (A(x))\,q$, and $B(y) \cap \mathrm{ran}(q) = B = (B(x))\,q$. Note that $\mathrm{ran}(q) \subseteq A \cup B \cup C$, and so $y \notin \mathrm{ran}(q)$. Therefore Lemma 3.3.1 implies that $q \cup \{(x, y)\} \in \mathrm{Aut}(\mathcal{P})^{<\omega}$. Moreover, if $y \in \mathrm{dom}(q)$, then $y \in C$, which is a contradiction, thus $y \notin \mathrm{dom}(q) \cup \mathrm{ran}(q)$. □

In the following lemma, we will show that there is an automorphism of the universal partially ordered set which satisfies a particular condition, which roughly speaking allows us to separate any two finite sets. This lemma is the essential part of proving that $\mathcal{P}$ has a 2-generated dense subgroup.

Lemma 3.3.3. *There is $f \in \mathrm{Aut}(\mathcal{P})$ and $x \in \mathcal{P}$ such that for all finite subsets Σ of $\mathcal{P}$ there are $n, m \in \mathbb{Z}$ with $(x)f^n \leq y \leq (x)f^m$ for all $y \in \Sigma$.*

A proof, alternative to the one which will be given here, can be obtained from a paper by Rubin [65]. For a partially ordered set $(A, \leq)$, Rubin defines $f \in \mathrm{Aut}(A, \leq)$ to be EMBRACING if for every $a, b \in A$ we have that $a < (a)f$ and there are $n, m \in \mathbb{Z}$ such that $(a)f^n < b < (a)f^m$. Note that an embracing automorphism of $\mathcal{P}$ satisfies the conclusion of Lemma 3.3.3. It is shown in [65, Lemma 4.6(a)] that for every $z \in \mathcal{P}$ there is $f \in \mathrm{Aut}(\mathcal{P})$ such that $(z)f = z$ and if A is one of the sets $\{x \in \mathcal{P} : x > z\}$, $\{x \in \mathcal{P} : x < z\}$, or $\{x \in \mathcal{P} : x \text{ is incomparable to } z\}$ then $f\,|_A$ is an embracing automorphism of A. Finally, note that using Lemma 3.2.3 it can be shown that $\{x \in \mathcal{P} : x > z\}$ is isomorphic to $\mathcal{P}$, and so there exists an embracing automorphism of $\mathrm{Aut}(\mathcal{P})$.

Proof. Since by Proposition 3.2.3 every countable partially ordered set satisfying Alice's restaurant property is isomorphic to $\mathcal{P}$, it is sufficient to construct a countable partially ordered set $\mathcal{S}$ and an automorphism f of $\mathcal{S}$ such that $\mathcal{S}$ has Alice's restaurant property, and for all finite subsets Σ of $\mathcal{S}$ there are $n, m \in \mathbb{Z}$ with $(u)f^n \leq y \leq (u)f^m$ for all $y \in \Sigma$.

Let $\Gamma_0 = \mathbb{Z}$ with the usual order of the integers, denoted by $\leq_0$, let $f_0 : \Gamma_0 \longrightarrow \Gamma_0$ given by $(x)f_0 = x + 1$, and fix $u \in \Gamma_0$. Let $n \geq 1$, and suppose that for all $m \in \{1, \ldots, n\}$, a countable partially ordered set $(\Gamma_m, \leq_m)$, and a partial isomorphism f_m of Γ_m which extends f_{m-1} if $m > 0$ are defined such that:

1. $\Gamma_{m-1} \subseteq \Gamma_m$;

2. if $x, y \in \Gamma_{m-1}$ then $x \leq_m y$ if and only if $x \leq_{m-1} y$;

3. we have that $\Gamma_{m-1} \subseteq \mathrm{dom}(f_m) \cap \mathrm{ran}(f_m)$;

4. for every $x \in \Gamma_m \setminus \Gamma_0$ we have that $(u)f_m^{-m} \leq_m x \leq_m (u)f_m^m$;

5. for every triple (A, B, C) of finite subsets of $\Gamma_{m-1} \setminus \{(u)f_m^k : |k| > m\}$ satisfying the condition of Alice's restaurant property, there is $w \in \Gamma_m$ such that $b \leq_m w \leq_m a$, and w is incomparable to c for all $a \in A$, $b \in B$, and $c \in C$.

Let $\mathcal{T}_n$ be the set of all triples (A, B, C), such that A, B, and C are finite subsets of $\Gamma_n \setminus \{(u)f_n^k : |k| > n+1\}$ such that $(u)f_n^{-(n+1)} \in B$, $(u)f_n^{n+1} \in A$, and the triple (A, B, C) satisfies the condition of Alice's restaurant property with respect to $\leq_n$. Then $\mathcal{T}_n$ is countable, since Γ_n is countable.

For every $T \in \mathcal{T}_n$, let v_T be a distinct element not in the set Γ_n, and let $\Gamma_{n+1} = \Gamma_n \cup \{v_T : T \in \mathcal{T}_n\}$. For $x, y \in \Gamma_{n+1}$, define $x \leq_{n+1} y$ if at least one of the following conditions is satisfied:

- $x = y$;
- $x, y \in \Gamma_n$ and $x \leq_n y$;
- $x = v_T$ for some $T = (A, B, C) \in \mathcal{T}_n$ and $y \in \Gamma_n$ such that $a \leq_n y$ for some $a \in A$;
- $x \in \Gamma_n$ and $y = v_T$ for some $T = (A, B, C) \in \mathcal{T}_n$ such that $x \leq_n b$ for some $b \in B$;
- $x = v_T$ and $y = v_{T'}$ for some $T = (A, B, C), T' = (A', B', C') \in \mathcal{T}_n$, and $a \leq_n b$ for some $a \in A$, $b \in B'$.

We will now show that $\leq_{n+1}$ is a partial order on Γ_{n+1}. It's easy to see that $\leq_{n+1}$ is reflexive and antisymmetric. For example, if $x \leq_{n+1} y$ and $x = v_T$ and $y = v_{T'}$ for some $T = (A, B, C), T' = (A', B', C') \in \mathcal{T}_n$, then $a \leq_n b$ for some $a \in A$ and $b \in B'$. Hence $y \leq_{n+1} x$ if and only if there is $a' \in A'$ and $b' \in B$ such that $a' \leq_n b'$. However, for all $a' \in A$ and all $b' \in B$ we have that $b' \leq_n a \leq_n b \leq_n a'$. The other cases follow by similar arguments. In order to show that $\leq_{n+1}$ is transitive, let $x, y, z \in \Gamma_{n+1}$ be such that $x \leq_{n+1} y$, and $y \leq_{n+1} z$. If $x, y, z \in \Gamma_n$ or any two elements from $\{x, y, z\}$ are equal, then $x \leq_{n+1} z$. Hence we may assume that at least one of x, y, z is equal to v_T for some $T \in \mathcal{T}_n$, and we may also assume that all three vertices x, y, and z are distinct. The following argument is a tedious case by case analysis.

Case 1. Suppose that $x = v_T$ for some $T = (A, B, C) \in \mathcal{T}_n$ and $y, z \in \Gamma_n$. Then since $y \in \Gamma_n$, by definition there is $a \in A$ with $a \leq_n y$. Also $y \leq_n z$ as both y and z are in Γ_n. Hence $a \leq_n z$ and $x \leq_{n+1} z$ by the definition.

Case 2. Suppose that $y = v_T$ for some $T = (A, B, C) \in \mathcal{T}_n$ and $x, z \in \Gamma_n$. Then since $x, z \in \Gamma_n$, there are $a \in A$ and $b \in B$ with $a \leq_n z$ and $x \leq_n b$. Since T satisfies the condition of Alice's restaurant property, it follows that $b \leq_n a$. Hence $x \leq_n z$ by transitivity of $\leq_n$, and so $x \leq_{n+1} z$.

Case 3. Suppose that $z = v_T$ for some $T = (A, B, C) \in \mathcal{T}_n$ and $x, y \in \Gamma_n$. Then there is $b \in B$ with $y \leq_n b$. Also $x \leq_n y$ as both x and y are in Γ_n. Hence $x \leq_n b$, and so $x \leq_{n+1} z$.

Case 4. Suppose that $x = v_T$, $y = v_{T'}$, and $z \in \Gamma_n$ for some $T, T' \in \mathcal{T}_n$. Let $T = (A, B, C)$, and let $T' = (A', B', C')$. Then by definition there are $a \in A$, $b' \in B'$, and $a' \in A'$ such that $a \leq_n b'$ and $a' \leq_n z$. Since $b' \leq_n a'$ it follows that $a \leq_n z$, and so $x \leq_{n+1} z$.

Case 5. Suppose that $x = v_T$, $y \in \Gamma_n$, and $z = v_{T'}$ for some $T, T' \in \mathcal{T}_n$. Let $T = (A, B, C)$, and let $T' = (A', B', C')$. Then there are $a \in A$, and $b' \in B'$ such that $a \leq_n y$ and $y \leq_n b'$. Hence $a \leq_n b'$ by transitivity of $\leq_n$, and so $x \leq_{n+1} z$.

Case 6. Suppose that $x \in \Gamma_n$, $y = v_T$, and $z = v_{T'}$ for some $T, T' \in \mathcal{T}_n$. Let $T = (A, B, C)$, and let $T' = (A', B', C')$. Then there are $b \in B$, $a \in A$, and $b' \in B'$ such that $a \leq_n b'$ and $x \leq_n b$. Since $b \leq_n a$ it follows that $x \leq_n b'$, and so $x \leq_{n+1} z$.

Case 7. Finally, suppose that $x = v_T$, $y = v_{T'}$, and $z = v_{T''}$ for some $T, T', T'' \in \mathcal{T}_n$. Let $T = (A, B, C)$, $T' = (A', B', C')$, and $T'' = (A'', B'', C'')$. Then there are $a \in A$, $b' \in B'$, $a' \in A'$ and $b'' \in B''$ such that $a \leq_n b'$ and $a' \leq_n b''$. Since also $b' \leq_n a'$, it follows that $a \leq_n b''$ and so $x \leq_{n+1} z$.

In each of the seven cases $x \leq_{n+1} z$, and thus $\leq_{n+1}$ is transitive. Therefore $\leq_{n+1}$ a partial order on Γ_{n+1}. Hence conditions 1 and 2 are satisfied. Next we will inductively define the isomorphism f_{n+1} between substructures of Γ_{n+1}.

Denote $\Gamma_n \setminus \mathrm{dom}(f_n)$ by $\{x_i : i \in I\}$, where $I = \{0, \ldots, m\}$ for some $m \in \mathbb{N}$ or $I = \mathbb{N}$ if $\Gamma_n \setminus \mathrm{dom}(f_n)$ is infinite. Let $g_0 = f_n$. Suppose that for $k \geq 1$ there is an extension g_k of g_{k-1} such that g_k is an isomorphism between substructures of Γ_{n+1} and that $x_i \in \mathrm{dom}(g_k)$ for all $i \in \{0, \ldots, k-1\}$.

If $x_k \in \mathrm{dom}(g_k)$, let $g_{k+1} = g_k$. Then g_{k+1} is an extension of g_k and $x_i \in \mathrm{dom}(g_{k+1})$ for all $i \in \{0, \ldots, k\}$. Suppose that $x_k \notin \mathrm{dom}(g_k)$. Let $A = \{y \in \Gamma_n : x_k \leq_{n+1} y\}$, let $B = \{y \in \Gamma_n : y \leq_{n+1} x_k\}$, and let $C = \Gamma_n \setminus (A \cup B)$. Then it is routine to show that the triple (A, B, C) satisfies the conditions of Alice's restaurant property. Since g_k is an isomorphism, it follows that the triple (A', B', C') where $A' = (A)g_k$, $B' = (B)g_k$, and $C' = (C)g_k$ also satisfies the condition of Alice's restaurant property. Since $u \in \Gamma_0$, and both g_k and f_n are extensions of f_0, it follows that $(u)g_k^{-n} = (u)f_0^{-n} = (u)f_n^{-n}$, and similarly $(u)g_k^n = (u)f_n^n$. Hence condition 4 of the inductive hypothesis together with the fact that $x_k \in \Gamma_n$ imply that

$$(u)g_k^{-n} \leq_{n+1} x_k \leq_{n+1} (u)g_k^n.$$

Hence $(u)g_k^n \in A$, and since $(u)g_k^{-(n+2)} = (u)f_n^{-(n+2)} \leq_n (u)g_k^{-n}$, it also follows that $(u)g_k^{-(n+2)} \in B$. Therefore, $(u)g_k^{-n-1} \in (B)g_k = B'$ and $(u)g_k^{n+1} \in (A)g_k = A'$. Hence $T = (A', B', C') \in \mathcal{T}_n$. Define $g_{k+1} = g_k \cup \{(x_k, v_T)\}$.

If $y \in \mathrm{dom}(g_k)$ such that $x_k \leq_{n+1} y$, then $y \in A$ by definition of $\leq_{n+1}$, and so $(y)g_{k+1} \in (A)g_k = A'$. Hence $v_T \leq_{n+1} (y)g_{k+1}$. A similar argument show that if $y \leq_{n+1} x_k$ or y is incomparable to x_k, then respectively $(y)g_{k+1} \leq_{n+1} v_T$ or $(y)g_{k+1}$ and v_T are incomparable. Then g_{k+1} is a partial isomorphism of Γ_{n+1} such that $x_i \in \text{dom}(g_{k+1})$ for all $i \in \{0, \dots, k\}$.

Therefore, by induction, for every $k \in I$ there is g_k an isomorphism extending g_{k-1} such that $x_i \in \text{dom}(g_k)$ for all $i \in \{0, \dots, k-1\}$. Let $g = \bigcup_{k \in I} g_k$. It is routine to show that g is a partial isomorphism of Γ_{n+1} such that $\Gamma_n \subseteq \text{dom}(g)$. Similarly, considering g^{-1} instead of f_n, we may extend g to an isomorphism f_{n+1} between substructures of Γ_{n+1} such that $\Gamma_n \subseteq \text{dom}(f_{n+1}^{-1}) = \text{ran}(f_{n+1})$. Hence $\Gamma_n \subseteq \text{dom}(f_{n+1}) \cap \text{ran}(f_{n+1})$, and so condition 3 is satisfied for $m = n+1$.

Let $x \in \Gamma_{n+1} \setminus \Gamma_0$. Suppose that $x \in \Gamma_n$. Then it follows from the inductive condition 4

$$(u)f_{n+1}^{-(n+1)} \leq_{n+1} (u)f_n^{-n} \leq_{n+1} x \leq_{n+1} (u)f_n^n \leq_{n+1} (u)f_{n+1}^{n+1}.$$

Suppose that $x = v_T$ for some $T = (A, B, C) \in \mathcal{T}_n$. Since f_{n+1} is an extension of f_n, it follows that $(u)f_{n+1}^{n+1} \in A$ and $(u)f_{n+1}^{-(n+1)} \in B$ by the definition of $\mathcal{T}_n$. Hence

$$(u)f_{n+1}^{-(n+1)} \leq_{n+1} v_T \leq_{n+1} (u)f_{n+1}^{n+1},$$

and so in both cases condition 4 is satisfied.

Finally, in order to show that $\leq_{n+1}$ satisfies condition 5, let $A, B, C \subseteq \Gamma_n \setminus \{f_{n+1}^k : |k| > n+1\}$ be a triple satisfying the condition of Alice's restaurant property. Then by condition 4, $(u)f_{n+1}^{-(n+1)} <_{n+1} y <_{n+1} (u)f_{n+1}^{n+1}$ for all $y \in A \cup B \cup C$, and so the triple $T = (A \cup \{(u)f_{n+1}^{n+1}\}, B \cup \{(u)f_{n+1}^{-(n+1)}\}, C)$ also satisfies the condition of Alice's restaurant property. Hence $T \in \mathcal{T}_n$. Then $v_T \leq_{n+1} a$, $b \leq_{n+1} v_T$ and c is incomparable to v_T for all $a \in A$, $b \in B$, and $c \in C$. Therefore, the partially ordered set $(\Gamma_{n+1}, \leq_{n+1})$, and the isomorphism f_{n+1} satisfy conditions 1–5.

Let $\mathcal{S} = \bigcup_{n \in \mathbb{N}} \Gamma_n$ be a partially ordered by $\bigcup_{n \in \mathbb{N}} \leq_{n+1}$, which we will denote by $\leq$, and let $f = \bigcup_{n \in \mathbb{N}} f_n$. It follows from the facts that each $\leq_n$ is a partial order and that $\leq_n$ is a restriction of $\leq_{n+1}$ onto $\Gamma_n \times \Gamma_n$ for all $n \in \mathbb{N}$, that $\leq$ is also a partial order. Similarly, since each f_n is a isomorphism, and $\bigcup_{n \in \mathbb{N}} \text{dom}(f_n) = \bigcup_{n \in \mathbb{N}} \text{ran}(f_n) = \mathcal{S}$, it follows that f_n is an automorphism of $(\mathcal{S}, \leq)$.

Let Σ be finite subset of $\mathcal{S}$. Then there is $n \in \mathbb{N}$ such that $\Sigma \subseteq \Gamma_n$. Hence $(u)f^{-n} \leq y \leq (u)f^n$ for all $y \in \Sigma$ by condition 4. Therefore, the conclusion of the lemma holds, it remains to show that $(\mathcal{S}, \leq)$ is isomorphic to the universal partially ordered set.

Suppose that A, B, and C are finite subsets of $\mathcal{S}$ such that the triple (A, B, C) satisfies the conditions of Alice's restaurant property. Then there is $n \in \mathbb{N}$ such that $A, B, C \subseteq \Gamma_n \setminus \{(u)f^k : |k| > n\}$, and so by condition 5 there is $w \in \mathcal{S}$

such that $w \leq a$, $b \leq w$, and w is incomparable to c for all $a \in A$, $b \in B$, and $c \in C$. Therefore $\mathcal{S}$ has Alice's restaurant property, thus $\mathcal{S}$ is isomorphic to $\mathcal{P}$ by Proposition 3.2.3, as required. □

Finally, Lemma 3.3.3 can be used to prove, Theorem 3.3.4, the main result of this section. We note that a stronger version of Theorem 3.3.4 was independently proved in [22]. An automorphism f of $\mathcal{P}$ satisfying the conclusion of Lemma 3.3.3 will be one of the two generators in the following proof.

Theorem 3.3.4. $\mathrm{Aut}(\mathcal{P})$ *has a* 2*-generated dense subgroup.*

Proof. Since $\mathcal{P}$ is countable, let $\{x_n : n \in \mathbb{N}\}$ be an arbitrary enumeration of $\mathcal{P}$. Let $\{p_n : n \in \mathbb{N}\}$ be the set of all isomorphisms between finite substructures of $\mathcal{P}$. Such an enumeration is possible since $\mathrm{Aut}(\mathcal{P})^{<\omega}$ is countable. Recall that $\{[p_n] : n \in \mathbb{N}\}$ is a basis for $\mathrm{Aut}(\mathcal{P})$. Hence it is sufficient to find $f, g \in \mathrm{Aut}(\mathcal{P})$ such that $\langle f, g\rangle \cap [p_n] \neq \varnothing$ for all $n \in \mathbb{N}$.

Let $f \in \mathrm{Aut}(\mathcal{P})$ and $x \in \mathcal{P}$ be as in the conclusion of Lemma 3.3.3, in other words for every finite subset Σ of $\mathcal{P}$ there are $n, m \in \mathbb{Z}$ such that

$$(x)f^n \leq y \leq (x)f^m$$

for all $y \in \Sigma$. We will now inductively construct g.

Let $n \in \mathbb{N}$. Suppose we have defined $q_0, \dots, q_n \in \mathrm{Aut}(\mathcal{P})^{<\omega}$ such that q_k is an extension of q_{k-1} for all $k \in \{1, \dots, n\}$, $x_k \in \mathrm{dom}(q_n) \cap \mathrm{ran}(q_n)$ for all $k \in \{0, \dots, n\}$, and $\langle f, g\rangle \cap [p_k] \neq \varnothing$ for all $k \in \{0, \dots, n\}$ and all $g \in [q_n]$.

If necessary, by extending q_n to $q_n' \in \mathrm{Aut}(\mathcal{P})^{<\omega}$ using Corollary 3.3.2, we may assume that $\mathrm{dom}(p_{n+1}) \cup \mathrm{ran}(p_{n+1}) \subseteq \mathrm{dom}(q_n')$ and $x_{n+1} \in \mathrm{dom}(q_n') \cap \mathrm{ran}(q_n')$. Since $\mathrm{dom}(q_n') \cup \mathrm{ran}(q_n')$ is finite it follows from Lemma 3.3.3 that there $k \in \mathbb{Z}$ such that for $(x)f^{-k} \leq y \leq (x)f^k$ for all $y \in \mathrm{dom}(q_n') \cup \mathrm{ran}(q_n')$. Let $u = f^{-2k}p_{n+1}f^{2k}$. Note that

$$\mathrm{dom}(u) \cup \mathrm{ran}(u) = (\mathrm{dom}(p_n) \cup \mathrm{ran}(p_n))f^{2k} \subseteq (\mathrm{dom}(q_n'))f^{2k},$$

and so if $y \in \mathrm{dom}(u) \cup \mathrm{ran}(u)$, then $y = (t)f^{2k}$ for some $t \in \mathrm{dom}(q_n')$. Since $(x)f^{-k} \leq t \leq (x)f^k$, it follows that $(x)f^k \leq (t)f^{2k} = y$. Hence, if $z \in \mathrm{dom}(q_n') \cup \mathrm{ran}(q_n')$, then

$$z \leq (x)f^k \leq y.$$

Therefore, $q_{n+1} = q_n' \cup u \in \mathrm{Aut}(\mathcal{P})^{<\omega}$. Moreover, it follows from the definition of u, that if $g \in [q_{n+1}]$, then $f^{2k}gf^{-2k} \in [f^{2k}uf^{-2k}] = [p_{n+1}]$, and so the inductive hypothesis is satisfied.

Therefore, by induction there are $q_0, q_1, \dots \in \mathrm{Aut}(\mathcal{P})^{<\omega}$ satisfying the inductive hypothesis. Let $g = \bigcup_{n\in\mathbb{N}} q_n$. Since each q_n is an isomorphism and $\mathrm{dom}(g) = \mathrm{ran}(g) = \mathcal{P}$, it follows that $g \in \mathrm{Aut}(\mathcal{P})$. Also $g \in [q_n]$, and so for every $n \in \mathbb{N}$ there is $k \in \mathbb{N}$ such that $f^{-k}gf^k \in [p_n]$. Hence $\langle f, g\rangle$ is dense in $\mathrm{Aut}(\mathcal{P})$. □

Looking back at the proof of Theorem 3.3.4, we may note that we only used products of a particular kind to obtain a dense subset of $\mathrm{Aut}(\mathcal{P})$. More precisely, there are $f, g \in \mathrm{Aut}(\mathcal{P})$ such that $\{f^{-n}gf^n : n \in \mathbb{Z}\}$ is a dense subset of $\mathrm{Aut}(\mathcal{P})$. Recall that groups satisfying aforementioned property are said to have a cyclically dense conjugacy class, as discussed in Section 3.1.

Theorem 3.3.5. $\mathrm{Aut}(\mathcal{P})$ *has a cyclically dense conjugacy class.*

3.4 Universal K_n-free graphs

In this section we will consider the ultrahomogeneous K_n-free graphs, denoted by H_n. The case $n = 2$ gives a graph with no edges and its automorphism group is just the symmetric group on countably many points, which was already considered in [12]. Throughout this section, let $n \geq 3$ be fixed.

For $x \in H_n$, let $N(x)$ be the set of all vertices adjacent to x in H_n. If $N(x)$ has a subgraph Γ isomorphic to K_{n-1}, then $\Gamma \cup \{x\}$ is isomorphic to K_n, which contradicts the fact that H_n is K_n-free. Hence $N(x)$ is K_{n-1}-free for every vertex $x \in H_n$. We will repeatedly make use of this fact without reference.

Let U and V be finite disjoint subsets of vertices of H_n such that U is K_{n-1}-free. Then, by the Alice's restaurant property for K_n-free graphs, there is a vertex $w \in H_n \setminus U \cup V$ such that there are no edges between w and V, and there is an edge between u and w for all $u \in U$. In other words, $N(w) \cap (U \cup V) = U$.

The purpose of this section is to prove Theorem 3.1.8(i), which we restate for the sake of convenience.

Theorem 3.4.1. *Let* $f \in \mathrm{Aut}(H_n)$ *be non-identity. Then* $D_f \cap \mathcal{I}(H_n)$ *is comeagre in* $\mathcal{I}(H_n)$.

Before giving the proof of Theorem 3.4.1 we will prove a number of technical results. First, we will show that the set $D_f \cap \mathcal{I}(H_n)$ can be written as a countable intersection of sets of a certain type. The rest of the argument will then be dedicated to showing that these sets are open and dense in H_n.

Lemma 3.4.2. *Let* $\mathcal{B} \subseteq \mathrm{Aut}(H_n)^{<\omega}$ *be such that* $b \in \mathcal{B}$ *if and only if* $\mathrm{dom}(b)$ *and* $\mathrm{ran}(b)$ *are disjoint, and there are no edges between* $\mathrm{dom}(b)$ *and* $\mathrm{ran}(b)$. *Then*

$$D_f \cap \mathcal{I}(H_n) = \bigcap_{b \in \mathcal{B}} \{g \in \mathcal{I}(H_n) : \langle f, g\rangle \cap [b] \neq \varnothing\}$$

for all $f \in \mathrm{Aut}(H_n)$.

Proof. Let $f \in H_n$. Since $D_f = \{g \in \text{Aut}(H_n) : \langle f, g\rangle \text{ is dense in } \text{Aut}(H_n)\}$, it follows that

$$D_f \cap \mathcal{I}(H_n) = \bigcap_{q \in \text{Aut}(H_n)^{<\omega}} \{g \in \mathcal{I}(H_n) : \langle f, g\rangle \cap [q] \neq \varnothing\}.$$

($\subseteq$) This direction follows immediately since $\mathcal{B} \subseteq \text{Aut}(H_n)^{<\omega}$.

($\supseteq$) Let $g \in \mathcal{I}(H_n)$ be such that $\langle f, g\rangle \cap [b]$ is non-empty for all $b \in \mathcal{B}$, and suppose that $q \in \text{Aut}(H_n)^{<\omega}$. By repeated application of the Alice's restaurant property we can find a subgraph Γ of H_n such that Γ is isomorphic to $\text{dom}(q)$, $\Gamma \cap (\text{dom}(q) \cup \text{ran}(q)) = \varnothing$, and such that there are no edges between Γ and $\text{dom}(q) \cup \text{ran}(q)$. Let b be an isomorphism from $\text{dom}(q)$ to Γ. Since H_n is ultrahomogeneous, we have that $b \in \text{Aut}(H_n)^{<\omega}$. Then $\text{dom}(b) = \text{dom}(q)$, $\text{ran}(b) = \text{dom}(b^{-1}q) = \Gamma$, and $\text{ran}(b^{-1}q) = \text{ran}(q)$. Hence $b, b^{-1}q \in \mathcal{B}$, and so by the choice of g there are $h_1, h_2 \in \langle f, g\rangle$ such that $h_1 \in [b]$ and $h_2 \in [b^{-1}q]$. Therefore $h_1h_2 \in [q]$ and $h_1h_2 \in \langle f, g\rangle$, thus $\langle f, g\rangle \cap [q] \neq \varnothing$. Since q was arbitrary,

$$g \in \bigcap_{q \in \text{Aut}(H_n)^{<\omega}} \{g \in \mathcal{I}(H_n) : \langle f, g\rangle \cap [q] \neq \varnothing\}$$

as required. □

In the following lemma we show that for every non-identity automorphism f of H_n, the support of f is infinite. The result follows from Corollary 2.10(ii) in [50]. We include the proof the keep the thesis self-contained.

Lemma 3.4.3. *Let $f \in \text{Aut}(H_n)$ be non-identity. Then support of f is infinite.*

Proof. Since f is non-identity, there is $x \in H_n$ such that $(x)f \neq x$. Suppose that $y \in H_n$ is such that y is adjacent to x, but not adjacent to $(x)f$. It then follows that $(y)f$ is adjacent to $(x)f$ as f is an automorphism. Hence $(y)f \neq y$ by definition of y, and so y is in the support of f. Finally, it is sufficient to show that there are infinitely many such vertices y. Define

$$\Sigma = \{y \in H_n : y \text{ is adjacent to } x \text{ but not adjacent to } (x)f\}.$$

Let $U = \{x\}$ and $V = \Sigma \cup \{(x)f\}$. Then if Σ is finite by Alice restaurant property there is $w \in H_n \setminus (U \cup V)$ such that w is adjacent to every vertex in U and not adjacent to every vertex in V. Hence w is adjacent to x and not adjacent to $(x)f$, and so $w \in \Sigma \subseteq U$, which is a contradiction. Therefore Σ is infinite, as required. □

The following lemma provides a condition under which it is possible to extend a given partial isomorphism of H_n to another partial isomorphism of H_n. Although we will only apply the following lemma to the graphs H_n, we state it for arbitrary ultrahomogeneous graphs, since the proof is no harder in the general case.

Lemma 3.4.4. *Let Γ be an ultrahomogeneous graph, let $q \in \mathrm{Aut}(\Gamma)^{<\omega}$, and let $x, y \in \Gamma$. Suppose that $x \notin \mathrm{dom}(q)$, $y \notin \mathrm{ran}(q)$, and $N(y) \cap \mathrm{ran}(q) = (N(x))\, q$. Then $q \cup \{(x, y)\} \in \mathrm{Aut}(\Gamma)^{<\omega}$.*

Proof. Since Γ is ultrahomogeneous, it is sufficient to show that $q \cup \{(x, y)\}$ an isomorphism between two subgraphs of Γ. By the hypothesis, q is an isomorphism, and so it suffices to show that there is an edge between vertices x and $z \in \mathrm{dom}(q)$ if and only if there is an edge between vertices y and $(z)q$. Let $z \in \mathrm{dom}(q)$. Then there is an edge between z and x if and only if $z \in N(x)$ which is equivalent to $(z)q \in N(y) \cap \mathrm{ran}(q)$, in other words there is an edge between $(z)q$ and y. □

The following easy corollary shows that any incomplete component of an isomorphism of H_n can be extended.

Corollary 3.4.5. *Let $q \in \mathrm{Aut}(H_n)^{<\omega}$, let $x \notin \mathrm{dom}(q)$, and let $\Sigma \subseteq H_n$ be finite. Then there is $y \in H_n \setminus (\{x\} \cup \Sigma)$ such that $q \cup \{(x, y)\} \in \mathrm{Aut}(H_n)^{<\omega}$.*

Proof. Let $U = (N(x))\, q$ and let $V = (\mathrm{ran}(q) \cup \{x\} \cup \Sigma) \setminus U$. Since $N(x)$ is K_{n-1}-free and q is a partial isomorphism, U is also K_{n-1}-free. Hence by the Alice's restaurant property there is $y \in H_n \setminus (\mathrm{ran}(q) \cup \{x\} \cup \Sigma)$ such that $N(y) \cap (\mathrm{ran}(q) \cup \Sigma \cup \{x\}) = (N(x))\, q$. Therefore $N(y) \cap \mathrm{ran}(q) = (N(x))\, q$, and so we are done by Lemma 3.4.4. □

We can now classify when a partial isomorphism of H_n is in $\mathcal{I}(H_n)^{<\omega}$.

Corollary 3.4.6. *Let $q \in \mathrm{Aut}(H_n)^{<\omega}$. Then $q \in \mathcal{I}(H_n)^{<\omega}$ if and only if q has no complete components.*

Proof. ($\Rightarrow$) Let $g \in \mathcal{I}(H_n)$ be the extension of q. Then every complete component is a finite orbit of g, and since g has no finite orbits, it follows that q has no complete components.

($\Leftarrow$) Let q be an isomorphism between finite subgraphs of H_n such that q has no complete components. Let $\{x_i : i \in \mathbb{N}\}$ an enumeration of H_n, i.e. $H_n = \{x_i : i \in \mathbb{N}\}$. If $x_0 \in \mathrm{dom}(q)$, let $h = q$. Suppose that $x_0 \notin \mathrm{dom}(q)$. Then by Corollary 3.4.5 there is $y \in H_n \setminus (\{x_0\} \cup \mathrm{dom}(q) \cup \mathrm{ran}(q))$ such that $h = q \cup \{(x_0, y)\} \in \mathrm{Aut}(H_n)^{<\omega}$. Then from the choice of y, it follows that h has no complete components and $x_0 \in \mathrm{dom}(h)$. If $x_0 \in \mathrm{ran}(h)$, let $q_0 = h$. Otherwise $x_0 \notin \mathrm{ran}(h)$, which implies that $x_0 \notin \mathrm{dom}(h^{-1})$ and by same argument there exist an extension $q_0^{-1} \in \mathrm{Aut}(H_n)^{<\omega}$ of h^{-1} such that q_0^{-1} has no complete components and $x_0 \in \mathrm{dom}(q_0^{-1})$. Hence q_0 has no complete components and $x_0 \in \mathrm{dom}(q_0) \cap \mathrm{ran}(q_0)$.

Suppose that for some $k \geq 1$ there is an extension $q_k \in \mathrm{Aut}(H_n)^{<\omega}$ of q_{k-1} such that q_k has no complete components and $x_j \in \mathrm{dom}(q_k) \cap \mathrm{ran}(q_k)$ for all

$j \in \{0, \ldots, k\}$. If $x_{k+1} \in \operatorname{dom}(q)$, then let $h = q_k$. Otherwise, $x_{k+1} \notin \operatorname{dom}(q)$. Then by Corollary 3.4.5 there is $y \in H_n \setminus (\{x_{k+1}\} \cup \operatorname{dom}(q_k) \cup \operatorname{ran}(q_k))$ such that $h = q_k \cup \{(x_{k+1}, y)\} \in \operatorname{Aut}(H_n)^{<\omega}$. Then from the choice of y, it follows that h has no complete components and $x_j \in \operatorname{dom}(h)$ for all $j \in \{0, \ldots, k+1\}$. If $x_{k+1} \in \operatorname{ran}(h)$, then let $q_{k+1} = h$. Otherwise, $x_{k+1} \notin \operatorname{ran}(h)$. Thus $x_{k+1} \notin \operatorname{dom}(h^{-1})$ and by same argument there exist an extension $q_{k+1}^{-1} \in \operatorname{Aut}(H_n)^{<\omega}$ of h^{-1} such that q_{k+1}^{-1}, has no complete components and $x_j \in \operatorname{dom}(q_{k+1}^{-1})$ for all $j \in \{0, \ldots, k+1\}$. Hence q_{k+1} has no complete components and $x_j \in \operatorname{dom}(q_{k+1}) \cap \operatorname{ran}(q_{k+1})$ for all $j \in \{0, \ldots, k+1\}$, and so q_{k+1} satisfies the inductive hypothesis.

Therefore, for every $k \in \mathbb{N}$ there is $q_k \in \operatorname{Aut}(H_n)^{<\omega}$ satisfying the inductive hypothesis. Define

$$g = \bigcup_{k \in \mathbb{N}} q_k.$$

Then $\operatorname{dom}(g) = \operatorname{ran}(g) = H_n$, and since every q_k was an isomorphism, it follows that $g \in \operatorname{Aut}(H_n)$. Similarly if g has a finite orbit, then there is k such that q_k has a complete component. Hence g has no finite orbits, and so $g \in \mathcal{I}(H_n)$, as required. □

The following two technical lemmas form the essential part of the proof of Theorem 3.4.1. Lemma 3.4.8 is the main result used to prove Theorem 3.4.1, and Lemma 3.4.7 is used to make the induction within the proof of Lemma 3.4.8 easier.

Lemma 3.4.7. *Let $q \in \mathcal{I}(H_n)^{<\omega}$ be such that $\operatorname{ran}(q) \cup \operatorname{dom}(q) = \Delta \cup \Gamma$ where $\Delta \cap \Gamma = \varnothing$ and Γ is the union of incomplete components of q of fixed length m, let $x, y \notin \operatorname{dom}(q) \cup \operatorname{ran}(q)$ be such that*

$$N(x) \cap \Delta \subseteq \operatorname{dom}(q^{2m}) \quad \text{and} \quad (N(x) \cap \Delta)\, q^{2m} = N(y) \cap \Delta$$

and let $\Sigma_1, \Sigma_2 \subseteq H_n \setminus \Gamma$ be finite subsets such that $\Sigma_1 \cap \operatorname{ran}(q) = \varnothing$ and $\Sigma_2 \cap \operatorname{dom}(q) = \varnothing$. Then there are $x_1, \ldots, x_{2m-1} \in H_n \setminus \Sigma_1 \cup \Sigma_2$ such that there are no edges between x_i and $\Sigma_1 \cup \Sigma_2$ for all $i \in \{1, \ldots, 2m-1\}$, and

$$q \cup \{(x_i, x_{i+1}) : 0 \leq i \leq 2m-1\} \in \mathcal{I}(H_n)^{<\omega}$$

where $x_0 = x$ and $x_{2m} = y$.

Proof. Define $q_0 = q$, $x_0 = x$ and $\Gamma_i = \operatorname{dom}(q_i) \cup \operatorname{ran}(q_i) \cup \Sigma_1 \cup \Sigma_2 \cup \{x, y\}$ for all i such that q_i is defined. Suppose that for $i \in \{0, \ldots, m-1\}$ there is an extension $q_i \in \mathcal{I}(H_n)^{<\omega}$ of q_0 such that $q_i = q_0 \cup \{(x_j, x_{j+1}) : 0 \leq j \leq i-1\}$

with $x_0 \notin \mathrm{ran}(q_i)$, $x_i \notin \mathrm{dom}(q_i)$, $y \notin \mathrm{ran}(q_i) \cup \mathrm{dom}(q_i)$, and

$$x_j \notin \Sigma_1 \cup \Sigma_2 \cup \Delta \tag{3.3}$$

$$N(x_j) \cap (\Sigma_1 \cup \Sigma_2 \cup \{x_0, \ldots, x_{j-1}, y\}) = \varnothing \tag{3.4}$$

$$N(x_i) \cap \Gamma_i = (N(x_0) \cap \Gamma_0)\, q_i^i \tag{3.5}$$

for all $j \in \{1, \ldots, i\}$.

If $i = 0$, then we have that $x_0, y \notin \mathrm{dom}(q_0) \cup \mathrm{ran}(q_0)$ and (3.3), (3.4), (3.5) are trivially satisfied.

Suppose that $i > 0$. Let $U = (N(x_i))\, q_i \subseteq \Gamma_i$ and $V = \Gamma_i \backslash U$. If $N(x_i)$ contains a subgraph isomorphic to K_{n-1}, then the subgraph together with x_i forms K_n, which is impossible. Hence $N(x_i)$ is K_{n-1}-free and since q_i is an isomorphism, U is also K_{n-1}-free. Therefore the sets U and V satisfy the hypothesis of the Alice's restaurant property and thus there is a vertex $x_{i+1} \in H_n \setminus \Gamma_i$ such that there is an edge between x_{i+1} and every vertex in U and there are no edges between x_{i+1} and V, i.e. $N(x_{i+1}) \cap \Gamma_i = U$. Also it follows from $\mathrm{ran}(q_i) \subseteq \Gamma_i$ that

$$N(x_{i+1}) \cap \mathrm{ran}(q_i) = (N(x_{i+1}) \cap \Gamma_i) \cap \mathrm{ran}(q_i) = U \cap \mathrm{ran}(q_i) = (N(x_i))\, q_i.$$

Then $q_{i+1} = q_i \cup \{(x_i, x_{i+1})\} = q_0 \cup \{(x_j, x_{j+1}) : 0 \leq j \leq i\} \in \mathrm{Aut}(H_n)^{<\omega}$ by Lemma 3.4.4, and so $q_{i+1} \in \mathcal{I}(H_n)^{<\omega}$ by Corollary 3.4.6. Since $x_{i+1} \notin \Gamma_i$, we have that $x_{i+1} \notin \{x_0, x_i, y\}$ implying that $x_0 \notin \mathrm{ran}(q_{i+1})$, $x_{i+1} \notin \mathrm{dom}(q_{i+1})$, and $y \notin \mathrm{ran}(q_{i+1}) \cup \mathrm{dom}(q_{i+1})$. It also follows from $\mathrm{dom}(q_i) \subseteq \Gamma_i$ and (3.5) that

$$N(x_{i+1}) \cap \Gamma_i = U = (N(x_i))\, q_i = (N(x_i) \cap \Gamma_i)\, q_i = (N(x_0) \cap \Gamma_0)\, q_i^{i+1}. \tag{3.6}$$

Since $\Sigma_1 \cup \Sigma_2 \cup \Delta \subseteq \Gamma_i$ and x_{i+1} is chosen outside the set Γ_i it follows that $x_{i+1} \notin \Sigma_1 \cup \Sigma_2 \cup \Delta$. Then $x_j \notin \Sigma_1 \cup \Sigma_2 \cup \Delta$ for all $j \in \{1, \ldots, i+1\}$ by (3.3).

We will now show that (3.4) holds for $j = i + 1$. First of all note that $x_0, y \notin \mathrm{ran}(q_i)$, and since $U \subseteq \mathrm{ran}(q_i)$ we have that $x_0, y \notin U$. From (3.4) we may deduce that $x_j \notin N(x_i)$, and thus $x_{j+1} \notin (N(x_i))\, q_i = U$, for all $j \in \{0, \ldots, i-1\}$, i.e. $\{x_0, \ldots, x_i, y\} \cap U = \varnothing$. It follows from the hypothesis that $\Sigma_1 \cap \mathrm{ran}(q_0) = \varnothing$, and so (3.3) implies that $\Sigma_1 \cap \mathrm{ran}(q_i) = \varnothing$. Since $U \subseteq \mathrm{ran}(q_i)$, we have that $(\Sigma_1 \cup \{x_0, \ldots, x_i, y\}) \cap U = \varnothing$.

It remains to show that $\Sigma_2 \cap U = \varnothing$. Suppose $z \in \Sigma_2 \cap U$. Then $z \in (N(x_0) \cap \Gamma_0)\, q_i^{i+1}$ by (3.6). Then $z \in \mathrm{ran}(q_i)$ and by above $z \neq x_j$ for all $j \in \{0, \ldots, i\}$, thus $z \in \mathrm{ran}(q_0) \subseteq \Gamma \cup \Delta$. However by the hypothesis of the lemma, $\Sigma_2 \subseteq H_n \setminus \Gamma$, implying that $z \in \Delta$. Since $x_j \notin \Delta$ for all $j \in \{1, \ldots, i\}$ by (3.3) and $x_0 \notin \Delta$ by the hypothesis of the lemma, it follows that the incomplete component of q_0 containing z was not extended in q_i. Moreover Δ is a union of incomplete components of q_0, hence $(z)q_i^{-(i+1)} \in N(x_0) \cap \Delta$. Also from

$\Sigma_2 \cap \mathrm{dom}(q_0) = \varnothing$ and (3.3) we may deduce that $\Sigma_2 \cap \mathrm{dom}(q_i) = \varnothing$ and so $z \notin \mathrm{dom}(q_i)$. It also follows from the hypothesis of the lemma that $(z)q_i^{-(i+1)} \in \mathrm{dom}(q_i^{2m})$. Then $z \in \mathrm{dom}(q_i^{2m-(i+1)})$, which is impossible since $i+1 < 2m$. Hence $U \cap (\Sigma_1 \cup \Sigma_2 \cup \{x_0, \ldots, x_i, y\}) = \varnothing$. Since $(\Sigma_1 \cup \Sigma_2 \cup \{x_0, \ldots, x_i, y\}) \subseteq \Gamma_{i+1}$ we have that

$$\begin{aligned} &N(x_{i+1}) \cap (\Sigma_1 \cup \Sigma_2 \cup \{x_0, \ldots, x_i, y\}) \\ &\qquad = (N(x_{i+1}) \cap \Gamma_{i+1}) \cap (\Sigma_1 \cup \Sigma_2 \cup \{x_0, \ldots, x_i, y\}) \\ &\qquad = U \cap (\Sigma_1 \cup \Sigma_2 \cup \{x_0, \ldots, x_i, y\}) = \varnothing. \end{aligned}$$

Then (3.3) and (3.4) are satisfied, and so it only remains to verify (3.5). It is routine to verify that $\mathrm{dom}(q_{i+1}^{i+1}) \setminus \mathrm{dom}(q_i^{i+1}) = \{x_0\}$. It follows from $x_0 \notin N(x_0)$ and (3.6) that $N(x_{i+1}) \cap \Gamma_i = (N(x_0) \cap \Gamma_0)\, q_{i+1}^{i+1}$. Since $\Gamma_{i+1} = \Gamma_i \cup \{x_{i+1}\}$ and $x_{i+1} \notin N(x_{i+1})$

$$N(x_{i+1}) \cap \Gamma_i = N(x_{i+1}) \cap \Gamma_{i+1} = (N(x_0) \cap \Gamma_0)\, q_{i+1}^{i+1}.$$

Therefore, q_{i+1} satisfies the inductive hypothesis. Thus by induction on i, there is $q_m \in \mathcal{I}(H_n)^{<\omega}$ such that $q_m = q_0 \cup \{(x_j, x_{j+1}) : 0 \leq j \leq m-1\}$, $x_0 \notin \mathrm{ran}(q_m)$, $x_m \notin \mathrm{dom}(q_m)$, $y \notin \mathrm{ran}(q_m) \cup \mathrm{dom}(q_m)$, q_m satisfies (3.3), (3.4) and (3.5).

Note that if $z \in \Sigma_1 \cup \Sigma_2 \setminus \{x, y\}$ then $z \notin \Gamma$ and by (3.3) either $z \notin \mathrm{dom}(q_m) \cup \mathrm{ran}(q_m)$ or $z \in \Delta$. Hence

$$\begin{aligned} N(x_m) \cap \Gamma_m &= (N(x_0) \cap \Gamma_0)\, q_m^m \\ &= (N(x_0) \cap (\Gamma \cup \{x, y\} \cup \Sigma_1 \cup \Sigma_2 \setminus \Delta))\, q_m^m \\ &\quad \cup (N(x_0) \cap \Delta)\, q_m^m \\ &= (N(x_0) \cap \Delta)\, q_m^m, \end{aligned} \tag{3.7}$$

since $x \notin N(x_0)$, $y \notin \mathrm{dom}(q_m)$ and all incomplete components on Γ of q are of length m.

The next step is to inductively construct an extension $h = q_{2m} \in \mathcal{I}(H_n)^{<\omega}$ of q_m. Suppose that for $i \in \{m, \ldots, 2m-2\}$ there is an extension $q_i \in \mathcal{I}(H_n)^{<\omega}$ of the form $q_i = q_m \cup \{(x_j, x_{j+1}) : m \leq j \leq i-1\}$ such that $x_0 \notin \mathrm{ran}(q_i)$, $x_i \notin \mathrm{dom}(q_i)$, $y \notin \mathrm{dom}(q_i) \cup \mathrm{ran}(q_i)$, and

$$x_j \notin \Sigma_1 \cup \Sigma_2 \cup \Delta \tag{3.3}$$

$$N(x_j) \cap (\Sigma_1 \cup \Sigma_2 \cup \{x_0, \ldots, x_{j-1}, y\}) = \varnothing \tag{3.4}$$

$$N(x_i) \cap \Gamma_i = \left(N(y) \cap \mathrm{dom}\left(q_i^{i-2m}\right)\right) q_i^{i-2m} \tag{3.8}$$

for all $j \in \{1, \ldots, i\}$.

We will now show that q_m satisfies the inductive hypothesis. Note that (3.3) and (3.4) are the same as before, so we only need to verify (3.8). Since no incomplete components of q_0, which intersect Δ, were extended in q_m, (3.3) implies that $(\Delta)\, q_0^k = (\Delta)\, q_m^k$ for any $k \in \mathbb{Z}$. It follows from the hypothesis of the lemma that $(N(x_0) \cap \Delta)\, q_m^m \subseteq \mathrm{dom}(q_m^m)$ and $(N(x_0) \cap \Delta)\, q_m^m = (N(y) \cap \Delta)\, q_m^{-m}$. Hence by (3.7)

$$N(x_m) \cap \Gamma_m = (N(x_0) \cap \Delta)\, q_m^m = (N(y) \cap \Delta)\, q_m^{-m}.$$

Suppose that $z \in N(y) \cap \mathrm{dom}(q_m^{-m})$. Then $z \in \mathrm{dom}(q_m) \cup \mathrm{ran}(q_m) = \Gamma \cup \Delta \cup \{x_0, \dots, x_m\}$. Note that all incomplete components of q_m, intersecting Γ not trivially, are of length m. Hence $z \in \Delta \cup \{x_m\}$ and by (3.4) we have that $x_m \notin N(y)$, thus $z \in \Delta$. Therefore $N(y) \cap \mathrm{dom}(q_m^{-m}) \subseteq N(y) \cap \Delta$, and so

$$N(x_m) \cap \Gamma_m = (N(y) \cap \Delta)\, q_m^{-m} = \left(N(y) \cap \mathrm{dom}(q_m^{-m})\right) q_m^{-m}.$$

Hence q_m satisfies (3.8) and the inductive hypothesis is satisfied for $i = m$.

Let $U = (N(x_i))\, q_i$ and $V = \Gamma_i \setminus U$. The sets U and V satisfy the hypothesis of the Alice's restaurant property and thus we can find $x_{i+1} \in H_n \setminus \Gamma_i$ with $N(x_{i+1}) \cap \Gamma_i = U = (N(x_i))\, q_i$. Then $N(x_{i+1}) \cap \mathrm{ran}(q_i) = (N(x_i))\, q_i$, and so $q_{i+1} = q_i \cup \{(x_i, x_{i+1})\} \in \mathcal{I}(H_n)^{<\omega}$ by Lemma 3.4.4 and Corollary 3.4.6. Since $x_{i+1} \notin \Gamma_i$, we have that $x_{i+1} \notin \{x_0, x_i, y\}$ implying that $x_0 \notin \mathrm{ran}(q_{i+1})$, $x_{i+1} \notin \mathrm{dom}(q_{i+1})$, and $y \notin \mathrm{dom}(q_{i+1}) \cup \mathrm{ran}(q_{i+1})$.

Since $\mathrm{dom}(q_i) \subseteq \Gamma_i$,

$$N(x_{i+1}) \cap \Gamma_i = U = (N(x_i))\, q_i = (N(x_i) \cap \Gamma_i)\, q_i.$$

It then follows from (3.8) that

$$N(x_{i+1}) \cap \Gamma_i = \left(N(y) \cap \mathrm{dom}\left(q_i^{i-2m}\right)\right) q_i^{i+1-2m}. \tag{3.9}$$

Since $\Sigma_1 \cup \Sigma_2 \cup \Delta \subseteq \Gamma_i$ and x_{i+1} is chosen outside the set Γ_i it follows that $x_{i+1} \notin \Sigma_1 \cup \Sigma_2 \cup \Delta$. Then $x_j \notin \Sigma_1 \cup \Sigma_2 \cup \Delta$ for all $j \in \{1, \dots, i+1\}$.

