



Farr, Robert and Mao, Yong (2016) Optimal Counter-current exchange networks. *Physical Review E*, 94 . 052410-052416. ISSN 1550-2376

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# Fractal counter-current exchange networks

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PACS 44.15.+a – Channel and internal heat flow

PACS 05.60.Cd – Classical transport

PACS 47.53.+r – Fractals in fluid dynamics

**Abstract** – We construct a general analysis for counter-current exchange devices, linking their efficiency to the (potentially fractal) geometry of the exchange surface and supply network. For certain parameter ranges, we show that the optimal exchanger consists of densely packed pipes which span a thin sheet of large area, which may be crumpled into a fractal surface and supplied with a fractal network of pipes. We present the efficiencies of such fractal exchangers, showing factor gains compared to regular exchangers, using parameters relevant for systems such as pigeon lungs and salmon gills.

**Introduction.** – The design of efficient exchange devices is an important problem in engineering and biology. A wide variety of heat exchangers, such as plate, coil and counter-current, are employed in industrial settings [1], while in nature, leaf venation, blood circulation networks, gills and lungs have evolved to meet multiple physiological imperatives. A distinctive feature of the biological examples is their complex, hierarchical (fractal) nature [2], with branching and usually anastomosing geometries [3, 4]. It is clear that one reason for this is the possibility to include a large surface for exchange within a compact volume (the human lungs for example comprise an alveolar area greater than 50m<sup>2</sup> [5]). However, maximal surface area is unlikely to be the only criterion for optimization. For example, West *et al* analysed biological circulatory systems on the basis that power is minimised with the constraint that a minimum flux of respiratory fluid is brought to every cell in the volume of an organism; they were able to explain the well known allometric scaling laws in biology [2].

With the advance of new fabrication technologies such as 3D printing [6], it will become possible to build structures of comparable complexity to biological systems, so there is a need not only to understand in detail the principles and compromises upon which natural systems are based, but also for that understanding to be constructive and accessible, mapping system parameters to actual designs.

The analytical literature in this area has focused on heat

transfer from a fluid to a solid body, with a particular emphasis on cooling of integrated circuits [7]. Branching fractal networks are much studied due to their ability to give good heat transfer with a low pressure drop [8, 9] (although sometimes simpler geometries can be more efficient [10]), and multiscale structures are also found to have a high heat transfer density [11].

In this Letter, we consider exchange as a general process, which includes gas and heat exchanges, and we look for the optimal designs which can ensure complete exchange (to be defined below) while requiring a minimum amount of mechanical power to generate the necessary fluid flows. We use the language of heat exchange since the relevant material properties have widely used its notation, and gather problem parameters into dimensionless groups, which span the space of possible exchange problems.

Suppose there are two counter-flowing (perhaps dissimilar) fluids with given properties: thermal conductivities  $\kappa_j$  ( $j \in \{1, 2\}$ ), heat capacities per unit volume  $C_j$  and viscosities  $\eta_j$ . Let there be an imposed difference  $\Delta T$  in the *inlet* temperatures, and an imposed volumetric flow rate  $Q_1$  of fluid 1 (while we are free to choose  $Q_2$ ). The streams are separated by walls of thickness  $w$  (taken to be the minimum consistent with biological or engineering constraints) and thermal conductivity  $\kappa_{\text{wall}}$  (again an imposed constraint). We assume that the exchanger needs to be compact, in that it fits inside a roughly cubical volume of side length  $L_{\text{max}}$ , and the pipes are each of length

57  $L \leq L_{\max}$ . Last, we wish the exchange process to go to  
58 completion, in that the total exchanged power is of order  
59  $E_{\text{end}} = C_1 Q_1 \Delta T$ . Our aim is to find an exchange network  
60 which satisfies all these constraints (which are a typical  
61 set for both engineering and biological systems), while re-  
62 quiring the minimum amount of power to drive the flow  
63 through the network.

