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On the quasi-yield surface concept in plasticity theory

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In this paper we provide deeper insights into the concept of the *quasi-yield surface* in plasticity theory. More specifically, in this work, unlike the traditional treatments of plasticity where special emphasis is placed on an unambiguous definition of a yield criterion and the corresponding loading-unloading conditions, we place emphasis on the study of a general rate equation which is able to enforce elastic-plastic behavior. By means of this equation we discuss the fundamental concepts of the elastic range and the elastic domain. The particular case in which the elastic domain degenerates into its boundary leads to the quasi-yield surface concept. We exploit this concept further by discussing several theoretical issues related to it and by introducing a simple material model. The ability of the model in predicting several patterns of the real behavior of metals is assessed by representative numerical examples.

1. Introduction

In a very recent paper, Xiao et al. [38] posed the question of whether one can construct rate equations which describe rate-independent elastic-plastic behavior, so that it's essential features, namely the yield criterion and the loading-unloading irreversibility, would not be introduced as extrinsic restrictive conditions, but instead will be derived directly by these equations. In the course of their analysis, Xiao et al. [38], derived a material model which had the ability of simulating several patterns of the real behavior of metals. These patterns comprised - but were not limited to - the prediction of plastic (irreversible) deformations at any stress level no matter how small the latter may be, and a continuous stress-deformation curve at the point of elastic-plastic transition. (For an alternative way of predicting a continuous stress-deformation curve where emphasis is placed in rate-dependent response see the recent works by Hollenstein et al. [11] and Jabareen [12]).

Our motivation for this paper is to provide deeper insights into the answer of the question posed by Xiao et al. in [38]. More specifically, in this work, on the basis of some ideas which go back to the classic paper by Lubliner [16] - see also Lubliner in [17,18] - we discuss a purely mathematical approach to elastic-plastic behavior, in which the basic ingredients of plasticity theory follow upon studying the properties of a suitably formulated differential equation. Within this context we pay special attention to a rather old concept, which has passed largely unnoticed within the literature of plasticity, namely the concept of the *quasi-yield surface*.

The basic steps of this study are as follows: In Section 2, we consider a general differential equation which aims to model rate-independent irreversible response and by means of it and an additional assumption underlying the loading-unloading behavior, we introduce the central concept of advanced plasticity theories, namely the *elastic range* (see Pipkin and Rivlin [30]; see also Lubliner [19,20]; Luchessi and Podio-Guidugli [24]; Bertram and Kraska [5]; Bertram [4]; Panoskaltsis et al. [27]). Several basic concepts of plasticity such as the loading rate, the elastic domain and the yield surface are also discussed within this framework. In Section 3, we deal with the particular case in which the basic equation is formulated in a way such as the elastic domain is degenerated to its boundary to form a surface; this surface is the aforementioned quasi-yield surface (Lubliner [18,19]). In Section 4, we provide additional insights to the quasi-yield surface concept upon introducing a material model. Finally, in Section 5, we demonstrate the ability of the model in predicting several patterns of the elastic-plastic behavior of metals by means of representative numerical examples.

2. Elastic and plastic processes; elastic range and domain

As a starting point we assume a homogeneous body undergoing finite deformation whose reference configuration - with points labeled by **X** - occupies a region B in the ambient space Ω. We define a motion of B in Ω as an one-parameter family of mappings $\varphi_t : B \rightarrow A$,

$$\mathbf{x}_t = \varphi_t(\mathbf{X}) = \varphi(\mathbf{X}, t), \ \mathbf{X} \in B, \ \mathbf{x} \in \Omega. \tag{1}$$

Then, the deformation gradient is the two-point tensor $\mathbf{F}$, defined as the tangent map of (1), that is

$$\mathbf{F} = T\varphi: \ T_{\mathbf{X}}B \to T_{\mathbf{x}}\Omega, \ \text{i.e.} \ \mathbf{F}_{iI} = \frac{\partial \varphi_i}{\partial X_I}(\mathbf{X}, t) \tag{2}$$

where $T_{\mathbf{X}}B$ and $T_{\mathbf{x}}\Omega$ stand for the tangent spaces at $\mathbf{X} \in B$ and $\mathbf{x} \in \Omega$, respectively.

If one assumes a referential description of the dynamical processes, the local mechanical state over the material point $\mathbf{X}$ can be determined by the second Piola-Kirchhoff stress tensor $\mathbf{S}$ and the internal variable vector $\mathbf{Q}$. We assume that the state (configuration) space S over the point $\mathbf{X}$ forms a local $(6+Q)$-dimensional manifold - where Q is the number of independent components of $\mathbf{Q}$ - with points denoted by $(\mathbf{S}, \mathbf{Q})$. A local process (at $\mathbf{X}$) is defined as a curve in S, that is as a mapping

$$\Psi: \ \mathrm{I} \in \mathbb{R} \to S, \ t \to (\mathbf{S}(t), \mathbf{Q}(t)),$$

where I is the time interval of interest. The direction and the speed of the process are determined by the tangent vector $\dot{\Psi}: S \to \mathrm{T}S$, with $\dot{\Psi}(t) = (\dot{\mathbf{S}}(t), \dot{\mathbf{Q}}(t))$, where $\mathrm{T}S$ is the tangent space of S. Since the stress rate $\dot{\mathbf{S}}(=\dot{\mathbf{S}}(t))$ is always known, the component $\dot{\mathbf{Q}}(=\dot{\mathbf{Q}}(t))$ of $\dot{\Psi}$ has to be determined. The latter may be assumed to be a function of the present values of the state variables and the stress rate, that is

$$\dot{\mathbf{Q}} = \mathbf{A}(\mathbf{S}, \mathbf{Q}, \dot{\mathbf{S}}), \tag{3}$$

where $\mathbf{A}: S \times \mathrm{T}S \to \mathrm{T}S$, is a vector field in S, which may be interpreted as a tensorial function of the state variables. In general, Eq. (3) introduces Q non-holonomic constraints (see, e.g., [1, pp. 624-629]) in S, a fact which from a physical stand point and since we deal with elastic-plastic (irreversible) response is desirable. However, from a mathematical stand point it may result in integrability problems. In order to surpass this, we further assume that the dependence of $\mathbf{A}$ on $\dot{\mathbf{S}}$ is linear, that is

$$\dot{\mathbf{Q}} = \mathbf{L}(\mathbf{S},\mathbf{Q}) : \dot{\mathbf{S}}, \tag{4}$$

where $\mathbf{L}$ is a tensor field in S. Motivated by the classical formulations of plasticity - see, e.g. [21, pp. 107,108] - we assume that the function $\mathbf{L}$ can be further decomposed as a tensor product as

$$\mathbf{L}(\mathbf{S},\mathbf{Q}) = \mathbf{A}(\mathbf{S},\mathbf{Q}) \otimes \boldsymbol{\Lambda}(\mathbf{S},\mathbf{Q}),$$

where $\mathbf{A}$ is a tensor field and $\Lambda : S \to T^{*}S$ is a one-form, so that Eq. (4) can be expressed as

$$\dot{\mathbf{Q}} = \mathbf{A}(\mathbf{S},\mathbf{Q})[\Lambda(\mathbf{S},\mathbf{Q}) : \dot{\mathbf{S}}] \tag{5}$$

