Capital allocation for portfolios with non-linear risk aggregation*

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Abstract

Existing risk capital allocation methods, such as the Euler rule, work under the explicit assumption that portfolios are formed as linear combinations of random loss/profit variables, with the firm being able to choose the portfolio weights. This assumption is unrealistic in an insurance context, where arbitrary scaling of risks is generally not possible. Here, we model risks as being partially generated by Lévy processes, capturing the non-linear aggregation of risk. The model leads to non-homogeneous fuzzy games, for which the Euler rule is not applicable. For such games, we seek capital allocations that are in the core, that is, do not provide incentives for splitting portfolios. We show that the Euler rule of an auxiliary linearised fuzzy game (non-uniquely) satisfies the core property and, thus, provides a plausible and easily implemented capital allocation. In contrast, the Aumann-Shapley allocation does not generally belong to the core. For the non-homogeneous fuzzy games studied, Tasche's (1999) criterion of suitability for performance measurement is adapted and it is shown that the proposed allocation method gives appropriate signals for improving the portfolio underwriting profit.

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1 Introduction

Financial firms often carry out a process of *capital allocation*, whereby the firm's total capital requirement is apportioned to different lines of business and sub-portfolios. The total capital is typically calculated using a risk measure, such as standard deviation, Tail-Value-at-Risk (TVaR) or Value-at-Risk (VaR), and reflects the diversification from risk pooling in the portfolio. Alternative allocation methods reflect the different ways in which individual risks and sub-portfolios contribute to the total capital. There are several streams in the literature, respectively motivated by arguments from: (a) cooperative game theory (Denault, 2001; Tsanakas and Barnett, 2003; Kalkbrener, 2005; Csóka et al., 2009; Hougaard and Smilgins, 2016); (b) performance and portfolio management (Tasche, 1999; Buch et al., 2011); (c) market valuation of assets and liabilities (Myers and Read, 2001; Sherris, 2006; Zanjani, 2010; Bauer and Zanjani, 2015); and (d) optimization (Dhaene et al., 2003; Dhaene et al., 2012).

A standard assumption in the literature is that portfolios are formed as linear combinations of random loss/profit variables, with the decision maker being able to choose the portfolio weights. As already noted by Mildenhall (2004, 2006), this assumption is not necessarily appropriate in an insurance context. Losses from an insurance portfolio arise from the aggregation of claims that are generally not perfectly dependent. Increasing the exposure in a line of business within an insurance portfolio does not correspond to linearly scaling up the loss, but to adding more policies to the portfolio (a similar lack of linear scalability is observed in credit risk portfolios). When insurance policies are independent, then claims can be modelled via Lévy processes; for instance the compound Poisson process is the canonical example in the actuarial literature. The risk capital is then determined by a risk measure evaluated at the aggregate claim. While the risk measure typically used is positively homogeneous (e.g. the standard deviation or a distortion risk measure such as TVaR), the risk capital is not homogeneous in the exposures; due to diversification effects, doubling the number of insurance policies written does not lead to a doubling of the required risk capital.

In this paper we address capital allocation using a model that incorporates both Lévy and linear portfolio components. Our main focus is on game theoretical arguments. A function r, mapping exposures to capital requirements, is called a *fuzzy game*. A fundamental question in this framework is whether candidate capital allocations belong to the *core* of r. An allocation that belongs to the core of r ensures that a lower amount of capital is allocated to any sub-portfolio, compared to it being operated on a stand-alone basis. When portfolios are linearly scalable, then the gradient of the fuzzy game, known as the *Euler rule* (Tasche, 1999), provides the unique core allocation (Aubin, 1979; 1981; Denault, 2001). However, in our case, as the fuzzy game r is not homogeneous, the Euler rule is no longer applicable.

We introduce an auxiliary homogeneous fuzzy game $\tilde{r}$, which can be seen as a linearisation of the original fuzzy game r. The values of r and $\tilde{r}$ coincide in the cases of full/no participation in individual lines of business. Furthermore, it is shown that, for risk measures such as TVaR or standard deviation, which preserve convex order, $\tilde{r}$ is dominated by r. As a consequence the Euler rule for the auxiliary fuzzy game $\tilde{r}$ belongs to the core of r. However, we note that for risk measures like VaR, which do not preserve convex order, the core may be empty.

Thus, our method gives a general construction of core capital allocations, applicable to insurance portfolios. Our finding is particularly relevant for the practice of insurance risk management. The Euler rule is often applied in the insurance industry, with the implicit but incorrect assumption of portfolio linearity. Our results show that using such a 'wrong' model with homogeneous risks turns out to give a risk allocation that is a core element of the 'correct' underlying fuzzy game.

Our proposed capital allocation method improves upon previous attempts to deal with risk portfolios that are non-linear in the exposures. In particular, Powers (2007) studies the Aumann-Shapley value (Aumann and Shapley, 1974), which is applicable in the case of non-homogeneous fuzzy games. However, the Aumann-Shapley value typically fails to produce computationally tractable risk capital allocations. Furthermore, we show the Aumann-Shapley value does not need to be in the core of r, except in the special case where the fuzzy game r is concave. Therefore, Aumann-Shapley allocations can produce incentives for portfolio fragmentation.

Finally, we consider implications for portfolio management. For linear portfolios, it is possible to derive appropriate signals for portfolio management, by evaluating for each line of business the return on allocated capital, as calculated by the Euler rule (Tasche, 1999). However, the diversification implicit in aggregating insurance risks necessitates the consideration of capital constraints. We adapt the arguments of Tasche (1999) and show that the proposed capital allocation method provides appropriate signals for increasing the aggregate underwriting profit.

Section 2 introduces the model and risk measures used. Section 3 contains the main contributions of this paper, including the proposed capital allocation method and the study of the core of the fuzzy game r. Signals for portfolio management are discussed in Section 4 and brief conclusions given in Section 5.

2 Model outline

We consider a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_w\}_{0\le w\le W})$ with $\mathcal{F} = \mathcal{F}_W$ for given $W > 1$. Throughout, (in-)equalities between random variables are understood in the $\mathbb{P}$-a.s. sense.

A financial firm, such as an insurance company, writes I lines of business. The exposure of the financial firm to the i^{th} line of business is described by $0 \le w_i \le W$, and the loss arising from that line of business is denoted by the random variable $X_i(w_i) \in L^1(\Omega, \mathcal{F}, \mathbb{P})$, $i = 1, \ldots, I$. The total loss of the financial firm as a function of the exposure $\boldsymbol{w} = (w_1, \ldots, w_I)'$ is denoted by

$$S(\boldsymbol{w}) = \sum_{i=1}^{I} X_i(w_i). \tag{1}$$

The current (base-line) exposure of the firm is $\boldsymbol{w} = \mathbf{1}_I = (1, \ldots, 1)'$, leading to the total loss $S(\mathbf{1}_I) = \sum_{i=1}^{I} X_i(1)$.

To provide a tractable structure for the ways in which changes in exposure $\boldsymbol{w}$ affect the joint law of the losses, we introduce the following model. Consider the random vector $\mathbf{Y}(\boldsymbol{w}) = (Y_1(w_1), \ldots, Y_I(w_I))'$, where $Y_i = (Y_i(w))_{0\le w\le W}$ are $\mathcal{F}_w$-adapted independent increasing Lévy processes for $i = 1, \ldots, I$. Let $\mathbf{Z} = (Z_1, \ldots, Z_I)'$ be an $\mathcal{F}_W$-measurable random vector (having possibly dependent components), and assume that $\mathbf{Z}$ and $(Y_i)_{i=1,\ldots,I}$ are independent. Then, we define for $i = 1, \ldots, I$ and exposure $w_i \in [0, W]$ the loss arising from the i^{th} line of business by

$$X_i(w_i) = Y_i(w_i) + w_i Z_i. \tag{2}$$

We also write $X_i = (X_i(w))_{0\le w\le W}$ and $\mathbf{X}(\boldsymbol{w}) = \mathbf{Y}(\boldsymbol{w}) + \boldsymbol{w} \cdot \mathbf{Z}$, where $\boldsymbol{w} \cdot \boldsymbol{Z}$ is the Hadamard (element-wise) product.[1]

We stress that the 'development' of the stochastic processes Y_i does *not* represent elapsed time, but increase in exposure. Thus, stopping the process at point w_i corresponds to placing a limit on the exposure of the i^{th} line of business. Losses from insurance portfolios can be modelled as aggregations of (typically independent) claim amounts from different policies, for which Lévy processes (with their connection to infinitely divisible distributions) provide an appropriate representation. Henceforth, the

[1] $\mathbf{X}$ is defined on the domain $[0, W]^I$ with $W > 1$, which extends the exposures beyond the base-line exposure $\mathbf{1}_I$. In part this is due to mathematical convenience, since in Section 3 derivatives of functions (fuzzy games) at $\boldsymbol{w} = \mathbf{1}_I$ will be taken. But also, in Section 4, the strategic behavior of the firm is considered, which includes the potential for portfolio expansion beyond base-line exposure $\boldsymbol{w} = \mathbf{1}_I$.

dynamics of the processes Y_i are not of particular interest in this paper; all distributions and moments evaluated are with respect to information $\mathcal{F}_0$.

The model allows two special cases:

- If $Y_i(w_i) \equiv 0$ for all $i = 1, \dots, I$, then $X_i(w_i) = w_i X_i(1)$, and the loss of the i^{th} line of business scales linearly in w_i. This is the common situation described e.g. by Tasche (1999), where $X_i(1)$ can be seen as the (negative) values of tradable assets and w_i are portfolio weights.

- If $Z_i = 0$ for all $i = 1, \dots, I$, then the line of business losses X_i are Lévy processes. In that case, X_i can represent a standard actuarial risk model. For example, if X_i is a Poisson process with unit intensity, then $X_i(w_i) \sim \text{Poisson}(w_i)$. Now, $X_i(w_i)$ is no longer linearly scalable with exposure, in particular, we have for its variance the property $\mathbb{V}(X_i(w_i)) = w_i \mathbb{V}(X_i(1))$.

The full model (2) can then be viewed as an insurance risk component $Y_i(w_i)$ augmented by common shocks $w_i Z_i$ that simultaneously affect all claims from the i^{th} line of business[2]. Dependence between losses of different lines of business is induced by the possible dependence between the elements of $\mathbf{Z}$; it is straightforward that for $i \neq j$ we have covariance $\mathbb{C}(X_i(w_i), X_j(w_j)) = w_i w_j \mathbb{C}(Z_i, Z_j)$.

In general, the processes X_i do not have independent increments; however the increments remain identically distributed.

Lemma 2.1. *Let* $0 \leq v_i < w_i \leq W$ *for all* $i = 1, \dots, I$. *Then,* $\mathbf{X}(\boldsymbol{w}) - \mathbf{X}(\boldsymbol{v}) \stackrel{d}{=} \mathbf{X}(\boldsymbol{w} - \boldsymbol{v})$.

Proof. The claim follows from the following identity

$$\begin{aligned} \mathbf{X}(\boldsymbol{w}) - \mathbf{X}(\boldsymbol{v}) &= \mathbf{Y}(\boldsymbol{w}) + \boldsymbol{w} \cdot \mathbf{Z} - (\mathbf{Y}(\boldsymbol{v}) + \boldsymbol{v} \cdot \mathbf{Z}) \\ &\stackrel{d}{=} \mathbf{Y}(\boldsymbol{w} - \boldsymbol{v}) + (\boldsymbol{w} - \boldsymbol{v}) \cdot \mathbf{Z} \\ &= \mathbf{X}(\boldsymbol{w} - \boldsymbol{v}), \end{aligned}$$

where we have used standard properties of Lévy processes and the independence between $\mathbf{Y}$ and $\mathbf{Z}$. □

Example 2.2. Let $I = 2$ and consider the model $X_i(w_i) = Y_i(w_i) + w_i \sigma_i Z_i$, $i = 1, 2$, where Y_i are Poisson processes with intensities λ_i, $\sigma_1, \sigma_2 > 0$, and $\mathbf{Z} = (Z_1, Z_2)$ is a

[2] An alternative interpretation of model (2) arises by considering $w_i Z_i$ as losses from a linear portfolio with weights $\boldsymbol{w}$. Then, the Lévy component Y_i can be seen as representing operational costs associated with the i^{th} line of business. In this interpretation, expected operational costs are increasing linearly with exposure, but become less volatile as the portfolio grows.

bivariate normally distributed random vector with standard margins and correlation $\mathbb{C}(Z_1, Z_2) = \eta \in [-1, 1]$.

