Extension of the Fuzzy Integral for General Fuzzy Set-Valued Information

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***Abstract*—The *fuzzy integral* (FI) is an extremely flexible aggregation operator. It is used in numerous applications, such as image processing, multi-criteria decision making, skeletal age-at-death estimation and multi-*source* (e.g., feature, algorithm, sensor, confidence) fusion. To date, a few works have appeared on the topic of generalizing Sugeno's original real-valued integrand and *fuzzy measure* (FM) for the case of higher-order uncertain information (both integrand and measure). For the most part, these extensions are motivated by, and are consistent with, Zadeh's *extension principle* (EP). Namely, existing extensions focus on *fuzzy number-* (FN), i.e., convex and normal *fuzzy set-* (FS), valued integrands. Herein, we put forth a new definition, called the *generalized FI* (gFI), and efficient algorithm for calculation for FS-valued integrands. In addition, we compare the gFI, numerically and theoretically, with our non EP-based FI extension called the *non-direct FI* (NDFI). Examples are investigated in the areas of skeletal age-at-death estimation in forensic anthropology and multi-source fusion. These applications help demonstrate the need and benefit of the proposed work. In particular, we show there is not one supreme technique. Instead, multiple extensions are of benefit in different contexts and applications.**

***Index Terms*—fuzzy integral, non-convex fuzzy set, sub-normal fuzzy set, discontinuous interval, skeletal age-at-death estimation, sensor data fusion**

I. Introduction

The *fuzzy integral* (FI) is a powerful nonlinear aggregation operator [1]. It has been generalized and applied to a number of areas such as image processing [2], multi-criteria decision making [3], skeletal age-at-death estimation in forensic anthropology [4–6], multi-*source* (e.g., feature, algorithm, sensor, confidence) fusion [7, 8], used as a distance metric [9], classification [10], and pattern recognition [11, 12]. The FI is most often used to combine the (objective) support in some hypothesis, e.g., algorithm outputs or confidences, from multiple sources with the (subjective) *worth* of the different subsets of sources, where the worth is encoded in a *fuzzy measure* (FM). Most applications rely on the real-valued integrand and FM. However, in many situations data are not of simple numeric form. Instead, higher-order uncertainty exists, e.g., intervals or fuzzy/probability sets. In the theme of David Marr and his Principle of Least Commitment [13], this article is an attempt to not disregard or type reduce important uncertainty information prematurely. Instead, the goal is to integrate with respect to all available information in its original full form. A system or individual can later decide to disregard such higher-order information, post-aggregation, or it can be used to help characterize and understand a decision and the confidence in such a decision.

To date, a number of papers have appeared regarding the extension of Sugeno's real-valued FI. However, a serious drawback is that these works focus on *fuzzy number-* (FN), i.e., convex and normal *fuzzy set-* (FS), valued information. Furthermore, most are motivated by, and are consistent with, Zadeh's *extension principle* (EP) [23] (which we will show the weakness of in Section V). Table II is a compilation of EP-based research on extending the FI.

While the primary focus of this article is exploring new mathematical extensions of the FI, it is ultimately driven by two needs (applications): fusion of uncertain evidence from multiple sources and forensic anthropology. These applications are of benefit as they help ground the different extensions and demonstrate their relative advantages. In particular, this article has four main contributions. First, we review different important FI extensions. Second, we put forth a new unrestricted FS-valued FI definition and an efficient algorithm, called the *generalized FI* (gFI), that subsumes most prior work. Third, we compare EP-based definitions to our non-EP FI generalization. Last, we explore various applications to demonstrate the benefit of having different generalizations.

The remainder of the article is organized as follows. First, we discuss the real-valued FI and classical higher-order FI extensions. This is followed by a review of our previous work on sub-normal, convex FS-valued integrands [6] and discontinuous interval and *interval FS-* (IFS) valued FIs [16]. Next, we put forth a new definition, gFI, for the unrestricted case of FS-valued integrands. These extensions are then compared to our *non-direct FI* (NDFI) extension. Ultimately, these extensions are explored in the context of two applications: skeletal age-at-death estimation and multi-sensor fusion.