We will now show that (3.4) holds for $j = i + 1$. First of all note that $x_0, y \notin \mathrm{ran}(q_i)$, and since $U \subseteq \mathrm{ran}(q_i)$ we have that $x_0, y \notin U$. From (3.4) we may deduce that $x_j \notin N(x_i)$, and thus $x_{j+1} \notin (N(x_i))\, q_i = U$, for all $j \in \{0, \dots, i-1\}$ in other words $\{x_0, \dots, x_i, y\} \cap U = \varnothing$. It follows from the hypothesis that $\Sigma_1 \cap \mathrm{ran}(q_0) = \varnothing$, and so (3.3) implies that $(\Sigma_1 \cup \{x_0, y\}) \cap \mathrm{ran}(q_i) = \varnothing$. Since $U \subseteq \mathrm{ran}(q_i)$, it follows that $(\Sigma_1 \cup \{x_0, \dots, x_i, y\}) \cap U = \varnothing$.

It remains to show that $\Sigma_2 \cap U = \varnothing$. Suppose $z \in \Sigma_2 \cap U$. Then $z \in \left(N(y) \cap \mathrm{dom}(q_i^{i-2m})\right) q_i^{i+1-2m}$ by (3.9). Note that $z \in U \subseteq \mathrm{ran}(q_i)$. Also it was shown in the previous paragraph that $z \neq x_j$ for all $j \in \{0, \dots, i\}$. Hence $z \in \mathrm{ran}(q_0) \subseteq \Gamma \cup \Delta$. However by the hypothesis of the lemma $\Sigma_2 \subseteq H_n \setminus \Gamma$, implying that $z \in \Delta$. Since $x_j \notin \Delta$ for all $j \in \{1, \ldots, i\}$ by (3.3) and $x_0 \notin \Delta$ by the hypothesis of the lemma, it follows that the incomplete component of q_0 containing z was not extended in q_i. Moreover, Δ is a union of incomplete components of q_0, and $z \in \mathrm{dom}(q_i^{2m-(i+1)})$, so $(z)q_i^{2m-(i+1)} \in N(y) \cap \Delta$. By the hypothesis of the lemma $(z)q_i^{2m-(i+1)} \in \mathrm{ran}(q_i^{2m})$. Then there is $u \in \mathrm{dom}(q_i^{2m})$ such that $(z)q_i^{2m-(i+1)} = (u)q_i^{2m}$, and so $z = (u)q_i^{i+1} \in \mathrm{dom}(q_i^{2m-(i+1)})$. Hence $z \in \mathrm{dom}(q_i)$, since $2m > i+1$. However, $z \in \Sigma_2$ and so $z \notin \mathrm{dom}(q_0)$, implying that $z \in \{x_0, \ldots, x_i\}$, which contradicts (3.3). Hence $U \cap (\Sigma_1 \cup \Sigma_2 \cup \{x_0, \ldots, x_i, y\}) = \varnothing$, and since $(\Sigma_1 \cup \Sigma_2 \cup \{x_0, \ldots, x_i, y\}) \subseteq \Gamma_{i+1}$ we have that for all $j \in \{1, \ldots, i+1\}$

$$N(x_j) \cap (\Sigma_1 \cup \Sigma_2 \cup \{x_0, \ldots, x_{j-1}, y\}) = \varnothing.$$

It is routine to verify that $\mathrm{dom}(q_{i+1}^{i+1-2m}) \setminus \mathrm{dom}(q_i^{i+1-2m}) = \{x_{i+1}\}$. Since $x_{i+1} \notin N(y)$ it follows from (3.9) that

$$N(x_{i+1}) \cap \Gamma_i = \left(N(y) \cap \mathrm{dom}\left(q_i^{i-2m}\right)\right) q_{i+1}^{i+1-2m}.$$

It is also routine to check that $\mathrm{dom}\left(q_{i+1}^{i+1-2m}\right) \setminus \mathrm{dom}\left(q_i^{i-2m}\right) \subseteq \{x_1, \ldots, x_{i+1}\}$. Then, by (3.4), we have that $x_j \notin N(y)$ for all $j \in \{x_0, \ldots, i+1\}$. Hence

$$N(x_{i+1}) \cap \Gamma_i = \left(N(y) \cap \mathrm{dom}\left(q_{i+1}^{i+1-2m}\right)\right) q_{i+1}^{i+1-2m}.$$

From the definition of Γ_{i+1} we obtain that $\Gamma_{i+1} = \Gamma_i \cup \{x_{i+1}\}$. However, $x_{i+1} \notin N(x_{i+1})$, and so

$$N(x_{i+1}) \cap \Gamma_{i+1} = \left(N(y) \cap \mathrm{dom}\left(q_{i+1}^{i+1-2m}\right)\right) q_{i+1}^{i+1-2m}.$$

Therefore q_{i+1} satisfies the inductive hypothesis and hence we obtain $q_{2m-1} = q_0 \cup \{(x_j, x_{j+1}) : 0 \leq j \leq 2m-2\} \in \mathrm{Aut}(H_n)^{<\omega}$ such that $y \notin \mathrm{dom}(q_{2m-1}) \cup \mathrm{ran}(q_{2m-1})$, $x_j \notin \Sigma_1 \cup \Sigma_2$, there are no edges between x_j and $\Sigma_1 \cup \Sigma_2$ for all $j \in \{1, \ldots, 2m-1\}$, and

$$N(x_{2m-1}) \cap \Gamma_{2m-1} = (N(y))\, q_{2m-1}^{-1}.$$

Therefore $h = q_{2m} = q_{2m-1} \cup \{(x_{2m-1}, y)\} \in \mathcal{I}(H_n)^{<\omega}$ by Lemma 3.4.4 and Corollary 3.4.6 is as required. □

Using Lemma 3.4.7 we can now prove the following result.

Lemma 3.4.8. *Let $q \in \mathcal{I}(H_n)^{<\omega}$, and let $b \in \mathcal{B}$ be such that the sets $\mathrm{dom}(q) \cup \mathrm{ran}(q)$ and $\mathrm{dom}(b) \cup \mathrm{ran}(b)$ are disjoint. Then there is an extension $h \in \mathcal{I}(H_n)^{<\omega}$ of q and $m \in \mathbb{N}$ such that h^{2m} extends b.*

Proof. If necessary by extending q, using Corollary 3.4.5, we may assume that all of the components of q have length m for some $m \in \mathbb{N}$.

Let $\mathrm{dom}(b) = \{x_1, \ldots, x_d\}$ for some $d \in \mathbb{N}$, let $q_0 = q$, $\Gamma = \mathrm{dom}(q_0) \cup \mathrm{ran}(q_0)$. We will now inductively define $q_k \in \mathcal{I}(H_n)^{<\omega}$, and once they are defined let $\Delta_k = \mathrm{dom}(q_k) \cup \mathrm{ran}(q_k) \setminus \Gamma$ for $k \in \{0, \ldots, d\}$. Suppose that for some $k \in \{0, \ldots, d-1\}$ we have defined $q_k \in \mathcal{I}(H_n)^{<\omega}$, such that q_k extends q_{k-1} for $k \geq 1$, both Γ and Δ_k are unions of incomplete components of q_k, that incomplete components of q_k contained in Δ_k are of length $2m+1$, and the following are true

$$x_j, (x_j)b \notin \mathrm{dom}(q_k) \cup \mathrm{ran}(q_k) \tag{3.10}$$

$$(x_i)q_k^{2m} = (x_i)b \tag{3.11}$$

$$N(x_j) \cap \Delta_k \subseteq \mathrm{dom}(q_k^{2m}) \tag{3.12}$$

$$(N(x_j) \cap \Delta_k)\, q_k^{2m} = N\left((x_j)b\right) \cap \Delta_k \tag{3.13}$$

for all $i \in \{1, \ldots, k\}$ and $j \in \{k+1, \ldots, d\}$.

Let $\Sigma_1 = \mathrm{dom}(b)$ and $\Sigma_2 = \mathrm{ran}(b)$. We will show that the hypothesis of Lemma 3.4.7 is satisfied by q_k, x_{k+1}, $(x_{k+1})b$, Σ_1, and Σ_2. First of all, note that $x_{k+1}, (x_{k+1})b \notin \mathrm{dom}(q_k) \cup \mathrm{ran}(q_k)$ by condition (3.10). Also by the hypothesis of the lemma $\Sigma_1, \Sigma_2 \subseteq H_n \setminus \Gamma$. Note that the conditions $N(x_{k+1})) \cap \Delta_k \subseteq \mathrm{dom}(q_k^{2m})$ and $(N(x_{k+1}) \cap \Delta_k)q_k^{2m} = N((x_{k+1})b) \cap \Delta_k$ of the hypothesis of Lemma 3.4.7 immediately follows from conditions (3.12) and (3.13). Hence to apply Lemma 3.4.7 we only need verify that $\Sigma_1 \cap \mathrm{ran}(q_k) = \Sigma_2 \cap \mathrm{dom}(q_k) = \varnothing$. We will do so in the next two paragraphs.

We will first show that $x_i \notin \mathrm{ran}(q_k)$ for all $i \in \{1, \ldots, d\}$. Suppose that $x_i \in \mathrm{dom}(q_k) \cup \mathrm{ran}(q_k)$, by the inductive hypothesis we can deduce that $i \leq k$. Since $\mathrm{dom}(b) \cap \Gamma = \varnothing$ by the hypothesis of the lemma, it then follows that $x_i \in \Delta_k$. Therefore, x_i is on an incomplete component of length $2m+1$ and $x_i \in \mathrm{dom}(q_k^{2m})$ by the inductive hypothesis, implying that $x_i \in \mathrm{dom}(q_k) \setminus \mathrm{ran}(q_k)$. Hence $\Sigma_1 \cap \mathrm{ran}(q_k) = \varnothing$.

The argument that $\Sigma_2 \cap \mathrm{dom}(q_k) = \varnothing$ is similar to above. Let $(x_i)b \in \Sigma_2$. Suppose that $(x_i)b \in \mathrm{dom}(q_k) \cup \mathrm{ran}(q_k)$. Then we can deduce that $i \leq k$. Since $\mathrm{ran}(b) \cap \Gamma = \varnothing$ by the hypothesis of the lemma, it then follows that $(x_i)b \in \Delta_k$. Therefore, $(x_i)b$ is on an incomplete component of length $2m+1$ and $(x_i)b \in \mathrm{ran}(q_k^{2m})$ by the inductive hypothesis, implying that $(x_i)b \in \mathrm{ran}(q_k) \setminus \mathrm{dom}(q_k)$.

Hence by Lemma 3.4.7 there is an extension $q_{k+1} \in \mathcal{I}(H_n)^{<\omega}$ of q_k such that $q_{k+1} = q_k \cup \{(y_i, y_{i+1}) : 0 \leq i \leq 2m-1\}$, $y_0 = x_{k+1}$, $y_{2m} = (x_{k+1})b$, there are no edges between y_i and $\Sigma_1 \cup \Sigma_2$, and $y_i \notin \Sigma_1 \cup \Sigma_2$ for $i \in \{1, \ldots, 2m-1\}$. Then by the choice of Σ_1, Σ_2, and the definition of q_{k+1}

$$x_j, (x_j)b \notin \mathrm{dom}(q_{k+1}) \cup \mathrm{ran}(q_{k+1})$$

$$(x_i)q_{k+1}^{2m} = (x_i)b$$

for all $i \in \{1, \ldots, k+1\}$ and $j \in \{k+2, \ldots, d\}$. It also follows from the definition of q_{k+1} that $\Delta_{k+1} = \Delta_k \cup \{y_i : 0 \leq i \leq 2m\}$ and thus Δ_{k+1} is a union of incomplete components of q_{k+1} each of length $2m+1$.

Let $j \in \{k+2, \ldots, d\}$, and let $z \in N(x_j) \cap \Delta_{k+1}$. If $z \in \Delta_k$, then by the inductive hypothesis $z \in \mathrm{dom}(q_k^{2m}) \subseteq \mathrm{dom}(q_{k+1}^{2m})$ and

$$(z)q_{k+1}^{2m} = (z)q_k^{2m} \in N((x_j)b) \cap \Delta_k \subseteq N((x_j)b) \cap \Delta_{k+1}.$$

Otherwise $z \in \Delta_{k+1} \setminus \Delta_k$. Hence $z = y_t$ for some $t \in \{0, \ldots, 2m\}$. However, y_t is such that there are no edges between y_t and $\mathrm{dom}(b)$ for $t \in \{1, \ldots, 2m-1\}$. Then z is either y_0 or y_{2m}. Since $b \in \mathcal{B}$ there are no edges between $x_j \in \mathrm{dom}(b)$ and $y_{2m} = (x_{k+1})b \in \mathrm{ran}(b)$. Hence $z = y_0$ and thus $z \in \mathrm{dom}(q_{k+1}^{2m})$. Since $z \in N(x_j)$ there is an edge between x_j and $z = y_0 = x_{k+1}$. Then it follows from the fact that b is an isomorphism that there is an edge between $(x_j)b$ and $(x_{k+1})b$. Hence $(z)q_{k+1}^{2m} = y_{2m} = (x_{k+1})b \in N((x_j)b) \cap \Delta_{k+1}$. Since z was arbitrary $N(x_j) \cap \Delta_{k+1} \subseteq \mathrm{dom}(q_{k+1}^{2m})$ and $(N(x_j) \cap \Delta_{k+1})\, q_{k+1}^{2m} \subseteq N((x_j)b) \cap \Delta_{k+1}$.

Let $z \in N((x_j)b) \cap \Delta_{k+1}$. If $z \in \Delta_k$ then it follows from the inductive hypothesis that

$$z \in N((x_j)b) \cap \Delta_k = (N(x_j) \cap \Delta_k)\, q_k^{2m} \subseteq (N(x_j) \cap \Delta_{k+1})\, q_{k+1}^{2m}.$$

Otherwise $z = y_j$ for some $j \in \{0, \ldots, 2m\}$. Similarly to above $z = y_{2m} = (x_{k+1})b$ and since b is an isomorphism $(z)q_{k+1}^{-2m} = y_0 = x_{k+1} \in N(x_j)$. Hence $z \in (N(x_j) \cap \Delta_{k+1})\, q_{k+1}^{2m}$, as $x_{k+1} \in \Delta_{k+1}$, and so

$$(N(x_j) \cap \Delta_{k+1})\, q_{k+1}^{2m} = N((x_j)b) \cap \Delta_{k+1}$$

for all $j \in \{k+2, \ldots, d\}$.

Therefore q_{k+1} satisfies the inductive hypothesis and by induction there is $h = q_d \in \mathcal{I}(H_n)^{<\omega}$ an extension of q such that h^{2m} is an extension of b. □

Finally, we can prove the main result of this section, we will restate the Theorem 3.4.1 for the benefit of the reader.

Theorem 3.4.1. *Let $f \in \mathrm{Aut}(H_n)$ be such that f is not the identity. Then $D_f \cap \mathcal{I}(H_n)$ is comeagre in $\mathcal{I}(H_n)$.*

Proof. By Lemma 3.4.2

$$D_f \cap \mathcal{I}(H_n) = \bigcap_{b \in \mathcal{B}} \{g \in \mathcal{I}(H_n) : \langle f, g\rangle \cap [b] \neq \varnothing\},$$

and $\{g \in \mathrm{Aut}(H_n) : \langle f, g\rangle \cap [b] \neq \varnothing\}$ is open by Lemma 3.2.7, thus it is enough to show that $\{g \in \mathcal{I}(H_n) : \langle f, g\rangle \cap [b] \neq \varnothing\}$ is dense in $\mathcal{I}(H_n)$ for all $b \in \mathcal{B}$.

Fix $b \in \mathcal{B}$, and let $q \in \mathcal{I}(H_n)^{<\omega}$. If necessary by extending q using Corollary 3.4.5, we may assume that all of the components of q have length m for some

$m \in \mathbb{N}$, and that $\operatorname{ran}(b) \cup \operatorname{dom}(b) \subseteq \operatorname{dom}(q)$. Suppose that $\operatorname{ran}(q) \setminus \operatorname{dom}(q) = \{x_{1,0}, x_{2,0}, \ldots, x_{d,0}\}$. Let $q_{1,0} = q$, and once $q_{i,j}$ is defined let $\Gamma_{i,j} = \operatorname{dom}(q_{i,j}) \cup \operatorname{ran}(q_{i,j})$ for all i, j. We will perform an induction on the elements of the set $\{1, \ldots, d\} \times \{0, \ldots, m\}$, ordered lexicographically, to construct $q_{d,m} \in \mathcal{I}(H_n)^{<\omega}$ of the form $q_{d,m} = q_{1,0} \cup \{(x_{i,j}, x_{i,j+1}) : 1 \le i \le d \text{ and } 0 \le j \le m-1\}$ such that $x_{i,j} \in \operatorname{supp}(f)$ and $(x_{i,j})f \notin \operatorname{ran}(q_{d,m}) \cup \operatorname{dom}(q_{d,m})$ for all i and all $j \ge 1$. In order to make the rest of the proof shorter, once we have defined $q_{i,m}$ for some $i < d$, we will set $q_{i+1,0} = q_{i,m}$, and similarly we denote $\Gamma_{i,-1} = \varnothing$ for all i.

Suppose that for $k \in \{1, 2, \ldots, d\}$ and $t \in \{0, 1, \ldots, m-1\}$ we defined $q_{k,t} = q_{1,0} \cup \{(x_{i,j}, x_{i,j+1}) : 1 \le i \le k \text{ and } 0 \le j \le t-1\} \in \mathcal{I}(H_n)^{<\omega}$ such that $x_{k,t} \in \operatorname{supp}(f)$ and

$$x_{k,t} \notin \Gamma_{k,t-1} \cup (\Gamma_{k,t-1})\, f \cup (\Gamma_{k,t-1})\, f^{-1}.$$

Choose $x \in \operatorname{supp}(f)$ such that $x \notin \Gamma_{k,t}$ which is possible since $\operatorname{supp}(f)$ is infinite. Then by the Alice's restaurant property there is a vertex $y \notin \Gamma_{k,t} \cup (\Gamma_{k,t})\, f^{-1} \cup \{x, (x)f\}$ such that there is an edge between x and y, and there are no edges between y and $\Gamma_{k,t} \cup (\Gamma_{k,t})\, f^{-1} \cup \{(x)f\}$. Let

$$U = (N(x_{k,t}))\, q_{k,t} \cup \{y\} \quad \text{and} \quad V = \big(\Gamma_{k,t} \cup (\Gamma_{k,t})\, f \cup (\Gamma_{k,t})\, f^{-1} \cup \{(y)f\}\big) \setminus U.$$

Since the subgraph $(N(x_{k,t}))\, q_{k,t}$ is K_{n-1}-free and there are no edges between y and $(N(x_{k,t}))\, q_{k,t}$, the set U is also K_{n-1}-free. Hence by Alice's restaurant property there is a vertex

$$x_{k,t+1} \in H_n \setminus \big(\Gamma_{k,t} \cup (\Gamma_{k,t})\, f \cup (\Gamma_{k,t})\, f^{-1} \cup \{y, (y)f\}\big)$$

such that $N(x_{k,t+1}) \cap (U \cup V) = U$. It follows from $\operatorname{ran}(q_{k,t}) \subseteq \Gamma_{k,t}$ and $y \notin \Gamma_{k,t}$ that

$$\begin{aligned} N(x_{k,t+1}) \cap \operatorname{ran}(q_{k,t}) &= U \cap \operatorname{ran}(q_{k,t}) \\ &= ((N(x_{k,t}))\, q_{k,t} \cup \{y\}) \cap \operatorname{ran}(q_{k,t}) \\ &= (N(x_{k,t}))\, q_{k,t}, \end{aligned}$$

and so $q_{k,t+1} = q_{k,t} \cup \{(x_{k,t}, x_{k,t+1})\} \in \mathcal{I}(H_n)^{<\omega}$ by Lemma 3.4.4 and Corollary 3.4.6.

It follows from f being an automorphism and the existence of an edge between x and y, that there is an edge between $(x)f$ and $(y)f$. However, there is no edge between y and $(x)f$, thus it follows that $y \in \operatorname{supp}(f)$. The vertex y was chosen so that $y \notin \Gamma_{k,t} \cup (\Gamma_{k,t})\, f^{-1}$, and so $y, (y)f \notin \Gamma_{k,t}$. Since $(N(x_{k,t}))\, q_{k,t} \subseteq \Gamma_{k,t}$ and $y \ne (y)f$, it follows that $(y)f \notin U$. By the choice of $x_{k,t+1}$ there is an edge between $x_{k,t+1}$ and y and there are no edges between $x_{k,t+1}$ and $(y)f$, thus

$x_{k,t+1} \in \text{supp}(f)$. Hence $q_{k,t+1}$ satisfies the inductive hypothesis.

This way we can obtain $q_{d,m} \in \mathcal{I}(H_n)^{<\omega}$ such that for all i and all $j \geq 1$

$$x_{i,j} \notin \Gamma_{i,j-1} \cup (\Gamma_{i,j-1})\, f \cup (\Gamma_{i,j-1})\, f^{-1}.$$

Hence $(x_{i,j})f \notin \Gamma_{i,j-1}$. Also if $(x_{i,j})f = x_{i',j'}$, where $(i,j) < (i',j')$ lexicographically, then $x_{i',j'} \in (\Gamma_{i,j})\, f$ which is impossible. Therefore, $(x_{i,j})f \notin \text{ran}(q_{d,m}) \cup \text{dom}(q_{d,m})$ and thus

$$\left((\text{dom}(q))\, q_{k,m}^m f\right) \cap (\text{dom}(q_{k,m}) \cup \text{ran}(q_{k,m})) = \varnothing.$$

Then since $b \in \mathcal{B}$ and $\mathcal{B}$ is closed under conjugation, $u = \left(q_{k,m}^m f\right)^{-1} b q_{k,m}^m f \in \mathcal{B}$. Recall that $\text{ran}(b) \cup \text{dom}(b) \subseteq \text{dom}(q)$, thus the partial isomorphisms $q_{k,m}$ and u satisfy the hypothesis of Lemma 3.4.8. Hence there is an extension $h \in \mathcal{I}(H_n)^{<\omega}$ of $q_{k,m}$ and $l \in \mathbb{Z}$ such that h^{2l} extends u. Therefore $h^m f h^{2l} (h^m f)^{-1}$ extends b and thus

$$\{g \in \mathcal{I}(H_n) : \langle f, g\rangle \cap [b] \neq \varnothing\} \cap [q] \neq \varnothing.$$

Since $q \in \mathcal{I}(H_n)^{<\omega}$ was arbitrary we get that $\{g \in \mathcal{I}(H_n) : \langle f, g\rangle \cap [b] \neq \varnothing\}$ is dense in $\mathcal{I}(H_n)$. □

3.5 Infinitely many finite complete graphs: ωK_n

In this section, we consider the ultrahomogeneous graphs ωK_n for $n \in \mathbb{N}$, $n > 0$. Throughout the section we assume that $n \in \mathbb{N}$, $n > 0$, is fixed and that the connected components of ωK_n are $\{L_i : i \in \mathbb{Z}\}$. We will first prove a couple of technical results.

We begin by characterising the elements of $\mathcal{I}_\Sigma(\omega K_n)$ in a lemma analogous to Corollary 3.4.6. Recall that if $f \in \text{Aut}(\omega K_n)$ or $f \in \text{Aut}(\omega K_n)^{<\omega}$ then $\overline{f}$ is a partial permutation on $\mathbb{Z}$ given by

$$(i)\overline{f} = j \quad \text{if} \quad (L_i)f = L_j.$$

Lemma 3.5.1. *Let $q \in \text{Aut}(\omega K_n)^{<\omega}$ be such that $\text{dom}(q)$ is a union of connected components of ωK_n, and there is $\Sigma \subseteq \text{dom}(q)$ which intersects every component of q in exactly one vertex. Then $q \in \mathcal{I}_\Sigma(\omega K_n)^{<\omega}$ if and only if $\overline{q}$ has no complete components.*

Proof. ($\Rightarrow$) Let $g \in \mathcal{I}_\Sigma(\omega K_n)$ be an extension of q, and let C be a component of q. Then by the hypothesis there is $x \in \Sigma \cap C$. The component of g containing x is infinite, and so C is not a complete component. Since C was arbitrary q has no complete components.

($\Leftarrow$) Let $\{x_0, x_1, \ldots\}$ be the vertices of ωK_n, and let $q_0 = q$. Suppose that for $m \geq 1$, $q_m \in \mathrm{Aut}((\omega K_n)^{<\omega}$ is an extension of q_{m-1}, $\mathrm{dom}(q_m)$ is a union of connected components of ωK_n, $\mathrm{dom}(q_m) \cap \mathrm{ran}(q_m) \supseteq \{x_0, \ldots, x_{m-1}\}$, and q_m has no complete components.

If $x_m \in \mathrm{dom}(q_m)$, let $q'_{m+1} = q_m$. Suppose that $x_m \in \omega K_n \setminus \mathrm{dom}(q_m)$. Let C be the connected component of ωK_n containing x_m, let C' be any connected component of ωK_n such that $C' \subseteq \omega K_n \setminus \mathrm{ran}(q_m)$, and let $\phi : C \longrightarrow C'$ be a bijection. Since $\mathrm{dom}(q_m)$ is a union of connected components, it follows that C is disjoint from $\mathrm{dom}(q_m)$. Hence $q'_{m+1} = q_m \cup \phi$ is an isomorphism between finite subgraphs of ωK_n, and since ωK_n is ultrahomogeneous, $q'_{m+1} \in \mathrm{Aut}(\omega K_n)^{<\omega}$. It follows from the definition that $\mathrm{dom}(q'_{m+1})$ is a union of connected components of ωK_n, $x_m \in \mathrm{dom}(q'_{m+1})$, and q'_{m+1} has no complete components.

If $x_m \in \mathrm{ran}(q'_{m+1})$, let $q_{m+1} = q'_{m+1}$. Suppose that $x_m \in \omega K_n \setminus \mathrm{ran}(q'_{m+1})$. Then $x_m \in \omega K_n \setminus \mathrm{dom}({q'_{m+1}}^{-1})$, and by the previous paragraph there is $q_{m+1} \in \mathrm{Aut}(\omega K_n)^{<\omega}$ such that q^{-1}_{m+1} extends ${q'_{m+1}}^{-1}$, $\mathrm{dom}(q^{-1}_{m+1})$, and thus $\mathrm{dom}(q_{m+1})$, are unions of connected components of ωK_n, $x_m \in \mathrm{ran}(q_{m+1}) = \mathrm{dom}(q^{-1}_{m+1})$, and q_{m+1} has no complete components.

Therefore, the required extension q_m exists for each $m \in \mathbb{N}$. Let $g = \bigcup_{m\in\mathbb{N}} q_m$. Then $g \in \mathrm{Aut}(\omega K_n)$, as $\mathrm{dom}(g) = \mathrm{ran}(g) = \omega K_n$. Since defining every extension q_m, no new components were created, and every component of q intersects Σ in exactly one point, it follows that Σ is a set of orbit representatives of g. Therefore, $q \in \mathcal{I}_\Sigma(\omega K_n)^{<\omega}$. □

In the next proposition we classify when the set $\mathcal{I}_\Sigma(\omega K_n)$ is non-empty.

Proposition 3.5.2. *Let Σ be a finite subset of ωK_n. Then $\mathcal{I}_\Sigma(\omega K_n)$ is non-empty if and only if $|\Sigma|$ is a multiple of n and if $r = |\Sigma|/n$, there is partition $\{P_1, \ldots, P_r\}$ of $\mathbb{Z}$ such that, P_i is infinite and*

$$\sum_{j\in P_i} |L_j \cap \Sigma| = n$$

for all $i \in \{1, \ldots, r\}$.

Proof. ($\Rightarrow$) Let $f \in \mathcal{I}_\Sigma(\omega K_n)$. If $x \in L_i$ and $(x)f \in L_j$, then, since f is an automorphism, $(L_i)f = L_j$. Moreover, if $(L_i)f^m = L_i$ for some $m \in \mathbb{Z}$, then $(L_i)f^{rm} = L_i$ for all $r \in \mathbb{Z}$, and since L_i is finite, f would have a finite cycle. Hence $(L_i)f^m \neq L_i$ for all $m \in \mathbb{Z}$, and so every vertex in L_i is on a separate orbit of f.

Let $k_1, \ldots, k_r \in \mathbb{Z}$ be orbit representatives of $\overline{f}$. Since for every orbit of $\overline{f}$ there are n orbits in f, it follows that $rn = |\Sigma|$. It follows from Lemma 3.5.1 that $\overline{f}$ has no complete component. So $(L_{k_i}) f^m = \left(L_{k_{i'}}\right) f^{m'}$ if and only if $i = i'$ and

$m = m'$. Hence

$$n = \left|\Sigma \cap \left(\bigcup_{m\in\mathbb{Z}} (L_{k_i})\, f^m\right)\right| = \left|\bigcup_{m\in\mathbb{Z}} (\Sigma \cap (L_{k_i})\, f^m)\right| = \sum_{m\in\mathbb{Z}} |\Sigma \cap (L_{k_i})\, f^m|$$

for all $i \in \{1, \ldots, r\}$. Let $P_i = \{(k_i)\,\overline{f}^m : m \in \mathbb{Z}\}$ where $i \in \{1, \ldots, r\}$. Then $\{P_1, \ldots, P_r\}$ is the required partition.

($\Leftarrow$) For $i \in \{1, \ldots, r\}$, let $P_i = \{k_{i,j} : j \in \mathbb{Z}\}$. Define $f \in \mathcal{I}_\Sigma(\omega K_n)$ to be such that

$$(k_{i,j})\overline{f} = k_{i,j+1}$$

for all $i \in \{1, \ldots, r\}$ and $j \in \mathbb{Z}$ by inductively defining f on $\bigcup_{j\in\mathbb{Z}} L_{k_{i,j}}$ for each i independently.

Let $i \in \{1, \ldots, r\}$ be arbitrary. Then $|L_{k_{i,0}} \cap \Sigma| + |L_{k_{i,1}} \cap \Sigma| \leq n$. Since $L_{k_{i,0}}$ and $L_{k_{i,1}}$ are both of size n, there exists a bijection $q_1 : L_{k_{i,0}} \longrightarrow L_{k_{i,1}}$ such that for every $x \in L_{k_{i,0}}$ at most one of the points x and $(x)q_1$ is in Σ. Suppose that for some $m \in \mathbb{N}$ we have defined a bijection

$$q_{2m+1} : \bigcup_{j=-m}^{m} L_{k_{i,j}} \to \bigcup_{k=-m+1}^{m+1} L_{k_{i,j}}$$

such that every incomplete component of q_{2m+1} intersects Σ in at most one point.

Let $t = \sum_{j=-m}^{m+1} |L_{k_{i,j}} \cap \Sigma|$. Then there are $n - t$ incomplete components of q_{2m+1} which have empty intersection with Σ. Since $\sum_{j=-m-1}^{m+1} |L_{i(j,r)} \cap \Sigma| \leq n$, it follows that $|L_{k_{i,-m-1}} \cap \Sigma| \leq n - t$. Hence there exists a bijection $\phi : L_{k_{i,-m-1}} \to L_{k_{i,-m}}$ such that for every $x \in L_{k_{i,-m-1}} \cap \Sigma$, the value $(x)\phi$ belongs to an incomplete component of q_{2m+1} which contains no points from Σ. If we set $q_{2m+2} = q_{2m+1} \cup \phi$, then every incomplete component of q_{2m+2} intersects Σ in at most one point. Similarly we can extend q_{2m+2} to q_{2m+3} by adding a bijection from $L_{k_{i,m+1}}$ to $L_{k_{i,m+2}}$.

Hence by induction

$$f_i = \bigcup_{m\in\mathbb{Z}} q_{2m+1}$$

is an automorphism of $\bigcup_{j\in\mathbb{Z}} L_{k_{i,j}}$ and every orbit of f_i intersects Σ exactly once. The required f is then just the function $\bigcup_{i=1}^{r} f_i$. □

Before proving the main result of this section we need a lemma analogous to Lemma 3.4.2.

Lemma 3.5.3. *Let $\Sigma \subseteq \omega K_n$ be finite, and let $\mathcal{F}$ consist of those $g \in \mathrm{Aut}(\omega K_n)^{<\omega}$ where the sets $\mathrm{dom}(g)$ and $\mathrm{ran}(g)$ are disjoint, both are unions of connected components of ωK_n, and $\overline{g}$ does not have any complete components. Then for every*

$f \in \mathrm{Aut}(\omega K_n)$

$$D_f \cap \mathcal{I}_\Sigma(\omega K_n) = \bigcap_{p\in\mathcal{F}} \{g \in \mathcal{I}_\Sigma(\omega K_n) : \langle f, g\rangle \cap [p] \neq \varnothing\}.$$

Proof. Recall that $D_f = \{g \in \mathrm{Aut}(\omega K_n) : \langle f, g\rangle$ is dense in $\mathrm{Aut}(\omega K_n)\}$. Then

$$D_f \cap \mathcal{I}_\Sigma(\omega K_n) = \bigcap_{q\in\mathrm{Aut}(\omega K_n)^{<\omega}} \{g \in \mathcal{I}_\Sigma(\omega K_n) : \langle f, g\rangle \cap [q] \neq \varnothing\}.$$

($\subseteq$) This follows immediately since $\mathcal{F} \subseteq \mathrm{Aut}(\omega K_n)^{<\omega}$.

($\supseteq$) Let $g \in \bigcap_{p\in\mathcal{F}}\{g \in \mathcal{I}_\Sigma(\omega K_n) : \langle f, g\rangle \cap [p] \neq \varnothing\}$, let $q \in \mathrm{Aut}(\omega K_n)^{<\omega}$, and let

$$\Gamma = \bigcup\{L_i : \mathrm{dom}(q) \cap L_i \neq \varnothing\}.$$

If $h \in \mathrm{Aut}(\omega K_n)$ is an extension of q, $x \in L_i$, and $(x)h \in L_j$, then $(L_i)h = L_j$. Hence $(\Gamma)h$ is a union of connected components of ωK_n. Let $r = h|_\Gamma$. Then $[r] \subseteq [q]$.

Let Δ be a subgraph of ωK_n such that Δ is isomorphic to $\mathrm{dom}(r)$, $\Delta \cap (\mathrm{dom}(r) \cup \mathrm{ran}(r)) = \varnothing$. Let p be any isomorphism between $\mathrm{dom}(r)$ and Δ. Note that since $\mathrm{dom}(r)$ is a union of connected components of ωK_n so is Δ. Since ωK_n is ultrahomogeneous, we have that $p \in \mathrm{Aut}(\omega K_n)^{<\omega}$. Then $\mathrm{dom}(p) = \mathrm{dom}(r)$, $\mathrm{ran}(p) = \mathrm{dom}(p^{-1}r) = \Delta$ and $\mathrm{ran}(p^{-1}r) = \mathrm{ran}(r)$. Hence $p, p^{-1}r \in \mathcal{F}$. By the choice of g there are $h_1, h_2 \in \langle f, g\rangle$ such that $h_1 \in [p]$ and $h_2 \in [p^{-1}r]$. Therefore $h_1h_2 \in [r] \subseteq [q]$ and $h_1h_2 \in \langle f, g\rangle$, thus $\langle f, g\rangle \cap [q] \neq \varnothing$. Since q was arbitrary, $g \in \bigcap_{q\in\mathrm{Aut}(\omega K_n)^{<\omega}}\{g \in \mathcal{I}_\Sigma(\omega K_n) : \langle f, g\rangle \cap [q] \neq \varnothing\}$. □

We will now prove Theorem 3.1.8(ii), which we restate for the sake of convenience.

Theorem 3.5.4. *Let $f \in \mathrm{Aut}(\omega K_n)$ be such that $\mathrm{supp}(\overline{f})$ is infinite, and let Σ be a finite subset of ωK_n. Then $D_f \cap \mathcal{I}_\Sigma(\omega K_n)$ is comeagre in $\mathcal{I}_\Sigma(\omega K_n)$.*

Proof. If $\mathcal{I}_\Sigma(\omega K_n)$ is empty, then the result holds trivially. So, for the remainder of the proof, we will suppose that $\mathcal{I}_\Sigma(\omega K_n)$ is non-empty.

By Lemma 3.5.3

$$D_f \cap \mathcal{I}_\Sigma(\omega K_n) = \bigcap_{p\in\mathcal{F}} \{g \in \mathcal{I}_\Sigma(\omega K_n) : \langle f, g\rangle \cap [p] \neq \varnothing\},$$

and by Lemma 3.2.7 the set $\{g \in \mathcal{I}_\Sigma(\omega K_n) : \langle f, g\rangle \cap [p] \neq \varnothing\}$ is open, so it suffices to show that the aforementioned set is dense in $\mathcal{I}_\Sigma(\omega K_n)$.

Let $p \in \mathcal{F}$ and let $q \in \mathcal{I}_\Sigma(\omega K_n)^{<\omega}$. We will show that there exists an extension $r \in \mathcal{I}_\Sigma(\omega K_n)^{<\omega}$ of q such that every extension $g \in \mathcal{I}_\Sigma(\omega K_n)$ of r satisfies $\langle f, g\rangle \cap [p] \neq \varnothing$. Since there exists $h \in \mathcal{I}_\Sigma(\omega K_n)$ an extension of q, there is a finite subset Γ of ωK_n such that $\mathrm{dom}(q) \cup \mathrm{dom}(p) \cup \mathrm{ran}(p) \subseteq \Gamma$,

Γ is a union of connected components of ωK_n, and that $h|_\Gamma$ has exactly $|\Sigma|$ components each intersecting Σ exactly once and each component is of the same length. If necessary, by considering $h|_\Gamma$ instead of q we may assume without loss of generality that $\mathrm{dom}(q)$ is a union of connected components of ωK_n, and that q has $|\Sigma|$ incomplete components each of some fixed length m, and that $\Sigma \cup \mathrm{dom}(p) \cup \mathrm{ran}(p) \subseteq \mathrm{dom}(q)$. Then $\mathrm{ran}(q) \setminus \mathrm{dom}(q)$ is a union of connected components $L_{1,0}, \ldots, L_{N,0}$ for some $N \in \mathbb{N}$.

Let $q_{1,0} = q$ and once $q_{i,j}$ is defined let $\Gamma_{i,j} = \mathrm{dom}(q_{i,j}) \cup \mathrm{ran}(q_{i,j})$. Suppose there is $i \in \{0, \ldots, m-1\}$ such that $q_{1,i} \in \mathcal{I}_\Sigma(\omega K_n)^{<\omega}$ is defined such that $\mathrm{dom}(q_{1,i})$ is a union of connected components, and $(x)q_{1,i}^j \in L_{1,j}$ for all $x \in L_{1,0}$ and $j \in \{1, \ldots, i\}$. Since $\overline{f}$ has infinite support, there exists a connected component $L_{1,i+1}$ of ωK_n such that $(L_{1,i+1})f \neq L_{1,i+1}$ and

$$L_{1,i+1} \cap \left(\Gamma_{1,i} \cup (\Gamma_{1,i})f \cup (\Gamma_{1,i})f^{-1}\right) = \varnothing.$$

Let $\phi : L_{1,i} :\longrightarrow L_{1,i+1}$ be a bijection, and let $q_{1,i+1} = q_{1,i} \cup \phi$. Then $q_{1,i+1} \in \mathcal{I}_\Sigma(\omega K_n)^{<\omega}$ by Lemma 3.5.1. Also by the definition of $q_{1,i+1}$ the set $\mathrm{dom}(q_{1,i+1}) = \mathrm{dom}(q_{1,i}) \cup L_{1,i}$ is a union of connected components, and $(x)q_{1,i+1}^{i+1} \in L_{1,i+1}$ for all $x \in L_{1,0}$. Hence by induction there is $q_{1,m} \in \mathcal{I}_\Sigma(\omega K_n)^{<\omega}$ such that $\mathrm{dom}(q_{1,m})$ is a union of connected components of ωK_n, and $(x)q_{1,m}^j \in L_{1,j}$ for all $x \in L_{1,0}$ and $j \in \{1, \ldots, m\}$.

Let $q_{2,0} = q_{1,m}$ and suppose for some $i \in \{2, \ldots N\}$ there is $q_{i,0} \in \mathcal{I}_\Sigma(\omega K_n)^{<\omega}$ an extension of q such that $\mathrm{dom}(q_{i,0})$ is a union of connected components of ωK_n, and $(x)q_{i,0}^k \in L_{j,k}$ for all $x \in L_{j,0}$, all $j \in \{1, \ldots i-1\}$, and all $k \in \{1, \ldots, m\}$. The same argument as before can be used to define $q_{i,m} \in \mathcal{I}_\Sigma(\omega K_n)^{<\omega}$ an extension of q such that $\mathrm{dom}(q_{i,m})$ is a union of connected components of ωK_n, and $(x)q_{i,m}^k \in L_{j,k}$ for all $x \in L_{j,0}$, all $j \in \{1, \ldots, i\}$, and all $k \in \{1, \ldots, m\}$. Hence by induction $\mathrm{dom}(q_{N,m})$ is a union of connected components of ωK_n, and $(x)q_{N,m}^k \in L_{j,k}$ for all $x \in L_{j,0}$, all $j \in \{1, \ldots, N\}$, and all $k \in \{1, \ldots, m\}$.

We will show that $q_{N,m}$ is the desired extension of q. Let $r = q_{N,m}$. If $x \in L_{i,0}$ for some $i \in \{1, \ldots, N\}$ and $j \in \{1, \ldots, m\}$, then

$$(x)r^j \in L_{i,j} \subseteq \Gamma_{i,j} \tag{3.14}$$

and so by the choice of $L_{i,j}$ we have $(x)r^j \notin \Gamma_{i,j-1} \cup (\Gamma_{i,j-1})f \cup (\Gamma_{i,j-1})f^{-1}$ for all $j \in \{1, \ldots, m\}$. In particular,

$$(x)r^j f \notin \Gamma_{i,j-1} \quad \text{and} \quad (x)r^j f^{-1} \notin \Gamma_{i,j-1} \tag{3.15}$$

for all $i \in \{1, \ldots, N\}$ and $j \in \{1, \ldots, m\}$.

Let $x \in L_{i,0}$ and $y \in L_{j,0}$ for any $i, j \in \{1, \ldots, N\}$. We will show that $\left((x)r^k\right) f \neq (y)r^l$ for all $k \in \{1, \ldots, m\}$ and $l \in \{-m+1, \ldots, m\}$. If $i = j$ and

$k = l$, then, since $(x)r^k, (y)r^l \in L_{i,k}$ by (3.14), and $(L_{i,k})f \neq L_{i,k}$ by the choice of $L_{i,j}$, it follows that $\left((x)r^k\right) f \neq (y)r^l$. Hence we may assume that $(i,k) \neq (j,l)$. There are three cases to consider.

If $l \leq 0$, then $(y)r^l \in \operatorname{dom}(q_{1,0}) \cup \operatorname{ran}(q_{1,0}) = \Gamma_{1,0} \subseteq \Gamma_{i,k}$ and $(x)r^k f \notin \Gamma_{i,k}$ by (3.15), and so $(x)r^k f \neq (y)r^l$.

Suppose that $i > j$ and $l > 0$, or $i = j$ and $k > l > 0$. Then $(y)r^l \in \Gamma_{j,l}$ by (3.14). By the assumption of this case, $\Gamma_{j,l} \subseteq \Gamma_{i,k-1}$ and $((x)r^k)f \notin \Gamma_{i,k-1}$ by (3.15). Thus $((x)r^k)f \neq (y)r^l$, in this case.

Suppose that $i < j$ and $l > 0$, or $i = j$ and $k < l$. Then $\Gamma_{i,k} \subseteq \Gamma_{j,l-1}$. Since $((y)r^l)f^{-1} \notin \Gamma_{j,l-1}$ by (3.15), it follows that $((y)r^l)f^{-1} \notin \Gamma_{i,k}$, and so $((x)r^k)f \neq (y)r^l$. Therefore, in all three cases $((x)r^k)f \notin \operatorname{ran}(r) \cup \operatorname{dom}(r)$.

Recall that $\operatorname{dom}(p) \cup \operatorname{ran}(p) \subseteq \operatorname{dom}(q)$ and that every point in $\operatorname{dom}(q)$ can be expressed as $(x)r^j$ for some $x \in \bigcup_{i=1}^N L_{i,0}$ and $j \in \{-m+1, \ldots, -1\}$. Define $u = (r^m f)^{-1} p (r^m f)$. Since $\overline{p}$ has no complete components, the same is true for $\overline{u}$. Also

$$\operatorname{dom}(u) \cup \operatorname{ran}(u) \subseteq \{\left((x)r^j\right) f : 1 \leq j \leq m, \text{ and } x \in L_{i,0} \text{ for some } i\}$$

and hence $(\operatorname{dom}(u) \cup \operatorname{ran}(u)) \cap (\operatorname{dom}(r) \cup \operatorname{ran}(r)) = \varnothing$.

Suppose $\operatorname{dom}(u) \setminus \operatorname{ran}(u) = \bigcup_{k=1}^M L_{i_k}$, and let n_k be the largest integer such that $(L_{i_k})u^{n_k}$ is defined for some $k \in \{1, \ldots, M\}$. Define v to be an extension of u by bijections $(L_{i_k})u^{n_k} \longrightarrow L_{i_{k+1}}$ for all $k \in \{1, \ldots, M-1\}$. Then the domain of v is a union of connected components of the graph, and v has no complete components, since neither p nor u do. Finally choose any bijection $\psi : L_{N,m} \longrightarrow L_{i_1}$ and define $h = r \cup \psi \cup v$. Then the number of components in h is $|\Sigma|$ and so $h \in \mathcal{I}_\Sigma(\omega K_n)^{<\omega}$ by Lemma 3.5.1. Let $g \in \mathcal{I}_\Sigma(\omega K_n)$ be an extension of h. By definition of u we have that $(h^m f)h(h^m f)^{-1}$ extends p, thus $\langle f, g\rangle \cap [p] \neq \varnothing$ and $g \in [q]$. Therefore the set $\{g \in \mathcal{I}_\Sigma(\omega K_n) : \langle f, g\rangle \cap [p] \neq \varnothing\}$ is dense in $\mathcal{I}_\Sigma(\omega K_n)$. □

The following is an immediate corollary of Lemma 3.2.8 and Theorem 3.5.4.

Corollary 3.5.5. *Let $f \in \operatorname{Aut}(\omega K_n)$ be such that $\operatorname{supp}(\overline{f})$ is infinite. Then $D_f \cap \mathcal{I}(\omega K_n)$ is comeagre in $\mathcal{I}(\omega K_n)$.*

3.6 Finitely many infinite complete graphs: nK_ω

In this section we will consider the ultrahomogeneous graph nK_ω for a fixed $n \in \mathbb{N}$ such that $n \geq 2$. Throughout this section let $L_1, L_2, \ldots, L_n$ be the connected components of nK_ω. Recall that, if $f \in \operatorname{Aut}(nK_\omega)$ and $\Sigma \subseteq nK_\omega$ is finite, then

$$\mathcal{A}_f = \{g \in \operatorname{Aut}(nK_\omega) : \langle \overline{f}, \overline{g}\rangle = S_n\}$$

and

$$\mathcal{A}_{f,\Sigma} = \{g \in \mathcal{A}_f : \Sigma \text{ is a set of orbit representatives of } g\}.$$

It follows from Theorem 3.1.3, that if $f \in \mathrm{Aut}(nK_\omega)$ such that $\overline{f} \neq \mathrm{id}$ and $n \geq 3$ then $\mathcal{A}_f \neq \varnothing$ if and only if $n \neq 4$, or $n = 4$ and $\overline{f} \notin \{(1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\}$.

We have shown in Section 3.2 that $\mathcal{A}_f$ and $\mathcal{A}_{f,\Sigma}$ are Baire spaces and thus we can consider their comeagre subsets.

The following lemma combined with Lemma 3.2.5 (i.e. the fact that $\mathcal{A}_f$ is closed) demonstrates that D_f is not dense, and thus not comeagre, in any set which is not contained in $\mathcal{A}_f$.

Lemma 3.6.1. *Let $f \in \mathrm{Aut}(nK_\omega)$. If $g \in \mathrm{Aut}(nK_\omega)$ is such that $\langle f, g\rangle$ is dense in $\mathrm{Aut}(nK_\omega)$, then $\langle \overline{f}, \overline{g}\rangle = S_n$. In other words, $D_f \subseteq \mathcal{A}_f$.*

Proof. Let $g \in \mathrm{Aut}(nK_\omega)$ be such that $\langle f, g\rangle$ is dense in $\mathrm{Aut}(nK_\omega)$. Let $\sigma \in S_n$ be arbitrary. Then it is straightforward to verify that there is $q \in \mathrm{Aut}(nK_\omega)^{<\omega}$ such that $\overline{q} = \sigma$. Since $\langle f, g\rangle$ is dense, it follows that there is an element $h \in \langle f, g\rangle$ which extends q. Therefore $\sigma = \overline{h} \in \langle \overline{f}, \overline{g}\rangle$ which implies that $g \in \mathcal{A}_f$. $\square$

Let $f \in \mathrm{Aut}(nK_\omega)$. Then f is called NON-STABILISING if for all $\Gamma \subsetneq nK_\omega$, all $x \in \Gamma$ and all $q \in \mathcal{A}_f^{<\omega}$ there is $g \in [q] \cap \mathcal{A}_f$ such that $(x)h \notin \Gamma$ for some $h \in \langle f, g\rangle$. We say that $f \in \mathrm{Aut}(nK_\omega)$ is STABILISING if it is not non-stabilising.

Proposition 3.6.2. *Let $f \in \mathrm{Aut}(nK_\omega)$ be such that $\mathcal{A}_f \neq \varnothing$. Then f is stabilising if and only if there is a finite subset Λ of nK_ω such that f stabilises Λ setwise and*

$$|L_i \cap \Lambda| = |L_j \cap \Lambda|$$

for all $i, j \in \{1, 2, \ldots, n\}$.

