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Approximations for quantiles of life expectancy and annuity values using the parametric improvement rate approach to modelling and projecting mortality

Abstract
In this paper, we develop accurate approximations for medians of life expectancy and life annuity pure premiums viewed as functions of future mortality trends as predicted by parametric models of the improvement rates in mortality. Numerical illustrations show that the comonotonic approximations perform well in this case, which suggests that they can be used in practice to evaluate the consequences of the uncertainty in future death rates. Prediction intervals based on 5% and 95% quantiles are also considered but appear to be wider compared to simulated ones. This provides the practitioner with a conservative shortcut, thereby avoiding the problem of simulations within simulations in, for instance, Solvency 2 calculations.

Key words and phrases: Life annuity, life expectancy, mortality projection, comonotonicity, simulation.

1 Introduction

Forecasting mortality in actuarial studies is generally based on extrapolation methods that capture the pattern in historical mortality rates by means of appropriate parametric predictor structures. Foremost among such structures is the Poisson log-bilinear specification proposed by Brouhns, Denuit and Vermunt (2002) and Renshaw and Haberman (2003) in line with the seminal paper by Lee and Carter (1992). Recently, Haberman and Renshaw (2012) have introduced and investigated parametric mortality projection methods based on mortality improvement rates (as opposed to mortality rates). This approach provides an efficient alternative to the direct parametric modelling and projecting of mortality rates.

In this paper, we consider present values of life annuity benefits as functions of the unknown life table applying in the future (as well as life expectancies, corresponding to zero interest rate). Deriving the exact distribution for this random variable requires extensive simulations of numerical evaluations. Therefore, we take the comonotonic approximations proposed by Denuit and Dhaene (2007) and Denuit (2007) in the random walk with drift case and extended to general ARIMA models by Denuit, Haberman and Renshaw (2010). Specifically, we adapt this approach to the parametric projection models targeting mortality improvement rates (rather than mortality rates) proposed by Haberman and Renshaw (2012). The approach developed in the present paper helps avoid the requirement to conduct simulations with simulations in, for instance, Solvency 2 reserving calculations. Numerical illustrations show that the comonotonic approximations perform well for medians (and other central quantiles), which suggests that they can be used in practice to evaluate the consequences of the uncertainty in future death rates. Prediction intervals based on 5% and 95% quantiles are also considered but appear to be wider compared to simulated ones. This provides the practitioner with a conservative shortcut avoiding simulations within simulations.

This paper is organised as follows. Section 2 describes the mortality projection method based on parametric improvement rates. The comonotonic approximations are derived in Section 3. Section 4 is devoted to numerical illustrations. The final Section 5 briefly discusses the results.

2 Mortality improvement rates

We consider a rectangular data array, partitioned into unit squares of size one year corresponding to ages $x = x_1, x_2, ..., x_k$ and periods $t = t_1, t_2, ...t_n$. Denote $m_x(t) \equiv m_{x,t}$ the central rate of mortality (or death rate) at age x in period t.

Referring to Haberman and Renshaw (2012), under their Route II approach we consider the period-based mortality improvement rates (MIR) given by

$$Z_{x,t} = 2\frac{1 - m_{x,t}/m_{x,t-1}}{1 + m_{x,t}/m_{x,t-1}}.$$

In this definition, we consider the ratio of the period one-step mortality improvements to the average of the two adjacent mortality rates. A more natural definition of improvement rate would involve the initial rate in the denominator i.e. the rate at time t-1. The definition here avoids the phase difference between the numerator and denominator that would otherwise be present and background calculations indicate that it leads to improved modelling results. Following modelling and extrapolation, the MIR are converted to mortality rates (MR) using the reverse relationship

$$m_{x,t_n+j} = m_{x,t_n+j-1}\frac{\left(2 - z_{x,t_n+j}\right)}{\left(2 + z_{x,t_n+j}\right)} = m_{x,t_n+j-1}g\left(z_{x,t_n+j}\right)$$

where the function g is defined as

$$g(z) = \frac{2 - z}{2 + z}.$$

Given the nature of z, which typically take values well within the range $(-0.5,\ 0.5)$ as can be seen for Figure 3 in Renshaw and Haberman (2012), we can safely restrict g to the domain $z \in (-1,\ 2)$ where g is positive and decreasing.

