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## **DISCUSSION PAPER PI-1107**

### **The Cost of Counterparty Risk and Collateralization in Longevity Swaps**

Enrico Biffis, David Blake, Lorenzo Pitotti, and Ariel Sun

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The Pensions Institute  
Cass Business School  
City University London  
106 Bunhill Row  
London EC1Y 8TZ  
UNITED KINGDOM

<http://www.pensions-institute.org/>

# The cost of counterparty risk and collateralization in longevity swaps\*

Enrico Biffis<sup>†</sup>    David Blake<sup>‡</sup>    Lorenzo Pitotti<sup>§</sup>    Ariel Sun<sup>†</sup>

## Abstract

Derivative longevity risk solutions, such as bespoke and indexed longevity swaps, allow pension schemes and annuity providers to swap out longevity risk, but introduce counterparty credit risk, which can be mitigated if not fully eliminated by collateralization. We examine the impact of bilateral default risk and collateral rules on the marking to market of longevity swaps, and show how longevity swap rates must be determined endogenously from the collateral flows associated with the marking-to-market procedure. For typical interest rate and mortality parameters, we find that the impact of collateralization is modest in the presence of symmetric default risk, but more pronounced when default risk and/or collateral rules are asymmetric. Our results suggest that the overall cost of collateralization is comparable with, and often much smaller than, that found in the interest-rate swaps market, which may then provide the appropriate reference framework for the credit enhancement of both indemnity-based and indexed longevity risk solutions.

*Keywords:* longevity swap, counterparty risk, default risk, collateral, marking-to-market.

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<sup>†</sup>Biffis (E.Biffis@imperial.ac.uk) and Sun (jingjing.sun08@imperial.ac.uk) are at Imperial College Business School, Imperial College London, South Kensington Campus, SW7 2AZ United Kingdom.

<sup>‡</sup>Blake (D.Blake@city.ac.uk) is at the Pensions Institute, Cass Business School, 106 Bunhill Row, London, EC1Y 8TZ, United Kingdom.

<sup>§</sup>Pitotti (Lorenzo.Pitotti@credit-suisse.com) is at Credit Suisse Securities Ltd, One Cabot Square, London E14 4QJ, United Kingdom.

# 1 Introduction

The market for longevity-linked securities and derivatives has recently experienced a surge in transactions in longevity swaps (Blake et al., 2013). These pure longevity hedges are agreements between two parties to exchange fixed payments against variable payments linked to the number of survivors in a reference population (see Dowd et al., 2006). Table 1 presents a list of recent deals that have been publicly disclosed. So far, transactions have mainly involved pension funds and annuity providers wanting to hedge their exposure to longevity risk<sup>1</sup> but without having to bear any basis risk. The variable payments in such longevity swaps are designed to match precisely the mortality experience of each individual hedger: hence the name bespoke longevity swaps. This is essentially a form of longevity risk insurance, similar to annuity reinsurance in reinsurance markets. Indeed, most of the longevity swaps executed to date have been bespoke, indemnity-based swaps of the kind familiar in reinsurance markets. This is true despite the fact that some of the swaps listed in table 1 have been arranged by investment banks: the banks have worked with insurance companies (in some cases insurance company subsidiaries) in order to deliver a solution in a format familiar to the counterparty. A fundamental difference from other forms of reinsurance, however, is that longevity swaps are typically collateralized, whereas typical insurance/reinsurance transactions are not.<sup>2</sup> The main reason is that longevity swaps are often part of a wider de-risking strategy involving other collateralized instruments (interest-rate and inflation swaps, for example), and also the fact that hedgers have been increasingly concerned with counterparty risk<sup>3</sup> in the wake of the Global Financial Crisis of 2008-09. In this article, we provide a framework to quantify the trade-off between the exposure to counterparty risk in longevity swaps and the cost of credit enhancement strategies such as collateralization.

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<sup>1</sup>By longevity risk we mean exposure to the systematic risk of mortality improvements, which cannot be mitigated by pooling together large numbers of lives.

<sup>2</sup>One rationale for this is that reinsurers aggregate several uncorrelated risks and pooling/diversification benefits compensate for the absence of collateral (e.g., Cummins and Trainar, 2009; Lakdawalla and Zanjani, 2012; Kessler, 2013). Insurers/reinsurers are still required by their regulators to post regulatory or solvency capital which plays a similar role to collateral but at aggregate level.

<sup>3</sup>Basel II (2006, Annex 4) defines counterparty risk as ‘the risk that the counterparty to a transaction could default before the final settlement of the transaction’s cash flows’. The recent Solvency II proposal makes explicit allowance for a counterparty risk module in its ‘standard formula’ approach; see CEIOPS (2009).

As there is no accepted framework yet for marking to market/model longevity swaps, hedgers and hedge suppliers look to other markets to provide a reference model for counterparty risk assessment and mitigation. In interest-rate swap markets, for example, the most common form of credit enhancement is the posting of collateral. According to the International Swap and Derivatives Association (ISDA) almost every swap at major financial institutions is ‘bilaterally’ collateralized (ISDA, 2010a), meaning that either party is required to post collateral depending on whether the market value of the swap is positive or negative.<sup>4</sup> The vast majority of transactions is collateralized according to the Credit Support Annex (CSA) to the Master Swap Agreement introduced by ISDA (1994). The Global Financial Crisis highlighted the importance of bilateral counterparty risk and collateralization for over-the-counter markets, spurring a number of responses (e.g, ISDA, 2009; Brigo and Capponi, 2009; Assefa et al., 2010; Brigo et al., 2011, 2012). The Dodd-Frank Wall Street Reform and Consumer Protection Act (signed into law by President Barack Obama on July 21, 2010) is likely to have a major impact on the way financial institutions will manage counterparty risk in the coming years.<sup>5</sup> The recently established Life and Longevity Markets Association (LLMA)<sup>6</sup> has counterparty risk at the center of its agenda, and will certainly draw extensively from the experience garnered in fixed-income and credit markets.

Collateralization strategies address the concerns aired by pension trustees regarding the efficacy of longevity swaps, but introduce another dimension in the traditional pricing framework used for insurance transactions. The ‘insurance premium’ embedded in a longevity swap rate reflects not only the aversion/appetite of the counterparties for the risk being transferred and the cost of regulatory capital involved in the transaction, but also the expected costs to be incurred from posting collateral during the life of the swap. To understand these costs, let us first take the perspective of a hedge supplier (reinsurer or investment bank) issuing a collateralized longevity swap to a counterparty (pension fund

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<sup>4</sup>Unlike a firm’s exposure to credit risk through a loan, where the exposure to credit risk is unilateral and only the lending bank faces the risk of loss, counterparty credit risk creates a bilateral risk of loss: the market value of the transaction can be positive or negative to either counterparty to the transaction. The market value is uncertain and can vary over time with the movement of underlying market factors.’ (Basel II, 2006, Annex 4).

<sup>5</sup>See, for example, ‘Berkshire may scale back derivative sales after Dodd-Frank’, *Bloomberg*, August 10, 2010.

<sup>6</sup>See <http://www.llma.org>.

or annuity provider). Whenever the swap is sufficiently out-of-the-money to the hedge supplier, the hedge supplier is required to post collateral, which can be used by the hedger to mitigate losses in the event of default. Although interest on collateral is typically rebated, there is both a funding cost and an opportunity cost, as the posting of collateral depletes the resources the hedge supplier can use to meet her capital requirements at aggregate level as well as to write additional business. On the other hand, whenever the swap is sufficiently in-the-money to the hedge supplier, the hedge supplier will receive collateral from the counterparty, thus benefiting from capital relief in regulatory valuations and freeing up capital that can be used to sell additional longevity protection. The benefits can be far larger if collateral can be re-pledged for other purposes, as in the interest-rate swaps market.<sup>7</sup> The same considerations can be made from the viewpoint of the hedger, but the funding needs and opportunity costs of the two parties are unlikely to offset each other exactly. This is particularly relevant for transactions involving parties subject to different regulatory frameworks. In the UK and several other countries, for example, longevity risk exposures are more capital intensive for hedge suppliers, such as insurers, than for pension funds.<sup>8</sup>

In the absence of collateral, and ignoring longevity risk aversion, swap rates<sup>9</sup> depend on best estimate survival probabilities for the hedged population and on the degree of covariation between the floating leg of the swap and the defaultable term structure of interest rates facing the hedger and the hedge supplier. This means that a proper analysis of a longevity swap cannot disregard the sponsor's covenant when the hedger is a pension plan (see section 3 below). In the presence of collateralization, longevity swap rates are also shaped by the expected collateral costs, and swap valuation formulae involve a discount rate reflecting the cost of collateral. As a result, default-free valuation formulae are not appropriate even in the presence of full collateralization and the corresponding absence of default losses.<sup>10</sup>

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<sup>7</sup>According to ISDA (2010a), the vast majority of collateral is rehypothecated for other purposes in interest-rate swap markets. Currently, collateral can be re-pledged under the New York Credit Support Annex, but not under the English Credit Support Deed (see ISDA, 2010b).

<sup>8</sup>This asymmetry is, in part, a by-product of rules allowing, for example, pension liabilities to be quantified by using outdated mortality tables or discount rates reflecting optimistic expected returns.

<sup>9</sup>Defined as the rates in the fixed legs of the swap zeroing its market value at inception.

<sup>10</sup>See Johannes and Sundareshan (2007) for the case of symmetric default risk and full collateralization in interest-rate swaps.

We quantify collateral costs in two ways: i) in terms of funding costs that are incurred or mitigated when collateral is posted or received, and ii) as the opportunity cost of selling additional longevity protection. In both cases, we find that, for typical interest rate and mortality parameters, the impact of collateralization on swap rates is modest when default risk and collateral rules are symmetric. The main reason is that longevity risk and interest-rate risk have countervailing effects that dilute the overall impact of collateralization on swap rates:

i) On the one hand, the receiver of the fixed survival rate (the hedge supplier) posts collateral when mortality is lower and hence longevity exposures are more capital intensive. On the other hand, she receives collateral when mortality is higher and longevity protection is less capital intensive. The overall effect is to push swap rates higher, to compensate the hedge supplier for the positive dependence between collateral posting and capital costs.

ii) When the hedger or hedge supplier is out-of-the-money, collateral outflows are larger in low interest rate environments (i.e., when liabilities are discounted at a lower rate), hence there is a negative relationship between the amount of collateral posted and the counterparties' funding/opportunity costs. This mitigates the overall impact of collateralization on longevity swap rates.

When default risk and/or collateral rules are asymmetric, the opposing effects are of different magnitudes and, as a result, the impact of collateral costs on longevity swap rates is larger. For example, we find that swap rates decrease substantially when the hedger has a lower credit standing (i.e., higher funding costs) and full collateralization is used, or when collateral rules are more favorable to the hedge supplier. Although collateralization introduces an explicit link between the individual risk exposures and the hedge supplier's funding risk (hence some of the pooling/diversification benefits used to substitute for collateralization in the standard insurance model may be lost), in our examples we find that the opposite effects of longevity and interest rate risk, and the different nature of the risk on which the swap is written (a floating rate in the case of interest-rate swaps, a smoother survival curve in the case of longevity swaps) make the overall impact of collateralization comparable with, and typically lower than, that observed in fixed-income markets (e.g., Johannes and Sundaresan, 2007). An important implication is that the interest-rate

swaps market might provide an appropriate framework for the collateralization of bespoke longevity solutions, even though such solutions lack of the transparency and standardization benefits associated with indexed-based instruments. Investment banks have sold index-based longevity swaps which have a structure that would be more familiar to capital markets investors, but they have so far been less popular than bespoke solutions to date.

On the methodological side, we show how longevity swap rates must be determined endogenously from the dynamic marking to market<sup>11</sup> of the swap and the collateral rules specified by the contract. To see why, note that the market value of the swap at each valuation date depends on the evolution of the relevant state variables (mortality, interest rates, credit spreads), as well as on the swap rate locked in at inception. On the other hand, the swap's market value will typically affect collateral amounts and, in a setting where collateral is costly, will embed the market value of the costs associated with future collateral flows. Hence, the swap rate can only be determined by explicitly taking into account the marking-to-market process and the dynamics of collateral posting. To avoid the computational burden of nested Monte Carlo simulations, we use an iterative procedure based on the Least-Squares Monte Carlo (LSMC) approach<sup>12</sup> (see Glassermann, 2004, and references therein). We provide several numerical examples showing how different collateralization rules shape longevity swap rates giving rise to margins in (best estimate) survival probabilities reflecting the cost of future collateral flows. Although our focus is on longevity risk solutions, the approach can be applied to other instruments, such as over-the-counter solutions for inflation and credit risk.

Our work contributes to the existing literature on longevity risk pricing in at least three ways: i) we introduce default risk in the pricing of longevity risk solutions, and properly address its bilateral nature; ii) we explicitly allow for collateralization rules, which are the backbone of any real-world hedging solution and materially affect the pricing of over-the-counter transactions; iii) we introduce a 'structural' dimension in an otherwise reduced-form

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<sup>11</sup>Here and in what follows, by 'market value' and 'marking to market' we mean that assets and liabilities are valued according to a market-consistent valuation model or regulatory standard.

<sup>12</sup>A similar approach is used by Bacinello et al. (2009, 2010) for surrender guarantees in life policies and by Bauer et al. (2009) for the computation of capital requirements within the Solvency II framework. The term American Monte Carlo is often used in financial engineering to refer to this approach. We stick to the term Least Squares Monte Carlo, as it is more common in the insurance industry (e.g., Hörig and Leitschkis, 2012).



pricing framework, by allowing for funding/opportunity costs associated with longevity risk exposures held by hedgers and hedge suppliers. As there is essentially no publicly available information on swap rates, our approach<sup>13</sup> has the advantage of using publicly available information on credit markets and regulatory standards, without having to rely exclusively on calibration to primary insurance market prices, approximate hedging methods or assumptions on agents' risk preferences (e.g., Dowd et al., 2006; Ludkovski and Young, 2008; Bauer et al., 2012, 2010; Biffis et al., 2010; Chen and Cummins, 2010; Cox et al., 2010; Deng et al., 2012; Wang et al., 2013, among others).