Proof. ($\Rightarrow$) Let f be a stabilising automorphism of nK_ω. By the definition, there is $\Delta \subsetneq nK_\omega$, $x \in \Delta$ and $q \in \mathcal{A}_f^{<\omega}$ such that for all $g \in [q] \cap \mathcal{A}_f$ and all $h \in \langle f, g\rangle$ we have that $(x)h \in \Delta$. If necessary by taking an extension of q, we may assume without loss of generality that $\overline{q} \in S_n$. Fix any $g \in [q] \cap \mathcal{A}_f$, and let $\Gamma = \{(x)h : h \in \langle f, g\rangle\} \subseteq \Delta$. Then the subgroup $\langle f, g\rangle$ stabilises Γ. Hence f also setwise stabilises Γ. Let $i, j \in \{1, \ldots, n\}$ be arbitrary. Since $g \in \mathcal{A}_f$ we may choose $h \in \langle f, g\rangle$ such that $(i)\overline{h} = j$. By definition Γ is setwise stabilised by h and thus

$$(L_i \cap \Gamma)\, h \subseteq L_j \cap \Gamma \quad \text{and} \quad (L_j \cap \Gamma)\, h^{-1} \subseteq L_i \cap \Gamma,$$

as both h and h^{-1} are bijections. It follows that $|L_i \cap \Gamma| = |L_j \cap \Gamma|$. Since $\langle f, g\rangle$ also setwise stabilises $nK_\omega \setminus \Gamma$, the same argument shows that $|L_i \cap (nK_\omega \setminus \Gamma)| = |L_j \cap (nK_\omega \setminus \Gamma)|$.

Finally, suppose that both Γ and $nK_\omega \setminus \Gamma$ are infinite. Then for every $i \in \{1, \ldots, n\}$ the sets $(\Gamma \cap L_i) \setminus (\mathrm{dom}(q) \cup \mathrm{ran}(q))$ and $((nK_\omega \setminus \Gamma) \cap L_i) \setminus (\mathrm{dom}(q) \cup$

ran(q)) are non-empty. Hence for every $i \in \{1, \ldots, n\}$ there are $x \in L_i \cap \Gamma$ and an extension $g \in \text{Aut}(nK_\omega)$ of q such that $(x)g \in nK_\omega \setminus \Gamma$, contradicting the choice of Γ. Therefore either Γ or $nK_\omega \setminus \Gamma$ is finite, and since both sets are stabilised setwise by f, one of them is the required set Λ.

($\Leftarrow$) Let $m = |L_i \cap \Lambda|$ for any, and all, $i \in \{1, 2, \ldots, n\}$ and let $L_i \cap \Lambda = \{\gamma(i, j) : 1 \leq j \leq m\}$. Since $\mathcal{A}_f$ is non-empty there is $\sigma \in S_n$ such that $\langle \overline{f}, \sigma \rangle = S_n$. Define a finite isomorphism $q : \Lambda \longrightarrow \Lambda$ such that $(\gamma(i, j))\, q = \gamma((i)\sigma, j)$ for all $j \in \{1, \ldots, m\}$. Then $\overline{q} = \sigma$ and so $q \in \mathcal{A}_f^{<\omega}$. Moreover, Λ is a union of cycles of q and hence $\langle f, g \rangle$ stabilises Λ for any $g \in [q]$. Therefore, f is stabilising. $\square$

The following theorem is a restatement of Theorem 3.1.8(iii), and it is the main result in this section.

Theorem 3.6.3. *Let $f \in \text{Aut}(nK_\omega)$. Then f is non-stabilising if and only if D_f is comeagre in $\mathcal{A}_f$. Furthermore, if f is non-stabilising and Σ is any finite subset of nK_ω, then $D_f \cap \mathcal{A}_{f,\Sigma}$ is comeagre in $\mathcal{A}_{f,\Sigma}$.*

If f is stabilising, and $D_f \cap \mathcal{A}_{f,\Sigma}$ is comeagre in $\mathcal{A}_{f,\Sigma}$ for all finite Σ, then by Lemma 3.2.8, $D_f \cap \mathcal{A}_f$ is comeagre in $\mathcal{A}_f$ and so, by Theorem 3.6.3, f is non-stabilising, which is a contradiction. Hence if f is stabilising, then there exists Σ such that $D_{f,\Sigma} \cap \mathcal{A}_{f,\Sigma}$ is not comeagre in $\mathcal{A}_{f,\Sigma}$. It is therefore natural to consider the following question.

Open question. *For which stabilising f and finite sets Σ, is $D_f \cap \mathcal{A}_{f,\Sigma}$ comeagre in $\mathcal{A}_{f,\Sigma}$?*

We will prove Theorem 3.6.3 in a series of lemmas. We begin by showing several ways to extend partial isomorphisms in $\mathcal{A}_{f,\Sigma}^{<\omega}$, which we will have to do ad infinitum in the proof of Theorem 3.6.3.

Lemma 3.6.4. *Let $q \in \text{Aut}(nK_\omega)^{<\omega}$ be such that $\overline{q} \in S_n$, and let $h = q \cup \{(x, y)\}$. Then $h \in \text{Aut}(nK_\omega)^{<\omega}$ if and only if there is $a \in \{1, \ldots, n\}$ such that $x \in L_a \setminus \text{dom}(q)$ and $y \in L_{(a)\overline{q}} \setminus \text{ran}(q)$.*

Proof. ($\Rightarrow$) Suppose that $h \in \text{Aut}(nK_\omega)^{<\omega}$ and let $a \in \{1, \ldots, n\}$ be such that $x \in L_a$. Since $q \cup \{(x, y)\}$ is a partial isomorphism, it follows that $x \notin \text{dom}(q)$ and $y \notin \text{ran}(q)$. Then $(x)h \in L_{(a)\overline{h}}$ by the definition of $\overline{h}$. Finally, since $\overline{h} = \overline{q}$, it follows that $x \in L_a \setminus \text{dom}(q)$ and $y = (x)g \in L_{(a)\overline{q}} \setminus \text{ran}(q)$.

($\Leftarrow$) Let $g \in \text{Aut}(nK_\omega)$ be an extension of q. Since g is an isomorphism, it follows that $(x)g, y \in L_{(a)\overline{q}}$. Hence the transposition $((x)g, y) \in \text{Aut}(nK_\omega)$ and thus $g \circ ((x)g, y) \in \text{Aut}(nK_\omega)$. Since $(x)g, y \notin \text{ran}(q)$, it then follows that $g \circ ((x)g, y)$ is an extension of h. Therefore $h \in \text{Aut}(nK_\omega)^{<\omega}$. $\square$

Roughly speaking, in the next lemma, we show how to extend a partial isomorphism with a set of orbit representatives to an automorphism with the same set of orbit representatives.

Lemma 3.6.5. *Let $q \in \operatorname{Aut}(nK_\omega)^{<\omega}$ be such that $\overline{q} \in S_n$, and let Σ be a finite subset of $\operatorname{dom}(q)$ such that $|\Sigma \cap C| \leq 1$ for every component C of q, with equality holding if C is complete. Suppose that for each $i \in \{1, \ldots, n\}$ there is $j \in \{1, \ldots, n\}$ such that $(j)\overline{q}^m = i$, for some $m \in \mathbb{Z}$, and $L_j \cap \Sigma$ contains a point in an incomplete component of q. Then there is an extension $g \in \operatorname{Aut}(nK_\omega)$ of q such that Σ is a set of orbit representatives of g, every incomplete component of q is contained in an infinite orbit of g, and $(x)g \notin \operatorname{dom}(q)$ for all $x \in \operatorname{ran}(q) \setminus \operatorname{dom}(q)$.*

Proof. For each $x \in \operatorname{ran}(q) \setminus \operatorname{dom}(q)$ there is $a \in \{1, \ldots, n\}$ such that $x \in L_a$, and there is $y \in L_{(a)\overline{q}} \setminus (\operatorname{dom}(q) \cup \operatorname{ran}(q))$. Then by Lemma 3.6.4 the mapping $q' = q \cup \{(x, y)\}$ is in $\operatorname{Aut}(nK_\omega)^{<\omega}$ and $(x)q' = y \notin \operatorname{dom}(q)$. Repeating this for each vertex in $\operatorname{ran}(q) \setminus \operatorname{dom}(q)$ we obtain an extension $q'' \in \operatorname{Aut}(nK_\omega)^{<\omega}$ of q such that $(x)q'' \notin \operatorname{dom}(q)$ for all $x \in \operatorname{ran}(q) \setminus \operatorname{dom}(q)$ and $(\operatorname{ran}(q) \setminus \operatorname{dom}(q)) \subseteq \operatorname{dom}(q'')$. Hence $(x)g = (x)q'' \notin \operatorname{dom}(q)$ for every extension $g \in \operatorname{Aut}(nK_\omega)$ of q'' and every $x \in \operatorname{ran}(q) \setminus \operatorname{dom}(q)$.

Suppose that O is an incomplete component of q'' such that $O \cap \Sigma = \varnothing$. Let $y \in O \cap \operatorname{dom}(q'') \setminus \operatorname{ran}(q'')$. Then there is $a \in \{1, \ldots, n\}$ such that $y \in L_a$. It follows from the hypothesis that there is $b \in \{1, \ldots, n\}$, $y_0 \in L_b \cap \operatorname{ran}(q'') \setminus \operatorname{dom}(q'')$ such that the component of q'' containing y_0 intersects Σ non-trivially, and $m \in \mathbb{N}$ such that $(b)\overline{q}^m = a$. Successively for each $i \in \{1, \ldots, m-1\}$ choose

$$y_i \in L_{(b)\overline{q}^i} \setminus (\operatorname{dom}(q'') \cup \operatorname{ran}(q'') \cup \{y_1, \ldots, y_{i-1}\}),$$

and let $y_m = y$. Then by repeated application of Lemma 3.6.4 we have that $q'' \cup \{(y_{i-1}, y_i) : 1 \leq i \leq m\} \in \operatorname{Aut}(nK_\omega)^{<\omega}$. If we repeat this for every incomplete component of q'' which has empty intersection with Σ, we obtain $q_0 \in \operatorname{Aut}(nK_\omega)^{<\omega}$ an extension of q'' such that every component of q_0 intersects Σ in exactly one point.

Let $nK_\omega = \{x_i : i \in \mathbb{N}\}$, and suppose that for some $j \in \mathbb{N}$ we have defined $q_j \in \operatorname{Aut}(nK_\omega)^{<\omega}$ such that incomplete components of q are contained in incomplete components of q_j, Σ consists of exactly one point from every component of q_j, and

$$\{x_1, \ldots, x_j\} \subseteq \operatorname{dom}(q_j) \cap \operatorname{ran}(q_j).$$

Suppose $x_{j+1} \notin \operatorname{dom}(q_j) \cap \operatorname{ran}(q_j)$. There are three cases to consider.

Suppose that $x_{j+1} \in \operatorname{ran}(q_j) \setminus \operatorname{dom}(q_j)$. Then by Lemma 3.6.4 there is an extension $q_{j+1} = q_j \cup \{(x_{j+1}, y)\} \in \operatorname{Aut}(nK_\omega)^{<\omega}$ for some $y \notin \operatorname{dom}(q_j) \cup \operatorname{ran}(q_j)$. Suppose that $x_{j+1} \in \operatorname{dom}(q_j) \setminus \operatorname{ran}(q_j)$. Then by Lemma 3.6.4 there is an extension $q_{j+1}^{-1} = q_j^{-1} \cup \{(x_{j+1}, y)\} \in \operatorname{Aut}(nK_\omega)^{<\omega}$ for some $y \notin \operatorname{dom}(q_j) \cup \operatorname{ran}(q_j)$.

Finally, suppose that $x_{j+1} \in L_a \setminus (\operatorname{dom}(q_j) \cup \operatorname{ran}(q_j))$ for some a. It follows from the hypothesis that there are $b \in \{1, \ldots, n\}$ and $y_0 \in L_b \cap \operatorname{ran}(q_j) \setminus \operatorname{dom}(q_j)$ such that the component of q_j containing y_0 intersects Σ non-trivially, such that

$(b)\overline{q}^m = a$ for some $m \in \mathbb{N}$. Successively for each $i \in \{1, \ldots, m-1\}$ choose

$$y_i \in L_{(b)\overline{q}^i} \setminus (\mathrm{dom}(q_j) \cup \mathrm{ran}(q_j) \cup \{y_1, \ldots, y_{i-1}\}).$$

Also let $y_m = x_{j+1}$. Then by repeated application of Lemma 3.6.4 we have that $q_j \cup \{(y_{i-1}, y_i) : 1 \le i \le m\} \in \mathrm{Aut}(nK_\omega)^{<\omega}$. Now, we fall into the first case and we can define q_{j+1} as before.

In all three cases, we have defined an extension q_{j+1} satisfying the inductive hypothesis. Let

$$g = \bigcup_{j \in \mathbb{N}} q_j.$$

Then $g \in \mathrm{Aut}(nK_\omega)$, the infinite orbits of g are in one to one correspondence with incomplete components of q_0, and finite orbits of g are in one to one correspondence with complete components. Recall that every component of q_0 intersects Σ in exactly one point. Hence Σ is a set of orbit representatives. □

We can now prove an easy corollary.

Corollary 3.6.6. *Let $f \in \mathrm{Aut}(nK_\omega)$ and let $\Sigma \subseteq nK_\omega$ be finite. Let $q \in \mathcal{A}_{f,\Sigma}^{<\omega}$ be such that $\Sigma \subseteq \mathrm{dom}(q)$ and $\overline{q} \in S_n$. Then there is an extension $g \in \mathcal{A}_{f,\Sigma}$ of q such that every incomplete component of q is contained in an infinite orbit of g, and $(x)g \notin \mathrm{dom}(q)$ for all $x \in \mathrm{ran}(q) \setminus \mathrm{dom}(q)$.*

Proof. Since $q \in \mathcal{A}_{f,\Sigma}^{<\omega}$, the set Σ intersects every incomplete component of q in at most one point, and every complete component in exactly one point.

If $i \in \{1, \ldots, n\}$ is arbitrary, then, since every extension $h \in \mathcal{A}_{f,\Sigma}$ of q has $|\Sigma|$ orbits, it follows that there is at least one infinite orbit C of h intersecting L_i non-trivially. Since Σ is a set of orbit representatives, there exists $x \in C$ such that $x \in \Sigma \cap L_j$ for some $j \in \{1, \ldots, n\}$ such that $(j)\overline{q}^m = i$ for some $m \in \mathbb{Z}$. In particular, since C is an infinite orbit of h, x is on an incomplete component of q, and so q satisfies the hypothesis of Lemma 3.6.5 from which the corollary follows. □

In the next lemma, as a further consequence of Lemma 3.6.5, we show that the left-to-right implication of the first part of Theorem 3.6.3, is a consequence of the second part.

Lemma 3.6.7. *Let $f \in \mathrm{Aut}(nK_\omega)$ be such that $D_f \cap \mathcal{A}_{f,\Sigma}$ is comeagre in $\mathcal{A}_{f,\Sigma}$ for all finite sets $\Sigma \subseteq nK_\omega$. Then D_f is comeagre in A_f.*

Proof. Let $q \in \mathcal{A}_f^{<\omega}$. We will show that there is an extension g of q with finitely many orbits, and so the result will follow from Lemma 3.6.5. First, note that there is $k \in \mathrm{Aut}(nK_\omega)$ extending q, since nK_ω is ultrahomogeneous. Hence there exists an extension $q' \in \mathrm{Aut}(nK_\omega)$ of q such that $\overline{q'} \in S_n$. If necessary by considering q' instead of q, we can assume that $\overline{q} \in S_n$. Then all extensions $h \in \mathrm{Aut}(nK_\omega)^{<\omega}$

of q are also in $\mathcal{A}_f^{<\omega}$. For all $i \in \{1, \ldots, n\}$, let $x_i \in L_i \setminus (\mathrm{dom}(q) \cup \mathrm{ran}(q))$. Then by applying Lemma 3.6.4 repeatedly we can construct $h \in \mathcal{A}_f^{<\omega}$ an extension of q such that each vertex x_i is on an incomplete component of h. Fix any finite $\Sigma \subseteq nK_\omega$ such that Σ intersects every component of h exactly once. Since $\overline{h} \in S_n$, for each $i \in \{1, \ldots, n\}$ there is an incomplete component containing x_i, and by the choice of Σ there is $j \in \{1, \ldots, n\}$ such that $\Sigma \cap L_j$ is non-empty and $(j)\overline{h}^m = i$ for some $m \in \mathbb{Z}$. Then by Lemma 3.6.5 there is g an extension of q with finitely many orbits. Therefore we are done by Lemma 3.2.8. □

The next result enables us to extend partial isomorphisms in $\mathcal{A}_{f,\Sigma}^{<\omega}$.

Lemma 3.6.8. *Let $f \in \mathrm{Aut}(nK_\omega)$ and let $\Sigma \subseteq nK_\omega$ be finite. Let $q \in \mathcal{A}_{f,\Sigma}^{<\omega}$ be such that $\Sigma \subseteq \mathrm{dom}(q)$ and $\overline{q} \in S_n$. Suppose $h = q \cup \{(x, y)\} \in \mathrm{Aut}(nK_\omega)^{<\omega}$ for some $x \notin \mathrm{dom}(q)$ and $y \notin \mathrm{dom}(q) \cup \mathrm{ran}(q)$ such that $x \neq y$. Then $h \in \mathcal{A}_{f,\Sigma}^{<\omega}$.*

Proof. Since $q \in \mathrm{Aut}(nK_\omega)^{<\omega}$ there is $r \in \mathrm{Aut}(nK_\omega)^{<\omega}$ extending q, such that $x \in \mathrm{ran}(r) \setminus \mathrm{dom}(r)$. By Corollary 3.6.6 there is $g \in \mathcal{A}_{f,\Sigma}$ such that every incomplete component of r is contained in an infinite orbit of g and $(x)g \notin \mathrm{dom}(r)$, and so $(x)g \notin \mathrm{dom}(q)$. Note that if $(x)g = x$, then $\{x\}$ is an orbit of g and therefore $x \in \Sigma$. However, $\Sigma \subseteq \mathrm{dom}(q)$, which contradicts the assumption that $x \notin \mathrm{dom}(q)$. Hence $(x)g \neq x$.

Since $x \notin \mathrm{dom}(q)$ and g is an extension of q, it follows that $(x)g \notin \mathrm{ran}(q)$. Then $(x)g, y \notin \mathrm{dom}(q) \cup \mathrm{ran}(q)$ and since $h \in \mathrm{Aut}(nK_\omega)^{<\omega}$ and $(x)g \in \mathrm{Aut}(nK_\omega)$ it follows that $(x)g$ and y are in the same connected component of nK_ω. Then the transposition $((x)g\ y)$ swapping $(x)g$ and y is in $\mathrm{Aut}(nK_\omega)$ and so

$$g' = ((x)g\ y)\, g\, ((x)g\ y) \in \mathrm{Aut}(nK_\omega).$$

It follows from $(x)g \neq x$, $(x)g \neq y$, and $(x)g, y \notin \mathrm{dom}(q) \cup \mathrm{ran}(q)$ that g' is an extension of h. Therefore $h \in \mathcal{A}_{f,\Sigma}^{<\omega}$. □

The following result gives a sufficient condition when two non-complete components of an element of $\mathcal{A}_{f,\Sigma}^{<\omega}$ can be combined.

Lemma 3.6.9. *Let $f \in \mathrm{Aut}(nK_\omega)$ and let $\Sigma \subseteq nK_\omega$ be finite, let $q \in \mathcal{A}_{f,\Sigma}^{<\omega}$ be such that $\Sigma \subseteq \mathrm{dom}(q)$ and suppose there exist A, B distinct incomplete components of q such that at most one of A and B intersects Σ non-trivially. Suppose that*

$$\overline{q|_{\mathrm{dom}(q)\setminus A}} = \overline{q|_{\mathrm{dom}(q)\setminus B}} \in S_n$$

and let $h = q \cup \{(x, y)\} \in \mathrm{Aut}(nK_\omega)^{<\omega}$, for some $x \in A \setminus \mathrm{dom}(q)$ and $y \in B \setminus \mathrm{ran}(q)$. Then $h \in \mathcal{A}_{f,\Sigma}^{<\omega}$.

Proof. Assume without loss of generality that $B \cap \Sigma = \varnothing$ and $B = \{y_1, \ldots, y_m\}$ for some $m \in \mathbb{N}$ such that $y_1 = y$ and $(y_i)q = y_{i+1}$ for all $i \in \{1, \ldots, m-1\}$.

The proof of the case when $B \cap \Sigma \neq \varnothing$ can be obtained by apply the argument below to q^{-1} and h^{-1}.

For $k \in \{1, \ldots, m\}$ we will define $h_k \in \mathcal{A}^{<\omega}_{f,\Sigma}$ such that h_k extends h_{k-1} for $k \geq 1$, $\Sigma \subseteq \mathrm{dom}(h_k)$, $\overline{h_k} \in S_n$,

$$(x)h_k^i = y_i \text{ for } 1 \leq i \leq k, \qquad y_k \notin \mathrm{dom}(h_k),$$

and

$$y_i \notin \mathrm{dom}(h_k) \cup \mathrm{ran}(h_k) \text{ for } k < i.$$

If $k = 1$, then we define $h_1 = h|_{\mathrm{dom}(h) \setminus B}$. By Lemma 3.6.8, it follows that $h_1 = q|_{\mathrm{dom}(q) \setminus B} \cup \{(x, y)\} \in \mathcal{A}^{<\omega}_{f,\Sigma}$, and so h_1 satisfies the required conditions.

Suppose $k > 1$. Then by Lemma 3.6.8 we have that $h_{k+1} = h_k \cup \{(y_k, y_{k+1})\} \in \mathcal{A}^{<\omega}_{f,\Sigma}$. Since $\mathrm{dom}(h_{k+1}) = \mathrm{dom}(h_k) \cup \{y_k\}$ and $\mathrm{ran}(h_{k+1}) = \mathrm{ran}(h_k) \cup \{y_{k+1}\}$, it follows that h_{k+1} satisfies the required conditions.

Therefore after repeating this process m times, we obtain $h_m \in \mathcal{A}^{<\omega}_{f,\Sigma}$ which extends h_1. It follows from the definition of h_m that $h_m = h$. □

Now we can characterise when the set $\mathcal{A}_{f,\Sigma}$ is non-empty.

Lemma 3.6.10. *Let $f \in \mathrm{Aut}(nK_\omega)$ and let Σ be a finite subset of nK_ω. Then $\mathcal{A}_{f,\Sigma}$ is non-empty if and only if $n \neq 4$, or $n = 4$ and $\overline{f}$ is not an element of $\{(1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\}$, and there exists $\sigma \in S_n$ such that $\langle \overline{f}, \sigma \rangle = S_n$ and for all $i \in \{1, \ldots, n\}$*

$$\left(\bigcup_{j \in \mathbb{Z}} L_{(i)\sigma^j} \right) \cap \Sigma \neq \varnothing.$$

Proof. ($\Rightarrow$) Suppose that $g \in \mathcal{A}_{f,\Sigma}$. Since $g \in \mathcal{A}_{f,\Sigma} \subseteq \mathcal{A}_f$, it follows from the definition of $\mathcal{A}_f$ that $\langle \overline{f}, \overline{g} \rangle = S_n$. Hence by Theorem 3.1.3 we have $n \neq 4$, or $n = 4$ and $\overline{f} \notin \{(1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\}$.

Finally, let $i \in \{1, \ldots, n\}$. Then there is $x \in \Sigma$ and $m \in \mathbb{N}$ such that $(x)g^m \in L_i$, since Σ is a set of orbit representatives. Then

$$x \in L_{(i)\overline{g}^{-m}} \subseteq \bigcup_{j \in \mathbb{Z}} L_{(i)\overline{g}^j}.$$

($\Leftarrow$) By Theorem 3.1.3 there is $\sigma \in S_n$ such that $\langle \overline{f}, \sigma \rangle = S_n$. It is routine to show that there is $q \in \mathrm{Aut}(nK_\omega)^{<\omega}$ such that $\Sigma \subseteq \mathrm{dom}(q)$, $\overline{q} = \sigma$ and q has precisely $|\Sigma|$ many components, all of which are incomplete, and Σ intersects them in precisely one point.

Since all components of q are incomplete, it satisfies the hypothesis of Lemma 3.6.5 and hence there is $g \in \mathcal{A}_{f,\Sigma}$ an extension of q. □

In the next lemma, we give a decomposition of $D_f \cap \mathcal{A}_{f,\Sigma}$ as an intersection of sets that we will later prove to be open and dense, under the hypothesis of Theorem 3.6.3.

Lemma 3.6.11. *Let $\mathcal{B} \subseteq \mathrm{Aut}(nK_\omega)^{<\omega}$ be such that $p \in \mathcal{B}$ if and only if $\mathrm{dom}(p)$ and $\mathrm{ran}(p)$ are disjoint, and $\overline{p} = \mathrm{id}$. Then*

$$D_f \cap \mathcal{A}_{f,\Sigma} = \bigcap_{p \in \mathcal{B}} \{g \in \mathcal{A}_{f,\Sigma} : \langle f, g\rangle \cap [p] \neq \varnothing\}.$$

Proof. Similarly to Lemmas 3.4.2 and 3.5.3,

$$D_f \cap \mathcal{A}_{f,\Sigma} = \bigcap_{q \in \mathrm{Aut}(nK_\omega)^{<\omega}} \{g \in \mathcal{A}_{f,\Sigma} : \langle f, g\rangle \cap [q] \neq \varnothing\}.$$

($\subseteq$) This follows immediately since $\mathcal{B} \subseteq \mathrm{Aut}(nK_\omega)^{<\omega}$.

($\supseteq$) Let $g \in \bigcap_{p\in\mathcal{B}}\{g \in \mathcal{A}_{f,\Sigma} : \langle f, g\rangle \cap [p] \neq \varnothing\}$, and let $q \in \mathrm{Aut}(nK_\omega)^{<\omega}$ be arbitrary. Since $g \in \mathcal{A}_{f,\Sigma}$ there is $h \in \langle f, g\rangle$ such that $\overline{h} = \overline{q}^{-1}$.

Let $p \in \mathrm{Aut}(nK_\omega)^{<\omega}$ be such that $\overline{p} = \mathrm{id}$, $\mathrm{dom}(p) = \mathrm{dom}(hq)$ and $\mathrm{ran}(p) \cap (\mathrm{dom}(hq) \cup \mathrm{ran}(hq)) = \varnothing$. Then $\mathrm{dom}(p^{-1}hq) = \mathrm{ran}(p)$ and $\mathrm{ran}(p^{-1}hq) = \mathrm{ran}(hq)$, so $p, p^{-1}hq \in \mathcal{B}$. Hence there are $h_1, h_2 \in \langle f, g\rangle$ such that $h_1 \in [p]$ and $h_2 \in [p^{-1}hq]$. Therefore $h^{-1}h_1h_2 \in [q]$, so

$$g \in \bigcap_{q \in \mathrm{Aut}(nK_\omega)^{<\omega}} \{g \in \mathcal{A}_{f,\Sigma} : \langle f, g\rangle \cap [q] \neq \varnothing\},$$

as required. □

Let w be a freely reduced word over the alphabet $\{\alpha, \beta\}$, in other words $w = \alpha^{n_1}\beta^{n_2}\cdots\beta^{n_{2N}}$ for some $N \in \mathbb{N}$ and $n_1, \ldots, n_{2N} \in \mathbb{Z}$ with $n_i \neq 0$ for all $i \in \{2, \ldots, 2N-1\}$. Also let $f \in \mathrm{Aut}(nK_\omega)$ be fixed and suppose that $p \in \mathrm{Aut}(nK_\omega)^{<\omega}$. Then define

$$w(p) = p^{n_1}f^{n_2}p^{n_3}\cdots p^{n_{2N-1}}f^{n_{2N}}$$

where the product on the right hand side is the usual product of partial permutations. Note that $\mathrm{Aut}(nK_\omega) \cup \mathrm{Aut}(nK_\omega)^{<\omega}$ forms a subsemigroup of the semigroup of all isomorphisms between induced subgraphs of nK_ω. Hence, if we denote by $F(\alpha, \beta)$ the free group on the alphabet $\{\alpha, \beta\}$, then $w(p)$ is simply the image of w under the semigroup homomorphism $\phi : F(\alpha, \beta) \longrightarrow \mathrm{Aut}(nK_\omega) \cup \mathrm{Aut}(nK_\omega)^{<\omega}$ such that $(\alpha)\phi = p$ and $(\beta)\phi = f$.

Lemma 3.6.12. *Let $n \in \mathbb{N}$ be such that $n \geq 2$ and let $f \in \mathrm{Aut}(nK_\omega)$ be non-stabilising. If $n = 2$ and $\overline{f} = \mathrm{id}$, then further suppose that $\mathrm{fix}(f)$ is finite. Let $\Gamma, \Delta \subseteq nK_\omega$ be finite and disjoint, and let $q \in \mathcal{A}^{<\omega}_{f,\Sigma}$ be such that $\overline{q} \in S_n$ and*

$\operatorname{ran}(q) \cap \Delta = \varnothing$. *Then there is an extension* $h \in \mathcal{A}_{f,\Sigma}^{<\omega}$ *of* q *and* $w \in F(\alpha, \beta)$ *such that*

$$\overline{w(h)} = \mathrm{id}, \qquad \operatorname{ran}(h) \cap \Delta = \varnothing, \qquad \Gamma \subseteq \operatorname{dom}(w(h)),$$

and

$$(\Gamma)\, w(h) \cap \operatorname{dom}(h) = \varnothing.$$

Moreover, $(\Gamma)\, w(h) h^m \cap \operatorname{dom}(q) = \varnothing$ *for all* $m \in \mathbb{Z}$*, i.e. no vertex in* $(\Gamma)\, w(h)$ *is on an incomplete component of* h*, which extends an incomplete component of* q*.*

The proof of Lemma 3.6.12 is rather involved, so before giving its proof we will demonstrate how the lemma can be used to prove Theorem 3.6.3.

We will first prove an easy special case of Theorem 3.6.3.

Lemma 3.6.13. *Let* $f \in \operatorname{Aut}(2K_\omega)$ *be non-stabilising such that* $\overline{f} = \mathrm{id}$ *and* $\operatorname{fix}(f)$ *is infinite, and let* $\Sigma \subseteq 2K_\omega$ *be finite. Then* $D_f \cap \mathcal{A}_{f,\Sigma}$ *is comeagre in* $\mathcal{A}_{f,\Sigma}$*.*

Proof. By Lemmas 3.2.7 and 3.6.11 we only need to show that $\{g \in \mathcal{A}_{f,\Sigma} : \langle f, g\rangle \cap [p] \neq \varnothing\}$ is dense in $\mathcal{A}_{f,\Sigma}$ for all $p \in \mathcal{B}$. Let $q \in \mathcal{A}_{f,\Sigma}^{<\omega}$ and suppose, without loss of generality, that $\operatorname{dom}(p) \cup \operatorname{ran}(p) \cup \Sigma \subseteq \operatorname{dom}(q)$ and $\overline{q} \in S_2$. Since $\overline{f} = \mathrm{id}$, it follows that $\overline{q} = (1\ 2)$.

Let L_1 and L_2 be the connected components of $2K_\omega$. If necessary by relabelling the connected components we may assume that $L_2 \cap \operatorname{fix}(f)$ is infinite. Note that $\mathcal{A}_f = \operatorname{Aut}(2K_\omega)$, since $\operatorname{Sym}(2)$ is a cyclic group. It follows from Proposition 3.6.2 that if f has a finite cycle contained in L_1, then f is stabilising. Hence all of the cycles of f contained in L_1 are infinite.

Let $m_1 \in \mathbb{Z}$ be such that $(L_1 \cap \operatorname{dom}(p))\, f^{m_1}$ is disjoint from $\operatorname{dom}(q) \cup \operatorname{ran}(q)$. By Lemmas 3.6.4 and 3.6.8 there is $q_1 \in \mathcal{A}_{f,\Sigma}^{<\omega}$ an extension of q such that $(\operatorname{dom}(p))\, f^{m_1} \subseteq \operatorname{dom}(q_1)$ and $(L_1 \cap \operatorname{dom}(p))\, f^{m_1} q_1 \subseteq \operatorname{fix}(f) \setminus \operatorname{dom}(q_1)$, which is possible since $L_2 \cap \operatorname{fix}(f)$ is infinite and $(L_1) f^{m_1} q_1 \subseteq L_2$. The extension q_1 can be chosen so that components of q_1 containing any vertices from $(L_1 \cap \operatorname{dom}(p))\, f^{m_1}$ do not extend any of the components of q. Since $(L_2 \cap \operatorname{dom}(p))\, f^{m_1} q_1 \subseteq L_1$, there is $m_2 \in \mathbb{Z}$ such that $(L_2 \cap \operatorname{dom}(p))\, f^{m_1} q_1 f^{m_2}$ is disjoint from $\operatorname{dom}(q_1) \cup \operatorname{ran}(q_1)$. Hence $(\operatorname{dom}(p))\, f^{m_1} q_1 f^{m_2} \cap \operatorname{dom}(q_1) = \varnothing$.

Let $m_3 \in \mathbb{Z}$ be such that $(L_1 \cap \operatorname{ran}(p))\, f^{m_3}$ is disjoint from $\operatorname{dom}(q_1) \cup \operatorname{ran}(q_1)$. By Lemmas 3.6.4 and 3.6.8 there is $q_2 \in \mathcal{A}_{f,\Sigma}^{<\omega}$ an extension of q_1 such that $(\operatorname{ran}(p))\, f^{m_3} \subseteq \operatorname{ran}(q_2)$, $(L_1 \cap \operatorname{ran}(p))\, f^{m_3} q_2^{-1} \subseteq \operatorname{fix}(f) \setminus \operatorname{ran}(q_2)$ and $(\operatorname{dom}(p))\, f^{m_1} q_2 f^{m_2}$ is disjoint from $\operatorname{dom}(q_2)$. The extension q_2 can be chosen so that components of q_2 containing any vertices from $(L_1 \cap \operatorname{ran}(p))\, f^{m_3}$ do not extend any of the components of q_1, and also that every vertex of $(L_1 \cap \operatorname{ran}(p)) f^{m_3}$ is on a different incomplete components of q_2. Then there is $m_4 \in \mathbb{Z}$ such that $(L_2 \cap \operatorname{ran}(p))\, f^{m_3} q_2^{-1} f^{m_4}$ is disjoint from $\operatorname{dom}(q_2) \cup \operatorname{ran}(q_2) \cup (\operatorname{dom}(p)) f^{m_1} q_2 f^{m_2}$.

Hence

$$(\mathrm{dom}(p))\, f^{m_1}q_2f^{m_2} \cap \mathrm{dom}(q_2) = \varnothing \quad \text{and} \quad (\mathrm{ran}(p))\, f^{m_3}q_2^{-1}f^{m_4} \cap \mathrm{ran}(q_2) = \varnothing.$$

Let $\mathrm{dom}(p) = \{x_1, \ldots, x_k\}$. Then for all $i \in \{1, \ldots, k\}$ there are

$$y_i \in 2K_\omega \setminus \big(\mathrm{dom}(q_2) \cup \mathrm{ran}(q_2) \cup (\mathrm{dom}(p))f^{m_1}q_2f^{m_2} \cup (\mathrm{ran}(p))f^{m_3}q_2^{-1}f^{m_4}\big)$$

such that $h' = q_2 \cup \{((x_i)f^{m_1}q_2f^{m_2}, y_i) : 1 \le i \le k\} \in \mathcal{A}_{f,\Sigma}^{<\omega}$ by Lemmas 3.6.4 and 3.6.8. Let A be the incomplete component of h' containing $(x_1)f^{m_1}q_2f_{m_2}$ and let B be the incomplete component of h' containing $(x_1)pf^{m_3}q_2^{-1}f^{m_4}$. Then $y_1 \in A$, and so $|A| \ge 2$. If $|B| = 1$, then $h' \cup \{(y_1, (x_1)pf^{m_3}q_2^{-1}f^{m_4})\} \in \mathcal{A}_{f,\Sigma}^{<\omega}$ by Lemmas 3.6.4 and 3.6.8, as $(x_1)f^{m_1}q_2f^{m_2}$ and $(x_1)pf^{m_3}q_2^{-1}f^{m_4}$ are in the same connected component of $2K_\omega$. If $(x_1)p \in L_2$, then by the choice of m_4, $(x_1)pf^{m_3}q_2^{-1}f^{m_4} \notin \mathrm{dom}(h') \cup \mathrm{ran}(h')$, and so $|B| = 1$, and we have already considered this case. Suppose that $|B| \ge 2$. Then $(x_1)p \in L_1$ and by the choice of q_2 the incomplete component of h' containing $(x_1)pf^{m_3}q_2^{-1}f^{m_4}$, in other words B, does not extend an incomplete component of q_1. Since A is an incomplete component of q_1 with y_1 adjoined, it follows that B intersects Σ trivially, and A and B are distinct. Hence $\overline{h'|_{\mathrm{dom}(h')\setminus A}} = \overline{h'|_{\mathrm{dom}(h')\setminus B}} = (1\ 2)$, and thus $h' \cup \{(y_1, (x_1)pf^{m_3}q_2^{-1}f^{m_4})\} \in \mathcal{A}_{f,\Sigma}^{<\omega}$ by Lemma 3.6.9. Repeating this argument for $i \in \{2, \ldots, k\}$, it can be shown that $h = q_2 \cup \{((x_i)f^{m_1}q_2f^{m_2}, y_i), (y_i, (x_i)pf^{m_3}q_2^{-1}f^{m_4}) : 1 \le i \le k\} \in \mathcal{A}_{f,\Sigma}^{<\omega}$. Hence

$$f^{m_1}gf^{m_2}g^2f^{-m_4}gf^{-m_3} \in [p]$$

for every $g \in [h] \cap \mathcal{A}_{f,\Sigma}$. Therefore $\{g \in \mathcal{A}_{f,\Sigma} : \langle f, g\rangle \cap [p] \neq \varnothing\}$ intersects $[q]$ non-trivially, and since q was arbitrary, is dense in $\mathcal{A}_{f,\Sigma}$. □

Next, we give the proof of Theorem 3.6.3 modulo the proof of Lemma 3.6.12, which is given in the next section.

Proof of Theorem 3.6.3. If $\mathcal{A}_f = \varnothing$, then f is non-stabilising and D_f is comeagre in $\mathcal{A}_f$. Hence we may assume that $\mathcal{A}_f \neq \varnothing$.

Suppose that f is stabilising. By the definition, there is $\Gamma \subsetneq nK_\omega$, $x \in \Gamma$ and $q \in \mathcal{A}_f^{<\omega}$ such that for all $g \in [q] \cap \mathcal{A}_f$ and all $h \in \langle f, g\rangle$ we have that $(x)h \in \Gamma$. Let $y \notin \Gamma$. Then $p = \{(x, y)\} \in \mathrm{Aut}(nK_\omega)^{<\omega}$. Then $\langle f, g\rangle \cap [p] = \varnothing$ and thus $g \notin D_f$ implying that D_f is not dense in $\mathcal{A}_f$. Hence $D_f \cap \mathcal{A}_f$ is not comeagre in $\mathcal{A}_f$.

If f is non-stabilising and Σ is a finite subset of nK_ω, then it suffices, by Lemma 3.6.7, to show that $D_f \cap \mathcal{A}_{f,\Sigma}$ is comeagre in $\mathcal{A}_{f,\Sigma}$. If $\mathcal{A}_{f,\Sigma} = \varnothing$, the result is trivial. Hence we may assume that $\mathcal{A}_{f,\Sigma} \neq \varnothing$. If $n = 2$, $\overline{f} = \mathrm{id}$, and $\mathrm{fix}(f)$ is infinite we are done by Lemma 3.6.13. Hence we may, additionally assume that $n \geq 2$, and that if $n = 2$ and $\overline{f} = \mathrm{id}$, then $\mathrm{fix}(f)$ is finite.

By Lemmas 3.2.7 and 3.6.11 we only need to show that $\{g \in \mathcal{A}_{f,\Sigma} : \langle f, g\rangle \cap [p] \neq \varnothing\}$ is dense in $\mathcal{A}_{f,\Sigma}$ for all $p \in \mathcal{P}$. Let $q \in \mathcal{A}^{<\omega}_{f,\Sigma}$ and suppose, without loss of generality, that $\mathrm{dom}(p) \cup \mathrm{ran}(p) \cup \Sigma \subseteq \mathrm{dom}(q)$ and $\overline{q} \in S_n$.

Apply Lemma 3.6.12 with $\Delta = \varnothing$ and $\Gamma = \mathrm{dom}(p)$. Then there is an extension $q_1' \in \mathcal{A}^{<\omega}_{f,\Sigma}$ of q and $\omega_1 \in F(\alpha, \beta)$ such that

$$\overline{\omega_1(q_1')} = \mathrm{id}, \qquad \mathrm{dom}(p) \subseteq \mathrm{dom}\left(\omega_1(q_1')\right),$$

and

$$(\mathrm{dom}(p))\, \omega_1(q_1') \cap \mathrm{dom}(q_1') = \varnothing.$$

Suppose $(\mathrm{dom}(p))\, \omega_1(q_1') \setminus \mathrm{ran}(q_1')$ is non-empty. Let $y \in (\mathrm{dom}(p))\, \omega_1(q_1') \setminus \mathrm{ran}(q_1')$ and let $a \in \{1, \ldots, n\}$ be such that $y \in L_a$. Then there is

$$x \in L_{(a)\overline{q_1'}^{-1}} \setminus \left(\mathrm{dom}(q_1') \cup \mathrm{ran}(q_1') \cup (\mathrm{dom}(p))\, \omega_1(q_1')\right).$$

It follows from Lemma 3.6.4 that $q_1'' = q_1' \cup \{(x, y)\} \in \mathrm{Aut}(nK_\omega)^{<\omega}$ and thus in $\mathcal{A}^{<\omega}_{f,\Sigma}$ by Lemma 3.6.8. Then

$$\overline{\omega_1(q_1'')} = \mathrm{id}, \qquad \mathrm{dom}(p) \subseteq \mathrm{dom}\left(\omega_1(q_1'')\right),$$

and

$$(\mathrm{dom}(p))\, \omega_1(q_1'') \cap \mathrm{dom}(q_1'') = \varnothing.$$

Moreover, $|(\mathrm{dom}(p))\, \omega_1(q_1') \setminus \mathrm{ran}(q_1')| > |(\mathrm{dom}(p))\, \omega_1(q_1'') \setminus \mathrm{ran}(q_1'')|$, and if we do this extension for every vertex in $(\mathrm{dom}(p))\, \omega_1(q_1') \setminus \mathrm{ran}(q_1')$, we can define an extension $q_1 \in \mathcal{A}^{<\omega}_{f,\Sigma}$ of q_1' such that

$$\begin{gathered} \overline{\omega_1(q_1)} = \mathrm{id}, \qquad \mathrm{dom}(p) \subseteq \mathrm{dom}\left(\omega_1(q_1)\right), \qquad \text{and} \\ (\mathrm{dom}(p))\, \omega_1(q_1) \subseteq \mathrm{ran}(q_1) \setminus \mathrm{dom}(q_1). \end{gathered} \tag{3.16}$$

Hence every vertex in $(\mathrm{dom}(p))\, \omega_1(q_1)$ is on a incomplete component of q_1.

If $\Delta = (\mathrm{dom}(p))\, \omega_1(q_1)$ and $\Gamma = \mathrm{ran}(p)$, then $\mathrm{ran}(q_1^{-1}) = \mathrm{dom}(q_1)$ and Δ are disjoint. Hence by Lemma 3.6.12, there is an extension $q_2^{-1} \in \mathcal{A}^{<\omega}_{f,\Sigma}$ of q_1^{-1} and $\omega_2' \in F(\alpha, \beta)$ such that $\overline{\omega_2'(q_2^{-1})} = \mathrm{id}$,

$$\begin{gathered} \mathrm{ran}(q_2^{-1}) \cap (\mathrm{dom}(p))\, \omega_1(q_1) = \varnothing, \\ \mathrm{ran}(p) \subseteq \mathrm{dom}\left(\omega_2'(q_2^{-1})\right), \\ (\mathrm{ran}(p))\, \omega_2'(q_2^{-1}) \cap \mathrm{dom}(q_2^{-1}) = \varnothing, \end{gathered}$$

and no vertex in $(\mathrm{ran}(p))\, \omega_2'(q_2^{-1})$ is on an incomplete component of q_2^{-1} extending an incomplete component of q_1^{-1}.

Since $\mathrm{dom}(p) \subseteq \mathrm{dom}\left(\omega_1(q_1)\right)$ by (3.16), and q_2 is an extension of q_1, it follows that $(\mathrm{dom}(p))\,\omega_1(q_1) = (\mathrm{dom}(p))\,\omega_1(q_2)$. Let $\omega_2 \in F(\alpha, \beta)$ be such that $\omega_2(q_2) = \omega_2'(q_2^{-1})$, more precisely replace every occurrence of α in ω_2' by α^{-1} and vica versa. Then $\overline{\omega_2(q_2)} = \mathrm{id}$,

$$\begin{gathered}\mathrm{dom}(q_2) \cap (\mathrm{dom}(p))\,\omega_1(q_2) = \varnothing,\\ \mathrm{ran}(p) \subseteq \mathrm{dom}\left(\omega_2(q_2)\right),\\ (\mathrm{ran}(p))\,\omega_2(q_2) \cap \mathrm{ran}(q_2) = \varnothing,\end{gathered}$$

and no vertex in $(\mathrm{ran}(p))\,\omega_2(q_2)$ is on an incomplete components of q_2 extending an incomplete component of q_1.

Let $\{\{i_{j,k} : k \in \{1, \ldots, m_j\}\} : j \in \{1, \ldots, l\}\}$ be the set of orbits of $\overline{q_2}$ such that $(i_{j,k})\overline{q_2} = i_{j,k+1}$ for all $j \in \{1, \ldots, l\}$ and all $k \in \{1, \ldots, m_j - 1\}$. For each $i \in \{1, \ldots, n\}$ choose

$$t_i \in L_i \setminus \Big(\mathrm{dom}(q_2) \cap \mathrm{ran}(q_2)\,(\mathrm{dom}(p))\,\omega_1(q_2) \cup (\mathrm{ran}(p))\,\omega_2(q_2)\Big),$$

and also for all $j \in \{1, \ldots, l\}$ choose

$$t_{i_{j,m_j+1}} \in L_{i_{j,1}} \setminus \left(\{t_{i_{j,1}}\} \cup \mathrm{dom}(q_2) \cap \mathrm{ran}(q_2) \cup (\mathrm{dom}(p))\,\omega_1(q_2) \cup (\mathrm{ran}(p))\,\omega_2(q_2)\right),$$

Then $h_0 = q_2 \cup \{(t_{i_{j,k}}, t_{i_{j,k+1}}) : j \in \{1, \ldots, l\} \text{ and } k \in \{1, \ldots, m_j\}\} \in \mathrm{Aut}(nK_\omega)^{<\omega}$ by Lemma 3.6.4 and also $h_0 \in \mathcal{A}_{f,\Sigma}^{<\omega}$ by Lemma 3.6.8. Let P be an arbitrary incomplete component of h_0. Since $t_{i_{j,k}} \notin \mathrm{dom}(q_2) \cup \mathrm{ran}(q_2)$ for all j and all k, it follows that P is either a subset of $K = \{t_{i_{j,k}} : j \in \{1, \ldots, l\} \text{ and } k \in \{1, \ldots, m_j + 1\}\}$ or disjoint from K. If $P \subseteq K$, then $q_2 \subseteq h_0|_{\mathrm{dom}(h_0) \setminus P}$, and so $\overline{h_0|_{\mathrm{dom}(h_0) \setminus P}} = \overline{q_2} \in S_n$. Otherwise $P \cap K = \varnothing$, and so $\{t_i : i \in \{1, \ldots, n\}\} \subseteq \mathrm{dom}(h_0) \setminus P$. Hence, $\{(t_{i_{j,k}}, t_{i_{j,k+1}})) : j \in \{1, \ldots, l\} \text{ and } k \in \{1, \ldots, m_j\}\} \subseteq h_0|_{\mathrm{dom}(h_0) \setminus P}$, which implies that $\overline{h_0|_{\mathrm{dom}(h_0) \setminus P}} = \overline{q_2} \in S_n$. It follows from the choice of vertices t_i and $t_{i_{j,m_j+1}}$, that $\overline{\omega_2(h_0)} = \mathrm{id}$,

$$\begin{gathered}\mathrm{dom}(h_0) \cap (\mathrm{dom}(p))\,\omega_1(h_0) = \varnothing,\\ \mathrm{ran}(p) \subseteq \mathrm{dom}\left(\omega_2(h_0)\right),\\ (\mathrm{ran}(p))\,\omega_2(h_0) \cap \mathrm{ran}(h_0) = \varnothing.\end{gathered}$$

Let k be the order of $\overline{q} \in S_n$. We will now inductively construct an extension $h \in \mathcal{A}_{f,\Sigma}^{<\omega}$ of h_0 (and hence of q) such that $(x)\omega_1(h)h^k\omega_2(h)^{-1} = (x)p$ for all $x \in \mathrm{dom}(p)$. Let $\mathrm{dom}(p) = \{x_1, \ldots, x_d\}$, and suppose that for some $j \in \{0, \ldots, k-2\}$ we have defined an extension $h_j \in \mathcal{A}_{f,\Sigma}^{<\omega}$ of h_0 such that

$$\mathrm{dom}(h_j) \cap (\mathrm{dom}(p))\,\omega_1(h_j)h_j^j = \varnothing, \qquad (\mathrm{ran}(p))\,\omega_2(h_j) \cap \mathrm{ran}(h_j) = \varnothing,$$

and $\mathrm{dom}(p)$ and $\mathrm{ran}(p)$ are contained in $\mathrm{dom}(\omega_1(h_j)h_j^j)$ and $\mathrm{dom}(\omega_2(h_j))$ respectively.

Note that if $j = 0$, the inductive hypothesis is satisfied since h_0^0 is an identity on $\mathrm{dom}(h_0)$, the $\mathrm{dom}(h_0)$ is disjoint from $(\mathrm{dom}(p))\,\omega_1(h_0)$, the set $\mathrm{ran}(h_0)$ is disjoint from $(\mathrm{ran}(p))\,\omega_2(h_j) \cap \mathrm{ran}(h_j)$, and $\mathrm{dom}(p)$ and $\mathrm{ran}(p)$ are contained in $\mathrm{dom}(\omega_1(h_0))$ and $\mathrm{dom}(\omega_2(h_0))$ respectively.

Suppose $j > 0$. For all $i \in \{1, \ldots, d\}$, let $y_i = (x_i)\omega_1(h_j)h_j^j$ and suppose that $a_i \in \{1, \ldots, n\}$ such that $y_i \in L_{a_i}$. Then for each successive $i \in \{1, \ldots, d\}$ choose

$$\begin{aligned} z_i \in L_{(a_i)\overline{h_j}} \setminus \Big(&\mathrm{dom}(h_j) \cup \mathrm{ran}(h_j) \cup \{y_1, \ldots, y_d\} \cup \\ &\{z_1, \ldots, z_{i-1}\} \cup (\mathrm{ran}(p))\,\omega_2(h_j) \Big). \end{aligned}$$

We define $h_{j+1} = h_j \cup \{(y_i, z_i) : 1 \leq i \leq d\}$. Since $z_i \in L_{(a_i)\overline{h_j}}$, it follows that $h_{j+1} \in \mathrm{Aut}(nK_\omega)^{<\omega}$ by Lemma 3.6.4 and hence $h_{j+1} \in \mathcal{A}_{f,\Sigma}^{<\omega}$ by Lemma 3.6.8. Note that the choice of z_i implies that none of the incomplete components of h_j are amalgamated in h_{j+1}.