In this paper, we consider

$$Z_{x+j,t_n+i} \equiv Z_{ji} = \beta_{x+j}\kappa_{t_n+i},$$

where Z_{ji} and κ_{t_n+i} are random variables and the β_{x+j} are considered as known constants. Henceforth, we assume that κ_{t_n+i} obeys some ARIMA time series model (and is therefore multivariate Normal). Then conditional on t_0 we have

$$Z_{ji} \sim N\left(\mu_{ji}, \sigma_{ji}^2\right)$$

with

$$\mu_{ji} = \beta_{x+j} E\left(\kappa_{t_n+i}\right),\ \sigma_{ji}^2 = \left(\beta_{x+j}\right)^2 \mathrm{Var}\left(\kappa_{t_n+i}\right).$$

3 Comonotonic approximations

In this section, we show that the theoretical arguments which formed the basis of Denuit, Haberman and Renshaw (2010) can be extended to provide approximations for quantiles of life expectancy and annuity predictions under parametric improvement rate modelling as defined in Section 2. There are, however some fundamental differences, as stressed below.

As in Section 2, we decompose the incremental mortality rate changes into

$$Z_{x+j,t_n+i} = \beta_{x+j}\kappa_{t_n+i} \sim N\left(\mu_{ji}, \sigma_{ji}^2\right).$$

The next result shows that assuming that the incremental mortality rate changes are perfectly correlated provides a conservative upper bound on future death rates. In this paper, we concentrate on u-type approximations for quantiles as the numerical study performed in Denuit, Haberman and Renshaw (2010) showed that they were more accurate than their l-type counterparts.

Before proceeding with this result, let us recall the definition of some useful stochastic order relations. For more details, we refer the interested reader to Denuit, Dhaene, Goovaerts and Kaas (2005). The increasing convex order, or stop-loss order (denoted as $\leq_{ICX}$) is defined for random variables X and Y as follows: $X \leq_{ICX} Y$ if $E[h(X)] \leq E[h(Y)]$ for all the non-decreasing convex functions h for which the expectations exist. In words, $X \leq_{ICX} Y$ means that X tends to be "smaller" and "less variable" than Y. The supermodular order (denoted as $\leq_{SM}$) is defined for random vectors $(X_1,...,X_n)$ and $(Y_1,...,Y_n)$ as $(X_1,...,X_n) \leq_{SM} (Y_1,...,Y_n)$ if $E[h(X_1,...,X_n)] \leq E[h(Y_1,...,Y_n)]$ for all the supermodular functions h for which the expectations exist. Recall that a (regular) supermodular function has a non-negative mixed partial derivative with respect to each pair of distinct components. In words, $(X_1,...,X_n) \leq_{SM} (Y_1,...,Y_n)$ means that the

components of $\left(X_1,...,X_n\right)$ are "less positively dependent" than the components of $\left(Y_1,...,Y_n\right)$.

Property. Let $Z \sim N\left(0,1\right)$. We then have the following upper bound on the death rate at age $x+j$ in calendar year $t_n + j$:

$$m_{x+j,t_n+j} \leq_{ICX} m_{x+j,t_n} \prod_{i=1}^{j} g\left(\mu_{ji} + \sigma_{ji} Z\right).$$

Proof. Whatever the dependent structure between the κ_{t_n+i}, $i = 1,2,3,...$, we have from Proposition 6.3.7 of Denuit, Dhaene, Goovaerts and Kaas (2005) that

$$\left(Z_{x+j,t_n+1}, Z_{x+j,t_n+2},..., Z_{x+j,t_n+j}\right) \leq_{SM} \left(\mu_{j1} + \sigma_{j1} Z, \mu_{j2} + \sigma_{j2} Z,..., \mu_{jj} + \sigma_{jj} Z\right).$$