We note that Eq. (5) is invariant under a replacement of t by $-t$ and accordingly enforces *reversible response* (see [29] for further details). On the other hand, *plastic behavior is an irreversible one*, a fact which calls for an appropriate modification of the rate equation (5). In order to accomplish this goal, we further assume this equation is able of simulating *two different types of possible material processes,* namely *quasi-static* and *dynamic* ones. More precisely, a material process Ψ may be defined as quasi-static if $\dot{\mathbf{Q}} = \mathbf{0}$, that is, if it lies entirely in a (6-dimensional) submanifold of S, defined by $\mathbf{Q} = const.$; a non quasi-static process is one which results in a change of the internal variable vector $(\dot{\mathbf{Q}} \neq \mathbf{0})$ and may be defined as a dynamic process. Herein, the terms *quasi-static* and *quasi-dynamic* are being used in complete analogy with classical thermodynamics, see, e.g., [40]. The concept of a quasi-static process leads to the concept of a *quasi-static range* which is defined at every material state of the material manifold Q

which comprises all material states $(\mathbf{S}^*, \mathbf{Q}^*)$ that can be reached from the current material state $(\mathbf{S}, \mathbf{Q})$ by a quasi-static process, that is

$$Q = \{(\mathbf{S}^*, \mathbf{Q}^*) \in S \,/\, \mathbf{S}^* = \mathbf{S} + d\mathbf{S}, \mathbf{Q}^* = \mathbf{Q}\},$$

where $d\mathbf{S}$ is an infinitesimal stress increment which can be interpreted as a one-form in *S*. In view of this definition, the quasi-static range can be determined as the union of the submanifolds Q_1 and Q_2 of *S*, which are defined as

$$Q_1 = \{(\mathbf{S}^*, \mathbf{Q}^*) \in S \,/\, \mathbf{A}(\mathbf{S}^*, \mathbf{Q}^*) = \mathbf{0} \text{ or } \boldsymbol{\Lambda}(\mathbf{S}^*, \mathbf{Q}^*) = \mathbf{0}\}, \tag{6}$$

where it is implied that the point $(\mathbf{S}^*, \mathbf{Q}^*)$ is attainable from $(\mathbf{S}, \mathbf{Q})$, and

$$Q_2 = \{(\mathbf{S}^*, \mathbf{Q}^*) \in S, \text{ where } (\mathbf{S}^*, \mathbf{Q}^*) \textit{ can be attained by a process with } \boldsymbol{\Lambda} : \dot{\mathbf{S}} = 0\} \tag{7}$$

As a first step, we disregard the (trivial) cases $\mathbf{A}(\mathbf{S}^*, \mathbf{Q}^*) = \mathbf{0}$ and $\boldsymbol{\Lambda}(\mathbf{S}^*, \mathbf{Q}^*) = \mathbf{0}$ i.e. we assume that $\mathbf{A}(\mathbf{S}^*, \mathbf{Q}^*) \neq \mathbf{0}$ and $\boldsymbol{\Lambda}(\mathbf{S}^*, \mathbf{Q}^*) \neq \mathbf{0}$, and we focus on the solutions of the equation

$$\boldsymbol{\Lambda}(\mathbf{S}, \mathbf{Q}) : \dot{\mathbf{S}} = 0 \tag{8}$$

which upon defining the Pfaffian form (see, e.g. [1, pp. 439-444])

$$\omega = \boldsymbol{\Lambda} : d\mathbf{S}, \tag{9}$$

results in the following Pfaffian equation

$$\omega = 0. \tag{10}$$

Then if the Pfaffian (one-form) (10) is completely integrable, there exists at the neighborhood of the current material state a scalar function (integrating factor) $\mu : S \to \mathbb{R}$ and a five-dimensional submanifold of *S*, defined by $F(\mathbf{S}, \mathbf{Q}) = const.$ - see [18,19] - such as

$$\Lambda(\mathbf{S},\mathbf{Q}) = \mu(\mathbf{S},\mathbf{Q})\frac{\partial F}{\partial \mathbf{S}}. \tag{11}$$

The submanifold $F(\mathbf{E},\mathbf{Q}) = const.$ is defined - see Eisenberg and Phillips [10]; see also [17,18] - as the *loading surface* (at $\mathbf{Q}$). Clearly, a process which lies entirely on the loading surface, that is one in which $\frac{\partial F}{\partial \mathbf{S}}:\dot{\mathbf{S}} = 0$ is a quasi-static one while processes with $\frac{\partial F}{\partial \mathbf{S}}:\dot{\mathbf{S}} \neq 0$ are dynamic ones.

Rate-independent plasticity - see, e.g., [16] - is closely tied to the concepts of loading and unloading. In order to involve these concepts in the analysis *we make the further assumption that a process with* $\frac{\partial F}{\partial \mathbf{S}}:\dot{\mathbf{S}} < 0,$ *results in quasi-static response* $(\dot{\mathbf{Q}} = \mathbf{0})$ *and may be defined as elastic unloading,* while a (dynamic) process with $\frac{\partial F}{\partial \mathbf{S}}:\dot{\mathbf{S}} > 0,$ which results in $\dot{\mathbf{Q}} \neq \mathbf{0},$ may be defined as *plastic loading*. The limiting case $\frac{\partial F}{\partial \mathbf{S}}:\dot{\mathbf{S}} = 0,$ may be defined as *neutral loading*. These concepts can be put together upon replacing the Pfaffian form ω in Eq. (5) by a fundamental concept in plasticity theory, namely that of the *loading rate* R - see, e.g., [16] - which may be defined as

$$R = \frac{\partial F}{\partial \mathbf{S}}:\dot{\mathbf{S}}.$$

Then Eq. (5) can be replaced by

$$\dot{\mathbf{Q}} = \mathbf{A}(\mathbf{S},\mathbf{Q})\langle R\rangle, \tag{12}$$

where $\langle\cdot\rangle$ stands for the Macauley bracket defined as $\langle x\rangle = \frac{x+|x|}{2}$ and it has been assumed that the function μ has been absorbed in $\mathbf{A}$.

We note that Eq. (12) is invariant under a replacement of t by $\varphi(t)$, where $\varphi(t)$ is any monotonically increasing, continuously differentiable function and accordingly enforces rate-independent response. Moreover, Eq. (12) constitutes the underlying equation of a general model of rate-independent elastic-plastic behavior called *generalized plasticity* (see Lubliner [20]; see also the later works given in [22,23,26,27,35]).

By means of Eq. (12) and by assuming that $\frac{\partial F}{\partial \mathbf{S}}(\mathbf{S},\mathbf{Q}) \neq \mathbf{0}$ in S we can define another fundamental concept, that of the elastic range E - see, e.g., [30,19,20,24] - as the submanifold of S which contains the points which can be reached from the current stress point as

$$E = \{(\mathbf{S},\mathbf{Q}) \in S \,/\, \mathbf{A}(\mathbf{S},\mathbf{Q})\big|_{\mathbf{Q}=const.} = \mathbf{0} \text{ or } R \leq 0\}.$$

REMARK 1: The present approach, which is based upon postulating a differential equation for the evolution of the internal variable vector and the *subsequent derivation of the elastic range by involving the concept of loading-unloading*, differs vastly from the standard approaches to the elastic range concept (see, e.g., [20,22,26,27]; see also [3,4]), where *the elastic range is considered as a primary concept* and the rate-equations are specified afterwards by imposing some regularity requirements and the rate-independence property. In this sense, the present approach resembles the general internal variable approach to material irreversible behavior discussed in Lubliner [16].

REMARK 2: It is stressed that Eq. (5), although is adequate to define the quasi-static range, it cannot define the elastic range unless a further assumption is made, that is a process with $\frac{\partial F}{\partial \mathbf{S}}:\dot{\mathbf{S}}<0$, is an (elastic) unloading process, that is in such a process the equality $\dot{\mathbf{Q}}=\mathbf{0}$ holds; as a matter of fact, this assumption introduces the concept of loading-unloading irreversibility in rate-independent plasticity.

REMARK 3: Since the present approach involves the stress tensor **S** and the stress-space loading rate $R=\frac{\partial F}{\partial \mathbf{S}}:\dot{\mathbf{S}}$, presupposes stability under stress control and accordingly is limited to work-hardening materials. Nevertheless, an equivalent approach which does not suffer from this limitation can be developed within the context of a strain (deformation) space formulation - see, e.g., [21, pp. 120-124]) - if **S** is replaced throughout by the right Cauchy-Green tensor **C,** which is defined in terms of the deformation gradient and the spatial metric $\mathbf{g}: T_{\mathbf{x}}\Omega \times T_{\mathbf{x}}\Omega \to \mathbb{R}$ as

$$\mathbf{C}=\mathbf{F}^{\mathrm{T}}\mathbf{g}\mathbf{F}.$$

It is also noted that the strain-space approach has been proven especially useful- see [27] - for a covariant formulation of the theory of rate-independent plasticity.

