# Double Chain Ladder and Bornhuetter-Ferguson

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**Abstract**

In this paper we propose a method close to Double Chain Ladder (DCL) introduced in Martínez-Miranda, Nielsen and Verrall (2012). The proposed method is motivated by the potential lack of stability of Double Chain Ladder (and of the classical Chain ladder method itself). We consider the implicit estimation of the underwriting year inflation in the classical Chain Ladder (CLM) method and the explicit estimation of it in DCL. This may represent a weak point for DCL and CLM because the underwriting year inflation might be estimated with significant uncertainty. A key feature of the new method is that the underwriting year inflation can be estimated from the less volatile incurred data and then transferred into the DCL model. We include an empirical illustration which illustrates the differences between the estimates of the IBNR and RBNS cash flows from DCL and the new method. We also apply bootstrap estimation to approximate the predictive distributions.

**Keywords:** Bootstrapping; Chain Ladder; Claims Reserves; Reserve Risk

# 1 Introduction

Double Chain Ladder (DCL) was recently introduced in Martínez-Miranda, Nielsen and Verrall (2012) and operates on a standard reserving triangle of aggregate paid claims with the addition of the triangle of the numbers of claims. DCL introduces a micro-model of the claims generating process in order to motivate the models used for estimation, which are applied to triangles of aggregated data. It predicts first the reported number of claims and then, through a delay function and a severity model, it models and predicts future payments. DCL has the attractive feature that when the observed reported claims are replaced by their theoretical expected values, and when one particular estimation procedure is chosen for one parameter, then the resulting prediction is exactly the prediction of the classical Chain Ladder method (CLM). In some senses, one can therefore interpret the DCL method as decomposing classical Chain Ladder into separate components which capture the separate sources of delay inherent in the way claims emerge. Thus, the interpretation of CLM through the framework of DCL shows how the delays determine when the claims emerge. It also shows that the severity of claims can depend on inflation in the underwriting year direction. This inflation in the underwriting year direction is estimated implicitly when CLM is applied: the estimation becomes explicit when DCL is used.

In this paper, we argue that the estimation of the underwriting year inflation in CLM and DCL may represent a weak point because it might be estimated with significant uncertainty. This has also been noticed in a practical setting, and it motivated the invention of the Bornhuetter-Ferguson technique: see Bornhuetter and Ferguson (1972). We suggest a different solution to that of Bornhuetter and Ferguson (1972), which we believe overcomes some of the difficulties associated with the Bornhuetter-Ferguson technique. In particular, the method we propose is less subjective and does not suffer from the criticism that can be made of the Bornhuetter-Ferguson technique that the numbers are chosen in order to produce the desired reserve.

Thus, this paper shows how to estimate the underwriting year inflation from the less volatile incurred data and then transfer this into the DCL model simply by replacing the DCL inflation estimates by those obtained from the incurred data. Because this method replaces the underwriting year parameters in a similar way to Bornhuetter-Ferguson, the title of this paper is Double Chain Ladder and Bourhuetter-Ferguson. However, we would emphasize that the method in this paper is much less subjective than that suggested by Bornhuetter and Ferguson (1972). For illustrative purposes we use the simple one-payment version of the DCL stochastic model suggested by Martínez-Miranda *et al.* (2012). The single payment assumption does not seem to be important when predicting the reserve or its distribution. An analysis of this was provided by Martínez-Miranda, Nielsen and Wüthrich (2012) considering both the original double chain ladder and its multi payment equivalent. However, it should be noted that the method can easily be generalized to more complicated structures as long as the first moment conditions remain unchanged. The same applies to DCL, and the explanation can be found in more detail in Martínez-Miranda *et al.* (2012).

The paper is set out as follows. In Section 2 we describe briefly the micro model which is used to motivate DCL. Section 3 summarizes the DCL estimation method and contains expressions for the point forecasts of the relevant quantities for claims reserving. This section also contains the description of the alterations made to DCL in order to produce the new method, which is referred to as BDCL (short for Bornhuetter Double Chain Ladder) hereafter. Finally in Section 4 we include an application to personal accident data from a major non-life insurer. We utilize bootstrap methods which are very similar to those proposed by Martínez-Miranda *et al.* (2012) in order to provide prediction errors and to make inferences about IBNR and RBNS claims.

# 2 The model for aggregated data

In this paper, we make the same distributional assumptions for the micro model as in Verrall, Nielsen and Jessen (2010) and Martínez-Miranda *et al.* (2011, 2012). This formulation allows us to estimate the settlement delay and therefore to predict RBNS and IBNR reserves separately. In contrast with other approaches which involve also micro models, our aim is not to perform the estimation using individual claims data. Instead, we build models which use some simplifying assumptions and which are applied to triangles of aggregated data. We first describe the micro model, and then show how this can be adapted in order to apply it to conventional triangles of data.

The micro model is constructed from three components: the settlement delay, the individual payments and the reported counts. Here we simply present some notation and the main points of the model (see the papers cited above for a full description).

We assume that two triangles of run-off data are available and that these both have dimension $m$ (i.e. they each have $m$ rows). These are a triangle of aggregated payments, $\Delta_m$, and a triangle of incurred counts, $\aleph_m$. We use $\mathcal{I} = \{(i,j) : i = 1,\ldots,m, j = 0,\ldots,m-1;\ i+j \leq m\}$ to denote the years for which data is available, where $i$ denotes the origin year and $j$ the delay year. The data triangles can then be written as follows:

- The aggregated incurred counts triangle: $\aleph_m = \{N_{ij} : (i,j) \in \mathcal{I}\}$, where $N_{ij}$ is the total number of claims which were incurred in year $i$, which are reported in year $i+j$ i.e. with $j$ periods delay from year $i$.

- The aggregated payments triangle: $\Delta_m = \{X_{ij}^{paid} : (i,j) \in \mathcal{I}\}$, with $X_{ij}^{paid}$ being the total payments from claims incurred in year $i$ and paid with $j$ periods delay from year $i$.