64 To proceed, we non-dimensionalise on  $L_{\max}$  and  $\kappa_{\text{wall}}$ :

$$\hat{w} \equiv w/L_{\max}, \quad \hat{r}_j \equiv r_j/L_{\max}, \quad \hat{L} \equiv L/L_{\max},$$

$$\hat{A} \equiv A/L_{\max}^2 \quad \text{and} \quad \hat{\kappa}_j \equiv \kappa_j/\kappa_{\text{wall}}.$$

65 The specification of the problem can be conveniently  
66 reduced to three non-dimensional parameters, the first two  
67 of which capture the asymmetry of the two fluids:

$$\beta \equiv (C_1/C_2)^2 (\eta_2/\eta_1) \quad \text{and} \quad \gamma \equiv \kappa_1/\kappa_2. \quad (1)$$

68 We note that if all the available volume were filled with  
69 pipes of the smallest possible radius, and the two fluids  
70 were set to uniform temperatures differing by  $\Delta T$ , then  
71 there would be a maximum possible exchanged power of  
72 order  $E_{\max} = \Delta T \kappa_{\text{wall}} L_{\max}^3 / w^2$ . Thus our last parameter  
73 is the ratio of the required exchange rate to this maximum:

$$\epsilon \equiv E_{\text{end}}/E_{\max} = Q_1 C_1 w^2 / (L_{\max}^3 \kappa_{\text{wall}}), \quad (2)$$

74 and we typically expect  $\epsilon \ll 1$ .

75  
76 **Regular exchangers.** – To begin, we consider a reg-  
77 ular array of counter-flowing streams in  $N_j$  straight pipes  
78 of radii  $r_j$  ( $j = 1, 2$ ) and length  $L$  [the same for both types;  
79 see fig. 1(b)], where we ignore any feed network to supply  
80 the individual pipes. Assuming roughly circular pipes, we  
81 approximate the total cross section (perpendicular to flow)  
82 of the array as  $A \approx \pi N_1 (r_1 + w/2)^2 + \pi N_2 (r_2 + w/2)^2$ .  
83 Let  $\alpha$  be the area across which exchange occurs, then if no  
84 clustering of one type occurs  $\alpha$  will be approximately the  
85 minimum of the two pipe perimeters, multiplied by  $L$ . We  
86 thus propose the simple approximation to the total area  
87 across which exchange occurs:

$$\alpha^{-1} \approx (2\pi L)^{-1} [(N_1 r_1)^{-1} + (N_2 r_2)^{-1}]. \quad (3)$$

88 When is exchange complete? We assume the pipes are  
89 slender, so that heat diffusion along the length of a pipe  
90 is negligible compared to advection, and that the temper-  
91 ature over a cross section perpendicular to its length is  
92 roughly uniform. Let  $z$  be the distance along a pipe, with  
93  $z = 0$  being the upstream end of fluid ‘1’. Then we have  
94 average temperatures  $T_j(z)$  over cross sections in each of  
95 the two types of pipe. We define  $\Delta T \equiv T_1(0) - T_2(L)$ .  
96 By considering the total heat flux per unit length  $J(z)$   
97 between the two sets of pipes, we can write down the ma-  
98 terial derivative of temperature as each fluid moves along  
99 its respective pipe:

$$\pi N_j r_j^2 C_j \frac{DT_j}{Dt} = (-)^j J(z), \quad (4)$$

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + (-)^{j+1} \frac{Q_j}{\pi N_j r_j^2} \frac{\partial}{\partial z}$$

where  $j \in \{1, 2\}$ , and if  $s$  is the thermal conductance per  
100 unit area between pipes we find: 101

$$J(z) \approx \alpha s [T_1(z) - T_2(z)]/L$$

$$s^{-1} \approx (w/\kappa_{\text{wall}}) + (r_1/\kappa_1) + (r_2/\kappa_2). \quad (5)$$

