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# **The Poisson log-bilinear Lee Carter model:** **Applications of efficient bootstrap methods to annuity** **analyses**

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## **Abstract**

Life insurance companies deal with two fundamental types of risks when issuing annuity contracts: financial risk and demographic risk. As regards the latter, recent work has focused on modelling the trend in mortality as a stochastic process. A popular method for modelling death rates is the Lee-Carter model. This methodology has become widely used and there have been various extensions and modifications proposed to obtain a broader interpretation and to capture the main features of the dynamics of mortality rates. In order to improve the measurement of uncertainty in survival probability estimates, in particular for older ages, the paper proposes an extension based on simulation procedures and on the bootstrap methodology. It aims to obtain more reliable and accurate mortality projections, based on the idea of obtaining an acceptable accuracy of the estimate by means of variance reducing techniques. In this way, the forecasting procedure becomes more efficient. The longevity question constitutes a critical element in the solvency appraisal of pension annuities. The demographic models used for the cash flow distributions in a portfolio impact on the mathematical reserve and surplus calculations and affect the risk management choices for a pension plan. The paper extends the investigation of the impact of survival uncertainty for life annuity portfolios and for a guaranteed annuity option in the case where interest rates are stochastic. In a framework in which insurance companies need to use internal models for risk management purposes and for determining their Solvency Capital Requirement, the authors consider the surplus value, calculated as the ratio between the market value of the projected assets to that of the liabilities, as a meaningful measure of the company's financial position, expressing the degree to which the liabilities are covered by the assets.

**Keywords:** life annuity, longevity risk, stochastic mortality, stochastic interest rates, cash flow analysis, bootstrap, Variance Reducing Techniques, Guaranteed Annuity Option.

## **§1. Longevity and disturbing effects on the risk management of annuity contracts.**

Life expectancy depends on several factors, the most important of which are age, gender, geographical region, and social class. Although some commentators

propose hypotheses for a long term change to the upward trend in life expectancy, due to causes such as obesity and worsening environmental conditions (as discussed by Loladze 2002 and Olshansky et al. 2005), it seems to the authors that life expectancies are following, and are likely to continue to follow, an increasing trend. Less clear is the shape of this trend, particularly in light of two noteworthy elements which emerge from empirical observations: life expectancies have increased at a faster rate than anticipated and the uncertainty in the projections of life expectancy increases as the time horizon increases. These ingredients are a source of concern to the financial institutions which provide financial products to the elderly such as pension annuities. If these trends are not considered, this type of contract will last longer than expected, bringing higher costs to these institutions. The situation can be summarized by saying that contracts like pension annuities are seriously exposed to longevity risk, and this exposure needs to be understood and managed.

Working out the risk quantification elements plays an important role in the Solvency II assessment. As stated in the E.U. Directive Draft (On the taking-up and pursuit of the business of insurance and reinsurance: Solvency II, amended February 2008) insurers can use (or the supervisor can require the use of) *internal models* in order to determine their Solvency Capital Required (SCR), rather than a standard formula. As the International Association of Insurance Supervisors (IAIS) defines in the Groupe Consultatif CEA Glossary “an *internal model* is a risk measurement system developed by an insurer to analyse the overall risk position, to quantify the risks and to determine the economic capital required to meet those risks”. Thus, IAIS (2008) states that an insurer’s internal model is effective in risk and capital management only if it is “fully embedded into the risk strategy and operational processes of the insurer”. An internal model conceived for risk assessment purposes provides a more realistic output for the insurer in the SCR framework. Further, it is well known that insurers generally use their internal models when major business decisions are to be made; for example pricing, product development and business planning all require the use of a pre-specified internal model. This means that internal models play a central role in current actuarial practice.

As Solvency II advocates, insurance companies usually make their valuations by means of present values of projected cash flows, measured in terms of their market value. Cash flow modelling is a crucial component of suitable internal models and the proper evaluation of the influence of the risk drivers on the cash flows is a key issue.

From a strict risk analysis viewpoint, the different risk aspects connected with the forecasting of future lifetimes arise from both accidental and systematic deviations of the actual data from their expected values. The longevity risk component is due to the systematic deviations of the deaths from the anticipated values, while the accidental variability in mortality causes a risk strongly influenced by the portfolio size, in the sense that large portfolios benefit from the well-known pooling effect, as shown by Coppola et al. (2000). If we work with a portfolio large enough to absorb the accidental demographic uncertainty, the demographic risk can be considered as constituting the systematic component and this is the main element that interacts with the risk arising from the variability in financial markets, the financial risk. The effect of both of these risks may be significant if we are referring to portfolios of life contracts of long duration such as pension annuities, which are characterized also by a multiplicity of payments.

The consideration of the systematic aspect of the demographic risk is deepened in this paper by our focusing on the evolution over time of mortality rates. As in Howse (2009), we observe that the longevity phenomenon originates from the

continuing improvement in adult age mortality and in particular, during the most recent 30 years, this decreasing trend has been accelerating. Potential discontinuities in longevity trends have been taken into account by some demographers and modellers but most forecasters seem to think that discontinuities or jumps with respect to recent trends are to be considered extremely unlikely.

The Lee-Carter model (Lee & Carter, 1992) has become a popular method for modelling and forecasting the trends over time in death rates: the main reasons are that it outperforms other models with respect to its prediction errors (e.g., Koissi et al., 2006; Melnikov & Romaniuk, 2006) and that it is easy to implement. This methodology has become widely used and there have been various extensions and modifications proposed in order to attain a broader interpretation and to capture the main features of the dynamics of mortality rates, e.g. the log-bilinear Poisson version of the Lee-Carter model as in Brouhns et al (2002) and Renshaw & Haberman (2003a, b, c). Further, in order to improve the survival probability outputs from the point of view of measuring uncertainty and hence longevity risk, recent approaches have used simulation procedures applied to the Lee-Carter family of models (Brouhns et al 2005, Renshaw & Haberman 2008, D'Amato et al 2010). These studies provide prediction intervals for the forecasted quantities that are derived by using simulation techniques: this is an important feature because of the non-linear nature of the quantities under consideration. The approaches are based on the bootstrap methodology for obtaining more reliable and accurate mortality projections and utilise the semi parametric bootstrap based on the Poisson distribution. More specifically, D'Amato et al (2010) adopt the idea of obtaining an acceptable accuracy of the estimate by solving the problem of reducing the variance estimator by means of one of the VRTs (Variance Reducing Techniques): the so-called Stratified Sampling technique.

The usual implementation of the Stratified Sampling is designed for Monte Carlo simulation. Instead, we propose to use it in the bootstrap context, in the light of a comparative study (Renshaw & Haberman 2008) of the simulation strategies applied to the Lee-Carter (LC) model, in which the authors show that the Monte Carlo simulation approach should not be used for risk measurement in LC modelling, since different choices of the parameter constraints in the model result in widely different simulated confidence and prediction intervals.

Furthermore, by nesting the original Poisson LC within the bootstrap with stratified sampling, we seek to reduce the approximation error for the statistics of interest. A sampling variance reduction approach for bootstrap mortality estimation is developed and analysed in comparison with the bootstrap procedure as in Brouhns et al., 2005 (referred to herein as the Standard Procedure or SP).