The submanifold D of S

$$D=\{(\mathbf{S},\mathbf{Q})\in S \,/\, \mathbf{A}(\mathbf{S},\mathbf{Q})=\mathbf{0}\}$$

which comprises only elastic processes may be defined as the *elastic domain D* and its boundary as the *yield hypersurface.* The intersection of the elastic domain with the submanifold of S, defined by $\mathbf{Q}=const.$, is defined as the elastic domain $D_{\mathbf{Q}}$ (at **Q**), while its boundary is defined to be the *yield surface*. Note that unlike the elastic range, which by definition is path connected and hence connected, the elastic domain can be a non-connected manifold. Accordingly, the yield surface in this case can be composed of several different independent submanifolds of *S*, which can be either disjoint, or intersect. We note also that the submanifold of Q_1 defined if $\Lambda(\mathbf{S}^*,\mathbf{Q}^*)\neq \mathbf{0}$ - recall Eq. (6) - is by construction a submanifold of $D_{\mathbf{Q}}$, but the converse is not necessarily true; this case may appear if the elastic domain is non-connected or contains isolated points which cannot be attained from the current material state.

Classical plasticity corresponds to the particular case when an additional constraint, namely the *invariance of the elastic domain under a plastic process* - see, e.g., [26,27] is introduced in the rate equation (12). If this is the case, the boundary of the elastic domain, i.e. the yield (hyper)surface, coincides with a unique loading (hyper)surface - say defined by $F(\mathbf{S},\mathbf{Q})=0$ - while the invariance condition (see, e.g., [1, pp. 256-257]) reads

$$\dot{\Psi}\cdot GRADF \leq 0. \tag{13}$$

where $(\cdot)$ stands for the inner product in *S*, and the gradient operator is defined as $GRAD(\cdot)=[\frac{\partial(\cdot)}{\partial\mathbf{S}},\frac{\partial(\cdot)}{\partial\mathbf{Q}}]$.

Then, the basic equations of classical rate-independent plasticity - see further [26,27] - can be derived upon assuming that the function $\mathbf{A}$ is of the form

$$\mathbf{A}(\mathbf{S},\mathbf{Q})=\frac{\langle F\rangle}{|F|}\mathbf{B}(\mathbf{S},\mathbf{Q})$$

and determining the limit

$$\lim_{F\to 0}\mathbf{A}(\mathbf{S},\mathbf{Q})=\lim_{F\to 0}\frac{\langle F\rangle}{|F|}\mathbf{B}(\mathbf{S},\mathbf{Q})R,$$

by means of the limiting case of Eq. (13) where the equality holds, that is

$$\dot{\Psi}:GRADF=\dot{F}(\mathbf{S},\mathbf{Q})=0 \tag{14}$$

which constitutes the *consistency condition* of classical plasticity.

An equivalent assessment of the theory in the spatial description can be derived upon performing a push-forward operation (see, e.g., [1, p. 355]; [36]) in Eq. (12). The resulting equation reads

$$L_{\mathrm{v}}\mathbf{q} = \mathbf{a}(\boldsymbol{\tau}, \mathbf{q}, \mathbf{F})\langle r \rangle, \tag{15}$$

where $\mathbf{q}$, $\mathbf{a}$ are the push-forwards of $\mathbf{Q}$ and $\mathbf{A}$ in the spatial configuration respectively, $\boldsymbol{\tau}$ is the Kirchhoff stress, i.e. $\boldsymbol{\tau} = \mathbf{F}\mathbf{S}\mathbf{F}^{\mathrm{T}}$, and r is the (scalar invariant) loading rate in the spatial configuration $r = \dfrac{\partial f}{\partial \boldsymbol{\tau}} : L_{\mathrm{v}}\boldsymbol{\tau}$ where $f = f(\boldsymbol{\tau}, \mathbf{q}, \mathbf{F})$ is the (spatial) expression for the loading surface. In component form, the push-forward operation for the arbitrary tensor $\mathbf{Q}$ reads

$$q^{i_1 \dots i_n}{}_{j_1 \dots j_m} = \frac{\partial x^{i_1}}{\partial X^{I_1}} \cdots \frac{\partial x^{i_n}}{\partial X^{I_n}} \frac{\partial X^{J_1}}{\partial x^{j_1}} \cdots \frac{\partial X^{J_m}}{\partial x^{j_m}} Q^{I_1 \dots I_n}{}_{J_1 \dots J_m}.$$

Finally, $L_{\mathrm{v}}(\cdot)$ stands for the (convected) Lie derivative (see further [1, pp. 359-369]; [36]) which is obtained by pulling back $\mathbf{q}$ to the reference configuration, taking its time derivative by keeping $\mathbf{X}$ fixed and pushing forward the result to the spatial configuration, that is:

$$\mathrm{L}_{\mathrm{v}}(\mathbf{q}) = \varphi_* \left(\frac{\partial}{\partial t} \varphi^*(\mathbf{q})_{\mathbf{X}=\mathrm{const.}} \Big| \right),$$

where $\varphi_*(\cdot)$ and $\varphi^*(\cdot) = \varphi_*^{-1}(\cdot)$ stand for the push-forward and the pull-back operations respectively.

3. The quasi-yield surface concept

An important particular case of the rate-equation (12) arises if the function $\mathbf{A}(\mathbf{S},\mathbf{Q})$ is non-vanishing in its arguments, so that the *elastic domain D vanishes.* In this case, there is no non-vanishing volume in S such as $R=\frac{\partial F}{\partial \mathbf{S}}(\mathbf{S},\mathbf{Q}):\dot{\mathbf{S}}=0.$ Nevertheless, as it is pointed out by Lubliner in [18] - see also [19] - since *loading can proceed in both the positive* $(R>0)$ *and the negative* $(R<0)$ *directions,* $\dot{\mathbf{S}}$ *has to take on both positive and negative values.* As a result, since $\dot{\mathbf{S}}\neq\mathbf{0}$, *there exists a surface on which* $\frac{\partial F}{\partial \mathbf{S}}=\mathbf{0}$; accordingly, the elastic domain degenerates to its boundary to form this surface, which may be defined as *a quasi-yield surface* (see further [18, 19]).

One may further assume that the function $\mathbf{A}$ is defined as

$$\mathbf{A}(\mathbf{S},\mathbf{Q})=\lambda(\mathbf{S},\mathbf{Q})\mathbf{M}(\mathbf{S},\mathbf{Q}), \tag{16}$$

where λ is a non-vanishing scalar function of the state variables and $\mathbf{M}:S\to \mathrm{T}S$ is a another (non-vanishing) tensorial function which accounts for the direction of plastic flow. Upon substitution of Eq. (16) into Eq. (12) we derive a rather general expression for the rate equations for a material possessing a quasi-yield surface as

$$\dot{\mathbf{Q}}=\lambda(\mathbf{S},\mathbf{Q})\mathbf{M}(\mathbf{S},\mathbf{Q})\langle R\rangle,\ (\lambda,\mathbf{M})\neq(0,\mathbf{0})\ \textit{for all}\ (\mathbf{S},\mathbf{Q})\in S. \tag{17}$$

From a physical point of view, for a material which possesses a quasi-yield surface any process with $R>0$ will result in a plastic process, irrespectively of the value of stress in the state in question; *accordingly plastic deformation appears upon loading at any stress level no matter how small it may be*. This response constitutes the very essence of the real elastic-plastic behavior of metals, especially at high rates of loading. Characteristic here is the following comment stated by J.F Bell in [2]: "*It was impossible to determine an elastic limit in the sense that all deformation was completely reversible … given sufficient accurate instrumentations one could always find permanent deformation associated with each elastic deformation.*". A similar like response is reported in the very recent paper by Chen et al. [8], who upon performing hundreds of high-precision loading-unloading-reloading tests conclude as: "*There is no significant linear elastic region, that is, the proportional limit is* 0 Mpa. *While the first increment of deformation shows a stress-strain slope equal to Young's modulus, progressive deviations of slope start immediately.*".

