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Generalized Bogoliubov transformations versus $\mathcal{D}$-pseudo-bosons

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Abstract

We demonstrate that not all generalized Bogoliubov transformations lead to $\mathcal{D}$-pseudo-bosons and prove that a correspondence between the two can only be achieved with the imposition of specific constraints on the parameters defining the transformation. For certain values of the parameters we find that the norms of the vectors in sets of eigenvectors of two related apparently non self-adjoint number-like operators possess different types of asymptotic behavior. We use this result to deduce further that they constitute bases for a Hilbert space, albeit neither of them can form a Riesz base. When the constraints are relaxed they cease to be Hilbert space bases, but remain $\mathcal{D}$-quasi bases.

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1 Introduction

In this manuscript we compare two concepts which have facilitated the study on non-Hermitian quantum systems in recent years, *generalized Bogoliubov transformations* (GBTs) and $\mathcal{D}$-*pseudo-bosons* ($\mathcal{D}$-PBs). Both ideas can be employed to address different aspects of the key questions in the study of non-Hermitian quantum systems: Under which circumstances are the spectra of non-Hermitian Hamiltonians real and what kind of metric needs to be employed to render the system physically meaningful? These issues have been the subject of investigations since the seminal papers [1, 2] on this topic and aspects of them have been answered for many different types of models. The underlying principle of both ideas make use of simple deformations of the canonical commutation relation $[c, c^\dagger] = \mathbb{1}$. This principle restricts, of course, our analysis to a subset of the class of PT-quantum mechanics which have been studied, in recent years, by several authors, see [3, 4] and references therein. However, this subset is rather large and contains several well known systems, [5], like the Swanson model, just to cite one.

Bogoliubov transformations are linear transformations mapping the operators c and $c^\dagger$ to a new canonical pair a and b. They were originally introduced to aid the study of pairing interaction in superconductivity [6] and have been generalized thereafter in various ways, e.g. [7]. When the operators in the new pair are not assumed to be mutually adjoint, i.e. $b^\dagger \neq a$, these maps are usually referred to as generalized Bogoliubov transformations. In the context of the study of non-Hermitian quantum systems they were employed to establish the reality of the spectra of certain non-Hermitian Hamiltonians and to identify well-defined metric operators which map the system to isospectral Hermitian counterparts [8, 9].

Domain issues are often left unaddressed in these constructions. They also constitute interesting mathematical problems in their own right and spectral properties of non self-adjoint operators can be quite intricate, see for example [10, 11] or the more recent volume [5]. One of the mathematical difficulties of non self-adjoint Hamiltonians is related to the eigenvectors φ_n of H and Ψ_n of $H^\dagger$ for $n \geq 0$, if they exist. One can not simply assume that the sets of these eigenvectors constitute biorthogonal bases of the Hilbert space $\mathcal{H}$ on which the models are defined. One needs to verify this property in detail and indeed in various models this assumption has turned out to be incorrect, as discussed in [12]-[15], for instance. In order to understand these aspects in depth a large series of investigations has been carried out in recent years [16, 17, 18, 14, 15] on so-called *pseudo-bosonic* systems [19]. The obtained results were recently reviewed and extended in [5]. In all the explicit examples studied so far, the eigenvectors of $H = ba$ and $H^\dagger = a^\dagger b^\dagger$ are biorthogonal, of course, but they are not bases for $\mathcal{H}$.

However, see below, they still produce a weaker version of the closure relation on some dense subspace $\mathcal{G}$ of $\mathcal{H}$ and are therefore coined $\mathcal{G}$-quasi bases.

At first sight GBTs and PBs appear to be quite similar and the natural question arises under which conditions they might be the same or more specifically: When do GBTs correspond to PBs? We will demonstrate here that the latter only happens under suitable conditions. Thus, in general these two notions are not equivalent. Interestingly, GBTs allow us to find examples of number-like pseudo-bosonic operators whose eigenstates form biorthogonal bases when certain requirements are fulfilled and examples in which this is not true, even if they still provide useful weaker versions of the resolution of the identity. A specific version of the Swanson model discussed in [18] is an example based on GBT which admits no bases.

Our manuscript is organized as follows: In section 2 we recall some definitions and general features on GBTs and PBs relevant for our investigations. In section 3 we analyze in detail under which conditions GBT give rise to PBs. By imposing suitable constraints on the parameters defining the GBT, we show that the eigenstates of the operators $N = ba$ and $N^\dagger$ are indeed biorthogonal, with simple eigenvalues, but they are bases only if these specific constraints are satisfied. The details are contained in section 3.1. In section 3.2 we show that, if these constraints are not satisfied, these eigenstates are $\mathcal{G}$-quasi bases but not bases. We state our conclusions and a further outlook into open problems in section 4.

2 Generalities and definitions

To establish our conventions let us first recall some well-known facts and definitions about GBTs and PBs.

2.1 Generalized Bogoliubov transformations

We consider two operators c and $c^\dagger$ satisfying the canonical commutation relation $[c, c^\dagger] = \mathbb{1}$. Taking for instance $c = \frac{1}{\sqrt{2}}\left(x + \frac{d}{dx}\right)$ and $c^\dagger = \frac{1}{\sqrt{2}}\left(x - \frac{d}{dx}\right)$ is a well-known possible realization of these operators in the Hilbert space $\mathcal{H} = \mathcal{L}^2(\mathbb{R})$ of the square integrable functions on $\mathbb{R}$. We use here the convention $\hbar = 1$ when comparing to a quantum mechanical setting.