In this framework, the aim of the paper is to perform a new approach for describing the survival phenomenon by means of an experimental simulation approach applied to the LC model: the Stratified Sampling Bootstrap (SSP). The approach provides good results compared with the Standard Bootstrap and the Iterative Procedure (IP) proposed by Renshaw & Haberman (2003c) in respect of the Italian population, as section 4 shows. The good performance leads us to prefer this procedure with respect to the SP and IP for the majority of cases (D'Amato et al. 2009b).

The longevity question is a critical element in the solvency appraisal of pension annuities, as a wide literature explains (see for example Hari et al. 2008, Olivieri & Pitacco 2003, Fornero & Luciano 2004, Pitacco et al. 2009). The demographic system for modelling a portfolio cash flow distribution impacts on key quantities such as technical reserves, surplus and funding ratios and it affects some of

the choices in the management of pension plans (Coppola et al. 2007, D'Amato et al. 2009a).

The current paper focuses on the funding ratio, defined as the ratio between the market value of the assets and that of the liabilities at a certain time, and is chosen as a measure of the solvency of the insurance portfolio at that time. In particular, the purpose is to study the financial implications of the improvement over time in mortality rates on the funding ratio, considering the results as measures of the cost of increased longevity. In this analysis, the financial risk is stochastically modelled and interacts with the demographic risk. Although recent econometric researchers highlight that a certain degree of correlation between these two sources of risk may exist, the risks are assumed to be independent in both the classic actuarial context of pricing and reserving in life insurance and in the Solvency 2 framework. We will make the assumption in this work.

In this paper, valuations are performed at the time of issue on the basis of the financial and demographic information flow at that time and are highly sensitive to the strength of the longevity phenomenon. Survival modelling is required, from the pension plan point of view, for the estimation of the premiums expected to be received during the accumulation period up to retirement age, and of the benefits expected to be paid during the period from retirement age until the death of the policyholder. Both premiums and benefits depend on the demographic factor in terms of their size and the number of payments to be paid or received.

In the numerical examples, we will discuss the funding ratio values using the SSP and compare them with those that result from the other two techniques under consideration: SP and IP, in a context in which interest rates are stochastically modelled. The analysis is completed with the calculation of the risk arising from the uncertainty in the choice of the simulation approach: this is allowed for by assigning probabilities to the choice of each of the three different approaches, in light of the subjective degree of reliability that the insurer attaches to each method.

## §2. The funding ratio in living benefit products

We focus on the funding ratio  $F_t$  at time  $t$ , defined as the ratio between  $A_t$ , the market value of assets and  $L_t$ , the market value of liabilities, projected from time  $0$  and valued at time  $t$ :

$$F_t = \frac{A_t}{L_t} \quad (1)$$

Formula (1) can be extended to a portfolio of identical policies, with benefits due in the case of survival to persons belonging to an initial group of  $c$  individuals aged  $x$ . We let  $w(t,j)$  be the value at time  $t$  of one monetary unit due at time  $j$  and  $X_j$  and  $Y_j$  be the stochastic cash flows respectively of assets coming into the portfolio and liabilities going out of it, both valued at time  $t$  on the basis of the information flow at the issue time. In particular, this leads to:

$$A_t = \sum_j N_j X_j w(t,j) \quad (2)$$

and

$$L_t = \sum_j N_j Y_j w(t, j) \quad (3)$$

with  $w(t, j) = v^{-\text{sign}(t-j)}$ ,  $v$  being the present value of 1 monetary unit due at time  $j$  and  $N_j$  being the number of survivors at time  $j$ .

The funding ratio at time  $t$ , as is clear in equation 1, may be usefully interpreted as the amount of assets for each unit of liability and, in this sense, it can be considered to be a useful solvency measure. It sums up, at time  $t$ , the full future cash inflow and outflow patterns and contains the longevity and financial information of the contract over future time periods.

### §3. The Poisson Lee Carter simulation approach for mortality projections

Mortality improvements force those responsible for the planning of public retirement systems and private life annuity business to forecast carefully future mortality rates. A variety of alternative methodologies for projecting mortality have been proposed in the literature. Considering a wide range of criteria, mainly based on the accuracy in representing the survival phenomenon, the Lee-Carter model, proposed in 1992, appears to be one of the most effective methods, for example, its use has been recommended by the US Social Security Technical Advisory Panel (Booth 2006; Booth & Tickle 2008).

According to Brillinger (1986) and Alho (2000), the assumption that the number of deaths has a Poisson distribution is plausible. In particular, the Poisson log-bilinear variant of the Lee Carter model overcomes the problems associated with the ordinary least squares method of fitting. In many cases, the Poisson version, based on heteroskedastic Poisson error structures, is consistently more accurate than the alternatives (as shown in Renshaw & Haberman 2003c).

The Lee Carter model in the Poisson setting is characterized by the following expressions:

$$D_{xt} \approx \text{Poisson}(E_{xt} \mu_{xt}) \quad (4)$$

in which

$$\mu_{xt} = \exp(\alpha_x + \beta_x k_t) \quad (5)$$

and  $E_{xt}$  represents the exposures to the risk of death (in other words the number of person years from which  $D_{xt}$  occurs) and where the parameters are subjected to the constraints:

$$\sum_t k_t = 0 \quad \sum_x \beta_x = 1$$

The force of mortality is thus assumed to have the log-bilinear form:

$$\ln(\mu_{xt}) = \alpha_x + k_t \beta_x$$

Because of the nonlinear nature of the quantities of interest, an analytical approach to the parameter estimation and the calculation of prediction intervals is not

tractable and hence we resort to a simulation approach, as advocated by Brouhns et al. (2005). With regard to the Poisson Lee Carter model, the objective of the simulation is represented by the sample estimate of the model parameter vector  $\mathcal{G} = (\alpha_x, \beta_x, k_t)$ , which is needed for the mortality projections.

The usefulness of the vector estimate is related to the variance of the estimator itself in the sense that a smaller estimator variance corresponds to a better estimate. As to the smaller variance, there are a variety of different methods which one can utilise for reducing the variance of the simulation estimate: the so-called variance reduction techniques (from herein VRTs). In general, the techniques use one or more of the following strategies for reducing variances: induce positive correlation, induce negative correlation; control randomness.

The most popular techniques include: Antithetic Variates, Control Variates, Stratified Sampling and Importance Sampling (Ross 2002).

All simulation studies use finite simulation runs and lengths and, as a consequence, the effectiveness of VRTs under finite simulation conditions needs to be investigated. In particular, the Stratified Sampling Procedure is concerned with the matrix of historical log mortality rates, which is naturally stratified on the basis of a homogeneity criterion for age groups or single ages. Our principal set of data relates to the Italian male population with annual age specific death counts from ages 0:100 ranging over the period 1950 to 2006. There are various ways of modelling the mortality trend, both at the aggregate level and at the level of individual age-sex groups. The level of individual age groups gives us another perspective on the mortality decline. It allows for recovering some information about infant, young and older ages, collecting data belonging the same age as in distinct *categories*. In this sense, the population dataset embraces a number of distinct *categories*, so that the sampling frame can be ordered by these *categories* into separate strata, by using stratified sampling. However, if we consider the dataset by single ages, the correlations between the residuals for adjacent age groups tend to be high (as noted in Denton et al 2005). This suggests that we are dealing with a certain class of dependent process. On the other hand, the stratified sampling technique treats each stratum as an independent population or cluster: this provides an indication for future research.