Another case of interest which is closely tied to the quasi-yield surface concept arises in metals at extremely high rates of loading - see, e.g., [21, pp. 108-109], [28] - where, during the various rate processes, different mechanisms within the same material respond in different characteristic times. These characteristic times may be very short and of the same order compared to a typical loading process. The first type of these mechanisms gives rise to instantaneous plastic strains and the second type to creep strains, which develop slowly. Such a response can be predicted upon combining a quasi-yield surface model with a rate-dependent (viscoplastic) model. In this case the basic rate equation (17) can be extended as

$$\dot{\mathbf{Q}} = \lambda(\mathbf{S},\mathbf{Q})\mathbf{M}(\mathbf{S},\mathbf{Q})\langle R\rangle + \mathbf{L}(\mathbf{S},\mathbf{Q}), \qquad (18)$$

where $\mathbf{L}: S \to \mathrm{T}S$ is another (non-vanishing) tensorial function of the internal variables which enforces the rate-dependent characteristics of the material. In general, the function $\mathbf{L}$ has to be determined in a manner such that for static and quasi-static rates the response is determined solely by the rate-dependent part of the model, while for dynamic ones the response is dominated by its dynamic (rate-independent) part. More information on this issue can be found in Panoskaltsis et al. [28].

To this end it is instructive to examine the integrability of the rate equation (17). By inspection we realize that the later is equivalent to the following Pfaffian system - see, e.g., [1, p. 443] - in S

$$d\mathbf{Q} = \lambda(\mathbf{S},\mathbf{Q})\mathbf{M}(\mathbf{S},\mathbf{Q})(\frac{\partial F}{\partial \mathbf{S}} : d\mathbf{S}). \qquad (19)$$

By assuming the general case, where the independent components of $\mathbf{Q}$, are tensors of type (N,M), with components $Q^{I_1 \ldots I_N}_{\;J_1 \ldots J_M}$ the system (19) reads

$$\omega^{I_1 \ldots I_N}_{\;J_1 \ldots J_M} = 0, \qquad (20)$$

where $\omega^{I_1 \ldots I_N}_{\;J_1 \ldots J_M}$ stands for the differential form

$$\omega^{I_1 \ldots I_N}_{\;J_1 \ldots J_M} = dQ^{I_1 \ldots I_N}_{\;J_1 \ldots J_M} - \lambda(S^{AB}, Q^{A_1 \ldots A_N}_{\;B_1 \ldots B_M}) M^{I_1 \ldots I_N}_{\;J_1 \ldots J_M}(S^{AB}, Q^{A_1 \ldots A_N}_{\;B_1 \ldots B_M}) \frac{\partial F}{\partial S^{IJ}} dS^{IJ}. \qquad (21)$$

By having Eqs. (20), (21) at hand, there are sufficient conditions for the application of the Frobenius theorem - see, e.g., [1, p. 443] - which states that (20) is completely integrable if and only if

$$C^{I_1 \ldots I_N}_{\;J_1 \ldots J_M IJKL} = 0, \qquad (22)$$

where $C^{I_1 \ldots I_N}_{\;J_1 \ldots J_M IJKL}$ are functions of the state variables which are given as

$$\begin{aligned} C^{I_1 \ldots I_N}_{\;J_1 \ldots J_M IJKL} = {} & \frac{\partial[\lambda M^{I_1 \ldots I_N}_{\;J_1 \ldots J_M} \frac{\partial F}{\partial S^{IJ}}]}{\partial S^{KL}} - \frac{\partial[\lambda M^{I_1 \ldots I_N}_{\;J_1 \ldots J_M IJ} \frac{\partial F}{\partial S^{KL}}]}{\partial S^{IJ}} + \\ & + \lambda M^{K_1 \ldots K_N}_{\;L_1 \ldots L_M} \frac{\partial F}{\partial S^{KL}} \frac{\partial[\lambda M^{I_1 \ldots I_N}_{\;J_1 \ldots J_M} \frac{\partial F}{\partial S^{IJ}}]}{\partial Q^{K_1 \ldots K_N}_{\;L_1 \ldots L_M}} - \lambda M^{K_1 \ldots K_N}_{\;L_1 \ldots L_M} \frac{\partial F}{\partial S^{IJ}} \frac{\partial[\lambda M^{I_1 \ldots I_N}_{\;J_1 \ldots J_M} \frac{\partial F}{\partial S^{KL}}]}{\partial Q^{K_1 \ldots K_N}_{\;L_1 \ldots L_M}}. \end{aligned} \qquad (23)$$

which constitute the desired integrability conditions.

REMARK4: The Pfaffian system (20) may be written equivalently - see further [1, p. 443] - as

$$\frac{\partial Q^{I_1 \dots I_N}{}_{J_1 \dots J_M}}{\partial S^{IJ}} = \lambda(S^{AB}, Q^{A_1 \dots A_N}{}_{B_1 \dots B_M}) M^{I_1 \dots I_N}{}_{J_1 \dots J_M}(S^{AB}, Q^{A_1 \dots A_N}{}_{B_1 \dots B_M}) \frac{\partial F}{\partial S^{IJ}}. \qquad (24)$$

This form has the advantage of allowing us a geometrical interpretation of the solutions. More precisely, if (24) is completely integrable, there exists a 6-dimensional submanifold *P of S*, with equation $\mathbf{Q} = \mathbf{G}(\mathbf{S})$, such that the vectors

$$\lambda(S^{AB}, Q^{A_1 \dots A_N}{}_{B_1 \dots B_M}) M^{I_1 \dots I_N}{}_{J_1 \dots J_M}(S^{AB}, Q^{A_1 \dots A_N}{}_{B_1 \dots B_M}) \frac{\partial F}{\partial S^{IJ}}.$$

are tangent to *P*, at every point $(\mathbf{S}, \mathbf{Q})$, with (local) coordinates $(S^{AB}, Q^{A_1 \dots A_N}{}_{B_1 \dots B_M})$.

REMARK 5: Wherever the integrability conditions (22) hold, *the constraints imposed in S by* (20) *are holonomic* and accordingly the (dynamical) system whose evolution is underlined by Eq. (17) is a holonomic one. Such a consequence plays a prominent role when one deals with stability postulates and/or invariance concepts within a Hamiltonian formulation of plasticity. For instance, the dissipation function $\widehat{D} : S \times TS \to \mathbb{R}$ of the system, that is

$$\widehat{D}(\mathbf{Q}, \mathbf{S}, \dot{\mathbf{S}}) = -\frac{\partial E}{\partial \mathbf{Q}} : \dot{\mathbf{Q}} = -\frac{\partial E}{\partial \mathbf{Q}} : \mathbf{M}(\mathbf{S}, \mathbf{Q})(\frac{\partial F}{\partial \mathbf{S}} : \dot{\mathbf{S}}),$$

where E is the internal energy density function, can be associated with a Lagrangian *L* in *S*, which *can be expressed solely in terms* of $\mathbf{S}$ and $\dot{\mathbf{S}}$, that is

$$D(\mathbf{Q}, \mathbf{S}, \dot{\mathbf{S}}) = L(\mathbf{G}(\mathbf{S}), \mathbf{S}, \dot{\mathbf{S}}).$$

REMARK 6: Another important consequence of the integrability of Eq. (17) appears if one considers the general case of combined models of rate-independent and rate dependent behavior discussed above (recall Eq. (18)). Then if the integrability conditions (22) hold, there exists a (local) transformation of the state space $\mathbf{R}: S \to S,\ \mathbf{R} = \mathbf{R}(\mathbf{S},\mathbf{Q})$ - see [16] - such as the rate Eq. (18) can be written in the form

$$\dot{\mathbf{R}} = \mathbf{P}(\mathbf{S},\mathbf{R}),$$

where $\mathbf{R}$ stands for the new (transformed) internal variable vector. This rate equation constitutes the basic state equation of a general class of models of highly non-linear rate-dependent response, which are usually termed within the literature as “unified” viscoplasticity models (see, e.g., [21, pp. 109, 110]). These models, besides being consistent with dislocation dynamics - see Bodner [6] - are extremely useful in the analysis of rate-sensitive materials, especially in cases of dynamic loadings. Models of this type have been proposed, among others, by Bodner and Partom [7] and Rubin [31,32].