While the expected values of $X_i(w_i)$ scale with exposures, $\mathbb{E}(X_i(w_i))) = w_i\lambda_i$, this is not the case when we consider the volatility (measured by the standard deviation) since we have $\sqrt{\mathbb{V}(X_i(w_i))} = \sqrt{w_i\lambda_i + w_i^2\sigma_i^2}$. The covariance of the total losses from the two lines of business is given by $\mathbb{C}(X_1(w_1), X_2(w_2)) = w_1w_2\sigma_1\sigma_2\eta$. ■

To determine the capital requirements for its portfolio, the financial firm uses a risk measure $\rho : L^1(\Omega, \mathcal{F}, \mathbb{P}) \to \overline{\mathbb{R}}$. Hence, the firm's capital requirement for the base-line exposure $\mathbf{1}_I$ is given by $\rho(S(\mathbf{1}_I))$. We assume throughout the paper that $\rho(S(\mathbf{1}_I))$ is finite.

Let V_1, V_2 be two elements of $L^1(\Omega, \mathcal{F}, \mathbb{P})$. Throughout the paper, we assume that the risk measure ρ satisfies the following properties:

- Law invariance: $V_1 \overset{d}{=} V_2 \implies \rho(V_1) = \rho(V_2)$.
- Positive homogeneity: $\rho(aV_1) = a\rho(V_1)$ for $a \geq 0$.
- Subadditivity: $\rho(V_1 + V_2) \leq \rho(V_1) + \rho(V_2)$.
- Consistency with convex order: $V_1 \preceq_{cx} V_2 \implies \rho(V_1) \leq \rho(V_2)$.

Recall the definition of convex order: $V_1 \preceq_{cx} V_2$ if $\mathbb{E}(u(V_1)) \leq \mathbb{E}(u(V_2))$ for all convex functions u such that the expectations exist (see, e.g., Denuit et al., 2006).

The standard deviation $\rho(V) = \sqrt{\mathbb{V}(V)}$ gives a simple example of a risk measure satisfying the above properties; note that to satisfy $\rho(V) < \infty$ one needs to restrict the space of loss positions considered to those with finite second moments, i.e. $V \in L^2(\Omega, \mathcal{F}, \mathbb{P})$. While the standard deviation is not easily interpreted as giving a reasonable capital requirement, typically, the *standard deviation principle* $\rho(V) = \mathbb{E}(V) + \beta\sqrt{\mathbb{V}(V)}$, $\beta > 0$, admits such an interpretation and still satisfies the required properties.

A further popular family of risk measures satisfying the stated properties is that of coherent *distortion risk measures* (Wang et al., 1997; Acerbi, 2002), defined by

$$\rho(V) = \int_0^1 F_V^{-1}(u)\zeta(u)du = \mathbb{E}\big(V\zeta(U_V)\big), \tag{3}$$

where

$$F_V^{-1}(u) = \inf\{x \in \mathbb{R} : \ \mathbb{P}(V \leq x) \geq u\} \tag{4}$$

is the quantile function of V (generalised inverse of the distribution function $F_V(\cdot) = \mathbb{P}(V \leq \cdot)$); ζ is an increasing non-negative weight function on $[0, 1]$ with normalization

$\int_0^1 \zeta(u)du = 1$; and U_V is a uniform random variable on $[0,1]$ increasing in (comonotonic to) V.[3]

An example of a coherent distortion risk measure is *Tail-Value-at-Risk (TVaR)*

$$\mathrm{TVaR}_p(V) = \frac{1}{1-p}\int_p^1 F_V^{-1}(u)du, \tag{5}$$

derived by setting $\zeta(u) = \frac{1}{1-p}\mathbf{1}_{\{u>p\}}$. When the distribution function of V is continuous, then we have that $\mathrm{TVaR}_p(V) = \mathbb{E}\big(V|V > F_V^{-1}(p)\big)$ (e.g. Dhaene et al., 2006).

The *Value-at-Risk (VaR)* risk measure, given by the quantile function,

$$\mathrm{VaR}_p(V) = F_V^{-1}(p), \tag{6}$$

is not subadditive or consistent with the convex order (Denuit et al., 2005, Section 2.3.2) and thus not considered further in this paper apart from Example 3.16.

Example 2.3. Following up Example 2.2, let the risk measure be the standard deviation $\rho(S(\boldsymbol{w})) = \sqrt{\mathbb{V}(S(\boldsymbol{w}))}$. Then, the risk measure of the aggregate loss of portfolio $\boldsymbol{w}$ is given by

$$\rho(S(\boldsymbol{w})) = \sqrt{w_1\lambda_1 + w_2\lambda_2 + w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\sigma_1\sigma_2\eta}.$$

To evaluate distortion risk measures it is necessary to use the whole distribution function of $S(\boldsymbol{w})$. Let $p_k(\boldsymbol{w}) = \frac{(w_1\lambda_1+w_2\lambda_2)^k \exp(-w_1\lambda_1-w_2\lambda_2)}{k!}$ be the probability mass in $k \in \mathbb{N}_0$ of a Poisson$(w_1\lambda_1 + w_2\lambda_2)$-distributed random variable. Then, the distribution of $S(\boldsymbol{w})$ is given by

$$F_{S(\boldsymbol{w})}(s) = \sum_{k=0}^{\infty} p_k(\boldsymbol{w})\Phi\left(\frac{s-k}{\sqrt{w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\sigma_1\sigma_2\eta}}\right),$$

where $\Phi(\cdot)$ is the standard normal distribution. For example, when $\rho(\cdot) = \mathrm{TVaR}_p(\cdot)$, numerical inversion of $F_{S(\boldsymbol{w})}$ yields $F_{S(\boldsymbol{w})}^{-1}(p)$, and subsequently the calculation of the risk measure follows by

$$\rho(S(\boldsymbol{w})) = \frac{1}{1-p}\sum_{k=0}^{\infty} p_k(\boldsymbol{w})\mathbb{E}\left(A_k 1_{\{A_k > F_{S(\boldsymbol{w})}^{-1}(p)\}}\right),$$

[3]Such a variable always exists. If the distribution $F_V(x) = \mathbb{P}(V \le x)$ is strictly increasing then $U_V = F_V(V)$; otherwise Rüschendorf's (2009) generalised distributional transform can be used.

where $A_k \sim N(k, w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\sigma_1\sigma_2\eta)$, with partial moments

$$\mathbb{E}\left(A_k 1_{\{A_k > F^{-1}_{S(\boldsymbol{w})}(p)\}}\right) = (1-\Phi(a_k))\left(k + \sqrt{w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\sigma_1\sigma_2\eta}\frac{\varphi(a_k)}{1-\Phi(a_k)}\right),$$

for $a_k = \frac{F^{-1}_{S(\boldsymbol{w})}(p)-k}{\sqrt{w_1^2\sigma_1^2+w_2^2\sigma_2^2+2w_1w_2\sigma_1\sigma_2\eta}}$ and $\varphi(\cdot)$ being the standard normal density. ∎

3 Capital allocations

3.1 Fuzzy games and their cores

We begin this section by introducing *fuzzy games*. A fuzzy game is a (typically non-linear) function, defined on vectors whose elements represent participation rates in different activities. The value of the fuzzy game stands for the resulting cost. In the setting of this paper, 'participation' corresponds to exposure in different lines of business and 'cost' stands for the capital requirement. Further interpretation of the properties of fuzzy games and allocations in the context of insurance risk management is provided in Section 3.2 below.

Following Aubin (1979, 1981), we call any function $g : [0, W]^I \to \mathbb{R}$, $W > 1$, a fuzzy game. The following properties of fuzzy games will be referred to in the sequel:

- Positive homogeneity: $g(a\boldsymbol{w}) = ag(\boldsymbol{w})$ for all $\boldsymbol{w} \in [0, W]^I$ and $a \in [0, 1]$.
- Subadditivity: for any $\boldsymbol{w}, \boldsymbol{w}^1, \boldsymbol{w}^2 \in [0, W]^I$ such that $\boldsymbol{w}^1 + \boldsymbol{w}^2 = \boldsymbol{w}$, we have $g(\boldsymbol{w}) \leq g(\boldsymbol{w}^1) + g(\boldsymbol{w}^2)$.

Allocations of fuzzy games represent apportionments of costs to different business activities.

Definition 3.1. *A vector $\boldsymbol{d} \in \mathbb{R}^I$ is called an* allocation *of the fuzzy game g if $\sum_{i=1}^I d_i = g(\mathbf{1}_I)$.*

The *core* of a fuzzy game is defined as follows (Aubin, 1979; 1981).

Definition 3.2. *The* core *of the fuzzy game $g : [0, W]^I \mapsto \mathbb{R}$, denoted by $\mathcal{C}(g)$, is the set of allocations $\boldsymbol{d} \in \mathbb{R}^I$ of g such that $\sum_{i=1}^I w_i d_i \leq g(\boldsymbol{w})$ for all $\boldsymbol{w} \in [0, 1]^I$.*

Core allocations thus produce apportionments of costs such that a subset of activities with incomplete participation is always assigned a lower cost than it would be the case if those activities were pursued on a stand-alone basis. A commonly encountered allocation is the *Euler allocation* (Aubin, 1979, 1981; Tasche, 1999).

Definition 3.3. *Let the fuzzy game g be positively homogeneous and partially differentiable at $\mathbf{1}_I$. Then, the* Euler allocation *is defined as the gradient of g at $\mathbf{1}_I$:*

$$\boldsymbol{d}^E(g) \in \mathbb{R}^I, \qquad \textit{with components } d_i^E(g) = \left.\frac{\partial g(\boldsymbol{w})}{\partial w_i}\right|_{\boldsymbol{w}=\mathbf{1}_I}, \textit{ for } i = 1, \ldots, I.$$

By positive homogeneity of g, it follows from Euler's homogeneous function theorem that $\sum_{i=1}^I d_i^E(g) = g(\mathbf{1}_I)$; thus $\boldsymbol{d}^E(g)$ is an allocation in the sense of Definition 3.1. If g is in addition subadditive, then the Euler allocation is the unique element of the core.

Lemma 3.4 (Aubin, 1979)**.** *Let $g : [0, W]^I \mapsto \mathbb{R}$ be positively homogeneous and subadditive. It holds that $\mathcal{C}(g) \neq \emptyset$. If g is partially differentiable at $\boldsymbol{w} = \mathbf{1}_I$, then $\mathcal{C}(g)$ is single-valued and $\boldsymbol{d}^E(g)$ is its unique element.*

Cases where g is not partially differentiable at $\boldsymbol{w} = \mathbf{1}_I$ are studied by, e.g., Mertens (1988) and Boonen et al. (2012). We do not discuss this situation in more detail here.

For general fuzzy games that are not necessarily positively homogeneous, the *Aumann-Shapley allocation* (Aumann and Shapley, 1974) for the game g is defined as follows.

Definition 3.5. *The* Aumann-Shapley allocation *of the fuzzy game g is defined by*

$$\boldsymbol{d}^{AS}(g) \in \mathbb{R}^I, \qquad \textit{with components } d_i^{AS}(g) = \int_0^1 \frac{\partial}{\partial w_i} g(\beta \mathbf{1}_I) d\beta, \textit{ for } i = 1, \ldots, I, \tag{7}$$

whenever the integral exists.

Note that for positively homogeneous fuzzy games, the Aumann-Shapley allocation reduces to the Euler allocation (Denault, 2001). A sufficient condition for (7) to exist, and to guarantee that it is an allocation, is continuous differentiability of g (Aumann and Shapley, 1974).[4] We will discuss this allocation rule for non-homogeneous fuzzy games in Section 3.4 below.

3.2 Construction of a core allocation for the insurance risk model

Consider the insurance risk model of Section 2. We define the fuzzy game $r : [0, W]^I \to \mathbb{R}$ as follows

$$r(\boldsymbol{w}) = \rho\left(\sum_{i=1}^I X_i(w_i)\right), \qquad \text{for all } \boldsymbol{w} \in [0, W]^I. \tag{8}$$

[4]This result is extended to a class of piece-wise continuously differentiable functions g by Samet et al. (1984).