The deaths count variable  $X$  represents the observations and is split into  $K$  populations, the so-called strata. Let us consider  $X_{ki}$  as the observations of the variable of interest,  $k$  indicating the single ages and  $i$  the time period. The variables  $X_{ki}$  and  $X_{kj}$ ,  $i \neq j$ , are independent within the  $k$ -th population, but they are dependent in the general population. Each stratum is then sampled as an independent sub-population (Hall 1986). There are several potential benefits to this approach. Dividing the population into distinct, independent strata can enable us to draw inferences about specific subgroups that may be lost in a more generalized random sample. Furthermore, it is sometimes the case that data are more readily available for individual, pre-existing strata within a population than for the overall population. In a certain sense, the same belief can be found in the *stratification* of female and male genders, which is widely used in the actuarial field particularly to analyze mortality rates and forecast the trends. Also, in this case, dependent data are treated as independent populations as in stratified sampling.

In particular, in our case, we single out the main features by bootstrapping from each stratum, referring to the perspective of single ages.

We recall that stratification is the process of grouping observations of the population into relatively homogeneous subgroups, the sub-populations, before sampling. The strata are mutually exclusive: each element belonging to the population is assigned to only one stratum. The strata are also collectively exhaustive: no population element is excluded.

Thus, in the Stratified Sampling technique, the region of interest is split into  $K$  disjoint subset, the so-called strata:

$$\bigcup_{k=1}^K Z_k = Z \quad (6)$$

where  $Z$  is the whole region corresponding to the population and  $K$  is the number of the strata. Moreover, on the basis of the stratification criterion, i.e. the age group or the single age category, each sub-population (stratum) comprises  $N_k$  observations (the size of the  $k^{\text{th}}$  stratum), where the size of the global population  $N$  is given by the following expression:

$$N = \sum_{k=1}^K N_k .$$

The next step is to draw a random sample from each stratum. Samples are selected independently from each stratum. Generally, when the sample is a simple random sample, it is referred to as *stratified random sampling*. Instead, we propose for each stratum a bootstrap simulation algorithm. The bootstrap method provides answers to a large class of statistical inference problems without making stringent structural assumptions on the underlying random process generating the data. From a theoretical standpoint, since its introduction by Efron and Tibshirani (1979), the bootstrap has been applied to a number of statistical problems, including many standard ones, where it has outperformed the existing methodology, as well as to many complex problems where conventional approaches have failed to provide satisfactory answers. Furthermore, we note that the basic bootstrap does not apply equally effectively to every type of random process.

Firstly the question is to determine the bootstrap sample size from the  $k^{\text{th}}$  stratum  $n_k$ , where

$$\sum_{k=1}^K n_k = n$$

and  $n$  is the total sample size. There are a number of different allocation schemes that can be used, as for instance the *equal allocation*, the *proportional allocation*, the *optimal allocation* (Hess et al. 1966). We adopt proportional allocation, where the size of the bootstrap samples drawn from each stratum is taken in proportion to the stratum size. In other words, the following expression defines the  $k^{\text{th}}$  stratum weight as

$$W_k = \frac{N_k}{N} = \frac{n_k}{n} .$$



According to the proportional allocation scheme, the sample is allocated proportionally to the stratum weights, so that

$$W_k = \frac{N_k}{N}$$

and using  $W_k$ , the size of the sample from  $k^{\text{th}}$  stratum is the following:

$$n_k = W_k n$$

and combining the previous formulae, we have:

$$\frac{n_k}{n} = \frac{N_k}{N}.$$

Consider a rectangular mortality data array  $(x_i, d_{x,t})$  comprising the numbers of deaths,  $d_{x,t}$ , modelled as independent Poisson responses in combination with the log-bilinear Lee Carter structure.

We consider the “natural” stratification of the dataset: strata per single ages are considered.

From each cell within each  $k$ -th stratum, we independently generate samples with size  $n_k$  from the Poisson distribution. For each stratum,  $B$  bootstrap frames are generated, drawing from the samples obtained in the previous step. We fit the log-bilinear structure for obtaining the model parameter vector  $\mathcal{g}_b = (\alpha_x^b, \beta_x^b, k_t^b)$  for each frames and then the  $k_t$  's are then projected on the basis of the re-estimated ARIMA model (D'Amato et al. 2010). Note that we do not select a new ARIMA model, instead we retain the model that was selected on the basis of the original data.

The estimate for the total sample mean  $E(y_{SS})$  is defined using the stratified formula as follows:

$$E(y_{SS}) = \sum_{k=1}^K W_k \bar{y}_k \tag{7}$$

where  $\bar{y}_k$  is the sample mean of the  $k^{\text{th}}$  stratum. Its variance is given by:

$$Var(\bar{y}_{SS}) = \sum_{k=1}^K W_k^2 Var(\bar{y}_k) \tag{8}$$

The stratified sampling method is mainly used to reduce population heterogeneity. In particular, stratification means division into groups or strata, where the strata should be formed so that each stratum is as homogeneous as possible. The number of groups (strata) to be considered is determined by the characteristics of the population, for example we can consider strata per age groups. In this way, by splitting the population into homogeneous strata, the procedure under consideration ensures greater accuracy, because of the decrease in the sample variance (D'Amato et al. 2010).

Stratification will achieve greater precision provided that the strata have been chosen so that members of the same stratum are as similar as possible in respect of the

characteristic of interest. Thus, the bigger the differences between the strata, the greater the gain in precision.

Furthermore, the results from each stratum may be of intrinsic interest and can be analysed separately. The proposed technique applied to Poisson Lee Carter simulations allows us to obtain more reliable mortality projections, as shown in D'Amato et al. (2009b).

#### §4. Residuals analysis comparison.

For the validation problem of our proposed SSP method, we have used the classical tools adopted in actuarial science. Dealing with an approach for projecting future mortality trends, we refer to the prevailing literature in this field (Pitacco et al. 2009). In particular, in order to assess the effectiveness of a demographic model, detailed residual analyses have been conducted, pointing out that none of the models considered in the main literature performs well in all sets of tests, and no model performs consistently better than the others (Dowd et al. 2010).

In our case, at the model fitting stage, diagnostic checks on the quality of the fit are accomplished as regards the three different IP, SP, SSP methods.