The more the κ_{t_n+i} are positively related, the closer is the incremental mortality rate random vector to the upper bound in the $\leq_{SM}$ sense. Now, we get from Property 3.4.61(ii) of Denuit et al. (2005) that

$$\left(g\left(Z_{x+j,t_n+1}\right),..., g\left(Z_{x+j,t_n+j}\right)\right) \leq_{SM} \left(g\left(\mu_{j1} + \sigma_{j1} Z\right),..., g\left(\mu_{jj} + \sigma_{jj} Z\right)\right)$$

also holds. From Proposition 6.3.9 of Denuit et al. (2005), we finally see that

$$\prod_{i=1}^{j} g\left(Z_{x+j,t_n+j}\right) \leq_{ICX} \prod_{i=1}^{j} g\left(\mu_{ji} + \sigma_{ji} Z\right)$$

from which the announced result follows since a ranking in the $\leq_{ICX}$ sense is not affected by scaling (i.e. multiplication by $m_{x+j,tn}$). This completes the proof.

Now, let us denote as ${}_d P_x\left(t_n \mid \boldsymbol{\kappa}\right)$ the random d-year survival probability for an individual aged x in calendar year t_n, that is, the conditional probability that this individual reaches age $x+d$ in year $t_n + d$, given the vector $\boldsymbol{\kappa}$ of the κ_t. It is formally defined as

$${}_d P_x\left(t_n \mid \boldsymbol{\kappa}\right) = \exp\left(-S_d\right)$$

where

$$S_d = m_{x,t_n} + \sum_{j=1}^{d-1} m_{x+j,t_n} \prod_{i=1}^{j} g\left(Z_{x+j,t_n+i}\right).$$

We know from Proposition 3.4.29 of Denuit et al. (2005) that

$$S_d \leq_{ICX} S_d^u = m_{x,t_n} + \sum_{j=1}^{d-1} m_{x+j,t_n} \prod_{i=1}^{j} g\left(\mu_{ji} + \sigma_{ji} Z\right).$$

Here, we take $\exp\left(-S_d^u\right)$ as an approximation to the d-year survival probability ${}_d P_x\left(t_n \mid \boldsymbol{\kappa}\right) = \exp\left(-S_d\right)$ and we investigate its accuracy in the next section, based on numerical illustrations.

As a final comment, let us mention that the approach developed in the present section also applies to alternative specifications for $Z_{x,t}$. For instance, the comonotonic approximations also hold for models with a cohort effect as long as the individual under interest belongs to a cohort whose effect can be estimated from the available historical data. Specifically, we can also consider

$$Z_{x+j,t_n+i} = \beta_{x+j} \kappa_{t_n+i} + \iota_{t_n-x+i-j} \sim N\left(\mu_{ji}, \sigma_{ji}^2\right).$$

with

$$\mu_{ji} = \beta_{x+j} E\left(\kappa_{t_n+i}\right) + \iota_{t_n-x+i-j},\ \sigma_{ji}^2 = \left(\beta_{x+j}\right)^2 Var\left(\kappa_{t_n+i}\right)$$

as long as the cohort effect ι can be considered as constant (i.e. estimated from past data).

4. Numerical illustrations

Let us consider a basic life annuity contract paying 1 unit of currency at the end of each year, as long as the annuitant survives. The random life annuity single premium, that is, the conditional expectation of the payments made to an annuitant aged x in the year t_n given $\kappa_{t_n}, \kappa_{t_n+1}, \kappa_{t_n+2}, \ldots$ is

$$a_x\left(t_n \mid \boldsymbol{\kappa}\right) = \sum_{d \geq 1} {}_d P_x\left(t_n \mid \boldsymbol{\kappa}\right) \nu\left(0, d\right),$$

where $\nu\left(.,.\right)$ is the discount factor (precisely, $\nu\left(s,t\right)$ is the present value at time s of a unit payment made at time t). Note that $a_x\left(t_n \mid \boldsymbol{\kappa}\right)$ corresponds to the generation aged x in calendar year t_n, and accounts for future mortality improvements experienced by this particular cohort. Clearly, $a_x\left(t_n \mid \boldsymbol{\kappa}\right)$ is a random variable that depends on the future

trajectory $\kappa_{t_n}, \kappa_{t_n+1}, \kappa_{t_n+2}, \ldots$. An analytical computation of the distribution function of $a_x(t_n \mid \boldsymbol{\kappa})$ is out of reach.