Both triangles consist of observed data which are usually available in practice. The micro model is based on some assumptions about the (unobserved) underlying individual claims data. In fact, we define a new (unobserved) triangle in between $\Delta_m$ and $\aleph_m$, which is the triangle of paid claims, $\aleph_m^{paid} = \{N_{ij}^{paid} : (i,j) \in \mathcal{I}\}$. Here $N_{ij}^{paid}$ is the number of payments which were incurred in year $i$ and settled with $j$ periods delay. Note that the settlement delay (or RBNS delay) is a stochastic component which arises by considering the number of the payments originating from the $N_{ij}$ reported claims which are paid with $l$ periods delay, $N_{ijl}^{paid}$. For simplicity we assume that the maximum period of delay is $m-1$. Then the number of paid claims is given by

$$N_{ij}^{paid} = \sum_{l=0}^{j} N_{i,j-l,l}^{paid}. \tag{1}$$

We emphasize that $N_{ij}^{paid}$ is not assumed to be part of the available data. However, by considering its relationship to the triangle of reported incurred counts, $\aleph_m$, and the distribution of individual claims, it is possible to build a model for the triangle of aggregated paid claims, $\Delta_m$. Thus, we denote the individual settled payments which arise from $N_{ij}^{paid}$ by $Y_{ij}^{(k)}$, $k = 1, \ldots, N_{ij}^{paid}$, $(i,j) \in \mathcal{I}$.

With these definitions, a distributional model for DCL and BDCL is described through the following assumptions (see Section 5 in Martínez-Miranda *et al.* (2012).

D1. The counts $N_{ij}$ are independent random variables from a Poisson distribution with multiplicative parametrization $\mathrm{E}[N_{ij}] = \alpha_i \beta_j$ and identification (Mack 1991), $\sum_{j=0}^{m-1} \beta_j = 1$.

D2. Given $N_{ij}$, the distribution of the numbers of paid claims follows a multinomial distribution, so that the random vector $(N_{ij0}^{paid}, \ldots, N_{ijd}^{paid}) \sim \mathrm{Multi}(N_{ij}; p_0, \ldots, p_{m-1})$, for each $(i,j) \in \mathcal{I}$. Let $\mathbf{p} = (p_0, \ldots, p_d)$ denote the delay probabilities such that $\sum_{l=0}^{m-1} p_l = 1$ and $0 \leq p_l \leq 1, \forall l = 0, \ldots, m-1$.

D3. The individual payments $Y_{ij}^{(k)}$ are mutually independent with distributions $f_i$. Let $\mu_i$ and $\sigma_i^2$ denote the mean and the variance for each $i = 1, \dots, m$. Assume that $\mu_i = \mu\gamma_i$, with $\mu$ being a mean factor and $\gamma_i$ the inflation in the accident years. Also the variances are $\sigma_i^2 = \sigma^2\gamma_i^2$ with $\sigma^2$ being a variance factor. And therefore we assume that the individual payments depend on the accident year $i$ but not on the total delay $j$.

D4. We assume also that the variables $Y_{ij}^{(k)}$ are independent of the counts $N_{ij}$, and also of the RBNS and IBNR delays. Also, it is assumed that the claims are settled with a single payment or maybe as "zero-claims".

Note that under the above assumptions the observed aggregated payments can be written as

$$X_{ij}^{paid} = \sum_{k=1}^{N_{ij}^{paid}} Y_{ij}^{(k)}, \quad \text{for each } (i,j) \in \mathcal{I},$$

which have conditional mean given by

$$\mathrm{E}[X_{ij}^{paid}|\aleph_m] = \mathrm{E}[N_{ij}^{paid}|\aleph_m]\mathrm{E}[Y_{ij}^{(k)}] = \sum_{l=0}^{j} N_{i,j-l}p_l\mu\gamma_i. \tag{2}$$

Verrall *et al.* (2010) showed that the conditional variance of $X_{ij}^{paid}$ is approximately proportional to the mean: $\mathrm{V}[X_{ij}^{paid}|\aleph_m] \approx \varphi_i\mathrm{E}[X_{ij}^{paid}|\aleph_m]$, where $\varphi_i = \gamma_i\varphi$ and $\varphi = \frac{\sigma^2+\mu^2}{\mu}$. Thus, the dispersion parameter, $\varphi_i$, depends on the accident year, $i$. This approximation, which is considered in more detail in Martínez-Miranda *et al.* (2012), justifies the use of an over-dispersed Poisson model to estimate the parameters $\sigma^2$ and $\varphi$.

# 3 Bornhuetter-Ferguson and Double Chain Ladder

In this section, we first outline the estimation method used in DCL and then show how to adapt this for BDCL. Note that, by its very nature, BDCL assumes that more information is available than is contained in the triangles of aggregated payments, $\Delta_m$, and incurred counts, $\aleph_m$. This is in line with the Bornhuetter-Ferguson technique where it is typically assumed that there is some external information available about the likely ratio of ultimate claims to the premium income (the so-called ultimate claims ratio). For BDCL, we assume that the extra information available consists of the triangle of incurred aggregated payments, which include the case reserves.

## 3.1 Model estimation: the DCL method

The Double Chain Ladder method proposed by Martínez-Miranda *et al.* (2012) considers the simple chain-ladder algorithm applied to the triangles of paid claims, $\Delta_m$, and incurred counts, $\aleph_m$, to estimate all the parameters in the model described in Section 2. Therefore, as implied by the name Double Chain Ladder, the classical technique CLM is applied twice and from this everything needed to estimate the outstanding claims is available. It was also shown that this estimation procedure can give identical results as the CLM for paid data when the observed counts are replaced by their fitted values.

An appealing feature of the DCL estimation method is that it uses the estimates of the chain ladder parameters from the triangle of counts and the triangle of payments. Assumption D2 in Section 2 defined a standard chain-ladder model for the counts data, $N_{ij}$. A similar model can be defined for the triangle of paid data, $X_{ij}^{paid}$, with parameters $\widetilde{\alpha}_i$ and $\widetilde{\beta}_j$. We denote the estimates of the parameters, using the chain-ladder model on each triangle, by $(\widehat{\alpha}_i, \widehat{\beta}_j)$ and $(\widehat{\widetilde{\alpha}}_i, \widehat{\widetilde{\beta}}_j)$, respectively, for $i = 1, \ldots, m, j = 0, \ldots, m-1$. Note that it is straightforward to obtain these estimates using the development factors provided by the chain ladder algorithm (Verrall, 1991). Consider the counts triangle (a similar approach can be used for the parameters of the paid triangle) and denote by $\widehat{\lambda}_j$, $j = 1, 2, \ldots, m-1$, the corresponding estimated development factors. Then the estimates of $\beta_j$ for $j = 0, \ldots, m-1$ can be calculated by

$$\widehat{\beta}_0 = \frac{1}{\prod_{l=1}^{m-1} \widehat{\lambda}_l} \tag{3}$$

and

$$\widehat{\beta}_j = \frac{\widehat{\lambda}_j - 1}{\prod_{l=j}^{m-1} \widehat{\lambda}_l} \tag{4}$$

for $j = 1, \ldots, m-1$ . The estimates of the parameters for the accident years can be derived from the latest cumulative entry in each row through the formula:

$$\widehat{\alpha}_i = \sum_{j=0}^{m-i} N_{ij} \prod_{j=m-i+1}^{m-1} \widehat{\lambda}_j. \tag{5}$$