It is easy to see that $\mathrm{dom}(h_{j+1}) = \mathrm{dom}(h_j) \cup \{y_1, \ldots, y_d\}$ and $\mathrm{ran}(h_{j+1}) = \mathrm{ran}(h_j) \cup \{z_1, \ldots, z_d\}$. Since $(x_i)\omega_1(h_{j+1})h_{j+1}^j = (x_i)\omega_1(h_j)h_j^j$ for all $i \in \{1, \ldots, d\}$

$$\begin{aligned} (x_i)\omega_1(h_{j+1})h_{j+1}^{j+1} &= (x_i)\omega_1(h_{j+1})h_{j+1}^j h_{j+1} \\ &= (x_i)\omega_1(h_j)h_j^j h_{j+1} \\ &= (y_i)h_{j+1} \\ &= z_i \notin \mathrm{dom}(h_{j+1}). \end{aligned}$$

Hence $\mathrm{dom}(h_{j+1}) \cap (\mathrm{dom}(p))\,\omega_1(h_{j+1})h_{j+1}^{j+1} = \varnothing$ and $\mathrm{dom}(p) \subseteq \omega_1(h_{j+1})h_{j+1}^{j+1}$.

It follows from $\mathrm{ran}(p) \subseteq \omega_2(h_0)$, that $(\mathrm{ran}(p))\,\omega_2(h_{j+1}) = (\mathrm{ran}(p))\,\omega_2(h_j)$, and so $(\mathrm{ran}(p))\,\omega_2(h_{j+1}) \cap \mathrm{ran}(h_j) = \varnothing$. Since $z_i \notin (\mathrm{ran}(p))\,\omega_2(h_j)$ for all $i \in \{1, \ldots, d\}$, it also follows that $(\mathrm{ran}(p))\,\omega_2(h_{j+1}) \cap \mathrm{ran}(h_{j+1}) = \varnothing$. Finally, $\mathrm{dom}(p)$ and $\mathrm{ran}(p)$ are contained in $\mathrm{dom}(\omega_1(h_{j+1})h_{j+1}^{j+1})$ and $\mathrm{dom}(\omega_2(h_{j+1}))$ respectively, and so h_{j+1} satisfies the inductive hypothesis.

By induction on j, we obtain an extension $h_{k-1} \in \mathcal{A}_{f,\Sigma}^{<\omega}$ of h_0 (and thus q) such that

$$\begin{aligned} \mathrm{dom}(h_{k-1}) \cap (\mathrm{dom}(p))\,\omega_1(h_{k-1})h_{k-1}^{k-1} &= \varnothing, \\ (\mathrm{ran}(p))\,\omega_2(h_{k-1}) \cap \mathrm{ran}(h_{k-1}) &= \varnothing, \end{aligned} \tag{3.17}$$

and $\mathrm{dom}(p)$ and $\mathrm{ran}(p)$ are contained in $\mathrm{dom}(\omega_1(h_{k-1})h_{k-1}^{k-1})$ and $\mathrm{dom}(\omega_2(h_{k-1}))$ respectively.

Define h to be

$$h_{k-1} \cup \left\{ \big((x_i)\omega_1(h_{k-1})h_{k-1}^{k-1}, ((x_i)p)\,\omega_2(h_{k-1}) \big) : 1 \leq i \leq d \right\}.$$

Recall that k is the order of $\overline{q}$. Since h_{k-1} is an extension of q and $\overline{q} \in S_n$, it follows that $\overline{h_{k-1}} = \overline{q}$, thus $\overline{h_{k-1}^k} = \text{id}$. Also $\omega_1(h_{k-1})$ and $\omega_2(h_{k-1})$ are extensions of $\omega_1(q_1)$ and $\omega_2(q_2)$ respectively, hence

$$\overline{\omega_1(h_{k-1})} = \overline{\omega_1(q_1)} = \text{id} = \overline{\omega_2(q_2)} = \overline{\omega_2(h_{k-1})}.$$

Then x_i, $(x_i)\omega_1(h_{k-1})h_{k-1}^k$, and $((x_i)p)\,\omega_2(h_{k-1})$ are in the same connected component of nK_ω for all i. Thus it follows from Lemma 3.6.4 and (3.17), that $h \in \text{Aut}(nK_\omega)^{<\omega}$.

We will now show that h can be obtained from h_{k-1} by repeated applications of Lemma 3.6.9, and so $h \in \mathcal{A}_{f,\Sigma}^{<\omega}$. First of all, note that $\Sigma \subseteq \text{dom}(q) \subseteq \text{dom}(h_{k-1})$, and that no incomplete components of h_0, and thus of q_2, were amalgamated in h_{k-1}. According to Lemma 3.6.12, q_2 was chosen so that $((x_i)p)\,\omega_2(q_2)$ is not on an incomplete component of q_2 extending an incomplete component of q_1 for all $i \in \{1, \ldots, d\}$. Hence the vertex $((x_i)p)\,\omega_2(h_{k-1})$ is not on an incomplete component of h_{k-1} extending an incomplete component of q_1 for all $i \in \{1, \ldots, d\}$. Also since $\Sigma \subseteq \text{dom}(q) \subseteq \text{dom}(q_1)$, it follows that the intersection of any incomplete component of h_{k-1} containing a vertex in $(\text{ran}(p))\,\omega_2(h_{k-1})$ and Σ is empty.

By (3.16) every vertex in $(\text{dom}(p))\,\omega_1(q_1)$ is on an incomplete component of q_1 and since $\omega_1(h_{k-1})h_{k-1}^{k-1}$ is defined on $\text{dom}(p)$ it follows that every vertex in $(\text{dom}(p))\,\omega_1(h_{k-1})h_{k-1}^{k-1}$ is on an incomplete component of h_{k-1} extending an incomplete component of q_1. Hence incomplete components of h_{k-1} containing vertices $(\text{ran}(p))\,\omega_2(h_{k-1})$ are distinct from the incomplete components of h_{k-1} containing the vertices $(\text{dom}(p))\,\omega_1(h_{k-1})h_{k-1}^{k-1}$. Also recall that for every incomplete component P of h_0 we have that

$$\overline{h_0|_{\text{dom}(h_0)\setminus P}} \in S_n.$$

Since h_{k-1} is an extension of h_0 and no incomplete components of h_0 were amalgamated, for any incomplete component Q of h_{k-1}

$$\overline{h_{k-1}|_{\text{dom}(h_{k-1})\setminus Q}} \in S_n.$$

Thus we can apply Lemma 3.6.9 to show that $h \in \mathcal{A}_{f,\Sigma}^{<\omega}$.

Finally h was defined so that

$$\omega_1(h)h^k\omega_2(h)^{-1} \in [p],$$

and thus any extension $g \in [h] \cap \mathcal{A}_{f,\Sigma}$ also satisfies $g \in \{r \in \mathcal{A}_{f,\Sigma} : \langle f, r\rangle \cap [p] \neq \varnothing\}$. Therefore, $\{g \in \mathcal{A}_{f,\Sigma} : \langle f, g\rangle \cap [p] \neq \varnothing\}$ is dense in $\mathcal{A}_{f,\Sigma}$ as required. □

Proof of Lemma 3.6.12

The purpose of this section is to prove Lemma 3.6.12. We will first prove a technical result relating to the behaviour of a non-stabilising automorphism f of nK_ω. Recall that $f \in \mathrm{Aut}(nK_\omega)$ is NON-STABILISING if for all $\Gamma \subsetneq nK_\omega$, all $x \in \Gamma$ and all $q \in \mathcal{A}_f^{<\omega}$ there is $g \in [q] \cap \mathcal{A}_f$ such that $(x)h \notin \Gamma$ for some $h \in \langle f, g\rangle$.

Let $f \in \mathrm{Aut}(nK_\omega)$ be non-stabilising and let $x \in nK_\omega$. Then for every $q \in \mathrm{Aut}(nK_\omega)^{<\omega}$ there is $g \in [q] \cap \mathcal{A}_f$ such that $(x)h \notin \mathrm{dom}(q)$ for some $h \in \langle f, g\rangle$. It follows that there is $N \in \mathbb{N}$ and $m_1, m_2, \ldots, m_{2N} \in \mathbb{Z}$ such that $(x)\prod_{i=1}^N g^{m_{2i-1}} f^{m_{2i}} \notin \mathrm{dom}(q)$. If we assume that the length of the product $\sum_{i=1}^{2N} |m_i|$ is minimal, then the image of x under any proper prefix of the product $\prod_{i=1}^N g^{m_{2i-1}} f^{m_{2i}}$ belongs to $\mathrm{dom}(q)$. Therefore

$$(x)\prod_{i=1}^N q^{m_{2i-1}} f^{m_{2i}} = (x)\prod_{i=1}^N g^{m_{2i-1}} f^{m_{2i}} \in nK_\omega \setminus \mathrm{dom}(q).$$

In the next lemma we show that the powers m_{2i-1} of q in the above equation, can be chosen to be positive.

Lemma 3.6.14. *Let f be non-stabilising and let $x \in nK_\omega$. Then for every $q \in \mathrm{Aut}(nK_\omega)^{<\omega}$ there is $N \in \mathbb{N}$ and $m_1, m_2, \ldots, m_{2N} \in \mathbb{Z}$ such that $m_1, m_3, \ldots m_{2N-1} > 0$ and*

$$(x)\prod_{i=1}^N q^{m_{2i-1}} f^{m_{2i}} \in nK_\omega \setminus \mathrm{dom}(q).$$

Proof. By the discussion above there are $K \in \mathbb{N}$ and $k_1, k_2, \ldots, k_{2K} \in \mathbb{Z}$ such that

$$(x)\prod_{i=1}^K q^{k_{2i-1}} f^{k_{2i}} \in nK_\omega \setminus \mathrm{dom}(q).$$

Suppose that $M \in \{0, 1, \ldots, K\}$ is the least value such that $(x)\prod_{i=1}^M q^{k_{2i-1}} f^{k_{2i}}$ is on an incomplete component of q, and $M = 0$ in the case that x is on an incomplete component. Then $y_t = (x)\prod_{i=1}^t q^{k_{2i-1}} f^{k_{2i}}$ is on a complete component of q for all $t \in \{0, \ldots, M-1\}$. It follows that there exist $m_{2t+1} > 0$ such that $(y_t)q^{m_{2t+1}} = (y_t)q^{k_{2t+1}}$ for all $t \in \{0, \ldots, M-1\}$. Additionally, define $m_{2i} = k_{2i}$ for all $i \in \{1, \ldots, M\}$.

By the choice of M, $y = (x)\prod_{i=1}^M q^{m_{2i-1}} f^{m_{2i}} = (x)\prod_{i=1}^M q^{k_{2i-1}} f^{k_{2i}}$ is in an incomplete component of q. Hence there is z in the incomplete component of y under q such that $z \in \mathrm{ran}(q) \setminus \mathrm{dom}(q)$ and there is $m_{2M+1} \geq 0$ such that

$(y)q^{m_{2M+1}} = z \notin \text{dom}(q)$. Therefore

$$(x)\left(\prod_{i=1}^{M} q^{m_{2i-1}} f^{m_{2i}}\right) q^{m_{2M+1}} \in nK_\omega \setminus \text{dom}(q),$$

as required. □

For the proofs of the next three lemmas we require the following notation. First of all, recall that for a fixed $f \in \text{Aut}(nK_\omega)$, if $p \in \text{Aut}(nK_\omega)^{<\omega}$ and $w = \alpha^{n_1}\beta^{n_2}\cdots\beta^{n_{2N}} \in F(\alpha,\beta)$ for some $N \in \mathbb{N}$ and $n_1,\ldots,n_{2N} \in \mathbb{Z}$, then

$$w(p) = p^{n_1} f^{n_2} p^{n_3} \cdots p^{n_{2N-1}} f^{n_{2N}}$$

where the product on the right hand side is the usual product of partial permutations. Let $\Gamma,\Theta,\Phi,\Delta \subseteq nK_\omega$ be finite subsets, let $p \in \mathcal{A}^{<\omega}_{f,\Sigma}$ and let $w \in F(\alpha,\beta)$. Suppose $x \in \Gamma$ and define $w_{p,x}$ to be the largest prefix of w such that $x \in \text{dom}(w_{p,x}(p))$ and let $w_{p,x}$ be the empty word if there is no such prefix. To make the notation less cluttered, whenever possible, we will identify the word $w_{p,x}$ with its realisation in $\text{Aut}(nK_\omega)^{<\omega}$, in other words with the partial isomorphism $w_{p,x}(p)$. To avoid confusion, if $w, w' \in F(\alpha,\beta)$, we denote that w and w' are equal by $w \equiv w'$. Note that if $w_{p,x}$ is a proper prefix of w (i.e. $|w_{p,x}| < |w|$), since f is an isomorphism we have that $(x)w_{p,x} \notin \text{dom}(p)$ and $w_{p,x}\alpha$ is a prefix of w.

Suppose that $\Theta \subseteq \Gamma$. Then we say that p satisfies $\mathcal{S}(\Gamma,\Theta,\Phi,\Delta,w)$ if the following conditions are satisfied:

1. $\overline{w(p)} = \text{id}$;
2. $\text{ran}(p) \cap \Delta = \varnothing$;
3. $\text{dom}\,(w(p)) \cap \Gamma = \Theta$;
4. the image of Θ under $w(p)$ is disjoint from $\text{dom}(p)$;
5. $(x)w_{p,x} \neq (y)w_{p,y}$ for all $x, y \in \Gamma$ such that $x \neq y$;
6. $(x)w_{p,x}p^m \in nK_\omega \setminus \Phi$ for all $x \in \Gamma$ and $m \in \mathbb{Z}$ such that $x \in \text{dom}(w_{p,x}p^m)$.

Finally, define $\mathbf{b}(w)$ to be the total number of occurrences of β and β^{-1} in the freely reduced word w.

Using the definition of $\mathcal{S}(\Gamma,\Theta,\Phi,\Delta,w)$ we can now restate Lemma 3.6.12.In the case that $\Gamma = \Theta$, it follows that $w_{p,x} = w$ for all $x \in \Gamma$. Hence 5, in this case, is a consequence of $w(p)$ being a finite isomorphism.

Lemma 3.6.15. *Let $n \in \mathbb{N}$ be such that $n \geq 2$ and let $f \in \text{Aut}(nK_\omega)$ be non-stabilising. If $n = 2$ and $\overline{f} = \text{id}$, then further suppose that $\text{fix}(f)$ is finite. Let*

$\Gamma, \Delta \subseteq nK_\omega$ *be finite and disjoint, and let* $q \in \mathcal{A}^{<\omega}_{f,\Sigma}$ *be such that* $\overline{q} \in S_n$ *and* $\mathrm{ran}(q) \cap \Delta = \varnothing$. *Then there is an extension* $h \in \mathcal{A}^{<\omega}_{f,\Sigma}$ *of* q *and* $w \in F(\alpha, \beta)$ *satisfying* $\mathcal{S}(\Gamma, \Gamma, \mathrm{dom}(q), \Delta, w)$.

The proof of Lemma 3.6.15 will be split into 3 parts. We start with a weaker version.

Lemma 3.6.16. *Let* $n \in \mathbb{N}$ *be such that* $n \geq 2$ *and let* $f \in \mathrm{Aut}(nK_\omega)$ *be non-stabilising. If* $n = 2$ *and* $\overline{f} = \mathrm{id}$, *then further suppose that* $\mathrm{fix}(f)$ *is finite. Let* $\Gamma, \Delta \subseteq nK_\omega$ *be finite, and let* $q \in \mathcal{A}^{<\omega}_{f,\Sigma}$ *be such that* $\mathrm{ran}(q) \cap \Delta = \varnothing$. *Then there is an extension* $h \in \mathcal{A}^{<\omega}_{f,\Sigma}$ *of* q *and* $w \in F(\alpha, \beta)$ *not containing* α^{-1} *and starting with* α *such that* h *satisfies* $\mathcal{S}(\Gamma, \varnothing, \mathrm{dom}(q), \Delta, w)$.

Proof. If necessary by extending q using Lemma 3.6.4 and Lemma 3.6.8, we may assume that $\overline{q} \in S_n$ and $\Sigma, \Gamma \subseteq \mathrm{dom}(q)$. In the case that $n = 2$ and $\overline{f} = \mathrm{id}$, we also assume that $\mathrm{fix}(f) \subseteq \mathrm{dom}(q)$.

Let $d = |\Gamma|$. We will now inductively define a sequence $q_0, \ldots, q_d \in \mathcal{A}^{<\omega}_{f,\Sigma}$ of extensions of q, and a sequence $\lambda^{(0)}, \ldots, \lambda^{(d)}$ of words in $F(\alpha, \beta)$ so that $h = q_d$ and $w = \lambda^{(d)}$ are as required. Let $q_0 = q$, let $\Gamma_0 = \varnothing$, and let $\lambda^{(0)} = \alpha$. Suppose that for some $j \in \{0, \ldots, d-1\}$ we have $\Gamma_j \subseteq \Gamma$, a word $\lambda^{(j)}$ in $F(\alpha, \beta)$ starting with α and not containing α^{-1}, and $q_j \in \mathcal{A}^{<\omega}_{f,\Sigma}$ such that $|\Gamma_j| = j$ and

(I) $\mathrm{ran}(q_j) \cap \Delta = \varnothing$;

(II) $(u)\lambda^{(j)}{}_{q_j,u} \neq (v)\lambda^{(j)}{}_{q_j,v}$ for all $u, v \in \Gamma_j$ with $u \neq v$;

(III) $(u)\lambda^{(j)}{}_{q_j,u} q_j^m \notin \mathrm{dom}(q)$ for all $m \in \mathbb{Z}$ such that $u \in \mathrm{dom}(\lambda^{(j)}{}_{q_j,u} q_j^m)$ and all $u \in \Gamma_j$;

(IV) $\lambda^{(j)}{}_{q_j,u} \not\equiv \lambda^{(j)}$ for all $u \in \Gamma_j$.

Let $x \in \Gamma \setminus \Gamma_j$ be arbitrary and let $\Gamma_{j+1} = \Gamma_j \cup \{x\}$. The first step in the proof is to find $\nu \in F(\alpha, \beta)$ so that $x \notin \mathrm{dom}\left(\lambda^{(j)} \nu \alpha(q_j)\right)$, and find $m \in \mathbb{N}$ such that $m > |\lambda^{(j)} \nu|$ and it so that we can define

$$\lambda^{(j+1)} \equiv \lambda^{(j)} \nu \alpha^m \beta \alpha. \tag{3.18}$$

In order to define ν consider two cases. If $x \in \mathrm{dom}\left(\lambda^{(j)}(q_j)\right)$, then by Lemma 3.6.14 there is $\nu \in F(\alpha, \beta)$ such that α^{-1} is not contained in ν and the image of x under $\lambda^{(j)} \nu(q_j)$ is in $nK_\omega \setminus \mathrm{dom}(q_j)$. Otherwise, $x \notin \mathrm{dom}\left(\lambda^{(j)}(q_j)\right)$, in which case let ν be the empty word. Hence in both cases

$$x \notin \mathrm{dom}\left(\lambda^{(j)} \nu \alpha(q_j)\right). \tag{3.19}$$

To define m we will again consider two separate cases. If $n = 2$ and $\overline{f} = \mathrm{id}$, let $m > |\lambda^{(j)} \nu|$ be arbitrary. Otherwise, either $n = 2$ and $\overline{f} = (1\ 2)$ or $n \geq 3$. Let

$L_1, \ldots, L_n$ be the connected components of nK_ω, and let $a \in \{1, \ldots, n\}$ so that $x \in L_a$. Consider any extension $g \in \operatorname{Aut}(nK_\omega)$ of q_j, and let b be the image of a under the permutation $\overline{\left(\lambda^{(j)}\nu\right)(g)}$. Since $\overline{q_j} \in S_n$, it follows that b is independent of the extension g. We will show that in this case we can choose $m > |\lambda^{(j)}\nu|$ to be such that

$$(b)\overline{q_j}^m \in \operatorname{supp}(\overline{f}). \tag{3.20}$$

If $n = 2$ and $\overline{f} = (1\ 2)$, then any $m > |\lambda^{(j)}\nu|$ satisfies (3.20). Let $n \geq 3$ be arbitrary, and let O be the orbit of $\overline{q_j}$ containing b. Suppose that $\overline{f}$ fixes O pointwise. If $|O| \leq 2$, then since $n \geq 3$, there is $c \in \{1, \ldots, n\} \setminus O$, and so $(b\ c) \notin \langle \overline{f}, \overline{q_j} \rangle$. If $|O| \geq 3$, then the symmetric group on $|O|$ is not cyclic, and so there is a $\sigma \in S_n$ such that $\operatorname{supp}(\sigma) \subseteq O$ and $\sigma \notin \langle \overline{q_j}|_O \rangle$. Then $\sigma \notin \langle \overline{f}, \overline{q_j} \rangle$. However, both cases are impossible since $\langle \overline{f}, \overline{q_j} \rangle = S_n$. Hence $\overline{f}$ does not fix O pointwise. Hence we may choose $m > |\lambda^{(j)}\nu|$ to satisfy (3.20). Let $\lambda^{(j+1)}$ be as in (3.18). For brevity, denote the prefix $\lambda^{(j)}\nu\alpha^m\beta$ of $\lambda^{(j+1)}$ by ρ.

Next we show how to construct $q_{j+1} \in \mathcal{A}^{<\omega}_{f,\Sigma}$ from q_j. In order to do so, we need to consider a possible complication, namely the existence of $y \in \Gamma_j$ such that $(y)\lambda^{(j+1)}{}_{q_j,y} = (x)\lambda^{(j+1)}{}_{q_j,x}$. The case where such y does not exist is slightly easier and can be proved in a very similar fashion, simply ignoring any mention of y in the following argument (to be more precise (i), (ii), (iv), and (v) are exactly the same, (vii) and (viii) are unnecessary, and in (iii) and (vi) the vertex u can be any vertex in the set Γ_j). Hence we will omit this case. Suppose there is $y \in \Gamma_j$ such that $(y)\lambda^{(j+1)}{}_{q_j,y} = (x)\lambda^{(j+1)}{}_{q_j,x}$. It follows from (II) that such y is unique. Since $\lambda^{(j+1)}{}_{q_j,x}$ is a partial isomorphism and $x \neq y$, it follows that $\lambda^{(j+1)}{}_{q_j,x} \neq \lambda^{(j+1)}{}_{q_j,y}$, and so $\lambda^{(j+1)}{}_{q_j,x} \not\equiv \lambda^{(j+1)}{}_{q_j,y}$. Condition (IV) implies that $\lambda^{(j)}{}_{q_j,y}$ is a proper prefix of $\lambda^{(j)}$, and so $y \notin \operatorname{dom}(\rho(q_j))$. Also from (3.19), we have that

$$|\lambda^{(j+1)}{}_{q_j,x}| \leq |\lambda^{(j)}\nu| < |\rho|. \tag{3.21}$$

Hence $|\lambda^{(j+1)}{}_{q_j,x}|, |\lambda^{(j+1)}{}_{q_j,y}| < |\rho|$. There are two cases to consider: either $|\lambda^{(j+1)}{}_{q_j,x}| > |\lambda^{(j+1)}{}_{q_j,y}|$ or $|\lambda^{(j+1)}{}_{q_j,x}| < |\lambda^{(j+1)}{}_{q_j,y}|$.

Consider $|\lambda^{(j+1)}{}_{q_j,x}| > |\lambda^{(j+1)}{}_{q_j,y}|$. We proceed by inductively constructing a sequence $r_0, \ldots, r_{|\rho|}$ of extensions of q_j, so that $r_0 = q_j$ and $r_{|\rho|}$ is the required q_{j+1}. Let $r_0 = q_j$. For $k \in \{0, \ldots, |\rho|\}$ let the inductive hypothesis be as follows: there is an extension $r_k \in \mathcal{A}^{<\omega}_{f,\Sigma}$ of r_{k-1} (or q_j if $k = 0$) such that

(i) $k \leq |\lambda^{(j+1)}{}_{r_k,x}| \leq |\rho|$;

(ii) $\operatorname{ran}(r_k) \cap \Delta = \varnothing$;

(iii) $\lambda^{(j+1)}{}_{r_k,u} \equiv \lambda^{(j)}{}_{q_j,u}$ for $u \in \Gamma_j \setminus \{y\}$;

(iv) $(u)\lambda^{(j+1)}{}_{r_k,u} \neq (v)\lambda^{(j+1)}{}_{r_k,v}$ for $u, v \in \Gamma_j$ with $u \neq v$;

(v) $(u)\lambda^{(j+1)}{}_{r_k,u}r_k^m \in nK_\omega \setminus \mathrm{dom}(q)$ for all $u \in \Gamma_j$ and for all $m \in \mathbb{Z}$ such that $u \in \mathrm{dom}(\lambda^{(j+1)}{}_{r_k,u}r_k^m)$;

(vi) $(x)\lambda^{(j+1)}{}_{r_k,x} \notin \mathrm{dom}(r_k) \cup \left\{(u)\lambda^{(j)}{}_{q_j,u} : u \in \Gamma_j \setminus \{y\}\right\}$. Moreover if $k > 0$ and we can write $\lambda^{(j+1)}{}_{r_k,x} \equiv \tau\beta^i$ for some $i \in \mathbb{Z} \setminus \{0\}$, and $\tau \in F(\alpha, \beta)$ such that τ ends with a letter α and the image of x under $\tau(r_k)$ is in $\mathrm{supp}(f^i)$, then $(x)\lambda^{(j+1)}{}_{r_k,x} \notin \mathrm{dom}(r_k) \cup \mathrm{ran}(r_k)$;

(vii) if $k > 0$ and the vertices$(x)\lambda^{(j+1)}{}_{r_{k-1},x}$ and $(y)\lambda^{(j+1)}{}_{r_{k-1},y}$ are not equal, then $(x)\lambda^{(j+1)}{}_{r_k,x} \neq (y)\lambda^{(j+1)}{}_{r_k,y}$.

(viii) $|\lambda^{(j+1)}{}_{r_k,x}| > |\lambda^{(j+1)}{}_{r_k,y}|$. Moreover, if $(x)\lambda^{(j+1)}{}_{r_k,x} = (y)\lambda^{(j+1)}{}_{r_k,y}$, then $|\lambda^{(j+1)}{}_{r_k,y}| \geq |\lambda^{(j+1)}{}_{q_j,y}| + k$;

We will first demonstrate that the base case, $k = 0$, holds. The condition (i) is satisfied by r_0 by (3.21), and condition (ii) is satisfied because $r_0 = q_j$ satisfies (I). Since q_j satisfies (IV) we have that $\lambda^{(j)}{}_{q_j,u} \not\equiv \lambda^{(j)}$ and thus $u \notin \mathrm{dom}\left(\lambda^{(j)}(q_j)\right)$ which then implies that $\lambda^{(j+1)}{}_{q_j,u} \equiv \lambda^{(j)}{}_{q_j,u}$ for all $u \in \Gamma_j$. Hence (iii) is satisfied by r_0. Since $\lambda^{(j+1)}{}_{q_j,u} \equiv \lambda^{(j)}{}_{q_j,u}$ for all $u \in \Gamma_j$, the conditions (iv) and (v) are the same as conditions (II) and (III) respectively. Recall that $(x)\lambda^{(j+1)}{}_{q_j,x} \in nK_\omega \setminus \mathrm{dom}(q_j)$ by the definition of $\lambda^{(j+1)}{}_{q_j,x}$, and that if $(x)\lambda^{(j+1)}{}_{q_j,x} = (u)\lambda^{(j+1)}{}_{q_j,u}$ where $u \in \Gamma_j$ then $u = y$ by (II). Hence r_0 satisfies the first part of (vi), while r_0 satisfies second part of (vi), (vii), and second part of (viii) trivially, since $k = 0$. Finally, the first part of (viii) is just the assumption of this case. Therefore r_0 satisfies the inductive hypothesis.

Next we show how to obtain r_{k+1} from r_k. Suppose that for some $k \in \{0, \ldots, |\rho| - 1\}$ we have $r_k \in \mathcal{A}_{f,\Sigma}^{<\omega}$ which satisfies (i) – (viii). We consider the case $\lambda^{(j+1)}{}_{r_k,x} \equiv \rho$ and $\lambda^{(j+1)}{}_{r_k,x}$ being a proper prefix of ρ separately.

Case 1: We begin by considering the case where $\lambda^{(j+1)}{}_{r_k,x}$ is a proper prefix of ρ. Let $z = (x)\lambda^{(j+1)}{}_{r_k,x}$. Since $\lambda^{(j+1)}{}_{r_k,x}$ is a proper prefix of $\lambda^{(j+1)}$, it follows that $z \notin \mathrm{dom}(r_k)$ and $\lambda^{(j+1)}{}_{r_k,x}\alpha$ is a prefix of $\lambda^{(j+1)}$. Recall that $\mathbf{b}(\lambda^{(j+1)})$ is the total number of occurrences of letters β and β^{-1} in the word $\lambda^{(j+1)} \in F(\alpha, \beta)$. Let $c \in \{1, \ldots, n\}$ be so that $z \in L_c$, and choose

$$z' \in L_{(c)\overline{r_k}} \setminus \bigcup_{i=-\mathbf{b}(\lambda^{(j+1)})}^{\mathbf{b}(\lambda^{(j+1)})} \left(\left\{(u)\lambda^{(j+1)}{}_{r_k,u} : u \in \Gamma_j\right\} \cup \Delta \cup \mathrm{dom}(r_k) \cup \mathrm{ran}(r_k) \cup \{z\} \right) f^{-i}.$$

Since $z \notin \mathrm{dom}(r_k)$ and $z' \notin \mathrm{dom}(r_k) \cup \mathrm{ran}(r_k)$ it follows from Lemmas 3.6.4 and 3.6.8 that $r_{k+1} = r_k \cup \{(z, z')\} \in \mathcal{A}_{f,\Sigma}^{<\omega}$. Then there is some $i \in \mathbb{Z}$ such that

$$\lambda^{(j+1)}{}_{r_{k+1},x} \equiv \lambda^{(j+1)}{}_{r_k,x}\alpha\beta^i. \tag{3.22}$$

Hence $|\lambda^{(j+1)}{}_{r_{k+1},x}| > |\lambda^{(j+1)}{}_{r_k,x}| \geq k$. We will now show that $\lambda^{(j+1)}{}_{r_{k+1},x}$ is a prefix of ρ. Suppose that $\lambda^{(j+1)}{}_{r_{k+1},x}$ is not a prefix of ρ. Since $\lambda^{(j+1)}{}_{r_{k+1},x}$ is a prefix of $\lambda^{(j+1)}$, it follows that $\lambda^{(j+1)}{}_{r_{k+1},x} = \lambda^{(j+1)}$. Hence the fact that $\lambda^{(j+1)} \equiv \rho\alpha$ and (3.22) imply that $\lambda^{(j+1)}{}_{r_k,x}\alpha\beta^i \equiv \lambda^{(j+1)}{}_{r_{k+1},x} = \lambda^{(j+1)} = \rho\alpha$, thus $i = 0$ and $\lambda^{(j+1)}{}_{r_k,x} = \rho$, which contradicts the assumption of this case. Therefore, $\lambda^{(j+1)}{}_{r_{k+1},x}$ is prefix of ρ, and so (i) is satisfied by r_{k+1}.

It follows from the definition of r_{k+1} that

$$\mathrm{dom}(r_{k+1}) = \mathrm{dom}(r_k) \cup \{z\} \quad \text{and} \quad \mathrm{ran}(r_{k+1}) = \mathrm{ran}(r_k) \cup \{z'\}. \tag{3.23}$$

Since the vertex z' was chosen outside Δ we have that (ii) is satisfied by r_{k+1}.

Let $u \in \Gamma_j \backslash \{y\}$. It follows from (vi) for r_k that $z = (x)\lambda^{(j+1)}{}_{r_k,x} \neq (u)\lambda^{(j)}{}_{q_j,u}$, and since r_k satisfies (iii), it follows that $z \neq (u)\lambda^{(j+1)}{}_{r_k,u}$. Also $\lambda^{(j+1)}{}_{r_k,u} = \lambda^{(j)}_{q_j,u}$ is a proper prefix of $\lambda^{(j)}$, and so a proper prefix of $\lambda^{(j+1)}$, by (iii) and (IV). Then $(u)\lambda^{(j+1)}{}_{r_k,u} \notin \mathrm{dom}(r_k)$ and $\lambda^{(j+1)}{}_{r_k,u}\alpha$ is a prefix of $\lambda^{(j+1)}$, and thus $(u)\lambda^{(j+1)}{}_{r_k,u} \notin \mathrm{dom}(r_{k+1})$ by (3.23). Hence $\lambda^{(j+1)}{}_{r_{k+1},u} \equiv \lambda^{(j+1)}{}_{r_k,u}$, and since r_k satisfies (iii)

$$\lambda^{(j+1)}{}_{r_{k+1},u} \equiv \lambda^{(j+1)}{}_{r_k,u} \equiv \lambda^{(j)}{}_{q_j,u}. \tag{3.24}$$

Therefore r_{k+1} satisfies (iii).

In order to prove that r_{k+1} satisfies (iv), we consider two cases. Suppose that $z = (x)\lambda^{(j+1)}{}_{r_k,x} \neq (y)\lambda^{(j+1)}{}_{r_k,y}$. It follows by (i) and (viii) that $|\lambda^{(j+1)}{}_{r_k,y}| < |\rho|$. Hence, $\lambda^{(j+1)}{}_{r_k,y}$ is a proper prefix of $\lambda^{(j+1)}$, and so $(y)\lambda^{(j+1)}{}_{r_k,y} \notin \mathrm{dom}(r_k)$ and $\lambda^{(j+1)}{}_{r_k,y}\alpha$ is a prefix of $\lambda^{(j+1)}$, and so $(y)\lambda^{(j+1)}{}_{r_k,y} \notin \mathrm{dom}(r_{k+1})$ by (3.23). Hence $\lambda^{(j+1)}{}_{r_{k+1},y} \equiv \lambda^{(j+1)}{}_{r_k,y}$, in other words

$$z \neq (y)\lambda^{(j+1)}{}_{r_k,y} \implies \lambda^{(j+1)}{}_{r_{k+1},y} \equiv \lambda^{(j+1)}{}_{r_k,y}. \tag{3.25}$$

Combining with the previous paragraph $\lambda^{(j+1)}{}_{r_{k+1},u} \equiv \lambda^{(j+1)}{}_{r_k,u}$ for all $u \in \Gamma_j$. Therefore, r_{k+1} satisfies (iv), since r_k does.

Otherwise, suppose that $z = (x)\lambda^{(j+1)}{}_{r_k,x} = (y)\lambda^{(j+1)}{}_{r_k,y}$. Since $(z')f^i \notin \mathrm{dom}(r_{k+1})$ for all $i \in \{-\mathbf{b}(\rho), \ldots, \mathbf{b}(\rho)\}$ by the choice of z' and (3.23), there exists $i \in \{-\mathbf{b}(\rho), \ldots, \mathbf{b}(\rho)\}$ such that $(y)\lambda^{(j+1)}{}_{r_{k+1},y} = (z')f^i$, and so $\lambda^{(j+1)}{}_{r_{k+1},y} \equiv \lambda^{(j+1)}{}_{r_k,y}\alpha\beta^i$, in other words

$$\begin{gathered}(x)\lambda^{(j+1)}{}_{r_k,x} = (y)\lambda^{(j+1)}{}_{r_k,y} \implies \\ \lambda^{(j+1)}{}_{r_{k+1},y} \equiv \lambda^{(j+1)}{}_{r_k,y}\alpha\beta^i \text{ for some } i \in \{-\mathbf{b}(\rho), \ldots, \mathbf{b}(\rho)\}.\end{gathered} \tag{3.26}$$

The vertex z' was chosen so that $(z')f^i \neq (u)\lambda^{(j+1)}{}_{r_k,u}$ for all $u \in \Gamma_j \setminus \{y\}$. Since $\lambda^{(j+1)}{}_{r_{k+1},u} \equiv \lambda^{(j+1)}{}_{r_k,u}$ for all $u \in \Gamma_j \setminus \{y\}$ and r_k satisfies (iv), it then follows that r_{k+1} satisfies (iv).

Let $u \in \Gamma_j \setminus \{y\}$ be arbitrary. Then $(u)\lambda^{(j+1)}{}_{r_{k+1},u} = (u)\lambda^{(j+1)}{}_{r_k,u}$ by (3.24). Since $z' \notin \mathrm{dom}(r_k)$, no two components of r_k become subsets of the same component of r_{k+1}. It follows that, for any $m \in \mathbb{Z}$, $(u)\lambda^{(j+1)}{}_{r_{k+1},u} r_{k+1}^m$ equals either $(u)\lambda^{(j+1)}{}_{r_k,u} r_k^m$ or z', neither of which belongs to $\mathrm{dom}(q)$. Hence (v) holds for all $u \in \Gamma_j \setminus \{y\}$.

By (3.25), if $z \neq (y)\lambda^{(j+1)}{}_{r_k,y}$ then $\lambda^{(j+1)}{}_{r_{k+1},y} \equiv \lambda^{(j+1)}{}_{r_k,y}$, and so using the argument of the previous paragraph, $(y)\lambda^{(j+1)}{}_{r_{k+1},y} r_{k+1}^m \notin \mathrm{dom}(q)$ for all $m \in \mathbb{Z}$. Hence to show that r_{k+1} satisfies (v) it remains to consider the case where $z = (x)\lambda^{(j+1)}{}_{r_k,x} = (y)\lambda^{(j+1)}{}_{r_k,y}$. It follows from (3.26) that $(y)\lambda^{(j+1)}{}_{r_{k+1},y} = (z')f^i$ for some $i \in \{-\mathbf{b}(\rho), \ldots, \mathbf{b}(\rho)\}$. If $(z')f^i \notin \mathrm{dom}(r_{k+1}) \cup \mathrm{ran}(r_{k+1})$, then no component of r_{k+1}, and thus q, contains the vertex $(z')f^i = (y)\lambda^{(j+1)}{}_{r_{k+1},y}$, and so r_{k+1} satisfies (v). Suppose that $(z')f^i \in \mathrm{dom}(r_{k+1}) \cup \mathrm{ran}(r_{k+1})$. But z' was chosen so that $(z')f^i \notin \mathrm{dom}(r_k) \cup \mathrm{ran}(r_k) \cup \{z\}$, which implies $(z')f^i = z'$ and so $(y)\lambda^{(j+1)}{}_{r_{k+1},y} = z'$. From its definition, the component of r_{k+1} containing $z' = (y)\lambda^{(j+1)}{}_{r_{k+1},y}$ is the component of r_k containing $z = (y)\lambda^{(j+1)}{}_{r_k,y}$ together with the vertex z'. In other words $(y)\lambda^{(j+1)}{}_{r_{k+1},y} r_{k+1}^m$, equals $(y)\lambda^{(j+1)}{}_{r_k,y} r_k^m$ or z', if defined. Since $(y)\lambda^{(j+1)}{}_{r_k,y} r_k^m \in nK_\omega \setminus \mathrm{dom}(q)$ for all $m \in \mathbb{Z}$, it follows that $(y)\lambda^{(j+1)}{}_{r_{k+1},y} r_{k+1}^m \notin \mathrm{dom}(q)$ for all $m \in \mathbb{Z}$. Thus r_{k+1} satisfies condition (v).

By (3.22), $\lambda^{(j+1)}{}_{r_{k+1},x} \equiv \lambda^{(j+1)}{}_{r_k,x} \alpha\beta^i$ for some $i \in \{-\mathbf{b}(\rho), \ldots, \mathbf{b}(\rho)\}$. Hence $(x)\lambda^{(j+1)}{}_{r_{k+1},x} = (z')f^i \notin \mathrm{dom}(r_k) \cup \{z\} \cup \big\{(u)\lambda^{(j+1)}{}_{r_k,u} : u \in \Gamma_j\big\}$ by the choice of z'. By (iii), $(u)\lambda^{(j+1)}{}_{r_k,u} = (u)\lambda^{(j+1)}{}_{q_j,u}$ for all $u \in \Gamma_j \setminus \{y\}$ and $\mathrm{dom}(r_{k+1}) = \mathrm{dom}(r_k) \cup \{z\}$, and so the first part of (vi) is satisfied by r_{k+1}. To check the second part of (vi), suppose that $\lambda^{(j+1)}{}_{r_{k+1},x} \equiv \tau\beta^i$ for some $i \in \mathbb{Z} \setminus \{0\}$ and $\tau \in F(\alpha, \beta)$ such that τ ends with a letter α and the image of x under $\tau(r_{k+1})$ is in $\mathrm{supp}(f^i)$. Then, by (3.22), $\tau = \lambda^{(j+1)}{}_{r_k,x}\alpha$ and the last part of the assumption from the previous sentence becomes $z' = (x)\lambda^{(j+1)}{}_{r_k,x} r_{k+1} \in \mathrm{supp}(f^i)$. Then $(x)\lambda^{(j+1)}{}_{r_{k+1},x} = (z')f^i \neq z'$. Since $(z')f^i \notin \mathrm{dom}(r_k) \cup \mathrm{ran}(r_k) \cup \{z\}$ by the choice of z', it follows from (3.23) that $(x)\lambda^{(j+1)}{}_{r_{k+1},x} \notin \mathrm{dom}(r_{k+1}) \cup \mathrm{ran}(r_{k+1})$. Therefore, r_{k+1} satisfies (vi).

By (3.25) if $z = (x)\lambda^{(j+1)}{}_{r_k,x} \neq (y)\lambda^{(j+1)}{}_{r_k,y}$, then $\lambda^{(j+1)}{}_{r_{k+1},y} \equiv \lambda^{(j+1)}{}_{r_k,y}$, so $(y)\lambda^{(j+1)}{}_{r_k,y} = (y)\lambda^{(j+1)}{}_{r_{k+1},y}$. It follows from (3.22) that there is $i \in \{-\mathbf{b}(\rho), \ldots, \mathbf{b}(\rho)\}$ so that $(x)\lambda^{(j+1)}{}_{r_{k+1},x} = (z')f^i$. Hence by the choice of z'

$$(x)\lambda^{(j+1)}{}_{r_{k+1},x} = (z')f^i \neq (y)\lambda^{(j+1)}{}_{r_k,y} = (y)\lambda^{(j+1)}{}_{r_{k+1},y},$$

and so (vii) holds for r_{k+1}.

Finally, we will show that r_{k+1} satisfies (viii). Suppose that $(y)\lambda^{(j+1)}{}_{r_k,y} \neq (x)\lambda^{(j+1)}{}_{r_k,x}$. Then $\lambda^{(j+1)}{}_{r_{k+1},y} \equiv \lambda^{(j+1)}{}_{r_k,y}$ by (3.25). Since $|\lambda^{(j+1)}{}_{r_k,x}| < |\lambda^{(j+1)}{}_{r_{k+1},x}|$ and r_k satisfies (viii), it follows that r_{k+1} satisfies (viii) as well. The other case is when $(y)\lambda^{(j+1)}{}_{r_k,y} = (x)\lambda^{(j+1)}{}_{r_k,x}$. Then $\lambda^{(j+1)}{}_{r_{k+1},y} \equiv \lambda^{(j+1)}{}_{r_k,y}\alpha\beta^i$ for some $i \in \{-\mathbf{b}(\rho), \ldots, \mathbf{b}(\rho)\}$ by (3.26). Since $\lambda^{(j+1)}{}_{r_k,y}$ is a proper prefix of $\lambda^{(j+1)}{}_{r_k,x}$ by (viii) applied to r_k, it follows that $\lambda^{(j+1)}{}_{r_k,y}\alpha$ is a prefix of $\lambda^{(j+1)}{}_{r_k,x}$, and so $\lambda^{(j+1)}{}_{r_k,y}\alpha\beta^{i'}$ is a prefix of $\lambda^{(j+1)}{}_{r_k,x}$ for some $i' \in \mathbb{Z}$. Assume that i' was picked so that $|i'|$ is maximal. Suppose that $\lambda^{(j+1)}{}_{r_{k+1},y}$ is not a prefix of $\lambda^{(j+1)}{}_{r_k,x}$, in other words either $i > 0$ and $i' \in \{0, \dots, i-1\}$; or $i < 0$ and $i' \in \{i+1, \dots, 0\}$. Then either $\lambda^{(j+1)}{}_{r_k,x}\beta$ or $\lambda^{(j+1)}{}_{r_k,x}\beta^{-1}$ must be a prefix of $\lambda^{(j+1)}$, which contradicts (3.22). Hence $\lambda^{(j+1)}{}_{r_{k+1},y}$ is a prefix of $\lambda^{(j+1)}{}_{r_k,x}$, and thus

$$|\lambda^{(j+1)}{}_{r_{k+1},y}| \leq |\lambda^{(j+1)}{}_{r_k,x}| < |\lambda^{(j+1)}{}_{r_{k+1},x}|.$$

Therefore, r_{k+1} satisfies first part of (viii).

In order to show the second part of (viii), suppose that $(x)\lambda^{(j+1)}{}_{r_{k+1},x} = (y)\lambda^{(j+1)}{}_{r_{k+1},y}$. Since (vii) holds for r_{k+1} we have that $(x)\lambda^{(j+1)}{}_{r_k,x} = (y)\lambda^{(j+1)}{}_{r_k,y}$ and thus $|\lambda^{(j+1)}{}_{r_k,y}| \geq |\lambda^{(j+1)}{}_{q_j,y}| + k$ by (viii) for r_k. Also (3.26) implies that $|\lambda^{(j+1)}{}_{r_k,y}| < |\lambda^{(j+1)}{}_{r_{k+1},y}|$. Therefore $|\lambda^{(j+1)}{}_{r_{k+1},y}| \geq |\lambda^{(j+1)}{}_{q_j,y}| + k + 1$ and thus r_{k+1} satisfies (viii) and hence this case is complete.

Case 2: Suppose $\lambda^{(j+1)}{}_{r_k,x} \equiv \rho$. It follows from (3.19) that $|\lambda^{(j+1)}{}_{r_0,x}| < |\rho|$, and so $k > 0$. Let $r_{k+1} = r_k$. Then r_{k+1} trivially satisfies conditions (i) – (vii) and the first part of condition (viii). To show second part of (viii) we will consider two cases. Suppose that $n = 2$ and $\overline{f} = \text{id}$. Since $\lambda^{(j+1)} \equiv \rho\alpha$ and $\lambda^{(j+1)}{}_{r_k,x} \equiv \rho$ it follows that the image of x under $\rho(r_k)$ is $(x)\lambda^{(j+1)}{}_{r_k,x} \in nK_\omega \setminus \text{dom}(r_k)$. Let $t \in nK_\omega$ be the image of x under $\lambda^{(j)}\nu\alpha^m(r_k)$. Then $\rho \equiv \lambda^{(j)}\nu\alpha^m\beta$ implies that $(t)f$ is the image of x under ρ, and so if $y \in \text{fix}(f)$

$$t = (t)f = (x)w_{r_k,x} \in nK_\omega \setminus \text{dom}(r_k)$$

by the assumption that $w_{r_k,x} = \rho$. However, we have assumed at the beginning of the proof that $\text{fix}(f) \subseteq \text{dom}(q)$, which is a contradiction since $\text{dom}(q) \subseteq \text{dom}(r_k)$. Hence $t \in \text{supp}(f)$. Otherwise, either $n = 2$ and $\overline{f} = (1\ 2)$, or $n \geq 3$. Recall that $a, b \in \{1, \dots, n\}$ are such that $x \in L_a$ and b is the image of a under $\overline{\lambda^{(j)}\nu(r_k)}$. Then the image of a under $\overline{\lambda^{(j)}\nu\alpha^m(r_k)}$ is in $\text{supp}(\overline{f})$ by (3.20), and so the imaged of x under $\lambda^{(j)}\nu\alpha^m(r_k)$ is in $\text{supp}(f)$ in both cases.Hence it follows from the second part of (vi) that

$$(x)\lambda^{(j+1)}{}_{r_k,x} \notin \text{dom}(r_k) \cup \text{ran}(r_k). \tag{3.27}$$

Next, using (3.27), will show that $(x)\lambda^{(j+1)}{}_{r_k,x} \neq (y)\lambda^{(j+1)}{}_{r_k,y}$, which then implies that r_{k+1} satisfies the second half of (viii), and this case will be complete. Suppose that $(x)\lambda^{(j+1)}{}_{r_k,x} = (y)\lambda^{(j+1)}{}_{r_k,y}$. Since $\lambda^{(j+1)}{}_{r_k,x} \equiv \rho \equiv \lambda^{(j)}\nu\alpha^m\beta$ and $|\lambda^{(j+1)}{}_{r_0,x}| \leq |\lambda^{(j)}\nu|$ by (3.19), the fact that at any inductive step incomplete components of q_j were extended by at most one point, implies that $k \geq m$. Since m was chosen so that $m > |\lambda^{(j)}\nu|$, and r_k satisfies (viii)

$$|\rho| = |\lambda^{(j+1)}{}_{r_k,x}| \geq |\lambda^{(j+1)}{}_{r_k,y}| \geq |\lambda^{(j+1)}{}_{q_j,y}| + k > m > |\lambda^{(j)}\nu|.$$

Hence $\lambda^{(j+1)}{}_{r_k,y}$ is a prefix of ρ, and $\lambda^{(j)}\nu$ is a prefix of $\lambda^{(j+1)}{}_{r_k,y}$. The former and the fact that $\lambda^{(j+1)} = \rho\alpha$ implies that ${\lambda^{(j+1)}}_{r_k,y}$ is a proper prefix of $\lambda^{(j+1)}$, and so ${\lambda^{(j+1)}}_{r_k,y}\alpha$ is a prefix of $\lambda^{(j+1)}$ and $y \notin \mathrm{dom}(\lambda^{(j+1)})$. Since $\lambda^{(j+1)} \equiv \lambda^{(j)}\nu\alpha^m\beta\alpha$, there is $i \in \{1, \ldots, m-1\}$ such that ${\lambda^{(j+1)}}_{r_k,y} \equiv \lambda^{(j)}\nu\alpha^i$. Hence $(y){\lambda^{(j+1)}}_{r_k,y} \in \mathrm{ran}(r_k)$. But this contradicts (3.27), and so we conclude that $(x){\lambda^{(j+1)}}_{r_k,x} \neq (y){\lambda^{(j+1)}}_{r_k,y}$. Therefore r_{k+1} satisfies the second part of (viii), since $r_{k+1} = r_k$, as required.