The article is organized as follows. In the next section, we introduce longevity swaps and formalize their payoffs. Although the setup covers the case of both bespoke and indexed swaps, we focus on the former to keep the paper focused. In section 2.1, we examine the marking to market of a longevity swap during its lifetime to demonstrate the impact of counterparty risk on the hedger's balance sheet. Section 3 introduces bilateral default risk in longevity swap valuation formulae, identifying the main channels through which default risk affects the market value of swaps. Section 4 introduces credit enhancement in the form of collateralization, and shows how longevity swap rates are affected even in the presence of full cash collateralization (and hence absence of default losses). We compute swap rates by using an iterative procedure based on the LSMC approach. In section 5, several stylized examples are provided to understand how different collateralization rules may affect longevity swap rates. Concluding remarks are offered in section 6. Further details and technical remarks are collected in an appendix.

< TABLE 1 ABOUT HERE >

## 2 Longevity swaps

We consider a hedger (insurer selling annuities, pension fund), referred to as party H, and a hedge supplier (reinsurer, investment bank), referred to as counterparty HS. Agent H has the obligation to pay amounts  $X_{T_1}, X_{T_2}, \dots$ , possibly dependent on interest rates

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<sup>13</sup>Similarly, Biffis and Blake (2010b, 2013) endogenize longevity risk premia by introducing asymmetric information and capital requirements in a risk-neutral setting.

and inflation, to each survivor at fixed dates  $0 < T_1 \leq T_2, \dots$  of an initial population of  $n$  individuals (annuitants or pensioners) alive at time zero. We are clearly restricting our attention to homogeneous liabilities for ease of exposition, more general situations requiring obvious modifications. Party H's liability at a generic payment date  $T > 0$  is given by the random variable  $(n - N_T)X_T$ , where  $N_T$  counts the number of deaths experienced by the population during the period  $[0, T]$ . Assuming that the individuals' death times have common intensity<sup>14</sup>  $(\mu_t)_{t \geq 0}$ , the expected number of survivors at time  $T$  can be written as  $E^{\mathbb{P}}[n - N_T] = np_T$ , with the survival probability  $p_T$  given by (see the appendix)

$$p_T := E^{\mathbb{P}} \left[ \exp \left( - \int_0^T \mu_t dt \right) \right]. \quad (2.1)$$

Here and in the following,  $\mathbb{P}$  denotes the real-world probability measure. The intensity could be modeled by using, for example, any of the stochastic mortality models considered in Cairns et al. (2009). For our examples, we will rely on the simple Lee-Carter mortality projection model (Lee and Carter, 1992).

Let us now consider a financial market and introduce the risk-free rate process  $(r_t)_{t \geq 0}$  (in practice, an overnight rate). We assume that a market-consistent price of the liabilities can be computed by using a risk-neutral measure  $\tilde{\mathbb{P}}$ , equivalent to  $\mathbb{P}$ , such that the death times have the same intensity process  $(\mu_t)_{t \geq 0}$  (with different dynamics, in general, under the two measures; see Biffis et al., 2010). The time-0 market value of the aggregate liability can then be written as

$$E^{\tilde{\mathbb{P}}} \left[ \sum_i \exp \left( - \int_0^{T_i} r_t dt \right) (n - N_{T_i}) X_{T_i} \right] = n \sum_i E^{\tilde{\mathbb{P}}} \left[ \exp \left( - \int_0^{T_i} (r_t + \mu_t) dt \right) X_{T_i} \right].$$

For the moment, we take the pricing measure as given: we will give it more structure later on.

We consider two instruments which H can enter into with HS to hedge its exposure: a bespoke longevity swap and an index-based longevity swap. In these swaps, in contrast with interest rate swaps, the fixed leg will be a series of fixed rates each one pertaining

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<sup>14</sup>As discussed more in detail in the appendix, for tractability we restrict our attention to the case of doubly stochastic (or Cox, conditionally Poisson) death times; see Biffis et al. (2010).

to an individual payment date. The reason is that mortality increases substantially at old ages and a single fixed rate would introduce a growing mismatch between the cashflows provided by the swap and those needed by the hedger. However, as with interest rate swaps, we can treat a longevity swap as a portfolio of forward contracts on the underlying floating (survival) rate.<sup>15</sup> In this section, we ignore default risk and focus on individual payments at maturity  $T > 0$ . Throughout the article, we always assume the perspective of the hedger.

A **bespoke longevity swap** allows party H to pay a fixed rate  $\bar{p}^N \in (0, 1)$  against the realized survival rate experienced by the population between time zero and time  $T$ . Assuming a notional amount equal to the initial population size,  $n$ , the net payout to the hedger at time  $T$  is<sup>16</sup>

$$n \left( \frac{n - N_T}{n} - \bar{p}^N \right),$$

i.e., the difference between the realized number of survivors and the pre-set number of survivors  $n\bar{p}^N$  agreed at inception. Letting  $S_0$  denote the market value of the swap at inception, we can write

$$\begin{aligned} S_0 &= nE^{\tilde{\mathbb{P}}} \left[ \exp \left( - \int_0^T r_t dt \right) \left( \frac{n - N_T}{n} - \bar{p}^N \right) \right] \\ &= nE^{\tilde{\mathbb{P}}} \left[ \exp \left( - \int_0^T (r_t + \mu_t) dt \right) \right] - nB(0, T)\bar{p}^N, \end{aligned} \quad (2.2)$$

with  $B(0, T)$  denoting the time-zero price of a zero-coupon bond with maturity  $T$ . By setting  $S_0 = 0$ , we obtain the swap rate as

$$\bar{p}^N = \tilde{p}_T + B(0, T)^{-1} \text{Cov}^{\tilde{\mathbb{P}}} \left( \exp \left( - \int_0^T r_t dt \right), \exp \left( - \int_0^T \mu_t dt \right) \right), \quad (2.3)$$

where the risk-adjusted survival probability  $\tilde{p}_T$  is defined as in (2.1) with expectations taken under  $\tilde{\mathbb{P}}$ . Expression (2.3) shows that if the intensity of mortality is uncorrelated with bond market returns (a reasonable first-order approximation), the longevity swap

<sup>15</sup>With a slight abuse of terminology, we use the term ‘swap rate’ for individual forward rates as well as for swap curves (a series of swap rates). We note that swap curves are often summarized by the improvement factor applied to the survival probabilities of a reference mortality table/model; see examples in section 5.

<sup>16</sup>For ease of exposition, here and in the following sections, we consider contemporaneous settlement only. Other settlement conventions (e.g. in arrears) have negligible effects, but make valuation formulae more involved when bilateral and asymmetric default risk is introduced.

curve just involves the survival probabilities  $\{\tilde{p}_{T_i}\}$  relative to the different maturities  $\{T_i\}$ . Several studies have recently addressed the issue of how to quantify risk-adjusted survival probabilities, for example, by calibration to annuity prices and books of life policies traded in secondary markets, or by use of approximate hedging methods (see references in Section 1). As there is essentially no publicly available information on swap rates, for our numerical examples we will suppose a baseline case in which  $\tilde{p}_{T_i} = p_{T_i}$  for each maturity  $T_i$  and focus on how counterparty default risk and collateral requirements might generate a positive or negative spread on best estimate survival rates. This is consistent with market practice where counterparties would agree on a real-world mortality model (and estimation methodology) to mark-to-model the swap at future dates. Although in what follows, we mainly concentrate on longevity risk, in practice, the floating payment of a longevity swap might involve an interbank rate component (e.g., LIBOR) or survival indexation rules different from the ones considered above. To keep the setup general, we will at times consider instruments making a generic variable payment,  $P$ , and write the corresponding swap rate  $\bar{p}$  as

$$\bar{p} = E^{\tilde{\mathbb{P}}} [P] + B(0, T)^{-1} \text{Cov}^{\tilde{\mathbb{P}}} \left( \exp \left( - \int_0^T r_t dt \right), P \right). \quad (2.4)$$

The setup can easily accommodate **index-based longevity swaps**, standardized instruments allowing the hedger to pay a fixed rate  $\bar{p}^I \in (0, 1)$  against the realized value of a survival index  $(I_t)_{t \geq 0}$  at time  $T$ . The latter might reflect the mortality experience of a reference population closely matching<sup>17</sup> that of the liability portfolio. Examples are represented by the LifeMetrics indices developed by J.P. Morgan, the Pensions Institute and Towers Watson,<sup>18</sup> or the Xpect indices developed by Deutsche Börse.<sup>19</sup> The relative advantages and disadvantages of index-based versus bespoke swaps are discussed, for example, in Biffis and Blake (2010a). Assuming that the index admits the representation  $I_t = \exp(-\int_0^t \mu_s^I ds)$ , with  $(\mu_t^I)_{t \geq 0}$  the intensity of mortality of a reference population, the swap rate  $\bar{p}^I$  is given again by expression (2.3), but with the process  $\mu$  replaced by  $\mu^I$ , and with  $\tilde{p}_T$  replaced by the corresponding risk-adjusted survival probability  $\tilde{p}_T^I$ .

<sup>17</sup>The risk of mismatch is called basis risk. See, for example, Coughlan et al. (2011), Stevens et al. (2011), Salhi and Loisel (2012), Gatzert and Wesker (2012), and Cairns (2013) for some results related to this risk dimension.

<sup>18</sup>See <http://www.lifemetrics.com>. The indices were transferred to LLMA in 2011.

<sup>19</sup>See <http://www.xpect-index.com>.

## 2.1 The marking-to-market (MTM) process

Longevity swaps are not currently exchange traded and there is no commonly accepted framework for counterparties to mark to market/model their positions.<sup>20</sup> The presence of counterparty default risk and collateralization rules, however, makes the MTM procedure a very important feature of these transactions for at least two reasons. First, at each payment date, the difference between the variable and pre-set payment generates a cash inflow or outflow to the hedger, depending on the evolution of mortality. In the absence of basis risk (which is the case for bespoke solutions), these differences show a pure ‘cashflow hedge’ of the longevity exposure in operation. However, as market conditions change (e.g., mortality patterns, counterparty default risk), the impact of the swap on the hedger’s balance sheet can evolve dramatically. For example, even if the swap payments are expected to provide a good hedge against longevity risk, the hedger’s position will weaken considerably if the expected present value of the net payments shrinks due to deterioration in the hedge supplier’s credit quality. Second, for solvency requirements, it is important to value a longevity swap under extreme market/mortality scenarios (‘stress testing’). This means, for example, that even if a longevity swap qualifies as a liability on a market-consistent basis, it might still provide considerable capital relief when valued on a regulatory basis due to its recognized effectiveness as a hedge.

To illustrate some of these points, let us consider the hypothetical situation of an insurer H with a liability represented by a group of ten thousand 65-year-old annuitants drawn from the population of England & Wales in 1980. We assume that party H entered a 25-year pure longevity swap in 1980 and we follow the evolution of the contract until maturity. The population is assumed to evolve according to the death rates reported in the Human Mortality Database (HMD) for England & Wales.<sup>21</sup> We assume that interest-rate risk is hedged away through interest rate swaps, locking in a rate of 5% throughout the life of the swap. The role of collateral is examined later on; here, we show how the hedging instrument operates from the point of view of the hedger. For this bespoke solution, the market value of each floating-for-fixed payment occurring at a generic date  $T$  can be computed by using

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<sup>20</sup>At the time of writing, LLMA was working on this issue.

<sup>21</sup>See <http://www.mortality.org>.

the valuation formula

$$S_t = nE_t^{\tilde{\mathbb{P}}} \left[ \exp \left( - \int_t^T r_s ds \right) \left( \frac{n - N_t}{n} \exp \left( - \int_t^T \mu_s ds \right) \right) \right] - nB(t, T)\bar{p}^N, \quad (2.5)$$

for each time  $t$  in  $[0, T]$  at which no default has yet occurred, with  $B(t, T)$  denoting the market value of a zero-coupon bond with time to maturity  $T - t$ , and  $E_t^{\tilde{\mathbb{P}}}[\cdot]$  the conditional expectation under a pricing measure  $\tilde{\mathbb{P}}$ , given the information available at time  $t$ . As a simple benchmark case, we assume that market participants receive information from the HMD and use the Lee-Carter model to value longevity-linked cashflows. In other words, at each MTM date (including inception), longevity swap rates are based on Lee-Carter forecasts computed using the latest HMD information available.<sup>22</sup> Figure 1 illustrates the evolution of swap survival rates for an England & Wales cohort tracked from age 65 in 1980 to age 90 in 2005. It is clear that the systematic underestimation of mortality improvements by the Lee-Carter model in this particular example will mean that the hedger's position will become increasingly in-the-money as the swap matures. This is shown in figure 2. In practice, the contract may allow the counterparty to cancel the swap or re-set the fixed leg for a nonnegative fee, but we ignore these features in this example. Figure 2 also reports the sequence of net cashflows generated by the swap. As interest rate risk is hedged away – and again ignoring default risk for the moment – cash inflows/outflows arising in the backtesting exercise only reflect the difference between the realized survival rates and the swap rates locked in at inception. On the other hand, the swap's market value reflects changes in market swap rates, which by assumption follow the updated Lee-Carter forecasts plotted in figure 1 and differ from the realized survival rates. As is evident from figure 2, the credit exposure of a longevity swap is close to zero at inception and at maturity, but may be sizable in between, depending on the trade-off between changes in market/mortality conditions and the residual swap payments (amortization effect). The credit exposure is quantified by the replacement cost, i.e., the cost that the nondefaulting counterparty would have to incur at the default time to replace the instrument at market prices then available. As a simple example which predicts the next section, let us introduce credit risk (but no

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<sup>22</sup>See Dowd et al. (2010a,b); Cairns et al. (2011) for a comprehensive analysis of alternative mortality models; see also Giroi and King (2008) and Pitacco et al. (2009).

default) and assume that in 1988 the credit spread of the hedge supplier widens across all maturities by 50 and 100 basis points. The impact of these two scenarios on the hedger's balance sheet is dramatic, as shown in figures 2-3, demonstrating how MTM profits and losses can jeopardize a successful cashflow hedge.