The same procedure can be used to produce $(\widehat{\widetilde{\alpha}}_i, \widehat{\widetilde{\beta}}_j)$, and the DCL method estimates the rest of the parameters in the model (formulated as D1-D4) using just the above estimates. Specifically, the reporting delay probabilities $\{p_0, \ldots, p_{m-1}\}$ can be estimated by solving the linear system given below to obtain estimates of $\{\pi_0, \ldots, \pi_{m-1}\}$ and then adjusting these.

$$\begin{pmatrix} \widetilde{\beta}_0 \\ \vdots \\ \vdots \\ \widetilde{\beta}_{m-1} \end{pmatrix} = \begin{pmatrix} \beta_0 & 0 & \cdots & 0 \\ \beta_1 & \beta_0 & \ddots & 0 \\ \vdots & \ddots & \ddots & 0 \\ \beta_{m-1} & \cdots & \beta_1 & \beta_0 \end{pmatrix} \begin{pmatrix} \pi_0 \\ \vdots \\ \vdots \\ \pi_{m-1} \end{pmatrix}. \tag{6}$$

Once the solution $\{\widehat{\pi}_0, \ldots, \widehat{\pi}_{m-1}\}$ is obtained, these preliminary delay parameters are adjusted to have the desired real probability vector, $(\widehat{p}_0, \ldots, \widehat{p}_{m-1})$ which satisfies the restrictions that $0 \leq \widehat{p}_l < 1$ and $\sum_{l=0}^{m-1} \widehat{p}_l = 1$. For more details of this estimation procedure, see Martínez-Miranda *et al.* (2012).

For the mean and variance of the distribution of individual payments DCL estimates the inflation parameters, $\gamma = \{\gamma_i : i = 1, \ldots, m\}$, and the mean factor, $\mu$, through the expression:

$$\widehat{\gamma}_i = \frac{\widehat{\widetilde{\alpha}}_i}{\widehat{\alpha}_i \widehat{\mu}} \quad i = 1, \ldots, m. \tag{7}$$

To ensure identifiability DCL sets $\gamma_1 = 1$, so that $\mu$ can be estimated by

$$\widehat{\mu} = \frac{\widehat{\widetilde{\alpha}}_1}{\widehat{\alpha}_1}. \tag{8}$$

The inflation parameters, $\widehat{\gamma}_i$, are estimated by substituting $\widehat{\mu}$ into equation (7). It only remains to adjust the final $\widehat{\mu}$ according to the estimates $\widehat{p}_l$ and in order to ensure Mack's identification holds. This is done by dividing $\widehat{\mu}$ by$\kappa$, where $\kappa = \sum_{j=0}^{m-1} \sum_{l=0}^{j} \widehat{\beta}_{j-l} \widehat{p}_l$. Hereafter, in a slight abuse of notation, we will retain the notation $\widehat{\mu}$ for the corrected estimator of $\mu$.

The estimates of the variances, $\sigma_i^2$ $(i = 1, \ldots, m)$, are obtained by first estimating the overdispersion parameter $\varphi$ (defined in Section 2) by

$$\widehat{\varphi} = \frac{1}{n-m} \sum_{i,j \in \mathcal{I}} \frac{(X_{ij}^{paid} - \widehat{X}_{ij}^{DCL})^2}{\widehat{X}_{ij}^{DCL} \widehat{\gamma}_i}, \tag{9}$$

where $n = m(m+1)/2$ and $\widehat{X}_{ij}^{DCL} = \sum_{l=0}^{m-1} N_{i,j-l} \widehat{p}_l \widehat{\mu} \widehat{\gamma}_i$. Then the variance factor of individual payment can be estimated by

$$\widehat{\sigma}_i^2 = \widehat{\sigma}^2 \widehat{\gamma}_i^2 \tag{10}$$

for each $i = 1, \ldots, m$, where $\widehat{\sigma}^2 = \widehat{\mu}\widehat{\varphi} - \widehat{\mu}^2$.

## 3.2 The BDCL method

The BDCL method follows identical steps as DCL but instead of using the estimates of the inflation parameters, $\gamma_i$, from the triangle of paid claims, these are identified using some extra deterministic information. There are a number of different sources that could be used for this, but in this paper it is done by first applying DCL to the triangles of reported counts and aggregated incurred claims. The parameters $\{\gamma_i : i = 1, \dots, m\}$ are estimated exactly as described in Section 3.1, except that the triangle of aggregate paid claims is replaced by the triangle of aggregated incurred claims. Thus, it is at this point that a third triangle of data is available, which includes the case reserves. In this way, the BDCL method then consists of the following two-step procedure:

- *Step 1: Parameter estimation.*
  Estimate the model parameters using DCL for the data in the triangles $\aleph_m$ and $\Delta_m$ and denote the parameter estimates by $(\widehat{p}_0, \dots, \widehat{p}_{m-1})$, $\widehat{\mu}$, $\widehat{\sigma}^2$ and $\{\widehat{\gamma}_{i,0} : i = 1, \dots, m\}$.
  Repeat this estimation using DCL but replacing the triangle of paid claims by the triangle of incurred data: $\breve{\Delta}_m = \{X_{ij} : (i,j) \in \mathcal{I}\}$, where $X_{ij}$ is incurred claims for accident year $i$ and development period $j$. Keep only the resulting estimated inflation parameters, denoted by $\{\widehat{\gamma}_{i,1} : i = 1, \dots, m\}$.
- *Step 2: BF adjustment.*
  Replace the inflation parameters $\{\widehat{\gamma}_{i,0} : i = 1, \dots, m\}$ from the paid data by the estimates from the incurred triangle, $\{\widehat{\gamma}_{i,1} : i = 1, \dots, m\}$. For simplicity of notation, these are denoted hereafter by $\{\widehat{\gamma}_i : i = 1, \dots, m\}$.

From Steps 1 and 2 the final parameter estimates will be $\widehat{\theta} = \{\widehat{p}_l, \widehat{\mu}, \widehat{\sigma}^2, \widehat{\gamma}_i, l = 0, \dots, m-1, i = 1, \dots, m\}$. In general, it would be possible to use other sources of information from those suggested here. Thus, Step 2 could be defined in a more arbitrary way, thereby mimicking more closely what is often done when the Bornhuetter-Ferguson technique is applied. In this way, the process described in this section could be viewed in a more general way. However, we believe that the use of the triangle of incurred claims is probably more justifiable in a regulatory environment.