GBTs [8, 9] are linear maps defined as

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \beta & -\delta \\ -\alpha & \gamma \end{pmatrix} \begin{pmatrix} c \\ c^\dagger \end{pmatrix}, \tag{1}$$

or more explicitly, the new operators are $a = \beta c - \delta c^\dagger$ and $b = -\alpha c + \gamma c^\dagger$. Here α, β, γ, $\delta \in \mathbb{C}$ and to ensure that $[a, b] = \mathbb{1}$ we require $\det(T) = \beta\gamma - \alpha\delta = 1$, where T is the two-by-two matrix in the defining relation for the transformation (1). In addition, we restrict here to the choices of α, β, γ and δ such that $b \neq a^\dagger$. Since $\det(T) = 1 \neq 0$, the inverse GBT always exists

$$T^{-1} = \begin{pmatrix} \gamma & \delta \\ \alpha & \beta \end{pmatrix}. \tag{2}$$

PBs are very similar objects as the operators a and b produced by the GBT. In fact, they were formally (i.e., with no care on the domains of the unbounded operators involved in their framework, as well as on other mathematical subtleties) introduced in [19], and then in a more rigorous way in [16], by taking the commutation relation $[a, b] = \mathbb{1}$ between two different operators a and b densely defined on a Hilbert space $\mathcal{H}$ as the primary object and it was found that when $b \neq a^\dagger$ interesting situations arose. When one wishes to apply these operators in building non-Hermitian quantum mechanical models further constraints are needed. To allow the usage of a and b for the construction of two families of biorthogonal vectors of $\mathcal{H}$ some specific scenarios were dealt with in [16]. Also when considering non-Hermitian models constructed from $\mathcal{PT}$-invariant combinations of c and $c^\dagger$, the requirement that they can be mapped by means of GBT into a form of a harmonic oscillator plus a Casimir operator [9] imposes further constraints on the constants α, β, γ, δ.

Next we recall some definitions and relevant facts for reference about PBs.

2.2 Pseudo-bosons

Definition $\mathcal{D}$-PB: *The pair of operators a and b are called $\mathcal{D}$-pseudo-bosons ($\mathcal{D}$-PB) if, for all $f \in \mathcal{D}$, we have*

$$a\, b\, f - b\, a\, f = f. \tag{3}$$

The domain $\mathcal{D}$ is a dense subspace of a Hilbert space $\mathcal{H}$ stable under the action of a, b, $a^\dagger$ and $b^\dagger$, that is $a^\sharp\, \mathcal{D} \subseteq \mathcal{D}$ and $b^\sharp\, \mathcal{D} \subseteq \mathcal{D}$, where $x^\sharp$ is x or $x^\dagger$.

Note that since $a^\sharp f$ is well defined and belongs to $\mathcal{D}$ for all $f \in \mathcal{D}$, it is clear that $\mathcal{D} \subseteq D(a^\sharp)$, the domain of the operator $a^\sharp$. The analogue holds also for $\mathcal{D} \subseteq D(b^\sharp)$. We often use a simplified notation and instead of (3) we only write $[a, b] = \mathbb{1}$, where, as before, $\mathbb{1}$ is the identity operator on $\mathcal{H}$, keeping in mind that both sides of this equation have to act on a certain $f \in \mathcal{D}$.

In addition we assume:

Assumption $\mathcal{D}$-PB 1: There exists a non-zero $\varphi_0 \in \mathcal{D}$ such that $a\, \varphi_0 = 0$.

Assumption $\mathcal{D}$-PB 2: There exists a non-zero $\Psi_0 \in \mathcal{D}$ such that $b^\dagger \Psi_0 = 0$.
Assumption $\mathcal{D}$-PB 3: The set $\mathcal{F}_\varphi := \{\varphi_n,\, n \geq 0\}$ is a basis for $\mathcal{H}$.

Note that the vectors

$$\varphi_n := \frac{1}{\sqrt{n!}}\, b^n \varphi_0, \qquad \Psi_n := \frac{1}{\sqrt{n!}}\, {a^\dagger}^n \Psi_0, \tag{4}$$

are well-defined for $n \geq 0$, since $\mathcal{D}$ is stable under the action of b and $a^\dagger$. In particular, it is obvious that $\varphi_0 \in D^\infty(b) := \cap_{k\geq 0} D(b^k)$ and that $\Psi_0 \in D^\infty(a^\dagger)$. Therefore we can introduce the sets $\mathcal{F}_\varphi$ and in addition $\mathcal{F}_\Psi = \{\Psi_n,\, n \geq 0\}$. By the same reasoning we also deduce that each φ_n and each Ψ_n belongs to $\mathcal{D}$ and therefore to the domains of $a^\sharp$, $b^\sharp$ and $N^\sharp$, where $N := ba$, $N^\dagger = a^\dagger b^\dagger$.

The following lowering and raising relations are then easily obtained

$$\begin{array}{lll} a\varphi_n = \sqrt{n}\,\varphi_{n-1},\ a\,\varphi_0 = 0, & b^\dagger\Psi_n = \sqrt{n}\,\Psi_{n-1},\ b^\dagger\Psi_0 = 0, & \text{for } n \geq 1, \\ a^\dagger\Psi_n = \sqrt{n+1}\,\Psi_{n+1}, & b\,\varphi_n = \sqrt{n+1}\,\varphi_{n+1}, & \text{for } n \geq 0, \end{array} \tag{5}$$

as well as the eigenvalue equations $N\varphi_n = n\varphi_n$ and $N^\dagger\Psi_n = n\Psi_n$ for $n \geq 0$. Notice that all the eigenvalues are simple. As a consequence of these last equations we derive that

$$\langle \varphi_n, \Psi_m \rangle = \delta_{n,m}, \tag{6}$$

for all $n, m \geq 0$, when choosing the normalization of φ_0 and Ψ_0 in such a way that $\langle \varphi_0, \Psi_0 \rangle = 1$. Then $\mathcal{F}_\varphi$ and $\mathcal{F}_\Psi$ are biorthonormal sets of eigenstates of N and $N^\dagger$, respectively.