The goodness of fit is checked by computing and monitoring the residuals:

$$\frac{d_{xt} - \hat{d}_{xt}}{\sqrt{\hat{d}_{xt}}}$$

where

$$\hat{d}_{xt} = e_{xt} \exp(\hat{\alpha}_x + \hat{\beta}_x k_t)$$

In the following three figures we show the results of the graphical analysis of the residuals based on the dataset referred to the Italian male population death rates collected in the period 1950-2006

Figure 1. Residuals for Italian male, IP method, age  $x=0,1,\dots,100$

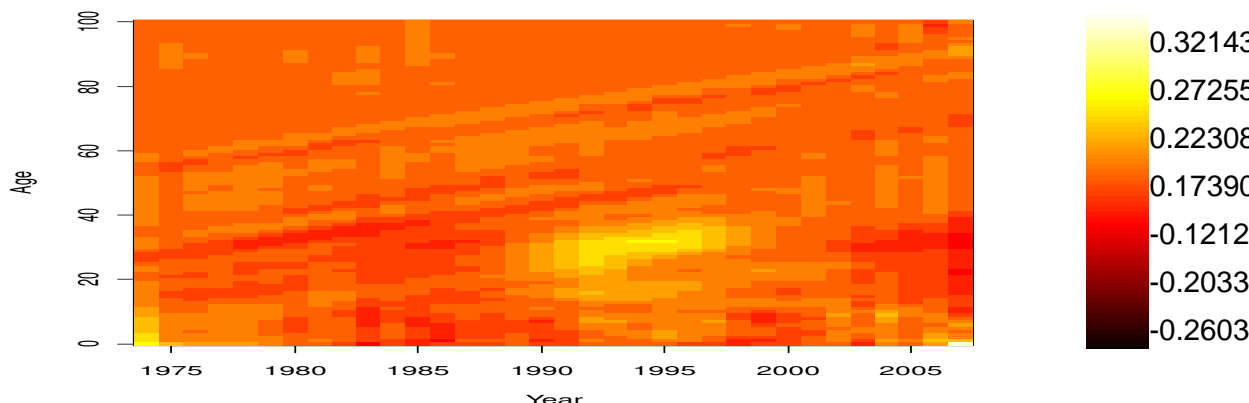


Figure 2. Residuals for Italian male, SP method, age  $x=0,1,\dots,100$

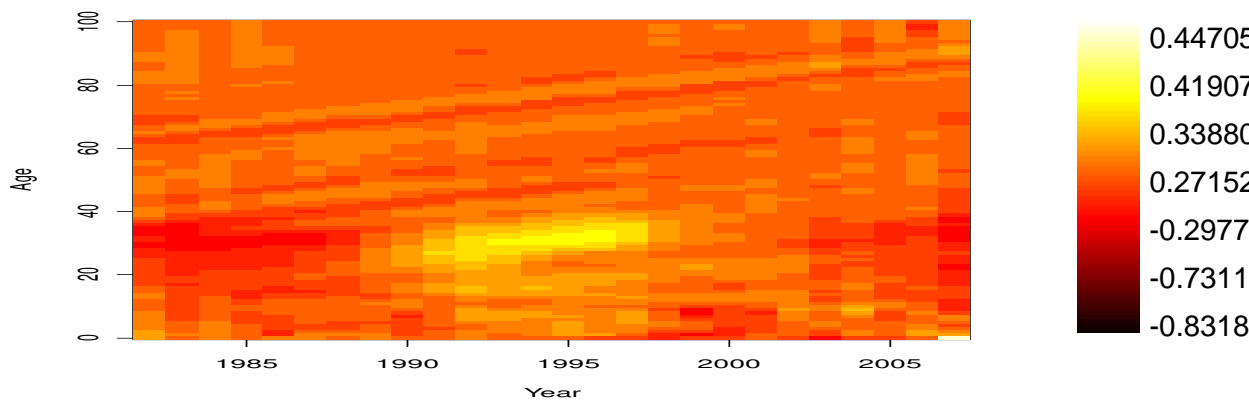
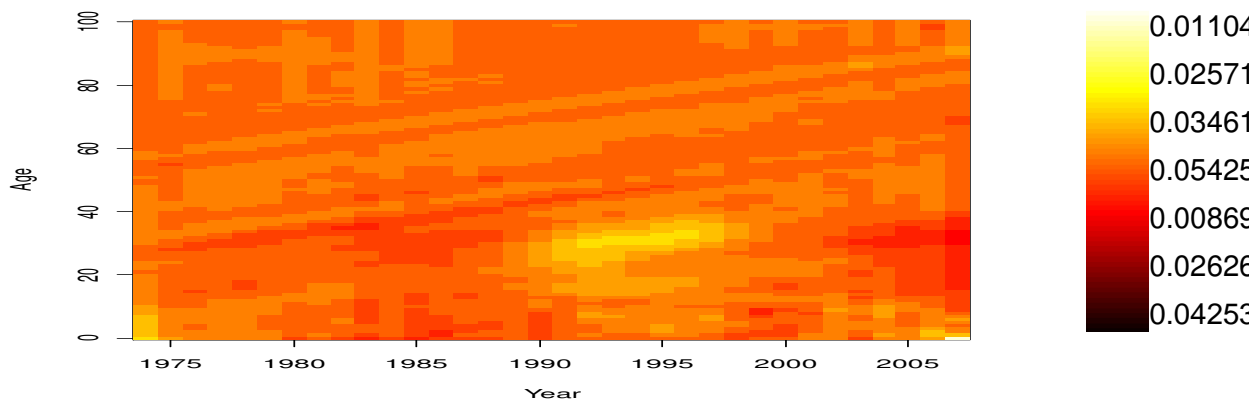


Figure 3. Residuals for Italian male, SSP method, age  $x=0,1,\dots,100$



In particular, the residuals are displayed with the help of these so-called “heat” maps, as a function of both age and calendar time. Looking at the graphical findings, we discover no systematic structure in terms of the residuals in the first and third maps, which ensures that the time trends have been appropriately captured by the IP and the SSP. On the other hand, the second map indicates a pattern in the residuals around the age interval 20-40 between years 1990 and 1998. In particular, a clustering of negative residuals is shown around this interval. Further, we note, in each figure a moderate cohort effect for different generations. We also note that the residuals are less dispersed in the SSP case.

## §5. Numerical Examples.

### 5.1 The case of a pension annuity portfolio.

The first numerical application that we propose concerns the funding ratio values calculated as in formula (1) for a portfolio of deferred annuity contracts under specific demographic and financial assumptions, with particular attention to the values emerging in the case of the SSP.

This application refers to a portfolio of  $c=1000$  deferred annuity contracts issued at age  $x=30$  with anticipated premiums paid during the accumulation period, extending from the issue time to age 65, and anticipated constant instalments  $R=100$  paid from age 65 until the insured dies. As regards the demographic scenario, the Poisson Lee Carter model is considered according to the three different approaches for forecasting mortality rates: the Iterative Procedure (IP) proposed by Renshaw & Haberman (2003c) to optimise the Poisson likelihood through the associated

deviance; the semi-parametric bootstrap (Brouhns et al. 2005) which we will refer to as the Standard Procedure (SP) and the Stratified Sampling Bootstrap (SSP), as proposed in §3.

The dataset that we refer to for estimating the survival probabilities is the Italian male population death rates collected in the period 1950-2006.

For the financial markets assumptions, we adopt the interest rate structure based on the Heath, Jarrow and Morton model (HJM, from here on – Heath et al. 1992). The HJM model describes the dynamics of instantaneous forward-rates which are wholly specified through their instantaneous volatility structures modelled as follows:

$$df(t,T) = \alpha(t,T, f(t,T))dt + \sigma(t,T, f(t,T))dW(t) \quad (9)$$

$$f(0,T) = f^W(0,T)$$

where:

$f^W(0,T)$  is the instantaneous-forward curve observed at time  $t = 0$  in the market

$W = (W_1, W_2, \dots, W_N)$  is an N-dimensional Brownian motion

$\sigma(t,T) = (\sigma_1(t,T), \sigma_2(t,T), \dots, \sigma_N(t,T))$  is a vector of adapted processes for  $\alpha(t,T)$ .