From the approximation S_d^u assumed for S_d, we get the following approximation for the random survival probabilities

$$ {}_d P_x(t_n \mid \boldsymbol{\kappa}) \approx \exp\left(-F_{S_d^u}^{-1}(1-U)\right) $$

where U is uniformly distributed on the interval $(0,1)$. Note that the same random variable U is used for all of the values of d, making the approximations to the conditional survival probabilities comonotonic. Hence, we obtain the following approximation for $a_x(t_n \mid \boldsymbol{\kappa})$

$$ a_x(t_0 \mid \boldsymbol{\kappa}) \approx \sum_{d \geq 1} \exp\left(-F_{S_d^u}^{-1}(1-U)\right) v(0,d). $$

Since this approximation is a sum of comonotonic random variables, its quantile functions is additive. So, we obtain the following approximations for the quantile function $F_{a_x(t_n \mid \boldsymbol{\kappa})}^{-1}(z)$ of $a_x(t_n \mid \boldsymbol{\kappa})$

$$ F_{a_x(t_0 \mid \boldsymbol{\kappa})}^{-1}(z) \approx \sum_{d \geq 1} \exp\left(-F_{S_d^u}^{-1}(1-z)\right) v(0,d) $$

where

$$ F_{S_d^u}^{-1}(z) \approx m_{x,t_n} + \sum_{j=1}^{d-1} m_{x+j,t_n} \prod_{i=1}^{j} g\left(\mu_{ji} + \sigma_{ji} \Phi^{-1}(z)\right). $$

The random cohort life expectancy $e_x^{\square}(t_n \mid \boldsymbol{\kappa})$ is the conditional expected remaining lifetime of an individual aged x in year t_n, given $\kappa_{t_n}, \kappa_{t_n+1}, \kappa_{t_n+2}, \ldots$. Keeping the assumption that deaths are uniformly distributed over each calendar year, this demographic indicator is given by

$$ e_x^{\square}(t_n \mid \boldsymbol{\kappa}) = \frac{1}{2} + \sum_{d \geq 1} {}_d P_x(t_n \mid \boldsymbol{\kappa}). $$

We use the $\square$ superscript to indicate that we work along a diagonal band in the Lexis diagram. Except for the additive constant $1/2$, $e_x^{\square}(t_n \mid \boldsymbol{\kappa})$ coincides with $a_x(t_n \mid \boldsymbol{\kappa})$ if we

let the interest rate tend to zero. As was the case for $a_x(t_n \mid \boldsymbol{\kappa})$, an analytic computation of the distribution function of $e^{\square}_x(t_n \mid \boldsymbol{\kappa})$ is out of reach.

From the approximation S_d^u assumed for S_d, we get the approximation for $e^{\square}_x(t_n \mid \boldsymbol{\kappa})$

$$e^{\square}_x(t_0 \mid \boldsymbol{\kappa}) \approx \frac{1}{2} + \sum_{d \geq 1} \exp(-S_d^u).$$

Since the S_d^u's are sums of comonotonic random variables, their quantile functions are additive. Moreover, the zth quantile of $\exp\left(-S_d^u\right)$ is $\exp\left(-F^{-1}_{S_d^u}(1-z)\right)$. This provides the following approximation for the quantile function $F^{-1}_{e^{\square}_x(t_n \mid \boldsymbol{\kappa})}(z)$ of $e^{\square}_x(t_n \mid \boldsymbol{\kappa})$

$$F^{-1}_{e^{\square}_x(t_n \mid \boldsymbol{\kappa})}(z) \approx \frac{1}{2} + \sum_{d \geq 1} \exp\left(-F^{-1}_{S_d^u}(1-z)\right).$$