The assumptions $\mathcal{D}$-PB 1 and $\mathcal{D}$-PB 2 do in principle not allow to conclude anything about the fact that $\mathcal{F}_\varphi$ and $\mathcal{F}_\Psi$ are also bases for $\mathcal{H}$, or even whether they are Riesz bases, which is the reason for making the assumption $\mathcal{D}$-PB 3. Notice that it automatically implies that $\mathcal{F}_\Psi$ is a basis for $\mathcal{H}$ as well [20]. However, during the years several examples in which this natural assumption is not satisfied have been found, see for instance [5] and references therein. For this reason a weaker version of assumption $\mathcal{D}$-PB 3 was introduced in [17].
Assumption $\mathcal{D}$-PBw 3: $\mathcal{F}_\varphi$ and $\mathcal{F}_\Psi$ are $\mathcal{G}$-*quasi bases* for some subspace $\mathcal{G}$ dense in $\mathcal{H}$.

Two biorthogonal sets $\mathcal{F}_\eta = \{\eta_n \in \mathcal{G},\, g \geq 0\}$ and $\mathcal{F}_\Phi = \{\Phi_n \in \mathcal{G},\, g \geq 0\}$ have been called $\mathcal{G}$-quasi bases when

$$\langle f, g \rangle = \sum_{n\geq 0} \langle f, \eta_n \rangle \langle \Phi_n, g \rangle = \sum_{n\geq 0} \langle f, \Phi_n \rangle \langle \eta_n, g \rangle\,, \tag{7}$$

holds for all $f, g \in \mathcal{G}$. It is clear that assumption $\mathcal{D}$-PB 3 implies its weaker version (7), but the reverse can not be inferred. However, when $\mathcal{F}_\eta$ and $\mathcal{F}_\Phi$ satisfy the relation (7) we still have some (weak) form of resolution of the identity and from a physical point of view we are still able to deduce interesting results, [17]. Incidentally we see that if $f \in \mathcal{G}$ is orthogonal to all the Φ_n's (or to all the η_n's), then f is necessarily zero. Hence, both $\mathcal{F}_\eta$ and $\mathcal{F}_\Phi$ are automatically complete in $\mathcal{G}$. For further results on $\mathcal{G}$-quasi bases we refer the reader to [17], where a discussion can be found in which sense these bases extend Riesz biorthogonal bases and additional results on the mathematical structure arising out of a, b and the various vectors introduced so far.

Here we will be mainly interested in demonstrating the interesting fact that depending on the choices of parameters involved in the GTBs they provide examples in which $\mathcal{F}_\varphi$ and $\mathcal{F}_\Psi$ are indeed bases such that assumption $\mathcal{D}$-PB 3 holds as well as examples in which they are just $\mathcal{G}$-quasi bases for some dense $\mathcal{G} \subset \mathcal{H}$, so that assumption $\mathcal{D}$-PBw 3 holds while assumption $\mathcal{D}$-PB 3 does not.

3 GBTs versus PBs

We are now in the position to address the following questions: *does a GBT always produce $\mathcal{D}$-PBs?* or more specifically *when does a GBT produce $\mathcal{D}$-PBs?* In fact, the first question can already be answered negatively by several simple counter examples. For instance, already in [19, 16], it was shown that specific operators of the form (1) satisfying $[a, b] = \mathbb{1}$ need not be pseudo-bosonic. Let us now make this more evident by treating the general case. We adopt the explicit realization of c and $c^\dagger$ quoted at the beginning of subsection 2.1. Hence we obtain

$$a = \frac{1}{\sqrt{2}}\left[(\beta-\delta)x + (\beta+\delta)\frac{d}{dx}\right], \qquad b = \frac{1}{\sqrt{2}}\left[(\gamma-\alpha)x - (\gamma+\alpha)\frac{d}{dx}\right], \tag{8}$$

$$a^\dagger = \frac{1}{\sqrt{2}}\left[(\overline{\beta}-\overline{\delta})x - (\overline{\beta}+\overline{\delta})\frac{d}{dx}\right], \qquad b^\dagger = \frac{1}{\sqrt{2}}\left[(\overline{\gamma}-\overline{\alpha})x + (\overline{\gamma}+\overline{\alpha})\frac{d}{dx}\right]. \tag{9}$$

We can easily verify whether the assumptions $\mathcal{D}$-PB 1 and $\mathcal{D}$-PB 2 are satisfied. In particular, the equations for the ground states $a\varphi_0(x) = 0$ and $b^\dagger\Psi_0(x) = 0$ admit the solutions

$$\varphi_0(x) = N_\varphi e^{-\frac{1}{2}x^2\frac{\beta-\delta}{\beta+\delta}}, \qquad \Psi_0(x) = N_\Psi e^{-\frac{1}{2}x^2\frac{\overline{\gamma}-\overline{\alpha}}{\overline{\gamma}+\overline{\alpha}}}, \tag{10}$$

where N_φ and N_Ψ are suitable normalization constants to be specified further below. The first crucial point to note is that these two functions do not always belong to $\mathcal{H} = \mathcal{L}^2(\mathbb{R})$. This is only true when the following constraints on the parameters in T are satisfied:

$$\Re\left(\frac{\beta-\delta}{\beta+\delta}\right) > 0, \qquad \Re\left(\frac{\gamma-\alpha}{\gamma+\alpha}\right) > 0. \tag{11}$$