The fuzzy game r represents the capital that the firm must hold as a function of its exposure $\boldsymbol{w}$. As before, the base-line exposure of the firm is the one of full participation in each line of business, $\boldsymbol{w} = \mathbf{1}_I$, which provides $r(\mathbf{1}_I) = \rho\left(S(\mathbf{1}_I)\right)$.

The fuzzy game r inherits the subadditivity property from the risk measure ρ.

Lemma 3.6. *The fuzzy game r defined in* (8) *is subadditive.*

Proof. For all $\boldsymbol{w}, \boldsymbol{w}^1, \boldsymbol{w}^2 \in [0, W]^I$ such that $\boldsymbol{w}^1 + \boldsymbol{w}^2 = \boldsymbol{w}$, we have:

$$
\begin{aligned}
r(\boldsymbol{w}) &= \rho\left(\sum_{i=1}^{I} X_i(w_i^1) + \sum_{i=1}^{I} \left(X_i(w_i) - X_i(w_i^1)\right)\right) \\
&\leq \rho\left(\sum_{i=1}^{I} X_i(w_i^1)\right) + \rho\left(\sum_{i=1}^{I} \left(X_i(w_i) - X_i(w_i^1)\right)\right) \\
&= \rho\left(\sum_{i=1}^{I} X_i(w_i^1)\right) + \rho\left(\sum_{i=1}^{I} \left(X_i(w_i - w_i^1)\right)\right) \\
&= r(\boldsymbol{w}^1) + r(\boldsymbol{w}^2).
\end{aligned}
$$

The inequality follows from the subadditivity of ρ, and the subsequent equality from Lemma 2.1 and the law invariance of ρ. □

The subadditivity of r implies that savings in capital occur when loss exposures are aggregated, due to diversification. A capital allocation $\boldsymbol{d}$ provides a mechanism for assessing the risk contribution of different sub-portfolios and lines of business, respectively. The elements of $\boldsymbol{d}$ can be seen as internal capital contribution rates, such that the contribution of a sub-portfolio (set of policies) $\boldsymbol{w} \in [0, 1]^I$ to the total capital $r(\mathbf{1}_I)$ can be quantified as $\sum_{i=1}^{I} w_i d_i$.

The focus of this section is on the stability of the portfolio. Given subadditivity of r, a capital allocation $\boldsymbol{d}$ should not give an incentive to split the portfolio of full exposures. This stipulation can be translated into the requirement that the allocation $\boldsymbol{d}$ should be in the core of r. If $\boldsymbol{d} \in \mathcal{C}(r)$, then the capital allocated to the sub-portfolio with exposures $\boldsymbol{w}$ satisfies $\sum_{i=1}^{I} w_i d_i \leq r(\boldsymbol{w})$, such that the risk contribution of any sub-portfolio is less than its stand-alone capital.

The same argument is used by Denault (2001), who deals with the special case of linear portfolios, i.e. Y_i=0 for all $i = 1, \ldots, I$, and proposes the use of Euler allocations. However, in our context, the Euler allocation is not applicable, since r does generally not satisfy positive homogeneity, unless $X_i(w_i)$ and $w_i X_i$ have the same distribution (which happens when Y_i=0, $i = 1,\dots,I$). Lack of homogeneity implies $\sum_{i=1}^I \frac{\partial r(\boldsymbol{w})}{\partial w_i}\Big|_{\boldsymbol{w}=\mathbf{1}_I} \neq r(\mathbf{1}_I)$.

To construct an element of $\mathcal{C}(r)$, we introduce an auxiliary fuzzy game. Consider the fuzzy game $\tilde{r} : \mathbb{R}^I \mapsto \mathbb{R}$, defined by

$$\tilde{r}(\boldsymbol{w}) := \rho\left(\sum_{i=1}^{I} w_i X_i(1)\right). \tag{9}$$

The fuzzy game $\tilde{r}$ can be seen as a linearised version of r. The fuzzy game $\tilde{r}$ is both positively homogeneous and subadditive; this follows directly from the corresponding properties of the risk measure ρ. The two fuzzy games r and $\tilde{r}$ represent different ways in which sub-portfolios may be formed: $r(\boldsymbol{w})$ represents the capital requirement for the exposure $\boldsymbol{w}$ as discussed in Section 2, while $\tilde{r}(\boldsymbol{w})$ gives the capital for a portfolio that would be composed of proportional shares of the full lines of business $X_i(1)$ (which might be interpreted as co-insurance or proportional re-insurance shares).

By positive homogeneity of $\tilde{r}$ we can derive the Euler allocation $\boldsymbol{d}^E(\tilde{r})$ (subject to differentiability). By Lemma 3.4, if $\boldsymbol{d}^E(\tilde{r})$ is well-defined, then $\boldsymbol{d}^E(\tilde{r}) \in \mathcal{C}(\tilde{r})$. We will show in the sequel that it is also the case that $\boldsymbol{d}^E(\tilde{r}) \in \mathcal{C}(r)$.

Firstly, the relation between r and $\tilde{r}$ is further characterised. While in general $\tilde{r}(\boldsymbol{w}) \neq r(\boldsymbol{w})$, it obviously holds that $\tilde{r}(\mathbf{1}_I) = r(\mathbf{1}_I)$ and $\tilde{r}(\mathbf{0}_I) = r(\mathbf{0}_I) = 0$; furthermore $\tilde{r}(\boldsymbol{w}) = r(\boldsymbol{w})$ for all $\boldsymbol{w} \in \{0,1\}^I$. For sub-portfolios of partial exposure, $\boldsymbol{w} \in [0,1]^I \setminus \{0,1\}^I$, we have the following result.

Proposition 3.7. *For all $\boldsymbol{w} \in [0,1]^I$, we have $\tilde{r}(\boldsymbol{w}) \leq r(\boldsymbol{w})$.*

Proof. First we show that for each $i = 1,\dots,I$, $w_i Y_i(1) \preceq_{cx} Y_i(w_i)$, where Y_i are the processes of model (2). For $w_i \in \{0,1\}$ this follows trivially. Assume initially that w_i is rational, with $w_i = \frac{p}{q}$, for $p,q \in \mathbb{N}_+$, $p < q$. Since Y_i is a Lévy process, it is $Y_i(1) \overset{d}{=} \sum_{k=1}^q U_k$, for some i.i.d. random variables $U_1,\dots,U_q$. Furthermore, $Y_i(p/q) \overset{d}{=} \sum_{k=1}^p U_k = \sum_{k=1}^q u_k U_k$, where $u_k = 1$, for $k = 1,\dots,p$, and $u_k = 0$, for $k = p+1,\dots,q$. Denote $\bar{u} = \frac{1}{q}\sum_{k=1}^q u_k = \frac{p}{q}$. Then, from Denuit et al. (2006, Corollary 3.4.24), we get that

$$\bar{u}\sum_{k=1}^{q} U_k \preceq_{cx} \sum_{k=1}^{q} u_k U_k \implies \frac{p}{q} Y_i(1) \preceq_{cx} Y_i\left(\frac{p}{q}\right).$$

For irrational w_i, let $p_q = \lfloor w_i q \rfloor$, such that $\frac{p_q}{q} \leq w_i$. Then,

$$\frac{p_q}{q} Y_i(1) \preceq_{cx} Y_i\left(\frac{p_q}{q}\right) \leq Y_i(w_i) \implies \frac{p_q}{q} Y_i(1) \preceq_{sl} Y_i(w_i)$$

by increasingness of the process Y_i, where '$\preceq_{sl}$' is the stop-loss order (Denuit et al., 2006). We have, as $q \to \infty$,

$$\begin{aligned} \tfrac{p_q}{q} Y_i(1) &\xrightarrow{d} w_i Y_i(1) \\ \mathbb{E}\left(\left(\tfrac{p_q}{q} Y_i(1)\right)_+\right) &\to \mathbb{E}\left((w_i Y_i(1))_+\right). \end{aligned}$$

Hence, by Denuit et al. (2006, Proposition 3.4.25), it is $w_i Y_i(1) \preceq_{sl} Y_i(w_i)$. Since $\mathbb{E}(w_i Y_i(1)) = \mathbb{E}(Y_i(w_i))$, we also have $w_i Y_i(1) \preceq_{cx} Y_i(w_i)$.

By the independence of the processes $Y_1, \ldots, Y_I$, it follows from Denuit et al. (2006, Proposition 3.4.25) that $\sum_{i=1}^I w_i Y_i(1) \preceq_{cx} \sum_{i=1}^I Y_i(w_i)$. Since $\boldsymbol{Z}$ is independent of $\boldsymbol{Y}$, we get for any convex function u:

$$\begin{aligned} \mathbb{E}\left(u\left(\sum_{i=1}^I w_i Y_i(1) + w_i Z_i\right)\right) &= \mathbb{E}\left[\mathbb{E}\left(u\left(\sum_{i=1}^I w_i Y_i(1) + w_i Z_i\right) \middle| \boldsymbol{Z}\right)\right] \\ &\leq \mathbb{E}\left[\mathbb{E}\left(u\left(\sum_{i=1}^I Y_i(w_i) + w_i Z_i\right) \middle| \boldsymbol{Z}\right)\right] \\ &= \mathbb{E}\left(u\left(\sum_{i=1}^I Y_i(w_i) + w_i Z_i\right)\right). \end{aligned}$$

Hence,

$$\begin{aligned} \sum_{i=1}^I w_i X_i(1) &= \sum_{i=1}^I w_i Y_i(1) + \sum_{i=1}^I w_i Z_i \\ &\preceq_{cx} \sum_{i=1}^I Y_i(w_i) + \sum_{i=1}^I w_i Z_i = \sum_{i=1}^I X_i(w_i). \end{aligned}$$

The result follows from the consistency of ρ with the convex order.

□

An implication of Proposition 3.7 is that, if an element of the core of $\tilde{r}$ is identified, then this will also belong to the core of r:

Proposition 3.8. *It holds that $\mathcal{C}(\tilde{r}) \subseteq \mathcal{C}(r)$.*

Proof. Let $\boldsymbol{d} \in \mathcal{C}(\tilde{r})$. Then, it holds for all $\boldsymbol{w} \in [0,1]^I$ that

$$\sum_{i=1}^{I} w_i d_i \leq \tilde{r}(\boldsymbol{w}) \leq r(\boldsymbol{w}),$$

where the last inequality follows from Proposition 3.7. Furthermore, $\sum_{i=1}^{I} d_i = \tilde{r}(\mathbf{1}_I) = r(\mathbf{1}_I)$. Hence, $\boldsymbol{d} \in \mathcal{C}(r)$. □

Lemma 3.4 and Proposition 3.8 indicate that the core of r is non-empty. In particular, as stated in the corollary below, $\boldsymbol{d}^E(\tilde{r})$ is an element of the core of r.

Corollary 3.9. *If $\boldsymbol{d}^E(\tilde{r})$ exists, then $\boldsymbol{d}^E(\tilde{r}) \in \mathcal{C}(r)$.*

Hence, the Euler allocation of $\tilde{r}$, which is well-studied in the literature, also provides a reasonable allocation for the fuzzy game r, which is the main object of interest in this paper. However, $\boldsymbol{d}^E(\tilde{r})$ is not necessarily the only element of $\mathcal{C}(r)$; as will be discussed in Section 3.3, $\mathcal{C}(r)$ is not necessarily single-valued.

These results matter for practical applications. When capital allocation exercises are performed by insurance firms, the allocation $\boldsymbol{d}^E(\tilde{r})$ is frequently used. However, $\tilde{r}$ does not generally correspond to a realistic situation in insurance, as linear portfolio weights cannot be arbitrarily changed; often one can only vary exposures in the way captured by r. In fact, the quantity $\sum_{i=1}^{I} w_i X_i(1)$ is only meaningful in an insurance context for $\boldsymbol{w} \in [0,1]^I$, since multiples of insurance contracts cannot be traded, while shares of contracts may be transferred via co-insurance and proportional re-insurance to a third party. Thus, allocations used in practice are based on the 'wrong game' $\tilde{r}$, rather than the 'game actually played' r. But our results show that this is not a serious problem, since via the Euler allocation of $\tilde{r}$, we end up with an element of the core of r.

We show in Appendix A that the Euler allocation of $\tilde{r}$ in closed-form in case ρ is the standard deviation or a distortion risk measure.