Hence by induction there is $q_{j+1} = r_{|\rho|} \in \mathcal{A}^{<\omega}_{f,\Sigma}$ satisfying conditions (i) – (viii). We will now show that q_{j+1} satisfies (I) – (IV).

It follows from (ii) that q_{j+1} satisfies (I). Suppose that $(x){\lambda^{(j+1)}}_{q_{j+1},x} = (y){\lambda^{(j+1)}}_{q_{j+1},y}$. Then by (i) and (viii) we have

$$|\rho| = |{\lambda^{(j+1)}}_{q_{j+1},x}| > |{\lambda^{(j+1)}}_{q_{j+1},y}| \geq |{\lambda^{(j+1)}}_{q_j,y}| + |\rho|.$$

which is a contradiction. Hence it follows from (iii), (iv), and (vi) that q_{j+1} satisfies (II). It follows from (v) that we only need to verify (III) for x. From (i) we have that ${\lambda^{(j+1)}}_{q_{j+1},x} \equiv \rho$, and so $(x){\lambda^{(j+1)}}_{q_{j+1},x} \notin \mathrm{dom}(q_{j+1}) \cup \mathrm{ran}(q_{j+1})$ by (vi) and the choice of ρ, and so (III) holds for q_{j+1}. Finally, condition (IV) follows from (i), (iii), (viii) and the fact that q_j satisfies (IV). Therefore, q_{j+1} satisfies the inductive hypothesis.

Consider the case where $|{\lambda^{(j+1)}}_{q_j,x}| < |{\lambda^{(j+1)}}_{q_j,y}|$. The above argument applies if we switch the roles of x and y, i.e. let $\Gamma'_j = \Gamma_j \cup \{y\} \setminus \{x\}$, and $\lambda^{(j)\prime} \equiv \lambda^{(j+1)}$. Then q_j, $\lambda^{(j)\prime}$ and Γ'_j satisfy conditions (I) – (IV) and we can proceed as before.

Hence by induction there is $h = q_d$ satisfying (I) – (IV). Since $\langle \overline{f}, \overline{h}\rangle = S_n$ there is $w \in F(\alpha, \beta)$ which does not contain α^{-1}, $\lambda^{(d)}$ is a prefix of w and $\overline{w(h)} = \mathrm{id}$. Then from (I) – (IV) it follows that conditions (2), (3), (5) and (6) of $\mathcal{S}(\Gamma, \varnothing, \mathrm{dom}(q), \Delta, w)$ are satisfied by h. Since $\Theta = \varnothing$, condition (4) of $\mathcal{S}(\Gamma, \varnothing, \mathrm{dom}(q), \Delta, w)$ follows trivially from (3) of $\mathcal{S}(\Gamma, \varnothing, \mathrm{dom}(q), \Delta, w)$. Hence h satisfies $\mathcal{S}(\Gamma, \varnothing, \mathrm{dom}(q), \Delta, w)$. □

The next lemma is the second step in the proof of Lemma 3.6.15.

Lemma 3.6.17. *Let $n \in \mathbb{N}$ be such that $n \geq 2$, let $f \in \mathrm{Aut}(nK_\omega)$ be non-stabilising, let $q \in \mathcal{A}^{<\omega}_{f,\Sigma}$ be such that $\overline{q} \in S_n$, and let $w \in F(\alpha, \beta)$ be a word which does not contain α^{-1} and which starts with α. Suppose $\Gamma, \Phi, \subseteq \mathrm{dom}(q)$, $\Theta \subseteq \Gamma$, and $x \in \Gamma \setminus \Theta$. If q satisfies $\mathcal{S}(\Gamma, \Theta, \Phi, \Delta, w)$, then there is an extension $h \in \mathcal{A}^{<\omega}_{f,\Sigma}$ of q such that h satisfies $\mathcal{S}(\Gamma, \Theta \cup \{x\}, \Phi, \Delta, w)$.*

Proof. For all $k \in \{0, \ldots, |w|\}$, define ρ_k to be a prefix of w of length k. Recall that for all $u \in \Gamma$ we identify the word $w_{q,u}$ with its realisation $w_{q,u}(u)$. In the same way, if q_k is a partial isomorphism, then we identify the word ρ_k with the partial isomorphism $\rho_k(q_k)$.

It follows from condition (3) of $\mathcal{S}(\Gamma, \Theta, \Phi, \mathrm{dom}(q), w)$ and the fact that $x \in \Gamma \setminus \Theta$, that $x \notin \mathrm{dom}\,(w(q))$, and so $w_{q,x}$ is a proper prefix of w. Let M be such that $M-1=|w_{q,x}|$, or in other words M is the smallest non-negative integer such that $x \notin \operatorname{dom}(\rho_M(q))$. Then $M \leq |w|$. Since $x \in \Gamma \subseteq \operatorname{dom}(q)$ and w starts with α, it follows that $M > 1$, and so $M \in \{2, \ldots, |w|\}$. Since $w_{q,x}$ is a proper prefix of w, it follows that $w_{q,x}\alpha$ is a prefix of w and $(x)w_{q,x} \in nK_\omega \setminus \operatorname{dom}(q)$. Hence $\rho_M = \rho_{M-1}\alpha$ and the image of x under $\rho_{M-1}(q)$ is in $nK_\omega \setminus \operatorname{dom}(q)$.

We will inductively construct a sequence $q_{M-1} = q, q_M, \ldots, q_{|w|} \in \mathcal{A}^{<\omega}_{f,\Sigma}$ such that if $j \in \{M, \ldots, |w|\}$ then q_j is an extension of q_{j-1} and the following conditions are satisfied

(i) $\operatorname{ran}(q_j) \cap \Delta = \varnothing$;

(ii) $w_{q_j,u} \equiv w_{q,u}$ and $(u)w_{q_j,u} \in nK_\omega \setminus \operatorname{dom}(q_j)$ for all $u \in \Gamma \setminus \{x\}$;

(iii) $(x)\rho_j f^i \in nK_\omega \setminus \operatorname{dom}(q_j)$ for all $i \in \{-\mathbf{b}(w)+\mathbf{b}(\rho_j), \ldots, \mathbf{b}(w)-\mathbf{b}(\rho_j)\}$;

(iv) $(x)w_{q_j,x} \neq (u)w_{q_j,u}$ for all $u \in \Gamma \setminus \{x\}$;

(v) $(u)w_{q_j,u}q_j^m \in nK_\omega \setminus \Phi$ for all $u \in \Gamma$ and for all $m \in \mathbb{Z}$ such that $u \in \operatorname{dom}(w_{q_j,u}q_j^m)$.

Then $h = q_{|w|}$ will be the required extension of q.

Let y be the image of x under $\rho_{M-1} = w_{q,x}$ and suppose $y \in L_a$ for some $a \in \{1, \ldots, n\}$. Recall that $\mathbf{b}(w)$ is the number of occurrences of letters β and β^{-1} in the word w. We may choose:

$$z \in L_{(a)\overline{q}} \setminus \bigcup_{i=-\mathbf{b}(w)}^{\mathbf{b}(w)} (\operatorname{dom}(q) \cup \operatorname{ran}(q) \cup \{y\} \cup \Delta \cup \{(u)w_{q,u} : u \in \Gamma\})\, f^{-i}.$$

and define $q_M = q \cup \{(y, z)\}$. Then $q_M \in \mathcal{A}^{<\omega}_{f,\Sigma}$ by Lemmas 3.6.4 and 3.6.8, since $y \notin \operatorname{dom}(q)$ and $z \notin \operatorname{dom}(q) \cup \operatorname{ran}(q)$.

First, we will show that q_M satisfies conditions (i) to (v). Since $\operatorname{ran}(q_M) = \operatorname{ran}(q) \cup \{z\}$ and z was chosen outside Δ, it follows that q_M satisfies (i). Let $u \in \Gamma \setminus \{x\}$. If $u \notin \Theta$, then, from (3) of $\mathcal{S}(\Gamma, \Theta, \Phi, \Delta, w)$, $u \notin \operatorname{dom}(w(q))$ and so $w_{q,u}$ is a proper prefix of w. It follows that $(u)w_{q,u} \in nK_\omega \setminus \operatorname{dom}(q)$. On the other hand, if $u \in \Theta$, then $w_{q,u} = w$ and $(u)w_{q,u} \in nK_\omega \setminus \operatorname{dom}(q)$ by (3) and (4) of $\mathcal{S}(\Gamma, \Theta, \Phi, \Delta, w)$. Hence in both cases $(u)w_{q,u} \in nK_\omega \setminus \operatorname{dom}(q)$. Since $\operatorname{dom}(q_M) \setminus \operatorname{dom}(q) = \{y\}$ and $(u)w_{q,u} \neq (x)w_{q,x} = y$ by part (5) of $\mathcal{S}(\Gamma, \Theta, \Phi, \Delta, w)$, it follows that $(u)w_{q,u} \in nK_\omega \setminus \operatorname{dom}(q_M)$, and so $w_{q_M,u} \equiv w_{q,u}$, proving (ii). Let $i \in \{-\mathbf{b}(w)+\mathbf{b}(\rho_M), \ldots \mathbf{b}(w)-\mathbf{b}(\rho_M)\}$. Since $\operatorname{dom}(q_M) = \operatorname{dom}(q) \cup \{y\}$, it follows from the choice of z that

$$(x)\rho_M f^i = (y)q_M f^i = (z)f^i \in nK_\omega \setminus \operatorname{dom}(q_M).$$

Hence q_M satisfies condition (iii). Let $u \in \Gamma \setminus \{x\}$. Note that since q_M satisfies (iii) there is $k \in \{-\mathbf{b}(w), \ldots, \mathbf{b}(w)\}$ such that $w_{q_M,x} = w_{q,x}\alpha\beta^k$, and so $(x)w_{q_M,x} =$

$(z)f^k$. It follows from the choice of z, and the fact that q_M satisfies (ii) that

$$(x)w_{q_M,x} = (z)f^k \neq (u)w_{q,u} = (u)w_{q_M,u}.$$

Hence q_M satisfies (iv).

Finally, to show that q_M satisfies (v) consider two cases: $u = x$ and $u \in \Gamma \setminus \{x\}$. Suppose that $u = x$ and $m \in \mathbb{Z}$ is such that $x \in \operatorname{dom}(w_{q_M,x}q_M^m)$. As shown before $(x)w_{q_M,x} = (z)f^k$ for some $k \in \{-\mathbf{b}(w), \ldots, \mathbf{b}(w)\}$. From the choice of z it follows that $(x)w_{q_M,x} = (z)f^k \notin \operatorname{dom}(q) \cup \operatorname{ran}(q) \cup \{y\}$. Suppose $(z)f^k \neq z$. Then $(x)w_{q_M,x} = (z)f^k \notin \operatorname{dom}(q_M) \cup \operatorname{ran}(q_M)$, and so $m = 0$. Since $\Phi \subseteq \operatorname{dom}(q) \subseteq \operatorname{dom}(q_M)$, this implies that

$$(x)w_{q_M,x}q_M^m = (z)f^k \in nK_\omega \setminus \Phi.$$

Suppose that $(z)f^k = z$, in other words $(x)w_{q_M,x} = z$. Since $z \notin \operatorname{dom}(q_M)$, it follows that $x \notin \operatorname{dom}(w_{q_M,x}q_M^m)$ for all $m > 0$. If $m = 0$ then $(x)w_{q_M,x}q_M^m = (z)f^k \in nK_\omega \setminus \Phi$ by the choice z and since $\Phi \subseteq \operatorname{dom}(q)$. Suppose that $m < 0$. Then $m + 1 \leq 0$ and it follows from the definition of q_M that $\operatorname{dom}(q_M^{m+1})$ is either $\operatorname{dom}(q^{m+1})$ or $\operatorname{dom}(q^{m+1}) \cup \{(z)q_M^{m+1}\}$. Note that $y \in \operatorname{dom}(q_M^{m+1})$ implies $y \in \operatorname{dom}(q^{m+1})$. It follows that from (6) of $\mathcal{S}(\Gamma, \Theta, \Phi, \Delta, w)$ that

$$(x)w_{q_M,x}q_M^m = (z)q_M^m = (y)q_M^{m+1} = (y)q^{m+1} = (x)w_{q,x}q^{m+1} \in nK_\omega \setminus \Phi.$$

Hence q_M satisfies (v) for $u = x$.

Suppose that $u \in \Gamma \setminus \{x\}$ and $m \in \mathbb{Z}$ is such that $u \in \operatorname{dom}(w_{q_M,u}q_M^m)$. Since q_M satisfies (ii), it follows that $(u)w_{q_M,u} = (u)w_{q,u}$. If $m \leq 0$, or $m > 0$ and there is no $m' \in \{0, \ldots, m-1\}$ with $(u)w_{q,u}q^{m'} = y$, then $(u)w_{q_M,u}q_M^m = (u)w_{q,u}q^m \in nK_\omega \setminus \Phi$ by (6) of $\mathcal{S}(\Gamma, \Theta, \Phi, \Delta, w)$. Otherwise, $m > 0$ and there is $m' \in \{0, \ldots, m-1\}$ such that $(u)w_{q,u}q^{m'} = y$, in which case $(u)w_{q_M,u}q_M^{m'+1} = z \notin \operatorname{dom}(q_M)$. Hence $m = m' + 1$, and since $\Phi \subseteq \operatorname{dom}(q) \subseteq \operatorname{dom}(q_M)$, it follows that $(u)w_{q_M,u}q_M^m \in nK_\omega \setminus \Phi$. Therefore, q_M satisfies (v) and thus the inductive hypothesis holds.

In the case where $M = |w|$, $q_{|w|}$ already satisfies conditions (i) to (v). Hence suppose that $M < |w|$ and suppose that for some $j \in \{M, \ldots, |w| - 1\}$ there is an extension $q_j \in \mathcal{A}_{f,\Sigma}^{<\omega}$ of q_{j-1} satisfying conditions (i) to (v). We have two cases to consider: either $\rho_{j+1} = \rho_j\beta^\varepsilon$ or $\rho_{j+1} = \rho_j\alpha^\varepsilon$ for some $\varepsilon \in \{-1, 1\}$.

First consider the case $\rho_{j+1} = \rho_j\beta^\varepsilon$, where $\varepsilon \in \{-1, 1\}$. Let $q_{j+1} = q_j$. Then conditions (i), (ii), (iv), and (v) are trivially satisfied by q_{j+1}. In order to show that q_{j+1} satisfies (iii), let $i \in \mathbb{Z}$ be such that $i \in \{-\mathbf{b}(w) + \mathbf{b}(\rho_{j+1}), \ldots, \mathbf{b}(w) - \mathbf{b}(\rho_{j+1})\}$. Then $|i + \varepsilon| \leq \mathbf{b}(w) - \mathbf{b}(\rho_{j+1}) + 1 = \mathbf{b}(w) - \mathbf{b}(\rho_j)$, and so

$$(x)\rho_{j+1}f^i = (x)\rho_j f^{i+\varepsilon} \in nK_\omega \setminus \operatorname{dom}(q_j) = nK_\omega \setminus \operatorname{dom}(q_{j+1}).$$

Hence q_{j+1} satisfies condition (iii), and so the induction hypothesis.

Otherwise $\rho_{j+1} = \rho_j\alpha^\varepsilon$ for some $\varepsilon \in \{-1, 1\}$, and so $\rho_{j+1} = \rho_j\alpha$ since w does not contain α^{-1}. Let $y = (x)\rho_j$, and let $a \in \{1, \dots, n\}$ be such that $y \in L_a$. Choose

$$z \in L_{(a)\overline{q_j}} \setminus \bigcup_{i=-\mathbf{b}(w)}^{\mathbf{b}(w)} \big(\mathrm{dom}(q_j) \cup \mathrm{ran}(q_j) \cup \{y\} \cup \Delta \cup \big\{(u)w_{q_j,u} : u \in \Gamma\big\}\big)\, f^{-i}.$$

Since $y \notin \mathrm{dom}(q_j)$ by (iii) and $z \notin \mathrm{dom}(q_j) \cup \mathrm{ran}(q_j)$, it follows from Lemmas 3.6.4 and 3.6.8 that $q_{j+1} = q_j \cup \{(y, z)\} \in \mathcal{A}^{<\omega}_{f,\Sigma}$. Observe that

$$\mathrm{dom}(q_{j+1}) = \mathrm{dom}(q_j) \cup \{y\} \quad \text{and} \quad \mathrm{ran}(q_{j+1}) = \mathrm{ran}(q_j) \cup \{z\}. \tag{3.28}$$

The vertex z was chosen so that $z \notin \Delta$, and so q_{j+1} satisfies (i).

It follows from (iii) that $x \in \mathrm{dom}\,(\rho_j)$ and $x \notin \mathrm{dom}\,(\rho_{j+1}(q_j))$, thus $w_{q_j,x} \equiv \rho_j$. Let $u \in \Gamma \setminus \{x\}$. Since q_j satisfies (iv)

$$(u)w_{q_j,u} \neq (x)w_{q_j,x} = (x)\rho_j = y.$$

It then follows from $(u)w_{q_j,u} \in nK_\omega \setminus \mathrm{dom}(q_j)$ and (3.28) that $(u)w_{q_j,u} \in nK_\omega \setminus \mathrm{dom}(q_{j+1})$, and so $w_{q_{j+1},u} \equiv w_{q_j,u}$. Then $(u)w_{q_{j+1},u} \in nK_\omega \setminus \mathrm{dom}(q_{j+1})$, and since q_j satisfies (ii) it follows that q_{j+1} also satisfies (ii).

Let $i \in \{-\mathbf{b}(w), \dots, \mathbf{b}(w)\}$. Then by (3.28) and the fact that z was chosen so that $(z)f^i \notin \mathrm{dom}(q_j) \cup \{y\}$

$$(x)\rho_{j+1}f^i = (z)f^i \in nK_\omega \setminus \mathrm{dom}(q_{j+1}).$$

Hence q_{j+1} satisfies (iii).

It follows from the fact that q_{j+1} satisfies (iii), that $w_{q_{j+1},x} \equiv w_{q_j,x}\alpha\beta^k$ for some $k \in \{-\mathbf{b}(w), \dots, \mathbf{b}(w)\}$, and so

$$(x)w_{q_{j+1},x} = (z)f^k. \tag{3.29}$$

By the choice of z and the fact that q_{j+1} satisfies (ii)

$$(x)w_{q_{j+1},x} = (z)f^k \neq (u)w_{q_j,u} = (u)w_{q_{j+1},u}$$

for every $u \in \Gamma \setminus \{x\}$. Hence q_{j+1} satisfies (iv).

Finally, to show that q_{j+1} satisfies (v) consider two cases — $u = x$ and $u \in \Gamma \setminus \{x\}$. Suppose that $u = x$ and $m \in \mathbb{Z}$ is such that $x \in \mathrm{dom}(w_{q_{j+1},x}q^m_{j+1})$. From the choice of z and (3.29) it follows that $(x)w_{q_{j+1},x} = (z)f^k \notin \mathrm{dom}(q) \cup \mathrm{ran}(q) \cup \{y\}$. Suppose $(z)f^k \neq z$. Then $(x)w_{q_{j+1},x} = (z)f^k \notin \mathrm{dom}(q_{j+1}) \cup \mathrm{ran}(q_{j+1})$, and so

$m = 0$, in which case $\Phi \subseteq \mathrm{dom}(q) \subseteq \mathrm{dom}(q_{j+1})$ implies that

$$(x)w_{q_{j+1},x}q_{j+1}^{m} = (z)f^k \in nK_\omega \setminus \Phi.$$

Suppose that $(z)f^k = z$, in other words $(x)w_{q_{j+1},x} = z$. Since $z \notin \mathrm{dom}(q_{j+1})$, it follows that $x \notin \mathrm{dom}(w_{q_{j+1},x}q_{j+1}^{m})$ for all $m > 0$. If $m = 0$, then $(x)w_{q_{j+1},x}q_{j+1}^{m} = (z)f^k \in nK_\omega \setminus \Phi$ by the choice z and since $\Phi \subseteq \mathrm{dom}(q)$. Suppose that $m < 0$. Then $m + 1 \leq 0$ and it follows from the definition of q_{j+1} that $\mathrm{dom}(q_{j+1}^{m+1})$ is either $\mathrm{dom}(q_j^{m+1})$ or $\mathrm{dom}(q_j^{m+1}) \cup \{(z)q_{j+1}^{m+1}\}$. Note that $y \in \mathrm{dom}(q_{j+1}^{m+1})$ implies $y \in \mathrm{dom}(q_j^{m+1})$. It follows that from (6) of $\mathcal{S}(\Gamma, \Theta, \Phi, \Delta, w)$ that

$$(x)w_{q_{j+1},x}q_{j+1}^{m} = (z)q_{j+1}^{m} = (y)q_{j+1}^{m+1} = (y)q_j^{m+1} = (x)w_{q_j,x}q^{m+1} \in nK_\omega \setminus \Phi.$$

Hence q_{j+1} satisfies (v) for $u = x$.

Suppose that $u \in \Gamma \setminus \{x\}$ and $m \in \mathbb{Z}$ such that $u \in \mathrm{dom}(w_{q_{j+1},u}q_{j+1}^{m})$. Since q_j and q_{j+1} satisfies (ii), it follows that $(u)w_{q_{j+1},u} = (u)w_{q,u} = (u)w_{q_j,u}$. If $m \leq 0$, or $m > 0$ and there is no $m' \in \{0, \ldots, m-1\}$ with $(u)w_{q_j,u}q^{m'} = y$, then $(u)w_{q_{j+1},u}q_{j+1}^{m} = (u)w_{q_j,u}q_j^{m} \in nK_\omega \setminus \Phi$ since q_j satisfies (v). Otherwise, $m > 0$ and there is $m' \in \{0, \ldots, m-1\}$ such that $(u)w_{q_j,u}q_j^{m'} = y$, in which case $(u)w_{q_{j+1},u}q_{j+1}^{m'+1} = z \notin \mathrm{dom}(q_{j+1})$. Hence $m = m' + 1$, and since $\Phi \subseteq \mathrm{dom}(q) \subseteq \mathrm{dom}(q_{j+1})$, it follows that $(u)w_{q_{j+1},u}q_{j+1}^{m} \in nK_\omega \setminus \Phi$. Therefore, q_{j+1} satisfies (v) and thus the inductive hypothesis.

By induction there is $h = q_{|w|} \in \mathcal{A}_{f,\Sigma}^{<\omega}$ satisfying (i) – (v). We will show that h satisfies $\mathcal{S}(\Gamma, \Theta \cup \{x\}, \Phi, \Delta, w)$ and will refer to parts (1) to (6) of this condition by writing (1) to (6), where appropriate, without reference to $\mathcal{S}(\Gamma, \Theta \cup \{x\}, \Phi, \Delta, w)$ in the rest of the proof.

Since h is an extension of q and $\overline{q} \in S_n$, it follows that $\overline{h} = \overline{q}$. Hence

$$\overline{w(h)} = \mathrm{id},$$

and so h satisfies (1). Since h satisfies (i) and (v), it also satisfies (2) and (6). Since $w = \rho_{|w|}$ condition (iii) implies that $x \in \mathrm{dom}(w(h))$, and so $x \in \mathrm{dom}(w(h)) \cap \Gamma$. If $u \in \Gamma \setminus \{x\}$, then $w_{h,u} \equiv w_{q,u}$ by (ii), and so $u \in \mathrm{dom}(w(h)) \cap \Gamma$ if and only if $u \in \mathrm{dom}(w(q)) \cap \Gamma$. Therefore, $\mathrm{dom}(w(h)) \cap \Gamma = \Theta \cup \{x\}$ as q satisfies $\mathcal{S}(\Gamma, \Theta, \Phi, \Delta, w)$, in other words h satisfies (3). By (iii) the image of x under $w(h)$ is in $nK_\omega \setminus \mathrm{dom}(h)$, and by (ii) the image of $u \in \Theta$ under $w(h) = w_{q,u}$ is also in $nK_\omega \setminus \mathrm{dom}(h)$. Hence h satisfies condition (4). It then follows from (ii), (iv) and the fact that q satisfies (5) of $\mathcal{S}(\Gamma, \Theta, \Phi, \Delta, w)$ that $(u)w_{h,u} = (v)w_{h,v}$ only if $u = v$ for all $u, v \in \Gamma$, and thus h satisfies (5). Hence h satisfies $\mathcal{S}(\Gamma, \Theta \cup \{x\}, \Phi, \Delta, w)$, as required. $\square$

Proof of Lemma 3.6.15. If necessary by extending q using Lemmas 3.6.4 and 3.6.8, we can assume that $\Gamma \subseteq \mathrm{dom}(q)$.

Let $d = |\Gamma|$. By Lemma 3.6.16, there is a freely reduced word $w \in F(\alpha, \beta)$ not containing α^{-1} and starting with α, and an extension $q_0 \in \mathcal{A}_{f,\Sigma}^{<\omega}$ of q satisfying $\mathcal{S}(\Gamma, \varnothing, \mathrm{dom}(q), \Delta, w)$. Suppose that for some $j \in \{0, 1, \dots, d-1\}$ we have already extended $q = q_0$ to $q_j \in \mathcal{A}_{f,\Sigma}^{<\omega}$ such that there is $\Gamma_j \subseteq \Gamma$, with $|\Gamma_j| = j$, and q_j satisfies $\mathcal{S}(\Gamma, \Gamma_j, \mathrm{dom}(q), \Delta, w)$. Let $x \in \Gamma \setminus \Gamma_j$ and let $\Gamma_{j+1} = \Gamma_j \cup \{x\}$. Then, by condition (3) of $\mathcal{S}(\Gamma, \Gamma_j, \mathrm{dom}(q), \Delta, w)$, $x \notin \mathrm{dom}\,(w(q_j))$. Hence if we let $\Theta = \Gamma_j$ and $\Phi = \mathrm{dom}(q)$ then by Lemma 3.6.17 there is $q_{j+1} \in \mathcal{A}_{f,\Sigma}^{<\omega}$ and extension of q_j satisfying $\mathcal{S}(\Gamma, \Gamma_{j+1}, \mathrm{dom}(q), \Delta, w)$.

Therefore, by induction on j we obtain $h = q_d \in \mathcal{A}_{f,\Sigma}^{<\omega}$ which satisfies $\mathcal{S}(\Gamma, \Gamma, \mathrm{dom}(q), \Delta, w)$, as required. □

Chapter 4

Make it universal: words, lists and sequences

This chapter is based on the research conducted in collaboration with J. Hyde. It is included in this thesis with the permission of my coauthor.

4.1 Introduction to universal words and sequences

In order to define the notions of universal words and universal sequences we first need to introduce some classical notations of free objects in semigroups, monoids and groups.

Let A be an alphabet, let S be a semigroup, and let $w \in A^+$. Then w is a SEMIGROUP UNIVERSAL WORD FOR S if for every $s \in S$ there is a semigroup homomorphism $\Phi : A^+ \longrightarrow S$ such that $(w)\Phi = s$. If $w = x_1 \ldots x_n$ where $x_i \in A$, then $s = (w)\Phi = (x_1)\Phi \ldots (x_n)\Phi$. So we may think of Φ as substituting elements of S for the letters from A in the word w.

Example 4.1.1. Let $A = \{a\}$, let S be a semigroup, and let $s \in S$. Define $\Phi : A^+ \longrightarrow S$ by $(a^i)\Phi = s^i$. Then Φ is clearly a homomorphism and $(a)\Phi = s$. Hence the word a is a semigroup universal word for S.

Example 4.1.2. A semigroup B is a BAND if $x^2 = x$ for all $x \in B$. Let A be a non-empty set, let B be a band, and let $x \in B$. Define $(w)\Phi = x$ for all $w \in A^+$. Then $(uv)\Phi = x = x^2 = (u)\Phi(v)\Phi$ for all $u, v \in A^+$, and so Φ is a homomorphism. Hence any word is a semigroup universal word for B.

We have seen in Example 4.1.1 that every word of length 1 is universal. It turns out that there are semigroups where the only universal words are words of length 1. A semigroup S is a ZERO SEMIGROUP if there is a distinguished element $0 \in S$ such that $xy = 0$ for all $x, y \in S$. Suppose that $|S| > 1$ and that $w \in A^+$ is of length at least 2. Then $(w)\Phi = 0$ for any homomorphism $\Phi : A^+ \longrightarrow S$, and so w is not a semigroup universal word for S.

It is worth noting that zero semigroups are not the only examples of semigroups with no word of length greater than 1 being universal. For example, free semigroups satisfy the same condition.

In the spirit of Example 4.1.2, we can ask, whether there are semigroups, other than bands, such that every word is a semigroup universal word. In the case of finite semigroups the answer is no.

Proposition 4.1.3. *Let A be an alphabet, let $a \in A$, and let S be a finite semigroup such that a^n is a semigroup universal word for S for every $n \in \mathbb{N}$. Then S is a band.*

Proof. Let $\mathcal{O} = \{\{s^m : m \geq 1\} : s \in S\}$. Then $\subseteq$ defines a partially order on $\mathcal{O}$. Since $\mathcal{O}$ is finite, it follows that there exists $s \in S$ such that $\{s^m : m \geq 1\}$ is a maximal element of $\mathcal{O}$. As a^2 is a semigroup universal word, there is a homomorphism $\Phi : A^+ \longrightarrow S$ such that $(a^2)\Phi = s$. Denote $t = (a)\Phi$. Then $t^2 = s$, and so

$$\{s^m : m \geq 1\} \subseteq \{t^m : m \geq 1\}.$$

Hence the two sets are equal, by the maximality of $\{s^m : m \geq 1\}$, and thus $t = s^m$ for some $m \geq 1$. Then $s = s^{2m}$. Since a^{2m-1} is also a semigroup universal word for S, there is $u \in S$ such that $u^{2m-1} = s$. Similarly as before, $\{u^m : m \geq 1\} = \{s^m : m \geq 1\}$ by the maximality of $\{s^m : m \geq 1\}$. Hence $u = s^k$ for some $k \geq 1$, and so $s = s^{k(2m-1)}$. If $k > 1$, then

$$s = s^{k(2m-1)} = s^{(k-1)(2m-1)-1} \cdot s^{2m} = s^{(k-1)(2m-1)}.$$

We can repeat this to show that $s = s^{2m-1}$, which together with $s = s^{2m}$, shows that $s = s^2$.

Finally, if $t \in S$ is arbitrary, then $\{t^m : m \geq 1\}$ is contained in a maximal $\{s^m : m \geq 1\} \in \mathcal{O}$ for some $s \in S$. By above $s^2 = s$, and so $\{s^m : m \geq 1\} = \{s\}$, implying that $t = s$. Therefore, $t^2 = t$ for every $t \in S$. □

Next we provide a concrete example of a non universal word.

Example 4.1.4. Let $A = \{a\}$. We will show that a^2 is not a universal word for $\mathrm{Sym}(\Omega)$ for any set Ω such that $|\Omega| \geq 2$. Suppose that a^2 is a semigroup universal word. Then there is $\Phi : A^+ \longrightarrow \mathrm{Sym}(\Omega)$ such that $(a^2)\Phi = (1\ 2)$. Hence $(1\ 2) = (a)\Phi(a)\Phi$. Note that if τ is a cycle of odd length, then τ^2 is a cycle of the same length, and similarly if τ is a cycle of length $2m$ for some $m \geq 1$, then τ^2 is a product of two disjoint cycles both of length m. Thus $|\Omega| \geq 4$ and we can write $(a)\Phi$ in disjoint cycle notation as $(1\ x\ 2\ y)\tau$ for some $x, y \in \{1, \ldots, n\}$ such that $x \neq y$ and a permutation τ in $\mathrm{Sym}(\Omega \setminus \{1, 2, x, y\})$.

However, $(a^2)\Phi = (1\ 2)(x\ y)\tau^2 \neq (1\ 2)$. Therefore, a^2 is not a semigroup universal word for $\text{Sym}(\Omega)$. A similar proof works if a^2 is replaced by a^n for any $n \geq 2$, provided that $|\Omega| \geq n$.

An analogous notion to that of the semigroup universal word can be defined for groups. Let A be an alphabet, let G be a group, and let w be an element of the free group $F(A)$, as defined in Section 1.2.2. Then w is a GROUP UNIVERSAL WORD FOR G if for every $g \in G$ there is a group homomorphism $\Phi : F(A) \longrightarrow G$ such that $(w)\Phi = g$.

The first result dealing with universal words dates back to Ore, see [58], where he proved the following theorem.

Theorem 4.1.5 (Ore's Theorem [58]). *Let $A = \{a, b\}$ and let Ω be countable. Then the word $a^{-1}b^{-1}ab$ is a group universal word for* $\text{Sym}(\Omega)$. *In other words, every element of* $\text{Sym}(\Omega)$ *is a commutator.*

Moreover, Ore also proves in the same paper that every element of the alternating group on $n \geq 5$ points is a commutator and raises the question whether this is also true for every finite non abelian simple group. The question subsequently became known as Ore's Conjecture. The conjecture was extensively studied in the literature, see [1, 18, 23, 24, 67, 71, 72, 73] and was finally proved in [47]. Universal words of other groups were also investigated, for example see [1, 54, 75].

It turns out that the non universal word for $\text{Sym}(n)$ or $\text{Sym}(\mathbb{N})$ in Example 4.1.4 is, in a sense, a canonical example of a non universal word for these groups.

Theorem 4.1.6 (see [15], [48], [57]). *Let $G = \text{Sym}(n)$ for some $n \in \mathbb{N}$ or $G = \text{Sym}(\mathbb{N})$, and let A be an alphabet. Then $w \in F(A)$ is a group universal word if and only if there is no $u \in F(A)$ and $m > 1$ such that $w = u^m$.*

In the same spirit as Theorem 4.1.6, it is conjectured that a free group word is a group universal word for the automorphism group of the random graph if and only if it is not a proper power of some other word. In [16], Droste and Truss prove some partial results to establish the conjecture. More precisely, they show that if w is an element of $F(A)$ for some alphabet A such that w is not a proper power of any element of $F(A)$, and $f \in \text{Aut}(R)$ is a certain type of element, called special in the paper, then there is a homomorphism $\Phi : F(A) \longrightarrow \text{Aut}(R)$ such that $(w)\Phi = f$. The property of being a special elements of $\text{Aut}(R)$ is strictly stronger than being a GENERIC ELEMENT (in other words, belonging to the comeagre conjugacy class). The aforementioned result is then used to show that free words of certain types are universal for $\text{Aut}(R)$, for example the word $a^{-n}b^{-1}a^m b$ for any $n, m \in \mathbb{N}$.

Let A be an alphabet, let S be a semigroup, and let $\{w_i : i \in I\} \subseteq A^+$ for some set I. Then $\{w_i : i \in I\}$ is a SEMIGROUP UNIVERSAL SEQUENCE if I

is countable and for every $\{s_i : i \in I\} \subseteq S$ there is a semigroup homomorphism $\Phi : A^+ \longrightarrow S$ such that $(w_i)\Phi = s_i$ for all $i \in I$. In the same way as with semigroup universal words, if $w_i = x_1 \dots x_n$ where $x_i \in A$, then $s_i = (w_i)\Phi = (x_1)\Phi \dots (x_n)\Phi$. So we may think of Φ as a substitution of letters from A by elements of S.

The following example shows that if Ω is a countable set, then the set of functions from Ω to Ω has a universal sequence over an alphabet of size 2. Originally the result was proven by Sierpiński [69] with Banach [5] and Hall [27] finding alternative universal sequences for Ω^Ω. Here we will include a short proof originally by James Hyde.

Example 4.1.7. Let $A = \{a, b, c\}$, and let Ω be the set of eventually constant sequences of integers, written from right to left, in other words

$$\Omega = \{(\dots, x_1, x_0) : x_i \in \mathbb{Z} \text{ and there is } K \in \mathbb{N} \text{ so that } x_i = x_K \text{ for all } i \geq K\}.$$

Let $\{f_n : n \in \mathbb{N}\} \subseteq \Omega^\Omega$ be a sequence. For every $(\dots, x_1, x_0)$ define $\alpha, \beta, \gamma \in \Omega^\Omega$ as follows

$$\begin{aligned}(\dots, x_1, x_0)\alpha &= (\dots, x_1, x_0, 0),\\ (\dots, x_1, x_0)\beta &= (\dots, x_1, x_0 + 1),\\ (\dots, x_1, x_0)\gamma &= (\dots, x_2, x_1)f_{x_0}.\end{aligned}$$

Then for any $n \in \mathbb{N}$ and arbitrary $(\dots, x_1, x_0) \in \Omega$

$$\begin{aligned}(\dots, x_1, x_0)\alpha\beta^n\gamma &= (\dots, x_1, x_0, 0)\beta^n\gamma\\ &= (\dots, x_1, x_0, n)\gamma\\ &= (\dots, x_1, x_0)f_n.\end{aligned}$$

Since $(\dots, x_1, x_0)$ was arbitrary, it follows that $\alpha\beta^n\gamma = f_n$ for all $n \in \mathbb{N}$. Let $\Phi : A^+ \longrightarrow \Omega^\Omega$ by the canonical homomorphism induced by $(a)\Phi = \alpha$, $(b)\Phi = \beta$, and $(c)\Phi = \gamma$. Then $(ab^nc)\Phi = f_n$ for all $n \in \mathbb{N}$, and so $\{ab^nc : n \in \mathbb{N}\}$ is a universal sequence for Ω^Ω over an alphabet of size 3. However, this argument can be extended further. Fix an arbitrary sequence $\{g_n : n \in \mathbb{N}\}$, let $f_0 = \beta$, and let $f_n = g_{n-1}$ for $n \geq 1$. If α, γ are defined as before, then $\alpha\gamma = \beta$, and so for all $n \in \mathbb{N}$

$$g_n = f_{n+1} = \alpha\,(\alpha\gamma)^{n+1}\,\gamma.$$

If we let $A = \{a, b\}$, and let $\Phi : A^+ \longrightarrow \Omega^\Omega$ by the canonical homomorphism induced by $(a)\Phi = \alpha$ and $(b)\Phi = \gamma$, then $(a\,(ab)^{n+1}\,b)\Phi = g_n$ for all $n \in \mathbb{N}$. Hence $\{a\,(ab)^{n+1}\,b : n \in \mathbb{N}\}$ is a semigroup universal sequence for Ω^Ω.

The definition of a semigroup universal sequence can be adapted to group in the same way as the definition of a semigroup universal word was adapted to groups. It is worth noting that if $\{w_n : n \in \mathbb{N}\} \subseteq A^+$ is a group universal sequence for a group G, then every countable subset of G is contained in $|A|$-generated subgroup of G. In was shown by Galvin in [20], that for any infinite Ω, every countable subset of $\mathrm{Sym}(\Omega)$ is contained in a 2-generated subgroup of $\mathrm{Sym}(\Omega)$. In [34], it was shown that the group of order automorphism of the rationals has a universal sequence over a 2 letter alphabet, and so every countable subset is contained in a 2-generated subgroup.

Similarly to the definition of semigroup universal sequence, we say that $\{w_i : i \in I\}$ is a SEMIGROUP UNIVERSAL LIST OF LENGTH $n \in \mathbb{N}$ if $|I| = n$ and for every $\{s_i : i \in I\} \subseteq S$ there is a semigroup homomorphism $\Phi : A^+ \longrightarrow S$ such that $(w_i)\Phi = s_i$ for all $i \in I$. Again the definition of a semigroup universal lists can be adapted to groups.

Example 4.1.8. For any $m \in \mathbb{N}$, it immediately follows from Example 4.1.7 that $\{a\,(ab)^n\,b : n \in \{1, \ldots, m\}\}$ is a semigroup universal list of length m for Ω^Ω.

The following is a very easy example of a universal list.

Example 4.1.9. Let $A = \{a_1, \ldots, a_n\}$, let $I = \{1, \ldots, m\}$ such that $m \leq n$, and let S be a semigroup. For an arbitrary $\{s_i \in S : i \in I\}$, let $\Phi : A^+ \longrightarrow S$ be the canonical homomorphism induced by $(a_i)\Phi = s_i$ for all $i \in I$ and $(a_i)\Phi = s_1$ for all $i \in \{m+1, \ldots, n\}$. Then $\{a_i : i \in I\}$ a semigroup universal list of length m for S.

For any semigroup S (or alternatively group) if the size of the alphabet A is n, then there always exists a universal list of length m for S for every $m \leq n$. In the next lemma we will show that if there is a universal list of length $n+1$ for S over an alphabet of size n, then there is a universal list of length m for S over the same alphabet for every $m \geq n+1$. Hence if we are only interested in the existence of a universal list for a given semigroup, we only need to find a universal list of length $n+1$.

Lemma 4.1.10. *Let S be a semigroup, and let A be an alphabet of size n. If S has a semigroup universal list of length at least $n+1$ over A for S, then S has a semigroup universal list of length m over A and every $m \in \mathbb{N}$.*

Proof. The existence of a semigroup universal list of length m over A for S follows immediately from Example 4.1.9 if $m \leq n$.

We will proceed to show the existence of universal lists of length $m \geq n+1$ by induction. By the hypothesis of the lemma there is $\{u_1, \ldots, u_k\}$ a semigroup universal list of length $k \geq n+1$ over A for S. Then $\{u_1, \ldots, u_{n+1}\}$ is a semigroup universal list of length $n+1$ over A for S. Suppose that for some $m \in \{n+1, n+2, \ldots\}$ and all $k \in \{n+1, \ldots, m\}$ there exists a semigroup universal list of length k over A for S.

Suppose that $\{w_1, \ldots, w_m\}$ is a universal list for S. If $A = \{a_1, \ldots, a_n\}$, for every $k \in \{1, \ldots, m\}$, we may find $i_{k,1}, \ldots, i_{k,t_k} \in \{1, \ldots, n\}$ such that $w_k = a_{i_{k,1}} \ldots a_{i_{k,t_k}}$. Define for all $k \in \{1, \ldots, m\}$

$$v_k = w_{i_{k,1}} \ldots w_{i_{k,t_k}},$$

and $v_{m+1} = w_{n+1}$. Then by definition $v_k \in A^+$ for all $k \in \{1, \ldots, m+1\}$.

Let $s_1, \ldots, s_{m+1} \in S$ be arbitrary. Then since $\{w_1, \ldots, w_m\}$ is a semigroup universal list over A for S, there is $\Phi : A^+ \longrightarrow S$ such that $(w_k)\Phi = s_k$ for $k \in \{1, \ldots, m\}$. Also there is $\Psi : A^+ \longrightarrow S$ such that $(w_k)\Psi = (a_k)\Phi$ for all $k \in \{1, \ldots, n\}$ and $(w_{n+1})\Psi = s_{m+1}$. Since Φ and Ψ are homomorphisms, it follows that for $k \in \{1, \ldots, m\}$

$$\begin{aligned}
(v_k)\Psi &= (w_{i_{k,1}})\Psi \ldots (w_{i_{k,t_k}})\Psi \\
&= (a_{i_{k,1}})\Phi \ldots (a_{i_{k,t_k}})\Phi \\
&= (w_k)\Phi = s_k,
\end{aligned}$$

and $(v_{m+1})\Psi = (w_{n+1})\Psi = s_{m+1}$. Hence $\{v_1, \ldots, v_{m+1}\}$ is a semigroup universal list of length $m+1$ over A for S, and therefore by induction there is a semigroup universal list of any length over A for S. □

In the proof of Lemma 4.1.10 the fact that the lemma was formulated for semigroups was not important. The same result (with almost identical proof) holds for groups.

The final piece of notation we will require for this chapter is defined as follows. If A is an alphabet, and $a, b \in A$, we will denote by aA^* all words in A^* starting with a, by A^*a all words in A^* ending with a, and by aA^*b all words in A^* starting with a and ending with b.

This chapter is organised in the following way: in Section 4.2 we investigate classes of semigroup universal words for Ω^Ω for a countable Ω; in Section 4.3 we prove results analogous to the ones in Section 4.2, but for semigroup universal sequences for Ω^Ω; and in Section 4.4 we prove that the automorphism group of the random graph has a universal list of any finite length over a 4 letter alphabet.

4.2 Universal words for Ω^Ω

In this section we will describe families of semigroup universal words for the transformation monoid on a countable set over a two letter alphabet. Throughout this section we assume that Ω is a countable set. However, most of the proofs can be easily adapted to higher cardinalities. We choose to only consider the countable case to keep the notation easier.

The question we are trying to answer in this section is as follows.

Question 4.2.1. *Let A be an alphabet. Is a word $w \in A^+$ a semigroup universal word for Ω^Ω?*

In Section 4.2.1 we give a sufficient condition for a word to be universal, and in Section 4.2.2 we analyse some of the words which do not satisfy the condition.

4.2.1 Classes of semigroup universal words for Ω^Ω

In Proposition 4.2.14 we will show that the only universal word for Ω^Ω over the alphabet $\{a\}$ is a. Hence in this section we will be mostly interested in universal words over a 2 letter alphabet. First of all, we prove a proposition allowing us to define a class of semigroup universal words for Ω^Ω. We note that the result was originally proved by Isbell, see [35]. We independently rediscovered the result in 2014.

Proposition 4.2.2. *Let A be an alphabet, and let $w \in A^+$ be such that every proper prefix of w is not a suffix of w. Then w is a universal word for Ω^Ω.*

Proof. Let Ω be the set of eventually constant sequences over A, written from right to left, that is

$$\Omega = \{(\dots, x_1, x_0) : x_i \in A \text{ and there is } K \in \mathbb{N} \text{ with } x_k = x_K \text{ for all } k > K\}.$$

If w is a single letter, the result is trivial, so we may assume that $|w| \geq 2$. Since no proper prefix of w is a suffix of w, it follows that the first and the last letters of w are different, and so $|A| \geq 2$. Suppose that the first letter of w is a and the last letter is b, namely $w \in aA^*b$.

Fix $f \in \Omega^\Omega$. Let $\alpha_x, \gamma \in \Omega^\Omega$, for $x \in A$, be defined as follows

$$(\dots, x_1, x_0)\alpha_x = (\dots, x_0, x)$$
$$(\dots, x_1, x_0)\gamma = \begin{cases} (\dots, x_{n+1}, x_n)f & \text{if } x_{n-1} \dots x_0 = w \\ (\dots, x_1, x_0) & \text{otherwise} \end{cases}.$$

Then define $\Phi : A^+ \longrightarrow \Omega^\Omega$ to be the canonical homomorphism induced by $(x)\Phi = \alpha_x$ for all $x \in A \setminus \{b\}$ and $(b)\Phi = \alpha_b \circ \gamma$. Let $w = y_1 \dots y_n$ where $y_i \in A$ for all $i \in \{1, \dots n\}$.

First note that $(\dots, x_1, x_0)(y_1)\Phi = (\dots, x_0, y_1)$, since $y_1 = a$. Suppose that for some $i \in \{1, \dots, n-2\}$ we have that $(\dots, x_1, x_0)(y_1 \dots y_i)\Phi = (\dots, x_0, y_1, \dots, y_i)$. Since Φ is a homomorphism

$$\begin{aligned}(\dots, x_1, x_0)(y_1 \dots y_{i+1})\Phi &= (\dots, x_1, x_0)(y_1 \dots y_i)\Phi(y_{i+1})\Phi \\ &= (\dots, x_1, x_0, y_1, \dots, y_i)(y_{i+1})\Phi.\end{aligned}$$

Suppose that $y_{i+1} \in A \setminus \{b\}$. Then $(y_{i+1})\Phi = \alpha_{y_{i+1}}$, and so

$$(\ldots, x_1, x_0)(y_1 \ldots y_{i+1})\Phi = (\ldots, x_1, x_0, y_1, \ldots, y_{i+1}),$$

as required. Suppose that $y_{i+1} = b$. Then $(y_{i+1})\Phi = \alpha_{y_{i+1}} \circ \gamma$, and so

$$(\ldots, x_1, x_0)(y_1 \ldots y_{i+1})\Phi = (\ldots, x_1, x_0, y_1, \ldots, y_{i+1})\gamma.$$

Since no proper prefix of w is equal to a suffix of w and $y_1 \ldots y_{i+1}$ is a prefix of w, it follows that $x_{n-i-2} \ldots x_0 y_1 \ldots y_{i+1} \neq w$. Hence γ acts as an identity on $(\ldots, x_1, x_0, y_1, \ldots, y_{i+1})$, and the inductive hypothesis is satisfied.

Therefore, $(\ldots, x_1, x_0)(y_1 \ldots y_{n-1})\Phi = (\ldots, x_1, x_0, y_1, \ldots, y_{n-1})$, and since $y_n = b$, it follows that

$$(\ldots, x_1, x_0)(w)\Phi = (\ldots, x_1, x_0, y_1, \ldots, y_n)\gamma = (\ldots, x_1, x_0)f.$$

Hence $(w)\Phi = f$, and we are done. □

In the next example we show that there exist words w, such that no proper prefix of w is equal to a suffix of w.

Example 4.2.3. For any $n \in \mathbb{N}$, consider the word $w = a(ab)^n b$. Then a proper prefix of w is either $a(ab)^k$ for some $k \in \{0, \ldots, n\}$ or $a(ab)^k a$ for some $k \in \{0, \ldots, n-1\}$. On the other hand, then proper suffixes are $(ab)^k b$ for some $k \in \{0, \ldots, n\}$, or $b(ab)^k b$ for some $k \in \{0, \ldots, n-1\}$. Therefore, no proper prefix is equal to a proper suffix, and so $w = a(ab)^n b$ is a semigroup universal word for Ω^Ω for all $n \in \mathbb{N}$.

However, the converse of Proposition 4.2.2 does not hold.

Example 4.2.4. Let $w = aba$. Then a is both a proper prefix and a proper suffix of w. Fix $f \in \Omega^\Omega$. If we define $\Phi : A^+ \longrightarrow \Omega^\Omega$ to be the canonical homomorphism induced by $(a)\Phi$ being the identity on Ω and $(b)\Phi = f$, then $(w)\Phi = f$.

We will now generalise Proposition 4.2.2 to a bigger class of universal words. In order to do so, we need to introduce some new concepts. For a given $w \in A^+$, consider a submonoid S of A^* such that the following two conditions are satisfied

$$\text{if there are } s, s' \in S \text{ and } u, v \in A^* \text{ such that } w = svuvs' \text{ then } v \in S; \quad (4.1)$$

$$\text{if there are } s, t, v \in A^* \text{ such that } w = svt \text{ and } sv, vt \in S \text{ then } w \in S. \quad (4.2)$$

For any given word w, there is at least one such submonoid, since A^* satisfies the conditions trivially.

Let I be a non-empty set, and for every $i \in I$ let S_i be a submonoid of A^* satisfying conditions (4.1) and (4.2). Suppose that there are $s, s' \in \bigcap_{i \in I} S_i$ and

$u, v \in A^*$ such that $w = svuvs'$. Then $v \in S_i$ by condition (4.1) for all $i \in I$. Hence $v \in \bigcap_{i \in I} S_i$, and so $\bigcap_{i \in I} S_i$ satisfies condition (4.1). Suppose that there are $s, t, v \in A^*$ such that $w = svt$ and $sv, vt \in \bigcap_{i \in I} S_i$. Then $w \in S_i$ by (4.2) for all $i \in I$. Therefore, there exists the smallest submonoid, with respect to containment, of A^* satisfying condition (4.1) and (4.2), we denote it by S_w.