< FIGURE 1 ABOUT HERE >

< FIGURE 2 ABOUT HERE >

< FIGURE 3 ABOUT HERE >

### 3 Counterparty default risk

The backtesting exercise of the previous section has demonstrated the importance of the hedge supplier's credit risk and the marking to market procedure in assessing the value of a longevity swap to the hedger. A correct approach, however, should allow for the fact that counterparty risk is bilateral. This is the case even when the hedger is a pension plan. Private sector defined benefit pension plans in countries such as the UK are founded on trust law and rely on a promise by (rather than a guarantee from) the sponsoring employer to pay the benefits to plan members. This promise is known as the 'sponsor covenant'. The strength of the sponsor covenant depends on both the financial strength of the employer and the employer's commitment to the scheme.<sup>23</sup> As a reasonable but imperfect proxy for the effect of the sponsor covenant, we use the sponsor's default intensity (party H's default intensity). For large corporate pension plans, the intensity can be derived/extrapolated from spreads observed in corporate bond and CDS markets. For smaller plans, an analysis of the funding level and strategy of the scheme is required.<sup>24</sup>

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<sup>23</sup>In the UK, for example, The Actuarial Profession (2005, par. 3.2) defined the sponsor covenant as: "the combination of (a) the ability and (b) the willingness of the sponsor to pay (or the ability of the trustees to require the sponsor to pay) sufficient advance contributions to ensure that the scheme's benefits can be paid as they fall due." See also The Pensions Regulator (2009).

<sup>24</sup>Along the same lines, Inkmann and Blake (2010) show how the discount rate for the valuation of pension liabilities should reflect funding risk.

Assume that both party H and HS may default at random times  $\tau^H, \tau^{HS}$ , admitting default intensities<sup>25</sup>  $(\lambda_t^H)_{t \geq 0}, (\lambda_t^{HS})_{t \geq 0}$ . Defining by  $\tau := \min(\tau^H, \tau^{HS})$  the default time of the swap transaction, we further assume that, on the event  $\{\tau \leq T\}$ , the nondefaulting counterparty, say party  $i$ , receives a fraction  $\psi^j \in [0, 1]$  ( $i \neq j$ , with  $i, j \in \{H, HS\}$ ) of the market value of the swap before default,  $S_{\tau-}$ , if she is in-the-money, otherwise she has to pay the full pre-default market value  $S_{\tau-}$  to the defaulting counterparty. Following Duffie and Huang (1996), we can then write the market value of a swap with notional amount  $n$  as

$$S_0 = nE^{\tilde{\mathbb{P}}} \left[ \exp \left( - \int_0^T (r_t + 1_{\{S_t < 0\}}(1 - \psi^H)\lambda_t^H + 1_{\{S_t \geq 0\}}(1 - \psi^{HS})\lambda_t^{HS}) dt \right) (P - \bar{p}^d) \right], \quad (3.1)$$

where  $P$  denotes the variable payment,  $\bar{p}^d$  the fixed rate, and the indicator function  $1_A$  takes the value of unity if the event  $A$  is true, zero otherwise. To understand the above formula, note that, in our setting, the risk-neutral valuation of a defaultable claim involves the use of a default-risk-adjusted short rate  $r_t + \lambda_t^H + \lambda_t^{HS}$  and dividend payment  $\lambda_t^H(\psi^H 1_{S_{t-} < 0} + 1_{S_{t-} \geq 0}) + \lambda_t^{HS}(\psi^{HS} 1_{S_{t-} \geq 0} + 1_{S_{t-} < 0})$  determined by the recovery rules described above. As a result, the valuation formula (3.1) entails discounting at a spread above the risk-free rate given by

$$\begin{aligned} \Lambda_t &:= \lambda_t^H + \lambda_t^{HS} - \lambda_t^H(\psi^H 1_{S_t < 0} + 1_{S_t \geq 0}) - \lambda_t^{HS}(\psi^{HS} 1_{S_t \geq 0} + 1_{S_t < 0}) \\ &= 1_{\{S_t < 0\}}(1 - \psi^H)\lambda_t^H + 1_{\{S_t \geq 0\}}(1 - \psi^{HS})\lambda_t^{HS}, \end{aligned}$$

showing a switching-type dependence on the characteristics of the counterparty that is out-of-the-money at each given time prior to default. The swap rate admits the representation

$$\bar{p}^d = E^{\tilde{\mathbb{P}}}[P] + \frac{\text{Cov}^{\tilde{\mathbb{P}}} \left( \exp \left( - \int_0^T (r_t + \Lambda_t) dt \right), P \right)}{E^{\tilde{\mathbb{P}}} \left[ \exp \left( - \int_0^T (r_t + \Lambda_t) dt \right) \right]}, \quad (3.2)$$

and hence depends in a complex way not only on the interaction between the variable

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<sup>25</sup>For tractability and symmetry with the mortality model of section 2, we work with doubly stochastic default times (see the appendix). The main drawback is that the occurrence of default does not affect the conditional default probability of the surviving counterparty, thus limiting the extent to which close-out risk can be properly modelled.



payments and risk factors such as interest rates, default intensities and recovery rates, but also on the path of the swap's market value itself. When  $P$  does not include a demographic component, as in the case of interest-rate swaps, the covariance term is typically negative. To see this, consider the case of the standard swap valuation formula obtained by assuming that both counterparties have the same default intensity ( $\lambda_t := \lambda_t^H = \lambda_t^{HS}$ ) and there is no recovery conditional on default ( $\psi^H$  and  $\psi^{HS}$  are simply zero). If the credit risk of the counterparties is equal to the average credit quality of the LIBOR panel, the discount rate in (3.2) is simply given by  $r + \lambda$ , where  $\lambda$  is just the LIBOR-Treasury (TED) spread. For a swap paying the LIBOR rate, we would then have a negative covariance term and hence  $\bar{p}^d \leq E^{\tilde{\mathbb{P}}}[P]$ . When  $P$  only includes a demographic component (as in expression (2.3) for example), which is uncorrelated with the other variables, we would still have a non-null covariance term, due to the regime-switching nature of the discount rate in formula (3.2), and the fact that switching is triggered by the value of the swap, which also depends on the floating rate of interest. More generally, one might expect the covariance term to be negative, as longevity-linked payments are likely to be positively correlated with the credit quality of hedge suppliers<sup>26</sup> and companies with significant pension liabilities. The case of floating payments linked to both mortality and interest rates would then suggest a swap rate satisfying  $\bar{p}^d \leq E^{\tilde{\mathbb{P}}}[P]$ . In the next section, we will show that this is not necessarily the case. To understand why, consider the case of full recovery as an example (set  $\psi^H$  and  $\psi^{HS}$  equal to one): expression (3.2) reduces to a default-free risk-neutral valuation formula, irrespective of both the default intensities of the counterparties and the costs involved by the credit enhancement tools needed to ensure that full recovery is indeed achieved upon default. This suggests that it is essential to consider explicitly counterparty risk mitigation tools in the pricing functional.

Counterparty risk can be mitigated in a number of ways, for example by introducing termination rights (e.g., credit puts and break clauses) or using credit derivatives (e.g., credit default swaps and credit spread options). We will focus on collateralization, a form of direct credit support requiring each party to post cash or securities when it is out-of-the-money. For simplicity, we consider the case of cash, which is by far the most

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<sup>26</sup>This is a reasonable assumption for monoline insurers such as pension buyout firms, but might be less so for well-diversified reinsurers.

common type of collateral (e.g., ISDA, 2010b) and allows us to disregard close-out risk, the risk that the value of collateral may change at default. In the interest-rate swaps market, Johannes and Sundaresan (2007) find evidence of costly collateral by comparing swap market data with swap values based on portfolios of futures and forward contracts, and by estimating a dynamic term structure model by using Treasury and swap data. We cannot carry out a similar exercise for longevity swaps, because there are no publicly available data on these transactions. On the other hand, we can quantify the funding/opportunity costs associated with the collateral flows originating from the MTM procedure, as will be shown in section 5.

## 4 Collateralization

Collateral agreements reflect the amount of acceptable credit exposure that each party agrees to take on. We consider simple collateral rules capturing the main features of the problem. Formally, let us introduce the pre-default collateral process<sup>27</sup>  $(C_t)_{t \geq 0}$ , which indicates how much cash,  $C_t$ , to post at each time  $t$  prior to default in response to changes in market conditions, including, in particular, the MTM value of the swap (we provide explicit examples below). Again, we develop our analysis from the point of view of the hedger, so that  $C_t > 0$  ( $C_t < 0$ ) means that party H is holding (posting) collateral. Using the notation  $a^+ := \max(a, 0)$  and  $a^- := \max(-a, 0)$ , we assume the recovery rules to take the following form:

- On the event  $\{\tau^H \leq \min(\tau^{\text{HS}}, T)\}$  (hedger's default), party HS recovers any collateral received by the hedger an instant prior to default,  $C_{\tau^H-}^-$ , and pays the full MTM value of the swap to party H if  $S_{\tau^H-} \geq 0$ . The net flow to party H is then  $S_{\tau^H-}^+ - C_{\tau^H-}^-$ .
- On the event  $\{\tau^{\text{HS}} \leq \min(\tau^H, T)\}$  (hedge supplier's default), party H pays the full MTM value of the swap to party HS if  $S_{\tau^{\text{HS}}-} < 0$ , and recovers any collateral received by HS an instant prior to default,  $C_{\tau^{\text{HS}}-}^+$ . The net flow to party H can then be written as  $-S_{\tau^{\text{HS}}-}^- + C_{\tau^{\text{HS}}-}^+$ .

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<sup>27</sup>In other words, the actual collateral process supporting the transaction is  $(1_{\{\tau > t\}} C_t)_{t \geq 0}$ ; hence, we are not concerned with the value taken by  $C_t$  after default.

- Whenever the nondefaulting counterparty, say H, is out-of-the-money, payment of the full MTM value of the swap is accomplished by party H recovering the extra amount  $(S_{\tau-}^+ - C_{\tau-}^+)^+$  in the case of overcollateralization, or by party H paying the extra amount  $(S_{\tau-}^- - C_{\tau-}^-)^+$  in case of undercollateralization. In case of full collateralization, party H simply loses any collateral posted with HS.

To obtain neater results, it is convenient to express the collateral before default of either party as a fraction of the MTM value of the swap,

$$C_t = (c_t^{\text{HS}} 1_{\{S_{t-} \geq 0\}} + c_t^{\text{H}} 1_{\{S_{t-} < 0\}}) S_{t-}, \quad (4.1)$$

where  $c^{\text{H}}, c^{\text{HS}}$  are two nonnegative left-continuous processes giving the fraction of the MTM value of the swap that is posted as collateral by party H or HS, respectively.<sup>28</sup> Finally, we introduce a nonnegative continuous process  $(\delta_t)_{t \geq 0}$  representing the yield on collateral, in the sense that holding/posting collateral of amount  $C_t$  yields/costs instantaneously the net amount  $\delta_t C_t$  (after rebate). We can introduce some asymmetry, by setting  $\delta_t = \delta_t^{\text{H}} 1_{\{S_{t-} < 0\}} + \delta_t^{\text{HS}} 1_{\{S_{t-} \geq 0\}}$ , so that  $\delta_t^{\text{H}}$  be interpreted as party H's net cost of posting collateral when she is out-of-the-money, and  $\delta_t^{\text{HS}}$  as the net yield on the collateral posted by party HS when party H is in-the-money. In general, one may regard the collateral costs embedded in swap market values as those of the marginal market participant. However, when considering individual longevity swap transactions with bespoke CSAs, it may be convenient to allow the pricing formula to take into account the cost of collateral of the counterparty: in this case  $\delta^{\text{H}}$  and  $\delta^{\text{HS}}$  may be regarded as the cost of posting collateral for party H and HS whenever they are out-of-the-money.

Denoting by  $\bar{p}^c$  the swap rate available in case of collateralization, we can write the MTM value of the swap as in (3.1), but with the spread  $\Lambda$  now replaced by (see the appendix for a proof)

$$\Gamma_t = \lambda_t^{\text{H}} (1 - c_t^{\text{H}}) 1_{\{S_t < 0\}} + \lambda_t^{\text{HS}} (1 - c_t^{\text{HS}}) 1_{\{S_t \geq 0\}} - (\delta_t^{\text{H}} c_t^{\text{H}} 1_{\{S_t < 0\}} + \delta_t^{\text{HS}} c_t^{\text{HS}} 1_{\{S_t \geq 0\}}). \quad (4.2)$$

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<sup>28</sup>Note that representation (4.1) comes at a cost: we cannot encompass the case when collateral is posted by a counterparty at inception (a form of overcollateralization), which may be the case for some transactions.

In the above expression, we recognize the typical features of valuation formulae for credit-risky securities (e.g., Bielecki and Rutkowski, 2002): the first two terms account for the fractional recovery of the swap MTM value in case of default of the counterparty, the third one for the costs incurred when posting collateral before default. An important difference, however, is that in (4.2) recovery rates depend on collateral rules, and collateral costs enter explicitly in the discount rate. We now examine simple special cases to understand better the role of collateral in shaping swap rates.

#### 4.1 Full collateralization

Consider the collateral rule obtained by setting  $c^H$  and  $c^{HS}$  equal to one, meaning that the full MTM value of the swap is received/posted as collateral depending on whether the marking-to-market procedure results in a positive/negative value for  $S_t$ . As we consider cash collateral, default is immaterial. In contrast with section 3, however, the expression for the swap MTM value does not reduce to the usual default-free, risk-neutral valuation formula in general, unless collateral costs are zero. In the case of symmetric collateral costs, for example, we obtain:

$$\bar{p}^c = E^{\tilde{\mathbb{P}}}[P] + \frac{\text{Cov}^{\tilde{\mathbb{P}}}\left(\exp\left(\int_0^T (\delta_t - r_t)dt\right), P\right)}{E^{\tilde{\mathbb{P}}}\left[\exp\left(\int_0^T (\delta_t - r_t)dt\right)\right]}. \quad (4.3)$$

If the cost of collateral is positively dependent on  $P$ , we expect the swap rate to be higher than  $\bar{p}^d$  in expression (3.2) (see Johannes and Sundaesan, 2007), reflecting the fact that (costly) collateralization results in the payer of the floating rate being compensated with a higher fixed rate. In the interest-rate swap market this happens for example if either the short rate or the TED spread are positively correlated with  $\delta$ . The intuition is that the floating rate payer will have to both post collateral and incur higher funding costs when the floating rate increases. In longevity space, one may expect the cost of collateral to be positively dependent on mortality improvements and negatively dependent on interest rates, as longevity-linked liabilities are more capital intensive in low mortality and low interest rate environments (due to lower discounting of future cashflows). The combined impact of these two effects is ambiguous, and is discussed in the examples of section 5.