In particular, the simulation experiment has been based on a panel data set referring to EURIBOR and EURIRS over the period January 2002 - March 2009 related to different maturities (0-year, 1 year, 3-year, 5-year, 10-year). For each of these, the corresponding implicit forward rates curves are calculated. In the rich family of HJM term structure models, where the choice of the volatility structure specifies a particular model, it is important to select a model that best describes the particular data set.

On the basis of the principal components analysis (PCA), as in Jarrow (1996), Rebonato (1998), Willmott (1998) and James and Weber (2000), the primary sources of randomness in the forward-rate curve are random changes in the slope of the curve followed by random twists in the curve and the curvature of the yield curve. When these forward rates are discretized by an Euler scheme, the forward rate equation for a three-factor HJM model is as follows:

$$\tilde{f}(t_{i+1}, T) = \tilde{f}(t_i, T) + \alpha(t_i, T)(t_{i+1} - t_i) + \sum_{j=1}^3 \sigma_j(t_i, T)(\tilde{W}_j(t_{i+1}) - \tilde{W}_j(t_i))$$

$$i = 0, 1, \dots, N$$

where  $\tilde{f}(t_i, T)$  is the value of the approximation of  $f(t_i, T)$  at the discretization time  $t_i$ . At time  $t_i$  the drift and the volatility functions of the forward rate are  $\alpha(t_i, T)$  and  $\sigma(t_i, T)$ , respectively. Under the equivalent martingale measure  $\tilde{W}_j(t_i)$  is a Wiener process.

The resulting model can then be used by practitioners, in their calibration procedures. Among the different methods for estimating the volatility parameters, the maximum likelihood estimator has favourable asymptotic properties as shown in Bhar et al. (2002). Under the selected three-factor HJM models, for the different maturities under consideration, the forward rates are simulated. In particular, we perform 10,000 Monte Carlo simulations, by using the previously selected model, with the starting values being the estimated volatilities. The results from the procedure are explained in 5.2.

In this framework, for the specific age at issue  $x=30$ , the funding ratio has been calculated at five different times of valuation ( $t=5, 20, 35, 40, 45$ ) with the aim of providing funding ratio values during the accumulation period ( $0 < t < 35$ ), exactly at the retirement date ( $t=35$ ) and during the annuitization period ( $t > 35$ ).

Table 1 shows the premium amounts for the different mortality forecasting methods, calculated using a constant interest rate of 4%:

**Table 1.** Premium amounts:  $x=30$ , fixed rate 4%.

<b>Survival Approach</b>	<b>Premium</b>
SP	35.83861
IP	36.18761
SSP	37.35852

In Table 2 and in Figure 4, we present the funding ratios calculated according to the three different approaches to modelling survival rates and valued at the five selected times, respectively reported in terms of their numerical values and as histograms.

**Table 2.** Funding Ratio:  $x=30$ , interest rates HJM

<b>Funding Ratio</b>					
<b>Survival Approach</b>	<b>Valuation Time</b>				
	$t=5$	$t=20$	$t=35$	$t=40$	$t=45$
SP	1.1347	1.2757	1.4515	1.6330	1.7936
IP	1.2214	1.3993	1.5538	1.6540	1.8182
SSP	1.2330	1.3879	1.5775	1.6646	1.8616

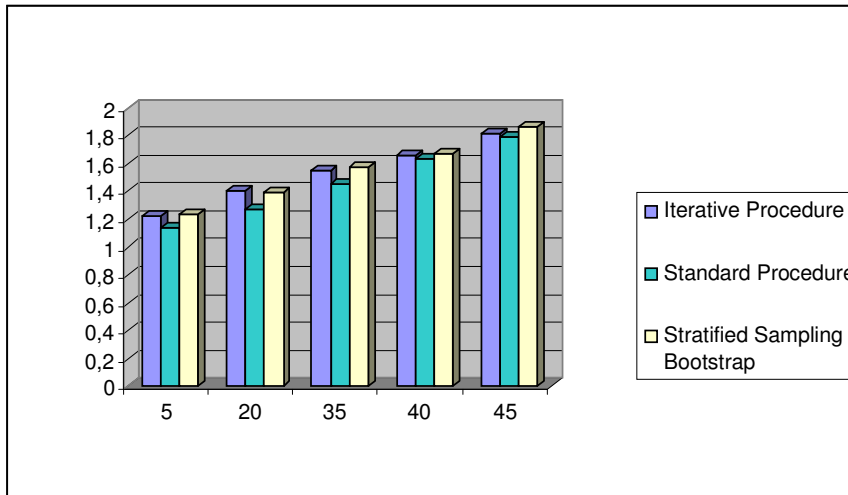
The values in Table 2 follow an increasing trend when the time of valuation increases, for each survival modelling approach. For each time of valuation, the funding ratio takes higher values as we move from the Standard Procedure to the Iterative one and finally to the Stratified Sampling Bootstrap.

In Table 3, we report the changes in the funding ratios with respect to the Standard Procedure in percentage terms.

**Table 3.** % change in funding ratio at time  $t$  relative to the SP approach

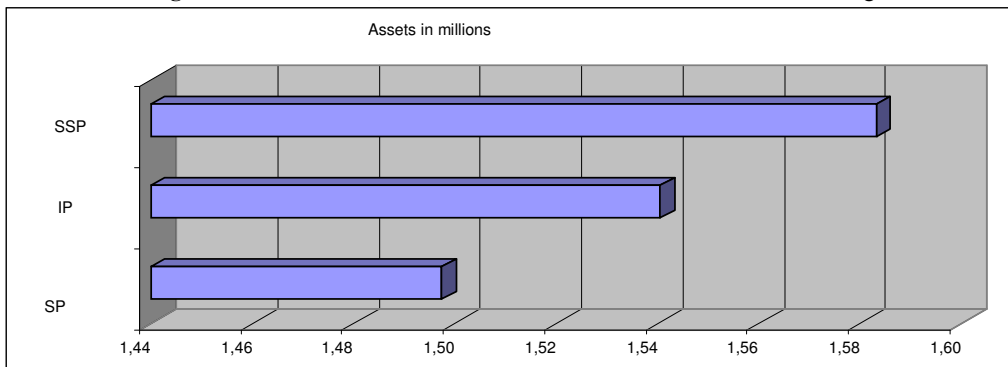
<b>Survival approach</b>	$t=5$	$t=20$	$t=35$	$t=40$	$t=45$
IP	7.64	9.69	7.05	1.29	1.37
SSP	8.66	8.80	8.68	1.94	3.79

**Figure 4.** Funding Ratio:  $x=30$ , Poisson Lee Carter with three different forecasting method,  $t=5, 20, 35, 40, 45$