Remark 3.10. It is argued in the literature that the lack of homogeneity of e.g. credit risk exposures (and thus of r) becomes negligible in large diversified portfolios (see McNeil et al., 2005, Section 8.4.3; Buch et al., 2011; consider also Mildenhall, 2006, Figure 5). In such portfolios, the impact of risk factors driving the dependence of the portfolio dominates that of idiosyncratic Lévy-type risk factors – indeed, then we have that the fuzzy games r and $\tilde{r}$ are close. However, this is *not* the argument of the present paper. The Euler rule, applied to $\tilde{r}$, provides an allocation in the core of r, regardless of whether $\tilde{r}$ gives a good approximation to r. This means that the proposed allocation is also useful in the case of less diversified portfolios where the Lévy component dominates, e.g. low-frequency catastrophic risks.

Remark 3.11. Requiring that allocations belong to the core is motivated by preserving the portfolio's stability. It could be argued that it is unrealistic to expect that allocations not in the core will actually produce portfolio fragmentation – it is after all not necessarily possible for a manager to withdraw her sub-portfolio and survive in the market. Nonetheless, allocations not in the core do present a viable business situation. Presenting a portfolio manager with an amount of allocated capital that is larger than what the stand-alone capital would be (or indeed creating a sub-portfolio with this property) would mean assigning a diversification penalty rather than a benefit. This would be organizationally untenable, leading to the capital allocation method not being applied at all.

3.3 Geometric structure of the core

The fuzzy game r defines a hyper-surface in $[0,1]^I \times \mathbb{R}$:

$$\mathcal{R} : [0,1]^I \to [0,1]^I \times \mathbb{R}; \qquad \boldsymbol{w} \mapsto (\boldsymbol{w}, r(\boldsymbol{w}))\,.$$

Similarly, $\tilde{r}$ defines a hyper-surface $\tilde{\mathcal{R}}$ in $[0,1]^I \times \mathbb{R}$. It follows from Proposition 3.7 that the hyper-surface $\mathcal{R}$ lies above the hyper-surface $\tilde{\mathcal{R}}$, with points of intersection including at least the set of corners $\{(\boldsymbol{w}, r(\boldsymbol{w})) : \ \boldsymbol{w} \in \{0,1\}^I\}$.

Let $\boldsymbol{d} \in \mathbb{R}^I$ be an allocation of r. Then $\boldsymbol{d}$ defines the hyper-plane

$$\mathcal{H}(\boldsymbol{d}) : [0,1]^I \to [0,1]^I \times \mathbb{R}; \qquad \boldsymbol{w} \mapsto \big(\boldsymbol{w}, \boldsymbol{w}'\boldsymbol{d}\big)\,.$$

The requirement $\boldsymbol{d} \in \mathcal{C}(r)$ is equivalent to asking that the hyper-plane $\mathcal{H}(\boldsymbol{d})$ is located below the hyper-surface $\mathcal{R}$. There are at least two intersections between the two hyper-surfaces given by the corner points $(\boldsymbol{0}_I, r(\boldsymbol{0}_I)) = (\boldsymbol{0}_I, 0)$ and $(\boldsymbol{1}_I, r(\boldsymbol{1}_I))$. In particular, in view of Corollary 3.9, the Euler allocation $\boldsymbol{d}^E(\tilde{r}) \in \mathcal{C}(\tilde{r}) \subseteq \mathcal{C}(r)$ is dominated by both hyper-surfaces $\mathcal{R}$ and $\tilde{\mathcal{R}}$, respectively.

The following example demonstrates these geometric ideas, and shows how they allow us to analyze graphically whether the core $\mathcal{C}(r)$ is single-valued.

Example 3.12. Let $I = 2$ and $Z_i = 0$ for $i = 1, 2$, so that we have a pure insurance risk model with two lines of business. Assume that Y_1 and Y_2 are Poisson processes with intensities λ_1 and λ_2, such that $S(\boldsymbol{w}) \sim \text{Poisson}(w_1\lambda_1 + w_2\lambda_2)$.

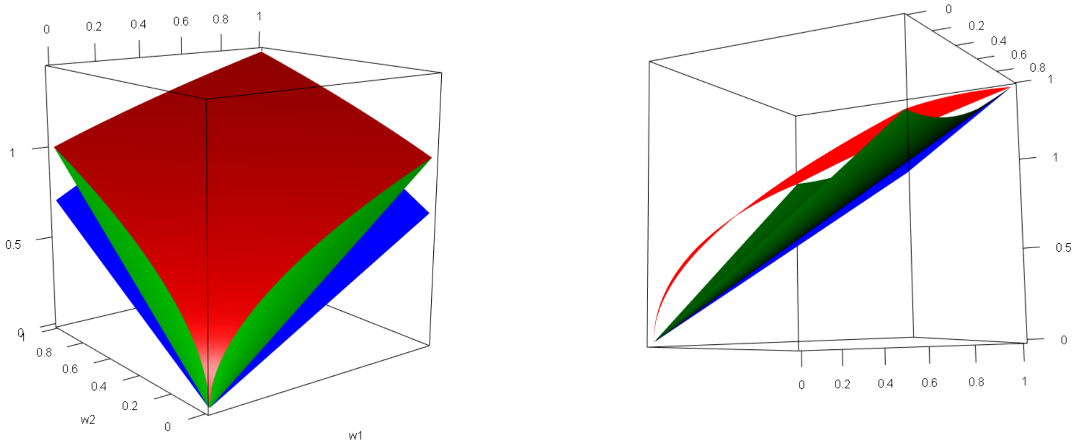

Figure 1: Two aspects from different angles of the hyper-surfaces $\mathcal{R}$ (red), $\tilde{\mathcal{R}}$ (green) and $\mathcal{H}(\boldsymbol{d}^E(\tilde{r}))$ (blue); Example 3.12.

Let the risk measure be given by the standard deviation. Then,

$$r(\boldsymbol{w}) = \sqrt{\mathbb{V}(S(\boldsymbol{w}))} = \sqrt{w_1\lambda_1 + w_2\lambda_2}.$$

The auxiliary fuzzy game $\tilde{r}$ is given by

$$\tilde{r}(\boldsymbol{w}) = \sqrt{\mathbb{V}(w_1 Y_1(1) + w_2 Y_2(1))} = \sqrt{w_1^2\lambda_1 + w_2^2\lambda_2}.$$

It is apparent that $\tilde{r}(\boldsymbol{w}) \leq r(\boldsymbol{w})$, since $w_i^2 \leq w_i$ for $\boldsymbol{w} \in [0,1]^2$. The Euler allocation is derived as

$$d_i^E(\tilde{r}) = \left.\frac{\partial \tilde{r}(\boldsymbol{w})}{\partial w_i}\right|_{\boldsymbol{w}=\boldsymbol{1}_I} = \frac{\lambda_i}{\sqrt{\lambda_1 + \lambda_2}}, \quad i = 1, 2.$$

The three surfaces $\mathcal{R}$ (red), $\tilde{\mathcal{R}}$ (green) and $\mathcal{H}(\boldsymbol{d}^E(\tilde{r}))$ (blue) are shown in Figure 1 for $\lambda_1 = \lambda_2 = 1$. It is seen that $\mathcal{R}$ dominates $\tilde{\mathcal{R}}$, which in turn dominates $\mathcal{H}(\boldsymbol{d}^E(\tilde{r}))$. The intersection of $\mathcal{R}$ and $\tilde{\mathcal{R}}$ at the corner points is also visible. Notably, while r is concave, $\tilde{r}$ is convex. ■

The visualisation of Example 3.12 demonstrates that the core of r may contain more elements than $\boldsymbol{d}^E(\tilde{r})$, in contrast to the core of $\tilde{r}$, which is single-valued, see Lemma 3.4. Alternative allocations $\boldsymbol{d}$ can be derived by rotating the hyper-plane $\mathcal{H}(\boldsymbol{d}^E(\tilde{r}))$ around axis ℓ, which is the diagonal in the cuboid $[0,1]^I \times [0, r(\mathbf{1}_I)]$ and is given by

$$\ell = \left\{\lambda(1,\dots,1,r(\mathbf{1}_I)) \in [0,1]^I \times [0, r(\mathbf{1}_I)];\ \lambda \in [0,1]\right\} \ \subset\ \mathcal{H}(\boldsymbol{d}^E(\tilde{r})).$$

These rotations are described by the normalised vectors $\boldsymbol{d} \in \mathbb{R}^I$, $\sum_{i=1}^I d_i = r(\mathbf{1})$ and provide the hyper-planes $\mathcal{H}(\boldsymbol{d})$, containing ℓ. If the resulting rotated hyper-plane $\mathcal{H}(\boldsymbol{d})$ is still dominated by $\mathcal{R}$, then $\boldsymbol{d} \in \mathcal{C}(r)$.

Following on from the previous example, it is shown that rotations of $\mathcal{H}(\boldsymbol{d}^E(\tilde{r}))$ can indeed produce core elements of r.

Example 3.13. Continuing from Example 3.12, consider allocations of the following form

$$d_1(\alpha) = \frac{\lambda_1 - \alpha}{\sqrt{\lambda_1 + \lambda_2}} \quad \text{and} \quad d_2(\alpha) = \frac{\lambda_2 + \alpha}{\sqrt{\lambda_1 + \lambda_2}},$$

for some $\alpha \geq 0$. The resulting planes $\mathcal{H}(\boldsymbol{d}(\alpha))$ are indeed rotations of $\mathcal{H}(\boldsymbol{d}^E(\tilde{r}))$; note that $\mathcal{H}(\boldsymbol{d}(\alpha))$ and $\mathcal{H}(\boldsymbol{d}^E(\tilde{r}))$ must intersect by construction at the points $(0,0,0)$ and $(1,1,\sqrt{\lambda_1+\lambda_2})$.

The situation is demonstrated in Figure 2, which shows the surfaces $\mathcal{R}$ (red) and $\mathcal{H}(\boldsymbol{d}(\alpha))$ (blue). On the left-hand side the plane is plotted for an allocation with rotation $\alpha = 0.4$. While the rotation is visible, compared to Figure 1, the hyper-plane $\mathcal{H}(\boldsymbol{d}(0.4))$ is still below the hyper-surface $\mathcal{R}$, and $\boldsymbol{d}(0.4)$ is in the core of r. On the right-hand side an allocation with rotation $\alpha = 0.6$ is shown. Here, it is seen that the rotation is excessive, in the sense that $\mathcal{H}(\boldsymbol{d}(0.6))$ is made to intersect at non-corner points (and thereby locally dominates) $\mathcal{R}$. For sub-portfolios $\boldsymbol{w}$ such that the blue hyper-plane is above the red hyper-surface, incentives are thus created to leave the portfolio, i.e. $\boldsymbol{d}(0.6)$ is not in the core of r.