## 3.3 Justification of BDCL

The CLM and Bornhuetter-Ferguson (BF) methods are among the easiest claim reserving methods, and due to their simplicity they are two of the most commonly used techniques in practice. Some recent papers on the BF method include Alai, Merz and Wüthrich (2009, 2010), Mack (2008), Schmidt and Zocher (2008), Verrall (2004). The BF method introduced by Bornhuetter and Ferguson (1972) aims to address one of the well known weaknesses of CLM which is the effect that outliers can have on the estimates of outstanding claims. To do this, the BF method incorporates prior knowledge from experts and is therefore more robust than the CLM method which relies completely on the data contained in a run-off triangle.

Specifically CLM method estimates of outstanding claims for accident year $i > 1$ by

$$\widehat{R}_i^{CLM} = C_{i,m-i}\left(\prod_{k=m-i+1}^{m-1} \widetilde{\lambda}_k - 1\right)$$

where $C_{i,m-i}$ denotes the latest observed cumulative claims and $\widetilde{\lambda}_1, \ldots, \widetilde{\lambda}_{m-1}$ are the development factors. Therefore the CLM reserve depends strongly on $C_{i,m-i}$, which means that very unstable (or even unusable) predictions can result. This often occurs for the latest origin years, where the triangle contains less information which is often more volatile. The BF method attempts to overcome this by replacing the latest cumulative claims by an external (prior) estimate. This estimate is obtained from an estimate of ultimate claims, $U_i^{prior}$. So BF replaces $C_{i,m-i}$ by $U_i^{prior}/(\prod_{k=m-i+1}^{m-1} \widetilde{\lambda}_k)$ and the BF estimate of outstanding claims is

$$\widehat{R}_i^{BF} = \frac{U_i^{prior}}{\prod_{k=m-i+1}^{m-1} \widetilde{\lambda}_k}\left(\prod_{k=m-i+1}^{m-1} \widetilde{\lambda}_k - 1\right).$$

Under the assumptions of the Poisson model with a multiplicative structure for the mean, $\mathrm{E}[X_{ij}] = \widetilde{\alpha}_i\widetilde{\beta}_j$, the relationship between CLM and BF reserve can be most clearly seen through the following expressions:

$$\widehat{R}_i^{CLM} = \widehat{\widetilde{\alpha}}_i \sum_{k=0}^{m-1} \widehat{\widetilde{\beta}}_k \left(\frac{\sum_{k=m-i+1}^{m-1} \widehat{\widetilde{\beta}}_k}{\sum_{k=0}^{m-1} \widehat{\widetilde{\beta}}_k}\right) = \widehat{U}_i^{CLM}\left(\frac{\sum_{k=m-i+1}^{m-1} \widehat{\widetilde{\beta}}_k}{\sum_{k=0}^{m-1} \widehat{\widetilde{\beta}}_k}\right)$$

and

$$\widehat{R}_i^{BF} = U_i^{prior}\left(\frac{\sum_{k=m-i+1}^{m-1} \widehat{\widetilde{\beta}}_k}{\sum_{k=0}^{m-1} \widehat{\widetilde{\beta}}_k}\right). \tag{11}$$

Here, $\widehat{U}_i^{CLM}$ and $\widehat{\widetilde{\alpha}}_i$ and $\widehat{\widetilde{\beta}}_k$ are the CLM estimation of the ultimate claims and the parameters in the model, respectively. Since $\sum_{k=0}^{m-1} \beta_k = 1$, $\widehat{U}_i^{CLM} = \widehat{\widetilde{\alpha}}_i$, and therefore BF replaces the estimated row parameters in the Poisson model.

The BDCL method follows the same spirit with the aim of stabilizing the row parameters with extra information. In this paper we do this using the incurred aggregated claims data, but other formulations would also be possible. Considering the assumed structure of the parameters in the DCL model, the row parameters are

$$\widetilde{\alpha}_i = \alpha_i \mu \gamma_i$$

for the paid data, and for the incurred data we also have the same row parameters as we will show in the next Section, $\breve{\alpha}_i = \alpha_i \mu \gamma_i$, where $\alpha_i$ is the row parameter in the model for the reported counts. This model assumption implies that the incurred reserves unconditionally have the correct means for the future RBNS. Therefore when BDCL uses the estimated inflation parameters from the incurred data, it is the more volatile parameter $\widetilde{\alpha}_i$ which is replaced. i.e. the inflation by accident year, $\gamma_i$, by the estimate derived from the triangles $(\aleph_m, \breve{\Delta}_m)$. In this way the predictions become more stable, and often they can be more realistic. This is illustrated in the example in Section 4.

## 3.4 Estimating the underwriting year inflation parameter from the incurred data

This paper shows that it is possible to stabilize the results by using the more stable incurred underwriting year inflation in place of the less stable paid data underwriting year inflation. To justify this, we develop in this section the equivalent equation to the equation (2) for the incurred data.

We define $X_{ij}^{(h)}$ as those payments stemming from underwriting year $i$ that are paid at time $j$, but belonging to claims already reported at time $h$ $(h = 0, \ldots, m-1)$. So by definition $X_{ij}^{(h)} = X_{ij}^{paid}$ when $h \geq j$ and $X_{ij}^{(h)} \to X_{ij}^{paid}$ as $h \to j$.

A very simple interpretation the incurred triangle is to define each element $X_{ik}$ as

$$X_{ih} = \sum_{j=0}^{m-1} \mathrm{E}\left[X_{ij}^{(h)} | \mathcal{F}_h\right],$$

where $\mathcal{F}_h$ is an increasing filtration illustrating our knowledge at time $h$. This is of course the ideal model of incurred data. In practice, incurred data will often include variability and uncertainty that can not be modelled after all. Note also that $\mathrm{E}[X_{ij}^{(h)} | \mathcal{F}_h] = X_{ij}^{paid}$ when $j \leq h$.

Now the corresponding equation to equation 2 above will be derived. Let consider first the sub-triangle $\aleph_h = \{N_{ij} : i = 1, \ldots, h, j = 0, \ldots, h;\ i + j \leq$

$h\}$

$$
\begin{aligned}
\mathrm{E}\left[\mathrm{E}\left[X_{ij}^{(h)}|\mathcal{F}_h\right]|\aleph_h\right] &= \mathrm{E}\left[X_{ij}^{(h)}|\aleph_h\right] \\
&= \begin{cases} \mathrm{E}\left[X_{ij}^{paid}|\aleph_h\right] & \text{for } j \leq h \\ \mathrm{E}\left[X_{ij}^{(h)}|\aleph_h\right] & \text{for } j > h \end{cases} \\
&= \begin{cases} \sum_{l=0}^{j} N_{i,j-l} p_l \mu \gamma_i & \text{for } j \leq h \\ \sum_{l=j-h}^{j} N_{i,j-l} p_l \mu \gamma_i & \text{for } j > h \end{cases}
\end{aligned}
$$

where we have used that $j - l \leq h \Leftrightarrow l \geq j - h$.