In the steady state regime,  $\partial/\partial t \equiv 0$  so eqs. (4) lead to  
102 an exchanged power  $E$  where 103

$$\frac{E}{s\alpha\Delta T} = \frac{\xi_1 \xi_2 (e^{1/\xi_1} - e^{1/\xi_2})}{\xi_2 e^{1/\xi_1} - \xi_1 e^{1/\xi_2}} \approx \min(1, \xi_1, \xi_2) \quad (6)$$

$$\xi_j \equiv Q_j C_j / (s\alpha). \quad (7)$$

Complete exchange means  $E \approx C_1 Q_1 \Delta T$ , which from  
104 eqn. (6) means  $\xi_1 \leq \xi_2$  and  $\xi_1 \leq 1$ . We note from the  
105 analysis accompanying eq. (6) that there is a special case  
106 of a ‘balanced’ exchanger, in which  $Q_1 C_1 = Q_2 C_2$  (so  
107  $\xi_1 = \xi_2$ ) and the change of temperature with  $z$  for both  
108 streams is linear, rather than being exponential. The op-  
109 timal exchanger should have this property, since otherwise  
110 some of the pipe length will contribute to dissipated power  
111 but not exchange. 112

Now we seek to minimise the total power  $P$  required to  
113 run the exchanger,  $P = Q_1 \Delta p_1 + Q_2 \Delta p_2$ , where  $\Delta p_j$  are  
114 the pressures dropped across the two types of pipes. For  
115 laminar (Poiseuille) flow, and using the ‘balanced’ condi-  
116 tion  $Q_1 C_1 = Q_2 C_2$  to eliminate  $Q_2$ , we obtain: 117

$$P = P_0 \hat{L} \left( \frac{1}{N_1 \hat{r}_1^4} + \frac{\beta}{N_2 \hat{r}_2^4} \right), \quad (8)$$

$$P_0 \equiv 8Q_1^2 \eta_1 / (\pi L_{\max}^3).$$

Our task is to minimise  $P$  in eq. (8) by choosing the five  
118 quantities  $N_j$ ,  $\hat{r}_j$  and  $\hat{L}$ , while also ensuring the exchanger  
119 is compact (fits in the required volume): 120

$$\max(\hat{r}_j) \leq \hat{L} \leq 1, \quad (9)$$

$$\hat{A} = \pi N_1 (\hat{r}_1 + \hat{w}/2)^2 + \pi N_2 (\hat{r}_2 + \hat{w}/2)^2 \leq 1, \quad (10)$$

and also that exchange is complete, which from  $\xi_1 \leq 1$   
121 and eqs. (3), (5), (7) leads to 122

$$\epsilon \left( \hat{w} + \frac{\hat{r}_1}{\hat{\kappa}_1} + \frac{\hat{r}_2}{\hat{\kappa}_2} \right) \left( \frac{1}{N_1 \hat{r}_1} + \frac{1}{N_2 \hat{r}_2} \right) \leq 2\pi \hat{L} \hat{w}^2. \quad (11)$$

The optimization can then be performed numerically by  
123 a simple downhill search. Table 1 shows the geometry  
124 of some optimised regular exchangers for real cases, and  
125 the optimised results are included in fig. 3 with the label  
126 ‘regular’. 127  
128

**Branched supply network.** – Now, consider the  
129 branched (and fractal) supply network shown in fig. 1(c),  
130 which brings the streams to the exchanger (‘active layer’).  
131 In contrast to Ref. [2], we do not need the supply network  
132 to pass close to every point in space; we only require that  
133 it does not dominate the power dissipated in driving the 134