For the numerical results which follow, we use the 1961-2009 USA male and female mortality experiences with deaths and matching exposures by individual calendar year for individual ages 20-104 (the full age range being 0-109) available through the Human Mortality Database (HMD). Preliminary analysis including an analysis of residuals (not reproduced) is supportive of the inclusion of the cohort effects terms ι_{t-x} for males but not for females. Hence, we report the results for the respective H_1 formulation (see below for a precise definition) for males and the LC formulation for females by depicting the fitted parameter values in Figure 1. Also included are the period component time series forecasts using the selected AR(1) process for males and the simple AR(0) process for females, fitted as Gaussian regression models. Thus, for males, we have chosen to model the MIR Gaussian structure using the so-called H_1 formulation:

$$Z_{x+j,t_n+i} = \beta_{x+j}\kappa_{t_n+i} + \iota_{t_n-x+i-j} \sim N\left(\mu_{ji}, \sigma^2_{ji}\right)$$

whereas for females, we have modelled the structure based on the LC formulation:

$$Z_{x+j,t_n+i} = \beta_{x+j}\kappa_{t_n+i} \sim N\left(\mu_{ji}, \sigma^2_{ji}\right)$$

where

$$\mu_{ji} = \beta_{x+j} E\left(\kappa_{t_n+i}\right) + \iota_{t_n-x+i-j} \text{ or } \mu_{ji} = \beta_{x+j} E\left(\kappa_{t_n+i}\right) \text{ and } \sigma^2_{ji} = \left(\beta_{x+j}\right)^2 \mathrm{Var}\left(\kappa_{t_n+i}\right).$$

Using these parameters estimates and forecasts we tabulate (Table 1: 1st and 3rd panels) details for life expectancy and 4% annuity predictions, computed by cohort trajectory for ages 40, 45, 50, …75 focused on the year 2009. For convenience, we have not used the topping-out procedure advocated by Haberman and Renshaw (2012) for dealing with extrapolating the life table to the oldest ages. For comparison, we also tabulate (Table 1: 2nd and 4th panels) the respective equivalent life expectancies and 4% annuity predictions generated by the simulation method described in Haberman and Renshaw (2012), using a total of 10,000 simulations for each age. Referring to Table 1 and Figure 1 we note the following points:

- On comparing like for like, there is an exceptionally close agreement between the matching theoretical and simulated median predictions. However, the interval prediction widths in the theoretical cases are much wider when compared with the matching simulated cases.
- We note the narrowness of the simulated males prediction intervals, which are appreciably narrower than equivalent simulated intervals for the England & Wales male mortality experience depicted in Figure 8 of Haberman & Renshaw (2012), where topping-out by age has been applied but this seems to have little effect on increasing the interval widths.
- With the exception of a few isolated ages in the male experience, the beta parameters are positive over the full age range for males and females and therefore for both modelling structures. It would be possible to adapt the algorithms so that the beta parameters are constrained to be positive. We note further that the period index forecasts for mortality improvement rates are negative for males using H_1 but positive for females using LC.
- In order to reach the 99.5% solvency probability required under Solvency 2, we assume that the policyholders are required to provide premiums adding up to the 75th quantile of the present value of annuity payments and the insurer pays for the difference between the 99.5th quantile of these payments and the aforementioned 75th quantile. Here, we make the assumption that the size of the portfolio is large enough to neglect diversifiable risk so that only the systematic risk matters. The latter is equal to the size of the portfolio multiplied by the expected present value of the annuity payments given future mortality. The difference in the 99.5th and 75th quantiles appears in the last column of each panel in Table 1. Comparing the differences based on the approximations derived in the present paper to the simulated ones, we see there that the amount of capital is over-estimated when the approximations are used.

In the standard model's module for longevity risk, the Value-at-Risk at probability level 99.5% is approximated by the change in net asset value due to a pre-specified longevity shock based on a 25% reduction in mortality rates at all ages. Let us now explore the accuracy of the approximations derived in the present paper in dealing with such a shock. Specifically, we compare the values obtained from the approximations derived in the present paper to those coming from the standard formula which consists of reducing death rates by 25%. Table 2 is the same as Table 1 subject to a reduction in all projected mortality rates by a factor of 25%. The close agreement between simulated and theoretical medians is preserved, as in Table 1. Also the medians in Table 2 are

consistently higher than their matching counter-parts in Table 1 as expected. The approximations to the 95^{th} quantiles listed in Table 1 are reasonably close to their simulated counterparts in Table 2 based on a reduction of death rates by 25%. This shows that the approximations derived in the present paper for high quantiles may be used as an alternative to the standard approach which involves decreasing all death rates by, say, 25%.