We can therefore write the simplified expression

$$
\mathrm{E}\left[\mathrm{E}\left[X_{ij}^{(h)}|\mathcal{F}_h\right]|\aleph_h\right] = \mathrm{E}\left[X_{ij}^{(h)}|\aleph_h\right] = \sum_{l=\max(0,j-h)}^{j} N_{i,j-l} p_l \mu \gamma_i.
$$

Now

$$
\begin{aligned}
\mathrm{E}\left[X_{ih}^{paid}|\aleph_h\right] &= \sum_{j=0}^{m-1} \mathrm{E}\left[\mathrm{E}\left[X_{ij}^{(h)}|\mathcal{F}_h\right]|\aleph_h\right] = \\
&= \sum_{j=0}^{m-1} \sum_{l=\max(0,j-h)}^{j} N_{i,j-l} p_l \mu \gamma_i.
\end{aligned}
$$

We therefore get the following unconditional mean for the incurred claims:

$$
\mathrm{E}\left[X_{ih}\right] = \mu \alpha_i \gamma_i \overline{\beta}_h,
$$

where $\overline{\beta}_h = \sum_{j=0}^{m-1} \sum_{l=\max(0,j-h)}^{j} \beta_{i,j-l} p_l.$

In its classical form, the chain ladder technique is primarily a first moment multiplicative model: see for example Kuang, Nielsen and Nielsen (2009). Therefore, applying the classical chain ladder method to incurred data will yield estimates of $\mu, \alpha_i, \gamma_i$ and $\overline{\beta}_h$, for $i = 1, \ldots, m; h = 0, \ldots, m-1$. The double chain ladder method adds information to the classical chain ladder approach by including counts data and allowing separate estimation of exposure and inflation parameters.

In common with the philosophy underlying the Bornhuetter-Ferguson method, the aim of this paper is to extract more stable estimates of the inflation parameter $\gamma_i$ from the incurred data. Hence, the only thing the BDCL method uses from the application of the double chain ladder method to the incurred data are the estimates of these parameters.

### 3.5 Forecasting outstanding claims: the RBNS and IBNR reserves and predictive distributions

In this section, the methods for producing estimates of outstanding claims are described. It is also noted that bootstrapping can be used to obtain the predictive distributions. The estimated parameters, $\widehat{\theta} = \{\widehat{p}_l, \widehat{\mu}, \widehat{\sigma}^2, \widehat{\gamma}_i, l = 0, \ldots, m-1, i = 1, \ldots, m\}$, derived from Steps 1-2 above can be used to calculate a point forecast of the RBNS and IBNR components of the reserve. Using the notation of Verrall *et al.* (2010) and Martínez-Miranda *et al.* (2011, 2012), we consider predictions and extend the model assumptions presented in Section 2 over the following triangles:

$$\mathcal{J}_1 = \{i = 2, \ldots, m; j = 0, \ldots, m-1 \text{ so } i + j = m+1, \ldots, 2m-1\}$$
$$\mathcal{J}_2 = \{i = 1, \ldots, m; j = m, \ldots, 2m-1 \text{ so } i + j = m+1, \ldots, 2m-1\}$$
$$\mathcal{J}_3 = \{i = 2, \ldots, m; j = m, \ldots 2m-1 \text{ so } i + j = 2m, \ldots, 3m-2\}.$$

As was pointed out in the above papers, the CLM method would produce forecasts over only $\mathcal{J}_1$. In contrast, DCL and consequently BDCL provide also estimates of the tail over $\mathcal{J}_2 \cup \mathcal{J}_3$.

For the RBNS reserve we follow the original suggestion of Verrall *et al.* (2010) and use the expression of the conditional mean in equation (2) i.e.

$$\widehat{X}_{ij}^{rbns} = \sum_{l=i-m+j}^{j} N_{i,j-l} \widehat{\pi}_l \widehat{\mu} \widehat{\gamma}_i, \tag{12}$$

with $(i,j) \in \mathcal{J}_1 \cup \mathcal{J}_2$.

For the IBNR forecast reserve the chain ladder predictions of future numbers of reported claims are used, $\widehat{N}_{i,j}$. i.e.

$$\widehat{X}_{ij}^{ibnr} = \sum_{l=0}^{i-m+j-1} \widehat{N}_{i,j-l} \widehat{\pi}_l \widehat{\mu} \widehat{\gamma}_i, \tag{13}$$

with $(i,j) \in \mathcal{J}_1 \cup \mathcal{J}_2 \cup \mathcal{J}_3$.

We also derive the bootstrap predictive distribution as Martínez-Miranda *et al.* (2011, 2012) proposed, using the data in the observed triangles $\aleph_m$ and $\Delta_m$, and the model with estimated parameter $\widehat{\theta}$.

# 4 An empirical illustration

In this paper we consider a personal accident data set from a major insurer. The data available consists of three incremental run-off triangles of dimension

$m = 19$: the reported counts, the aggregated paid data and the incurred data. We use a yearly aggregation of data in our empirical illustration below. However, in practice the insurer would often use quarterly data, and we therefore consider quarterly data when validating the model in Section 4.1.

The data are presented in Figures 1 and 3. In Figure 1 the cumulative payments are shown as a function of development year. Also shown are the maximum likelihood estimates of the underwriting year effect when a multiplicative Poisson model is applied to the number of reported claims. Since this gives an indication of the total number of claims expected, we use the term "exposure" to label this. It can be seen from this initial view that it appears that these data arise from a significant growth in business, with the exposure rapidly increasing the first 10 years. As a result of this, it is difficult to draw inference from the paid claims alone. In Figure 3, we therefore shows illustrations of the data scaling first by the exposure, but also by scaling by the underwriting year inflation from the DCL method applied to paid data and also from the BDCL method, which extracts the inflation from the incurred data. Both estimated inflations are showed in Figure 2. Figure 3 shows that adjusting for exposure can help with the understanding of the data, and that the curves for different accident years really become comparable when we scale for the underwriting year estimated from either paid data (DCL) or incurred data (BDCL). Figure 3 also presents a graph of the coefficient of variation as a function of development year. It is the coefficient of variation of the cumulative paid curves for each development year. It can be seen that adjusting for underwriting year inflation significantly reduces the coefficient of variation, thereby providing some support for the methodology. Note that the DCL adjustment gives a lower coefficient of variation than the BDCL adjustment for the early development years. A reason for this is that DCL uses just the paid data, and may therefore provide a better goodness-of-fit, even though its forecasts may be less credible in practice.