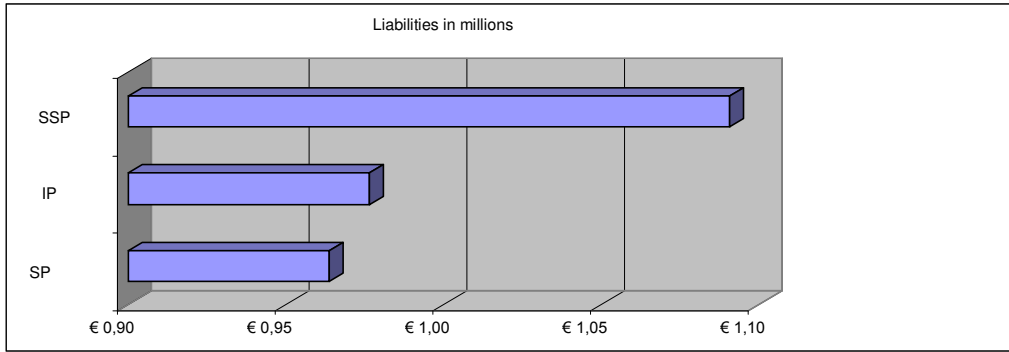


The plots reported in Figures 5 and 6 represent an example of the values of the assets and liabilities calculated at the retirement age  $x=65$  corresponding to the valuation time  $t=35$ :

**Figure 5.** Asset in  $t=35$ , Poisson Lee Carter with three different forecasting method



**Figure 6.** Liabilities in  $t=35$ , Poisson Lee Carter with three different forecasting method



Figures 5 and 6 clearly show how the SSP produces higher values for assets and liabilities with respect to the other two methods.

We next extend the analysis by considering the model risk arising from the choice of the survival function in the financial valuations. As in Di Lorenzo & Sibillo (2002), we measure the effect of this risk component arising from the choice of the “right model” to be used as a random variable assuming different forms (in this case three: IP, SP, SSP) with probabilities based on the subjective reliability that the insurer assigns to each of them on the basis of their own individual performance. We calculate the Demographic Model Risk Measure (DMRM) quantifying the impact of the demographic model risk on the value  $F_t$  of the funding ratio in  $t$ , by means of the following equation (Coppola et al. 2007):

$$DMRM = \sqrt{\text{Var}(E[F_t | K_t])}$$

in which the entity in square brackets is conditioned on the choice of the survival model. The DMRM value is to be interpreted as a standard deviation estimated in a framework of stochastic interest rates, whose effects have been averaged out.

As an example of its application, the DMRM values are now calculated in the case of the funding ratios of the portfolio of pension annuities under consideration. Thus, based on the insurer’s experience, we propose (taking also into account the results of the validation test carried out on the three procedures) assigning the following set of three probabilities 0.6, 0.3 and 0.1 to the choice of respectively SSP, IP and SP and proceed to the calculation of the DMRM at the five valuation times (Table 4).

**Table 4.** DMRM: an example of application with probabilities 0.6, 0.3 and 0.1 to choose respectively SSP, IP and SP.

$t=5$	$t=20$	$t=35$	$t=40$	$t=45$
4.7%	5.3%	6%	3%	2.5%

Table 4 shows that the magnitude of the risk connected to the choice of the demographic model (measured relative to the funding ratios) increases up to age 65 ( $t=35$ ) and then decreases, reaching lower values for higher values of  $t$  (see Di Lorenzo & Sibillo (2002) for a discussion on the trend in this risk measure as a function of valuation time and age at issue).

## 5.2 The case of a Guaranteed Annuity Option

In this section, we extend the analysis using the different survival approaches to more complex annuity contracts, involving embedded options.

There are several common types of pension contracts in the market, that offer a convenient way to convert part or all of an individual's pension savings into a regular income stream in retirement. Here, we consider an example of a complex contract which has become very popular in Europe and US, the so-called guaranteed annuity option (GAO). The GAO standard contract is designed as follows: at time  $t = 0$  the policyholder agrees to pay a single premium  $U_{GAO}$  for an insurance policy maturing at  $T$ . The single premium payment assures a guaranteed benefit  $B_T$  at a specified age reached at  $T$ , which, in this application, is the retirement age 65. The guaranteed benefit may be:

- a cash payment, corresponding to the value of the fund at time  $T$  which arises from investing a part of the premium received at time 0 in a money market or other type of account; or
- an annuity at a guaranteed rate  $g$  payable throughout the policyholder's remaining lifetime.

At time  $T$ , a rational policyholder will exercise the option if this is profitable in terms of the annuity rates prevailing in the market at that time. This means that the choice will be made after a comparison between the market annuity rates and the guaranteed rate  $g$ .

The literature concerning the GAO pricing has developed in several directions. For an overview of the GAO valuation methodology that has been proposed, see Bacinello & Ortu (1993), Bacinello (2001), Bacinello & Persson (2002), Ballotta & Haberman (2003, 2006) and Grasselli & Silla (2009).

As this wide literature shows, the valuation of the GAO is crucial. Its underestimation has caused many solvency problems to life insurance companies (e.g. the failure of Equitable Life in the U.K.): the reductions in market interest rates and the unanticipated decreasing trend in mortality rates have increased the value of the guarantee relative to the cash benefit.

As with other long-term products, GAO's are significantly exposed to unanticipated changes over time in the mortality rates of the reference population and in interest rates. In this respect, the assessment of the fair value of the GAO depends on having available suitable models for longevity risk and for the dynamics of the term structure of interest rates.

In this section, we calculate the funding ratio values as in formula (1) for a GAO contract. The application is referred to a policy issued at age  $x=30$  with a single premium  $U_{GAO}$  paid at  $t=0$  and anticipated constant instalments  $R=100$  paid from age  $x=65$  until the insured dies, if the insured buys the annuity, or otherwise the market value of the fund accumulated from time 0 to time 35, paid at age 65. As regards the demographic scenario, we consider the Poisson Lee Carter model according to the three abovementioned approaches.

The strong dependency of the value of the option at time  $T$  on the comparison between the market interest rates at that time and the guaranteed annuity rate means that it is particularly interesting to study the funding ratios in more than one interest rate stochastic scenario. The application developed in this section provides for three different hypotheses for describing the evolution in time of the term structure of interest rates. We will consider the Vasicek model, the Heath, Jarrow and Morton model (HJM) and the Hull and White model (HW). This approach will allow us to observe the differences in the funding ratio behaviour arising from the use of a model



that is not arbitrage free (Vasicek), in comparison with the arbitrage free ones (HJM and HW), with all calibrated to the current structure of interest rates.

In the ensuing paragraphs, we briefly recall the stochastic differential equations underpinning the three interest rate models, leaving the reader to refer to the literature for an in-depth analysis.

The Vasicek model is one of the most widely used models for bond pricing, the SDE being the following:

$$dr_t = \alpha(\gamma - r_t)dt + \sigma dW_t \quad (10)$$

where  $\{W_t\}$  is a standard Wiener process and  $\alpha, \gamma$  and  $\sigma$  are positive constants. In particular,  $\alpha$  represents the force bringing the process back towards its long term mean  $\gamma$  and  $\sigma^2$  is the constant instantaneous variance, which determines fluctuations around the level  $\gamma$ . In this application the Vasicek rates are obtained from a Monte Carlo simulation.