Furthermore, one can see from Figure 2 that concavity of r implies that, in order to have $\boldsymbol{d}(\alpha) \in \mathcal{C}(r)$, one needs to check only whether the (rotated) hyper-plane $\mathcal{H}(\boldsymbol{d}(\alpha))$ lies below the hyper-surface $\mathcal{R}$ at the corner points $\boldsymbol{w} = (1,0)$ and $\boldsymbol{w} = (0,1)$. Hence we require

$$\begin{aligned} d_1(\alpha) \leq r(1,0) &\implies \frac{\lambda_1 - \alpha}{\sqrt{\lambda_1+\lambda_2}} \leq \sqrt{\lambda_1}, \\ d_2(\alpha) \leq r(0,1) &\implies \frac{\lambda_2 + \alpha}{\sqrt{\lambda_1+\lambda_2}} \leq \sqrt{\lambda_2}, \end{aligned}$$

which for $\lambda_1 = \lambda_2 = 1$ implies $|\alpha| \leq \sqrt{2} - 1 \simeq 0.41$. All rotations α satisfying this

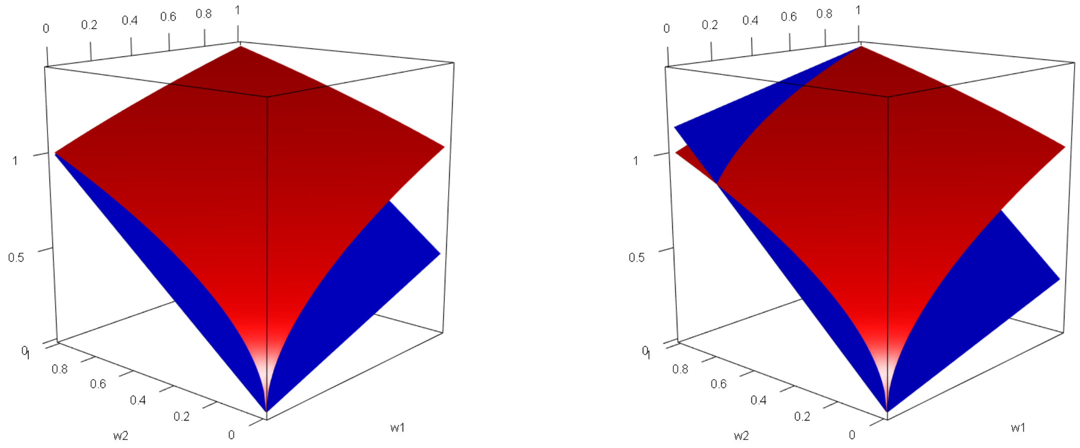

Figure 2: Hyper-surfaces $\mathcal{R}$ (red) and $\mathcal{H}(\boldsymbol{d}(\alpha))$ (blue), for rotations $\alpha = 0.4$ (left-hand side) and $\alpha = 0.6$ (right-hand side); Example 3.13.

requirement provide allocations $\boldsymbol{d}(\alpha) \in \mathcal{C}(r)$. Figure 2 shows that a rotation with $\alpha = 0.4$ yields an allocation close to the boundary of the core $\mathcal{C}(r)$. ■

Example 3.13 demonstrates geometrically that concavity of r ensures that $\mathcal{C}(r)$ will contain more elements than the Euler allocation $\boldsymbol{d}^E(\tilde{r})$, since there exist rotations of $\mathcal{H}(\boldsymbol{d}^E(\tilde{r}))$ remaining below $\mathcal{R}$. Furthermore, to characterise core allocations, it suffices to focus on the corner points $\boldsymbol{w} \in \{0,1\}^I$. The following results formalise this insight.

Here and in the sequel denote $\mathcal{I} = \{1, \ldots, I\}$. For any $S \subseteq \mathcal{I}$, denote by $e_S \in \mathbb{R}^I$ the vector with elements $e_{S,i} = 1$ if $i \in S$ and $e_{S,i} = 0$ otherwise.

Proposition 3.14. *Let r be concave. Then, $\mathcal{C}(r) = \mathcal{CC}(r)$, where the crisp core $\mathcal{CC}(r)$ is the set of allocations $\boldsymbol{d}$ of r such that $\sum_{i=1}^I w_i d_i \leq r(\boldsymbol{w})$ for all $\boldsymbol{w} \in \{0,1\}^I$.*

Proof. The inclusion $\mathcal{C}(r) \subseteq \mathcal{CC}(r)$ is trivial, so we only show $\mathcal{CC}(r) \subseteq \mathcal{C}(r)$. Let $a \in \mathcal{CC}(r)$, i.e., it holds that $\sum_{i \in S} a_i \leq r(e_S)$ for all $S \subseteq \mathcal{I}$. It is shown by Owen (1972, below Equation (8) therein) that

$$\sum_{S \subseteq \mathcal{I} \setminus \{i\}} \prod_{j \in S} w_j \prod_{j \in \mathcal{I} \setminus (S \cup \{i\})} (1 - w_j) = 1, \tag{10}$$

for any $i \in \mathcal{I}$ and $\boldsymbol{w} \in [0,1]^{\mathcal{I}\setminus\{i\}}$. Then, we get for any $\boldsymbol{w} \in [0,1]^I$ that

$$\begin{aligned}
\sum_{i=1}^{I} w_i a_i &= \sum_{S\subseteq\mathcal{I}} \left\{ \prod_{i\in S} w_i \prod_{j\in\mathcal{I}\setminus S} (1-w_j) \right\} \sum_{i\in S} a_i \\
&\leq \sum_{S\subseteq\mathcal{I}} \left\{ \prod_{i\in S} w_i \prod_{j\in\mathcal{I}\setminus S} (1-w_j) \right\} r(e_S) \\
&\leq r\left(\sum_{S\subseteq\mathcal{I}} \left\{ \prod_{i\in S} w_i \prod_{j\in\mathcal{I}\setminus S} (1-w_j) \right\} e_S \right) \\
&= r\left(\sum_{i=1}^{I} w_i e_i \sum_{S\subseteq\mathcal{I}\setminus\{i\}} \left\{ \prod_{j\in S} w_j \prod_{j\in\mathcal{I}\setminus(S\cup\{i\})} (1-w_j) \right\} \right) \\
&= r(\boldsymbol{w}),
\end{aligned}$$

where the first equality is due to Owen (1972, Theorem 2), the second inequality is due to the concavity of r, and the last equality is due to (10). □

The crisp core was originally defined for cooperative transferable utility games by Gillies (1953). If the fuzzy game r is concave, it is known (Shapley, 1971) that the crisp core can be characterised by its marginal vectors. Let $\Pi(I)$ be the set of all permutations of the set $\mathcal{I} = \{1,\ldots,I\}$. For some $\sigma \in \Pi(I)$, we write $\sigma = \{\sigma(1),\ldots,\sigma(I)\}$. The core is precisely characterised by the following proposition; in fact, it is a convex polytope.

Proposition 3.15. *(Shapley, 1971, Theorems 3 and 5) Let r be concave. Then, we have*

$$\mathcal{CC}(r) = \operatorname{conv}\{m^\sigma \in \mathbb{R}^I : \sigma \in \Pi(I)\},$$

where

$$\begin{aligned}
m^\sigma_{\sigma(1)} &= r(e_{\{\sigma(1)\}}) \\
m^\sigma_{\sigma(i)} &= r(e_{\{\sigma(1),\ldots,\sigma(i)\}}) - r(e_{\{\sigma(1),\ldots,\sigma(i-1)\}}),\ i = 2,\ldots,I,
\end{aligned}$$

and conv *is the convex hull of its arguments.*

For $I = 2$, we get from Proposition 3.15 that the core is given by the set $\mathcal{C}(r) = \operatorname{conv}\{(r(1,0), r(1,1) - r(1,0)), (r(1,1) - r(0,1), r(0,1))\}$.

While concavity of r allows a complete characterization of $\mathcal{C}(r)$, most risk measures will actually not yield a concave r for the insurance risk model (2); even when the standard deviation risk measure is used, concavity is only guaranteed by $Z_i = 0$ for all $i = 1, \ldots, I$. However, $\mathcal{C}(r)$ will generally still not be single-valued, as the next example demonstrates.

Example 3.16. Consider the same independent Poisson model as in Examples 3.12 and 3.13, but with the two different risk measures TVaR (see (5)) and VaR (see (6)), each evaluated with $p = 0.9$. For $\lambda_1 = \lambda_2 = 1$, we plot in Figure 3 the hyper-surfaces $\mathcal{R}$ and $\mathcal{H}(\boldsymbol{d})$ for $\boldsymbol{d}$ with $d_1 = d_2 = \frac{r(1,1)}{2}$. Observe that by symmetry in the model we have $\boldsymbol{d} = \boldsymbol{d}^E(\tilde{r})$. The plot on the left-hand side in Figure 3 corresponds to the TVaR risk measure. It is seen directly that r is not concave. But since TVaR preserves convex order, as Corollary 3.9 implies, the hyper-plane $\mathcal{H}(\boldsymbol{d})$ lies below the hyper-surface $\mathcal{R}$, such that $\boldsymbol{d}$ is in the core of r. Furthermore, it is clear that rotations such that $\mathcal{H}(\boldsymbol{d})$ remains below $\mathcal{R}$ are possible.

In contrast, the plot on the right-hand side of Figure 3, corresponding to the VaR risk measure (see (6)), which does not preserve convex order, shows that $\boldsymbol{d}$ is not in the core of r, since there are areas where the hyper-plane $\mathcal{H}(\boldsymbol{d})$ lies above the hyper-surface $\mathcal{R}$. Actually, the core is empty for any specification of λ_1, λ_2 and p. The random variable $S(\boldsymbol{w})$ has a positive probability mass at zero: $\mathbb{P}(S(\boldsymbol{w}) = 0) = \exp(-w_1\lambda_1 - w_2\lambda_2) > 0$. Therefore, we can always choose w_1 and w_2 sufficiently small such that $\mathbb{P}(S(\boldsymbol{w}) = 0) > p$, which implies that $\mathrm{VaR}_p(S(\boldsymbol{w})) = 0$. This corresponds to the flat area of the hyper-surface $\mathcal{R}$ for $\boldsymbol{w}$ close to $\mathbf{0}$. ■

3.4 Aumann-Shapley allocations

In this section we discuss Aumann-Shapley allocations, which have been proposed for non-homogeneous fuzzy games by Billera and Heath (1982) and Mirman and Tauman (1982). Recall its definition in (7). It is originally characterised in the context of cost-sharing and the axiomatizations used by Billera and Heath (1982) and Mirman and Tauman (1982) are based on an additivity axiom that is hard to interpret in the context of risk capital allocations (see Denault, 2001). The motivation for considering Aumann-Shapley allocations for risk capital allocations relies on an asymptotic argument of Aumann and Shapley (1974). Suppose a partition is imposed on every line of business' participation level, and consider every fractional line of business as a separate entity to which either full or no participation is possible. Then, the Shapley value (Shapley, 1953) of this ('crisp' rather than 'fuzzy') game leads to an allocation method. If we let the partition be infinitesimally fine, this allocation converges uniformly to the Aumann-Shapley value under appropriate differentiability conditions.

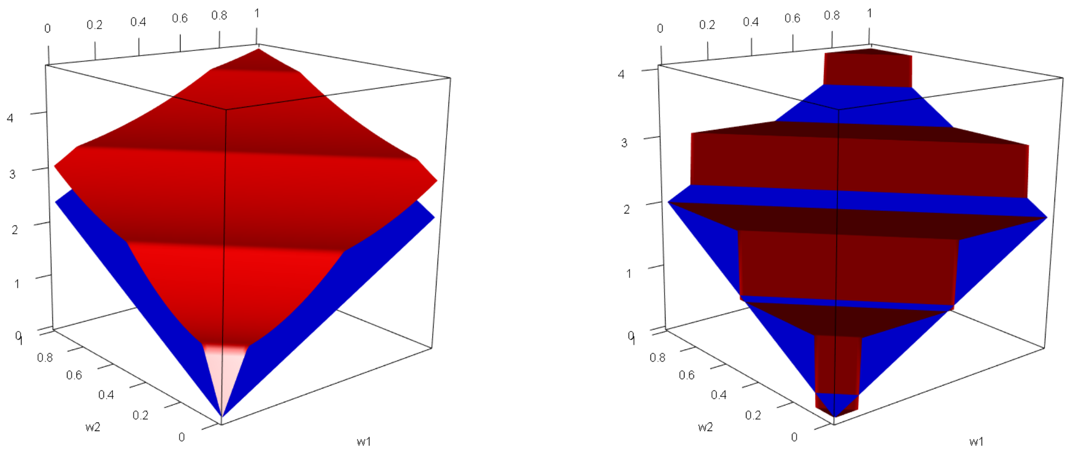

Figure 3: Hyper-surfaces $\mathcal{R}$ (red) and $\mathcal{H}(\boldsymbol{d}^E(\tilde{r}))$ (blue) for risk measures $\mathrm{TVaR}_{0.9}$ (left-hand side) and $\mathrm{VaR}_{0.9}$ (right-hand side); Example 3.16.

Aumann-Shapley allocations have appeared in the actuarial literature considering non-homogeneous risk measures (Tsanakas, 2009) and, similar to this paper, the situation when portfolios cannot be represented as linear combinations (Powers, 2007). While it is appealing that the Aumann-Shapley allocations use directly the gradient of r (as opposed to that of $\tilde{r}$), the evaluation of (7) can be numerically expensive. For example, when distortion risk measures are used, calculating the integral of $\frac{\partial}{\partial w_i} r(\beta \mathbf{1}_I)$ requires nested simulation.

Furthermore, the Aumann-Shapley allocation $\boldsymbol{d}^{AS}(r)$ is not necessarily in the core $\mathcal{C}(r)$, unless the fuzzy game r is concave.