Is is evident that (11) is distinct from the necessary condition $\det(T) = 1$. It is easily seen that assuming the latter we can produce all possible scenarios a) $\varphi_0(x) \in \mathcal{H}$, $\Psi_0(x) \notin \mathcal{H}$, b) $\varphi_0(x) \notin \mathcal{H}$, $\Psi_0(x) \in \mathcal{H}$, c) $\varphi_0(x)$, $\Psi_0(x) \notin \mathcal{H}$ and d) $\varphi_0(x)$, $\Psi_0(x) \in \mathcal{H}$. Explicit examples for parameter choices for these cases are for instance a) $\alpha = \gamma = \delta = 1$, $\beta = 2$, b) $\alpha = \beta = \delta = 1$, $\gamma = 2$, c) $\alpha = -3/2$, $\beta = 1/4$, $\gamma = 1$, $\delta = 1/2$ and d) $\alpha = 2/3$, $\beta = 2$, $\gamma = 1$, $\delta = 3/2$. Thus on the basis of assumptions $\mathcal{D}$-PB 1 and $\mathcal{D}$-PB 2 alone we conclude already that the GBT described by the cases a), b) and c) can not be $\mathcal{D}$-PB. However, the case d) demonstrates that we have GBTs that might also produce $\mathcal{D}$-PBs. Thus we need to verify whether the remaining assumption $\mathcal{D}$-PB 3 in section 2.2 also holds for those cases and of course we also need to fix $\mathcal{D}$.

Thus the next step is to compute

$$\varphi_n(x) = \frac{1}{\sqrt{n!}} b^n \varphi_0(x), \qquad \Psi_n(x) = \frac{1}{\sqrt{n!}} {a^\dagger}^n \Psi_0(x), \tag{12}$$

for $n \geq 0$. It suffices to determine the expression for $\varphi_n(x)$ as those for $\Psi_n(x)$ can be obtained from the former simply by replacing δ with $\overline{\alpha}$ and β with $\overline{\gamma}$ when noting that b and $\Psi_0(x)/N_\Psi$ algebraically coincide with $a^\dagger$ and $\varphi_0(x)/N_\varphi$, respectively after this requirements. Moreover, with the use of condition (11) we can fix the value for the product of $\overline{N}_\varphi$ and N_Ψ

$$\langle \Psi_0, \varphi_0 \rangle = 1 \quad \Rightarrow \quad \overline{N}_\varphi N_\Psi = \frac{1}{\sqrt{\pi(\alpha+\gamma)(\beta+\delta)}}. \tag{13}$$

From the definition (12) and (10) it is easy to see that

$$\begin{aligned} \varphi_n(x) &= \frac{N_\varphi}{\sqrt{n! 2^n}} \left[(\gamma - \alpha)\, x - (\gamma + \alpha) \frac{d}{dx} \right]^n e^{-\frac{1}{2} x^2 \frac{\beta - \delta}{\beta + \delta}}, & (14) \\ &= \frac{N_\varphi}{\sqrt{n! 2^n}} \left(\frac{\alpha + \gamma}{\beta + \delta} \right)^{n/2} H_n \left[\frac{x}{\sqrt{(\alpha+\gamma)(\beta+\delta)}} \right] e^{-\frac{1}{2} x^2 \frac{\beta - \delta}{\beta + \delta}}, & (15) \end{aligned}$$

for all $n \geq 0$, with $H_n(x)$ denoting the n-th Hermite polynomial. The constraint needed for these functions to be square integrable is the same as (11), since they are simply polynomials times the same Gaussian that already appeared in $\varphi_0(x)$. The functions $\Psi_n(x)$ are then readily deduced by using the aforementioned replacement rule

$$\Psi_n(x) = \frac{N_\Psi}{\sqrt{n! 2^n}} \left(\frac{\overline{\beta} + \overline{\delta}}{\overline{\alpha} + \overline{\gamma}} \right)^{n/2} H_n \left[\frac{x}{\sqrt{(\overline{\alpha}+\overline{\gamma})(\overline{\beta}+\overline{\delta})}} \right] e^{-\frac{1}{2} x^2 \frac{\overline{\gamma} - \overline{\alpha}}{\overline{\gamma} + \overline{\alpha}}}. \tag{16}$$

We have now constructed our sets $\mathcal{F}_\varphi$ and $\mathcal{F}_\Psi$.

A special case of this general treatment was previously discussed in [18], where the pseudo-bosonic operators a and b were denoted as $A_\theta = \cos\theta\, c + i\sin\theta\, c^\dagger$ and $B_\theta = \cos\theta\, c^\dagger + i\sin\theta\, c$, depending on a real parameter $\theta \in I := (-\frac{\pi}{4}, \frac{\pi}{4}) \setminus \{0\}$. Thus the parameters in T are identified as $\beta = \gamma = \cos\theta$ and $\delta = \alpha = -i\sin\theta$, clearly satisfying $\det(T) = 1$ and the two constraints in (11) equal each other reducing to $\Re\left(e^{2i\theta}\right) = \cos(2\theta) > 0$ for $\theta \in I$. Furthermore, the general solutions (15) and (16) for $\varphi_n(x)$ and $\Psi_n(x)$ simplify to

$$\varphi_n^\theta(x) = \frac{N_\varphi}{\sqrt{2^n\, n!}}\, H_n\left(e^{i\theta}x\right) \exp\left[-\frac{1}{2}\, e^{2i\theta}\, x^2\right] \qquad \Psi_n^\theta(x) = \frac{N_\Psi}{N_\varphi}\, \varphi_n^{-\theta}(x). \tag{17}$$

constructed in [18] for the specific choice of parameters as given above.