4. A model problem

Up to now our formulation has been discussed largely in an abstract manner, by leaving the kinematics of the problem and the kind and the number of the internal variables entirely unspecified. In this section we present a material model to clarify the application of the quasi-yield surface concept for the constitutive modeling of solid materials.

As a basic kinematic assumption we consider a local multiplicative decomposition of the deformation gradient (see, e.g., [25,14,13]; see also [19,20,33]) into elastic $\mathbf{F_e}$ and plastic parts $\mathbf{F_p}$ parts, i.e.

$$\mathbf{F} = \mathbf{F_e F_p}.$$

Consistently, with the developments given in section 2 - see also [33], [34, pp. 302-311] - the formulation of the model may, in principle, be given equivalently with respect to the reference or the spatial configuration. Since we deal with large scale plastic flow, kinematical arguments together with the concept of spatial covariance - see e.g. [33,27] - suggest that, a formulation of the model in the spatial configuration is more fundamental. Thus, by following Simo in [33] we define the left elastic Cauchy-Green tensor $\mathbf{b_e}$ as

$$\mathbf{b_e} = \mathbf{F_e F_e^T}.$$

Since $\mathbf{b_e}$ is symmetric and positive-definite, it can serve as a primary measure (metric) of plastic deformation and accordingly a flow rule can be formulated in terms of its Lie derivative (see further [33]; see also the recent developments given in [11,12]).

The internal variable vector $\mathbf{q}$ is assumed to be composed by $\mathbf{b_e}$, a scalar internal variable κ which serves as a measure of the isotropic hardening of the loading surfaces and a deviatoric tensorial internal variable $\mathbf{a}$ (back-stress), which serves as a measure of their directional hardening. In component form the internal variable vector reads

$$\mathbf{q} = \begin{bmatrix} b_e^{ij} \\ \kappa \\ a^{ij} \end{bmatrix}.$$

Motivated by classical metal plasticity we introduce a von-Mises type expression for the loading surfaces, that is

$$f(\boldsymbol{\tau},\mathbf{g},\kappa,\mathbf{a})=\sqrt{(\tau'^{ij}-a^{ij})(\tau'^{kl}-a^{kl})g_{ik}g_{jl}}-\sqrt{\frac{2}{3}}K\kappa=const.,$$

where g_{jl} are components of the spatial metric, τ'^{ij} are the components of the deviatoric Kirchhoff stress tensor, i.e.

$$\tau'^{ij}=\tau^{ij}-\frac{1}{3}(\tau^{kl}g_{kl})(g^{-1})^{ij},$$

and K is a model parameter designating (isotropic) hardening.

The evolution of plastic flow is considered to be normal to the loading surfaces as per

$$\begin{aligned}&L_{\mathbf{v}}\mathbf{b}_{\mathbf{e}}=\lambda(\boldsymbol{\tau},\mathbf{g},\kappa,\mathbf{a})\frac{\partial f}{\partial\boldsymbol{\tau}}(\mathbf{g}^{-1}\otimes\mathbf{g}^{-1})\langle r\rangle,\ \text{i.e.,}\\&(L_{\mathbf{v}}b_{e})^{ij}=\lambda(\tau^{ab},g_{ab},\kappa,a^{ab})\frac{\partial f}{\partial\tau^{kl}}g^{ki}g^{lj}\langle r\rangle,\end{aligned}\tag{25}$$

where the function λ is assumed to be an isotropic function in all its arguments, so that the principle of material frame-indifference - see, e.g., [33, p. 272], [35] - is satisfied.

In accordance with the infinitesimal theory - see, e.g., [33, pp. 90-91; 310-311] - we adopt the following evolution equations for the remaining internal variables

$$\dot{\kappa}=\sqrt{\frac{2}{3}}\lambda(\boldsymbol{\tau},\mathbf{g},\kappa,\mathbf{a})\langle r\rangle,\tag{26}$$

$$L_{\mathbf{v}}\mathbf{a}=\frac{2}{3}HL_{\mathbf{v}}\mathbf{b}_{\mathbf{e}},\tag{27}$$

where H is the (linear kinematic) hardening modulus.

Finally, the stress response is assumed to be hyperelastic, governed by an isotropic strain energy function in terms of the first (i_1) and the third (i_3) invariants of $\mathbf{b}_{\mathbf{e}}$ - see e.g. [34, pp. 258,259] which reads

$$\rho e(i_1,i_3)=\frac{\lambda'}{4}(i_3-1)-(\frac{\lambda'}{2}+\mu')\ln\sqrt{i_3}+\frac{\mu'}{2}(i_1-3),$$

where ρ is the density in the spatial configuration and λ', μ' are the (elastic) material parameters to be the Lame$'$ parameters, which are related to the standard elastic constants E and ν by

$$\lambda' = \frac{\nu E}{(1+\nu)(1-2\nu)}, \quad \mu' = \frac{E}{2(1+\nu)}.$$

Then, the Cauchy-stress tensor $\boldsymbol{\sigma}$ is determined by the Doyle-Ericksen formula $\boldsymbol{\sigma} = 2\rho \frac{\partial e}{\partial \mathbf{g}}$ - see, e.g., [36,27,29] - which yields

$$\boldsymbol{\sigma} = \frac{\lambda'}{2}(i_3 - 1)\mathbf{g}^{-1} + \mu'(\mathbf{b_e} - \mathbf{g}^{-1}). \tag{28}$$

In order to close the model equations it remains to determine the form of the function λ but before we address this issue, we present several ideas underlying its importance.

REMARK 7: We consider the particular case where the ambient space is Euclidean so that the spatial metric coincides with the Euclidean one $\mathbf{i}$ and the material is elastic-perfectly plastic. In this case, the von-Mises loading surface is expressed in the following remarkably simple form

$$f(\boldsymbol{\tau}) = \|\boldsymbol{\tau}'\| = const.,$$

with normal vector $\frac{\partial f}{\partial \boldsymbol{\tau}} = \frac{\boldsymbol{\tau}'}{\|\boldsymbol{\tau}'\|}$, where $\|\cdot\|$ stands for the Euclidean norm. Then the flow rule (25) reads

$$(L_{\mathrm{v}} b_e)_{ij} = \frac{\lambda(\tau_{ab}, b_{ecd})}{(\tau'_{mn}\tau'_{nm})^2} \tau'_{ij}\tau'_{kl}(L_{\mathrm{v}}\tau)_{kl} \tag{29}$$

Upon noting that the Lie derivative operator $L_{\mathrm{v}}(\cdot)$ shares the same properties with the standard differential operator $d(\cdot),$ the solutions of Eq. (29) will be identical with those of following differential equation

$$\frac{db_{eij}}{d\tau_{kl}} = \frac{\lambda(\tau_{ab}, b_{ecd})}{(\tau'_{mn}\tau'_{nm})^2}\tau'_{ij}\tau'_{kl}$$

which means that, in this case, the function λ controls directly the *shape of the stress - (plastic) deformation curve.*