System:	T.E.G.	Pigeon	Salmon
Exchanged:	Heat	Oxygen	Oxygen
$L_{\max}/m$	$2.0 \cdot 10^{-1}$	$5.0 \cdot 10^{-2}$	$2.0 \cdot 10^{-2}$
$w/m$	$5.0 \cdot 10^{-4}$	$5.0 \cdot 10^{-7}$	$5.0 \cdot 10^{-7}$
$Q_1/m^3s^{-1}$	$5.0 \cdot 10^{-2}$	$2.0 \cdot 10^{-5}$	$1.0 \cdot 10^{-6}$
$C_1/S.I.$	$1.0 \cdot 10^3$	$2.0 \cdot 10^{-6}$	$2.0 \cdot 10^{-6}$
$C_2/S.I.$	$1.0 \cdot 10^3$	$1.3 \cdot 10^{-5}$	$1.0 \cdot 10^{-7}$
$\kappa_1/S.I.$	$4.0 \cdot 10^{-2}$	$1.8 \cdot 10^{-16}$	$1.6 \cdot 10^{-16}$
$\kappa_2/S.I.$	$4.0 \cdot 10^{-2}$	$2.3 \cdot 10^{-10}$	$1.6 \cdot 10^{-16}$
$\kappa_{\text{wall}}/S.I.$	$1.0 \cdot 10^1$	$1.8 \cdot 10^{-16}$	$1.6 \cdot 10^{-16}$
$\eta_1/Pa s$	$4.0 \cdot 10^{-5}$	$4.0 \cdot 10^{-3}$	$4.0 \cdot 10^{-3}$
$\eta_2/Pa s$	$4.0 \cdot 10^{-5}$	$4.0 \cdot 10^{-5}$	$1.0 \cdot 10^{-3}$
$\beta$	$1.0 \cdot 10^0$	$2.4 \cdot 10^{-4}$	$1.0 \cdot 10^2$
$\gamma$	$1.0 \cdot 10^0$	$7.8 \cdot 10^{-7}$	$1.0 \cdot 10^0$
$\epsilon$	$1.6 \cdot 10^{-4}$	$4.4 \cdot 10^{-4}$	$3.9 \cdot 10^{-4}$
$r_{1,\text{reg}}/m$	$1.0 \cdot 10^{-3}$	$2.5 \cdot 10^{-5}$	$5.2 \cdot 10^{-6}$
$r_{2,\text{reg}}/m$	$1.0 \cdot 10^{-3}$	$2.2 \cdot 10^{-6}$	$2.1 \cdot 10^{-5}$
$A_{\text{reg}}/m^2$	$4.0 \cdot 10^{-2}$	$2.5 \cdot 10^{-3}$	$4.0 \cdot 10^{-4}$
$L_{\text{reg}}/m$	$2.0 \cdot 10^{-1}$	$5.0 \cdot 10^{-2}$	$2.0 \cdot 10^{-2}$
$P_{\text{reg}}/W$	$2.4 \cdot 10^1$	$6.2 \cdot 10^{-1}$	$7.7 \cdot 10^{-1}$
$r_{1,\text{frac}}/m$	$5.1 \cdot 10^{-4}$	$5.0 \cdot 10^{-6}$	$5.0 \cdot 10^{-6}$
$r_{2,\text{frac}}/m$	$5.1 \cdot 10^{-4}$	$5.4 \cdot 10^{-7}$	$7.1 \cdot 10^{-6}$
$A_{\text{frac}}/m^2$	$6.6 \cdot 10^{-2}$	$1.0 \cdot 10^{-2}$	$7.7 \cdot 10^{-4}$
$L_{\text{frac}}/m$	$4.4 \cdot 10^{-2}$	$7.1 \cdot 10^{-4}$	$2.8 \cdot 10^{-3}$
$P_{\text{frac}}/W$	$1.8 \cdot 10^1$	$6.0 \cdot 10^{-2}$	$4.0 \cdot 10^{-1}$

Table 1: Estimated parameters for various real systems. ‘S.I.’ refers to the international system of units; so for thermal systems  $C$  will have units  $Jm^{-3}K^{-1}$  and  $\kappa$  will have units  $Wm^{-1}K^{-1}$ . For gas exchange,  $C$  will have units kilogram of relevant gas per  $m^3$  of fluid, per Pascal of partial pressure, and  $\kappa$  will have units  $kg s^{-1}m^{-1}Pa^{-1}$  (so that  $\kappa/C$  is a diffusivity). ‘T.E.G.’ is thermo-electric generation from internal combustion engine exhaust (we have chosen values corresponding to a car/personal automobile). For the animal respiratory systems we assume that transport across the exchange membrane is similar to that of water. For blood, we assume that oxygen can exist in a mobile form (dissolved in the water-like serum) and an immobile form (bound to haemoglobin). Thus the oxygen conductivity  $\kappa_1$  for blood is the same as for water, while  $C_1$  is increased over that of water by the carrying capacity of haem. Data are from Refs. [14–18]. Results for a regular exchange network are indicated by the subscript ‘reg’; while the results for the fractal exchange surfaces (denoted by a subscript ‘frac’) use a Hausdorff dimension  $d = 2.33$ . For the cases of pigeon and salmon respiration, we impose the additional constraint that  $r_1 > 5\mu m$ , in order to allow erythrocytes to pass through. This only affects the fractal case, and without this requirement, the optimised value of  $r_1$  for the fractal case would be  $1.5\mu m$  and  $0.4\mu m$  respectively.