There is also a more constructive way of defining S_w. Let $S_0 = \langle 1 \rangle$, and suppose that we have defined S_n, a submonoid of A^*, for some $n \in \mathbb{N}$. Let

$$\begin{gathered} X_n = \{v : w = svuvs' \text{ for some } s, s' \in S_n \text{ and } u, v \in A^*\}; \\ Y_n = \{w\} \text{ if } w = svt \text{ for some } s, v, t \in A^* \text{ so that } sv, vt \in S_n, \\ \text{and } Y_n = \varnothing \text{ otherwise}; \\ S_{n+1} = \langle S_n, X_n, Y_n \rangle. \end{gathered} \tag{4.3}$$

Then $S_0 \leq S_1 \leq S_2 \leq \ldots$ by definition of S_{n+1}. Let $S = \bigcup_{n \in \mathbb{N}} S_n$. If $s, t \in S$, then there are $n, m \in \mathbb{N}$ such that $s \in S_n$ and $t \in S_m$. Then $s, t \in S_{\max(s,t)}$, and so $st \in S_{\max(s,t)} \subseteq S$. Hence S is a submonoid of A^*.

Suppose that $w = svuvs'$ for some $s, s' \in S$ and $u, v \in A^*$. Then by definition of S and the fact that $S_0 \leq S_1 \leq \ldots$ there is some $n \in \mathbb{N}$ such that $s, s' \in S_n$. Hence $v \in X_n \subseteq S_{n+1} \subseteq S$, and so S satisfies condition (4.1). Suppose that $w = svt$ for some $s, v, t \in A^*$ and $sv, vt \in S$. Then by definition of S and the fact that $S_0 \leq S_1 \leq \ldots$ there is some $n \in \mathbb{N}$ such that $sv, vt \in S_n$. Hence $w \in Y_n \subseteq S_{n+1} \subseteq S$, and so S satisfies condition (4.2).

Let T be a submonoid of A^* satisfying conditions (4.1) and (4.2). Since $S_0 = \langle 1 \rangle$, it follows that $S_0 \leq T$. Suppose that $S_n \leq T$ for some $n \in \mathbb{N}$. If $w = svuvs'$ such that $s, s' \in S_n$ and $u, v \in A^*$, then $v \in T$, since T satisfies condition (4.1) and $s, s' \in T$. Hence $X_n \subseteq T$. If $w = svt$ such that $s, v, t \in A^*$ and $sv, vt \in S_n$, then it follows from the fact that T satisfies condition (4.2), that $w \in T$, and so $Y_n \subseteq T$, and so $S_{n+1} \subseteq T$. Then by induction $S_n \leq T$ for all $n \in \mathbb{N}$, and thus $S \leq T$. Therefore S is the minimal submonoid of A^* satisfying conditions (4.1) and (4.2), in other words $S = S_w$.

It follows from the definition of S_n that if $S_k = S_{k+1}$ for some $k \in \mathbb{N}$, then $S_k = S_n$ for all $k \leq n$. Let $K \in \mathbb{N}$ be the number of distinct non-empty subwords of w. It follows from the definition of S_{n+1} that if $S_{n+1} \gneq S_n$, then there is a non-empty subword v of w such that $v \in S_{n+1} \setminus S_n$. Therefore, $S_K = S_n$ for all $n \geq K$, and so

$$S_w = \bigcup_{n=1}^{K} S_n = S_K.$$

Note that K is bounded from above by $\sum_{i=1}^{|w|} |w| - (i-1) = |w|(|w|+1)/2$. Hence $S_w = S_{|w|(|w|+1)/2}$. Thus there is an algorithm which given word w computes S_w, more precisely a set generating S_w, in finite time.

Proposition 4.2.5. *The monoid S_w is finitely generated.*

Proof. By (4.3) the monoid S_w is generated by

$$\bigcup_{n\in\mathbb{N}} X_n \cup Y_n.$$

For all $n \in \mathbb{N}$, both X_n and Y_n are sets of non-empty subwords of w. However, there are only finitely many subwords of w, and so the generating set is finite. □

Next we present a couple of examples where we compute S_w for a given w.

Example 4.2.6. Let $w \in A^+$ be word such that no proper prefix of w is a proper suffix. Suppose that $w = svuvs'$ such that $s, s' \in S_0$ and $u, v \in A^*$. By definition $S_0 = \langle 1 \rangle$, and so $s = s' = 1$. Since any proper prefix of w is not a suffix of v, it also follows that $v = 1$. Hence $X_0 = \{1\}$.

Suppose that $w = svt$ such that $s, v, t \in A^*$ and $sv, vt \in S_0$. Then $sv = vt = 1$, and so $s = v = t = 1$, which is only possible if $w = 1$. Hence $Y_n = \varnothing$, and so $S_1 = S_0$. Therefore $S_w = \langle 1 \rangle$.

Example 4.2.7. Let $w = aba^2b$. Suppose that $w = svuvs'$ such that $s, s' \in S_0$ and $u, v \in A^*$. Since $S_0 = \langle 1 \rangle$, we have that $s = s' = 1$, and so $aba^2b = vuv$. Hence $v \in \{1, ab\}$, implying that $X_0 = \{1, ab\}$. Suppose that $w = svt$ such that $s, v, t \in A^*$ and $sv, vt \in S_0$. Then $sv = vt = 1$, and so $s = v = t = 1$, which is only possible if $w = 1$. Hence $Y_0 = \varnothing$, and so $S_1 = \{(ab)^n : n \in \mathbb{N}\}$.

Let $s = 1$, $s' = ab$, $v = a$, and $u = b$. Then $svuvs' = w$, and so $v = a \in S_2$, by (4.1) since $s, s' \in S_1$.

Finally, let $s = a$, $s' = 1$, $v = b$, and $u = a^2$. Then $svuvs' = w$, and so $b \in S_3$ by (4.1) since $s, s' \in S_2$. Therefore $S_w = A^*$.

Example 4.2.8. Let $w = aba^2b^2ab$. Suppose that $w = svuvs'$ such that $s, s' \in S_0$ and $u, v \in A^*$. Since $S_0 = \langle 1 \rangle$, we have that $s = s' = 1$, and so $aba^2b^2ab = vuv$. Hence $v \in \{1, ab\}$, implying that $X_0 = \{1, ab\}$. If $w = svt$ such that $s, v, t \in A^*$ and $sv, vt \in S_0$, then $s = v = t = 1$, which is impossible. Hence $Y_0 = \varnothing$, and so $S_1 = \langle ab \rangle = \{(ab)^n : n \in \mathbb{N}\}$.

Suppose that $w = svuvs'$ such that $s, s' \in S_1$ and $u, v \in A^*$. Then $s, s' \in \{1, ab\}$. If we consider the four different possibilities, it is easy to see that $v \neq 1$ only if $s = s' = 1$, in which case $v = ab$. Hence $X_2 = X_1$. Suppose that $w = svt$ such that $s, v, t \in A^*$ and $sv, vt \in S_1$. It then follows that $sv, vt \in \{1, ab\}$. If $sv = 1$, then $v = 1$ and $t = w$, and so $vt = w \neq ab$, which is impossible. If $sv = ab$, then $t = a^2b^2ab$, and so $vt \notin \{1, ab\}$, again contradicting the fact that $vt \in \{1, ab\}$. Hence $Y_1 = \varnothing$, thus $S_2 = S_1$, and so $S_w = \{(ab)^n : n \in \mathbb{N}\}$.

In Theorem 4.2.12, we will use the notion of S_w to provide a sufficient condition for a word w to be universal for Ω^Ω. Before stating the main result of this section, we will prove a couple of technical results concerning S_w.

Lemma 4.2.9. *Let A be an alphabet, and let $w \in A^+$ be such that $S_w \neq \langle 1 \rangle$. Then there is $b \in S_w$ such that b is a proper prefix and a proper suffix of w.*

Proof. Recall that $S_w = \bigcup_{n\in\mathbb{N}} S_n$ where $S_0 = \{1\}$, and S_n is as in (4.3). Let $s, s' \in S_0$ and $u, v \in A^*$ such that $w = svuvs'$. Then $s = s' = 1$, and so $w = vuv$. If $v \neq 1$, then $v \in S_w$ and it is both a proper prefix and a proper suffix of w.

Otherwise, $v = 1$, and so $X_0 = \{1\}$. Let $w = svt$ such that $s, v, t \in A^*$ and $sv, vt \in S_0$. Then $sv = vt = 1$, and so $s = v = t = 1$, which is only possible if $w = 1$. Hence $Y_0 = \varnothing$, implying that $S_1 = \{1\}$. □

Lemma 4.2.10. *Let $A = \{a, b\}$, and let $w \in aA^*b$ be such that $a, b \notin S_w$. Then $S_w \subseteq aA^*b \cup \{1\}$.*

Proof. Recall that $S_w = \bigcup_{n\in\mathbb{N}} S_n$, where S_n is as in (4.3). Then $S_0 = \{1\} \subseteq aA^*b \cup \{1\}$. Suppose that for some $n \in \mathbb{N}$ we have that $S_n \subseteq aA^*b \cup \{1\}$.

Suppose that $w = svuvs'$ such that $s, s' \in S_n$ and $u, v \in A^*$. If $v \in A^*a$, then we may write $v = v'a$ for some $v' \in A^*$, and since w starts with a, $w = atv'as'$ for some $t \in A^*$. Hence $a \in X_n$ by (4.1) which contradicts the fact that $a \notin S_w$. If $v \in bA^*$, then similarly $w = sbv'tb$ for some $v', t \in A^*$. Since $s \in S_w$, it follows that $b \in X_n \subseteq S_w$ which is a contradiction. Hence $v \in aA^*b$, and so $X_n \subseteq aA^*b \cup \{1\}$.

By the definition Y_n is either $\{w\}$ or $\varnothing$. In both cases $Y_n \subseteq aA^*b \cup \{1\}$. Hence $S_{n+1} \subseteq aA^*b \cup \{1\}$. Therefore, by induction on n, $S_n \subseteq aA^*b \cup \{1\}$ for all $n \in \mathbb{N}$, implying that $S_w \subseteq aA^*b \cup \{1\}$. □

The last technical result we need, shows that if the word w has a shared prefix and suffix, then there is a shared prefix and suffix which is at most half the length of the original word.

Lemma 4.2.11. *Let A be an alphabet, and let $w \in A^+$ and suppose there exists $v \in A^+$ such that v is both a proper prefix and a proper suffix of w. Then there are $q \in A^+$ and $u \in A^*$ such that $w = quq$.*

Proof. Let q be the shortest subword of w which is both a prefix and a suffix of w. Such word must exist by the hypothesis. Suppose that there is no $u \in A^*$ such that $w = quq$. Then there are $r, s, t \in A^+$ such that $w = rst$ and $q = rs = st$. Hence s is both a prefix and a suffix of w. However, s is a word shorter than q, contradicting the assumption. Hence $w = quq$, for some $u \in A^*$. □

Suppose that A is an alphabet and $w \in A^+$ is such that $w \notin S_w$. Let $p \in A^*$ be the longest prefix of w such that $p \in S_w$, and let $s \in A^*$ be the longest suffix of w such that $s \in S_w$. Then either there is $u \in A^+ \setminus S_w$ such that $w = pus$, or there are $u, v, t \in A^*$ such that $w = uvt$, $p = uv$ and $s = vt$. In the latter case, $w \in S_w$ by (4.2). Therefore,

$$w = pus,$$

for some $u \in A^+ \setminus S_w$.

The following is the main result of the section.

Theorem 4.2.12. *Let $w \in \{a,b\}^+$ be such that $w \notin S_w$. Let $p, u, s \in \{a,b\}^*$ be such that $w = pus$, and p and s are respectively the longest prefix and the longest suffix of w such that $p, s \in S_w$. Suppose that u is not a subword of p. Then w is a semigroup universal word for Ω^Ω.*

Proof. Denote the alphabet $\{a,b\}$ by A. Suppose that $a \in S_w$. Then $w \neq a^i$ for any $i \in \mathbb{N}$, since $w \notin S_w$. So either $w = a^iba^j$ for some $i,j \in \mathbb{N}$, or $w = a^ibuba^j$ for some $i,j \in \mathbb{N}$ and $u \in A^*$. In the latter case (4.1) implies that $b \in S_w$. Hence $S_w = A^*$, which contradicts the assumption that $w \notin S_w$. Suppose that $w = a^iba^j$. Then $a \in S_w$ by (4.1), and so $S_w \supseteq \{a^n : n \in \mathbb{N}\}$. Moreover $\{a^n : n \in \mathbb{N}\}$ satisfies both (4.1) and (4.2), and so is the minimal such monoid. Hence $S_w = \{a^n : n \in \mathbb{N}\}$. Fix $f \in \Omega^\Omega$ and define $\Phi : A^* \longrightarrow \Omega^\Omega$ to be the canonical homomorphism induced by $(a)\Phi$ being the identity on Ω and $(b)\Phi = f$. Then $(w)\Phi = f$, and so w is a universal word for Ω^Ω. The proof for $b \in S_w$ is identical.

Thus it is sufficient to consider only the case where $a, b \notin S_w$. It follows from (4.1) that the first and the last letters of w must be different, in other words $w \in aA^*b$ or $w \in bA^*a$. Without the loss of generality, assume that $w \in aA^*b$. Then $S_w \subseteq aA^*b \cup \{1\}$ by Lemma 4.2.10.

Let Ω be the set of eventually constant sequences over $F(A)$, written from right to left, namely

$$\Omega = \{(\ldots, x_1, x_0) : x_i \in F(A) \text{ and there is } K \in \mathbb{N} \text{ such that } x_K = x_k \text{ for all } k \geq K\}.$$

Note that by Lemma 4.2.9 either both $p \neq 1$ and $s \neq 1$, or $S_w = \langle 1 \rangle$, and so $p = s = 1$. We proceed by proving a series of claims.

Claim 1. $u \in aA^*b$.

Proof. If $p = s = 1$, then $u = w \in aA^*b$. Suppose that $p \neq 1$ and $s \neq 1$. Since $w \in aA^*b$, there are $p', s' \in A^*$ such that $p = ap'$ and $s = s'b$. If $u = bv$ for some $v \in A^*$, then $w = pbvs'b$, and so $b \in S_w$ by (4.1) and the fact that $p \in S_w$. Similarly, if $u = va$ for some $v \in A^*$, then $w = ap'vas$, and so $a \in S_w$ by (4.1) and the fact that $s \in S_w$. However, both cases are impossible since $a, b \notin S_w$. Therefore $u \in aA^*b$. □

By Proposition 4.2.5 there is a finite generating set X' of S_w. Let $X \subseteq X'$ be an irredundant generating set of S_w, that is if $v \in X$, then $\langle X \setminus \{v\} \rangle \neq \langle X \rangle$.

Claim 2. For each $v \in X$, there are $t, t' \in S_w$ such that tv is a prefix of p, and vt' is a suffix of s.

Proof. Let X_n and Y_n be as in (4.3). Note that $Y_n = \varnothing$ for all $n \in \mathbb{N}$, as $w \notin S_w$. Then it follows from the proof of Proposition 4.2.5 that $X \subseteq \bigcup_{n\in\mathbb{N}} X_n$.

Let $n \in \mathbb{N}$. If $v \in X \cap X_n$, then $w = tvqvt'$ where $t, t' \in S_n$ and $q \in A^*$. Hence $tv, vt' \in S_w$, and so it then follows from the maximality of p and s that tv is a prefix of p, and vt' is a suffix of s. □

Claim 3. For all $v \in X$, a prefix of v is not a suffix of u, and a suffix of v is not a prefix of u.

Proof. Let $v \in X$ be arbitrary. Then by Claim 2 there are $t, t' \in S_w$ such that tv is a prefix of p and vt' is a suffix of s. Hence there are $r, r' \in A^*$ such that $w = tvrur'vt'$. If q is a prefix of v and a suffix of u, or a suffix of v and prefix of u, then $q \in S_w$, which contradicts the maximality of p and s. □

Claim 4. For every $v, v' \in X$, if a non-trivial prefix q of v is a suffix of v', then $q = v = v'$.

Proof. Let $v, v' \in X$ be arbitrary. Suppose that $v = qr$ and $v' = r'q$ for some $r, r', q \in A^*$ and $q \neq 1$. By Claim 2 there are $t, t' \in S_w$ such that tv is a prefix of p, and $v't'$ is a suffix of s. Then there is $h \in A^*$ such that $w = tvhv't' = tqrhr'qt'$, implying that $q \in S_w$. Since $v \in X$, by Claim 2 there are $l, l' \in S_w$ and $d \in A^*$ such that $w = lvdvl' = lqrdqrl'$. It follows from the fact that $lq, l' \in S_w$ that $r \in S_w$. Since X is irredundant and $q \neq 1$, it follows that $q = v$ and $r = 1$. The same argument for v' implies that $r' = 1$, and so $q = v = v'$. □

Let $f \in \Omega^\Omega$. We will construct a homomorphism $\Phi : A^+ \to \Omega^\Omega$ such that $(w)\Phi = f$. In order to do that we will need some auxiliary functions $\alpha, \beta, \gamma \in \Omega^\Omega$ defined as follows

$$(\dots, x_1, x_0)\alpha = (\dots, x_0, a),$$
$$(\dots, x_1, x_0)\beta = (\dots, x_0, b),$$

and

$$(\dots, x_1, x_0)\gamma = \begin{cases} (\dots, x_{i+1}, x_i v) & \text{if } x_{i-1}\dots x_0 = v \in X \text{ for some } i \geq 1 \\ & \text{and } x_j \in A^+ \text{ for all } j \in \{0, \dots i-1\} \\ (\dots, x_1, x_0) & \text{otherwise} \end{cases}.$$

Suppose there are $i, i' \in \mathbb{N}$ such that $x_{i-1}\dots x_0 = v$ and $x_{i'-1}\dots x_0 = v'$ for some $v, v' \in X$, where $x_j \in A^+$ for all $j \in \{0, \dots, \max(i, i') - 1\}$. Then either v' is a suffix of v or v is a suffix of v'. By Claim 4 it is only possible if $v = v'$, and so γ is well-defined. In order to define the homomorphism Φ, we will first define another homomorphism. Let $\Psi : A^+ \longrightarrow \Omega^\Omega$ be the canonical homomorphism induced by $(a)\Psi = \alpha$ and $(b)\Psi = \beta \circ \gamma$.

Claim 5. For any $v \in aA^*$ such that no prefix of v is a suffix of a word in X there are $z_1, \ldots, z_k \in A^+$ such that $z_1 \ldots z_k = v$ and for every $(\ldots, x_1, x_0) \in \Omega$

$$(\ldots, x_1, x_0)\,((v)\Psi) = (\ldots, x_1, x_0, z_1, \ldots, z_k).$$

Proof. Let $v \in aA^*$such that no prefix of v is a suffix of a word in X, and write $v = y_1 \ldots y_m$ for some $m \in \mathbb{N}$ and $y_1, \ldots, y_m \in A$. Since $v \in aA^*$, $(y_1)\Psi = \alpha$, and so for all $(\ldots, x_1, x_0) \in \Omega$

$$(\ldots, x_1, x_0)\alpha = (\ldots, x_1, x_0, a) = (\ldots, x_1, x_0, y_1).$$

Suppose that for some $i \in \{1, \ldots, m-2\}$ there are $j \in \mathbb{N}$ and $z_1, \ldots z_j \in A^+$ such that $y_1 \ldots y_i = z_1 \ldots z_j$, and for every $(\ldots, x_1, x_0) \in \Omega$

$$(\ldots, x_1, x_0)\,((y_1 \ldots y_i)\Psi) = (\ldots, x_1, x_0, z_1, \ldots, z_j).$$

If $y_{i+1} = a$, then since Ψ is a homomorphism

$$\begin{aligned}(\ldots, x_1, x_0)\,((y_1 \ldots y_{i+1})\Psi) &= (\ldots, x_1, x_0, z_1, \ldots, z_j)\,((y_{i+1})\Psi) \\ &= (\ldots, x_1, x_0, z_1, \ldots, z_j)\alpha \\ &= (\ldots, x_1, x_0, z_1, \ldots, z_j, a),\end{aligned}$$

and $z_1 \ldots z_j a = y_1 \ldots y_{i+1}$. Hence the inductive hypothesis is satisfied in this case. If $y_{i+1} = b$, then since Ψ is a homomorphism

$$\begin{aligned}(\ldots, x_1, x_0)\,((y_1 \ldots y_{i+1})\Psi) &= (\ldots, x_1, x_0, z_1, \ldots, z_j)\,((y_{i+1})\Psi) \\ &= (\ldots, x_1, x_0, z_1, \ldots, z_j)\beta \circ \gamma \\ &= (\ldots, x_1, x_0, z_1, \ldots, z_j, b)\gamma,\end{aligned}$$

and $z_1 \ldots z_j b = y_1 \ldots y_{i+1}$, and so $z_1 \ldots z_j b$ is a prefix of v. By the assumption $z_1 \ldots z_j b$ is not a suffix of any word in X, thus $z_1 \ldots z_j b \notin X$, and if $x_0, \ldots, x_t \in A^+$ then $x_t \ldots x_0 z_1 \ldots z_j b \notin X$ for all $t \in \mathbb{N}$. Hence either γ acts as identity on $(\ldots, x_1, x_0, z_1, \ldots, z_j, b)$, or there is $k > 1$ such that $z_k \ldots z_j b \in X$. In the latter case

$$\begin{aligned}(\ldots, x_1, x_0)\,((y_1 \ldots y_{i+1})\Psi) &= (\ldots, x_1, x_0, z_1, \ldots, z_j, b)\gamma \\ &= (\ldots, x_1, x_0, z_1, \ldots, z_{k-1} z_k \ldots z_j b),\end{aligned}$$

and so in both cases the inductive hypothesis is satisfied. The claim holds by induction. ∎

Claim 6. If $v \in S_w$, then $(v)\Psi$ is a bijection on Ω, in particular

$$(\ldots, x_1, x_0)(v)\Psi = (\ldots, x_1, x_0 v)$$

for all $(\ldots, x_1, x_0) \in \Omega$.

Proof. Let $v \in X$. Since $S_w \subseteq aA^*b \cup \{1\}$ there is $v' \in aA^*$ such that $v = v'b$. By Claim 4 any prefix of v' is not a suffix of any word in X. Hence by Claim 5 there exists $j \in \mathbb{N}$ and $z_1, \ldots, z_j \in A^+$ such that $z_1 \ldots z_j = v'$ and for all $(\ldots, x_1, x_0) \in \Omega$

$$(\ldots, x_1, x_0)\,((v')\Psi) = (\ldots, x_1, x_0, z_1, \ldots, z_j).$$

It follows from $v = v'b$ and the fact that Ψ is a homomorphism, that

$$\begin{aligned}(\ldots, x_1, x_0)\,((v)\Psi) &= (\ldots, x_1, x_0, z_1, \ldots, z_j, b)\gamma \\ &= (\ldots, x_1, x_0 v).\end{aligned} \tag{4.4}$$

In order to show that $(v)\Psi$ is injective, suppose that $(\ldots, x_1, x_0 v) = (\ldots, x_1', x_0' v)$ where $x_i, x_i' \in F(A)$ for all $i \in \mathbb{N}$. Then $x_i = x_i'$ for all $i \geq 1$, and $x_0 v = x_0' v$. Since $x_0 v, x_0' v \in F(A)$, it follows that $x_0 = x_0'$. Hence $(v)\Psi$ is injective by (4.4). Let $(\ldots, x_1, x_0) \in \Omega$. Then by (4.4)

$$(\ldots, x_1, x_0 v^{-1})\,((v)\Psi) = (\ldots, x_1, x_0)$$

so $(v)\Psi$ is surjective, and thus bijective on Ω. Since $v \in X$ was arbitrary and X generates S_w, it follows that

$$(\ldots, x_1, x_0)\,((v)\Psi) = (\ldots, x_1, x_0 v),$$

and $(v)\Psi$ is a bijection for all $v \in S_w$. □

In order to define the required homomorphism Φ, we need one more function on Ω. Define $\delta \in \Omega^\Omega$ as follows:

$$(\ldots, x_1, x_0)\delta = \begin{cases} (\ldots, x_{i+1}, x_i p^{-1})f \circ (s)\Psi^{-1} & \text{if } x_{i-1} \ldots x_0 = u \text{ for some } i \geq 1 \text{ and } x_j \in A^+ \text{ for all } j \in \{0, \ldots i-1\} \\ (\ldots, x_1, x_0) & \text{otherwise} \end{cases}.$$

Note that $(s)\Psi^{-1}$ is defined by Claim 6. Suppose there are $i, i' \in \mathbb{N}$ such that $x_{i-1} \ldots x_0 = u = x_{i'-1} \ldots x_0$, where $x_j \in A^+$ for all $j \in \{0, \ldots, \max(i, i') - 1\}$. Then $i = i'$, and so δ is well-defined. Let Φ be the canonical homomorphism induced by $(a)\Phi = \alpha$ and $(b)\Phi = \beta \circ \gamma \circ \delta$.

Claim 7. $(v)\Phi = (v)\Psi$ for all $v \in S_w$.

Proof. Let $v \in X$ be arbitrary, and suppose that $v = y_1 \dots y_m$ where $y_i \in A$ for all $i \in \{1, \dots, m\}$. It follows from $S_w \subseteq aA^*b \cup \{1\}$ and $v \neq 1$, that $y_1 = a$, implying that $(y_1)\Phi = \alpha = (y_1)\Psi$. Suppose $(y_1 \dots y_i)\Phi = (y_1 \dots y_i)\Psi$ for some $i \in \{1, \dots, m-1\}$. Then since Φ is a homomorphism

$$(y_1 \dots y_{i+1})\Phi = (y_1 \dots y_i)\Psi \circ (y_{i+1})\Phi.$$

If $y_{i+1} = a$ then $(y_{i+1})\Phi = (y_{i+1})\Psi$, and so the inductive hypothesis is satisfied. Suppose that $y_{i+1} = b$. Then $(y_{i+1})\Phi = (y_{i+1})\Psi \circ \delta$. Hence by the inductive hypothesis

$$(y_1 \dots y_{i+1})\Phi = (y_1 \dots y_i)\Phi \circ (y_{i+1})\Phi = (y_1 \dots y_i)\Psi \circ (y_{i+1})\Phi = (y_1 \dots y_{i+1})\Psi \circ \delta.$$

If $i + 1 < m$, then $y_1 \dots y_{i+1}$ is a proper prefix of v. By Claim 4 for any $j \in \{1, \dots, i+1\}$ the proper prefix $y_1 \dots y_j$ of v is a not a suffix of any word in X. Since $y_1 \dots y_{i+1} \in aA^*$, by Claim 5 there exists $j \in \mathbb{N}$ and $z_1, \dots, z_j \in A^+$ such that $z_1 \dots z_j = y_1 \dots y_{i+1}$ and for all $(\dots, x_1, x_0) \in \Omega$

$$(\dots, x_1, x_0)\,((y_1 \dots y_{i+1})\Psi) = (\dots, x_1, x_0, z_1, \dots, z_j).$$

If $i + 1 = m$, then $y_1 \dots y_{i+1} = v$, and so

$$(\dots, x_1, x_0)\,((y_1 \dots y_{i+1})\Psi) = (\dots, x_1, x_0 y_1 \dots y_{i+1})$$

for all $(\dots, x_1, x_0) \in \Omega$ by Claim 6. Hence in both cases there are $z_0, \dots, z_j \in A^+$ such that $z_0 \dots z_j = y_1 \dots y_{i+1}$ and for all $(\dots, x_1, x_0) \in \Omega$

$$(\dots, x_1, x_0)\,((y_1 \dots y_{i+1})\Psi) = (\dots, x_1, x_0 z_0, z_1, \dots, z_j). \tag{4.5}$$

We will now show that δ acts as the identity on $(\dots, x_1, x_0)\,((y_1 \dots y_{i+1})\Psi)$ for all $(\dots, x_1, x_0) \in \Omega$. Fix $(\dots, x_1, x_0) \in \Omega$, and let $z_0, \dots, z_j \in A^+$ be as in (4.5). Suppose that there is $i \geq 0$ such that $x_i, \dots, x_1, x_0 z_0 \in A^+$ and $x_i \dots x_0 z_0 \dots z_j = u$. Then $z_0 \dots z_j = y_1 \dots y_{i+1}$ is both a prefix of v and a suffix of u, contradicting Claim 3. By Claim 2 there is $t \in S_w$ such that tv is a prefix of p, and by the hypothesis of the theorem, it follows that u is not a subword of v. If $k > 0$ then $z_k \dots z_j$ is a subword of v, and so not equal to u. Hence δ acts as identity on $(\dots, x_1, x_0 z_0, z_1, \dots, z_j)$. Therefore, the inductive hypothesis is satisfied, and so by induction $(v)\Phi = (v)\Psi$ for all $v \in X$, and so $(v)\Phi = (v)\Psi$ for all $v \in S_w$. □

Claim 8. $(u)\Phi = (u)\Psi \circ \delta$.

Proof. Let $u = y_1 \dots y_m$ for some $m \in \mathbb{N}$ and $y_i \in A$ for all $i \in \{1, \dots, m\}$. First, we will show inductively that $(y_1 \dots y_{m-1})\Phi = (y_1 \dots y_{m-1})\Psi$. Since $y_1 = a$ by Claim 1, it follows that $(y_1)\Phi = \alpha = (y_1)\Psi$. Suppose $(y_1 \dots y_i)\Phi = (y_1 \dots y_i)\Psi$ for some $i \in \{1, \dots, m-2\}$. Then since Φ is a homomorphism

$$(y_1 \dots y_{i+1})\Phi = (y_1 \dots y_i)\Psi \circ (y_{i+1})\Phi.$$

If $y_{i+1} = a$ then $(y_{i+1})\Phi = (y_{i+1})\Psi$, and so the inductive hypothesis is satisfied.

Suppose that $y_{i+1} = b$, then $(y_{i+1})\Phi = (y_{i+1})\Psi \circ \delta$. Hence $(y_1 \dots y_{i+1})\Phi = (y_1 \dots y_{i+1})\Psi \circ \delta$. It follows from Claim 3 that no prefix of $y_1 \dots y_{i+1}$ is a suffix of any word $v \in X$, and since $y_1 = a$, Claim 5 implies that there is $j \in \mathbb{N}$ and $z_1, \dots, z_j \in A^+$ such that $z_1 \dots z_j = y_1 \dots y_{i+1}$ and

$$(\dots, x_1, x_0)\left((y_1 \dots y_{i+1})\Psi\right) = (\dots, x_1, x_0, z_1, \dots, z_j).$$

Since $|y_1 \dots y_{i+1}| < |u|$, it follows that u is not equal to $z_k \dots z_j$ for any $k \geq 1$. Suppose that $u = x_k \dots x_0 z_1 \dots z_j$ for some $k \geq 0$ such that $x_0, \dots, x_k \in A^+$. Then $y_1 \dots y_{i+1}$ is both a proper prefix and a suffix of u, and so by Lemma 4.2.11 there is a $q \in A^+$ and $t \in A^*$ such that $u = qtq$. Hence $q \in S_w$ by (4.1) contradicting maximality of p and s. Then the inductive hypothesis is satisfied.

Hence $(y_1 \dots y_{m-1})\Phi = (y_1 \dots y_{m-1})\Psi$, and since $y_m = b$

$$(u)\Phi = (u)\Psi \circ \delta.$$

□

Finally, we show that $(w)\Phi = f$. It follows from Claim 6, Claim 7, Claim 8, and the fact that Φ is a homomorphism, that for all $(\dots, x_1, x_0) \in \Omega$

$$\begin{aligned}(\dots, x_1, x_0)(w)\Phi &= (\dots, x_1, x_0)\left((p)\Psi \circ (u)\Psi \circ \delta \circ (s)\Psi\right)\\ &= (\dots, x_1, x_0p)\left((u)\Psi \circ \delta \circ (s)\Psi\right)\end{aligned}$$

It follows from Claim 3 and Claim 5 that there are $z_1, \dots, z_k \in A^+$ such that $z_1 \dots z_k = u$ and

$$\begin{aligned}(\dots, x_1, x_0)(w)\Phi &= (\dots, x_1, x_0p)\left((u)\Psi \circ \delta \circ (s)\Psi\right)\\ &= (\dots, x_1, x_0p, z_1, z_2, \dots, z_k)\delta \circ (s)\Psi\end{aligned}$$

Then by the definition of δ

$$\begin{aligned}(\dots, x_1, x_0)(w)\Phi &= (\dots, x_1, x_0p, z_1, z_2, \dots, z_k)\delta \circ (s)\Psi\\ &= (\dots, x_1, x_0)f \circ (s)\Psi^{-1} \circ (s)\Psi\\ &= (\dots, x_1, x_0)f.\end{aligned}$$

Therefore $(w)\Phi = f$ as required. $\square$

Note that if $w \in \{a, b\}^+$ is such that no proper prefix of w is a suffix of w, then $S_w = \langle 1 \rangle$ by Example 4.2.6. Hence $w \notin S_w$ and Proposition 4.2.2 is now an easy corollary of Theorem 4.2.12. The following example shows that Theorem 4.2.12 is actually stronger than Proposition 4.2.2.

Example 4.2.13. Let $w = aba^2b^2ab$. It was shown in Example 4.2.8 that $S_w = \{(ab)^n : n \in \mathbb{N}\}$. Hence w is a universal word for Ω^Ω by Theorem 4.2.12, but the word ab is both a prefix and a suffix of w.

In the next section, in Proposition 4.2.15, we will demonstrate that the word $(ab)^n a$ is universal for Ω^Ω. However, note that $S_w = \{a, b\}^*$, and so the converse of Theorem 4.2.12 does not hold.

4.2.2 Analysis of some the words not covered by Theorem 4.2.12

In this section we will analyse some of the words w over the alphabet $A = \{a, b\}$ such that $w \in S_w$. In order to do that, we will use the notions of collapse and defect defined in Section 1.3.2. Recall that if $f \in \Omega^\Omega$ and if Σ is any transversal of f, then

$$c(f) = |\Omega \setminus \Sigma|$$
$$d(f) = |\Omega \setminus (\Omega)\, f|$$

First consider the case where $w = a^n$ for some $n \in \mathbb{N}$.

Proposition 4.2.14. *The word a^n is universal for Ω^Ω if and only if $n = 1$.*

Proof. Let $f \in \Omega^\Omega$, and let $\Phi : \{a\}^+ \longrightarrow \Omega^\Omega$ be given by $(a^m)\Phi = f^m$ for all $m \geq 1$. Then Φ is a homomorphism and $(a)\Phi = f$. Hence a is a semigroup universal word for Ω^Ω.

On the other hand, suppose that a^n is universal for $n \geq 1$. Let $f \in \Omega^\Omega$ be such that $c(f) = 1$ and $d(f) = 0$, which exists by Proposition 1.3.4. Let $\Phi : \{a\}^+ \longrightarrow \Omega^\Omega$ by the homomorphism such that $(a^n)\Phi = f$, and let $(a)\Phi = \alpha$. Then $\alpha^n = f$. It follows from parts (i) and (iii) of Lemma 1.3.5 that $c(\alpha) \leq c(f) = 1$ and $d(\alpha) \leq d(f) = 0$. Hence by Lemma 1.3.7

$$1 = c(f) - d(f) = n\,(c(\alpha) - d(\alpha))\,.$$

Therefore, $n \mid 1$, and so $n = 1$. $\square$

The next step is to consider all the words w which start and finish with the same letter. That is, words $w \in \{a, b\}^+$ such that $a \in S_1$ using the notation of (4.3).

Proposition 4.2.15. *Let $A = \{a, b\}$ and let $w \in aA^+a$ be such that $w \neq a^n$ for any $n \in \mathbb{N}$. Then w is a universal word for Ω^Ω if and only if $w = (ab)^n a$ or $w = a^n ba^m$ for some $n, m \in \mathbb{N}$.*

Proof. ($\Rightarrow$) Suppose that w is a universal word for Ω^Ω such that $w \neq a^n ba^m$ for any $n, m \in \mathbb{N}$. Then there are at least two occurrences of the letter b in the word w.

Fix any $f \in \Omega^\Omega$ such that $c(f), d(f)$ are finite and $|c(f) - d(f)| = 1$. Then there is $\Phi : A^+ \longrightarrow \Omega^\Omega$ a homomorphism such that $(w)\Phi = f$. Let $\alpha = (a)\Phi$ and $\beta = (b)\Phi$. Since w starts and ends with the letter a, there is $\gamma \in \Omega^\Omega$ such that $f = \alpha \circ \gamma \circ \alpha$. It follows from parts (i) and (iii) of Lemma 1.3.5 that $c(\alpha) \leq c(f)$ and $d(\alpha) \leq d(f)$. Hence both $c(\alpha)$ and $d(\alpha)$ are finite. Since the letter b occurs at least twice in the word w, there are $n, m \in \mathbb{N}$ and $u \in A^*$ such that $w = a^n buba^m$. Then there is $\gamma' \in \Omega^\Omega$ such that $f = \alpha^n \circ \beta \circ \gamma' \circ \beta \circ \alpha^m$. It follows from Lemma 1.3.5(vi) that $c(\beta \circ \gamma' \circ \beta \circ \alpha^m)$ is finite, and $d(\alpha^n \circ \beta \circ \gamma' \circ \beta)$ is finite by Lemma 1.3.5(v). Hence by parts (i) and (iii) of Lemma 1.3.5 $c(\beta)$ and $d(\beta)$ are also finite.

Let $N_w(a)$ be the number occurrences of the letter a in the word w, and $N_w(b)$ be the number of occurrences of the letter b in w. By Lemma 1.3.7

$$c((w)\Phi) - d((w)\Phi) = N_w(a)\left(c((a)\Phi) - d((a)\Phi)\right) + N_w(b)\left(c((b)\Phi) - d((b)\Phi)\right),$$

and so

$$c(f) - d(f) = N_w(a)\left(c(\alpha) - d(\alpha)\right) + N_w(b)\left(c(\beta) - d(\beta)\right). \tag{4.6}$$

Suppose that $c(\alpha) = d(\alpha)$. Since $|c(f) - d(f)| = 1$ it follows that $N_w(b) \mid 1$, which is impossible since $N_w(b) \geq 2$. In the same way, if $c(\beta) = d(\beta)$, then $N_w(a) \mid 1$, which again is a contradiction. Hence $c(\alpha) \neq d(\alpha)$ and $c(\beta) \neq d(\beta)$. We have shown that if $c(f) = 1$ and $d(f) = 0$; or $c(f) = 0$ and $d(f) = 1$, then all the parameters $c(\alpha)$, $d(\alpha)$, $c(\beta)$, and $d(\beta)$ are finite, and $c(\alpha) \neq d(\alpha)$ and $c(\beta) \neq d(\beta)$.

Consider the case where $c(f) = 1$ and $d(f) = 0$. Since $f = \alpha \circ \gamma \circ \alpha$ for some $\gamma \in \Omega^\Omega$, it follows that $c(\alpha) \leq c(f)$ and $d(\alpha) \leq d(f)$ from parts (i) and (iii) of Lemma 1.3.5. Hence $d(\alpha) = 0$ and $c(\alpha) \leq 1$. It follows from $c(\alpha) \neq d(\alpha) = 0$, that $c(\alpha) = 1$. Since $d(\alpha^k) = 0$ for all $k \in \mathbb{N}$ by Lemma 1.3.5(i), Lemma 1.3.7 implies that

$$c(\alpha^n) = c(\alpha^n) - d(\alpha^n) = n\left(c(\alpha) - d(\alpha)\right) = n.$$

If $f = \alpha^n \circ \gamma$ for some $\gamma \in \Omega^\Omega$, then $n = c(\alpha^n) \leq c(f) = 1$. Hence $n = 1$, and so the word w starts with ab.

Now we consider different f in order to obtain more information about w. Suppose that $c(f) = 0$ and $d(f) = 1$. As in the previous paragraph $c(\alpha) \leq c(f)$

and $d(\alpha) \leq d(f)$ by parts (i) and (iii) of Lemma 1.3.5. Hence $c(\alpha) = 0$ and $d(\alpha) \leq 1$. It follows from the fact $c(\alpha) \neq d(\alpha)$ that $d(\alpha) = 1$. Suppose that k is the biggest integer such that $w = (ab)^k ua$ for some $u \in A^*$.Then $k \geq 1$ by the previous paragraph. It follows from parts (i), (ii), and (iii) of Lemma 1.3.5 that $c(\alpha) \leq c(\alpha \circ \beta) \leq c(f)$ and

$$d(\alpha) \leq d(\beta \circ \alpha) + d(\alpha^{m-1}) = d(\beta \circ \alpha^m) \leq d(f),$$

since ba^m is a suffix of w and α^{m-1} is injective. Hence $c(\alpha\circ\beta) = 0$ and $d(\beta\circ\alpha) = 1$. Since α is injective, Lemma 1.3.5(ii) implies that $d(\beta \circ \alpha) = d(\beta) + d(\alpha)$, and so $d(\beta) = 0$. It follows from Lemma 1.3.7 that

$$-d(\alpha\circ\beta) = c(\alpha\circ\beta) - d(\alpha\circ\beta) = c(\alpha) - d(\alpha) + c(\beta) - d(\beta) = -1 + c(\beta). \quad (4.7)$$

Hence $c(\beta) - 1 \leq 0$, and since $c(\beta) \neq d(\beta)$, we have that $c(\beta) = 1$. It follows from (4.7) that $d(\alpha \circ \beta) = 0$.

Since $w = (ab)^k ua$, there is $\gamma \in \Omega^\Omega$ such that $f = (\alpha \circ \beta)^k \circ \gamma$. Then Lemma 1.3.7 implies that

$$-1 = c(f) - d(f) = k\,(c(\alpha\circ\beta) - d(\alpha\circ\beta)) + c(\gamma) - d(\gamma) = c(\gamma) - d(\gamma).$$

Also $d(\gamma) \leq d(f) = 1$ by Lemma 1.3.5(i), thus $c(\gamma) = 0$ and $d(\gamma) = 1$. If ua contains at least two occurrences of both letters a and b, the above argument shows that ua must start with ab which contradicts maximality of k. Hence either $ua = b^t a$, or $ua = ba^t$ for some $t \in \mathbb{N}$.

Suppose that $ua = b^t a$, then by Lemma 1.3.7 and the fact that $\gamma = (ua)\Phi$

$$-1 = c(\gamma) - d(\gamma) = c(\alpha) - d(\alpha) + t\,(c(\beta) - d(\beta)) = -1 + t.$$

Hence $t = 0$, and $w = (ab)^k a$. If $ua = ba^t$, then by Lemma 1.3.7

$$-1 = c(\gamma) - d(\gamma) = t\,(c(\alpha) - d(\alpha)) + c(\beta) - d(\beta) = -t + 1.$$

Hence $t = 2$, and so $w = (ab)^k ba^2$. It follows from Lemma 1.3.5(ii) that $2 = d(\alpha^2)$ and from part (i) of the same lemma that $2 = d(\alpha^2) \leq d(f) = 1$, which is impossible. Hence $w = (ab)^k a$.

($\Leftarrow$) Fix $f \in \Omega^\Omega$. Suppose that $w = a^n ba^m$. Let α be the identity on Ω, let $\beta = f$, and let $\Phi : A^+ \longrightarrow \Omega^\Omega$ be the homomorphism induced by $(a)\Phi = \alpha$ and $(b)\Phi = \beta$. Then $(w)\Phi = f$. Hence w is a universal word for Ω^Ω.

Suppose that $w = (ab)^n a$. Choose any $g \in \Omega^\Omega$ such that $(x)g \in (x)f^{-1}$ for every $x \in (\Omega)f$. Then $f \circ g \circ f = f$. Let $\Phi : A^+ \longrightarrow \Omega^\Omega$ be the homomorphism induced by $(a)\Phi = \alpha$ and $(b)\Phi = \beta$. Then $(w)\Phi = (fg)^n f = f$. Hence w is a universal word for Ω^Ω. □

Note that the word $(ab)^n a$ is universal for Ω^Ω because for each $f \in \Omega^\Omega$ there exists $g \in \Omega^\Omega$ such that $f = f \circ g \circ f$. Semigroups which satisfy this property are called REGULAR. Hence the word $(ab)^n a$ is a universal word for every regular semigroup.

Theorem 4.2.16. *Let $A = \{a, b\}$, let $u \in aA^*b$, and let $w = u^n v u^m$ for some $n, m \geq 1$ and $v \in aA^*a \cup \{a\}$. Then w is a semigroup universal word for Ω^Ω if and only if $w = a^k ba(a^k b)^m$ for some $k, m \in \mathbb{N}$.*

Proof. ($\Rightarrow$) Suppose that $w = u^n v u^m$ for some $n, m \in \mathbb{N}$ such that $v \in aA^*a \cup \{a\}$, and suppose that w is a semigroup universal word for Ω^Ω. For any $x \in A$ and any $q \in A^*$, let $N_q(x)$ be the number of the occurrences of the letter x in the word q.

Let $f \in \Omega^\Omega$ be such that $c(f) = 1$ and $d(f) = 0$ which exists by Proposition 1.3.4. Since w is a universal word, there is a homomorphism $\Phi : A^+ \longrightarrow \Omega^\Omega$ such that $(w)\Phi = f$. Let $\alpha = (a)\Phi$, $\beta = (b)\Phi$, $\mu = (u)\Phi$, and $\gamma = (v)\Phi$. Then $f = \mu^n \circ \gamma \circ \mu^m$. It follows from parts (i) and (iii) of Lemma 1.3.5 that $c(\mu) \leq c(f) = 1$ and $d(\mu) \leq d(f) = 0$.

If $c(\mu) = 0$, then μ is a bijection and so $c(\gamma) = 1$ and $d(\gamma) = 0$. Recall that $\gamma = (v)\Phi$ and $v \in aA^*a \cup \{a\}$, and so there is $\gamma' \in \Omega^\Omega$ such that $\gamma = \gamma' \circ \alpha$. Hence $d(\alpha) = 0$ by Lemma 1.3.5(i). Since $u \in aA^*b$, there is some $\mu' \in \Omega^\Omega$ such that $\mu = \alpha \circ \mu'$. Then $c(\alpha) \leq c(\mu) = 0$ by Lemma 1.3.5(iii). Hence α is a bijection. Since $u = a^k bu'$ for some $k \geq 1$ and $u' \in A^*$, there is $\mu'' \in \Omega^\Omega$ such that $\mu = \alpha^k \circ \beta \circ \mu''$. It follows from the fact that α is bijective that $c(\beta \circ \mu'') = c(\mu) = 0$. Hence $c(\beta) \leq c(\beta \circ \mu'') = 0$ by Lemma 1.3.5(iii), and so both α and β are injections. This is a contradiction since $f = (w)\Phi$ is not an injection, and so $c(\mu) = 1$.

It follows from parts (i) and (iv) of Lemma 1.3.5 that $d(\mu^k) = 0$ and $c(\mu^k) = k$ for every $k \in \mathbb{N}$. However, $c(\mu^n) \leq c(f) = 1$ by Lemma 1.3.5(iii), and so $n = 1$.

Recall that there is $\gamma \in \Omega^\Omega$ such that $f = \mu \circ \gamma \circ \mu^m$. By above $c(\mu^m) = m$ and by Lemma 1.3.5(v) if $\aleph_0 \leq d(\mu \circ \gamma)$ then $d(f) = d(\mu \circ \gamma \circ \mu^m) \geq \aleph_0$, which is impossible, and so $d(\mu \circ \gamma)$ is finite. Similarly if $\aleph_0 \leq c(\gamma \circ \mu^m)$, then $c(f) = c(\mu \circ \gamma \circ \mu^m) \geq \aleph_0$, by Lemma 1.3.5(vi), which is impossible. Hence $c(\gamma \circ \mu^m)$ is also finite. Then $c(\gamma) \leq c(\gamma \circ \mu^m)$ and $d(\gamma) \leq d(\mu \circ \gamma)$ by parts (i) and (iii) of Lemma 1.3.5, implying that $c(\gamma)$ and $d(\gamma)$ are both finite.

It follows from Lemma 1.3.5 that $c(\mu \circ \gamma) \leq c(f) = 1$ and $c(\mu \circ \gamma) = c(\mu) + c(\gamma) = c(\gamma) + 1$. Hence $c(\gamma) = 0$. Lemma 1.3.7 implies that

$$1 = c(f) - d(f) = (m + 1)\,(c(\mu) - d(\mu)) + c(\gamma) - d(\gamma) = m + 1 - d(\gamma),$$

and so $d(\gamma) = m$. Also $c(\alpha) \leq c(\gamma) = 0$, $d(\alpha) \leq d(\gamma) = m$ and $d(\beta) \leq d(\mu) = 0$. Since $c(f), d(f), c(\alpha), d(\alpha)$ are all finite, $f \in \langle \alpha, \beta \rangle$, and $f \neq \alpha^n$ for all $n \in \mathbb{Z}$, it follows form Lemma 1.3.6 that $c(\beta)$ and $d(\beta)$ are finite, and so by Lemma 1.3.7

$$c(\mu) - d(\mu) = N_u(a)\,(c(\alpha) - d(\alpha)) + N_u(b)\,(c(\beta) - d(\beta))$$
$$c(\gamma) - d(\gamma) = N_v(a)\,(c(\alpha) - d(\alpha)) + N_v(b)\,(c(\beta) - d(\beta))$$

Let $x = c(\beta)$, and let $y = d(\alpha)$. Hence

$$1 = xN_u(b) - yN_u(a) \tag{4.8}$$

and

$$-m = xN_v(b) - yN_v(a). \tag{4.9}$$

Suppose now that $c(f) = 0$ and $d(f) = 1$. As before let $\Phi : A^+ \longrightarrow \Omega^\Omega$ be a homomorphism such that $(w)\Phi = f$, and also let $\alpha = (a)\Phi$, $\beta = (b)\Phi$, $\mu = (u)\Phi$, and $\gamma = (v)\Phi$. Since $f = \mu \circ \gamma \circ \mu^m$, Lemma 1.3.5(i) and (iii) imply that $c(\mu) \leq c(f) = 0$ and $d(\mu^m) \leq d(f) = 1$. Hence $c(\mu) = 0$ and by part (ii) of the same lemma $md(\mu) = d(\mu^m) \leq 1$. Hence either $d(\mu) = 0$, or $d(\mu) = 1$ and $m = 1$.