Theorem 3.17. *If r is concave and continuously differentiable, the Aumann-Shapley allocation $\boldsymbol{d}^{AS}(r)$ is in the core $\mathcal{C}(r)$.*

Proof. Due to Proposition 3.15, it is sufficient to check that the Aumann-Shapley value is in the crisp core $\mathcal{CC}(r)$. Here, we use the Aumann-Shapley method of Moulin (1995); see also Calvo and Santos (2000) and Sprumont (2005). We impose a specific equidistant partition on the domain $[0, 1]$ of w_i, and index the intervals by $\mathcal{I}_i^n = \{1, 2, \dots, 2^n\}$ for all $i \in \mathcal{I} = \{1, \dots, I\}$, where $n \in \mathbb{N}$. The corresponding Transferable Utility (TU) game is then defined as $v^n(S) := \rho(S(\{\frac{1}{2^n}|\mathcal{I}_i^n \cap S| : i \in \mathcal{I}\})) = r(\{\frac{1}{2^n}|\mathcal{I}_i^n \cap S| : i \in \mathcal{I}\})$ for $S \subseteq \mathcal{I}_n = \bigcup_{i=1}^{I} \mathcal{I}_i^n$, where $|\cdot|$ denotes the cardinality of a set. Then, we apply the Shapley value (Shapley, 1953) to the problem with the hypothetical agents in $\mathcal{I}_n$ for a given $n \in \mathbb{N}$. This is called the Aumann-Shapley method allocation (Moulin, 1995), and is defined as

$$\hat{a}_i^n := \sum_{j \in \mathcal{I}_i^n} \Phi_j(\mathcal{I}_n, v^n),$$

where Φ is the Shapley value (Shapley, 1953).

A TU game v^n is called concave when $v^n(S \cup T) + v^n(S \cap T) \leq v^n(S) + v^n(T)$ for all $S, T \subseteq \mathcal{I}_n$. Moreover, the core of a TU game v^n is defined as $\{a \in \mathbb{R}^{\mathcal{I}_n} : \sum_{j \in \mathcal{I}_n} a_j = v^n(\mathcal{I}_n), \sum_{j \in S} a_j \leq v^n(S), \text{ for all } S \subset \mathcal{I}_n\}$. Since the fuzzy game r is concave, we have that the TU game v^n is concave for every $n \in \mathbb{N}$. Therefore, it follows from Shapley (1971, Theorem 4 and Theorem 7) that the core of v^n is non-empty, and that the Shapley value is in the core of v^n. For any vector a^n in the core of v^n, we have by construction for $\hat{a}_i^n = \sum_{j \in \mathcal{I}_i^n} a_j^n, i \in \mathcal{I}$, that $\hat{a}^n \in \mathcal{CC}(r)$. The limit of $\hat{a}^n$ for $n \to \infty$ exists, and equals the Aumann-Shapley value due to Calvo and Santos (2000, Proposition 3.1 and Theorem 4.1), i.e., $\lim_{n\to\infty} \hat{a}_n = \boldsymbol{d}^{AS}(r)$. Since the crisp core $\mathcal{CC}(r)$ is a closed set (in fact a convex polytope), we have that $\boldsymbol{d}^{AS}(r) \in \mathcal{CC}(r)$. This concludes the proof. □

Aumann-Shapley allocations for concave and non-concave fuzzy games r are illustrated by the next two examples.

Example 3.18. As in Example 3.12, let $Z_i = 0$, and $X_i(w_i) = Y_i(w_i)$ be independent Poisson processes with intensities λ_i, $i = 1, \ldots, I$. Let ρ be given by the standard deviation risk measure. Then,

$$r(\boldsymbol{w}) = \sqrt{\sum_{i=1}^{I} w_i \lambda_i} \quad \Longrightarrow \quad \frac{\partial r(\boldsymbol{w})}{\partial w_i} = \frac{1}{2} \frac{\lambda_i}{\sqrt{\sum_{j=1}^{I} w_j \lambda_j}}.$$

It follows that

$$d_i^{AS}(r) = \int_0^1 \frac{1}{2} \frac{\lambda_i}{\sqrt{\sum_{j=1}^{I} \beta \lambda_j}} d\beta = \frac{\lambda_i}{\sqrt{\sum_{j=1}^{I} w_j \lambda_j}},$$

which coincides with $\boldsymbol{d}^E(\tilde{r})$ in Example 3.12. Hence, for this particular example, it holds in fact that $\boldsymbol{d}^{AS}(r) = \boldsymbol{d}^E(\tilde{r}) \in \mathcal{C}(r)$. ■

Example 3.19. Consider the case where $I = 2$, Y_1 and Y_2 are Poisson processes with intensities λ_1, λ_2, and Z_1, Z_2 are independent zero-mean normal random variables with standard deviations σ_1 and σ_2. The risk measure ρ used is the standard deviation.

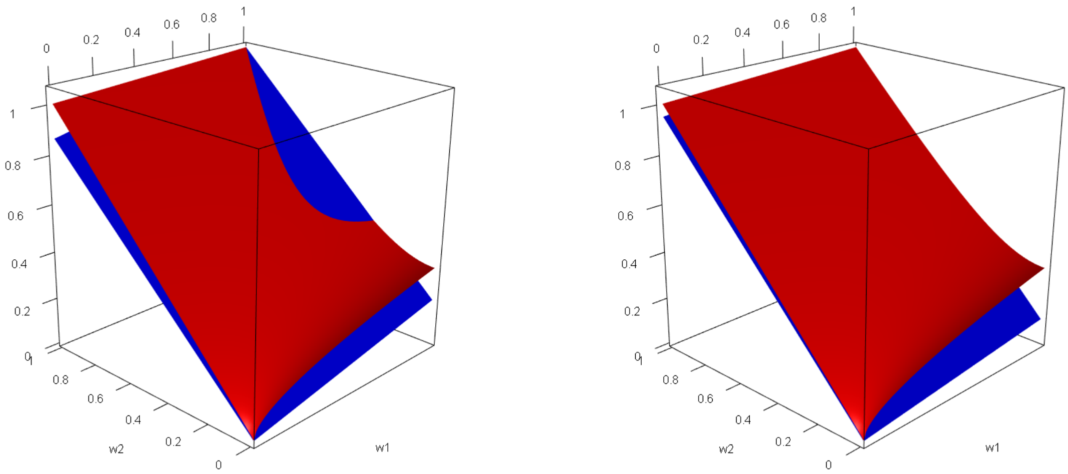

Figure 4: On the left-hand side, hyper-surfaces $\mathcal{R}$ (red) and $\mathcal{H}(\boldsymbol{d}^{AS}(r))$ (blue); on the right-hand side, hyper-surfaces $\mathcal{R}$ (red) and $\mathcal{H}(\boldsymbol{d}^{E}(\tilde{r}))$ (blue); Example 3.19.

For this model, we get

$$r(\boldsymbol{w}) = \sqrt{w_1\lambda_1 + w_2\lambda_2 + w_1^2\sigma_1^2 + w_2^2\sigma_2^2},$$
$$\frac{\partial r(\boldsymbol{w})}{\partial w_i} = \frac{\lambda_i + 2w_i\sigma_i^2}{2\sqrt{w_1\lambda_1 + w_2\lambda_2 + w_1^2\sigma_1^2 + w_2^2\sigma_2^2}},$$

for $i = 1, 2$. This implies

$$d_i^{AS} = \int_0^1 \frac{\lambda_i + 2\beta\sigma_i^2}{2\sqrt{\beta(\lambda_1 + \lambda_2) + \beta^2(\sigma_1^2 + \sigma_2^2)}} d\beta, \quad i = 1, 2.$$

For $\lambda_1 = 0.1$, $\lambda_2 = 0.01$, $\sigma_1 = 0.1$, and $\sigma_2 = 1$, the Aumann-Shapley allocation is evaluated numerically to give $d_1^{AS}(r) = 0.190$, $d_2^{AS}(r) = 0.868$. We also evaluate the Euler allocation of the linearised fuzzy game $\tilde{r}$ which, using the formulas of Appendix A, is $d_1^E(\tilde{r}) = 0.104$, $d_2^E(\tilde{r}) = 0.954$.

On the left-hand side of Figure 4, the hyper-surfaces $\mathcal{R}$ (red) and $\mathcal{H}(\boldsymbol{d}^{AS}(r))$ (blue), the latter generated by the Aumann-Shapley allocation, are plotted. On the right-hand side, $\mathcal{R}$ is plotted along with the hyper-plane corresponding to the Euler allocation $\boldsymbol{d}^E(\tilde{r})$. It is seen that the function r is not concave. Furthermore, $\boldsymbol{d}^{AS}(r)$ is not in the core $\mathcal{C}(r)$ (the assumption of Theorem 3.17 does not hold), while the Euler allocation $\boldsymbol{d}^E(\tilde{r})$ is; the latter follows from Corollary 3.9. ■

4 Signals for portfolio management

In this section we study the extent to which, for an insurance risk model of the form (2), capital allocation can provide signals that are useful for re-balancing a portfolio in order to improve its performance. Here, we additionally require that risk measures satisfy the property of:

- Translation invariance: $\rho(V + a) = \rho(V) + a$, for all V and $a \in \mathbb{R}$.

Distortion risk measures satisfy this property; the standard deviation risk measure can be adapted to satisfy it, by considering the standard deviation principle $\rho(V) = \mathbb{E}(V) + \beta\sqrt{\mathbb{V}(V)}$ for some $\beta > 0$.

A financial firm's performance is often assessed using a Return-on-Capital (RoC) measure. Assume that the premium that is received by insuring $X_i(w_i)$ is linear in the exposure w_i. In particular, for $\boldsymbol{p} = (p_1, \ldots, p_I)$, the premium earned for writing $S(\boldsymbol{w})$ is equal to $\boldsymbol{w}'\boldsymbol{p}$. Let $\mathbb{E}(X_i(1)) = x_i$ and assume $p_i - x_i > 0$, $i = 1, \ldots, I$, implying that all lines of business are expecting a profit. The *underwriting profit* for the whole portfolio of $S(\boldsymbol{w})$ is given by $\boldsymbol{w}'(\boldsymbol{p} - \boldsymbol{x})$ and the *risk capital* is $r(\boldsymbol{w}) - \boldsymbol{w}'\boldsymbol{x}$. The RoC is then defined as

$$\mathrm{RoC}(\boldsymbol{w}) = \frac{\boldsymbol{w}'(\boldsymbol{p} - \boldsymbol{x})}{r(\boldsymbol{w}) - \boldsymbol{w}'\boldsymbol{x}}. \tag{11}$$

Tasche (1999) introduced the idea of *suitability* of capital allocations for performance management which in our setting translates as follows.

Definition 4.1. *A vector $\boldsymbol{d}(r) \in \mathbb{R}^I$ is called suitable for performance measurement with fuzzy game r if the inequalities*

$$\frac{p_i - x_i}{d_i(r) - x_i} \underset{(<)}{>} \frac{\boldsymbol{1}_I'(\boldsymbol{p} - \boldsymbol{x})}{r(\boldsymbol{1}_I) - \boldsymbol{1}_I'\boldsymbol{x}},$$

imply that there is an $\varepsilon > 0$ such that for all $t \in (0, \varepsilon)$ we have

$$\mathrm{RoC}(-te_i + \boldsymbol{1}_I) \underset{(>)}{<} \mathrm{RoC}(\boldsymbol{1}_I) \underset{(>)}{<} \mathrm{RoC}(te_i + \boldsymbol{1}_I),$$

where e_i is the i^{th} canonical unit vector in $\mathbb{R}^I$.

If the performance of a particular line of business, measured by $\frac{p_i - x_i}{d_i(r) - x_i}$, is higher (lower) than the one of the overall portfolio at $\boldsymbol{w} = \boldsymbol{1}_I$, then increasing (decreasing) the exposure in that line of business will lead to an improvement in the portfolio's RoC.

Note that in the definition of suitability, $\boldsymbol{d}(r)$ is not required to be an allocation here, i.e. it is not assumed that its elements add up to $r(\mathbf{1}_I)$. The next result, which slightly adapts Tasche (1999, Theorem 1), shows that for a vector to be suitable for performance measurement it must be an Euler allocation.