5. Discussion

Combining a conservative shift with non necessary conservative ones, the approximations derived in the present paper appear to be very accurate in the centre of the distribution (around the median) but tend to over estimate the tails (left and right). Using the proposed easy-to-compute approximation may thus be a good strategy for the calculation of the percentiles in the centre of the distribution (for example, the valuation of the median) as it would considerably reduce the computational burden and save time.

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		life expectancy				4% annuity value			
age|	quantile	0.05	0.5	0.95	0.995-0.75	0.05	0.5	0.95	0.995-0.75
40		36.64	45.31	56.21	8.929	18.38	19.92	21.40	1.217
45		32.70	40.27	50.22	8.687	17.34	18.91	20.52	1.397
50		28.86	35.37	44.28	8.295	16.17	17.76	19.48	1.577
55		25.11	30.61	38.40	7.720	14.88	16.44	18.23	1.740
60		21.46	25.98	32.55	6.910	13.43	14.92	16.72	1.843
65		17.97	21.59	26.91	5.927	11.85	13.24	14.97	1.871
70		14.68	17.46	21.56	4.797	10.17	11.41	12.98	1.792
75		11.64	13.67	16.62	3.587	8.43	9.46	10.80	1.579

USA females- LC: theoretical predictions

		life expectancy				4% annuity value			
age|	quantile	0.05	0.5	0.95	0.995-0.75	0.05	0.5	0.95	0.995-0.75
40		43.56	45.22	46.94	2.002	19.64	19.91	20.17	0.288
45		38.70	40.21	41.76	1.746	18.62	18.90	19.18	0.310
50		33.95	35.32	36.74	1.635	17.44	17.75	18.06	0.352
55		29.36	30.57	31.85	1.448	16.11	16.43	16.76	0.373
60		24.87	25.95	27.06	1.328	14.58	14.91	15.25	0.396
65		20.65	21.56	22.50	1.132	12.89	13.23	13.57	0.395
70		16.70	17.44	18.22	0.914	11.07	11.39	11.73	0.375
75		13.04	13.64	14.28	0.749	9.15	9.45	9.76	0.356

USA females- LC: simulated predictions (10,000 simulations)

		life expectancy				4% annuity value			
age|	quantile	0.05	0.5	0.95	0.995-0.75	0.05	0.5	0.95	0.995-0.75
40		40.38	44.61	49.14	4.909	18.87	19.62	20.33	0.760
45		34.71	38.13	41.97	4.330	17.51	18.24	18.97	0.803
50		29.84	32.58	35.72	3.636	16.12	16.80	17.51	0.801
55		25.46	27.61	30.12	2.941	14.65	15.28	15.95	0.765
60		22.05	23.77	25.76	2.348	13.35	13.92	14.53	0.707
65		18.66	19.99	21.52	1.811	11.90	12.41	12.95	0.633
70		15.10	16.05	17.16	1.305	10.19	10.60	11.06	0.531
75		11.68	12.31	13.04	0.869	8.31	8.63	8.97	0.411

USA males-H1: theoretical predictions

		life expectancy				4% annuity value			
age|	quantile	0.05	0.5	0.95	0.995-0.75	0.05	0.5	0.95	0.995-0.75
40		43.88	44.54	45.18	0.729	19.49	19.61	19.73	0.131
45		37.52	38.10	38.66	0.618	18.11	18.24	18.36	0.133
50		32.06	32.54	33.03	0.566	16.67	16.80	16.92	0.145
55		27.18	27.59	28.00	0.444	15.15	15.28	15.40	0.137
60		23.39	23.74	24.09	0.418	13.79	13.91	14.03	0.140
65		19.67	19.96	20.25	0.335	12.28	12.40	12.51	0.131
70		15.81	16.03	16.25	0.267	10.49	10.59	10.69	0.122
75		12.13	12.29	12.46	0.193	8.53	8.62	8.70	0.100