## 4.2 Common collateral rules

According to ISDA (2010b), it is typical for collateral agreements to specify collateral triggers based on the market value of the swap or other relevant variables (credit ratings, credit spreads, etc.) crossing pre-specified threshold levels. In longevity swaps, the CSA may define collateral rules that depend on the underlying mortality experience, involve path dependence (with respect to mortality experience/expectations for instance), and monitor different variables at different frequency. For example, the CSA may allow for daily collateral adjustments for financial conditions, quarterly adjustments for death experience, and annual adjustments for changes in future mortality improvements. The following examples illustrate some of these aspects:

- a) Set  $c_t^{\text{HS}} = 1_{\{S_{t-} \geq \underline{s}(t)\}}$  and  $c_t^{\text{H}} = 1_{\{S_{t-} \leq \bar{s}(t)\}}$  (for continuous functions  $\underline{s}, \bar{s}$  defined on  $[0, T]$  and satisfying  $\underline{s} \leq \bar{s}$ ), meaning that the hedge supplier (hedger) is required to post full collateral if the swap's MTM value is above (below) the appropriate time-dependent threshold. More general collateral rules can be obtained by setting  $c_t^{\text{HS}} = \gamma_t^{\text{HS}} 1_{\{S_{t-} \geq \underline{s}(t)\}}$  and  $c_t^{\text{H}} = \gamma_t^{\text{H}} 1_{\{S_{t-} \leq \bar{s}(t)\}}$ , for suitable processes  $\gamma^{\text{H}}, \gamma^{\text{HS}}$  depending on prevailing market conditions or expectations about future mortality.
- b) In longevity swaps, however, it is more common to define collateral thresholds in terms of mortality forecasts based on a model agreed at contract inception, and monitor the deaths in the hedger's population instead of the market value of the swap. This is due to both the re-estimation risk affecting any given mortality model and the presence of substantial model risk, which most likely would prevent the counterparties from agreeing on a common model at future dates. We can set, for example,  $c_t^{\text{HS}} = 1_{\{N_{t-} \leq \alpha(t)\}}$  and  $c_t^{\text{H}} = 1_{\{N_{t-} \geq \beta(t)\}}$ , for continuous functions  $\alpha$  and  $\beta$  satisfying  $0 \leq \alpha \leq \beta \leq n$ , meaning that the hedge supplier (hedger) is required to post full collateral if realized deaths are below (above) the relevant threshold.
- c) For an index-based swap, it may be more convenient to work with the mortality intensity  $\mu^I$  of the reference population (see section 2) and set  $c_t^{\text{HS}} = 1_{\{\int_0^t \mu_s^I ds \leq a(t)\}}$  and  $c_t^{\text{H}} = 1_{\{\int_0^t \mu_s^I ds \geq b(t)\}}$  for (say) continuous functions  $a, b$  satisfying  $0 \leq a \leq b$ . This means that collateral posting is triggered at each time  $t$  if the realized value of the longevity

index,  $\exp(-\int_0^t \mu_s^I ds)$ , falls outside the open interval  $(\exp(-b(t)), \exp(-a(t)))$ .

- d) As was emphasized in section 2.1, the severity of counterparty risk depends on the credit quality of the counterparties. This is why collateralization agreements may set collateral thresholds that explicitly depend on credit ratings or CDS spreads. A simple example of this practice can be obtained as a special case of (a) by setting  $c_t^{\text{HS}} = 1_{\{N_{t-} \leq \alpha(t)\} \cup \{\lambda_t^{\text{HS}} \geq \underline{\lambda}\}}$ ,  $c_t^{\text{H}} = 1_{\{N_{t-} \geq \beta(t)\} \cup \{\lambda_t^{\text{H}} \geq \underline{\lambda}\}}$ , meaning that, at each time  $t$ , the hedger (hedge supplier) receives collateral when either realized deaths fall below the level  $\alpha(t)$  (respectively  $\beta(t)$ ) or the hedge supplier's (respectively hedger's) default intensity overshoots a given threshold  $\underline{\lambda} \geq 0$ . Note that both  $c^{\text{H}}$  and  $c^{\text{HS}}$  can be non zero at the same time (for example on the event  $\{N_{t-} \leq \alpha(t)\} \cap \{\lambda_t^{\text{H}} \geq \underline{\lambda}\}$ ), but expression (4.1) ensures that only the party out-of-the-money will have to post collateral.

### 4.3 Computing the swap rate

The recursive nature of swap valuation formulae in the case of bilateral and asymmetric counterparty risk (in a doubly stochastic setting) was already noted by Duffie and Huang (1996). By modeling the recovery rates and the difference in counterparties' credit spreads in reduced form, however, they could use a simple iterative procedure to determine the swap rate.<sup>29</sup> Here, we explicitly allow for the impact of collateral and the MTM procedure in the pricing functional: working in a high-dimensional Markov setting, we use a Least-Squares Monte Carlo approach. Exploiting the properties of the doubly stochastic setup, we do not model death/default times explicitly, but just rely on the mortality/default intensities (see algorithm 2 in Bacinello et al., 2010, for example). The procedure involves the following steps (we focus on the individual forward rates for convenience):

*Step 1.* For an arbitrary maturity  $T_i$  and fixed rate  $\bar{p}_i^c := \bar{p}_{T_i}^c \in (0, 1)$ , generate  $M$  simulated paths of the state variable process,  $X$ , under  $\tilde{\mathbb{P}}$  along the time grid  $\mathcal{T}_i := \{0 < t_1, t_2, \dots, t_{n_i} = T_i\}$ . Denote by  $S_{t_j}^{i,m}$  the (ex-dividend) MTM value of the swap, and by

<sup>29</sup>Johannes and Sundaesan (2007) sidestep recursivity issues by considering full collateralization and symmetric default risk and collateral costs.

$f_{t_j}^{i,m}$  the net payments<sup>30</sup> from the swap (excluding collateral flows) at time  $t_j$ , on path  $m$ , and for given forward rate  $\bar{p}_i^c$ .

*Step 2.* Compute recursively the approximate<sup>31</sup> value  $\bar{S}_{t_j}^{i,m}$  of the swap at time  $t_j$  (for  $j = n_i - 1, \dots, 0$  with  $t_0 = 0$ ) as  $\bar{S}_{t_j}^{i,m} = \beta_{i,j}^* \cdot e(X_{t_j}^m)$ , where  $e(x) := (e_1(x), \dots, e_H(x))^\top$  and  $\{e_1, \dots, e_H\}$  is a finite set of functions taken from a suitable basis of  $L^2(\Omega)$ , and  $\beta_{i,j}^*$  is given by

$$\beta_{i,j}^* = \arg \min_{\beta \in \mathbb{R}^H} \sum_{m=1}^M \left\{ \left( S_{t_{j+1}}^{i,m} + f_{t_{j+1}}^{i,m} \right) \exp \left( - \int_{t_j}^{t_{j+1}} r_u^m du \right) - \beta e \left( X_{t_j}^m \right) \right\}^2,$$

where  $S_{t_j}^{i,m} = \sum_{h=j+1}^{n_i} \exp \left( - \int_{t_j}^{t_h} \left( r_u^m + \Gamma_u^m(\bar{S}_{t_j}^{i,m}, \dots, \bar{S}_{t_{h-1}}^{i,m}) \right) du \right) f_{t_h}^{i,m}$  is the discounted cumulative value of the cash-flows originating from the swap allowing for collateral costs, where the spread  $\Gamma^m$  depends on the approximations  $\{\bar{S}_{t_{j+u}}^{i,m}\}_{u=0}^{n_i-1}$  determined in the previous iterations, which are used to check at each time  $t_{h-1}$  whether the collateral thresholds are triggered and determine the corresponding amount/cost of collateral over the time interval  $[t_{h-1}, t_h)$ . The value of the swap at time  $t_0$  is then given by  $\frac{1}{M} \sum_{m=1}^M S_{t_0}^{i,m}$ .

*Step 3.* Iterate<sup>32</sup> the above procedure over different values for  $\bar{p}_i^c$  until a candidate rate  $\bar{p}_i^{c*}$  is found, such that the initial price,  $S_{t_0}^{i,m}$ , is close enough to zero.

Of course, the procedure relies on knowledge of the dynamics of the state variable process under the pricing measure. To this end, in the next section, we outline a calibration approach based on the joint use of fixed-income data and funding costs / capital requirements for longevity-linked liabilities.

## 5 Examples

We use a continuous-time model for the risk-free yield curve, the LIBOR and mortality rates, as well as for the cost of collateral. The credit risk of party HS is assumed to be equal

<sup>30</sup>In the case of a  $T_i$ -forward contract, we have  $f_{t_j}^{i,m} = 0$  for  $0 \leq j < n_i$ , and  $f_{t_j}^{i,m} = n(n^{-1}(n - N_{T_i}^m) - \bar{p}_i^c)$  for  $j = n_i$ ; compare with expression (2.2).

<sup>31</sup>The approximation does not take into account default risk and collateral costs over  $[t_j, t_{j+1})$ , but is used to determine which party is in-the-money and out-of-the-money (and hence default risk and collateral costs) in the following step of the recursion.

<sup>32</sup>In the numerical examples of section 5, we use a combination of bisection, secant, and inverse quadratic interpolation methods to compute  $\bar{p}_i^{c*}$  (see Forsythe et al., 1976).

to the average credit quality of the LIBOR panel, so that the TED spread would be party HS's default intensity if there were zero recovery upon default (see section 3). We then set  $\lambda^H = \lambda^{HS} + \Delta$  and consider two cases: party H is either of the same credit quality as party HS ( $\Delta = 0$ ) or is more credit-risky ( $\Delta > 0$ ).

We describe the evolution of uncertainty by a six-dimensional state variable vector  $X$  with the Gaussian dynamics reported in appendix B. The first four components are: the short rate,  $r = X^{(1)}$ , assumed to revert to the long-run central tendency factor  $X^{(2)}$ , representing the slope of the risk-free yield curve; the TED spread  $X^{(3)}$ , so that the LIBOR rate is given by  $X^{(1)} + X^{(3)}$ ; and the net yield on collateral in the interest-rate swap market,  $X^{(4)}$ . The latter factor is used to draw a comparison with the cost of collateral in the longevity swap market. The remaining two components describe the opportunity cost of longevity swap dealers,  $X^{(5)}$ , and the log-intensity of mortality of a given population,  $\log \mu = X^{(6)}$ . Under the assumption of independence between  $(X^{(1)}, X^{(2)}, X^{(3)}, X^{(4)})$  and  $X^{(6)}$ , we can estimate separately the dynamics of the two groups of factors. For the first vector, we rely on the estimates of Johannes and Sundaesan (2007), who use weekly Treasury and swap data from 1990 to 2002 to obtain the parameter values reported in table 2. For the intensity  $\exp(X_t^{(6)})$ , we use a continuous-time version of the Lee-Carter mortality projection model for a cohort of 65-year olds; see appendix B for details.

As a first example, we focus on funding costs and simply take  $\delta^{HS} = X^{(3)}$  and  $\delta^H = X^{(3)} + \Delta$ , meaning that the hedger's net cost of collateral coincides with its funding costs net of the short rate (assuming it is rebated), whereas the hedger's net yield on the collateral amounts posted by party HS coincides with the TED spread. Assuming that the pricing formula uses information on the collateral costs of the counterparty, an alternative interpretation is that each party's net collateral costs coincide with their borrowing costs net of the risk-free rate. In the case of asymmetric default risk, we consider values of 100 and 200 basis points for  $\Delta$ .

We compute the longevity swap rates for a 25-year swap written on a population of 10,000 US males aged 65 at the beginning of 2008. In figure 4, we plot the underlying forward rates obtained for different collateralization rules against the percentiles of survival rate improvements based on Lee-Carter forecasts. We see that margins are positive and



increasing with payment maturity in the case of symmetric default risk, for both uncollateralized and fully collateralized transactions. As soon as we introduce asymmetry in default risk ( $\Delta > 0$ ), however, margins widen in the case of no collateralization, reflecting the fact that the hedger needs to pay an additional premium on account of its higher credit risk. In the case of full collateralization, counterparty risk is neutralized, but the hedger is compensated for her higher funding costs and the positive dependence between funding costs and collateral amounts discussed before: equilibrium swap rates are pushed lower and produce a negative margin on best estimate swap rates.

< TABLE 2 ABOUT HERE >

In figure 5, we examine the swap margins induced by one-way collateralization in the case of asymmetric default risk. When only the hedge supplier has to post full collateral, forward rates are higher than best estimate survival probabilities, meaning that the hedger has to compensate the hedge supplier for bearing both the cost of risk mitigation and the hedger's higher default risk. The opposite is true when it is the hedger who has to post full collateral when out-of-the money. In this case, swap margins are clearly negative, and decreasing in payment maturity. These effects are amplified when the asymmetry in counterparties' credit quality is greater, as can be seen from the spreads reported in table 3 for some key maturities and collateralization rules.

Plotting the swap rate *margins* against best estimate mortality improvements allows one to interpret the swap rates as outputs of a pricing functional based on adjustments to a reference mortality model (which is common practice in longevity space; see Biffis et al., 2010). On the other hand, longevity swap *spreads* are easier to compare with those emerging in other transactions. In table 4, we make a comparison with the interest-rate swap spreads implied by our parameterization of the state vector  $(X^{(1)}, X^{(2)}, X^{(3)}, X^{(4)})$ . In particular, we report the difference between interest-rate futures prices (obtained by considering full collateralization and setting the cost of collateral equal to the risk-free rate) and interest-rate forward rates for collateralized transactions with collateral costs equal to the funding costs of the counterparties. Spreads are negative, in line with the intuition that interest rate risk leads to a discount for the payer of the fixed rate, as discussed in the introduction,

and are of a magnitude consistent with the findings of Johannes and Sundaresan (2007). The results show that longevity swap spreads are comparable with, and often much smaller (in absolute value) than, those found in the interest-rate swap market. For example, in the case of bilateral full collateralization, longevity forward rates for 15- to 25-year maturities embed a spread substantially smaller than that of interest-rate forwards of corresponding maturity. In the case of one-way collateralization on the hedger's side, in interest-rate forward rates we find a discount (negative spread) that turns into a premium (positive spread) of comparable size in the corresponding longevity swap, due to the additional and opposite effect of longevity risk on swap rates. Our findings are robust to the choice of maturity, collateralization rules, and counterparty credit quality, and are mainly driven by two effects: i) the different nature of the risk underlying the swap, a survival curve in the case of longevity swaps, and a floating rate in the case of interest-rate swaps; ii) the fact that interest rate risk and longevity risk impact longevity swap margins in opposite directions, thus diluting the overall effect of collateralization on longevity swap rates.