exchanger. Suppose that the pipes comprising the supply network branch into  $b$  smaller pipes at each hierarchical level  $k$  of the tree, (where pipes with higher values of  $k$  are smaller, and closer to the active layer – the regular array of pipes – where exchange occurs). Let the ratio of pipe radii between neighbouring levels be  $\rho < 1$ , and the ratio of pipe lengths be  $\lambda$ . The ratio of power dissipated

between hierarchical levels is therefore

$$P_{k+1}/P_k = \lambda/(b\rho^4). \quad (12)$$

Since the active layer is densely covered with pipes, the condition to fit the supply network into space is  $\rho \geq b^{-1/2}$ . Therefore, provided  $\lambda > b\rho^4$ , the power will increase exponentially with  $k$  and the overall power dissipation in the supply network will be of order that in the last layer; and therefore of the same order as in the active layer. The supply network will therefore not dominate the power dissipation.

**Double fractal exchange networks.** – From the solutions to the optimum regular exchange networks, the lateral cross section  $A$  always expands to its maximum value  $L_{\max}^2$ . If this restriction were lifted, a more efficient exchanger would likely be possible. This can be achieved by allowing the active layer (provided it is thin enough, and can still be provided with a branching supply network) to become corrugated, while still fitting within the prescribed roughly cubical volume  $L_{\max}^3$  available. One way to do this is to turn the active layer into an approximation to a fractal surface. Suppose the active layer is corrugated into a fractal surface over a range of lateral length scales down to a scale  $x \geq L$ , such that in the limit  $x \rightarrow 0$  the Hausdorff dimension [19] of the surface would be  $d$ . Fig. 2 shows schematically an example in which the surface is the type I quadratic Koch surface with (in the limit) Hausdorff dimension  $d_{\text{Koch}} = \ln 13/\ln 3 \approx 2.33$ . Let the area of the active layer be  $A(x)$ , where  $A(L_{\max}) = L_{\max}^2$ , then from Hausdorff’s definition of dimension, we see that  $A(x) = L_{\max}^2(x/L_{\max})^{2-d}$ . We can therefore replace the inequality  $A \leq 1$  in eq. (10) by

$$\hat{A} = \pi N_1(\hat{r}_1 + \hat{w}/2)^2 + \pi N_2(\hat{r}_2 + \hat{w}/2)^2 \leq \hat{L}^{2-d}. \quad (13)$$

Fig. 3 shows the effect of  $\epsilon$  (varied through altering  $Q_1$ ) on the power dissipation for fractal exchangers corresponding to the scenarios in table 1, compared to that of the regular exchanger. Corrugating the exchange layer into a type I quadratic Koch surface leads to a significant reduction in the dissipated power for the two biological cases (factor gain of 10 for pigeon lungs and 2 for salmon gills). However, the small size of the optimum pipe radii  $r_1$  may mean that this degree of optimization is precluded by other considerations. For instance, erythrocytes need to be able to pass through these type 1 (blood carrying) vessels.

**Conclusions.** – Exchange networks of the class we show here exhibit broadly power-law dependence of the dissipated power with the quantity  $\epsilon$  (which measures the required throughput: the rate of exchange of heat or gas needed). This is true both for a fractally corrugated or a simple regular array of exchange pipes. However, the fractal exchangers demonstrate factor gain in efficiency when compared to regular exchangers for parameters relevant

for pigeon lungs and salmon gills. The exchangers exhibit a crossover between regimes as a function of the required throughput, where different constraints (geometrical or completeness of exchange) are limiting; and if other parameters were changed, such as conductivities, one could expect regimes in which either wall or fluid conductivity would be limiting.