A direct computation shows that the two sets of functions $\mathcal{F}_\varphi$ and $\mathcal{F}_\Psi$ are indeed biorthogonal sets satisfying (7). In previous analysis on $\mathcal{D}$-PBs [5] a particular relevant role was played by the norms of φ_n and Ψ_n. For several concrete models it has been proved that $\lim_{n\to\infty} \|\varphi_n\| = \lim_{n\to\infty} \|\Psi_n\| = \infty$, which is enough to conclude that $\mathcal{F}_\varphi$ and $\mathcal{F}_\Psi$ do not constitute bases for $\mathcal{H}$, [11], Lemma 3.3.3. However, this property does not exclude the possibility that they are $\mathcal{D}$-quasi bases for some dense subspace $\mathcal{D}$ in $\mathcal{H}$. Thus we will next compute those limits in order to be able to decide whether we encounter $\mathcal{D}$-PB or no PB at all and for which choices of the parameters in T any of these situations might occur.

We proceed by imposing some constraints on T rather than considering the complete generic case and compute $\|\varphi_n\|$ together with the appropriate limit. This will make computations transparent at first. Subsequently we investigate the consequences of relaxing some of the constraints. In this manner we obtain unexpected and interesting conclusions about the sets $\mathcal{F}_\varphi$ and $\mathcal{F}_\Psi$.

3.1 Real valued GBT with constraint $\alpha\beta = \gamma\delta$

We assume here that T is real valued and its parameters are ordered as $\beta > \delta > 0$ and that $\gamma > \alpha > 0$. This choice guarantees that the constraints in (11) are automatically satisfied, whereas $\det(T) = 1$ must still be imposed. An explicit example for this choice of the parameters is case d) provided after (11). To compute $\|\varphi_n\|$ we use the general formula

$$\int_0^\infty e^{-px^2}\, H_n(bx)\, H_n(cx)\, dx = \frac{2^{n-1}\, n!\, \sqrt{\pi}}{p^{(n+1)/2}}\, (b^2 + c^2 - p)^{n/2}\, P_n\left(\frac{bc}{\sqrt{p(b^2+c^2-p)}}\right), \tag{18}$$

which holds for all p with positive real part [21] and $P_n(x)$ denotes the n-th Legendre polynomial. From (18) and (15) follows

$$\|\varphi_n\|^2 = \frac{\sqrt{\pi}|N_\varphi|^2}{2}\left(\frac{\alpha+\gamma}{\beta+\delta}\right)^n\left(\frac{\beta+\delta}{\beta-\delta}\right)^{\frac{n+1}{2}}\left(\frac{\gamma-\alpha}{\gamma+\alpha}\right)^{\frac{n}{2}} P_n\left(\frac{1}{\sqrt{(\beta^2-\delta^2)(\gamma^2-\alpha^2)}}\right). \tag{19}$$

We observe that $(\beta^2-\delta^2)(\gamma^2-\alpha^2) = 1-(\alpha\beta-\gamma\delta)^2$, which implies that the argument of the Legendre polynomial is always greater or equal to one. It is suggestive to take, to begin with, $\alpha\beta = \gamma\delta$ as for that choice the expression in (19) simplifies drastically due to the fact that $P_n(1) = 1$ for all n. Thus (19) collapses to

$$\|\varphi_n\|^2 = \frac{\sqrt{\pi}|N_\varphi|^2}{2}\sqrt{\frac{\beta+\delta}{\beta-\delta}}\left(\frac{\gamma}{\beta}\right)^n, \tag{20}$$

and by using the aforementioned replacement rule we also obtain

$$\|\Psi_n\|^2 = \frac{\sqrt{\pi}|N_\Psi|^2}{2}\sqrt{\frac{\gamma+\alpha}{\gamma-\alpha}}\left(\frac{\beta}{\gamma}\right)^n. \tag{21}$$

Then the conclusions are clear. We distinguish three cases:

$$\begin{array}{lll} \gamma=\beta: & \lim_{n\to\infty}\|\varphi_n\| = \text{const}, & \lim_{n\to\infty}\|\Psi_n\| = \text{const}, \\ \gamma>\beta: & \lim_{n\to\infty}\|\varphi_n\| = \infty, & \lim_{n\to\infty}\|\Psi_n\| = 0, \\ \gamma<\beta: & \lim_{n\to\infty}\|\varphi_n\| = 0, & \lim_{n\to\infty}\|\Psi_n\| = \infty. \end{array} \tag{22}$$

For $\gamma = \beta$ we simply have $\varphi_n(x) = \Psi_n(x)N_\varphi/N_\Psi$, since this choice also implies $\alpha = \delta$ and therefore the GBT reduces to the standard Bogoliubov transformation with $a = b^\dagger$. The two sets $\mathcal{F}_\varphi$ and $\mathcal{F}_\Psi$ essentially simply collapse.

The situation $\gamma \neq \beta$ is more interesting. The fact that the norms of the elements in $\mathcal{F}_\varphi$ and $\mathcal{F}_\Psi$ behave differently in the large n asymptotic limit constitutes a new result when compared with the many examples previously considered in the literature. For instance, for the special case of the Swanson model, dealt with in [18], it was found that both norms diverge $\lim_{n\to\infty}\|\varphi_n\| = \lim_{n\to\infty}\|\Psi_n\| = \infty$.