< TABLE 3 ABOUT HERE >

< TABLE 4 ABOUT HERE >

< FIGURE 3 ABOUT HERE >

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In a second example, we focus on the opportunity cost of selling additional longevity protection. As we do not have any publicly available transaction data from the longevity swap market to calibrate  $X^{(5)}$ , we simulate the capital charges arising from holding a representative longevity-linked liability in response to changes in the evolution of the factors  $(X^{(1)}, \dots, X^{(4)})$  and  $X^{(6)}$ . We therefore 'synthesize' the realizations of  $X^{(5)}$  by using information on regulatory requirements to quantify the capital charges accruing to the

counterparties during the life of the swap. In particular, we use the following bottom-up procedure:

*Step 1:* We simulate several paths of the factors  $X^{(1)}, \dots, X^{(4)}$  and  $X^{(6)}$  along a time grid  $\hat{\mathcal{T}} := \{t_1, t_2, \dots, t_k\}$  (with  $t_k = \hat{T} > t_1 > 0$ ) and under the pricing measure  $\tilde{\mathbb{P}}$ . Again, for our example, we assume the  $\tilde{\mathbb{P}}$ -dynamics of  $X^{(6)}$  to be the same as under the physical measure.

*Step 2:* The paths simulated in the previous step are used to compute, at each date  $t \in \hat{\mathcal{T}}$ , the regulatory capital needed by an insurer to hold the liability  $n - N_{t+\bar{T}}$ , where  $\bar{T} < \hat{T}$  is a representative maturity proxying the average duration of longevity-linked liabilities in the longevity swap market. We use  $\bar{T} = 15$  and  $\hat{T} = 40$  (years) for our example. To compute the capital requirements, we use the Solvency II framework, which is based on the 99.5% value-at-risk of the net assets over a one-year horizon. For simplicity, we assume holders of longevity exposures to be invested in cash. The distribution of the one-year-ahead market-consistent value of the liability usually requires nested simulation, unless a simplified approach is adopted. In our setting, market-consistent discount factors can be computed analytically based on the one-year-ahead simulated realizations, as the pair  $(X^{(1)}, X^{(2)})$  is an affine process. We use the LSMC approach (see section 4.3) to determine the expected number of survivors.<sup>33</sup>

*Step 3:* We use the simulated capital charges obtained in the previous step to compute the gains/costs incurred to reduce/increase capital at each time step along each simulated path. We assume that capital charges are funded at the counterparties' funding cost, plus a spread of 6%<sup>34</sup> to reflect the opportunity cost of diverting to an individual liability funds that could be used to support insurance business at the aggregate level. The simulated realizations of the opportunity cost of capital (see figure 7 for an example) are used to estimate the dynamics of  $X^{(5)}$  reported in the appendix. The parameter estimates are included in table 2.

In the case of symmetric collateralization, we find results comparable with those obtained by using the counterparties' funding costs for the process  $\delta$ . However, figure 6 shows

<sup>33</sup>See Stevens et al. (2010) for other approximation methods in the context of Lee-Carter forecasts.

<sup>34</sup>This is a reasonable, conservative value for the return on capital of longevity swaps dealers: anecdotal evidence suggests that it can be twice as large.

that margins increase (decrease) considerably when one-way collateralization on the hedge supplier's (hedger's) side is considered. This is because the party required to post collateral explicitly takes into account tail events in computing collateral costs, whereas in figure 5 funding costs were computed on the basis of the market value of the longevity swap.

Finally, we study the sensitivity of longevity forward spreads to the volatility of the net collateral cost  $X^{(5)}$ . To close off the interest-rate risk channel, we fix the factors  $X^{(1)}, X^{(2)}$  equal to their long-run means. Table 5 reports the results obtained for different values of the volatility parameter  $\sigma_5$  in the case of symmetric default risk and bilateral full collateralization. We see that spreads increase dramatically for large values of the volatility parameter, but are comparable with those found in the previous examples for reasonable volatility levels (i.e., below 5%).

< TABLE 4 ABOUT HERE >

< FIGURE 5 ABOUT HERE >

< FIGURE 6 ABOUT HERE >

## 6 Conclusion

In this study, we have provided a framework for understanding and quantifying the cost of bilateral default risk and collateral strategies on longevity risk solutions. The results address the concerns aired by potential hedgers regarding how to measure the trade-off between the hedge effectiveness of longevity-linked instruments and the counterparty risk they entail. We have described a methodology for pricing longevity swaps that explicitly takes into account the dynamics of the marking-to-market process, the collateral flows it generates, and the costs associated with the posting of collateral. We have shown how collateral strategies can mitigate if not eliminate counterparty risk, but inevitably introduce an extra cost that must be borne by the hedge supplier or by the hedger, depending on how their credit quality and collateral costs compare with each other. Our most significant and

useful finding is that the overall cost of the collateralization strategies in the longevity swap market is comparable with, and often smaller than, that found in the much more liquid interest-rate swap market. Hence, there is no reason to suppose that counterparty risk will provide an insurmountable barrier to the further development of the longevity swap market. Our analysis accordingly provides a robust framework for comparing the costs of credit enhancement in bespoke longevity swaps with the benefits offered by competing solutions such as securitization and indexed swaps.

## References

Assefa, S., Bielecki, T. R., Crépey, S., and Jeanblanc, M. (2010). CVA computation for counterparty risk assessment in credit portfolios. In Bielecki, T., Brigo, D., and Patras, F., editors, *Recent Advancements in the Theory and Practice of Credit Derivatives*. Bloomberg Press.

Bacinello, A., Biffis, E., and Millosovich, P. (2009). Pricing life insurance contracts with early exercise features. *Journal of Computational and Applied Mathematics*, 233(1):27–35.

Bacinello, A., Biffis, E., and Millosovich, P. (2010). Regression-based algorithms for life insurance contracts with surrender guarantees. *Quantitative Finance*, 10:1077–1090.

Basel II (2006). *International Convergence of Capital Measurement and Capital Standards: A Revised Framework*. Basel Committee on Banking Supervision.

Bauer, D., Benth, F., and Kiesel, R. (2012). Modeling the forward surface of mortality. *SIAM Journal on Financial Mathematics*, 3(1):639–666.

Bauer, D., Bergmann, D., and Reuß, A. (2009). Solvency II and nested simulations - A Least-Squares Monte Carlo approach. Technical report, University of Ulm and Georgia State University, [http://www.cb.wsu.edu/aria2009/ARIA2009Papers/FullPapers/session6D\\_Bauer\\_Bergmann\\_Reuss.pdf](http://www.cb.wsu.edu/aria2009/ARIA2009Papers/FullPapers/session6D_Bauer_Bergmann_Reuss.pdf).

Bauer, D., Börger, M., and Ruß, J. (2010). On the pricing of longevity-linked securities. *Insurance: Mathematics & Economics*, 46(1):139 – 149.

Bielecki, T. and Rutkowski, M. (2002). *Credit Risk: Modeling, Valuation and Hedging*. Springer.

- Biffis, E. and Blake, D. (2010a). Mortality-linked securities and derivatives. In Bertocchi, M., Schwartz, S., and Ziemba, W., editors, *Optimizing the Aging, Retirement and Pensions Dilemma*. John Wiley & Sons.
- Biffis, E. and Blake, D. (2010b). Securitizing and tranching longevity exposures. *Insurance: Mathematics & Economics*, 46(1):186–197.
- Biffis, E. and Blake, D. (2013). Informed intermediation of longevity exposures. *Journal of Risk and Insurance*, 80(3):559–584.
- Biffis, E. and Denuit, M. (2006). Lee-Carter goes risk-neutral: An application to the Italian annuity market. *Giornale dell’Istituto Italiano degli Attuari*, LXIX:33–53.
- Biffis, E., Denuit, M., and Devolder, P. (2010). Stochastic mortality under measure changes. *Scandinavian Actuarial Journal*, 2010:284–311.
- Blake, D., Cairns, A., Coughlan, G., Dowd, K., and MacMinn, R. (2013). The new life market. *Journal of Risk and Insurance*, 80(3):501–558.
- Brigo, D. and Capponi, A. (2009). Bilateral counterparty risk valuation with stochastic dynamical models and application to CDSs. Technical report, King’s College London and California Institute of Technology, <http://arxiv.org/abs/0812.3705>.
- Brigo, D., Capponi, A., and Pallavicini, A. (2012). Arbitrage-free bilateral counterparty risk valuation under collateralization and application to credit default swaps. *Mathematical Finance*. to appear, <http://onlinelibrary.wiley.com/doi/10.1111/j.1467-9965.2012.00520.x/pdf>.
- Brigo, D., Pallavicini, A., and Papatheodorou, V. (2011). Arbitrage-free valuation of bilateral counterparty risk for interest-rate products: Impact of volatilities and correlations. *International Journal of Theoretical and Applied Finance*, 14(6):773802.
- Cairns, A., Blake, D., Dowd, K., Coughlan, G., Epstein, D., and Khalaf-Allah, M. (2011). Mortality density forecasts: An analysis of six stochastic mortality models. *Insurance: Mathematics & Economics*, 48:355–367.
- Cairns, A., Blake, D., Dowd, K., Coughlan, G., Epstein, D., Ong, A., and Balevich, I. (2009). A quantitative comparison of stochastic mortality models using data from England & Wales and the United States. *North American Actuarial Journal*, 13(1):1–35.

Cairns, A. J. G. (2013). Robust hedging of longevity risk. *Journal of Risk and Insurance*, 80(3):621–648.

CEIOPS (2009). SCR standard formula: Further advice on the counterparty default risk module. Technical report, Committee of European Insurance and Occupational Pension Supervisors' Consultation Paper no. 51.

Chen, H. and Cummins, J. (2010). Longevity bond premiums: The extreme value approach and risk cubic pricing. *Insurance: Mathematics & Economics*, 46(1):150–161.

Coughlan, G., Khalaf-Allah, M., Ye, Y., Kumar, S., Cairns, A., Blake, D., and Dowd, K. (2011). Longevity hedging 101: A framework for longevity basis risk analysis and hedge effectiveness. *North American Actuarial Journal*, 15(2):150–176.

Cox, S., Lin, Y., and Pedersen, H. (2010). Mortality risk modeling: Applications to insurance securitization. *Insurance: Mathematics & Economics*, 46(1):242–253.

Cummins, J. and Trainar, P. (2009). Securitization, insurance, and reinsurance. *Journal of Risk and Insurance*, 76(3):463–492.

Deng, Y., Brockett, P. L., and MacMinn, R. D. (2012). Longevity/mortality risk modeling and securities pricing. *Journal of Risk and Insurance*, 79(3):697–721.

Dowd, K., Blake, D., Cairns, A., and Dawson, P. (2006). Survivor swaps. *Journal of Risk and Insurance*, 73(1):1–17.

Dowd, K., Cairns, A., Blake, D., Coughlan, G., Epstein, D., and Khalaf-Allah, M. (2010a). Backtesting stochastic mortality models: An ex-post evaluation of multi-period-ahead density forecasts. *North American Actuarial Journal*, 14(3):281–298.

Dowd, K., Cairns, A., Blake, D., Coughlan, G., Epstein, D., and Khalaf-Allah, M. (2010b). Evaluating the goodness of fit of stochastic mortality models. *Insurance: Mathematics & Economics*, 47:255–265.

Duffie, D. and Huang, M. (1996). Swap rates and credit quality. *Journal of Finance*, 51(3):921–949.

Forsythe, G., Malcolm, M., and Moler, C. (1976). *Computer Methods for Mathematical Computations*. Prentice-Hall.

Gatzert, N. and Wesker, H. (2012). Mortality risk and its effect on shortfall and risk

- management in life insurance. *Journal of Risk and Insurance*. In press, <http://dx.doi.org/10.1111/j.1539-6975.2012.01496.x>.
- Girosi, F. and King, G. (2008). *Demographic Forecasting*. Princeton University Press.
- Glassermann, P. (2004). *Monte Carlo Methods in Financial Engineering*. Springer.
- Hörig, M. and Leitschkis, M. (2012). Solvency ii proxy modelling via least squares monte carlo. Technical report, Milliman, <http://publications.milliman.com/publications/life-published/pdfs/solvency-II-proxy-modelling.pdf>.
- Inkman, J. and Blake, D. (2010). Managing financially distressed pension plans in the interest of beneficiaries. Technical report, Pensions Institute Discussion Paper PI-0709, <http://www.pensions-institute.org/workingpapers/wp0709.pdf>.
- ISDA (1994). *Credit Support Annex*. International Swaps and Derivatives Association.
- ISDA (2009). *Big Bang Protocol*. International Swaps and Derivatives Association.
- ISDA (2010a). *2010 Margin Survey*. International Swaps and Derivatives Association.
- ISDA (2010b). *Market Review of OTC Derivative Bilateral Collateralization Practices*. International Swaps and Derivatives Association.
- Johannes, M. and Sundaresan, S. (2007). The impact of collateralization on swap rates. *Journal of Finance*, 62(1):383–410.
- Kessler, D. (2013). Why (re)insurance is not systemic. *Journal of Risk and Insurance*. In press, <http://dx.doi.org/10.1111/j.1539-6975.2013.12007.x>.
- Lakdawalla, D. and Zanjani, G. (2012). Catastrophe bonds, reinsurance, and the optimal collateralization of risk transfer. *Journal of Risk and Insurance*, 79(2):449–476.
- Lee, R. and Carter, L. (1992). Modeling and forecasting US mortality. *Journal of the American statistical association*, 87(419):659–671.
- Ludkovski, M. and Young, V. (2008). Indifference pricing of pure endowments and life annuities under stochastic hazard and interest rates. *Insurance: Mathematics and Economics*, 42(1):14–30.
- Pitacco, E., Denuit, M., Haberman, S., and Olivieri, A. (2009). *Modelling Longevity Dynamics for Pensions and Annuity Business*. Oxford University Press.