This has provided some initial exploratory data analysis, and we now illustrate the BDCL method and compare it to the simple DCL method and classical chain ladder, using annual data. In this illustration, the effect of BDCL is only significant in the last 4 inflation parameters, $\gamma_{16}, \gamma_{17}, \gamma_{18}, \gamma_{19}$. However, these parameters have a large effect on the estimates of outstanding claims, and as a result the DCL and classical chain ladder estimates of outstanding claims are almost double as high as those from the BDCL method. The effect is greater when considering the IBNR reserve than the RBNS reserve. When analyzing this data set and providing a best estimate of the reserve the actuary therefore faces a considerable challenge. Is the incurred information to be trusted? Is the underwriting year inflation we find in the paid data really that wrong? Or should we find the answer in some irregularities around the practical principles underlying the incurred data at hand? It is of course not sufficient for an actuary to dismiss one of these two estimates of outstanding claims when setting the reserves simply from preference for one of the two principles over the other (paid data to incurred data). Both are estimated from data, one from real data (paid) another from real data combined with collected expert data (incurred). The paid data is more clean because we know exactly where it comes from, but it is also often more unstable than the incurred data: the incurred data is often more stable, but contains opaque elements that could be open to challenge in the process of setting reserves.

Table 1 gives the estimates of the parameters for the model from BDCL and from DCL. Figure 2 presents the underwriting year inflation when estimated from paid data (DCL) and incurred data (BDCL). As mentioned above, various adjustments of the paid data are plotted in Figure 3.

Table 2 shows the predicted RBNS and IBNR reserve and also the total (RBNS + IBNR) reserve for BDCL and DCL methods. The total reserve of the BDCL method is less than 60% of the DCL and CLM reserve, with the biggest difference being between the IBNR reserves. One interpretation of this is could be that the DCL and the CLM methods overestimate the underwriting year inflation dramatically the last years, and therefore the predicted reserve increases significantly. However, the assessment of a result such as this will depend greatly on other information that the insurer has about the underlying changes driving the data. The reserve based on the classical chain ladder technique applied to the incurred data is around 20% lower that the reserve estimated by the BDCL method. Hence, the BDCL reserve is actually closer to the estimate from the CLM applied to paid data than the reserve resulting from the CLM applied to the incurred data. One interpretation of this is, as observed by previous authors, that the BDCL method finds a pragmatic solution to what is a complicated problem. The final choice in setting the reserve should result from the insight of the actuary, who is advised to ask many questions about the way the incurred information is collected. In this way, the actuary has to verify whether the strong and dominating inflation conclusions from the incurred data are to be trusted - even when they are so much out of line with the observed paid data.

The predictive distributions relating to the RBNS and IBNR claims can be estimated using the bootstrap methods proposed by Martínez-Miranda *et al.* (2012). This can be done for both BDCL and DCL methods and the corresponding cash-flows simulated. As a comparison, the CLM predictive distribution can be estimated using the bootstrap method introduced by England and Verrall (1999). Note that this does not provide the split between the RBNS and IBNR reserves. The summary statistics for the RBNS, IBNR and total (RBNS + IBNR) cash-flows are shown in Table 3, and these results provide a numerical comparison among the cash-flows derived from the BDCL, DCL and CLM methods.

Again, it can be seen that DCL and CLM provide upper quantiles for the total reserve which are about the double those given by the BDCL method. Figure 4 shows box plots of the predictive distribution of the total reserve in the future from BDCL, DCL and CLM. Also shown are histograms of the overall total reserve for future years which represent the predictive distribution of the three compared methods. Note that the BDCL cash-flows move around smaller values than those by the DCL method. The chain ladder cash-flows derived by the England and Verrall (1999)'s bootstrap method move close to the DCL cash-flows. However we can see that the England and Verrall (1999) distribution fails in the tails, providing inadequate cash-flows to cover the full range of reserving variability.

## 4.1 Further discussion and validation of the results

Traditionally, the actuary often relies on the incurred data and tries to use this to obtain more reliable estimates where the observed information is scarce. As we have discussed in this paper a more controlled way to introduce such information is to simply incorporate the inflation parameters from the incurred data as prior knowledge. Even though these parameters should, in theory, be the same as those from the paid data, the estimation can often be more stable from the incurred data since they contain more information.

As was shown in Table 2 DCL, which uses the underwriting year inflation from the paid data, seems to over-estimate the outstanding claims. On the other hand, when we consider the BDCL method and we then use the inflation parameters estimated from the incurred data, the estimates differ notably. So a natural question is which method is actually the better for the data being considered. In order to assess the accuracy of the BDCL and DCL estimation methods we have performed a back-test on the observed data. Such a test should be part of a wider validation process which should be carried out in practice. The back-test was carried out as follows:

First the data was considered in its original quarterly format instead of considering the data aggregated by years, so that it consisted of run-off triangles with dimension $m$=79. From these larger triangles there is more data with which it is possible to make comparisons of the predictions. The back-test was performed by removing one-by-one the last calendar years (diagonals) in the triangles and estimating the model to obtain predictions from both DCL and BDCL using the expressions (12) and (13). The difference between the actual data and the estimate can then be calculated. This was done for the last 5 calendar years and the (square root) mean squared error calculated. Table 4 shows the results from these sub-triangles for $m_c = 78, 77, 76, 75, 74$. From these results it can be seen that indeed BDCL is significantly more accurate than DCL. The error in the total reserve is about the 30% of the error provided by DCL estimates. The same pattern is also observed by removing more diagonals and also considering other subsets of the observed data.

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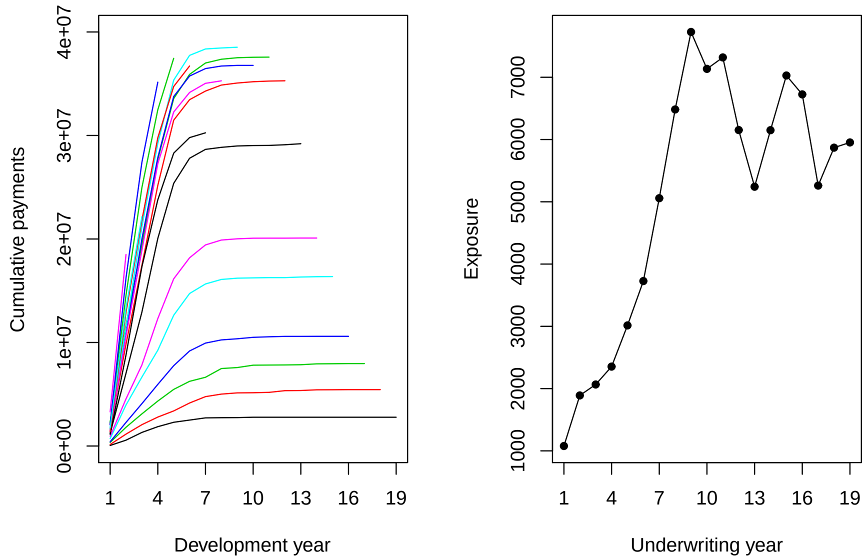


Figure 1: Cumulative paid data and the exposure.