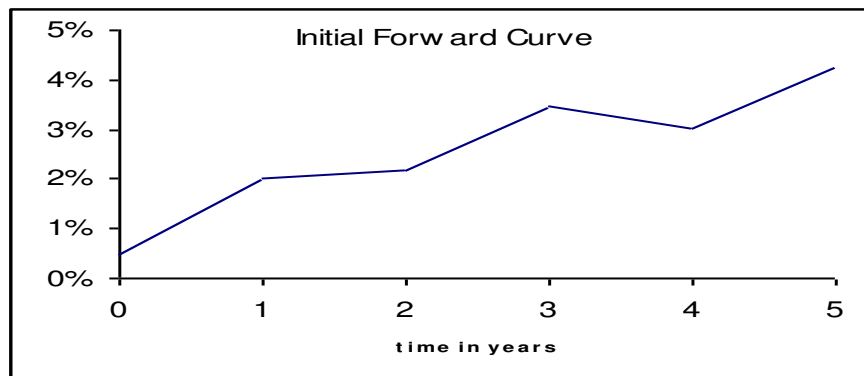
The HW version implemented in the paper is governed by the following SDE:

$$dr = (\theta(t) - ar)dt + \sigma dz \quad (11)$$

where  $\theta(t)$  is chosen to match the current term structure of interest rates and  $z$  is a Wiener process. Although for the HW model in equation (11), analytical solutions for discount bonds and plain-vanilla options exist, here the model has been implemented on the basis of simulation algorithms. In fact, it is sometimes necessary to represent term structure models by numerical approximations. This is the case if we have to deal with path-dependency or more complex payoff functions. In particular, a tree-building procedure is computationally effective for the type of HW model (Hull & White 1994a, 1994b) represented in equation (11). For the numerical procedures we refer to the efficient methods proposed by Kijima & Nagayama (1994). The HJM model has been introduced in section 5.1.

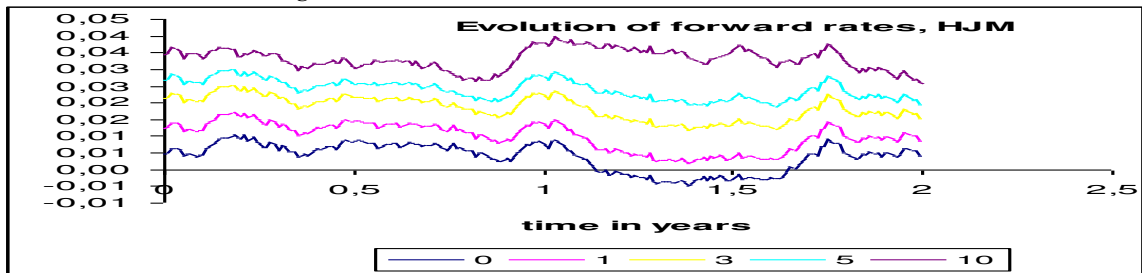
The simulation algorithms have been implemented on the basis of a panel of EURIBOR and EURIRS as at December 2009. The following figures show the evolution of implicit forward rates on the basis of the term structure of spot interest rates:

**Figure 7.** The forward curve of interest rates, Euribor and Eurirs 2009



Figures 8, 9 and 10 show the projected evolution of the interest rates by means of simulation algorithms applied to the three models being considered.

Figure 8. Simulated HJM interest rate curve



In particular, in figure 8, we can visualize the forward curves related to different forward periods (0-year forward that corresponds to the initial forward curve, 1 year forward, 3-year forward, 5-year forward, 10-year forward).

Figures 9 and 10 show the evolution of the yield curves respectively for the HW and the Vasicek models.

Figure 9. Simulated HW interest rates curve

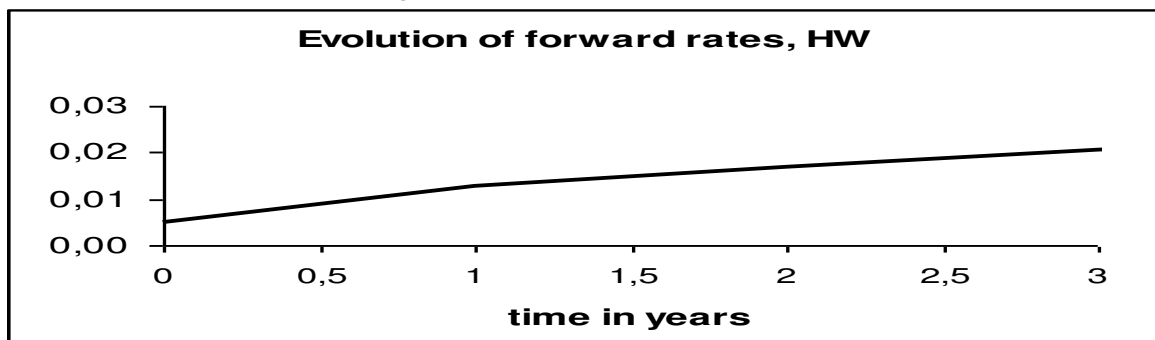
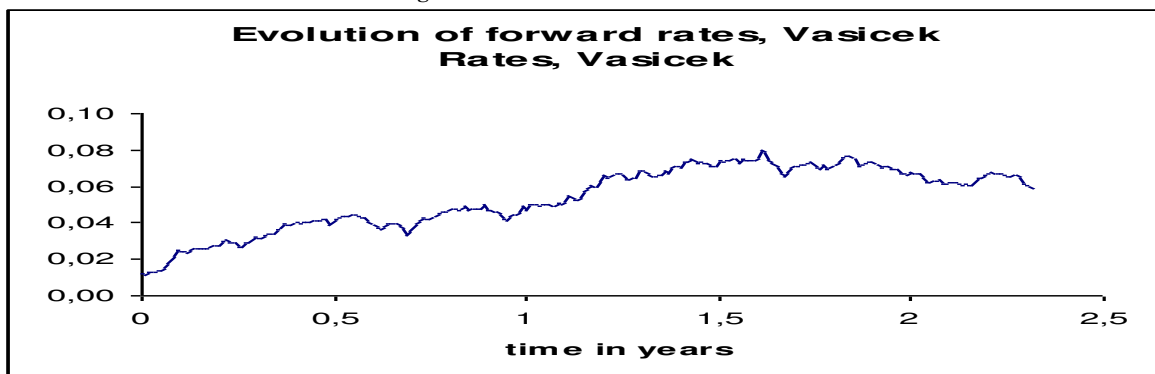


Figure 10. Simulated Vasicek interest rates curve



The above figures 8, 9 and 10 refer to the evolution of forward rates as regards the different interest rate models, and we have decided to represent the obtained forward curves in respect of the same future time horizon: 3 years. The time-step length is a user-configurable run-time parameter that remains constant throughout the simulation model run. Unfortunately, as is well known, the parameter estimates obtained are rather unstable, when changing the time-step length.

As previously mentioned, the actuarial premium collected at time  $0$  by the insurer is the amount involved in the accumulation process for constituting the fund at the maturity date  $T$ .