REMARK 8: Several choices of the function $\lambda,$ may be made if one starts by postulating *that the quasi-yield (hyper)surface is invariant under a plastic process,* with the invariance condition (recall section 2) being $\dot{f} = 0,$ that is

$$\frac{\partial f}{\partial \boldsymbol{\tau}} : L_{\mathrm{v}}\boldsymbol{\tau} + \frac{\partial f}{\partial \kappa}\dot{\kappa} + \frac{\partial f}{\partial \mathbf{a}} : L_{\mathrm{v}}\mathbf{a} = 0. \tag{30}$$

Upon substituting from Eqs. (25) to (27) and defining $h = -\sqrt{\frac{2}{3}}\frac{\partial f}{\partial \kappa} - \frac{2}{3}H\frac{\partial f}{\partial \mathbf{a}} : \frac{\partial f}{\partial \boldsymbol{\tau}},$ Eq. (30) reads

$$r(1 - h\lambda) = 0,$$

which, for a plastic process $(r > 0),$ yields $\lambda = \frac{1}{h},$ so that the flow rule (25) takes the form

$$L_{\mathrm{v}}\mathbf{b}_{\mathbf{e}} = \frac{1}{h}\frac{\partial f}{\partial \boldsymbol{\tau}}\langle r \rangle. \tag{31}$$

By means of the flow rule (31), one can derive *a large class of models upon scaling it by a non-vanishing function* (e.g. exponential, hyperbolic) $y = y(\boldsymbol{\tau}, \kappa, \mathbf{a})$, as far as the integrability conditions (23) hold. The resulting flow rule reads

$$L_{\mathrm{v}}\mathbf{b}_{\mathbf{e}} = y(\boldsymbol{\tau}, \kappa, \mathbf{a}) \frac{1}{h} \frac{\partial f}{\partial \boldsymbol{\tau}} \langle r \rangle. \tag{32}$$

Note that the scaling function y determines the relative placing of the material state with respect to the quasi-yield (hyper)surface, in the course of plastic deformation.

REMARK 9: The idea discussed in Remark 6 is the one appearing within a somewhat different kinematic context in Xiao et al [38]; see also example 3.4 in [35]. More specifically these authors determine the function y upon making a shift of emphasis from the quasi-yield (hyper)surface to a particular loading surface - termed therein as the “bounding surface” - which is defined as

$$f(\boldsymbol{\tau}, \kappa, \mathbf{a}) = g(\boldsymbol{\tau}, \mathbf{a}) - j(\kappa) = 0,$$

where

$$g(\boldsymbol{\tau}, \mathbf{a}) = \|\boldsymbol{\tau}' - \mathbf{a}\|, \; j(\kappa) = \sqrt{\frac{2}{3}}(K\kappa + c_0),$$

and c_0 is identified as a model parameter. Then, the function y can be specified upon demanding that this loading surface plays the role of *a yield-like* (hyper)surface, so that for large scale plastic flow defined by $f > 0$, the material state remains close to it; accordingly the authors suggest an exponential type of function which fulfills this requirement, that is

$$y = -\frac{g}{j} \exp[-m(1 - \frac{g}{j})],$$

where m is an additional model parameter.

In this work, for the function λ we assume an expression discussed within the context of the infinitesimal theory in [23], which within the present (large deformation) formulation is expressed in in the following somewhat surprising format

$$\lambda = -\frac{1}{2}\frac{f}{\beta(H+K)+R(\beta-f)},$$

in which β and R (from now on) are two model parameters.

5. One-component loadings

In this section we implement the proposed model numerically - see, e.g., [34, pp. 311-320, 26] for computational details - in order to show its ability in predicting several patterns of some complex phenomena which appear in metallic alloys. In particular, we consider two cases of one-component loadings: one of a simple shear and another one of uniaxial tension.

5.1 Simple shear

The simple shear problem constitutes a standard test within the context of large deformation plasticity - see, e.g. [2,15,9,35] - and is defined (recall Eq. (1)) - as:

$$x^1 = X^1 + \gamma X^2, \;\; x^2 = X^2, \;\; x^3 = X^3,$$

where $\gamma = \gamma(t)$ is the applied shear. Our purpose in this example is to present the monotonic curves predicted by the model for different values of the parameter β. The remaining model parameters are set equal to

$$E = 300.00,\ \nu{=}0.3,\ \text{R=30.00, Perfect plasticity } K = H = 0.$$

The results are shown in Fig. 1 and Fig. 2 for the shear τ_{12} and the normal τ_{11} stress components, respectively. By referring to Fig.1, we observe that the model predicts continuous stress-deformation curves, *with a non-unambiguously specified elastic portion and a non-well defined yield stress*, which as the deformation increases converge to a (constant) stress which may be defined as the material ultimate strength; we note that the higher the value of β, the higher is the predicted ultimate strength. Such a response is in absolute accordance with the one exhibited by almost all advanced metallic alloys; compare for instance the predicted behavior with the ones reported by Chen et al. in [8].

5.2 Tension-compression tests

As a second example we discuss the predictions of the model for some tension-compression tests. These tests, in general, are defined as

$$x^1 = (1+\chi)X^1,\ \ x^2 = (1+\psi)X^2,\ \ x^3 = (1+\psi)X^3,$$

where $1+\chi(t)$ and $1+\psi(t)$ are the principal stretches along the longitudinal and the transverse directions respectively. By means of this example we'll demonstrate the ability of the model in predicting several patterns of the real response of metals which cannot be predicted by the conventional plasticity models.

As a first simulation we consider a loading history comprising loading-unloading-reloading. The results for two different values of the parameter R and a constant value of β (β=5), are shown in Fig. 3. In this case we verify the ability of the model to predict the real response of metals - recall section 3 - according to which, the reloading, following (plastic) loading and subsequent (elastic) unloading, results at plastic deformation at any stress level. Moreover, depending on the value of R, the reloading curve may or may not converge (asymptotically) to the monotonic loading curve. The later pattern of response corresponds to the so-called *long-term* or *permanent softening effect* (see, e.g., [39,37]), which plays an important role in the numerical simulation and design of metal sheets in forming processes. This phenomenon appears alike in a (two-sided) tension-compression test (see Fig. 4).

As a second simulation we study the (low cycle) fatigue behavior at low stress levels (see Fig. 5). For this purpose we perform a loading-unloading-reloading test at a small stress level, by selecting a value for R (R=30), such as the reloading curve convergences to the corresponding loading curve. Next, we perform a loading-unloading test, but now the specimen is subjected to a cyclic loading with stress amplitude equal to the stress level where the (first) unloading began. Upon referring to the results of Fig. 5, we note the ability of the model in predicting (real) material behavior, which consists of the appearance of residual strains - apparently plastic - and accumulation of plastic work. Moreover, due to the material fatigue, permanent softening phenomena appear in a rather profound manner.

As a final simulation, we consider the case where the material is subjected to (two-sided) cyclic loading. For this problem we consider that the kinematic hardening law (27) may be replaced by the standard (non-linear) Armstrong-Frederic hardening law, i.e.

$$L_{\mathrm{v}}\mathbf{a}=\frac{2}{3}HL_{\mathrm{v}}\mathbf{b}_{\mathbf{e}}-L\mathbf{a}\dot{\kappa},$$

where L is the non-linear (kinematic) hardening modulus. The remaining model parameters are set equal to

$$E=300.00,\ \ \nu=0.3,\ \mathrm{R}=30.00,\ K=0.10,\ H=0.3,\ L=30.$$

The results of this test, for two different values of the parameter β are shown in Figs. 6 and 7. The model predicts stresses which are increasing as the number of cycles increases and eventually stabilize at a constant value after a few cycles. This response constitutes the very essence of the cyclic behavior of mild steels (see, e.g., Fig. 6a in [39]). The predictions of the model in the absence of hardening mechanisms $(K=H=L=0)$ are also presented in Fig. 8 (β=3). In this case, the model has the ability to predict almost stabilized stress-deformation curves from the first cyclic of strain. This response is identical to the one exhibited by dual-phase high strength steel specimens (see, e.g., figure 6b in [39])

6. Concluding remarks

The basic impact of this paper relies crucially in providing deeper insights into the quasi-yield surface concept in plasticity theory. In particular in this paper:

i. Motivated by a question posed in a very recent paper by Xiao et al. [37], we have shown how the basic concepts in plasticity theory can be introduced in a purely mathematical manner, upon studying the properties of a suitably formulated differential equation and involving the basic concepts of loading and unloading. The proposed formulation is rather general and includes classical plasticity as a special case.

ii. We have revisited the quasi-yield surface concept by clarifying some basic theoretical issues related to it.

iii. We have shown how the concept can be applied in the constitutive modeling of solid materials and in particular in metals, upon developing a rather simple material model.