Theorem 4.2. *Let r be continuously differentiable in a neighborhood of $\mathbf{1}_I$. There exists an allocation $\boldsymbol{d}$ that is suitable for performance measurement with r if and only if $\nabla r(\mathbf{1}_I)'\mathbf{1}_I = r(\mathbf{1}_I)$. If such an allocation exists, it must be $\boldsymbol{d} = \nabla r(\mathbf{1}_I)$.*

Proof. Due to translation invariance, we get $\rho(S(\boldsymbol{w}) - \boldsymbol{w}'\boldsymbol{p}) = \rho(S(\boldsymbol{w})) - \boldsymbol{w}'\boldsymbol{p} = r(\boldsymbol{w}) - \boldsymbol{w}'\boldsymbol{p}$. From this and the fact that $r(\boldsymbol{w}) - \boldsymbol{w}'\boldsymbol{p} + \boldsymbol{w}'(\boldsymbol{p} - \boldsymbol{x}) = r(\boldsymbol{w}) - \boldsymbol{w}'\boldsymbol{x}$, we get that our definition of RoC coincides with the function g_r in Tasche (1999) applied to the risk $S(\boldsymbol{w}) - \boldsymbol{w}'\boldsymbol{p}$. Then, Tasche (1999, Theorem 1) shows that the only suitable vector is given by $\frac{\partial}{\partial w_i}\, \rho(S(\boldsymbol{w}) - \boldsymbol{w}'\boldsymbol{p})|_{\boldsymbol{w}=\mathbf{1}_I}$, $i = 1, \ldots, I$. Since ρ is translation invariant, this suitable vector equals $\nabla r(\mathbf{1}_I)$. The gradient $\nabla r(\mathbf{1}_I)$ is an allocation if and only if $\nabla r(\mathbf{1}_I)'\mathbf{1}_I = r(\mathbf{1}_I)$. □

Thus, for a vector to be suitable, it must be equal to the gradient *and* provide a capital allocation. Since for functions $r(\boldsymbol{w})$ that are not homogeneous, as it is the case in our setting, it is $\nabla r(\mathbf{1}_I)'\mathbf{1}_I \neq r(\mathbf{1}_I)$, and a suitable $\boldsymbol{d}$ cannot generally be found. Of course, in the special case where $Y_i(1) = 0$ for all $i = 1, \ldots, I$ and r is continuously differentiable around $\mathbf{1}_I$, then r is also homogeneous and a suitable vector is given by the Euler allocation $\boldsymbol{d} = \boldsymbol{d}^E(r) = \nabla r(\mathbf{1}_I)$.

More generally, the concept of suitability as defined above is problematic in our context, as the diversification in increasing exposures implicit in the Lévy component of model (2) implies that portfolio performance can always be improved by writing more insurance policies, as shown below.

Lemma 4.3. *For $\boldsymbol{w} \in [0, W]^I$ and $0 \leq \lambda \leq 1$, we have* $\mathrm{RoC}(\lambda\boldsymbol{w}) \leq \mathrm{RoC}(\boldsymbol{w})$.

Proof. A slight modification in the argument used to prove Proposition 3.7 yields that for all $\lambda \in [0, 1]$ and $\boldsymbol{w} \in [0, 1]^I$, we have $r(\lambda\boldsymbol{w}) \geq \rho\left(\sum_{i=1}^I \lambda X_i(w_i)\right) = \lambda r(\boldsymbol{w})$. This implies

$$\frac{\lambda\boldsymbol{w}'(\boldsymbol{p} - \boldsymbol{x})}{r(\lambda\boldsymbol{w}) - \lambda\boldsymbol{w}'\boldsymbol{x}} \leq \frac{\boldsymbol{w}'(\boldsymbol{p} - \boldsymbol{x})}{r(\boldsymbol{w}) - \boldsymbol{w}'\boldsymbol{x}}.$$

□

Hence, an insurer will generally have an incentive to grow its portfolio. Limitations to such growth are typically due to capital constraints (we do not investigate here limitations in the demand for insurance). For that reason, it is meaningful to consider the case where $r(\boldsymbol{w}) - \boldsymbol{w}'\boldsymbol{x}$ is fixed and the capital allocation is only considered as giving possible signals for re-balancing the portfolio. It is seen that the Euler allocation $\boldsymbol{d}^E(\tilde{r})$ still plays a role in this context. Proposition 4.4 shows that this allocation gives appropriate signals for improving the portfolio underwriting profit under capital constraints.

Proposition 4.4. *Let $\{1, \ldots, I\} = \mathcal{I}^+ \cup \mathcal{I}^-$ with $\mathcal{I}^+ \cap \mathcal{I}^- = \emptyset$ such that*

$$\begin{aligned} p_i - x_i &> \left(d_i^E(\tilde{r}) - x_i\right) \mathrm{RoC}(\mathbf{1}_I), \quad \text{for } i \in \mathcal{I}^+, \\ p_i - x_i &< \left(d_i^E(\tilde{r}) - x_i\right) \mathrm{RoC}(\mathbf{1}_I), \quad \text{for } i \in \mathcal{I}^-. \end{aligned}$$

Consider any $\boldsymbol{v} \in \mathbb{R}^I$ with $v_i \geq 0$ for $i \in \mathcal{I}^+$ and $v_i \leq 0$ for $i \in \mathcal{I}^-$ such that

$$\frac{\partial}{\partial t} \left(r(\mathbf{1}_I + t\boldsymbol{v}) - (\mathbf{1}_I + t\boldsymbol{v})'\boldsymbol{x}\right)\Big|_{t=0} = 0.$$

Then, there exists an $\varepsilon > 0$ such that for all $t \in (0, \varepsilon)$ we get

$$(\mathbf{1}_I + t\boldsymbol{v})'(\boldsymbol{p} - \boldsymbol{x}) \geq \mathbf{1}_I'(\boldsymbol{p} - \boldsymbol{x}).$$

Proof. We need to show $\boldsymbol{v}'(\boldsymbol{p} - \boldsymbol{x}) \geq 0$. Denote $\frac{\partial r(\boldsymbol{w})}{\partial w_i} = r_i(\boldsymbol{w})$. Note that

$$\frac{\partial}{\partial t} \left(r(\mathbf{1}_I + t\boldsymbol{v}) - (\mathbf{1}_I + t\boldsymbol{v})'\boldsymbol{x}\right)\Big|_{t=0} = \sum_{i=1}^{I} v_i (r_i(\mathbf{1}_I) - x_i) = 0.$$

We have

$$\sum_{i \in \mathcal{I}^+} v_i (p_i - x_i) \geq \sum_{i \in \mathcal{I}^+} v_i \left(d_i^E(\tilde{r}) - x_i\right) \mathrm{RoC}(\mathbf{1}_I),$$

and

$$\sum_{i \in \mathcal{I}^-} |v_i| (p_i - x_i) \leq \sum_{i \in \mathcal{I}^-} |v_i| \left(d_i^E(\tilde{r}) - x_i\right) \mathrm{RoC}(\mathbf{1}_I).$$

These two inequalities imply

$$\sum_{i=1}^{I} v_i (p_i - x_i) \geq \sum_{i=1}^{I} v_i \left(d_i^E(\tilde{r}) - x_i\right) \mathrm{RoC}(\mathbf{1}_I).$$

Then the result follows, as long as $d_i^E(\tilde{r}) \geq r_i(\mathbf{1}_I),\ i = 1, \dots, I$.

By Proposition 3.7 we have for $\delta \in [-1, 0)$

$$r(\mathbf{1}_I + \delta e_i) \geq \tilde{r}(\mathbf{1}_I + \delta e_i) \implies \frac{r(\mathbf{1}_I + \delta e_i) - r(\mathbf{1}_I)}{\delta} \leq \frac{\tilde{r}(\mathbf{1}_I + \delta e_i) - \tilde{r}(\mathbf{1}_I)}{\delta}.$$

A slight adaptation of the proof of Proposition 3.7 yields $r(\boldsymbol{w}) \leq \tilde{r}(\boldsymbol{w})$ for every $\boldsymbol{w} \in [1, W]^I$. So, for $\delta \in (0, W - 1]$, we get

$$r(\mathbf{1}_I + \delta e_i) \leq \tilde{r}(\mathbf{1}_I + \delta e_i) \implies \frac{r(\mathbf{1}_I + \delta e_i) - r(\mathbf{1}_I)}{\delta} \leq \frac{\tilde{r}(\mathbf{1}_I + \delta e_i) - \tilde{r}(\mathbf{1}_I)}{\delta}.$$

Consequently

$$r_i(\mathbf{1}_I) = \lim_{\delta \to 0} \frac{r(\mathbf{1}_I + \delta e_i) - r(\mathbf{1}_I)}{\delta} \leq \lim_{\delta \to 0} \frac{\tilde{r}(\mathbf{1}_I + \delta e_i) - \tilde{r}(\mathbf{1}_I)}{\delta} = d_i^E(\tilde{r}),$$

which completes the proof. □

5 Conclusions

We address a fundamental shortcoming of the literature on capital allocation, as applied to insurance portfolios. Contrary to what most capital allocation methods assume, insurance portfolios are not linearly scalable in exposure; in that, they are unlike portfolios of liquidly traded investments such as stocks and bonds. The lack of scalability extends beyond insurance, for instance, to operational and credit risks. As a consequence, the popular Euler allocation rule is not directly applicable in order to derive capital allocations.

We show that the Euler allocation rule can still be used in an insurance context, if applied to a portfolio that is linearised around its base-line exposure. The resulting allocation belongs to the core and, thus, provides disincentives for splitting the portfolio. Furthermore, the Aumann-Shapley allocation, which is sometimes proposed as an allocation principle for non-homogeneous fuzzy games, does not have appealing properties, as it does not generally belong to the core, except in the special case where the risk measure is concave in exposures. Finally, we show how the Euler allocation rule for the linearised portfolio can be used to derive appropriate signals for portfolio management.

The results derived are contingent on the model used. Our proposed model generalises the linear portfolios that are standard in the literature, through the introduction of independent Lévy processes capturing the non-linear aggregation of insurance risk.

Independence of the Lévy processes could be relaxed, e.g. by considering conditionally independent processes, as long as Proposition 3.7 remains true. Such an investigation is beyond the scope of this paper.

Our study of capital allocation in insurance is of potential interest beyond the field of financial risk management. The literature on cooperative fuzzy games rarely considers fuzzy games that are not homogeneous and/or convex. For the particular class of fuzzy games r, which are neither homogeneous nor convex (but are subadditive), we introduce the auxiliary fuzzy game $\tilde{r}$, which do satisfy these properties. Thus, we show that the core is not empty, by explicitly constructing a core element via $\tilde{r}$.

A The Euler allocation $\boldsymbol{d}^E(\tilde{r})$ for particular risk measures

In this appendix, we provide general formulas for the evaluation of the Euler allocation $\boldsymbol{d}^E(\tilde{r})$ for the fuzzy game $\tilde{r}$ defined in Definition 3.3 and (9), in the case that the risk measure is either the standard deviation or a distortion risk measure.