Salhi, Y. and Loisel, S. (2012). Basis risk modelling: A co-integration based approach. Technical report, ISFA Lyon, <http://hal.archives-ouvertes.fr/hal-00746859>.

Stevens, R., De Waegenaere, A., and Melenberg, B. (2010). Calculating capital requirements for longevity risk in life insurance products: Using an internal model in line with Solvency II. Technical report, Tilburg University, <http://www.netspar.nl/files/Evenementen/20100610/067ralphstevens.pdf>.

Stevens, R., De Waegenaere, A., and Melenberg, B. (2011). Longevity risk and hedge effects in a portfolio of life insurance products with investment risk. Technical report, Tilburg University, [http://papers.ssrn.com/sol3/papers.cfm?abstract\\_id=1799325](http://papers.ssrn.com/sol3/papers.cfm?abstract_id=1799325).

The Actuarial Profession (2005). *Allowing for the Sponsor Covenant in Actuarial Advice*. Sponsor Covenant Working Party Final Report, <http://www.actuaries.org.uk/sites/all/files/documents/pdf/sponsorcovenantrep.pdf>.

The Pensions Regulator (2009). *Scheme Funding and the Employer Covenant*. <http://www.thepensionsregulator.gov.uk/docs/employer-covenant-statement-june-2009.pdf>.

Wang, C.-W., Huang, H.-C., and Liu, I.-C. (2013). Mortality modeling with non-Gaussian innovations and applications to the valuation of longevity swaps. *Journal of Risk and Insurance*, 80(3):775–798.

## A Details on the setup

We take as given a filtered probability space  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$ , and model the death times in a population of  $n$  individuals (annuitants or pensioners) as stopping times  $\tau^1, \dots, \tau^n$ . This means that at each time  $t$  the information carried by  $\mathcal{F}_t$  allows us to state whether each individual has died or not. The hedger's liability is given by the random variable  $\sum_{i=1}^n 1_{\{\tau^i > T\}}$ , which can be equivalently written as  $n - \sum_{i=1}^n 1_{\{\tau^i \leq T\}} = n - N_T$ . We assume that death times coincide with the first jumps of  $n$  conditionally Poisson processes with common random intensity of mortality  $(\mu_t)_{t \geq 0}$  under both  $\mathbb{P}$  and an equivalent martingale measure  $\tilde{\mathbb{P}}$  (see Biffis et al., 2010, for details). The expected number of survivors over  $[0, T]$  under the two measures can then be expressed as  $E^{\mathbb{P}}[\sum_{i=1}^n 1_{\{\tau^i > T\}}] = np_T$  and  $E^{\tilde{\mathbb{P}}}[\sum_{i=1}^n 1_{\{\tau^i > T\}}] = n\tilde{p}_T$ , with  $p_T$  and  $\tilde{p}_T$  given by the expectation (2.1) computed under

the relevant probability measure.

Consider any stopping time  $\tau^i$  satisfying the above assumptions, an integrable random variable  $Y \in \mathcal{F}_T$  and a bounded process  $(X_t)_{t \in [0, T]}$  such that each  $X_t$  is measurable with respect to  $\mathcal{F}_{t-}$ , the information available up to, but not including, time  $t$ . Then a security paying  $Y$  at time  $T$  in case  $\tau^i > T$  and  $X_{\tau^i}$  at time  $\tau^i$  in case  $\tau^i \leq T$  has time-zero price

$$E^{\tilde{\mathbb{P}}} \left[ \int_0^T \exp \left( - \int_0^s (r_t + \mu_t) dt \right) X_s \mu_s ds + \exp \left( - \int_0^T (r_t + \mu_t) dt \right) Y \right].$$

Consider now two stopping times  $\tau^i, \tau^j$ , with intensities  $\mu^i, \mu^j$ , jointly satisfying the above assumptions (i.e., they are the first jump times of the components of a bivariate conditionally Poisson process). A security paying  $Y$  at time  $T$  in case neither stopping time has occurred (i.e.,  $\min(\tau^i, \tau^j) > T$ ) and  $X_t$  in case the first occurrence is at time  $t \in (0, T]$  (i.e.,  $t = \min(\tau^i, \tau^j)$ ) has time-zero price given by the same formula, with  $\mu_t$  replaced by  $\mu_t^i + \mu_t^j$ . This follows from the fact that the stopping time  $\min(\tau^i, \tau^j)$  is the first jump time of a conditionally Poisson process with intensity  $(\mu_t^i + \mu_t^j)_{t \geq 0}$  (e.g., Bielecki and Rutkowski, 2002). The expressions presented in sections 2-4 all follow from these simple results.

**Proof of expression (4.2).** Let  $(\delta_t^H)_{t \geq 0}$  denote the hedger's net cost of posting collateral and  $(\delta_t^{\text{HS}})_{t \geq 0}$  the net yield on the collateral amounts received from party HS, meaning that holding collateral of amount  $C_t$  provides the hedger with an instantaneous yield equal to  $\delta_t^{\text{HS}} C_t^+ - \delta_t^H C_t^-$ . We assume that collateral is bounded and  $C_t$  is measurable with respect to  $\mathcal{F}_{t-}$  for all  $t \in [0, T]$ . Parties H and HS are assumed to have death (default) times satisfying the properties reviewed above, in particular having intensities  $\lambda^H, \lambda^{\text{HS}}$ . Recalling the recovery rules described in section 4, we can then write:

$$\begin{aligned} S_0 = & E^{\tilde{\mathbb{P}}} \left[ \exp \left( - \int_0^T (r_t + \lambda_t^H + \lambda_t^{\text{HS}}) dt \right) (P - \bar{p}^d) \right] \\ & + E^{\tilde{\mathbb{P}}} \left[ \int_0^T \exp \left( - \int_0^s (r_t + \lambda_t^H + \lambda_t^{\text{HS}}) dt \right) (\lambda_s^H (S_s^+ - C_s^-) + \lambda_s^{\text{HS}} (C_s^+ - S_s^-)) ds \right] \\ & + E^{\tilde{\mathbb{P}}} \left[ \int_0^T \exp \left( - \int_0^s (r_t + \lambda_t^H + \lambda_t^{\text{HS}}) dt \right) (\delta_s^{\text{HS}} C_s^+ - \delta_s^H C_s^-) ds \right]. \end{aligned}$$

Using representation (4.1), the amount recovered by the nondefaulting counterparty at

time  $\tau = \min(\tau^H, \tau^{\text{HS}}) \leq T$  is

$$1_{\{\tau=\tau^H\}}S_{\tau-}(c_\tau^H 1_{\{S_{\tau-}<0\}} + 1_{\{S_{\tau-}\geq 0\}}) + 1_{\{\tau=\tau^{\text{HS}}\}}S_{\tau-}(c_\tau^{\text{HS}} 1_{\{S_{\tau-}\geq 0\}} + 1_{\{S_{\tau-}<0\}}),$$

where we see that  $c^H, c^{\text{HS}}$  replace the recovery rates  $\psi^H, \psi^{\text{HS}}$  introduced in section 3. We can then write

$$\begin{aligned} S_0 &= E^{\tilde{\mathbb{P}}} \left[ \exp \left( - \int_0^T (r_t + \lambda_t^H + \lambda_t^{\text{HS}}) dt \right) (P - \bar{p}^d) \right] \\ &\quad + E^{\tilde{\mathbb{P}}} \left[ \int_0^T \exp \left( - \int_0^s (r_t + \lambda_t^H + \lambda_t^{\text{HS}}) dt \right) (\lambda_s^H + (\lambda_s^{\text{HS}} + \delta_s^{\text{HS}}) c_s^{\text{HS}}) S_s^+ - (\lambda_s^{\text{HS}} + (\lambda_s^H + \delta_s^H) c_s^H) S_s^- \right] ds \\ &= E^{\tilde{\mathbb{P}}} \left[ \exp \left( - \int_0^T (r_t + \Gamma_t) dt \right) (P - \bar{p}^d) \right], \end{aligned}$$

which is nothing other than the usual risk-neutral valuation formula for a security with terminal payoff  $S_T = P - \bar{p}^d$  paying continuously a dividend equal to a fraction

$$(\lambda_s^H + (\lambda_s^{\text{HS}} + \delta_s^{\text{HS}}) c_s^{\text{HS}}) 1_{\{S_{t-} \geq 0\}} + (\lambda_s^{\text{HS}} + (\lambda_s^H + \delta_s^H) c_s^H) 1_{\{S_{t-} < 0\}}$$

of the security's market value an instant before each  $t \in [0, T]$ . Subtracting the dividend rate from  $\lambda^H + \lambda^{\text{HS}}$  and rearranging terms we obtain expression (4.2) for  $\Gamma$ .

## B Details on the numerical examples

The numerical examples are based on a six-dimensional state variable process  $X = (X^{(1)}, \dots, X^{(6)})^\top$  having  $\tilde{\mathbb{P}}$ -dynamics

$$\begin{aligned} dX_t^{(1)} &= \left( k_1(X_t^{(2)} - X_t^{(1)}) - \eta_1 \right) dt + \sigma_1 dW_t^{(1)} \\ dX_t^{(2)} &= \left( k_2(\theta_2 - X_t^{(2)}) - \eta_2 \right) dt + \sigma_2 dW_t^{(2)} \\ dX_t^{(3)} &= \left( \kappa_3(\theta_3 - X_t^{(3)}) + \kappa_{3,1}(X_t^{(1)} - \theta_2) + \kappa_{3,4}(X_t^{(4)} - \theta_4) - \eta_3 \right) dt + \sigma_3 dW_t^{(3)} \\ dX_t^{(4)} &= \left( \kappa_4(\theta_4 - X_t^{(4)}) + \kappa_{4,1}(X_t^{(1)} - \theta_2) + \kappa_{4,2}(X_t^{(2)} - \theta_2) - \eta_4 \right) dt + \sigma_4 dW_t^{(4)} \\ dX_t^{(5)} &= \left( \kappa_5(\theta_5 - X_t^{(5)}) + \kappa_{5,1}(X_t^{(1)} - \theta_2) + \kappa_{5,2}(X_t^{(2)} - \theta_2) + \kappa_{5,3}(X_t^{(3)} - \theta_3) \right. \\ &\quad \left. + \kappa_{5,4}(X_t^{(4)} - \theta_4) + \kappa_{5,6}(X_t^{(6)} - E_0[X_t^{(6)}]) - \eta_5 \right) dt + \sigma_5 dW_t^{(5)} \\ dX_t^{(6)} &= \left( A_x(t) + B_x(t)(X_t^{(6)} - a_x(t)) \right) dt + \sigma_6(t, x) dW_t^{(6)}, \end{aligned}$$

where  $W = (W^{(1)}, \dots, W^{(6)})^\top$  is a standard  $\tilde{\mathbb{P}}$ -Brownian motion, the constants  $\eta_i$  represent market prices of risk,  $x$  is the age of a reference cohort of individuals at time 0, and  $A_x(\cdot), B_x(\cdot), \sigma_6(\cdot, x)$  are functions characterizing the dynamics of  $X_t^{(6)} = X_{t,x}^{(6)}$  (see below for explicit definitions). The  $\mathbb{P}$ -dynamics are obtained by removing the market prices of risk from the drifts of the relevant factors and replacing the innovations with the corresponding  $\mathbb{P}$ -Brownian innovations. We assume that  $X^{(6)}$  has the same dynamics under the physical and the pricing probability measures, consistent with our baseline case of a swap rate equal to  $p_T$  for each  $T$  in the absence of collateral. The Brownian innovations are uncorrelated, with the exception of the pair  $(W^{(1)}, W^{(2)})$ , whose instantaneous correlation is denoted by  $\rho_{1,2}$ .

For the first four factors, we use data from Johannes and Sundaesan (2007) who rely on a two-stage maximum likelihood procedure based on weekly data sampled on Wednesdays, from 1990 to 2002, and set the long-run mean of  $X^{(3)}$  equal to the average of the 3-month TED spread over the sampling period. For the log-intensity  $X^{(6)}$ , we use the mortality model described below, and assume that the Brownian component  $W^{(6)}$  is uncorrelated with the other ones. The intensity of mortality is modeled using a continuous-time version of the Lee-Carter model (see Biffis and Denuit, 2006; Biffis et al., 2010). We first use the

annual central death rates  $\{m_{y,s}\}$  for US males from the Human Mortality Database to estimate the model  $m_{y,s} = \exp(\alpha(y) + \beta(y)K_s)$  for dates  $s = 1961, 1962, \dots, 2007$  and ages  $y = 20, 21, \dots, 89$  with Singular Value Decomposition (see Lee and Carter, 1992). The resulting estimates for  $K$  are then fitted with the process  $K_{s+1} = \delta_K K_s + \sigma_K \varepsilon$ , with  $\varepsilon \sim N(0, 1)$ . For fixed age  $x = 65$ , the estimates for  $\{\hat{\alpha}(x+h), \hat{\beta}(x+h)\}_{h=0,1,\dots}$  are interpolated with differentiable functions  $a_x(t), b_x(t)$ . The functions  $A_x, B_x, \sigma_6$  are finally obtained by setting  $A_x(t) = a'_x(t) + b_x(t)\delta_K$ ,  $B_x(t) = b'_x(t)b_x(t)^{-1}$  and  $\sigma_6(t, x) = b_x(t)\sigma_K$ . As we consider a single cohort aged  $x$  at the reference date 0, here and throughout the paper we simply write  $X_t^{(6)} := X_{t,x}^{(6)}$ . The extension to multiple (say  $l$ ) cohorts, would require the analysis of the vector of log-intensities  $(X_{t,x_1}^{(6)}, \dots, X_{t,x_l}^{(6)})$ . Although the drift and volatility parameters would be different for each  $X_{t,x_i}^{(6)}$ , the Lee-Carter specification assumes that all cohorts are affected by the same Brownian component  $W^{(6)}$ . Other models may instead require the introduction of additional sources of uncertainty.