Case 1: Suppose that $d(\mu) = 1$. Then $f = \mu \circ \gamma \circ \mu$. It follows from Lemma 1.3.5(v) if $\aleph_0 \leq d(\mu \circ \gamma)$ then $d(f) = d(\mu \circ \gamma \circ \mu) \geq \aleph_0$, which is impossible, and so $d(\mu \circ \gamma)$ is finite. Similarly if $\aleph_0 \leq c(\gamma \circ \mu)$, then $c(f) = c(\mu \circ \gamma \circ \mu) \geq \aleph_0$, by Lemma 1.3.5(vi), which is impossible. Hence $c(\gamma \circ \mu)$ is finite. Then $c(\gamma) \leq c(\gamma \circ \mu)$ and $d(\gamma) \leq d(\mu \circ \gamma)$ by parts (i) and (iii) of Lemma 1.3.5. Therefore, $c(\gamma)$ and $d(\gamma)$ are both finite. Then Lemma 1.3.7 implies that

$$-1 = c(f) - d(f) = c(\gamma) - d(\gamma) + 2(c(\mu) - d(\mu)) = c(\gamma) - d(\gamma) - 2,$$

and so $c(\gamma) = 1 + d(\gamma)$. Also $1 = d(f) \geq d(\gamma \circ \mu) = d(\gamma) + d(\mu) = d(\gamma) + 1$, by parts (i) and (ii) of Lemma 1.3.5. So $d(\gamma) = 0$ and $c(\gamma) = 1$. Recall that $\gamma = \kappa \circ \alpha$ for some $\kappa \in \Omega^\Omega$, thus $d(\alpha) \leq d(\gamma) = 0$, and $c(\alpha) \leq c(f) = 0$. Hence α is bijection. Since w contains at least two occurrences of the letter b, we may write $w = a^k q$ where $q \in bA^*b$. Let $\delta = (q)\Phi$. Since α^k is a bijection, $c(\delta) = 0$ and $d(\delta) = 1$, so $c(\beta) \leq c(\delta) = 0$ and $d(\beta) \leq d(\delta) = 1$ by parts (i) and (iii) of Lemma 1.3.5. If β was a bijection then f, a product of α and β, would also be a bijection, which is a contradiction. Then $d(\beta) = 1$. It follows from Lemma 1.3.7 that $1 = N_w(b)$, however the letter b occurs at least twice in the word w, which again is a contradiction.

Case 2: Suppose that $d(\mu) = 0$. Since μ is a bijection $c(\gamma) = 0$ and $d(\gamma) = 1$. Then $c(\alpha) \leq c(\gamma) = 0$ and $d(\alpha) \leq 1$. If α is a bijection, we may conclude, as in the previous case, that there is only one occurrence of the letter b in the word w, which is a contradiction. Hence $d(\alpha) = 1$. It follows from Lemma 1.3.6 that $c(\beta)$

and $d(\beta)$ are finite, and so Lemma 1.3.7 implies

$$c(\mu) - d(\mu) = N_u(a)\,(c(\alpha) - d(\alpha)) + N_u(b)\,(c(\beta) - d(\beta))$$
$$c(\gamma) - d(\gamma) = N_v(a)\,(c(\alpha) - d(\alpha)) + N_v(b)\,(c(\beta) - d(\beta))$$

Let $z = c(\beta) - d(\beta)$. Then

$$0 = zN_u(b) - N_u(a) \tag{4.10}$$

and

$$-1 = zN_v(b) - N_v(a). \tag{4.11}$$

Finally, combining equations (4.8) with (4.10) and (4.9) with (4.11) we get that there exists $x, y, z \in \mathbb{N}$ such that $y \leq m$,

$$1 = N_u(b)(x - yz) \quad \text{and} \quad y - m = N_v(b)(x - yz).$$

Then $N_u(b) = x - yz = 1$ and $0 \geq y - m = N_v(b)(x - yz) = N_v(b) \geq 0$. Hence $y = m$ and $N_v(b) = 0$, implying that $N_v(a) = 1$. Hence $u = a^k b$ for some $k \geq 1$, and $v = a$, and so $w = a^k ba(a^k b)^m$.

($\Leftarrow$) Let $w = a^k ba(a^k b)^m$ for some $k, m \in \mathbb{N}$, and let $f \in \Omega^\Omega$. We will show that there is a homomorphism $\Phi : \{a, b\}^+ \longrightarrow \Omega^\Omega$ such that $(w)\Phi = f$. The proof consists of two steps. First, we will show that there is $g \in \Omega^\Omega$ such that g is surjective; that every kernel class of g is contained in a kernel class of f; and g satisfies the additional condition that if K is a kernel class of f and $K = \bigcup_{i \in I} L_i$ where L_i is a kernel class of g for all $i \in I$, then $|\,((K)f)\, g^{-m}| \geq |I|$. We will use g to define the homomorphism Φ.

Suppose that $|\,(\Omega)\, f| = |\Omega|$. Let $g \in \Omega^\Omega$ be such that kernels of f and g are the same. Since the number of kernel classes of g is $|\,(\Omega)\, f|$ we may also choose g to be surjective. The additional condition is easily satisfied since the kernels of f and g are the same.

In the case where $(\Omega)\, f$ is finite, there is at least one infinite kernel class K. Hence we can partition K into countably many countable sets $\{L_n : n \in \mathbb{N}\}$, which together with the remaining kernel classes of f gives a partition of Ω. Then define $g \in \Omega^\Omega$ to be a surjection with the kernel being the aforementioned partition and such that $(L_i)g \in L_{i-1}$ for all $i \in \{1, \dots m-1\}$ and $(L_0)g = (K)f$. Then $((K)f)\, g^{-m} = L_{m-1}$, and since K is the only kernel class of f which is not a kernel class of g, it follows that the additional condition is satisfied.

Since g is surjective, g^m is also surjective for any $m \in \mathbb{N}$. Hence for each $x \in \Omega$ the set $(x)g^{-1} \circ f \circ g^{-m}$ is non-empty. If $x \neq y$ and $(x)g^{-1} \circ f \circ g^{-m} = (y)g^{-1} \circ f \circ g^{-m} = X$ then $(x)g^{-1}$ and $(y)g^{-1}$ are disjoint subsets of the same kernel class of f. By the additional condition the set X is bigger than the number of $x \in \Omega$ such that $(x)g^{-1} \circ f \circ g^{-m} = X$. Hence we can define $\alpha \in \Omega^\Omega$ to be an injection such that

$$(x)\alpha \in (x)g^{-1} \circ f \circ g^{-m}$$

for all $x \in \Omega$. Then $g \circ \alpha \circ g^m = f$.

Since α is injective, so is α^k. Fix $z \in \Omega$ and define $\beta \in \Omega^\Omega$ by

$$(x)\beta = \begin{cases} (x)\alpha^{-k}g & \text{if } x \in (\Omega)\,\alpha^k, \\ z & \text{otherwise.} \end{cases}$$

Then $\alpha^k \circ \beta = g$, and therefore $f = g \circ \alpha \circ g^m = \alpha^k \circ \beta \circ \alpha \circ (\alpha^k \circ \beta)^m$. Hence if $\Phi : A^+ \longrightarrow \Omega^\Omega$ is the canonical homomorphism induced by $(a)\Phi = \alpha$ and $(b)\Phi = \beta$, then $(a^k ba(a^k b)^m)\Phi = f$. □

4.3 Universal sequences of Ω^Ω

In this section we will show that similar results to the ones in Section 4.2.1 also hold for sequences. Throughout this section we will assume that Ω is countable, again for the sake of simplicity.

The following result is analogous to Proposition 4.2.2 for semigroup universal sequences.

Proposition 4.3.1. *Let $\{w_n : n \in \mathbb{N}\}$ be a sequence of words in $\{a, b\}^+$ such that w_n is a subword of w_m only if $n = m$. Suppose that for any $n, m \in \mathbb{N}$ if a prefix p of w_n is a suffix of w_m, then $n = m$ and $p = w_n = w_m$. Then $\{w_n : n \in \mathbb{N}\}$ is a semigroup universal sequence for Ω^Ω.*

Proof. Let $A = \{a, b\}$, and let Ω be the set of eventually constant sequences over A, written from right to left, that is

$$\Omega = \{(\ldots, x_1, x_0) : x_i \in A \text{ and there is } K \in \mathbb{N} \text{ with } x_k = x_K \text{ for all } k > K\}.$$

Note that if $w_n = a$ for some $n \in \mathbb{N}$, then the first and the last letter of w_m has to be b for all $m \in \mathbb{N} \setminus \{n\}$, which contradicts the hypothesis. Hence we may assume that $|w_n| \geq 2$ for all $n \in \mathbb{N}$. If there exists $i \in \mathbb{N}$ such that a is the first letter of the word w_i, then for every $j \in \mathbb{N}$ the letter a cannot be the last letter of w_j. Hence w_j finishes with b for all $j \in \mathbb{N}$, and so b can not be the first letter of w_i for any $i \in \mathbb{N}$. Then $w_i \in aA^*b$ for all $i \in \mathbb{N}$. In the case, where there is $i \in \mathbb{N}$ such that the word w_i starts with b, the same argument shows that $w_i \in bA^*a$ for all $i \in \mathbb{N}$. Therefore, without loss of generality we may assume that, $w_i \in aA^*b$ for all $i \in \mathbb{N}$.

Fix $\{f_n : n \in \mathbb{N}\} \subseteq \Omega^\Omega$. Let $\alpha, \beta, \gamma \in \Omega^\Omega$ be defined as follows

$$(\ldots, x_1, x_0)\alpha = (\ldots, x_0, a)$$
$$(\ldots, x_1, x_0)\beta = (\ldots, x_0, b)$$
$$(\ldots, x_1, x_0)\gamma = \begin{cases} (\ldots, x_{i+1}, x_i)f_n & \text{if } x_{i-1}\ldots x_0 = w_n \\ (\ldots, x_1, x_0) & \text{otherwise} \end{cases}.$$

Suppose there are $i, j, n, m \in \mathbb{N}$ such that $x_{i-1}\ldots x_0 = w_n$ and and $x_{j-1}\ldots x_0 = w_m$. Then either the word w_n is a prefix of w_m or w_m is a prefix of w_n, in both cases the hypothesis implies that $w_n = w_m$. Hence γ is well-defined.

Define $\Phi : A^+ \longrightarrow \Omega^\Omega$ to be the canonical homomorphism induced by $(a)\Phi = \alpha$ and $(b)\Phi = \beta \circ \gamma$. Let $n \in \mathbb{N}$, and let $w_n = y_1 \ldots y_k$ where $y_i \in A$ for all $i \in \{1, \ldots k\}$. We will inductively show that $(\ldots, x_1, x_0)(y_1 \ldots y_{k-1})\Phi = (\ldots, x_0, y_1, \ldots, y_{k-1})$ for every $(\ldots, x_1, x_0) \in \Omega$. First note that

$$(\ldots, x_1, x_0)(y_1)\Phi = (\ldots, x_1, x_0)\alpha = (\ldots, x_1, x_0, y_1)$$

for every $(\ldots, x_1, x_0) \in \Omega$. Suppose that for some $i \in \{1, \ldots, k-2\}$ we have that

$$(\ldots, x_1, x_0)(y_1 \ldots y_i)\Phi = (\ldots, x_0, y_1, \ldots, y_i).$$

Since Φ is a homomorphism, it follows that

$$\begin{aligned}(\ldots, x_1, x_0)(y_1 \ldots y_{i+1})\Phi &= (\ldots, x_1, x_0)(y_1 \ldots y_i)\Phi \circ (y_{i+1})\Phi \\ &= (\ldots, x_1, x_0, y_1, \ldots, y_i)(y_{i+1})\Phi.\end{aligned}$$

If $y_{i+1} = a$, then $(y_{i+1})\Phi = \alpha$, and so

$$(\ldots, x_1, x_0)(y_1 \ldots y_{i+1})\Phi = (\ldots, x_1, x_0, y_1, \ldots, y_{i+1})$$

for all $(\ldots, x_1, x_0) \in \Omega$ proving the inductive hypothesis. Suppose that $y_{i+1} = b$. Then $(y_{i+1})\Phi = \beta \circ \gamma$, and so for all $(\ldots, x_1, x_0) \in \Omega$

$$(\ldots, x_1, x_0)(y_1 \ldots y_{i+1})\Phi = (\ldots, x_1, x_0, y_1, \ldots, y_{i+1})\gamma.$$

Since $y_1 \ldots y_{i+1}$ is a proper prefix of w_n, it follows that $y_1 \ldots y_{i+1}$ cannot be a suffix of w_m for any $m \in \mathbb{N}$. Then $x_j \ldots x_0 y_1 \ldots y_{i+1} \neq w_m$ for all $j \in \mathbb{N}$. Similarly, if there is $j \in \mathbb{N}$ such that $y_j \ldots y_{i+1} = w_m$ for some $m \in \mathbb{N}$, then w_m is a subword of w_n, contradicting the hypothesis of the lemma. Hence γ acts as an identity on $(\ldots, x_1, x_0, y_1, \ldots, y_{i+1})$, and the inductive hypothesis is satisfied.

It then follows by the above inductive argument that

$$(\dots, x_1, x_0)(y_1 \dots y_{k-1})\Phi = (\dots, x_1, x_0, y_1, \dots, y_{k-1}),$$

and since $y_n = b$,

$$(\dots, x_1, x_0)(w_n)\Phi = (\dots, x_1, x_0, y_1, \dots, y_k)\gamma = (\dots, x_1, x_0)f_n.$$

Hence $(w_n)\Phi = f_n$, and since n was arbitrary, $\{w_n : n \in \mathbb{N}\}$ is a universal sequence. □

Next we will show that the sequence we have seen in Example 4.1.7 can be proved to be universal using Proposition 4.3.1.

Example 4.3.2. Let $w_n = a(ab)^n b$ for all $n \in \mathbb{N}$ as in Example 4.1.7. The possible prefixes of w_n are w_n, $a(ab)^i$ for some $i \in \{0, \dots, n\}$, and $a(ab)^i a$ for some $i \in \{0, \dots, n-1\}$. Similarly the possible suffixes of w_m are w_m, $(ab)^i b$ for some $i \in \{0, \dots, n\}$, and $b(ab)^i b$ for some $i \in \{0, \dots, n-1\}$. It is then easy to see that a prefix of w_n is equal to a suffix of w_m if and only if $w_n = w_m$. Hence by Proposition 4.3.1 the sequence $\{a(ab)^n b : n \in \mathbb{N}\}$ is a semigroup universal sequence for Ω^Ω, which agrees with the conclusion of Example 4.1.7.

We will proceed by generalising Proposition 4.3.1 in the same way Proposition 4.2.2 was generalised to Theorem 4.2.12. In order to do that, we need to introduce notions similar to that of S_w. Let $A = \{a, b\}$, and let $\mathbf{w} = \{w_n : n \in \mathbb{N}\} \subseteq A^+$ be given, and consider a submonoid S of A^* such that

$$\text{if there are } n \in \mathbb{N}, s, s' \in S, \text{ and } u, v \in A^* \text{ such that } w_n = svuvs' \text{ then } v \in S; \tag{4.12}$$

$$\text{if there are distinct } n, m \in \mathbb{N} \text{ such that } w_n = svt \text{ and } w_m = t'vs' \text{ with } s, s' \in S \text{ and } t, t', v \in A^* \text{ then } v \in S; \tag{4.13}$$

$$\text{if there are } n \in \mathbb{N} \text{ and } s, t, v \in A^* \text{ such that } w_n = svt \text{ and } sv, vt \in S \text{ then } w_n \in S. \tag{4.14}$$

For every $\mathbf{w}$ there is at least one such submonoid, since A^* satisfies the condition trivially.

Let I be a non-empty set, and let S_i be a submonoid of A^* satisfying conditions (4.12), (4.13), and (4.14) for every $i \in I$. Suppose that there are $s, s' \in \bigcap_{i \in I} S_i$ and $u, v \in A^*$ such that $w_n = svuvs'$ for some $n \in \mathbb{N}$. Then $v \in S_i$ for all $i \in I$ by (4.12). Hence $v \in \bigcap_{i \in I} S_i$, and so $\bigcap_{i \in I} S_i$ satisfies condition (4.12). Suppose that there are distinct $n, m \in \mathbb{N}$, $s, s' \in \bigcap_{i \in I} S_i$ and $t, t', v \in A^*$ such that $w_n = svt$ and $w_m = t'vs'$. Then $v \in S_i$ by condition (4.13) for all $i \in I$. Hence $v \in \bigcap_{i \in I} S_i$, and so $\bigcap_{i \in I} S_i$ satisfies condition (4.13). Suppose that there are $s, t, v \in A^*$

and $n \in \mathbb{N}$ such that $w_n = svt$ and $sv, vt \in \bigcap_{i \in I} S_i$. Then $w_n \in S_i$ by (4.14) for all $i \in I$. Therefore, there exists the smallest submonoid, with respect to containment, of A^* satisfying conditions (4.12), (4.13), and (4.14), we denote it by $S_{\mathbf{w}}$.

There is also a more constructive way of defining $S_{\mathbf{w}}$. Let $S_0 = \langle 1 \rangle$, and suppose that we have defined S_n, a submonoid of A^*, for some $n \in \mathbb{N}$. Let

$$\begin{aligned}
X_n &= \{v : w_i = svuvs' \text{ for some } i \in \mathbb{N}, s, s' \in S_n \text{ and } u, v \in A^*\};\\
Y_n &= \{v : w_i = svt, w_j = t'vs' \text{ for some distinct } i, j \in \mathbb{N},\\
&\qquad s, s' \in S_n \text{ and } t, t', v \in A^*\};\\
Z_n &= \{w_i : w_i = svt \text{ for some } i \in \mathbb{N} \text{ and } s, v, t \in A^*\\
&\qquad \text{so that } sv, vt \in S_n\};\\
S_{n+1} &= \langle S_n, X_n, Y_n, Z_n \rangle.
\end{aligned} \tag{4.15}$$

Then $S_0 \leq S_1 \leq S_2 \leq \ldots$ by definition of S_{n+1}. Let $S = \bigcup_{n \in \mathbb{N}} S_n$. If $s, t \in S$, then there are $n, m \in \mathbb{N}$ such that $s \in S_n$ and $t \in S_m$. Then $s, t \in S_{\max(s,t)}$, and so $st \in S_{\max(s,t)} \subseteq S$. Hence S is a submonoid of A^*.

Suppose that $w_i = svuvs'$ for some $i \in \mathbb{N}$, $s, s' \in S$, and $u, v \in A^*$. By definition of S and the fact that $S_0 \leq S_1 \leq \ldots$ there is some $n \in \mathbb{N}$ such that $s, s' \in S_n$, and so $v \in X_n \subseteq S_{n+1} \subseteq S$. Hence S satisfies condition (4.12). Suppose that $w_i = svt$ and $w_j = s'vt'$ for some distinct $i, j \in \mathbb{N}$, $s, s' \in S$ and $u, v \in A^*$. Then there is some $n \in \mathbb{N}$ such that $s, s' \in S_n$. Hence $v \in Y_n \subseteq S$, and so S satisfies condition (4.13). Suppose that $w_i = svt$ for some $i \in \mathbb{N}$, $s, v, t \in A^*$ and $sv, vt \in S$. Again there is some $n \in \mathbb{N}$ such that $sv, vt \in S_n$. Then $w_i \in Z_n \subseteq S$, and so S satisfies condition (4.14).

Let T be any submonoid of A^* satisfying conditions (4.12), (4.13), and (4.14). Since $S_0 = \langle 1 \rangle$, it follows that $S_0 \leq T$. Suppose that $S_n \leq T$ for some $n \in \mathbb{N}$. Suppose that $w_i = svuvs'$ for some $i \in \mathbb{N}$, $s, s' \in S_n$, and $u, v \in A^*$. Then $v \in T$ by (4.12), and so $X_n \subseteq T$. If $w_i = svt$ and $w_j = t'vs'$ for some distinct $i, j \in \mathbb{N}$ such that $s, s' \in S_n$ and $t, t', v \in A^*$, then $v \in T$, since T satisfies condition (4.13) and $s, s' \in T$. Hence $Y_n \subseteq T$. If $w_i = svt$ for some $i \in \mathbb{N}$ such that $s, v, t \in A^*$ and $sv, vt \in S_n$, then it follows from the fact that T satisfies condition (4.14), that $w_i \in T$. Then $Z_n \subseteq T$, and so $S_{n+1} \subseteq T$. Then by induction $S_n \leq T$ for all $n \in \mathbb{N}$, and thus $S \leq T$. Therefore S is the minimal submonoid of A^* satisfying conditions (4.12), (4.13), and (4.14), in other words $S = S_{\mathbf{w}}$.

Note that unlike in the case of S_w where w is a word over A, the monoid $S_{\mathbf{w}}$ does not have to be finitely generated.

Before presenting the main result of this section we prove a technical result.

Lemma 4.3.3. *Let $\mathbf{w} = \{w_n : n \in \mathbb{N}\} \subseteq aA^*b$. Suppose that $a, b \notin S_{\mathbf{w}}$. Then $S_{\mathbf{w}} \subseteq aA^*b \cup \{1\}$.*

Proof. Recall that $S_{\mathbf{w}} = \bigcup_{n\in\mathbb{N}} S_n$, where S_n is as in (4.15). Then $S_0 = \{1\} \subseteq aA^*b \cup \{1\}$. Suppose that for some $n \in \mathbb{N}$ we have that $S_n \subseteq aA^*b \cup \{1\}$.

Let $m \in \mathbb{N}$, and suppose that $w_m = svuvs'$ such that $s, s' \in S_n$ and $u, v \in A^*$. If $v \in A^*a$ then we may write $v = v'a$ and $w_m = aqv'as'$ for some $q \in A^*$ since the first letter of w_m is a. Hence $a \in S_{\mathbf{w}}$ by (4.12), which contradicts the hypothesis. Suppose that $v \in bA^*$. Then $v = bv'$ for some $v' \in A^*$ and there is $q \in A^*$ so that $w_m = sbv'qb$ since b is the last letter of w_m. Hence $b \in S_{\mathbf{w}}$ by (4.12), which again is a contradiction. Therefore $v \in aA^*b$, and so $X_n \subseteq aA^*b \cup \{1\}$.

Let $m, k \in \mathbb{N}$ be distinct and suppose that $w_m = svt$ and $w_k = t'vs'$ such that $s, s' \in S_n$ and $v, t, t' \in A^*$. If $v \in A^*a$, then we may write $v = v'a$ for some $v' \in A^*$, and since w_m starts with a, it follows that $w_m = aq$ for some $q \in A^*$ and $w_k = t'v'as'$. Hence $a \in S_{\mathbf{w}}$ by (4.13), which contradicts the hypothesis. Suppose $v \in bA^*$. Then there is $v' \in A^*$ such that $v = bv'$, and since w_k finishes with b, it follows that $w_m = sbv't$ and $w_k = bq$ for some $q \in A^*$. Hence $b \in S_{\mathbf{w}}$ by (4.13), which again is a contradiction. Therefore $v \in aA^*b$, and so $Y_n \subseteq aA^*b \cup \{1\}$.

By the definition Z_n is a subset of $\{w_n : n \in \mathbb{N}\}$, and so by the hypothesis $Z_n \subseteq aA^*b \cup \{1\}$. Hence $S_{n+1} \subseteq aA^*b \cup \{1\}$. Therefore by induction $S_n \subseteq aA^*b \cup \{1\}$ for all $n \in \mathbb{N}$, implying that $S_{\mathbf{w}} \subseteq aA^*b \cup \{1\}$. □

Suppose that $\mathbf{w} = \{w_n : n \in \mathbb{N}\}$ is such that $w_n \notin S_{\mathbf{w}}$ for all $n \in \mathbb{N}$. Let $p_n \in A^*$ be the longest prefix of w_n such that $p_n \in S_{\mathbf{w}}$, and let $s_n \in A^*$ be the longest suffix of w_n such that $s_n \in S_{\mathbf{w}}$. Then either there is $u_n \in A^+$ such that $w_n = p_n u_n s_n$, or there are $s, t, v \in A^*$ such that $w_n = stv$, $p_n = st$, and $s_n = tv$. In the latter case $w_n \in S_{\mathbf{w}}$ by (4.14), which contradicts the assumption. Therefore for each n there is $u_n \in A^+$ such that

$$w_n = p_n u_n s_n.$$

The following theorem is analogous to Theorem 4.2.12.

Theorem 4.3.4. *Let $\mathbf{w} = \{w_n : n \in \mathbb{N}\} \subseteq \{a, b\}^+$ such that $w_n \notin S_{\mathbf{w}}$ for all $n \in \mathbb{N}$. Let $p_n, s_n, u_n \in \{a, b\}^*$ be such that $w_n = p_n u_n s_n$, and p_n and s_n are respectively the longest prefix and the longest suffix of w_n so that $p_n, s_n \in S_{\mathbf{w}}$. Suppose that u_n is a subword of w_m if and only if $n = m$ and that u_n is not a subword of p_n for all n. Then $\{w_n : n \in \mathbb{N}\}$ is a semigroup universal sequence for Ω^Ω.*

Proof. Denote by A the set $\{a, b\}$. Let $n \in \mathbb{N}$ and suppose that $a \in S_{\mathbf{w}}$. Then $w_n \neq a^i$ for any $i \in \mathbb{N}$ since $w_n \notin S_{\mathbf{w}}$. So either $w_n = a^i ba^j$ for some $i, j \in \mathbb{N}$, or $w_n = a^i buba^j$ for some $i, j \in \mathbb{N}$ and $u \in A^*$. In the latter cases $b \in S_{\mathbf{w}}$ by (4.12), and so $S_{\mathbf{w}} = A^*$, which contradict the assumption that $w_n \notin S_{\mathbf{w}}$. Suppose that $w_n = a^{i_n} ba^{j_n}$ for some $i_n, j_n \in \mathbb{N}$ and all $n \in \mathbb{N}$. Then $b \in S_{\mathbf{w}}$ by (4.13), which again is a contradiction. Therefore $a \notin S_{\mathbf{w}}$ and the symmetric argument shows that $b \notin S_{\mathbf{w}}$.

For the rest of the proof we assume that $a, b \notin S_{\mathbf{w}}$. Then for any n and m in $\mathbb{N}$, the first letter of w_n and the last letters of w_m have to be different, thus $\{w_n : n \in \mathbb{N}\} \subseteq aA^*b$ or $\{w_n : n \in \mathbb{N}\} \subseteq bA^*a$. Without the loss of generality, we may assume that $w_n \in aA^*b$ for all $n \in \mathbb{N}$. Then $S_{\mathbf{w}} \subseteq aA^*b \cup \{1\}$ by Lemma 4.3.3.

Let Ω be the set of eventually constant sequences over $F(A)$, written from right to left, namely

$$\Omega = \{(\ldots, x_1, x_0) : x_i \in F(A) \text{ and there is } K \in \mathbb{N} \text{ such that } x_K = x_k \text{ for all } k \geq K\}.$$

We proceed by proving a series of claims.

Claim 1. $u_n \in aA^*b$ for all $n \in \mathbb{N}$.

Proof. Let $n \in \mathbb{N}$, and let $m \in \mathbb{N}$ be such that $n \neq m$. Since $w_m \in aA^*b$ there is some $v \in A^*$ such that $w_m = avb$. Suppose that $u_n \in bA^*$. Then $u_n = bu$ for some $u \in A^*$, and so $w_n = p_n bus_n$. Since $w_m = avb$, condition (4.13) implies that $b \in S_{\mathbf{w}}$, which is a contradiction. Suppose that $u_n \in A^*a$. Then $u_n = ua$ for some $u \in A^*$. Hence $w_n = p_n uas_n$, and so (4.13) implies that $a \in S_{\mathbf{w}}$, which again is a contradiction. Hence $u_n \in aA^*b$. □

Claim 2. Let $n, m \in \mathbb{N}$ be such that $n \neq m$. Then u_n is not a suffix of u_m.

Proof. Suppose that u_n is a suffix of u_m, in other words there is $v \in A^*$ such that $u_m = vu_n$. Then $w_n = p_n u_n s_n$ and $w_m = p_m vu_n s_m$. Since $p_n, s_m \in S_{\mathbf{w}}$ condition (4.13) implies that $u_n \in S_{\mathbf{w}}$. Hence $w_n = p_n u_n s_n \in S_{\mathbf{w}}$, contradicting the hypothesis. □

By (4.15), there is a generating set G for $S_{\mathbf{w}}$ consisting of subwords of words in $\mathbf{w}$. Let G_n be the set of all words in G of length at most n. Recall that we say that a generating set T is irredundant if $\langle T \setminus \{v\}\rangle \neq \langle T\rangle$ for every $v \in T$. Let $T_1 = G_1$. Then T_1 is irredundant. For some $n \in \mathbb{N}$, suppose that there is T_n an irredundant set such that $\langle T_n\rangle = \langle G_n\rangle$ and $T_{n-1} \leq T_n$ if $n \geq 2$. Define T_{n+1} to be the maximal irredundant subset of G_{n+1} containing T_n, which is possible since G_{n+1} is finite. Hence $T_n \leq T_{n+1}$ and $\langle G_{n+1}\rangle = \langle T_{n+1}\rangle$, thus the induction hypothesis is satisfied. Therefore such T_n exists for all $n \in \mathbb{N}$. Let $X = \bigcup_{n\in\mathbb{N}} T_n$. Then it is routine to verify that X is an irredundant generating set for $S_{\mathbf{w}}$.

Claim 3. For each $v \in X$, there are $t, t' \in S_{\mathbf{w}}$ and $n, m \in \mathbb{N}$ such that tv is a prefix of p_n, and vt' is a suffix of s_m.

Proof. Let X_k, Y_k, and Z_k be as in (4.15). Note that if $Z_k \neq \varnothing$, then there is $m \in \mathbb{N}$ such that $w_m \in Z_k$, and so $w_m \in S_w$, contradicting the hypothesis. Hence $Z_n = \varnothing$ for all $n \in \mathbb{N}$, and $X \subseteq \bigcup_{n\in\mathbb{N}} X_n \cup Y_n$ by the choice of X.

Let $v \in X$. Suppose there is $k \in \mathbb{N}$ such that $v \in X \cap X_k$. Then $w_n = tvuvt'$ for some $n \in \mathbb{N}$, $t, t' \in S_k$, and $u, v \in A^*$. Hence $tv, vt' \in S_{\mathbf{w}}$, and so it then follows from the maximality of p_n and s_n that tv is a prefix of p_n, and vt' is a suffix of s_n. On the other hand, suppose there is $k \in \mathbb{N}$ such that $v \in X \cap Y_k$. Then $w_n = qvt$ and $w_m = t'vq'$ for some $n, m \in \mathbb{N}$, $t, t' \in A^*$, and $q, q' \in S_k$. Hence $tv, vt' \in S_{\mathbf{w}}$, and so tv is a prefix of p_n, and vt' is a suffix of s_m. □

Claim 4. For all $v \in X$ and all $n \in \mathbb{N}$, a prefix of v is not a suffix of u_n, and a suffix of v is not a prefix of u_n.

Proof. Let $v \in X$ and $n \in \mathbb{N}$ be arbitrary. Then by Claim 3 there are $t, t' \in S_{\mathbf{w}}$ such that tv is a prefix of p_m and vt' is a suffix of s_k for some $m, k \in \mathbb{N}$. Then there are $r, r' \in A^*$ so that $w_m = tvru_ms_m$ and $w_k = p_ku_kr'vt'$. Suppose that q is a non-trivial prefix of v which is also a suffix of u_n. First, consider the case where $m = n$. Then $q \in S_{\mathbf{w}}$ by (4.12) and the fact that $w_m = tqhqs_m$ for some $h \in A^*$.

Suppose that $m \neq n$. Then $w_m = tvru_ms_m$ and $w_n = p_nu_ns_n$, and so $q \in S_{\mathbf{w}}$ by (4.13) as $t, s_n \in S_{\mathbf{w}}$. Hence in both cases $q \in S_{\mathbf{w}}$, which contradicts the maximality of s_n.

The case where q is non-trivial suffix of v which is a prefix of u_n follows in almost identical way, using $w_k = p_ku_kr'vt'$ instead of $w_m = tvru_ms_m$. □

Claim 5. For every $v, v' \in X$, if a non-trivial prefix q of v is a suffix of v', then $q = v = v'$.

Proof. Let $v, v' \in X$ be arbitrary. Suppose that $v = qr$ and $v' = r'q$ for some $r, r' \in A^*$ and $q \in A^+$. By Claim 3 there are $t, t' \in S_{\mathbf{w}}$ and $n, m \in \mathbb{N}$ such that tv is a prefix of p_n, and $v't'$ is a suffix of s_m. If $n = m$ then there is $x \in A^*$ such that

$$w_n = tvxv't' = tqrxr'qt',$$

and so $q \in S_{\mathbf{w}}$ by (4.12) since $t, t' \in S_{\mathbf{w}}$. If $n \neq m$ we may write $w_n = tvx = tqrx$ and $w_m = x'v't' = x'r'qt'$ for some $x, x' \in A^*$. Since $t, t' \in S_{\mathbf{w}}$, (4.13) implies that $q \in S_{\mathbf{w}}$. Hence $q \in S_{\mathbf{w}}$ in both cases.

Since $v \in X$, by Claim 3 there are $n', m' \in \mathbb{N}$, $l, l' \in S_{\mathbf{w}}$ so that lv is a prefix of $p_{n'}$ and vl' is a suffix of $s_{m'}$. As in the previous paragraph, if $n = m$ then there is $x \in A^*$ such that

$$w_n = lvxvl' = lqrxqrl',$$

and so $r \in S_{\mathbf{w}}$ by (4.12) since $lq, l' \in S_{\mathbf{w}}$. If $n \neq m$ we may write $w_n = lvx = lqrx$ and $w_m = x'v'l' = x'qrl'$ for some $x, x' \in A^*$. Since $lq, l' \in S_{\mathbf{w}}$, (4.13) implies that $r \in S_{\mathbf{w}}$. Hence $r \in S_{\mathbf{w}}$ in both cases. Since X is irredundant, $q, r \in S_{\mathbf{w}}$, and $qr \in X$, it follows that $r = 1$. The same argument for v' implies that $r' = 1$, and so $q = v = v'$. □

Let $\{f_n : n \in \mathbb{N}\} \subseteq \Omega^\Omega$. We will construct a homomorphism $\Phi : A^+ \to \Omega^\Omega$ such that $(w_n)\Phi = f_n$ for all $n \in \mathbb{N}$. In order to do that we will need some auxiliary functions $\alpha, \beta, \gamma \in \Omega^\Omega$ defined as follows

$$(\ldots, x_1, x_0)\alpha = (\ldots, x_0, a),$$
$$(\ldots, x_1, x_0)\beta = (\ldots, x_0, b),$$

and

$$(\ldots, x_1, x_0)\gamma = \begin{cases} (\ldots, x_{i+1}, x_i v) & \text{if } x_{i-1}\ldots x_0 = v \in X \text{ for some } i \geq 1 \\ & \text{and } x_j \in A^+ \text{ for all } j \in \{0, \ldots i-1\} \\ (\ldots, x_1, x_0) & \text{otherwise} \end{cases}.$$

If there are $i, i' \in \mathbb{N}$, such that $i > i'$, $x_{i-1}\ldots x_0 = v$, and $x_{i'-1}\ldots x_0 = v'$ for some $v, v' \in X$, and so that $x_j \in A^+$ for all $j \in \{0, \ldots, i\}$. Then v' is a suffix of v. By Claim 5 it is only possible if $v = v'$. Hence γ is well-defined. Let $\Psi : A^+ \longrightarrow \Omega^\Omega$ be the canonical homomorphism induced by $(a)\Psi = \alpha$ and $(b)\Psi = \beta \circ \gamma$. We will later use Ψ to define the required Φ.

Claim 6. For $v \in aA^*$ such that no prefix of v is a suffix of a word in X, there are $z_1, \ldots, z_k \in A^+$ such that $z_1 \ldots z_k = v$ and

$$(\ldots, x_1, x_0)\left((v)\Psi\right) = (\ldots, x_1, x_0, z_1, \ldots, z_k)$$

for every $(\ldots, x_1, x_0) \in \Omega$.

Proof. Let $v \in aA^*$ be such that no prefix of v is a suffix of a word in X, and write $v = y_1 \ldots y_m$ for some $m \in \mathbb{N}$ and $y_1, \ldots, y_m \in A$. Since $v \in aA^*$, $(y_1)\Psi = \alpha$, and so for all $(\ldots, x_1, x_0) \in \Omega$

$$(\ldots, x_1, x_0)\alpha = (\ldots, x_1, x_0, a) = (\ldots, x_1, x_0, y_0).$$

Suppose that for some $i \in \{1, \ldots, m-2\}$ there are $j \in \mathbb{N}$ and $z_1, \ldots z_j \in A^+$ such that $y_1 \ldots y_i = z_1 \ldots z_j$ and

$$(\ldots, x_1, x_0)\left((y_1 \ldots y_i)\Psi\right) = (\ldots, x_1, x_0, z_1, \ldots, z_j)$$

for every $(\ldots, x_1, x_0) \in \Omega$.

There are two cases to consider, either $y_{i+1} = a$, or $y_{i+1} = b$. Suppose that $y_{i+1} = a$. Since Ψ is a homomorphism, for all $(\dots, x_1, x_0) \in \Omega$

$$\begin{aligned}(\dots, x_1, x_0)\,((y_1 \dots y_{i+1})\Psi) &= (\dots, x_1, x_0, z_1, \dots, z_j)\,((y_{i+1})\Psi)\\ &= (\dots, x_1, x_0, z_1, \dots, z_j)\alpha\\ &= (\dots, x_1, x_0, z_1, \dots, z_j, a),\end{aligned}$$

and $z_1 \dots z_j a = y_1 \dots y_{i+1}$. Hence the condition is satisfied in this case. Suppose that $y_{i+1} = b$. Again since Ψ is a homomorphism, for all $(\dots, x_1, x_0) \in \Omega$

$$\begin{aligned}(\dots, x_1, x_0)\,((y_1 \dots y_{i+1})\Psi) &= (\dots, x_1, x_0, z_1, \dots, z_j)\,((y_{i+1})\Psi)\\ &= (\dots, x_1, x_0, z_1, \dots, z_j)\beta \circ \gamma\\ &= (\dots, x_1, x_0, z_1, \dots, z_j, b)\gamma,\end{aligned}$$

and $z_1 \dots z_j b = y_1 \dots y_{i+1}$. Since $y_1 \dots y_{i+1}$ is a prefix of v, by the assumption it cannot be a suffix of any word in X. Thus $z_1 \dots z_j b \notin X$ and if $x_0, \dots, x_t \in A^+$ then $x_t \dots x_0 z_1 \dots z_j b \notin X$ for all $t \in \mathbb{N}$. Hence either γ acts as identity on $(\dots, x_1, x_0, z_1, \dots, z_j, b)$, or there is $k > 1$ such that $z_k \dots z_j b \in X$, in which case

$$\begin{aligned}(\dots, x_1, x_0)\,((y_1 \dots y_{i+1})\Psi) &= (\dots, x_1, x_0, z_1, \dots, z_j, b)\gamma\\ &= (\dots, x_1, x_0, z_1, \dots, z_{k-1} z_k \dots z_j b).\end{aligned}$$

Since $z_1 \dots z_j b = y_1 \dots y_{i+1}$, the inductive hypothesis is satisfied in both cases. Hence the claim holds by induction. ☐

Claim 7. Let $v \in S_{\mathbf{w}}$. Then $(v)\Psi \in \Omega^\Omega$ is a bijection, in particular $(\dots, x_1, x_0)\,((v)\Psi) = (\dots, x_1, x_0 v)$ for all $(\dots, x_1, x_0) \in \Omega$.

Proof. Let $v \in X$. Since $v \in S_{\mathbf{w}} \subseteq aA^*b \cup \{1\}$ and $v \neq 1$, we may write $v = v'b$ for some $v' \in aA^*$. By Claim 5 any proper prefix of v' is not a suffix of any word in X. Hence by Claim 6 there exists $j \in \mathbb{N}$ and $z_1, \dots, z_j \in A^+$ such that $z_1 \dots z_j = v'$ and for all $(\dots, x_1, x_0) \in \Omega$

$$(\dots, x_1, x_0)\,((v')\Psi) = (\dots, x_1, x_0, z_1, \dots, z_j).$$

It follows from $v = z_1 \dots z_j b$ and the fact that Ψ is a homomorphism that

$$\begin{aligned}(\dots, x_1, x_0)\,((v)\Psi) &= (\dots, x_1, x_0, z_1, \dots, z_j, b)\gamma\\ &= (\dots, x_1, x_0 v).\end{aligned} \tag{4.16}$$

Suppose that $(\dots, x_1, x_0 v) = (\dots, x_1', x_0' v)$ where $x_i, x_i' \in F(A)$ for all $i, i' \in \mathbb{N}$. Then $x_i = x_i'$ for all $i \geq 1$ and $x_0 v = x_0' v$. Since $x_0 v$ and $x_0' v$ are both elements of the free group $F(A)$ it follows that $x_0 = x_0'$. Hence $(v)\Psi$ is injective by (4.16).

Let $(\dots, x_1, x_0) \in \Omega$. Then by (4.16)

$$(\dots, x_1, x_0v^{-1})\,((v)\Psi) = (\dots, x_1, x_0),$$

so $(v)\Psi$ is surjective, and hence bijective on Ω. Since X is a generating set for $S_{\mathbf{w}}$, it follows that $(\dots, x_1, x_0)\,((v)\Psi) = (\dots, x_1, x_0v)$, and $(v)\Psi \in \mathrm{Sym}(\Omega)$ for all $v \in S_{\mathbf{w}}$. $\square$

In order to define the required homomorphism Φ, we need one more function on Ω. Define $\delta \in \Omega^\Omega$ a follows:

$$(\dots, x_1, x_0)\delta = \begin{cases} (\dots, x_{i+1}, x_ip_n^{-1})f \circ (s_n)\Psi^{-1} & \text{if } x_{i-1}\dots x_0 = u_n \text{ for some} \\ & n \in \mathbb{N},\ i \geq 1 \text{ where } x_j \in A^+ \\ & \text{for all } j \in \{0, \dots i-1\} \\ (\dots, x_1, x_0) & \text{otherwise} \end{cases}.$$

Note that $(s_n)\Psi^{-1}$ is defined by Claim 7. Suppose there are $i, i', n, n' \in \mathbb{N}$ such that $x_{i-1}\dots x_0 = u_n$ and $x_{i'-1}\dots x_0 = u_{n'}$ where $x_j \in A^+$ for all $j \in \{0, \dots, \max(i, i') - 1\}$. Then either u_n is a suffix of u_m or u_m is a suffix of u_n. Then $n = n'$ by Claim 2, and so $i = i'$. Therefore, δ is well-defined.

Let Φ be the canonical homomorphism induced by $(a)\Phi = \alpha$ and $(b)\Phi = \beta \circ \gamma \circ \delta$.

Claim 8. If $v \in S_{\mathbf{w}}$, then $(v)\Phi = (v)\Psi$.

Proof. Suppose that $v = y_1 \dots y_m$ such that $y_i \in A$ for all $i \in \{1, \dots, m\}$. Since $S_{\mathbf{w}} \subseteq aA^*b \cup \{1\}$ and $v \neq 1$, it follows that $y_1 = a$, and so $(y_1)\Phi = \alpha = (y_1)\Psi$. Suppose $(y_1 \dots y_i)\Phi = (y_1 \dots y_i)\Psi$ for some $i \in \{1, \dots, m-1\}$. Since Φ is a homomorphism

$$(y_1 \dots y_{i+1})\Phi = (y_1 \dots y_i)\Psi \circ (y_{i+1})\Phi.$$

In the case when $y_{i+1} = a$, it follows that $(y_{i+1})\Phi = (y_{i+1})\Psi$, and so the inductive hypothesis is satisfied. Suppose that $y_{i+1} = b$, then $(y_{i+1})\Phi = (y_{i+1})\Psi \circ \delta$, and so $(y_1 \dots y_{i+1})\Phi = (y_1 \dots y_{i+1})\Psi \circ \delta$. If $i + 1 < m$, then $y_1 \dots y_{i+1}$ is a proper prefix of v. By Claim 5 for any $j \in \{1, \dots, i+1\}$ the proper prefix $y_1 \dots y_j$ of v is a not a suffix of any word in X. Since $y_1 \dots y_{i+1} \in aA^*$, by Claim 6 there exists $j \in \mathbb{N}$ and $z_1, \dots, z_j \in A^+$ such that $z_1 \dots z_j = y_1 \dots y_{i+1}$ and for all $(\dots, x_1, x_0) \in \Omega$

$$(\dots, x_1, x_0)\,((y_1 \dots y_{i+1})\Psi) = (\dots, x_1, x_0, z_1, \dots, z_j).$$

If $i + 1 = m$, then $y_1 \dots y_{i+1} = v$, and so

$$(\dots, x_1, x_0)\,((y_1 \dots y_{i+1})\Psi) = (\dots, x_1, x_0y_1 \dots y_{i+1})$$

for all $(\dots, x_1, x_0) \in \Omega$ by Claim 7. Hence in both cases there are $z_0, \dots, z_j \in A^+$ such that $z_0 \dots z_j = y_1 \dots y_{i+1}$ and for all $(\dots, x_1, x_0) \in \Omega$

$$(\dots, x_1, x_0)\,((y_1 \dots y_{i+1})\Psi) = (\dots, x_1, x_0 z_0, z_1, \dots, z_j). \qquad (4.17)$$

We will show that δ acts as the identity on $(\dots, x_1, x_0)\,((y_1 \dots y_{i+1})\Psi)$ for all $(\dots, x_1, x_0) \in \Omega$. Fix $(\dots, x_1, x_0) \in \Omega$, let $n \in \mathbb{N}$, and let $z_0, \dots, z_j \in A^+$ be as in (4.17). Suppose that there are $i \geq 0$ and $n \in \mathbb{N}$ such that $x_i, \dots, x_1, x_0 z_0 \in A^+$ and $x_i \dots x_0 z_0 \dots z_j = u_n$. Then $z_0 \dots z_j = y_1 \dots y_{i+1}$ is both a prefix of v and a suffix of u_n, contradicting Claim 4. By Claim 3 there are $t \in S_w$ and $m \in \mathbb{N}$ such that tv is a prefix of p_m, and by the hypothesis of the theorem, it follows that u_n is not a subword of p_m and so not a subword of v for all $n \in \mathbb{N}$. If $k > 0$ then $z_k \dots z_j$ is a subword of v, and so not equal to u_n. Hence δ acts as identity on $(\dots, x_1, x_0 z_0, z_1, \dots, z_j)$. Hence the inductive hypothesis is satisfied and by induction $(v)\Phi = (v)\Psi$ for all $v \in X$. Since X is a generating set for $S_{\mathbf{w}}$, it follows that $(v)\Phi = (v)\Psi$ for all $v \in S_{\mathbf{w}}$. □

Claim 9. $(u_n)\Phi = (u_n)\Psi \circ \delta$ for all $n \in \mathbb{N}$.

Proof. Let $n \in \mathbb{N}$, and let $u_n = y_1 \dots y_m$. We will now show that $(y_1 \dots y_{m-1})\Phi = (y_1 \dots y_{m-1})\Psi$. Since $y_1 = a$, it follows that $(y_1)\Phi = \alpha = (y_1)\Psi$. Suppose $(y_1 \dots y_i)\Phi = (y_1 \dots y_i)\Psi$ for some $i \in \{1, \dots, m-2\}$. Then

$$(y_1 \dots y_{i+1})\Phi = (y_1 \dots y_i)\Psi \circ (y_{i+1})\Phi.$$

If $y_{i+1} = a$ then $(y_{i+1})\Phi = (y_{i+1})\Psi$, and so the inductive hypothesis is satisfied in this case. Suppose $y_{i+1} = b$. Then $(y_{i+1})\Phi = (y_{i+1})\Psi \circ \delta$. Hence $(y_1 \dots y_{i+1})\Phi = (y_1 \dots y_{i+1})\Psi \circ \delta$. By Claim 4, for every $j \in \{1, \dots, i+1\}$ the proper prefix $y_1 \dots y_j$ of u_n is not a suffix of any word in X. Hence by Claim 6 there exists $j \in \mathbb{N}$ and $z_1, \dots, z_j \in A^+$ such that $z_1 \dots z_j = y_1 \dots y_{i+1}$ and for all $(\dots, x_1, x_0) \in \Omega$

$$(\dots, x_1, x_0)\,((y_1 \dots y_{i+1})\Psi) = (\dots, x_1, x_0, z_1, \dots, z_j).$$

Suppose that $z_k \dots z_j = u_t$ for some $k \in \{1, \dots, j\}$ and $t \in \mathbb{N}$. Then u_t is a subword of u_n, and so of w_n. Hence $t = n$ by the hypothesis of the theorem, and thus u_n is a proper suffix of u_n, which is a contradiction. Suppose that $u_t = x_k \dots x_0 z_1 \dots z_j$ for some $k \geq 0$ and $t \in \mathbb{N}$ such that $x_0, \dots, x_k \in A^+$. Then $z_1 \dots z_j$ is a prefix of u_n and a suffix of u_t, and so $z_1 \dots z_j \in S_{\mathbf{w}}$ by the definition of $S_{\mathbf{w}}$, which contradicts the choice of u_n. So δ acts as identity on $(\dots, x_1, x_0, z_1, \dots, z_j)$. Hence the inductive hypothesis is satisfied, and by induction $(y_1 \dots y_{m-1})\Phi = (y_1 \dots y_{m-1})\Psi$. Since $y_m = b$

$$(u_n)\Phi = (u_n)\Psi \circ \delta.$$ □

Let $n \in \mathbb{N}$. It follows from Claim 7, Claim 8, Claims 9, and the fact that Φ is a homomorphism, that for all $(\dots, x_1, x_0) \in \Omega$

$$
\begin{aligned}
(\dots, x_1, x_0)(w_n)\Phi &= (\dots, x_1, x_0)\left((p_n)\Psi \circ (u_n)\Psi \circ \delta \circ (s_n)\Psi\right) \\
&= (\dots, x_1, x_0 p_n)\left((u_n)\Psi \circ \delta \circ (s_n)\Psi\right)
\end{aligned}
$$

It follows from Claims 4 and 6 there are $z_1, \dots, z_k \in A^+$ such that $z_1 \dots z_k = u_n$ and

$$
\begin{aligned}
(\dots, x_1, x_0)(w_n)\Phi &= (\dots, x_1, x_0 p_n)\left((u_n)\Psi \circ \delta \circ (s_n)\Psi\right) \\
&= (\dots, x_1, x_0 p_n, z_1, z_2, \dots, z_k)\delta \circ (s_n)\Psi
\end{aligned}
$$

Finally, by the definition of δ

$$
\begin{aligned}
(\dots, x_1, x_0)(w_n)\Phi &= (\dots, x_1, x_0 p_n, z_1, z_2, \dots, z_k)\delta \circ (s_n)\Psi \\
&= (\dots, x_1, x_0) f_n \circ (s)\Psi^{-1} \circ (s)\Psi \\
&= (\dots, x_1, x_0) f_n.
\end{aligned}
$$

Therefore $(w_n)\Phi = f_n$, and since n was arbitrary, $\{w_n : n \in \mathbb{N}\}$ is a universal sequence. □

It is worth noting that Proposition 4.3.1 is a consequence of Theorem 4.3.4. Suppose that $\mathbf{w} = \{w_n : n \in\} \subseteq A^+$ is such that p is a prefix of w_n and a suffix of w_m if and only if $p = w_n = w_m$, also suppose that w_n is a subword of w_m only if $n = m$. Then using the notion of (4.15), it follows that $X_1 = Y_1 = Z_1 = \varnothing$. Hence $S_{\mathbf{w}} = \langle 1 \rangle$, and so by Theorem 4.3.4 the sequence $\{w_n : n \in \mathbb{N}\}$ is universal for Ω^Ω.