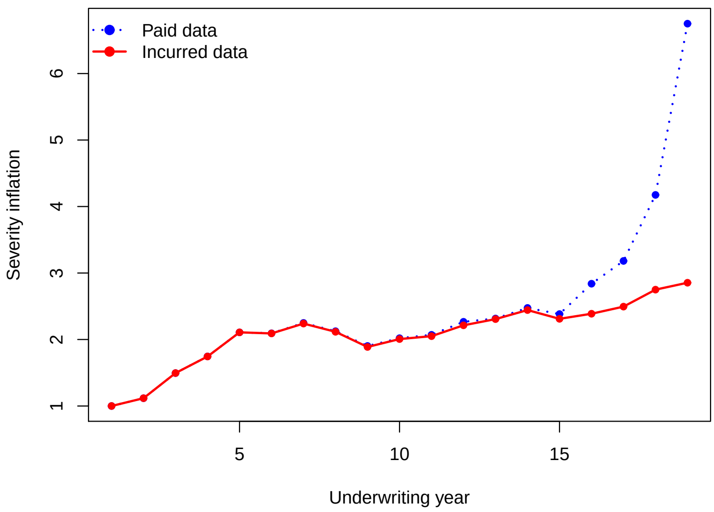


Figure 2: Estimated inflation by DCL from the paid data (blue-dotted curve) and the incurred data (red-solid curve).

| $\widehat{p}_l$ | $\widehat{\gamma}_{i,BDCL}$ | $\widehat{\gamma}_{i,DCL}$ |
|---|---|---|
| 0.0592 | 1.00 | 1.00 |
| 0.3097 | 1.12 | 1.12 |
| 0.2032 | 1.50 | 1.49 |
| 0.1996 | 1.74 | 1.75 |
| 0.1388 | 2.11 | 2.11 |
| 0.0440 | 2.09 | 2.09 |
| 0.0227 | 2.24 | 2.25 |
| 0.0095 | 2.12 | 2.13 |
| 0.0017 | 1.89 | 1.90 |
| 0.0029 | 2.01 | 2.02 |
| 0.0002 | 2.05 | 2.07 |
| 0.0026 | 2.21 | 2.27 |
| 0.0019 | 2.31 | 2.32 |
| 0.0031 | 2.44 | 2.47 |
| 0.0006 | 2.31 | 2.38 |
| 0.0000 | 2.39 | 2.84 |
| 0.0000 | 2.49 | 3.18 |
| 0.0000 | 2.75 | 4.17 |
| 0.0000 | 2.85 | 6.75 |
| $\widehat{\mu} = 2579.064$ | | |
| $\widehat{\sigma}^2_{BDCL} = 350497302$ | | |
| $\widehat{\sigma}^2_{DCL} = 286808926$ | | |

Table 1: Estimated parameters: the delay probabilities $\widehat{p}_l$ ($l = 0, \ldots, 18$), the inflation parameters $\widehat{\gamma}_i$ ($i = 1, \ldots, 19$) and the estimates of the mean and variance parameters, $\mu$ and $\sigma^2$, from the BDCL and DCL methods.

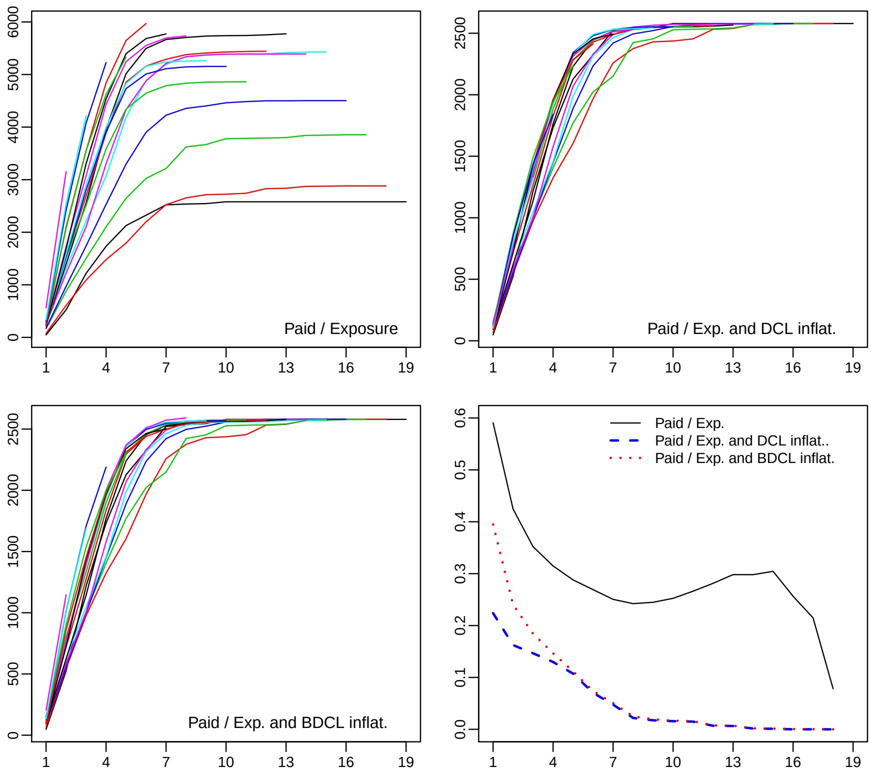


Figure 3: Cumulative paid data adjusted by the exposure and the inflation. The top left panel shows the cumulative payments adjusted by the exposure. Each curve corresponds to a different accident year and it shows the cumulative payments across the development years. The top right and the bottom left panels show the same adjusted data but adjusted also by the inflation estimated using DCL and BDCL, respectively. The bottom right panel shows the coefficient of variation of the cumulative paid curves for each development year.