To estimate the dynamics of  $X^{(5)}$ , the component of collateral costs related to longevity risk, we implement the procedure discussed in section 5, setting the duration  $\bar{T}$  of the representative liability equal to 15. We simulate forward all of the other state variables, and at each time step we compute the opportunity cost of capital arising from the capital charges accruing to the hedge supplier based on the simulated mortality and market conditions. The expectation appearing in the drift of  $X^{(5)}$  ensures that the longevity capital charges react to departures of realized mortality from the term structure of survival rates estimated at inception. We assume that funding occurs at the LIBOR rate plus a fixed spread of 6%, a conservative value for the cost of internal capital. To obtain the net cost of collateral, we take into account the rebate of the risk-free rate. We estimate the parameters for the dynamics of  $X^{(5)}$  based on the simulated realizations of  $X^{(5)}$  (an example is depicted in figure 7). The parameter estimates are obtained by regressing the simulated dynamics of  $X^{(5)}$  on the simulated vector of state variables  $(X^{(1)}, X^{(2)}, X^{(3)}, X^{(4)}, X^{(6)})$ . We simulate 10,000 paths over 40 years on a semi-annual grid. For each simulation, we set the parameter  $\theta_5$  equal to the average of  $X^{(5)}$  along the simulated path. The regression estimates obtained for each simulated path are averaged across all simulations to obtain the final values reported in table 2.

## C Tables and figures

Date	Hedger	Size	Term (yrs)	Type	Interm./supplier
Jan 2008	Lucida	N.A.	10	indexed	JP Morgan ILS funds
Jul 2008	Canada Life	GBP 500m	40	bespoke	JP Morgan ILS funds
Feb 2009	Abbey Life	GBP 1.5bn	run-off	bespoke	Deutsche Bank ILS funds / Partner Re
Mar 2009	Aviva	GBP 475m	10	bespoke	Royal Bank of Scotland
Jun 2009	Babcock International	GBP 750m	50	bespoke	Credit Suisse Pacific Life Re
Jul 2009	RSA	GBP 1.9bn	run-off	bespoke	Goldman Sachs (Rothesay Life)
Dec 2009	Berkshire Council	GBP 750m	run-off	bespoke	Swiss Re
Feb 2010	BMW UK	GBP 3bn	run-off	bespoke	Deutsche Bank Paternoster
Dec 2010	Swiss Re (Kortis bond)	USD 50m	8	indexed	ILS funds
Feb 2011	Pall (UK) Pension Fund	GBP 70m	10	indexed	JP Morgan
Aug 2011	ITV	GBP 1.7bn	N.A.	bespoke	Credit Suisse
Nov 2011	Rolls Royce	GBP 3bn	N.A.	bespoke	Deutsche Bank
Dec 2011	British Airways	GBP 1.3bn	N.A.	bespoke	Goldman Sachs (Rothesay Life)
Jan 2012	Pilkington	GBP 1bn	N.A.	bespoke	Legal & General
Apr 2012	Berkshire Council	GBP 100m	run-off	bespoke	Swiss Re

Table 1: Publicly announced longevity swap transactions 2008-2012.

$\kappa_1$	0.969	$\eta_1$	-0.053	$\sigma_1$	0.008	UK	
$\kappa_2$	0.832	$\eta_3$	-0.014	$\sigma_2$	0.155	$\delta_K$	-0.888
$\kappa_3$	1.669	$\eta_4$	0.007	$\sigma_3$	0.009	$\sigma_K$	1.156
$\kappa_4$	0.045	$\eta_5$	0.055	$\sigma_4$	0.010	US	
$\kappa_5$	0.990	$\kappa_{5,1}$	0.147	$\sigma_5$	0.690	$\delta_K$	-0.761
$\kappa_{3,1}$	-0.163	$\kappa_{5,2}$	1.340	$\theta_2$	0.046	$\sigma_K$	1.078
$\kappa_{4,1}$	0.114	$\kappa_{5,3}$	2.509	$\theta_3$	0.003		
$\kappa_{3,4}$	0.804	$\kappa_{5,4}$	-0.133	$\theta_4$	0.007		
$\kappa_{4,2}$	-0.038	$\kappa_{5,6}$	-0.002	$\theta_5$	0.115	$\rho_{1,2}$	-0.036

Table 2: Parameter values for the dynamics of  $X$  given in Appendix B. As in Johannes and Sundaesan (2007), we set the parameter  $\eta_2$  equal to zero. The estimates for  $X^{(5)}$  are based on the assumption that capital increases are funded by counterparties at 6% plus the LIBOR rate.

$\lambda^H = \lambda^{HS} + \Delta$	Maturity	$c^H = 0$	$c^H = 0$	$c^H = 1$	$c^H = 1$
$\delta^H = \delta^{HS} + \Delta$	payment	$c^{HS} = 0$	$c^{HS} = 1$	$c^{HS} = 0$	$c^{HS} = 1$
	(yrs)	(bps)	(bps)	(bps)	(bps)
$\Delta = 0$	15	0.03	11.34	-11.76	0.05
	20	1.11	19.93	-17.94	0.86
	25	1.50	21.25	-18.35	1.24
$\Delta = 100$ bps	15	5.45	16.79	-17.29	-5.84
	20	10.16	28.95	-27.08	-8.23
	25	10.96	30.75	-27.76	-9.19
$\Delta = 200$ bps	15	11.30	22.29	-22.90	-11.25
	20	19.26	38.06	-36.16	-17.42
	25	19.46	40.27	-37.02	-18.38

Table 3: Forward rate spreads  $\bar{p}_{T_i}^c - p_{T_i}$  (in basis points) for different collateralization rules, maturities and spread  $\Delta \in \{0, 0.01, 0.02\}$ . The LSMC procedure uses 5000 paths over a quarterly grid with polynomial basis functions of order 3, and is repeated for 100 seeds.

$\lambda^H = \lambda^{HS} + \Delta$	Maturity	IRS ( $\delta^{HS} = X^{(4)}$ )				Longevity ( $\delta^{HS} = X^{(3)}$ )		
		$c^H = 0$	$c^H = 1$	$c^H = 1$	$c^H = 1$	$c^H = 0$	$c^H = 1$	$c^H = 1$
$\delta^H = \delta^{HS} + \Delta$	payment	$c^{HS} = 1$	$c^{HS} = 0$	$c^{HS} = 1$	$c^{HS} = 1$	$c^{HS} = 0$	$c^{HS} = 1$	
	(yrs)	(bps)	(bps)	(bps)	(bps)	(bps)	(bps)	
$\Delta = 0$	15	-7.96	-44.97	-52.86	11.34	-11.76	0.05	
	20	-12.68	-42.64	-56.22	19.93	-17.94	0.86	
	25	-17.94	-40.98	-58.92	21.25	-18.35	1.24	
$\Delta = 100$ bps	15	-8.00	-67.87	-75.23	16.79	-17.29	-5.84	
	20	-12.65	-63.84	-77.42	28.95	-27.08	-8.23	
	25	-17.65	-60.63	-77.64	30.75	-27.76	-9.19	

Table 4: Comparison of interest-rate swaps (IRSs) with longevity swaps. The IRS spreads represent the difference between the futures prices (the opportunity cost of collateral coincides with the risk-free rate for both parties) and the forward rate for a collateralized transaction (for different collateralization rules, maturities, and credit risk).

$\sigma_5$	$p_{25}$	$\bar{p}^c$	spread (bps)
0.0005	0.201425	0.201469	2.15
0.0100	0.201425	0.201822	19.68
0.0150	0.201425	0.202009	28.96
0.0200	0.201425	0.202196	38.26
0.1000	0.201425	0.205237	189.24
0.1500	0.201425	0.207184	285.90

Table 5: Sensitivity with respect to parameter  $\sigma_5$ : we compute 25-year forward rates and spreads (in basis points) under full collateralization by setting  $X^{(1)}, X^{(2)}$  equal to their long run means. The baseline estimated parameter values for the dynamics of  $X^{(5)}$  are  $\theta_5 = 0.000254$ ,  $\kappa_5 = 1.005073$ ,  $\sigma_5 = 0.000542$ ,  $\eta_5 = 0.000269$ ,  $\kappa_{53} = 0.003648$ ,  $\kappa_{54} = 0.000018$ ,  $\kappa_{56} = 0.000261$ .

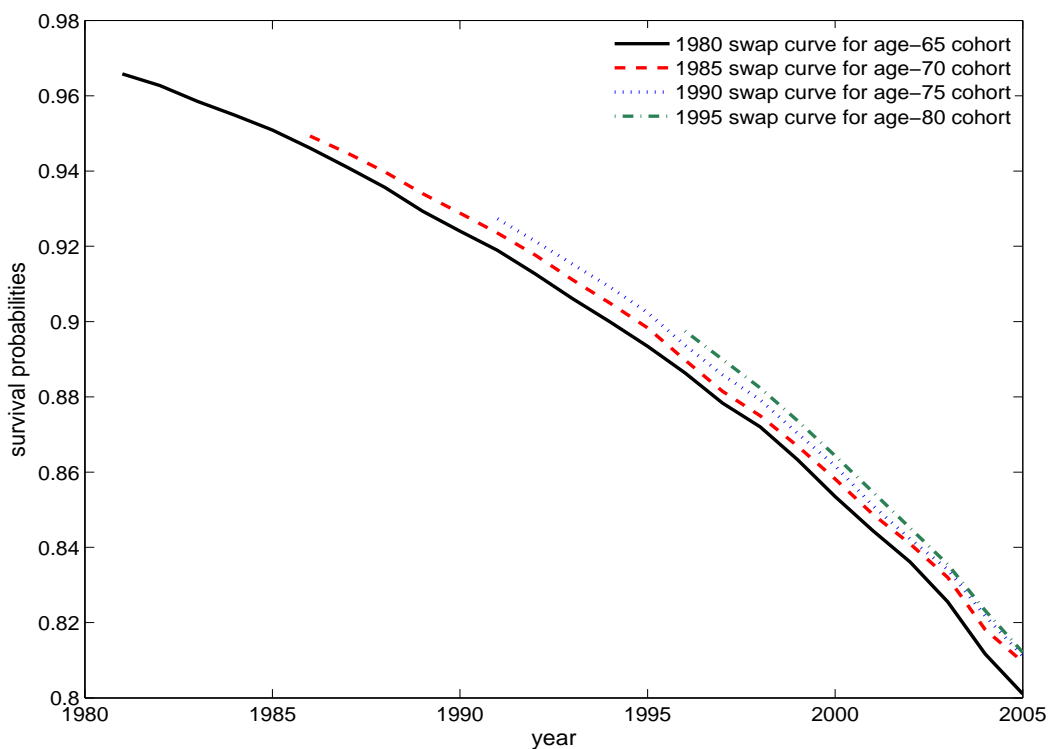


Figure 1: Survival curves computed at the beginning of year  $t = 1980, 1985, 1990, 1995$  for England & Wales males aged  $65 + t - 1980$  in year  $t$ . Forecasts are based on the Lee-Carter mortality projection model using the latest Human Mortality Database data available at the beginning of each year  $t$ .



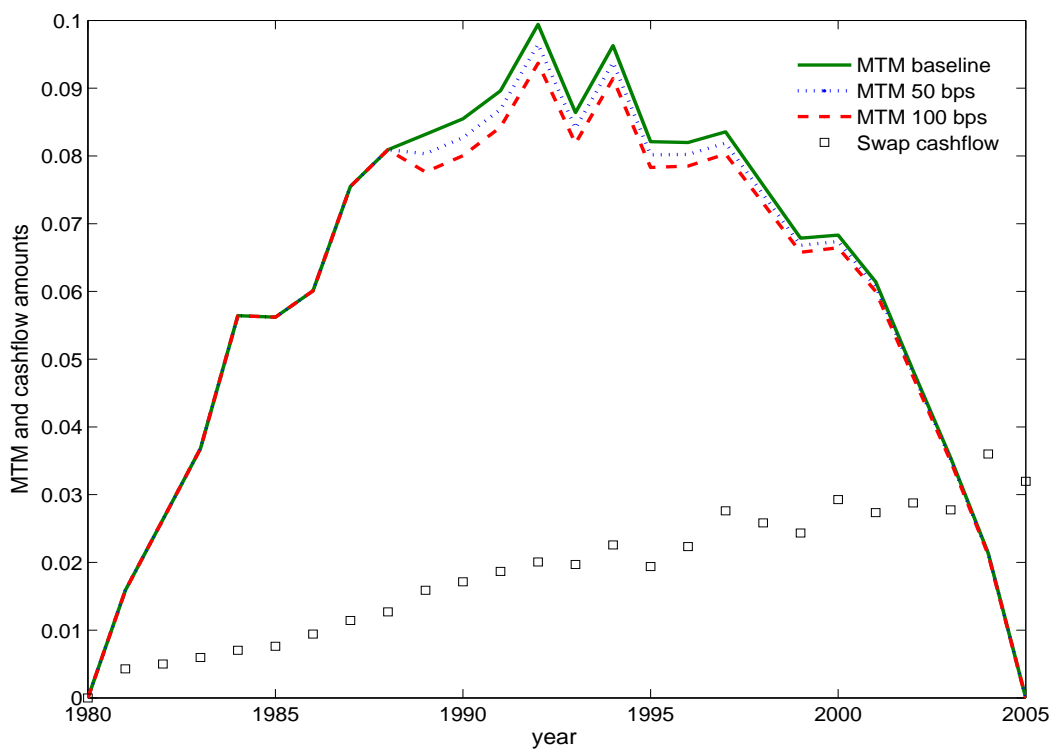


Figure 2: Mark-to-market value of the longevity swap in the baseline case and with the credit spread of counterparty HS widening by 50 and 100 basis points over 1988-2005. In the absence of default, the net payments from the swap are insensitive to credit spread changes.