The first clear conclusion is: *Neither $\mathcal{F}_\varphi$ nor $\mathcal{F}_\Psi$ can be Riesz bases when $\gamma \neq \beta$.* The reason is simple. For definiteness we take $\gamma > \beta$. Then, since $\lim_{n\to\infty}\|\Psi_n\| = 0$ one may imagine the existence of an bounded operator V and an orthonormal basis $\{e_n\}$ of $\mathcal{H}$ such that $\Psi_n = Ve_n$. However, due to the uniqueness of the biorthogonal basis, we must necessarily have $\varphi_n = (V^{-1})^\dagger e_n$. Now, since $\lim_{n\to\infty}\|\varphi_n\| = \infty$, the operator V^{-1} cannot be bounded, which in turn implies our statement[1].

[1] We recall that in the definition of a Riesz basis the operator V is required to be invertible.

Our second conclusion is: *$\mathcal{F}_\varphi$ and $\mathcal{F}_\Psi$ constitute two biorthogonal bases for $\mathcal{H}$.* Notice that this, of course, is not in contrast with the fact that they are not Riesz bases. The reason is because $\|\varphi_n\|$ and $\|\Psi_n\|$ have a different asymptotic behavior, so that $\|\varphi_n\|\|\Psi_n\|$ is uniformly bounded in n, as required in [11]. In fact, our second conclusion is not difficult to prove. Using $\det(T)=1$ and our constraint $\alpha\beta=\gamma\delta$ we can eliminate α and γ from the expressions $\varphi_n(x)$ in (15) and $\Psi_n(x)$ in (16) and express them entirely in terms of the parameters β and δ

$$\varphi_n(x)=\frac{N_\varphi}{\sqrt{2^n\,n!}}\left(\frac{1}{\beta^2-\delta^2}\right)^{n/2}H_n\left(\sqrt{\frac{\beta-\delta}{\beta+\delta}}\,x\right)e^{-\frac{1}{2}x^2\frac{\beta-\delta}{\beta+\delta}} \tag{23}$$

and

$$\Psi_n(x)=\frac{N_\Psi}{\sqrt{2^n\,n!}}\left(\beta^2-\delta^2\right)^{n/2}H_n\left(\sqrt{\frac{\beta-\delta}{\beta+\delta}}\,x\right)e^{-\frac{1}{2}x^2\frac{\beta-\delta}{\beta+\delta}}, \tag{24}$$

with $\overline{N}_\varphi N_\Psi=\sqrt{(\beta-\delta)/(\beta+\delta)\pi}$. Note that by our initial assumption the arguments of all the square roots are positive. Then we have $\Psi_n(x)=(\beta^2-\delta^2)^n\,\varphi_n(x)N_\Psi/N_\varphi$ for all $n\geq 0$, i.e. the two sets only differ by a constant, albeit n-dependent, factor. Taking now a generic function $f(x)\in\mathcal{L}^2(\mathbb{R})$ simple manipulations show that

$$\sum_{n=0}^{\infty}\langle\varphi_n,f\rangle\,\Psi_n(x)=\sum_{n=0}^{\infty}\frac{\overline{N}_\varphi N_\Psi\sqrt{\pi}}{\mu}\left(\int_{\mathbb{R}}e_n(s)f_\mu(s)ds\right)e_n(t), \tag{25}$$

where we have introduced the positive quantity $\mu=\sqrt{(\beta-\delta)/(\beta+\delta)}$, the variable $t=x\mu$, the shorthand notation $f_\mu(s)=f\left(s/\mu\right)$ and the function $e_n(s)=H_n(s)e^{-\frac{1}{2}s^2}/\sqrt{2^n\,n!\sqrt{\pi}}$, which all together (i.e. for $n=0,1,2,\ldots$) form an orthonormal basis for $\mathcal{L}^2(\mathbb{R})$. This implies that

$$\sum_{n=0}^{\infty}\langle\varphi_n,f\rangle\,\Psi_n(x)=f_\mu(t)=f(x), \tag{26}$$

which is what we had to prove. Therefore $\mathcal{F}_\Psi$ is a basis. Analogously, we can show that $\mathcal{F}_\varphi$ is a basis too. Moreover, they are clearly both $\mathcal{H}$-quasi bases.

Remark: This is not very different from what happens if we start with a generic orthonormal basis $\mathcal{E}=\{e_n\}$ of $\mathcal{H}$ and construct out of it two sets $\mathcal{F}_\varphi=\{\varphi_n=\lambda_n e_n\}$ and $\mathcal{F}_\Psi=\{\Psi_n=\lambda_n^{-1}e_n\}$, using a sequence $\{\lambda_n\}$ of non zero complex numbers. Then, if for instance $|\lambda_n|\leq M$, for some $0<M<\infty$, with divergent λ_n^{-1}, it is clear that: (i) $\mathcal{F}_\varphi$ and $\mathcal{F}_\Psi$ are not Riesz bases, (ii) They are biorthogonal and they are both bases for $\mathcal{H}$ and (iii) they are both $\mathcal{H}$-quasi bases.

It is easy to understand why, when $\det(T)=1$ and $\alpha\beta=\gamma\delta$, the sets $\mathcal{F}_\varphi$ and $\mathcal{F}_\Psi$ essentially collapse and became bases for $\mathcal{H}$. The reason is that, under these assumptions on the coefficients, the number operator $N=ba$ is self-adjoint[2]. In fact, we find that

$$N=\frac{1}{\beta^2-\delta^2}\,\left(\beta c-\delta c^\dagger\right)^\dagger\left(\beta c-\delta c^\dagger\right).$$

So, we are just working with a sort of rescaled self-adjoint harmonic oscillator. This will not be so in the next Section, where something completely different will be deduced.