| | BDCL | | | DCL | | | |
|---|---|---|---|---|---|---|---|
| Future | RBNS | IBNR | Total | RBNS | IBNR | Total | CLM |
| 1 | 37813 | 615 | 38428 | 59845 | 1387 | 61232 | 61091 |
| 2 | 25878 | 3294 | 29172 | 41447 | 7406 | 48853 | 48061 |
| 3 | 17804 | 2537 | 20341 | 31016 | 5611 | 36627 | 36266 |
| 4 | 9485 | 2495 | 11980 | 17542 | 5502 | 23044 | 22990 |
| 5 | 3699 | 1867 | 5566 | 6443 | 4069 | 10512 | 10439 |
| 6 | 1839 | 821 | 2660 | 3192 | 1720 | 4912 | 4914 |
| 7 | 905 | 462 | 1366 | 1446 | 945 | 2391 | 2380 |
| 8 | 512 | 246 | 759 | 675 | 487 | 1162 | 1174 |
| 9 | 457 | 113 | 571 | 642 | 210 | 853 | 848 |
| 10 | 329 | 87 | 416 | 424 | 169 | 592 | 600 |
| 11 | 337 | 40 | 377 | 536 | 72 | 608 | 594 |
| 12 | 242 | 49 | 292 | 404 | 99 | 504 | 496 |
| 13 | 163 | 37 | 200 | 335 | 74 | 409 | 397 |
| 14 | 28 | 46 | 73 | 60 | 97 | 157 | 136 |
| 15 | 0 | 18 | 18 | 0 | 37 | 37 | 109 |
| 16 | 0 | 7 | 7 | 0 | 12 | 12 | 0 |
| 17 | 0 | 4 | 4 | 0 | 7 | 7 | 0 |
| 18 | 0 | 2 | 2 | 0 | 4 | 4 | 0 |
| 19 | 0 | 1 | 1 | 0 | 2 | 2 | |
| 20 | 0 | 1 | 1 | 0 | 1 | 1 | |
| 21 | 0 | 0 | 0 | 0 | 1 | 1 | |
| 22 | 0 | 0 | 0 | 0 | 0 | 0 | |
| Total | 99492 | 12741 | 112233 | 164007 | 27911 | 191918 | 190496 |

Table 2: Point forecasts by calendar year. Columns 2-4 show the predictions from BDCL. Columns 5-7 show the predictions by DCL, and column 8 the classical Chain Ladder predictions (CLM). The quantities are given in thousands.

| | Bootstrap predictive distribution | | | | | | |
|---|---|---|---|---|---|---|---|
| | BDCL | | | DCL | | | |
| | RBNS | IBNR | Total | RBNS | IBNR | Total | CLM |
| mean | 97900 | 12509 | 110409 | 163534 | 28246 | 191780 | 193149 |
| pe | 18671 | 6121 | 23160 | 37387 | 13876 | 48439 | 18206 |
| 1% | 61032 | 3144 | 68131 | 100702 | 7857 | 112348 | 157572 |
| 5% | 71695 | 4960 | 78153 | 114031 | 11233 | 128032 | 165976 |
| 50% | 96706 | 11512 | 108451 | 158880 | 25662 | 184786 | 191578 |
| 95% | 130606 | 23887 | 149298 | 232537 | 53395 | 280328 | 225784 |
| 99% | 155128 | 32733 | 185638 | 276334 | 73146 | 343335 | 245683 |

Table 3: Predictive distribution of RBNS, IBNR and total (RBNS + IBNR) reserve. The six first columns give the summary of the distribution from the bootstrap method for BDCL. The following three columns show the results from DCL. The last column shows the England and Verrall (1999) distribution. The quantities are given in thousands.

| $m_c$ | DCL | BDCL | Rerr |
|---|---|---|---|
| 78 | 221665.5 | 99071.9 | 0.4469 |
| 77 | 210708.1 | 98297.5 | 0.4665 |
| 76 | 233875.4 | 84232.2 | 0.3602 |
| 75 | 317434.6 | 77075.6 | 0.2428 |
| 74 | 283276.9 | 87542.4 | 0.3090 |

Table 4: Results of the back-test to evaluate of the discrepancy between estimates and actual numbers. The second and third columns show the (square root) mean squared error of the estimates by DCL and BDCL, respectively. The discrepancies have been evaluated on the last $m - m_c$ diagonals in the original quarterly paid triangle. The last column shows the relative error defined as the ratio of the BDCL and the DCL errors.

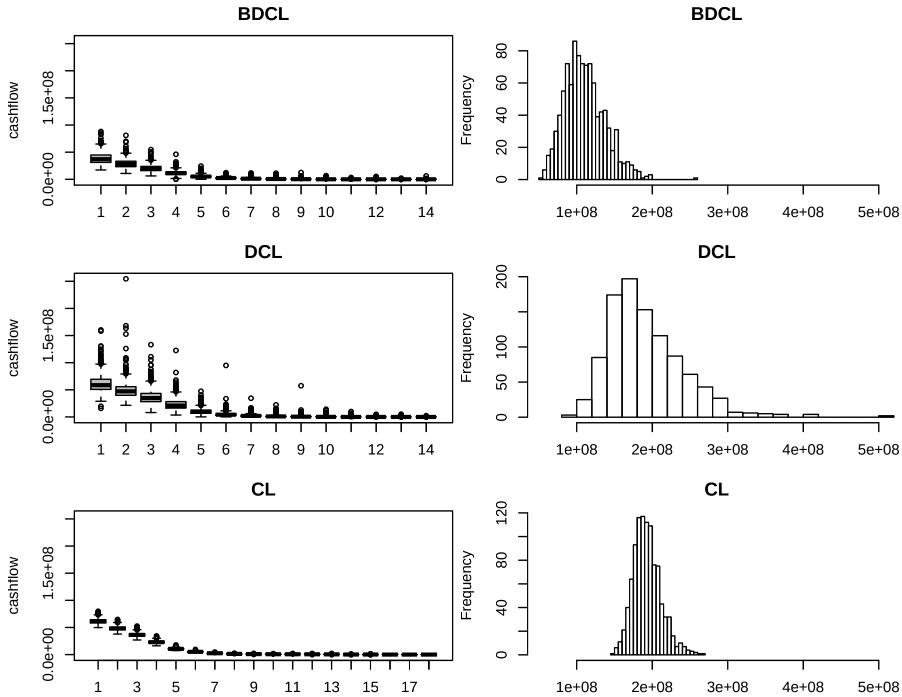


Figure 4: Box plots representing the predictive distribution of the total (RBNS+IBNR) reserve in the future from BDCL, DCL and CLM (rows 1,2 and 3 respectively). Right panels show the histograms of the total reserve (the overall total for the next years) by the three methods.