Moreover, we have implemented the model numerically and we have demonstrated its ability in predicting several patterns of the complex response of metals which cannot be predicted by the conventional plasticity models.

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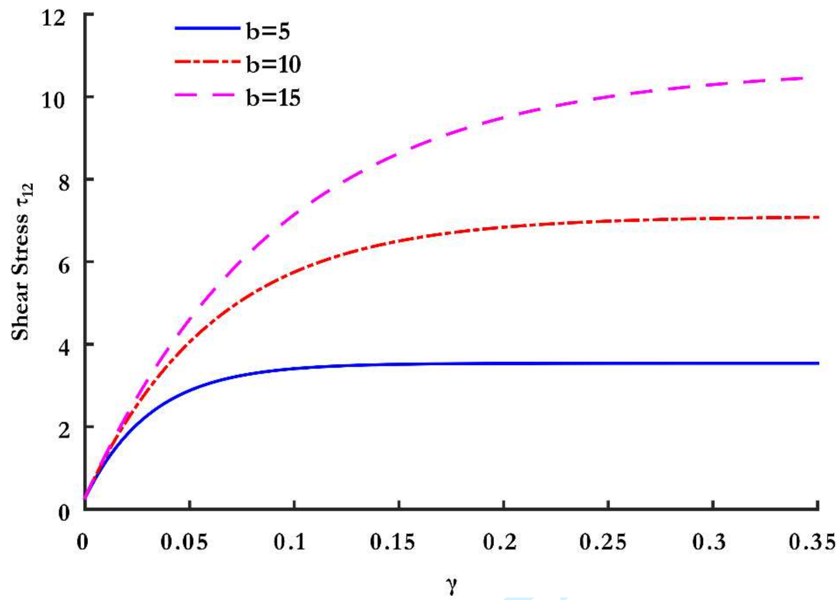

Fig. 1: Simple shear: Shear stress τ_{12} vs. shear strain (γ).

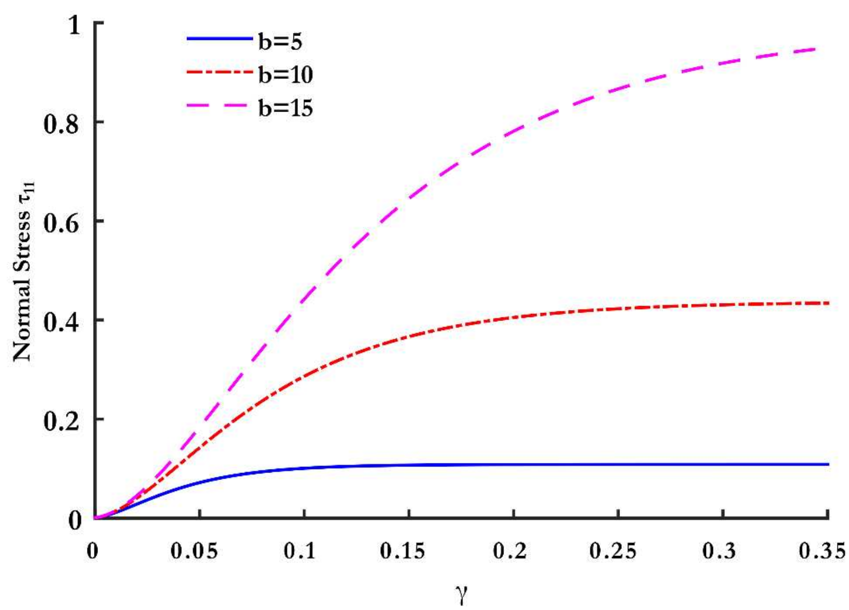

Fig. 2: Simple shear: Normal stress τ_{11} vs. shear strain (γ).

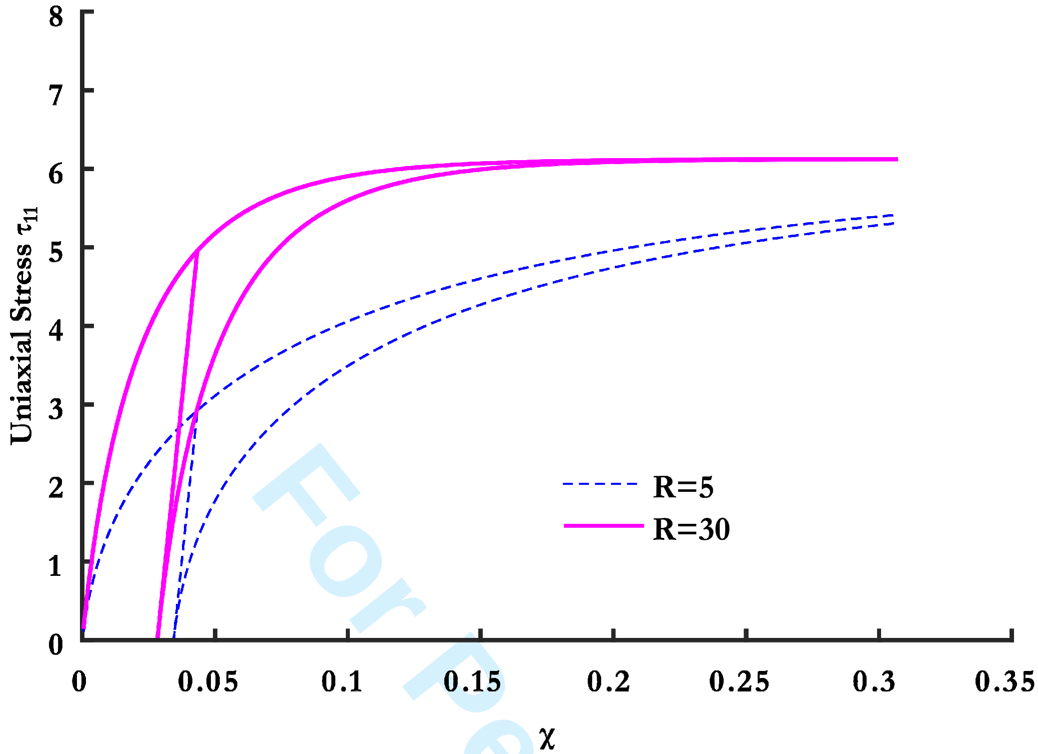

Fig. 3: Tension-compression: Loading-unloading-reloading (one-sided).

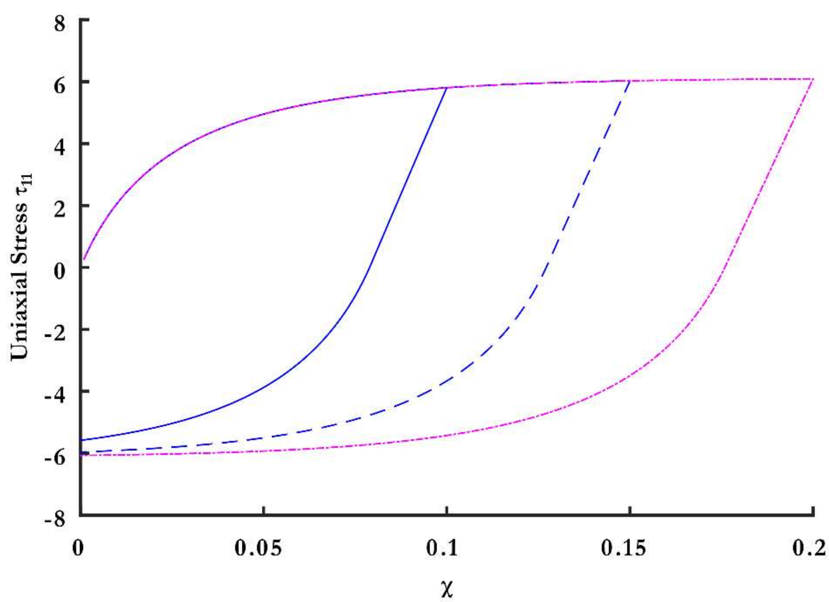

Fig. 4: Tension-compression: Loading-unloading-reloading (two-sided).