3.2 Removing constraint $\alpha\beta=\gamma\delta$

From the previous subsection it appears at first sight that the constraint $\alpha\beta=\gamma\delta$ only facilitated our computations. We will now demonstrate that it actually describes a very special situation and when it is relaxed the properties of our PBs change severely, i.e. we find that *neither $\mathcal{F}_\varphi$ nor $\mathcal{F}_\Psi$ constitute bases for $\mathcal{H}$ when the coefficients in the GBT are such that $\alpha\beta\neq\gamma\delta$.* The proof of this claim makes use of the fact that in this case the argument of the Legendre polynomial in (19) is always larger or equal to one. Then, to deduce the asymptotic behavior of $\|\varphi_n\|$, where $\varphi_n(x)$ are now those in (15), for large n we can employ the following formula, see [22],

$$P_n(x)\simeq\frac{1}{\sqrt{2\pi n}}\,\frac{1}{(x^2-1)^{1/4}}\left\{x+\sqrt{x^2-1}\right\}^{n+1/2}, \tag{27}$$

which holds if $x>1$. The asymptotic behavior of (19) and the analogous one for Ψ_n

$$\|\Psi_n\|^2=\frac{\sqrt{\pi}|N_\Psi|^2}{2}\left(\frac{\delta+\beta}{\gamma+\alpha}\right)^n\left(\frac{\gamma+\alpha}{\gamma-\alpha}\right)^{\frac{n+1}{2}}\left(\frac{\beta-\delta}{\beta+\delta}\right)^{\frac{n}{2}}P_n\left(\frac{1}{\sqrt{(\beta^2-\delta^2)(\gamma^2-\alpha^2)}}\right), \tag{28}$$

are described by

$$\|\varphi_n\|^2\simeq A^\varphi\,\frac{x^n}{\sqrt{n}},\qquad \|\Psi_n\|^2\simeq A^\Psi\,\frac{y^n}{\sqrt{n}}. \tag{29}$$

We introduced here the quantities

$$x:=\frac{1+|\alpha\beta-\gamma\delta|}{\beta^2-\delta^2},\qquad y:=\frac{1+|\alpha\beta-\gamma\delta|}{\gamma^2-\alpha^2}, \tag{30}$$

and

$$A^\varphi:=\frac{A\sqrt{\pi}|N_\varphi|^2}{2}\sqrt{\frac{\beta+\delta}{\beta-\delta}},\qquad A^\Psi=\frac{A\sqrt{\pi}|N_\Psi|^2}{2}\sqrt{\frac{\gamma+\alpha}{\gamma-\alpha}}, \tag{31}$$

[2] It might be useful to observe that condition $\gamma=\beta$ needs not to be satisfied, here, since it is a sufficient but not a necessary condition to have $N=N^\dagger$.

with

$$A=\frac{1}{\sqrt{2\pi}}\frac{1}{(s^2-1)^{1/4}}\left[s+\sqrt{s^2-1}\right]^{1/2}, \tag{32}$$

where $s=\frac{1}{\sqrt{(\beta^2-\delta^2)(\gamma^2-\alpha^2)}}$.

From (29) it is clear that $\lim_{n\to\infty}\|\varphi_n\|=0$ for $0<x\leq 1$ and that it diverges when $x>1$. Analogously, we have $\lim_{n\to\infty}\|\Psi_n\|=0$ for $0<y\leq 1$ and $\lim_{n\to\infty}\|\Psi_n\|=\infty$ when $y>1$. Therefore, at a first sight, we might expect to recover the same situation as in the previous section where the product $\|\varphi_n\|\|\Psi_n\|$ turned out to be independent of n, see (20) and (21). In particular, this product was uniformly bounded in n, which is a necessary (but not sufficient) condition for $\mathcal{F}_\varphi$ and $\mathcal{F}_\Psi$ to be bases for $\mathcal{H}$, see [11], Lemma 3.3.3. In contrast, we will see that this condition is never satisfied in the present setting. The proof of this behavior is indeed simple. From (29) follows that

$$\|\varphi_n\|^2\|\Psi_n\|^2=A^\varphi A^\Psi\frac{(xy)^n}{n}=A^\varphi A^\Psi\frac{1}{n}\left(\frac{1+|\alpha\beta-\gamma\delta|}{1-|\alpha\beta-\gamma\delta|}\right)^n, \tag{33}$$

which diverges with $n\to\infty$ whenever $\alpha\beta-\gamma\delta\neq 0$. Thus in this case we recover the behavior already encountered for the special case of the Swanson model, where both $\mathcal{F}_\varphi$ and $\mathcal{F}_\Psi$ were shown not to be bases. Nonetheless, we are left with the possibility that they are $\mathcal{D}$-quasi bases. And indeed, this is what happens.

To prove the latter, we begin by introducing the set

$$\mathcal{D}=\left\{f(x)\in\mathcal{L}^2(\mathbb{R}):\, e^{\frac{1}{2}x^2|\alpha\beta-\gamma\delta|}f(x)\in\mathcal{L}^2(\mathbb{R})\right\}. \tag{34}$$

This set is dense in $\mathcal{L}^2(\mathbb{R})$, since it contains the set $D(\mathbb{R})$ of the C^∞ functions with bounded support. Moreover, if $\alpha\beta\neq\gamma\delta$, it clearly does not coincide with $\mathcal{L}^2(\mathbb{R})$. Now, if $f(x)$ and $g(x)$ belong to $\mathcal{D}$, we can check that

$$\langle f,\varphi_n\rangle=A_n\sqrt{(\alpha+\gamma)(\beta+\delta)}\int_{\mathbb{R}}\overline{f_1(x)}\,H_n(x)e^{-x^2/2}dx \tag{35}$$

and

$$\langle\Psi_n,g\rangle=B_n\sqrt{(\alpha+\gamma)(\beta+\delta)}\int_{\mathbb{R}}H_n(x)e^{-x^2/2}g_1(x)\,dx, \tag{36}$$

where, to simplify the notation, we have introduced

$$A_n=\frac{N_\varphi}{\sqrt{2^n\,n!}}\left(\frac{\alpha+\gamma}{\beta+\delta}\right)^n,\qquad B_n=\frac{\overline{N_\Psi}}{\sqrt{2^n\,n!}}\left(\frac{\beta+\delta}{\alpha+\gamma}\right)^n, \tag{37}$$

and

$$f_1(x)=f\left(x\sqrt{(\alpha+\gamma)(\beta+\delta)}\right)\,e^{\frac{1}{2}x^2(\gamma\delta-\alpha\beta)},\quad g_1(x)=g\left(x\sqrt{(\alpha+\gamma)(\beta+\delta)}\right)\,e^{\frac{1}{2}x^2(\alpha\beta-\gamma\delta)}.$$