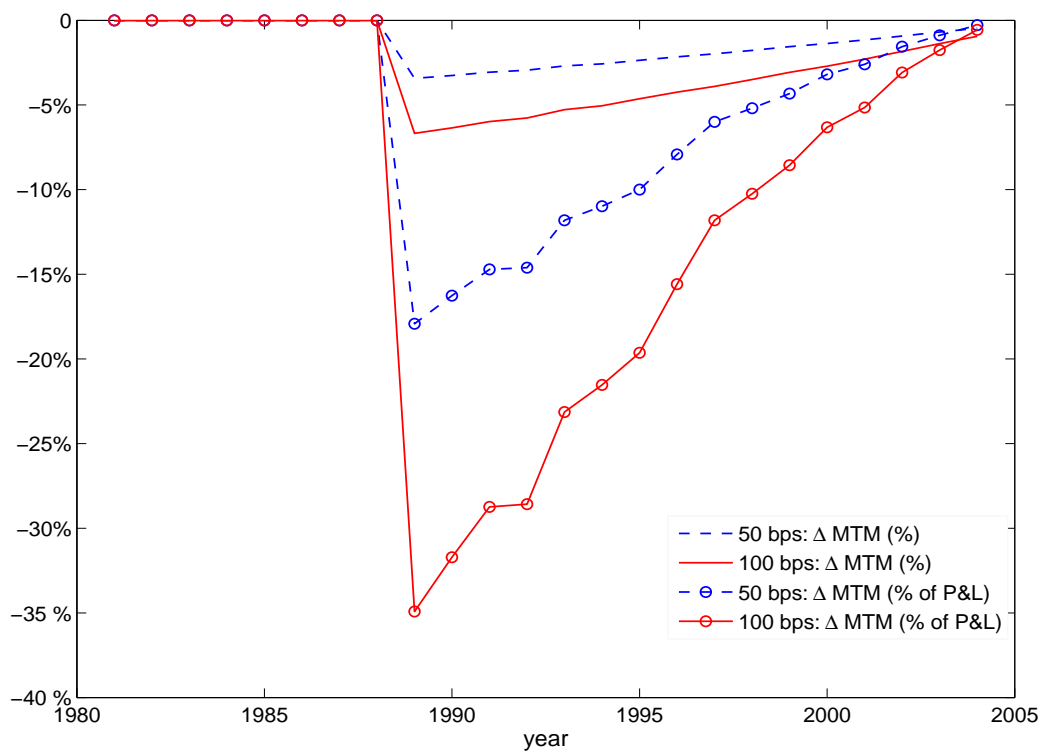


Figure 3: Change in longevity swap MTM relative to the baseline case and to the net payments from the swap, when the credit spread of counterparty HS widens by 50 and 100 basis points over 1988-2005.

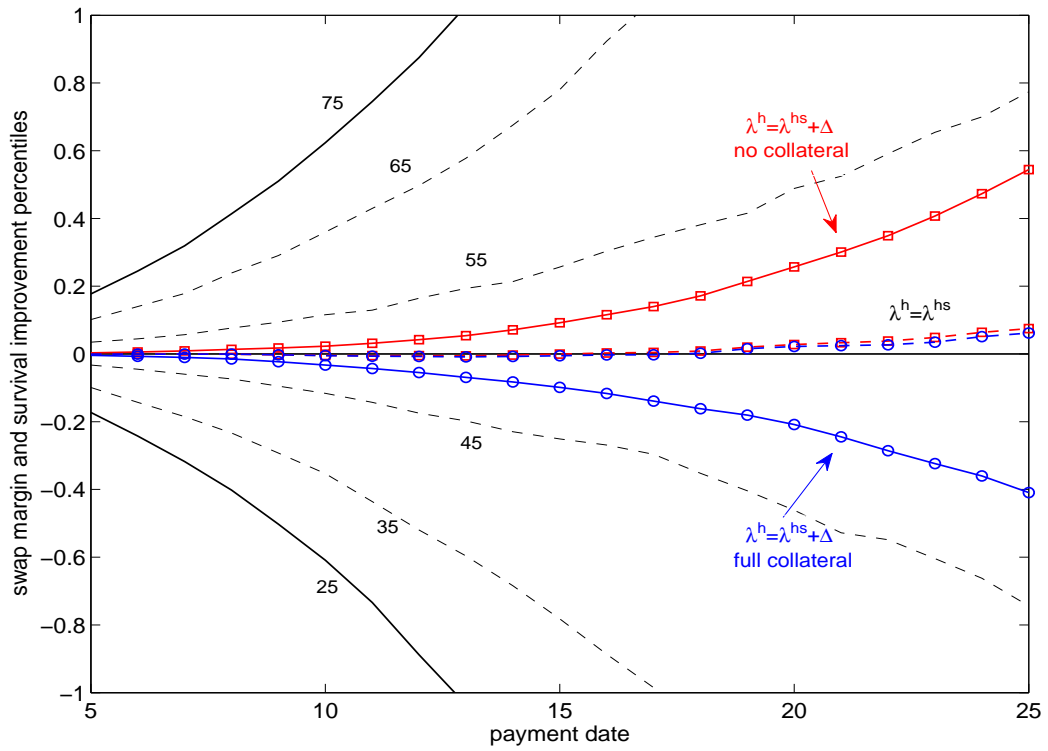


Figure 4: Swap margins  $\bar{p}_{T_i}^c / p_{T_i} - 1$  computed for different maturities  $\{T_i\}$  and collateral rules, with  $\delta^h = \lambda^h$ ,  $\delta^{hs} = \delta^h + \Delta$ , and  $\lambda^h = \lambda^{hs} + \Delta$ , with  $\Delta = 0$  (dashed lines) or  $\Delta = 0.01$  (solid lines): no collateral (squares), full collateralization (circles). The underlying is a cohort of 10,000 US males aged 65 at the beginning of 2008. Forward rates are plotted against the percentiles of improvements in survival rates based on Lee-Carter forecasts.

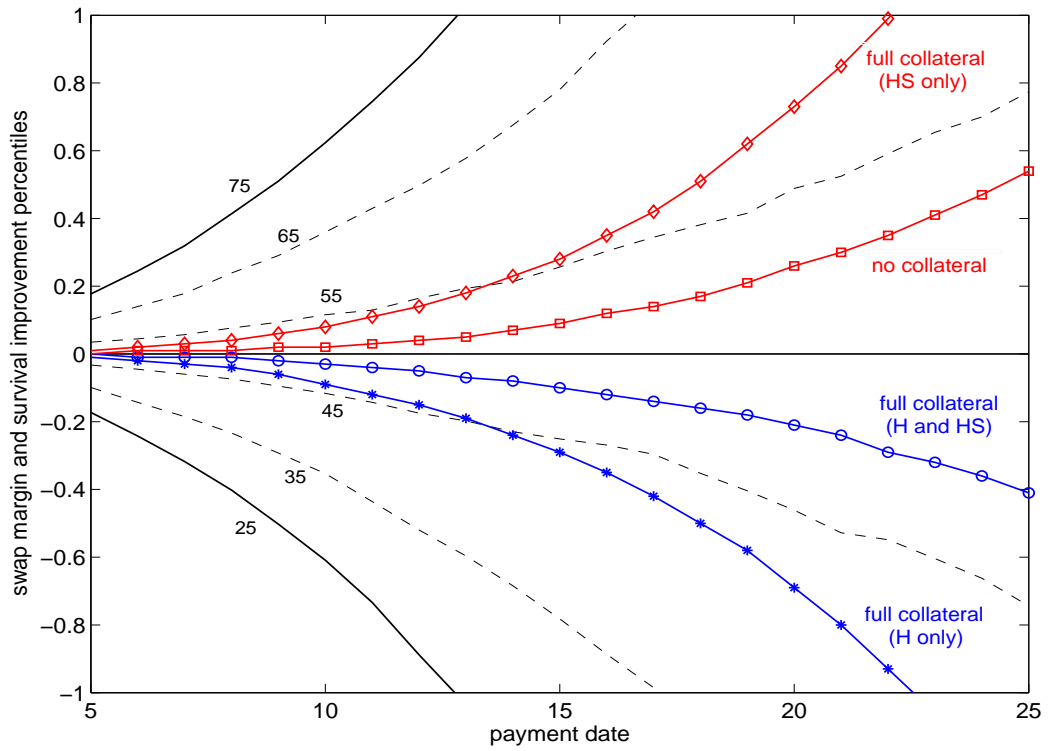


Figure 5: Swap margins  $\bar{p}_{T_i}^c / p_{T_i} - 1$  computed for different maturities  $\{T_i\}$  and collateral rules, with  $\delta^H = \lambda^H$ ,  $\delta^{HS} = \delta^H + 0.01$ , and  $\lambda^H = \lambda^{HS} + 0.01$ : no collateral (squares), full collateralization (circles), full collateral posted only by party H (stars) or party HS (diamonds). The underlying is a cohort of 10,000 US males aged 65 at the beginning of 2008. Forward rates are plotted against the percentiles of improvements in survival rates based on Lee-Carter forecasts.

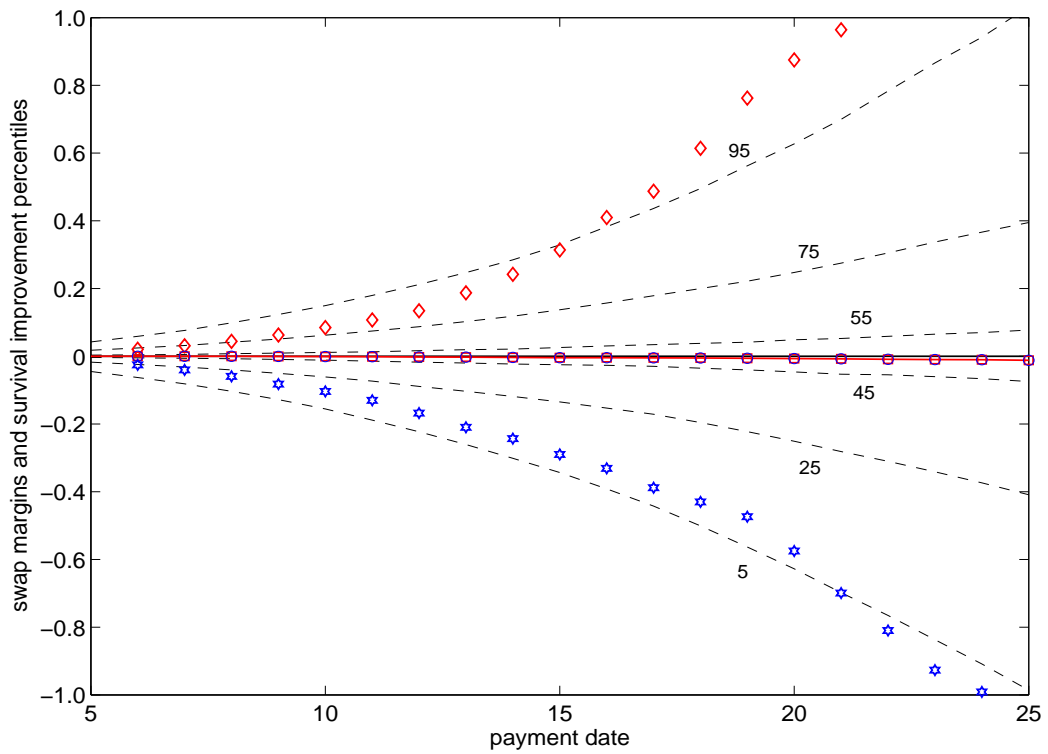


Figure 6: Swap margins  $\bar{p}_{T_i}^c/p_{T_i} - 1$  computed for different maturities  $\{T_i\}$  and collateral rules, with  $\lambda^H = \lambda^{HS}$  and  $\delta^H = \delta^{HS} = X^{(5)}$ , where the parameter estimates for the dynamics of  $X^{(5)}$  are given in table 2. Collateral rules: no collateral (squares), full collateralization (circles), full collateral posted only by party H (stars) or HS (diamonds). Forward rates are plotted against the percentiles of improvements in survival rates based on Lee-Carter forecasts (65-year old US males in 2008).

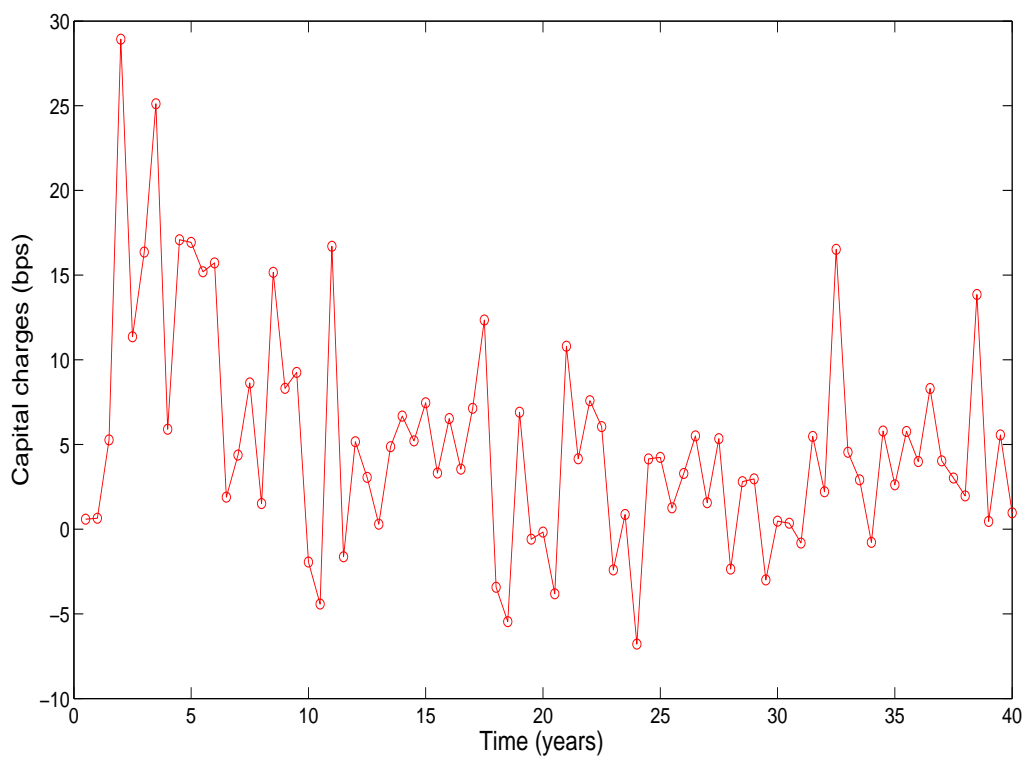


Figure 7: A simulated path of the capital charges accruing to the longevity swap dealer holding a representative longevity-linked liability  $n - N_{t+\bar{T}}$  under the Solvency II regulatory framework.