In this respect, we consider the results for the different GAO prices obtained on the basis of the different stochastic models for interest rates and different approaches to estimating the survival probabilities: these are shown in Table 5. In this application, we have assumed that the amount accumulated for 35 years is a part of the premium  $U_{\text{GAO}}$  that corresponds to the actuarial premium of the life annuity provided by the contract.

**Table 5.** Price of the GAO

	SP	IP	SSP
HJM	89.58	105.88	167.75
HW	86.00	132.29	88.55
Vasicek	43.27	52.59	92.48

We observe a regular increasing trend of the prices in Table 5 when the projection level of the survival scenario increases, confirming the behaviour shown in Ballotta & Haberman (2003), except for the cell HW - IP. The values obtained when assuming a Vasicek model for interest rates are markedly lower than those one coming from the two market consistent models, HW and HJM, except for the case where the survival assumption is not the strongest one (SSP).

As in the case of the traditional deferred annuity treated in section 5.1, here we calculate the funding ratios in the GAO case. The times of valuation are chosen to be in the interval preceding the option maturity  $T$ . Tables 6, 7, 8 show the funding ratios calculated at time  $t=5, 20, 30$  and  $35$ , all belonging to the accumulation period and preceding the option time.

**Table 6.** Funding Ratio:  $x=30$ , HJM interest rates

Survival Approach	Valuation Time			
	$t=5$	$t=20$	$t=30$	$t=35$
IP	0,7858	1,2777	1,0616	0,8968
SP	0,7930	1,2992	1,0772	0,9097
SSP	0,7940	1,2218	1,0091	0,8521

**Table 7.** Funding Ratio:  $x=30$ , HW interest rates

Survival Approach	Valuation Time			
	$t=5$	$t=20$	$t=30$	$t=35$
IP	0,2929	0,7015	0,7319	0,7501
SP	0,3021	0,8149	0,9884	1,1578
SSP	0,3702	0,8945	1,0160	1,8663

**Table 8.** Funding Ratio:  $x=30$ , Vasicek interest rates

Survival Approach	Valuation Time			
	$t=5$	$t=20$	$t=30$	$t=35$
IP	0,3182	0,6473	0,5700	0,7129
SP	0,7878	0,9254	0,7407	0,7018
SSP	0,8891	1,1475	0,9601	1,1489

Tables 6, 7 and 8 show that the funding ratio is more sensitive to the choice of interest rate model, than to the choice of survival model, this being particularly notbale for the Vasicek case. This feature is confirmed by the DMRM calculated in the case of the GAO in Table 9 where we have fixed the interest rate model and varied the survival model. In this case, as in the preceding example, we assign probabilities 0.6, 0.3 and 0.1 for choosing respectively the SSP, the IP or the SP procedure.

**Table 9.** DMRM on Funding Ratio:  $x=30$

Interest rate model	Valuation Time			
	$t=5$	$t=20$	$t=30$	$t=35$
HJM	0.37%	3.06%	2.8%	2.37%
HW	3.68%	8.64%	12.86%	50.93%
Vasicek	25.67%	22.43%	17.65%	21.50%

The demographic model risk is seen to be higher in the Vasicek case and the lowest values are connected to the HJM model for interest rates. The behaviour of the DMRM is generally less regular than for the pension annuity example of section 5.1. The regular increasing trend is apparent in the HW case, with a very high value corresponding to  $t=35$ , and less so in the HJM case.

## Conclusions

The paper deals with the accuracy in mortality projection procedures based on the Lee Carter model. With this aim in mind, we have described the survival dynamics taking into account three different approaches, namely the Iterative Procedure proposed by Renshaw and Haberman, the semi-parametric bootstrap to which we have referred as Standard Procedure, proposed by Brouhns et al. and the proposed procedure named Stratified Sampling Bootstrap. We emphasize that the bootstrap that we propose is quite different from the first version introduced by Efron and Tibshirani (1979). From its genesis until now, different types of bootstrap methods have been applied effectively and we should look at those situations where these methods have run into problems and point out possible remedies, if there is one that is known. The basic principle underlying the bootstrap method in its various settings and different forms is the following: it attempts to recreate the relation between the population and the sample, by considering the sample as an epitome of the underlying population and by re-sampling from it to generate the bootstrap sample.

Noting that there is no unique way of re-sampling, we are looking, in terms of future work, at the further development of our algorithm based on the block bootstrap in the context of dependency (Kunsch 1989).

In order to investigate the SSP performance in comparison with the other two methods, we have used the classical tools adopted in actuarial science. The validation test proposed in the paper has shown the good performance of the SSP approach. Ongoing studies are focused on setting out a framework for evaluating the adequacy of the proposed method, with the initial tests suggesting that the SSP performs better than the other methods. Further in-depth analyses will constitute the subject of a future paper. We have used the three procedures in order to quantify the impact of each of them on the *funding ratio* index, which we have chosen as it is a good indicator of the financial solvency of an insurance portfolio. The funding ratio indicates the degree to which pension liabilities are covered by assets, in other words it measures the relative size of the pension assets compared to the pension liabilities. If the funding ratio is greater than 100%, then the pension fund is overfunded, otherwise it is underfunded or exactly fully funded if it is respectively less than or equal to 100%. It follows that, within the context of an internal model setting, the information from the values of the funding ratio is very valuable for an insurance company seeking to measure its financial strength.

The effect that different survival projection approaches have on the funding ratio values is studied in the applications proposed in the last section of the paper. In interpreting the results, we note that the projected survival rates simultaneously affect the values of the assets and the liabilities. Each of these values sums up the future cash flows underpinning the assets and liabilities, as evaluated at the selected time.

The first illustrative numerical example considers a homogeneous portfolio of traditional pension annuities with anticipated premiums paid during the accumulation period, extending from the issue time to the retirement age, and anticipated constant instalments, paid from the date of retirement until the time of the policyholder's death. The financial scenario is described by the HJM interest rate model.

Making use of forecasting procedures taking into account the longevity trend, i.e. the SP, IP and SSP, we note that the funding ratios are bigger than 1, showing an overfunding situation in the case in study, which improves as time increases.

The Demographic Model Risk Measure shows that the funding ratio is more sensitive to the randomness of the choice of the survival model during the accumulation period, reaching its highest sensitivity at ages around age 65, confirming the results obtained in Di Lorenzo & Sibillo (2002). The DMRM values appear less sensitive to the model choice at older ages.

The second example is related to the Guaranteed Annuity Contract (GAO). In this case, the option at time  $T$  strongly depends on the interest rate model chosen to describe the behaviour of the financial framework. We study the funding ratios in three different financial hypotheses: the Hull and White model, the HJM model and the Vasicek model. Here, the situation is different from the first example, involving the traditional pension annuity. In the HJM case, the funding ratio increases until  $t=20$ , that is insured's age 40, and then decreases, taking values smaller than 1 at the option maturity date. In the HW case, the funding ratio is almost always smaller than 1, assuming highest values at the maturity date, where in some cases (SP and SSP) is bigger than 1. The trend in the Vasicek case is less clear. It is evident that, in the HJM case, the funding ratios are not so sensitive to the choice of the survival projection procedure, but the behaviour of the funding ratio is different in the HW case and particularly in the Vasicek case. The numbers obtained for the DMRM confirm these

observations, showing the highest values in the Vasicek case and the lowest in the HJM case.

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