We note that the analysis we have performed here aims specifically to minimise required power while ensuring complete exchange has taken place. In practice, other design constraints may need to be included, for example a requirement that the network be robust [3] or repairable [20,21] under external attack [22,23], or the cost of building the network may be significant compared to its operating costs [24,25].

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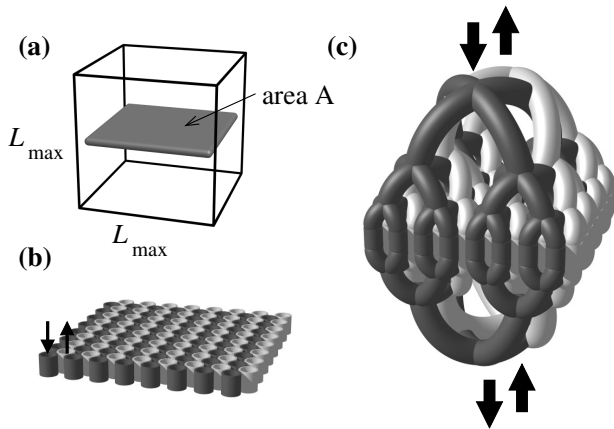


Fig. 1: (a) Schematic of the geometry of a counter-current heat exchanger ('active layer') fitting inside a prescribed cubic volume of side length  $L_{\max}$ . (b) Detail of the active layer, showing a regular array of pipes carrying alternately counter-flowing streams. (c) The active layer connected to a branching and (on the other side) anastomosing fractal supply network.

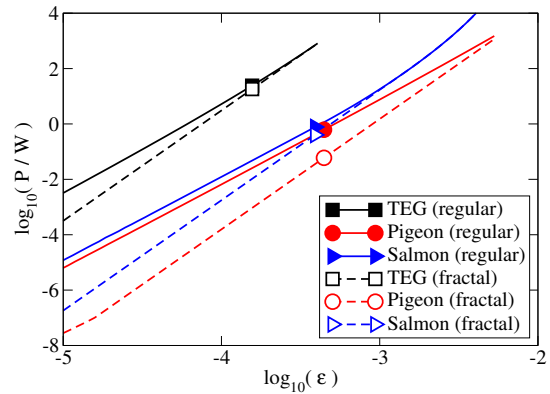


Fig. 3: Plots of power dissipated in exchange for the three cases of table 1. Here we change  $Q_1$  to achieve different values of  $\epsilon$ . The actual cases in table 1 are shown as symbols, and for some cases, a change of regime is observed, witnessed by a change in the slope of the line; although the curve visible in the top right part of the curve for 'Salmon (fractal)' are due to the constraint that blood vessels (type 1 pipes) should be large enough to carry erythrocytes (taken as the condition  $r_1 > 5\mu\text{m}$ ).

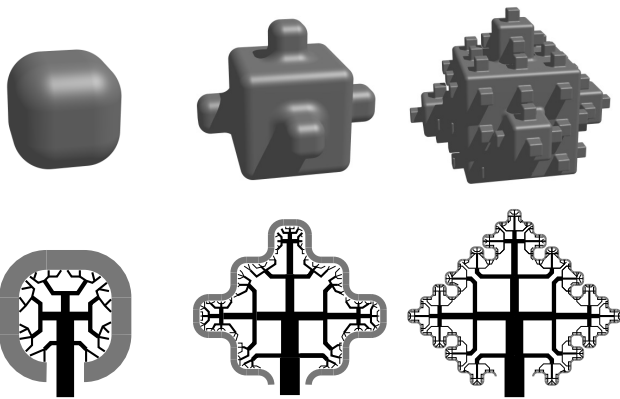


Fig. 2: Top row: schematic of the active layer of fig. 1(a), corrugated into a hierarchical (fractal) surface, comprising (left to right) greater area and more iterations of the fractal. Bottom row: Schematic section through these surfaces showing the fractal supply network in the interior (the corresponding network outside is not shown, and will require a more complex design to ensure equal flow to all parts of the active layer).