Standard deviation risk measure

Let the risk measure ρ be the standard deviation risk measure. Then, direct calculation yields

$$r(\boldsymbol{w}) = \sqrt{\mathbb{V}\left(\sum_{i=1}^{I} X_i(w_i)\right)} = \left(\sum_{i=1}^{I} w_i \mathbb{V}(Y_i(1)) + \sum_{i=1}^{I}\sum_{j=1}^{I} w_i w_j \mathbb{C}(Z_i, Z_j)\right)^{1/2}, \tag{12}$$

$$\tilde{r}(\boldsymbol{w}) = \sqrt{\mathbb{V}\left(\sum_{i=1}^{I} w_i X_i(1)\right)} = \left(\sum_{j=1}^{I} w_i^2 \mathbb{V}(Y_i(1)) + \sum_{i=1}^{I}\sum_{j=1}^{I} w_i w_j \mathbb{C}(Z_i, Z_j)\right)^{1/2}. \tag{13}$$

The inequality $\tilde{r}(\boldsymbol{w}) \leq r(\boldsymbol{w})$ for $\boldsymbol{w} \in [0,1]^I$ follows directly from $w_i^2 \leq w_i$. The Euler allocation for a linearised model $\sum_{i=1}^{I} w_i X_i(1)$ is provided by (Tasche, 1999), and reads as

$$d_i^E(\tilde{r}) = \frac{\mathbb{C}\left(X_i(1), \sum_{j=1}^{I} X_j(1)\right)}{\sqrt{\mathbb{V}\left(\sum_{j=1}^{I} X_j(1)\right)}} = \frac{\mathbb{V}(Y_i(1)) + \sum_{j=1}^{I} \mathbb{C}(Z_i, Z_j)}{\left(\sum_{j=1}^{I} \mathbb{V}(Y_j(1)) + \sum_{k=1}^{I}\sum_{j=1}^{I} \mathbb{C}(Z_k, Z_j)\right)^{1/2}}. \tag{14}$$

Distortion risk measures

Recall that $S(\boldsymbol{w}) = \sum_{i=1}^{I} X_i(w_i)$ and define $\tilde{S}(\boldsymbol{w}) = \sum_{i=1}^{I} w_i X_i(1)$; it holds that $S(\mathbf{1}_I) = \tilde{S}(\mathbf{1}_I)$. For a coherent distortion risk measure with weight function ζ we have

$$r(\boldsymbol{w}) = \mathbb{E}\left(\sum_{i=1}^{I} X_i(w_i)\zeta\big(U_{S(\boldsymbol{w})}\big)\right), \tag{15}$$

$$\tilde{r}(\boldsymbol{w}) = \mathbb{E}\left(\sum_{i=1}^{I} w_i X_i(1)\zeta\big(U_{\tilde{S}(\boldsymbol{w})}\big)\right). \tag{16}$$

It is known from e.g. Tsanakas and Barnett (2003) that, subject to differentiability[5], it holds that

$$d_i^E(\tilde{r}) = \mathbb{E}\left(X_i(1)\zeta\big(U_{\tilde{S}(\mathbf{1}_I)}\big)\right) = \mathbb{E}\left(X_i(1)\zeta\big(U_{S(\mathbf{1}_I)}\big)\right). \tag{17}$$

This allocation is easily interpreted as the expected value of the loss $X_i(1)$, weighted by $\zeta\big(U_{S(\mathbf{1}_I)}\big)$, reflecting a higher emphasis on those states of the world where the total loss $S(\mathbf{1}_I)$ is high. Typically, (17) is calculated by Monte-Carlo simulation or importance sampling.

It may appear counter-intuitive that the capital allocated to a sub-portfolio with incomplete participation, by equation (17), involves the weighted expectation of $w_i X_i(1)$ rather than $X_i(w_i)$. However, as the following lemma shows, this is not an issue.

Lemma A.1. *Let ρ be a coherent distortion risk measure with weight function ζ. Then,*

$$w_i \mathbb{E}\left(X_i(1)\zeta\big(U_{S(\mathbf{1}_I)}\big)\right) = \mathbb{E}\left(X_i(w_i)\zeta\big(U_{S(\mathbf{1}_I)}\big)\right), \tag{18}$$

for any $w_i \in [0,1]$.

Proof. For the Lévy process Y_i and $w_i \leq 1$ it holds that $\mathbb{E}(Y_i(w_i)|Y_i(1)) = w_i Y_i(1)$ (see

[5] Tasche's (1999) requirement of continuous conditional densities implies differentiability of $F^{-1}_{\tilde{S}(\boldsymbol{w})}$ and, thereby, of a distortion risk measure $\rho(\tilde{S}(\boldsymbol{w}))$. Carlier and Dana (2003) show that in L^∞ differentiability of coherent distortion risk measures is implied by strict increasingness of the quantile function $F^{-1}_{\tilde{S}(\mathbf{1}_I)}(u)$ in u.

e.g. Hoyle et al., 2011). Then we have

$$\begin{aligned}
\mathbb{E}\left(X_i(w_i)\zeta\big(U_{S(\mathbf{1}_I)}\big)\right) &= \mathbb{E}\left(Y_i(w_i)\zeta\big(U_{S(\mathbf{1}_I)}\big)\right) + \mathbb{E}\left(w_i Z_i \zeta\big(U_{S(\mathbf{1}_I)}\big)\right) \\
&= \mathbb{E}\left(\mathbb{E}\Big(Y_i(w_i)\zeta\big(U_{S(\mathbf{1}_I)}\big)|\mathbf{Y}(\mathbf{1}_I),\mathbf{Z}\Big)\right) + \mathbb{E}\left(w_i Z_i \zeta\big(U_{S(\mathbf{1}_I)}\big)\right) \\
&= \mathbb{E}\left(\mathbb{E}\Big(Y_i(w_i)|Y_i(1)\Big)\zeta\big(U_{S(\mathbf{1}_I)}\big)\right) + \mathbb{E}\left(w_i Z_i \zeta\big(U_{S(\mathbf{1}_I)}\big)\right) \\
&= \mathbb{E}\left(w_i Y_i(1)\zeta\big(U_{S(\mathbf{1}_I)}\big)\right) + \mathbb{E}\left(w_i Z_i \zeta\big(U_{S(\mathbf{1}_I)}\big)\right) \\
&= w_i \mathbb{E}\left(X_i(1)\zeta\big(U_{S(\mathbf{1}_I)}\big)\right).
\end{aligned}$$

□

Remark A.2. Note that even when $\tilde{r}$ is not differentiable, the expression on the right-hand side of equation (17) is an allocation. Moreover, it still belongs to $\mathcal{C}(\tilde{r})$ and consequently (by Proposition 3.8) to $\mathcal{C}(r)$. This follows from the following argument:

$$\begin{aligned}
\sum_{i=1}^{I} w_i \mathbb{E}\left(X_i(1)\zeta\big(U_{\tilde{S}(\mathbf{1}_I)}\big)\right) &= \mathbb{E}\left(\tilde{S}(\boldsymbol{w})\zeta\big(U_{\tilde{S}(\mathbf{1}_I)}\big)\right) \\
&\leq \mathbb{E}\left(\tilde{S}(\boldsymbol{w})\zeta\big(U_{\tilde{S}(\boldsymbol{w})}\big)\right) \\
&= \tilde{r}(\boldsymbol{w}) \ \leq \ r(\boldsymbol{w}).
\end{aligned}$$

The first inequality follows from the fact that the two random vectors $\left(\tilde{S}(\boldsymbol{w}), \zeta\big(U_{\tilde{S}(\mathbf{1}_I)}\big)\right)$ and $\left(\tilde{S}(\boldsymbol{w}), \zeta\big(U_{\tilde{S}(\boldsymbol{w})}\big)\right)$ have the same marginal distributions, with the second being comonotonic (Denuit et al., 2006, Proposition 6.2.6). The second inequality follows from Proposition 3.7.

Remark A.3. The allocation rule as it appears on the right-hand side of (18) in Lemma A.1 is in fact the price in a competitive equilibrium. It is shown by Aubin (1981) that for homogeneous fuzzy games, the set of allocations in competitive equilibria coincides with the core $\mathcal{C}(\tilde{r})$. For differentiable and homogeneous fuzzy games, the core is single-valued, and given by the Euler rule (Lemma 3.4).

References

Acerbi, C. (2002). Spectral measures of risk: a coherent representation of subjective risk aversion. *Journal of Banking & Finance 26*(7), 1505–1518.

Aubin, J. (1979). *Mathematical Methods of Game and Economic Theory.* Amsterdam: North-Holland.

Aubin, J. (1981). Cooperative fuzzy games. *Mathematics of Operations Research 6*, 1–13.

Aumann, R. J. and L. S. Shapley (1974). *Values of Non-Atomic Games.* Princeton University Press, Princeton, New Jersey.

Bauer, D. and G. Zanjani (2015). The marginal cost of risk, risk measures, and capital allocation. *Management Science 62*(5), 1431–1457.

Billera, L. J. and D. Heath (1982). Allocation of shared costs: a set of axioms yielding a unique procedure. *Mathematics of Operations Research 7*, 32–39.

Boonen, T. J., A. De Waegenaere, and H. Norde (2012). A generalization of the Aumann-Shapley value for risk capital allocation problems. On SSRN: http://papers.ssrn.com/sol3/papers.cfm?abstract_id=2181938.

Buch, A., G. Dorfleitner, and M. Wimmer (2011). Risk capital allocation for RORAC optimization. *Journal of Banking & Finance 35*(11), 3001–3009.

Calvo, E. and J. C. Santos (2000). A value for multichoice games. *Mathematical Social Sciences 40*(3), 341–354.

Carlier, G. and R. A. Dana (2003). Core of convex distortions of a probability. *Journal of Economic Theory 113*, 199–222.

Csóka, P., P. J. J. Herings, and L. Á. Kóczy (2009). Stable allocations of risk. *Games and Economic Behavior 67*, 266–276.

Denault, M. (2001). Coherent allocation of risk capital. *Journal of Risk 4*, 1–34.

Denuit, M., J. Dhaene, M. Goovaerts, and R. Kaas (2006). *Actuarial Theory for Dependent Risks: Measures, Orders and Models.* John Wiley & Sons.

Dhaene, J., M. Goovaerts, and R. Kaas (2003). Economic capital allocation derived from risk measures. *North American Actuarial Journal 7*, 44–59.

Dhaene, J., A. Tsanakas, E. A. Valdez, and S. Vanduffel (2012). Optimal capital allocation principles. *Journal of Risk and Insurance 79*(1), 1–28.

Dhaene, J., S. Vanduffel, M. Goovaerts, R. Kaas, Q. Tang, and D. Vyncke (2006). Risk measures and comonotonicity: a review. *Stochastic Models 22*, 573–606.

Gillies, D. (1953). *Some Theorems on n-Person Games.* Ph. D. thesis, Princeton University.

Hougaard, J. L. and A. Smilgins (2016). Risk capital allocation with autonomous subunits: The Lorenz set. *Insurance: Mathematics and Economics 67*, 151–157.

Hoyle, E., L. P. Hughston, and A. Macrina (2011). Lévy random bridges and the modelling of financial information. *Stochastic Processes and their Applications 121*, 856–884.

Kalkbrener, M. (2005). An axiomatic approach to capital allocation. *Mathematical Finance 15*, 425–437.

McNeil, A., R. Frey, and P. Embrechts (2005). *Quantitative Risk Management* (2 ed.). Princeton Series in Finance, Princeton University Press.

Mertens, J.-F. (1988). The Shapley value in the non differentiable case. *International Journal of Game Theory 17*, 1–65.

Mildenhall, S. J. (2004). A note on the Myers and Read capital allocation formula. *North American Actuarial Journal 8*, 32–44.

Mildenhall, S. J. (2006). Actuarial geometry. Paper presented at the 2006 Risk Theory Seminar.

Mirman, L. and I. Tauman (1982). Demand compatible equitable cost sharing prices. *Mathematics of Operations Research 7*, 40–56.

Moulin, H. (1995). On additive methods to share joint costs. *Japanese Economic Review 46*(4), 303–332.

Myers, S. C. and J. A. Read (2001). Capital allocation for insurance companies. *Journal of Risk and Insurance 68*, 545–580.

Owen, G. (1972). Multilinear extensions of games. *Management Science 18*, 64–79.

Powers, M. R. (2007). Using Aumann-Shapley values to allocate insurance risk. *North American Actuarial Journal 11*, 113–127.

Rüschendorf, L. (2009). On the distributional transform, Sklar's theorem, and the empirical copula process. *Journal of Statistical Planning and Inference 139*(11), 3921–3927.

Samet, D., Y. Tauman, and I. Zang (1984). An application of the Aumann-Shapley prices for cost allocation in transportation problems. *Mathematics of Operations Research 9*(1), 25–42.

Shapley, L. (1953). A value for n-person games. In *Contributions to the Theory of Games*, pp. 307–317. Princeton: Princeton University Press.

Shapley, L. S. (1971). Cores of convex games. *International Journal of Game Theory 1*(1), 11–26.

Sherris, M. (2006). Solvency, capital allocation, and fair rate of return in insurance. *Journal of Risk and Insurance 73*, 71–96.

Sprumont, Y. (2005). On the discrete version of the Aumann-Shapley cost-sharing method. *Econometrica*, 1693–1712.

Tasche, D. (1999). Risk contributions and performance measurement. Preprint, TU Munich.

Tsanakas, A. (2009). To split or not to split: Capital allocation with convex risk measures. *Insurance: Mathematics and Economics 44*, 268–277.

Tsanakas, A. and C. Barnett (2003). Risk capital allocation and cooperative pricing of insurance liabilities. *Insurance: Mathematics and Economics 33*, 239–254.

Wang, S. S., V. R. Young, and H. H. Panjer (1997). Axiomatic characterization of insurance prices. *Insurance: Mathematics and Economics 21*, 173–183.

Zanjani, G. (2010). An economic approach to capital allocation. *Journal of Risk and Insurance 77*, 523–549.