USA males- H1: simulated predictions (10,000 simulations)

Table 1. USA female & male 2009 life expectancy and 4% annuity quantile predictions, ages 40(05)75: comparison of theoretical & simulated predictions.

		life expectancy				4% annuity value			
age	quantile	0.05	0.5	0.95	0.995-0.75	0.05	0.5	0.95	0.995-0.75
40		38.80	48.32	58.29	7.493	18.92	20.48	21.78	1.000
45		34.82	43.21	52.47	7.410	17.95	19.56	21.00	1.167
50		30.91	38.21	46.67	7.209	16.85	18.49	20.06	1.340
55		27.09	33.34	40.88	6.856	15.62	17.26	18.93	1.508
60		23.35	28.57	35.10	6.602	14.23	15.83	17.55	1.638
65		19.75	24.00	29.45	5.575	12.70	14.22	15.92	1.714
70		16.33	19.67	24.00	4.681	11.06	12.44	14.04	1.700
75		13.14	15.63	18.86	3.654	9.33	10.52	11.92	1.563

USA females- LC: theoretical predictions

		life expectancy				4% annuity value			
age	quantile	0.05	0.5	0.95	0.995-0.75	0.05	0.5	0.95	0.995-0.75
40		46.51	48.22	49.94	1.876	20.22	20.47	20.71	0.265
45		41.58	43.14	44.65	1.670	19.28	19.55	19.80	0.277
50		36.72	38.16	39.60	1.623	18.18	18.48	18.78	0.323
55		32.02	33.28	34.58	1.489	16.94	17.25	17.56	0.353
60		27.41	28.54	29.68	1.437	15.49	15.83	16.15	0.393
65		23.01	23.99	24.97	1.144	13.88	14.22	14.55	0.383
70		18.86	19.66	20.52	0.997	12.11	12.44	12.78	0.397
75		14.98	15.64	16.34	0.769	10.21	10.52	10.84	0.352

USA females- LC: simulated predictions (10,000 simulations)

		life expectancy				4% annuity value			
age	quantile	0.05	0.5	0.95	0.995-0.75	0.05	0.5	0.95	0.995-0.75
40		43.47	47.88	52.22	4.477	19.55	20.26	20.90	0.660
45		37.72	41.40	45.23	4.122	18.30	19.01	19.69	0.723
50		32.73	35.74	39.00	3.593	17.00	17.69	18.37	0.746
55		28.20	30.63	33.30	3.011	15.62	16.27	16.93	0.737
60		24.62	26.59	28.76	2.450	14.38	14.98	15.60	0.697
65		21.03	22.59	24.30	1.937	12.97	13.52	14.08	0.641
70		17.23	18.39	19.66	1.454	11.27	11.74	12.23	0.561
75		13.53	14.34	15.22	1.021	9.37	9.75	10.15	0.458

USA males-H1: theoretical predictions

		life expectancy				4% annuity value			
age	quantile	0.05	0.5	0.95	0.995-0.75	0.05	0.5	0.95	0.995-0.75
40		47.16	47.78	48.39	0.693	20.14	20.24	20.35	0.117
45		40.79	41.34	41.91	0.646	18.89	19.00	19.12	0.128
50		35.22	35.73	36.21	0.548	17.57	17.69	17.80	0.128
55		30.20	30.63	31.06	0.485	16.15	16.27	16.39	0.133
60		26.23	26.59	26.95	0.427	14.86	14.98	15.09	0.135
65		22.28	22.59	22.89	0.351	13.40	13.52	13.63	0.127
70		18.16	18.41	18.65	0.282	11.64	11.75	11.85	0.118
75		14.19	14.37	14.56	0.211	9.68	9.77	9.86	0.103

USA males- H1: simulated predictions (10,000 simulations)

Table 2. USA female & male 2009 life expectancy and 4% annuity quantile predictions, ages 40(05)75: comparison of theoretical & simulated predictions subject to a 25% reduction in projected mortality rates.