Notice that, since both $f(x)$ and $g(x)$ are taken into $\mathcal{D}$, $f_1(x)$ and $g_2(x)$ are square integrable, even if $\gamma\delta \neq \alpha\beta$. Now, using again the orthonormal eigenfunctions of the harmonic oscillator

$$e_n(x) = \frac{1}{\sqrt{2^n\, n!\sqrt{\pi}}}\, H_n(x)\, e^{-x^2/2}, \tag{38}$$

for $n = 0, 1, 2, \ldots$, we deduce that

$$\begin{aligned} \sum_{n=0}^{\infty} \langle f, \varphi_n\rangle \langle \Psi_n, g\rangle &= \sqrt{\pi}\,(\alpha+\gamma)(\beta+\delta) N_\varphi \overline{N_\Psi} \sum_{n=0}^{\infty} \langle f_1, e_n\rangle \langle e_n, g_1\rangle & (39)\\ &= \sqrt{\pi}\,(\alpha+\gamma)(\beta+\delta) N_\varphi \overline{N_\Psi}\, \langle f_1, g_1\rangle = \langle f, g\rangle\,. & (40)\end{aligned}$$

Analogously one can prove that $\sum_{n=0}^{\infty} \langle f, \Psi_n\rangle \langle \varphi_n, g\rangle = \langle f, g\rangle$, for all $f, g \in \mathcal{D}$. The conclusion is that $\mathcal{F}_\varphi$ and $\mathcal{F}_\Psi$, <u>even if they are not bases for $\mathcal{H}$</u>, are $\mathcal{D}$-quasi bases. This is what happens also for the aforementioned special case of the Swanson model, which, however, differs from the situation considered here since in that case some of the coefficients were complex valued. Therefore, apparently, having real or complex parameters in T does not prevent the sets $\mathcal{F}_\varphi$ and $\mathcal{F}_\Psi$ to be $\mathcal{D}$-quasi bases, while among all the real possibilities, there exists just a particular family of choices which reproduces biorthogonal bases for $\mathcal{H}$. Other choices of real parameters produce not bases, but $\mathcal{D}$-quasi bases.

4 Conclusions

In this manuscript we have studied the relations between GBTs and $\mathcal{D}$-PBs. We have found the interesting possibility that GBTs may produce examples of biorthogonal sets that are in addition bases for a Hilbert space $\mathcal{H}$. When the map T in (1) that defines the GBT is taken to be real valued and its parameters are ordered as $\beta > \delta > 0$ and $\gamma > \alpha > 0$ we found two qualitatively different situation. Imposing in addition the constraint $\alpha\beta = \gamma\delta$ we found the hitherto unobserved behavior that the norms of the vectors in sets of the eigenvectors of two related number operators, $\mathcal{F}_\varphi$ and $\mathcal{F}_\Psi$, respectively, possess different types of asymptotic behavior. We concluded from this that they do not form Riesz bases, but still they constitute two biorthogonal bases for $\mathcal{H}$, and $\mathcal{H}$-quasi bases as a consequence. In contrast, when we relax the constraint and take $\alpha\beta \neq \gamma\delta$ instead we proved that neither $\mathcal{F}_\varphi$ nor $\mathcal{F}_\Psi$ are bases for $\mathcal{H}$. Nonetheless, even in this case the sets $\mathcal{F}_\varphi$ and $\mathcal{F}_\Psi$ are still $\mathcal{D}$-quasi bases for a suitable $\mathcal{D}$ dense in $\mathcal{H}$ as specified in (34). The latter behavior was previously observed for specific complex choices of the parameters in T related to a particular version of the Swanson model. In fact, see [5], in this case the sets $\mathcal{F}_\varphi$ and $\mathcal{F}_\Psi$ are $\mathcal{G}$-quasi bases, where $\mathcal{G}$ is the linear span of the $e_n(x)$'s, which is obviously dense in $\mathcal{H}$.

One may also consider the reverse construction, i.e. the possibility of constructing a GBT out of a family of $\mathcal{D}$-PBs. Indeed, this is either trivial as in the example provided with operators A_θ and B_θ, where one simply has to read off the values for the complex-valued parameters α, β, γ and δ to define the map T that represents the GBT or it is not possible at all. The latter case emerges for instance when the $\mathcal{D}$-PBs are constructed by adding complex constants to the standard representations of the operators c and $c^\dagger$, see for instance [15]. Clearly such a construction can not be cast into the form of a GBT of the form as specified in (1). In summary, not all GBT correspond to PBs and vice versa not all versions PBs may be cast into the form of a GBT.

There are clearly some challenges left. Obviously to complete the picture it would be interesting to study the behavior for the remaining choices of α, β, γ and δ not covered in our treatment. In addition, it would be interesting to study these aspects in more complicated models based on GBT, such as the non-Hermitian Hamiltonians of Lie algebraic type investigated in [9].

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