Next we present an example of a universal sequence which can be proved to be universal with Theorem 4.3.4, but does not satisfy the hypothesis of Proposition 4.3.1.

Example 4.3.5. Let $w_n = aba(ab)^{n+1}bab \in \{a, b\}^+$ for all $n \in \mathbb{N}$, and let $\mathbf{w} = \{w_n : n \in \mathbb{N}\}$. Then for any n and $m \in \mathbb{N}$, the possible prefixes of w_n are a, ab, $aba(ab)^k$ for $k \in \{0, \dots, n+1\}$, $aba(ab)^k a$ for $k \in \{0, \dots, n\}$, $aba(ab)^i n + 1b$, $aba(aba)^{n+1}ba$, and w_n. On the other hand, the possible suffixes of w_m are b, ab, bab, and some other words which finish with b^2ab. Note that the only proper prefix of w_n which is a suffix of w_m is ab. Hence $X_0 = Y_0 = \{ab\}$, and since $Z_0 = \varnothing$, it follows that $S_1 = \{(ab)^n : n \in \mathbb{N}\}$.

Suppose now that for any $n \in \mathbb{N}$, $w_n = svuvs'$ for some $s, s' \in S_1$ and $v, u \in \{a, b\}^*$. Then $s, s' \in \{1, ab\}$, and so vuv is one of the words $a(ab)^{n+1}b$, $aba(ab)^{n+1}b$, $a(ab)^{n+1}bab$, or w_n. Observe that neither of the first three words have a proper prefix which is also a suffix, and so $X_1 = X_0$.

If for some distinct $n, m \in \mathbb{N}$, $w_n = svt$ and $w_m = t'v's$ where $s, s' \in S_1$, $t, t', v \in \mathbb{N}$. Then vt is either w_n or $a(ab)^{n+1}bab$ and similarly tv' is either w_m or $aba(ab)^{m+1}b$. Then v is either 1 or ab, and so $Y_1 = Y_0$. It is also easy to see that $Z_1 = \varnothing$. Hence $S_2 = S_1$, implying that $S_{\mathbf{w}} = \{(ab)^n : n \in \mathbb{N}\}$.

Finally, let $n \in \mathbb{N}$. Then $w_n \notin S_{\mathbf{w}}$ for all $n \in \mathbb{N}$. Then the longest prefix p_n of w_n such that $p_n \in S_{\mathbf{w}}$ is $p_n = ab$, and similarly the longest suffix s_n of w_n such that $s_n \in S_{\mathbf{w}}$ is $s_n = ab$. Hence if $w_n = p_n u_n s_n$ then $u_n = a(ab)^{n+1}b$, and so if u_n is a subword of w_m, it follows that $m = n$, also u_n is not a subword of $p_n = ab$. Therefore $\{w_n : n \in \mathbb{N}\}$ is a universal sequence for Ω^Ω by Theorem 4.3.4.

4.4 Universal lists of automorphism group of the random graph

Let R be the random graph, namely the Fraïssé limit of the class of all finite graphs. In this section we will show that $\text{Aut}(R)$ has a group universal list over an alphabet of size four.

In this section we will use a rather different approach to find a universal list. The approach is inspired by an alternative proof of Ore's Theorem, namely that $a^{-1}b^{-1}ab$ is a universal word for $\text{Sym}(\Omega)$ for a countable Ω.

Proposition 4.4.1 (see [42])**.** *Let $A = \{a, b\}$ be an alphabet and let G be a topological group such that G is a Baire space and has a comeagre conjugacy class. Then $a^{-1}b^{-1}ab$ is a group universal word for G.*

Proof. Let $f \in G$ be arbitrary, and let $\mathcal{C}$ be the comeagre conjugacy class of G. Since G is a topological group, multiplication on the right by an element of G is a homeomorphism by Corollary 1.4.13. Hence $\mathcal{C}f = \{gf : g \in \mathcal{C}\}$ is also comeagre by Proposition 1.4.14. Then the set $\mathcal{C} \cap \mathcal{C}f$ is also comeagre, and since G is a Baire space, non-empty. So there are $g, h \in \mathcal{C}$ such that $g = hf$. Since g and h belong to the same conjugacy class, there is $k \in G$ such that $g = k^{-1}hk$. Then

$$f = h^{-1}g = h^{-1}k^{-1}hk.$$

Let $\Phi : F(a, b) \longrightarrow G$ be the canonical homomorphism induced by $(a)\Phi = h$ and $(b)\Phi = k$. Then $(a^{-1}b^{-1}ab)\Phi = f$. □

As discussed in Examples 1.4.11 and 1.4.18, for countable Ω, $\text{Sym}(\Omega)$ and any closed subgroup of $\text{Sym}(\Omega)$ are topological groups which are also Baire spaces. Hence Proposition 4.4.1 implies that, in order to show that $a^{-1}b^{-1}ab$ is a group universal word for $\text{Sym}(\Omega)$ it is sufficient to show, well known result by Truss [74], that $\text{Sym}(\Omega)$ has a comeagre conjugacy class.

Proposition 4.4.2 (Theorem 3.1 [74])**.** *Let Ω be countable. Then* $\text{Sym}(\Omega)$ *has a comeagre conjugacy class.*

Proof. For $n, m \in \mathbb{N}$ and $x \in \Omega$, define the following sets

$$A_{n,m} = \{f \in \mathrm{Sym}(\Omega) : f \text{ has at least } n \text{ cycles of length } m\}$$

and

$$B_x = \{f \in \mathrm{Sym}(\Omega) : x \text{ is on a finite orbit of } f\}.$$

We will show that $A_{n,m}$ and B_x are open and dense for all $n, m \in \mathbb{N}$ and $x \in \Omega$.

Let $f \in A_{n,m}$ for some $n, m \in \mathbb{N}$. Then there is a finite subset Γ of Ω such that $f|_\Gamma$ is a product of n disjoint cycles of length m, and so $[f|_\Gamma] \subseteq A_{n,m}$. Hence $A_{n,m}$ is a union of basic open sets, and thus an open set. Similarly, if $f \in B_x$ for some $x \in \Omega$, then there is a finite orbit Γ of Ω containing x. Then $[f|_\Gamma] \subseteq B_x$, and so B_x is an open set for every $x \in \Omega$.

Suppose that q a bijection between finite subsets of Ω, in other words $q \in \mathrm{Sym}(\Omega)^{<\omega}$. Let $n, m \in \mathbb{N}$, and let $x \in \Omega$. Then there is an extension $f \in \mathrm{Sym}(\Omega)$ of q such that f at least n cycles of length m, and so $[q] \cap A_{n,m} \neq \varnothing$. Similarly, there is $f \in \mathrm{Sym}(\Omega)$ such that x is on a finite orbit of f. Hence $[q] \cap B_x \neq \varnothing$, thus both $A_{m,n}$ and B_x are dense in $\mathrm{Sym}(\Omega)$. Therefore, the set

$$\mathcal{C} = \bigcap_{n,m \in \mathbb{N}} A_{n,m} \cap \bigcap_{x \in \Omega} B_x$$

is comeagre.

Note that if $f \in \mathcal{C}$, then f has no infinite cycles and countably many cycles of any length $n \geq 1$. By a classical result, two permutations are conjugate if and only if they have the same number of cycles of length n for every $n \in \mathbb{N} \cup \{\aleph_0\}$. Therefore, $\mathcal{C}$ is the comeagre conjugacy class. □

Ore's Theorem then follows immediately from Propositions 4.4.1 and 4.4.2.

For the remainder of this section, if G is a group and $g, h \in G$ by g^h we denote the product $h^{-1}gh$. The main result of this section is stated in the following theorem.

Theorem 4.4.3. *Let $n \in \mathbb{N}$, $n > 4$, and let $w_i = a^{b^{i-1}} c^{d^{i-1}} \in F(a, b, c, d)$ for $i \in \{1, \ldots, n\}$. Then $w_1, \ldots, w_n$ is a group universal list for* $\mathrm{Aut}(R)$.

In order to make the notation easier throughout this section we will assume that the underlying vertex set of R is $\mathbb{N}$. In order to prove Theorem 4.4.3 we will need the notion of a Banach-Mazur game for a subset of a topological space.

Let X be a non-empty topological space, and let $C \subseteq X$. The Banach-Mazur game of C is a two player game, in which Player I and Player II alternate to choose non-empty open sets $U_0, V_0, U_1, V_1, \ldots$, such that $U_0 \supseteq V_0 \supseteq U_1 \supseteq V_1 \supseteq \ldots$. Player II wins the game if $\bigcap_n V_n = \bigcap_n U_n \subseteq C$, and Player I wins the game if Player II does not win. Note that the $\bigcap_n V_n = \bigcap_n U_n$ follows immediately from the fact that $U_0 \supseteq V_0 \supseteq U_1 \supseteq V_1 \supseteq \ldots$.

If X is a Baire space, a Banach-Mazur game can be used to show that a subset of X is comeagre. In order to establish this connection, we need to formally define what it means for a player to have a winning strategy. If X is a non-empty topological space, let T be the set consisting of all finite sequences $(W_0, \ldots, W_n)$ such that W_i is a non-empty open subset of X for all $i \in \{0, \ldots, n\}$ and $W_0 \supseteq \ldots \supseteq W_n$. A STRATEGY S for Player II for a Banach-Mazur game is then a subset of T such that

(i) S is non-empty;

(ii) if $(U_0, V_0, \ldots, V_n) \in S$, then $(U_0, V_0, \ldots, V_n, U_{n+1}) \in S$ for every non-empty open set U_{n+1} such that $U_{n+1} \subseteq V_n$;

(iii) if $(U_0, V_0, \ldots, U_n) \in S$, there is a unique V_n such that $(U_0, V_0, \ldots, V_n) \in S$.

Roughly speaking, a strategy is a rule which tells Player II which non-empty open set V_n to choose at any stage of the game. A strategy S is a WINNING STRATEGY FOR PLAYER II for a Banach-Mazur game of C, if for any infinite sequence $(U_0, V_0, U_1, \ldots)$ so that $(U_0, V_0, \ldots, V_n) \in S$ for all $n \in \mathbb{N}$, it follows that $\bigcap_{n \in \mathbb{N}} U_n \subseteq C$.

Theorem 4.4.4 (see Theorem 8.33 in [41]). *Let X be a non-empty Baire space, and let $C \subseteq X$. Then C is comeagre if and only if Player II has a winning strategy for the Banach-Mazur game of C.*

Proof. ($\Rightarrow$) Suppose that C is comeagre, and for $n \in \mathbb{N}$ let A_n be open and dense subsets of X so that $\bigcap_{n \in \mathbb{N}} A_n \subseteq C$. If for $n \in \mathbb{N}$, the non-empty open sets $U_0 \supseteq V_0 \supseteq \ldots \supseteq U_n$ were chosen, let Player II choose $V_n = U_n \cap A_n$. Since A_n is dense, V_n is a non-empty open set. Then

$$\bigcap_{n \in \mathbb{N}} V_n = \bigcap_{n \in \mathbb{N}} U_n \cap A_n = \bigcap_{n \in \mathbb{N}} U_n \cap \bigcap_{n \in \mathbb{N}} A_n \subseteq C,$$

and so this is a winning strategy for Player II.

($\Leftarrow$) Suppose that Player II has a winning strategy S. We will construct a non-empty subset $Q \subseteq S$ such that $\varnothing \in Q$, and for any $p = (U_0, V_0, \ldots, V_n) \in Q$, the set $\mathcal{V}_p = \{V_{n+1} : (U_0, V_0, \ldots, V_n, U_{n+1}, V_{n+1}) \in Q\}$ consists of pairwise disjoint open subsets so that $\bigcup \mathcal{V}_p$ is dense in V_n. Before showing that such Q exists we will demonstrate how to use it to prove that C is comeagre. For $n \in \mathbb{N}$, let $W_n = \bigcup \{V_n : (U_0, V_0, \ldots, V_n) \in Q\}$. Then W_n is open for all $n \in \mathbb{N}$. We will inductively show that W_n is dense in X for all $n \in \mathbb{N}$.

First, note that $W_0 = \bigcup \mathcal{V}_{\varnothing}$, and so W_0 is dense in X. Suppose that W_n is dense in X, for some $n \in \mathbb{N}$. Let U be a non-empty open subset of X. Since W_n is dense, there is V_n such that V_n intersects U non-trivially and $p = (U_0, V_0, \ldots, V_n) \in Q$. Then $\bigcup \mathcal{V}_p \subseteq W_{n+1}$, and since $\bigcup \mathcal{V}_p$ is dense in V_n, it follows that W_{n+1} is dense in V_n. It follows form the fact that $U \cap V_n$ is a non-empty open subset of V_n that W_{n+1} intersects U non-trivially. Since U is arbitrary, W_{n+1} is dense in X. Therefore, by induction W_n is open and dense for all $n \in \mathbb{N}$. If $x \in \bigcap_{n\in\mathbb{N}} W_n$, then there is a unique sequence of non-empty open sets $(U_0, V_0, U_1, \ldots)$ such that any finite prefix of the sequence is in Q, an $x \in V_n$ for all $n \in \mathbb{N}$. Since S is a winning strategy for Player II, it follows that

$$x \in \bigcap_{n\in\mathbb{N}} V_n \subseteq C,$$

and so $\bigcap_{n\in\mathbb{N}} W_n \subseteq C$, proving that C is comeagre. Therefore, it only remains to show that such collection Q exists.

Let $Q_0 = \{\varnothing\}$. Let $n \in \mathbb{N}$, and suppose that for every $k \in \{0, \ldots, n\}$ we have defined $Q_k \subseteq S$ such that every sequence in Q_k is of length $2k$. Fix $p = (U_0, V_0, \ldots, V_{n-1}) \in Q_n$. Then for every non-empty open subset $U_n \subseteq V_{n-1}$, define U_n^* to be the unique choice of Player II in the strategy S. Let $\mathcal{A}$ to be the collection of all sets A consisting of non-empty open sets $U_n \subseteq V_{n-1}$ such that $\{U_n^* : U_n \in A\}$ is a set of pairwise disjoint sets. The set $\mathcal{A}$ is partially ordered set under inclusion, and for any chain of elements in $\mathcal{A}$ the union of these elements is an upper bound of the chain. Hence by Theorem 1.1.1 (Kuratowski-Zorn Lemma), there is $\mathcal{U}_p$, a maximal collection of non-empty open subsets $U_n \subseteq V_{n-1}$ such that $\{U_n^* : U_{n-1} \in \mathcal{U}_p\}$ is a set of pairwise disjoint sets. Let Q_{n+1} the set of sequences of the form $(U_0, V_0, \ldots, V_{n-1}, U_n, U_n^*)$ for each $p = (U_0, V_0, \ldots, V_{n-1}) \in Q_n$ and $U_n \in \mathcal{U}_p$. Let $Q = \bigcup_{n\in\mathbb{N}} Q_n$.

For $p = (U_0, V_0, \ldots, V_{n-1}) \in Q$, it follows that $p \in Q_n$, and so

$$\mathcal{V}_p = \{V_n : (U_0, V_0, \ldots, V_{n-1}, U_n, V_n) \in Q_{n+1}\} = \{U_n^* : U_n \in \mathcal{U}_p\}$$

is a set of pairwise disjoint subsets. Let U be an open subset of V_{n-1}, and let U^* be the unique subset of U such that $(U_0, V_0, \ldots, V_{n-1}, U, U^*) \in S$. By maximality of $\mathcal{U}_p$ there is $U_n \in \mathcal{U}_p$ so that U^* intersects U_n^* non-trivially. Hence U intersects $\bigcup \mathcal{V}_p$ non-trivially, and so $\bigcup \mathcal{V}_p$ is dense in V_{n-1}. □

The following result by Hrushovski will be used in the proof of the main result of this section.

Theorem 4.4.5 (Hrushovski's Property, see [32])**.** *Let G be a finite graph. Then there exists a finite graph H containing G as an induced subgraph, such that every isomorphism between induced subgraphs of G extends to an automorphism of H.*

We first observe that if G is a topological group which is Baire, then every element can be obtained as a product of an element from a comeagre set with the inverse of another element from the same comeagre set.

Lemma 4.4.6 (see Theorem 7.4 in [74])**.** *Let G be a topological group and suppose that G is Baire. If $\mathcal{C}$ is a comeagre subset of G, then $\mathcal{C}^{-1}\mathcal{C} = G$.*

Proof. Let $g \in G$. Since multiplication on the right by a group element is a homeomorphism by Corollary 1.4.13, Proposition 1.4.14 implies that $\mathcal{C}g$ is comeagre. Then $\mathcal{C} \cap \mathcal{C}g$ is also comeagre and thus non-empty. Hence there are $x, y \in \mathcal{C}$ such that $x = yg$, or equivalently $g = y^{-1}x \in \mathcal{C}^{-1}\mathcal{C}$. $\square$

We will prove Theorem 4.4.3 in a series of lemmas. The key part of the proof of Theorem 4.4.3 is proving that the set

$$\{(f, f^g, \dots, f^{g^{n-1}}) : f, g \in \mathrm{Aut}(R)\}$$

is comeagre in $\mathrm{Aut}(R)^n$, which we will prove in Lemma 4.4.9 using the Banach-Mazur game. Then Lemma 4.4.9 can be used to show that the list given in Theorem 4.4.3 is indeed universal.

Before proving Lemma 4.4.9, we prove a technical result which enables us to extend isomorphisms of finite subgraphs of R in a certain way. Recall that if $x \in R$, we denote by $N(x)$ the set of all the vertices in R adjacent to x. It was brought to our attention by H. D. Macpherson that Lemma 4.4.7 has been shown in the proof of Theorem 3.1 in [36] for any Fraïssé limit of a free amalgamation class, which covers the case of the random graph.

Lemma 4.4.7 (see Theorem 3.1 in [36])**.** *Let $f \in \mathrm{Aut}(R)^{<\omega}$, and let Γ be a finite subset of R such that $\mathrm{dom}(f) \cup \mathrm{ran}(f) \subseteq \Gamma$. Then there is an extension $g \in \mathrm{Aut}(R)^{<\omega}$ of f such that the following hold:*

(i) $\mathrm{dom}(g) = \Gamma$;

(ii) $(\Gamma \setminus \mathrm{dom}(f))g \cap \Gamma = \varnothing$;

(iii) *for $x \in (\Gamma \setminus \mathrm{dom}(f))g$ and $y \in \Gamma$ we have that x and y are adjacent if and only if $y \in \mathrm{ran}(f)$ and $(x)g^{-1}$ is adjacent to $(y)f^{-1}$.*

Proof. Let $\Gamma = \{x_i : 1 \leq i \leq m\}$, for some $m \in \mathbb{N}$, and let $g_0 = f$. We will construct the required g inductively. Suppose that for some $k \in \{0, \dots, m-1\}$ we have defined $g_k \in \mathrm{Aut}(R)^{<\omega}$ such that g_k extends g_{k-1} if $k \geq 1$, $x_i \in \mathrm{dom}(g_k)$ for all $i \in \{1, \dots, k\}$, the sets $(\mathrm{dom}(g_k) \setminus \mathrm{dom}(f))g_k$ and Γ are disjoint, and if $x \in (\mathrm{dom}(g_k) \setminus \mathrm{dom}(f))g_k$ and $y \in \Gamma$ then x and y are adjacent if and only if $y \in \mathrm{ran}(f)$ and $(x)g_k^{-1}$ is adjacent to $(y)f^{-1}$.

If $x_{k+1} \in \mathrm{dom}(g_k)$, let $g_{k+1} = g_k$. Then induction hypothesis is satisfied by g_{k+1}. Suppose $x_{k+1} \notin \mathrm{dom}(g_k)$. Let $U = (N(x_{k+1}))\, g_k$ and $V = \Gamma \cup \mathrm{ran}(g_k) \setminus U$. Then by Alice's restaurant property there is $y \in R \setminus (\Gamma \cup \mathrm{ran}(g_k))$ such that y is adjacent to every vertex in U and not adjacent to every vertex in V. Hence $g_{k+1} = g_k \cup \{(x_{k+1}, y)\}$ is an isomorphism between subgraphs of R, and since R is ultrahomogeneous $g_{k+1} \in \mathrm{Aut}(R)^{<\omega}$.

It follows from the definition of g_{k+1} that

$$(\mathrm{dom}(g_{k+1}) \setminus \mathrm{dom}(f))\, g_{k+1} = (\mathrm{dom}(g_k) \setminus \mathrm{dom}(f))\, g_k \cup \{y\}. \tag{4.18}$$

Hence the sets $(\mathrm{dom}(g_{k+1}) \setminus \mathrm{dom}(f))\, g_{k+1}$ and Γ are disjoint by the inductive hypothesis and the fact that $y \notin \Gamma$.

Let $z \in \Gamma$, and let $y' \in (\mathrm{dom}(g_{k+1}) \setminus \mathrm{dom}(f))\, g_{k+1}$. By (4.18) either $y' \in (\mathrm{dom}(g_k) \setminus \mathrm{dom}(f))\, g_k$, or $y' = y$. Suppose that $y' = y$. Then since $z \in \Gamma \subseteq U \cup V$, y' is adjacent to z if and only if $z \in U$ which is equivalent $z \in \mathrm{ran}(g_k)$ and $(z)g_k^{-1} \in N(x_{k+1})$. Note that $(\mathrm{dom}(g_k) \setminus \mathrm{dom}(f))g_k = \mathrm{ran}(g_k) \setminus \mathrm{ran}(f)$ is disjoint from Γ, as g_k is an extension of f, and since $\mathrm{ran}(f) \subseteq \Gamma$, it follows that $\mathrm{ran}(g_k) \cap \Gamma = \mathrm{ran}(f)$. Hence y' is adjacent to z if and only if $z \in \mathrm{ran}(f)$ and $x_{k+1} = (y')g_{k+1}^{-1}$ is adjacent to $(z)f^{-1}$. The case where $y' \in (\mathrm{dom}(g_k) \setminus \mathrm{dom}(f))g_k$ follows immediately from the hypothesis. Hence g_{k+1} satisfies the inductive hypothesis.

Therefore there is $g = g_m \in \mathrm{Aut}(R)^{<\omega}$ such that $\mathrm{dom}(g) = \Gamma$; the sets $(\mathrm{dom}(g) \setminus \mathrm{dom}(f))g_k$ and Γ are disjoint; and if $x \in (\mathrm{dom}(g) \setminus \mathrm{dom}(f))g_k$ and $y \in \Gamma$ then x and y are adjacent if and only if $y \in \mathrm{ran}(f)$ and $(x)g^{-1}$ is adjacent to $(y)f^{-1}$, as required. □

In order to describe a winning strategy of the Banach-Mazur game we will use some extra notation. Recall that we assume that the underlying vertex set of R is $\mathbb{N}$. Let $a, b, g_1, \ldots, g_n$ be isomorphisms between finite subgraphs of R, and let $k \geq 0$. We say that $a, b, g_1, \ldots, g_n$ satisfy the property $\mathcal{G}(k)$ if the following are true:

(i) there is a finite subset $\Gamma \subseteq R$ such that $g_1, \ldots, g_n \in \mathrm{Aut}(\Gamma)$;

(ii) $\mathrm{dom}(b) \subseteq \Gamma$ is stabilised by $g_1, \ldots, g_{n-1}$ setwise;

(iii) $\mathrm{ran}(b) \subseteq \Gamma$ is stabilised by $g_2, \ldots, g_n$ setwise;

(iv) for $i \in \{1, \ldots, n-1\}$ if $x \in \mathrm{dom}(b)$ then $(x)g_i b = (x)bg_{i+1}$;

(v) $a = g_1$ and $a^{b^{i-1}} \subseteq g_i$ for all $i \in \{2, \ldots, n\}$;

(vi) $k \in \Gamma$, and $k - 1 \in \mathrm{dom}(b) \cap \mathrm{ran}(b)$ if $k > 0$.

Observe that if $a, b, f_1, \ldots, f_n \in \mathrm{Aut}(R)^{<\omega}$ are such that $a, b, f_1, \ldots, f_n$ satisfy $\mathcal{G}(k)$ for some $k \in \mathbb{N}$, and $g_1, \ldots, g_n \in \mathrm{Aut}(R)^{<\omega}$ are extensions of $f_1, \ldots, f_n$ respectively, then $a, b, g_1, \ldots, g_n$ satisfy conditions (ii) – (iv) and condition (vi) of $\mathcal{G}(k)$.

In the following lemma, we show how to obtain $a, b, h_1, \ldots, h_n \in \mathrm{Aut}(R)^{<\omega}$ which satisfy $\mathcal{G}(k + 1)$. This will serve as a step in the Banach-Mazur game.

Lemma 4.4.8. *Suppose that $a, b, f_1, \ldots, f_n \in \mathrm{Aut}(R)^{<\omega}$ satisfy $\mathcal{G}(k)$, and let $g_1, \ldots, g_n \in \mathrm{Aut}(R)^{<\omega}$ be extensions of $f_1, \ldots, f_n$ respectively. Then there are extensions $c, d, h_1, \ldots, h_n \in \mathrm{Aut}(R)^{<\omega}$ of $a, b, g_1, \ldots, g_n$ satisfying $\mathcal{G}(k+1)$.*

Proof. By Theorem 4.4.5 there is Γ' a finite subgraph of R and extensions $h_1, \ldots, h_n \in \mathrm{Aut}(\Gamma')$ of $g_1, \ldots, g_n$ respectively. Hence without loss of generality we may assume that there is a finite subgraph Γ' of R such that $g_1, \ldots, g_n \in \mathrm{Aut}(\Gamma')$. Then $a, b, g_1, \ldots, g_n$ satisfy conditions (i) – (iv) and condition (vi) of $\mathcal{G}(k)$ by the observation above.

The proof consists of two major steps. First, we will show that are $p, q, r_1, \ldots, r_n \in \mathrm{Aut}(R)^{<\omega}$ extending $a, b, g_1, \ldots, g_n$ respectively, such that $p, q, r_1, \ldots, r_n$ satisfy conditions (i) – (v) of $\mathcal{G}(k+1)$ and $k \in \mathrm{ran}(q)$. The next step is to extend $p, q, r_1, \ldots, r_n$ to $c, d, h_1, \ldots, h_n \in \mathrm{Aut}(R)^{<\omega}$ respectively, so that $c, d, h_1, \ldots, h_n$ satisfy $\mathcal{G}(k+1)$.

Since $\mathrm{dom}(b^{-1}) \cup \mathrm{ran}(b^{-1}) = \mathrm{dom}(b) \cup \mathrm{ran}(b) \subseteq \Gamma \subseteq \Gamma'$, we may apply Lemma 4.4.7 to b^{-1} and Γ'. Then there is $q^{-1} \in \mathrm{Aut}(R)^{<\omega}$ extending b^{-1} such that $\mathrm{ran}(q) = \mathrm{dom}(q^{-1}) = \Gamma'$, $\Delta \cap \Gamma' = \varnothing$ where $\Delta = (\Gamma' \setminus \mathrm{ran}(b))\, q^{-1}$, and

$$x \in \Delta \text{ is adjacent to } y \in \Gamma' \text{ if and only if } y \in \mathrm{dom}(b) \text{ and } (x)q \text{ is adjacent to } (y)b. \tag{4.19}$$

Since q is injective, it follows that $\Delta = (\Gamma')q^{-1} \setminus (\mathrm{ran}(b))q^{-1} = (\Gamma')q^{-1} \setminus \mathrm{dom}(b)$. Hence Δ is disjoint from $\mathrm{dom}(b)$. Also note that $\mathrm{dom}(b) = (\mathrm{ran}(b))q^{-1} \subseteq (\Gamma')q^{-1}$, and so

$$\mathrm{dom}(q) = (\mathrm{ran}(q))q^{-1} = (\Gamma')q^{-1} = \Delta \cup \mathrm{dom}(b). \tag{4.20}$$

Let $i \in \{1, \ldots, n-1\}$, and let $\phi_i : \Delta \longrightarrow R$ be defined by $(x)\phi_i = (x)qg_{i+1}q^{-1}$. Then ϕ_i is an isomorphism between finite subgraphs of R. Since $g_2, \ldots, g_n$ all stabilise both Γ' and $\mathrm{ran}(b)$ setwise by condition (iii) of $\mathcal{G}(k)$, it follows that $g_2, \ldots, g_n$ stabilise $\Gamma' \setminus \mathrm{ran}(b)$ setwise. Hence ϕ_i stabilises $\Delta = (\Gamma' \setminus \mathrm{ran}(b))q^{-1}$ setwise, in other words $\phi_i \in \mathrm{Aut}(\Delta)$. We will now show that $g_i \cup \phi_i$ is an isomorphism between finite subgraphs of R. Since both g_i and ϕ_i are isomorphisms, it is enough to consider $x \in \mathrm{dom}(\phi_i) = \Delta$ and $y \in \mathrm{dom}(g_i)$ and show that x and y are adjacent if and only if $(x)\phi_i$ and $(y)g_i$ are adjacent. First, suppose that $y \in \mathrm{dom}(g_i) \setminus \mathrm{dom}(b)$. Then $(y)g_i \notin \mathrm{dom}(b)$, since g_i stabilises $\mathrm{dom}(b)$. Hence x is not adjacent to y and $(x)\phi_i$ is not adjacent to $(y)g_i$ by (4.19).

Suppose that $y \in \mathrm{dom}(b)$, then $(y)g_i \in \mathrm{dom}(b)$, and so

$$\begin{aligned}
x \sim y &\iff (x)q \sim (y)b && \text{by (4.19)}\\
&\iff (x)qg_{i+1} \sim (y)bg_{i+1} && \text{since } (x)q, (y)b \in \Gamma'\\
&\iff (x)\phi_i q \sim (y)g_i b && \text{by (iv) and the definition of } \phi_i\\
&\iff (x)\phi_i \sim (y)g_i && \text{by (4.19).}
\end{aligned}$$

Hence $g_i' = g_i \cup \phi_i$ is an isomorphism between finite subgraphs of R. Let $g_n' = g_n$. Then $g_i' \in \mathrm{Aut}(R)^{<\omega}$ for all $i \in \{1, \ldots, n\}$. By Theorem 4.4.5 there is a finite subset Γ'' of R and $r_1, \ldots, r_n \in \mathrm{Aut}(\Gamma'')$ such that $\Gamma' \subseteq \Gamma''$ and $r_1, \ldots, r_n$ extend $g_1', \ldots, g_n'$ respectively.

Let $p = r_1$. Next we will show that $p, q, r_1, \ldots, r_n$ satisfy conditions (i) – (v) of $\mathcal{G}(k+1)$. First, note that $p, q, r_1, \ldots, r_n$ satisfy condition (i) of $\mathcal{G}(k+1)$ by the choice of $r_1, \ldots, r_n$. It follows from (4.20) that $\mathrm{dom}(q) = \mathrm{dom}(b) \cup \Delta$ and from the choice of q that $\mathrm{ran}(q) = \Gamma'$. Since $g_1, \ldots, g_{n-1}$ stabilise $\mathrm{dom}(b)$ setwise by condition (ii) of $\mathcal{G}(k)$, and $g_1', \ldots, g_{n-1}'$ stabilise Δ setwise by the definition, it follows that $r_1, \ldots, r_{n-1}$ stabilise $\mathrm{dom}(q)$ setwise. Also $g_1, \ldots, g_n \in \mathrm{Aut}(\Gamma')$, thus $r_1, \ldots, r_n$ stabilise $\mathrm{ran}(q)$ setwise. Hence conditions (ii) and (iii) of $\mathcal{G}(k+1)$ hold for $p, q, r_1, \ldots, r_n$.

Let $x \in \mathrm{dom}(q)$, and let $i \in \{1, \ldots, n-1\}$. If $x \in \mathrm{dom}(b)$, then $x \in \mathrm{dom}(g_i)$ and $(x)g_i \in \mathrm{dom}(b)$, since $a, b, g_1, \ldots, g_n$ satisfy condition (ii) of $\mathcal{G}(k)$. Hence

$$(x)r_i q = (x)g_i b = (x)b g_{i+1} = (x)q r_{i+1}$$

by condition (iv) of $\mathcal{G}(k)$. Suppose that $x \in \mathrm{dom}(q) \setminus \mathrm{dom}(b) = \Delta$. Then from the definition of ϕ_i, it follows that

$$(x)r_i q = (x)\phi_i q = (x)q g_{i+1} q^{-1} q = (x)q r_{i+1}$$

and thus (iv) of $\mathcal{G}(k+1)$ is satisfied.

Let $i \in \{2, \ldots, n\}$. Note that $p = r_1$, and suppose that $p^{q^{j-1}} \subseteq r_j$ for $j \in \{1, \ldots, i-1\}$. If $x \in \mathrm{ran}(q)$, then by condition (iv) of $\mathcal{G}(k+1)$ for $p, q, r_1, \ldots, r_n$

$$(x)r_i = (x)q^{-1} r_{i-1} q = (x)q^{-1} p^{q^{i-2}} q = (x)p^{q^{i-1}}.$$

If $x \notin \mathrm{ran}(q)$, then $x \notin \mathrm{dom}(p^{q^{i-1}})$, and so $p^{q^{i-1}} \subseteq r_i$. Therefore, induction on i shows that condition (v) of $\mathcal{G}(k+1)$ holds for $p, q, r_1, \ldots, r_n$, as required. Since $k \in \Gamma$ by condition (vi) of $\mathcal{G}(k)$, the fact that $\Gamma \subset \mathrm{ran}(q)$ implies that $k \in \mathrm{ran}(q)$.

Now we will extend q to $d \in \mathrm{Aut}(R)^{<\omega}$ so that $k \in \mathrm{dom}(d)$. By Lemma 4.4.7 applied to q and Γ'' there is an extension $d \in \mathrm{Aut}(R)^{<\omega}$ of q such that if $\Sigma = (\Gamma'' \setminus \mathrm{dom}(q))\, d$ then $\mathrm{dom}(d) = \Gamma''$, $\Sigma \cap \Gamma'' = \varnothing$, and

$$x \in \Sigma \text{ is adjacent to } y \in \Gamma'' \text{ if and only if } y \in \mathrm{ran}(q) \text{ and } (x)d^{-1} \text{ is adjacent to } (y)q^{-1}. \tag{4.21}$$

Since d is injective $\Sigma = (\Gamma'')d \setminus (\mathrm{dom}(q))d = (\Gamma'')d \setminus \mathrm{ran}(q)$. Hence Σ is disjoint from $\mathrm{ran}(q)$. Note that $\mathrm{ran}(q) = (\mathrm{dom}(q))d \subseteq (\Gamma'')d$, and so

$$\mathrm{ran}(d) = (\mathrm{dom}(d))d = (\Gamma'')d = \Sigma \cup \mathrm{ran}(q). \tag{4.22}$$

Let $i \in \{2, \ldots, n\}$, and let $\psi_i : \Sigma \longrightarrow R$ be defined by $(x)\psi_i = (x)d^{-1}r_{i-1}d$. Then ψ_i is an isomorphism between finite subgraphs of R. Since r_i stabilises both Γ'' and $\mathrm{dom}(q)$ setwise, it follows that r_i stabilises $\Gamma'' \setminus \mathrm{dom}(q)$ setwise. Hence ψ_i stabilises Σ setwise, in other words $\psi_i \in \mathrm{Aut}(\Sigma)$. We will now show that $r_i \cup \psi_i$ is an isomorphism. Since both r_i and ψ_i are isomorphisms, it is enough to show that $x \in \Sigma$ is adjacent to $y \in \Gamma'' = \mathrm{dom}(r_i)$ if and only if $(x)\psi_i$ is adjacent to $(y)r_i$. Let $x \in \Sigma$. First, suppose that $y \in \mathrm{dom}(r_i) \setminus \mathrm{ran}(q)$. Then $(y)r_i \notin \mathrm{ran}(q)$, since r_i stabilises $\mathrm{ran}(q)$ by condition (iii) of $\mathcal{G}(k+1)$. Hence x is not adjacent to y and $(x)\psi_i$ is not adjacent to $(y)r_i$ by (4.21).

Suppose that $y \in \mathrm{ran}(q)$. Then there is $z \in \mathrm{dom}(q)$ such that $y = (z)q$. It follows from condition (iv) of $\mathcal{G}(k+1)$ that $(z)r_{i-1}q = (z)qr_i$. Hence

$$(y)q^{-1}r_{i-1} = (y)r_iq^{-1}, \tag{4.23}$$

and since $(y)r_i \in \mathrm{ran}(q)$

$$\begin{aligned}
x \sim y \iff & (x)d^{-1} \sim (y)q^{-1} && \text{by (4.21)}\\
\iff & (x)d^{-1}r_{i-1} \sim (y)q^{-1}r_{i-1} && \text{since } (x)q, (y)d \in \Gamma'' = \mathrm{dom}(r_{i-1})\\
\iff & (x)\psi_i d^{-1} \sim (y)r_iq^{-1} && \text{by (4.23) and the definition of } \psi_i\\
\iff & (x)\psi_i \sim (y)r_i && \text{by (4.21).}
\end{aligned}$$

Hence $r'_i = r_i \cup \psi_i$ is an isomorphisms between finite subgraphs of R. Let $r'_1 = r_1$. Then $r'_i \in \mathrm{Aut}(R)^{<\omega}$ for all $i \in \{1, \ldots, n\}$. By Theorem 4.4.5 there are $\Gamma''' \subseteq R$ and $h_1, \ldots, h_n \in \mathrm{Aut}(\Gamma''')$ such that Γ''' is finite, $\Gamma'' \subseteq \Gamma'''$, and $h_1, \ldots, h_n$ extends $r'_1, \ldots, r'_n$ respectively. Moreover, if necessary, by first extending r'_1 to some finite isomorphism r''_1 such that $k+1 \in \mathrm{dom}(r''_1)$, we may assume that $k+1 \in \Gamma'''$.

Let $c = h_1$. We will now show that $c, d, h_1, \ldots, h_n$ satisfy $\mathcal{G}(k+1)$. By the choice of d we have that $\mathrm{dom}(d) = \Gamma''$ and $\mathrm{ran}(d) = \mathrm{ran}(q) \cup \Sigma$ by (4.22). Since $r_2, \ldots, r_n$ all stabilise $\mathrm{ran}(q)$ setwise by condition (iii) of $\mathcal{G}(k+1)$, and all $r'_1, \ldots, r'_n$ stabilise Σ by the definition, it follows that $h_2, \ldots, h_n$ stabilise $\mathrm{ran}(d)$ setwise. Also $r_i \in \mathrm{Aut}(\Gamma'')$ for all $i \in \{1, \ldots, n\}$, and so r_i stabilise $\Gamma'' = \mathrm{dom}(d)$ setwise. Hence conditions (i), (ii), and (iii) of $\mathcal{G}(k+1)$ hold for $c, d, h_1, \ldots, h_n$.

Let $x \in \mathrm{dom}(d)$, and let $i \in \{1, \ldots, n-1\}$. If $x \in \mathrm{dom}(q) \subseteq \Gamma''$, then $x \in \mathrm{dom}(r_i)$ and $(x)r_i \in \mathrm{dom}(q)$, since r_i stabilise $\mathrm{dom}(q)$ by condition (ii) of $\mathcal{G}(k+1)$ for $p, q, r_1, \ldots, r_n$. Hence by the condition (iv) of $\mathcal{G}(k+1)$

$$(x)h_id = (x)r_iq = (x)qr_{i+1} = (x)dh_{i+1}.$$

Otherwise $x \in \mathrm{dom}(d) \setminus \mathrm{dom}(q) = (\Sigma)d^{-1}$, thus $(x)d \in \Sigma$. Then by the definition of ψ_i we have

$$(x)dh_{i+1} = (x)dd^{-1}r_id = (x)h_id$$

and thus (iv) of $\mathcal{G}(k+1)$ is satisfied.

Let $i \in \{2, \dots, n\}$, and recall that $c = h_1$. Suppose that $c^{d^{j-1}} \subseteq h_j$ for all $j \in \{1, \dots, i-1\}$. If $x \in \operatorname{ran}(d)$, then by condition (iv) of $\mathcal{G}(k+1)$

$$(x)h_i = (x)d^{-1}h_{i-1}d = (x)d^{-1}c^{d^{i-2}}d = (x)c^{d^{i-1}}.$$

If $x \notin \operatorname{ran}(d)$, then $x \notin \operatorname{dom}(c^{d_{i-1}})$, and so $c^{d^{i-1}} \subseteq h_i$. Therefore, by induction condition (v) of $\mathcal{G}(k+1)$ holds for $c, d, h_1, \dots, h_n$.

Finally, $k \in \Gamma' \subseteq \operatorname{dom}(d) \cap \operatorname{ran}(d)$, and we have chosen Γ''' in such a way that $k+1 \in \Gamma'''$. Therefore $c, d, h_1, \dots, h_n$ satisfy condition (vi) of $\mathcal{G}(k+1)$ and therefore $\mathcal{G}(k+1)$. □

Next we will use $\mathcal{G}(k)$ to show that $\{(f, f^g, \dots, f^{g^{n-1}}) : f, g \in \operatorname{Aut}(R)\}$ is comeagre.

Lemma 4.4.9. *The set* $\mathcal{C} = \{(f, f^g, \dots, f^{g^{n-1}}) : f, g \in \operatorname{Aut}(R)\}$ *is comeagre in* $\operatorname{Aut}(R)^n$.

Proof. We will show that Player II has a winning strategy for a Banach-Mazur game of $\mathcal{C}$.

Let U_0 be an open set, and let B be any basic open set such that $B \subseteq U_0$. Then there are $h_1, \dots, h_n \in \operatorname{Aut}(R)^{<\omega}$ such that $B = [h_1] \times \dots \times [h_n]$. By Theorem 4.4.5 there is a finite $\Gamma_0 \subseteq R$ and $g_{0,1}, \dots, g_{0,n} \in \operatorname{Aut}(\Gamma_0)$ extending $h_1, \dots, h_n$ respectively such that $0 \in \Gamma_0$. Let $V_0 = [g_{0,1}] \times \dots \times [g_{0,n}]$, let $a_0 = g_{0,1}$, and let $b_0 = \varnothing$. Since $\operatorname{dom}(b_0) = \operatorname{ran}(b_0) = \varnothing$, it follows that $a_0, b_0, g_{0,1}, \dots, g_{0,n}$ satisfy $\mathcal{G}(0)$.

Suppose that $U_0, V_0, \dots, V_{k-1} \subseteq \operatorname{Aut}(R)^n$ are chosen such that $U_0 \supseteq V_0 \supseteq \dots \supseteq V_{k-1}$; and for all $i \in \{1, \dots, k-1\}$ there are $a_i, b_i, g_{i,1}, \dots, g_{i,n} \in \operatorname{Aut}(R)^{<\omega}$ such that $V_i = [g_{i,1}] \times \dots \times [g_{i,n}]$, a_i and b_i extensions of a_{i-1} and b_{i-1} respectively, and $a_i, b_i, g_{i,1}, \dots, g_{i,n}$ satisfy $\mathcal{G}(i)$. Let $U_k \subseteq V_{k-1}$ be an arbitrary open set, and let $B \subseteq U_k$ be a basic open set. Then there are $h_1, \dots, h_n \in \operatorname{Aut}(R)^{<\omega}$ such that $B = [h_1] \times \dots \times [h_n]$. Since $[g_{k-1,1}] \times \dots \times [g_{k-1,n}] = V_{k-1} \supseteq U_k$, it follows that $h_1, \dots, h_n$ extend $g_{k-1,1}, \dots, g_{k-1,n}$ respectively. By Lemma 4.4.8 there are $a_k, b_k, g_{k,1}, \dots, g_{k,n} \in \operatorname{Aut}(R)^{<\omega}$, extensions of $a_{k-1}, b_{k-1}, h_1, \dots, h_n$ respectively, satisfying $\mathcal{G}(k)$. Let $V_k = [g_{k,1}] \times \dots \times [g_{k,n}]$.

Suppose now that we have a sequence of open sets $U_0 \supseteq V_0 \supseteq U_1 \supseteq \dots$, where V_i are chosen using the strategy described above. Let

$$a = \bigcup_{i \in \mathbb{N}} a_i, \quad b = \bigcup_{i \in \mathbb{N}} b_i, \quad \text{and} \quad g_j = \bigcup_{i \in \mathbb{N}} g_{i,j} \quad \text{for all } j \in \{1, \dots, n\}.$$

It follows from condition (i) of $\mathcal{G}(k)$ that $\operatorname{dom}(g_j) = \operatorname{ran}(g_j) = \bigcup_{i \in \mathbb{N}} \Gamma_i$. By (vi) of $\mathcal{G}(k)$ we have that $k \in \Gamma_k$, and so $\bigcup_{i \in \mathbb{N}} \Gamma_i = R$. Since $g_{i,j} \in \operatorname{Aut}(R)^{<\omega}$ for all

i and j, it follows that $g_j \in \mathrm{Aut}(R)$ for all $j \in \{1, \ldots, n\}$. Then

$$\bigcap_{i\in\mathbb{N}} V_i = \bigcap_{i\in\mathbb{N}} [g_{i,1}] \times \ldots \times [g_{i,n}] = \left(\bigcap_{i\in\mathbb{N}} [g_{i,1}]\right) \times \ldots \times \left(\bigcap_{i\in\mathbb{N}} [g_{i,n}]\right) = \{(g_1, \ldots, g_n)\}.$$

Hence to show that the set $\mathcal{C}$ is comeagre, it is sufficient to demonstrate that $(g_1, \ldots, g_n) \in \mathcal{C}$.

By condition (vi) of $\mathcal{G}(k)$ we have that $k-1 \in \mathrm{dom}(b_k) \cap \mathrm{ran}(b_k)$. Since $b_i \in \mathrm{Aut}(R)^{<\omega}$ for all $i \in \mathbb{N}$, it follows that $b \in \mathrm{Aut}(R)$. Also $(x)a = (x)g_1$ for all $x \in \bigcup_{i\in\mathbb{N}} \Gamma_i$ by (v) of $\mathcal{G}(k)$, thus $a = g_1 \in \mathrm{Aut}(R)$.

Finally, let $x \in R$, and let $i \in \{1, \ldots, n\}$. We will show that $(x)a^{b^{i-1}} = (x)g_i$. Let

$$\Delta = \left\{x, (x)b^{-1}, \ldots, (x)b^{-i+1}, (x)b^{-i+1}a, (x)b^{-i+1}ab, \ldots, (x)b^{-i+1}ab^{i-1}\right\}.$$

Since $\Delta \subseteq R$ is finite, we may find $k \in \mathbb{N}$ such that $\max(\Delta) < k$. Hence $\Delta \subseteq \Gamma_k$ and $\Delta \subseteq \mathrm{dom}(b_k) \cap \mathrm{ran}(b_k)$. Then by the choice of Δ, $x \in \mathrm{dom}(a_k^{b_k^{i-1}})$, and thus by (v) of $\mathcal{G}(k)$

$$(x)a^{b^{i-1}} = (x)a_k^{b_k^{i-1}} = (x)g_{k,i} = (x)g_i.$$

Hence $g_i = a^{b^{i-1}}$ for all $i \in \{1, \ldots, n\}$, and so

$$\bigcap_{i\in\mathbb{N}} V_i = \{(g_1, \ldots, g_n)\} \subseteq \mathcal{C}.$$

Therefore, $\mathcal{C}$ is comeagre by Theorem 4.4.4. □

Finally, we can prove the main result of this section.

Proof of Theorem 4.4.3. First of all, recall that $\mathrm{Aut}(R)$ is a closed subgroup of $\mathrm{Sym}(\Omega)$ by Theorem 1.6.3. Hence $\mathrm{Aut}(R)$ is a Polish group by Example 1.4.11, and so by Proposition 1.4.5, the product $\mathrm{Aut}(R)^n$ is a Polish group for all $n \in \mathbb{N}$. Therefore $\mathrm{Aut}(R)^n$ is a Baire space by Theorem 1.4.17.

Let $\mathcal{C} = \{(f, f^g, \ldots, f^{g^{n-1}}) : f, g \in \mathrm{Aut}(R)\}$. Then $\mathcal{C}$ is comeagre in $\mathrm{Aut}(R)^n$ by Lemma 4.4.9. It is easy to see that $\mathcal{C}^{-1} = \mathcal{C}$. Since $\mathrm{Aut}(R)^n$ is a topological group and also a Baire space, Lemma 4.4.6 implies that

$$\mathrm{Aut}(R)^n = \mathcal{C}\mathcal{C} = \{(fh, \ldots, f^{g^{n-1}}h^{r^{n-1}}) : f, g, h, r \in \mathrm{Aut}(R)\}. \tag{4.24}$$

Let $w_i = a^{b^{i-1}}c^{d^{i-1}}$ for all $i \in \{1, \ldots, n\}$. Let $p_1, \ldots, p_n \in \mathrm{Aut}(R)$. By (4.24) there are $f, g, h, r \in \mathrm{Aut}(R)$ such that $p_i = f^{g^{i-1}}h^{r^{i-1}}$ for all $i \in \{1, \ldots, n\}$. Let $\Phi : F(a, b, c, d) \longrightarrow \mathrm{Aut}(R)$ to be the canonical homomorphism induced by

$$(a)\Phi = f, \quad (b)\Phi = g, \quad (c)\Phi = h, \quad \text{and} \quad (d)\Phi = r.$$

Then $(w_i)\Phi = p_i$ for all $i \in \{1, \ldots, n\}$, and so $w_1, \ldots, w_n$ is a group universal list for $\text{Aut}(R)$. □

The proof of Lemma 4.4.8 made extensive use of Hrushovski property to find a finite set which was stabilised setwise by a collection of n isomorphisms between finite subgraphs of R. If we tried to adapt the same argument to find a universal sequence for $\text{Aut}(R)$, we would need to find a set X stabilised setwise by countably many isomorphism between finite subgraphs of R. However, in this case it is impossible to ensure that X is finite. Hence the following question remains open.

Question 4.4.10. *Is there a finite alphabet A such that there exists a universal sequence $\{w_n : n \in \mathbb{N}\} \subseteq F(A)$ for* $\text{Aut}(R)$*?*

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