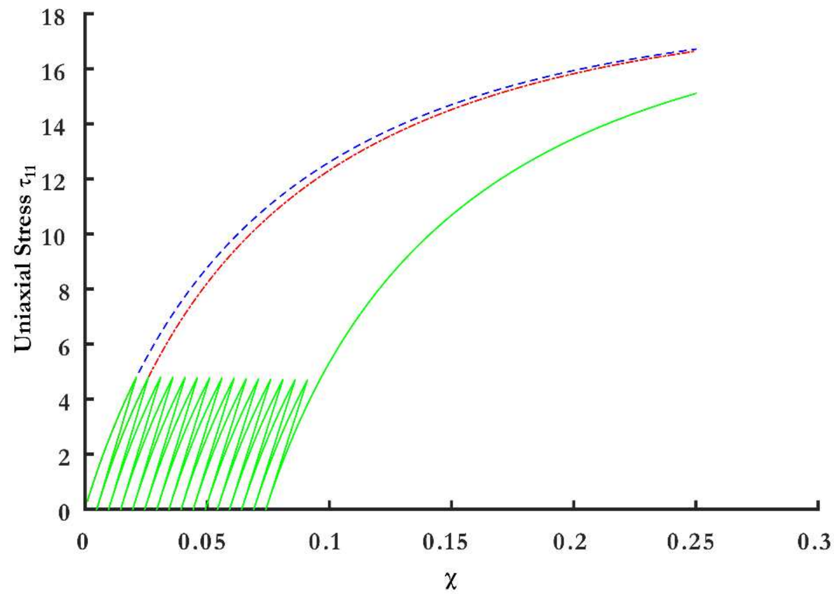

Fig. 5: Tension-compression: Low cycle fatigue behavior.

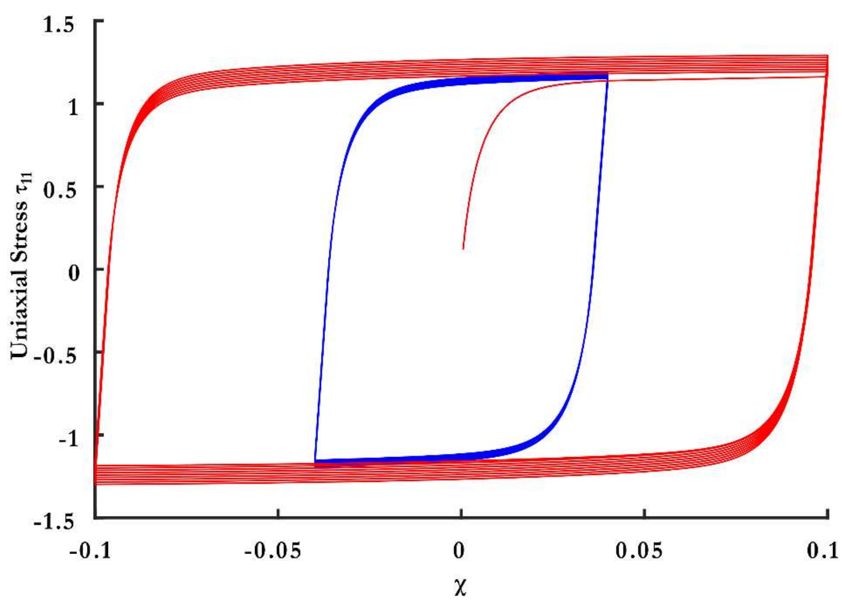

Fig. 6: Tension-compression: Two- sided cyclic loading; non-linear kinematic hardening (β=1).

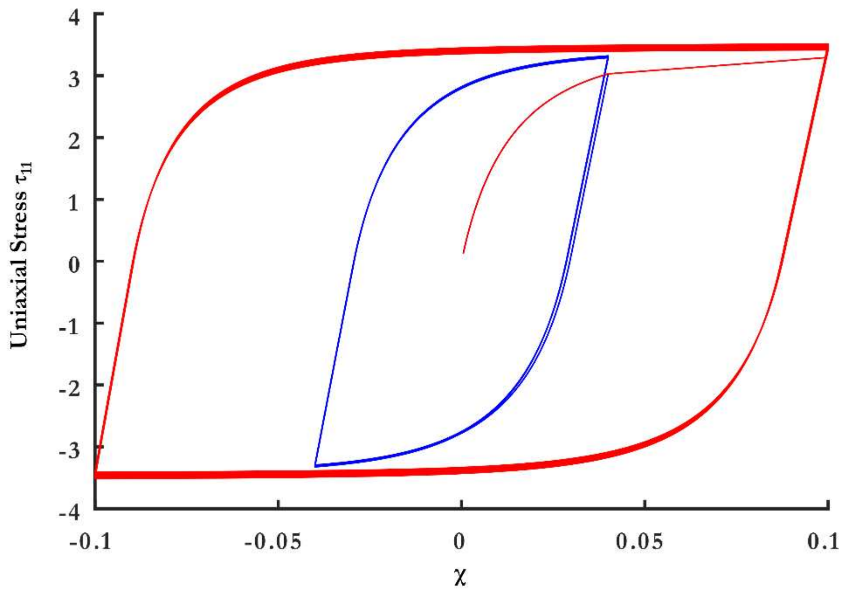

Fig. 7: Tension-compression: Two- sided cyclic loading; non-linear kinematic hardening (β=3).

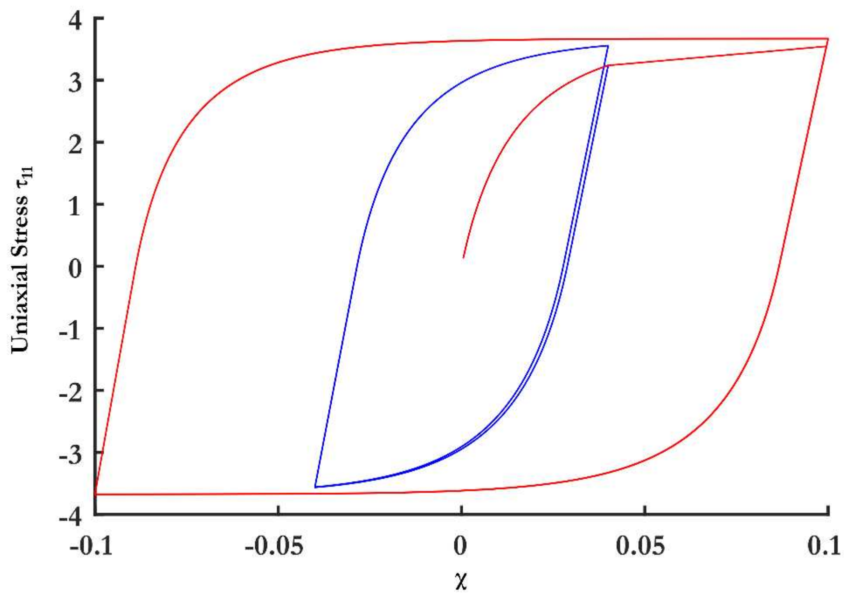

Fig. 8: Tension-compression: Two- sided cyclic loading; perfect plasticity.