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TABLE I
ACRONYMS AND NOTATION

CFI	Choquet fuzzy integral
EP	Extension principle
FI	Fuzzy integral
FM	Fuzzy measure
FN	Fuzzy number, i.e., a convex, normal fuzzy set
FS	Fuzzy set
gFI	Generalized fuzzy set-valued fuzzy integral
IFS	Interval fuzzy set, e.g., A, where $\mu_A(x) \in \{0, 1\}$
SFI	Sugeno fuzzy integral
X	Set of sources
x_i	ith source
$h(x_i)$, h_i	$\Re$-valued evidence offered by x_i
I	Set of intervals, $\{\bar{u} \subseteq \Re : \bar{u} = [u^-, u^+], u^- \leq u^+\}$
$\bar{h}(x_i)$, $\bar{h}_i$	interval-valued evidence
$\bar{\bar{h}}(x_i)$, $\bar{\bar{h}}_i$	Discontinuous interval-valued evidence
$H(x_i)$, H_i	fuzzy set-valued evidence
$\hat{H}(x_i)$, $\hat{H}_i$	fuzzy number-valued evidence
$\hat{\hat{H}}(x_i)$, $\hat{\hat{H}}_i$	Sub-normal, convex fuzzy set-valued evidence
$\bar{H}(x_i)$, $\bar{H}_i$	Convex, normal interval-valued fuzzy set
$\bar{\bar{H}}(x_i)$, $\bar{\bar{H}}_i$	Non-convex, normal interval-valued fuzzy set
${}^{\alpha}[H_i]$, $[H_i]_{\alpha}$, ${}^{\alpha}H_i$	α-cut of H_i, i.e., $\{a \in \Re : [H_i](a) \geq \alpha\}$
g	$\Re$-valued fuzzy measure, $g : 2^X \rightarrow [0, 1]$
g_{λ}	Sugeno λ-valued fuzzy measure
$g(x_i)$, g_i	$\Re$-valued worth of source i
π	Permutation function
$\int h \circ g$	Fuzzy integral of h with respect to g

TABLE II
SELECTED WORKS ON GENERALIZATIONS OF THE FI AND FM. WORK PUT FORTH BY OUR GROUP IS INDICATED WITH A *.

			Integrand			
			Numeric	**Interval**		**Fuzzy Set**
				Continuous	**Discontinuous**	
FM	**Numeric**		[1, 2, 14, 15]	[15]	* [16]	Convex, normal: [7, 15, 17] * Convex, non-normal: [4, 6] * Non-convex, normal *(for IFS)*: [16] * **Unrestricted FS case: *this paper*** * Type-II FNs: [18–20]
	Interval	**Continuous**	[21]	[21]		[21]
		Discontinuous				
	Fuzzy Set		Convex, normal: [21]	Convex, normal: [21]		Convex, normal: [21, 22]

II. RELATED WORK: SUGENO AND CHOQUET FUZZY INTEGRALS

The aggregation of information using the *classical Sugeno FI* (SFI), i.e., real-valued integrand (h) and FM (g), and the *Choquet FI* (CFI) has a rich history. Several applications and core theory can be found in [2, 14]. First, consider a non-empty finite set $X = \{x_1, ..., x_N\}$. Depending on the problem domain, X can be a set of experts, evidence, sensors, features, pattern recognition algorithms, etc. Both the SFI and the CFI take (typically *objective*) partial support for some hypothesis from the standpoint of each x_i and fuse it with the (perhaps *subjective*) worth (or *reliability*), encoded in a FM [1], of each subset of X in a nonlinear fashion. In particular, the FM is the important driving force behind the FI. The FI is a flexible aggregation operator. The specific FM dictates how the aggregation behaves. We can select (or learn) a particular g to achieve different combination strategies. In the following subsections we review different FI formulations and extensions.

A. Real-valued FM

Measure theory is a fundamental concept in mathematics. A famous example is the Lebesgue measure and the integral with respect to that measure. A key aspect of a FM is that it requires the property of monotonicity with respect to set inclusion, a weaker property than the additive property of a probability measure. Initial definitions [1] focused on $h : X \rightarrow [0, 1]$ and $g : 2^X \rightarrow [0, 1]$. However, these can, and have been, defined more generally. For example, it is convenient to think of h and g on the unit interval, $[0, 1]$, for scenarios such as confidence aggregation. However, we define the function h more generally as $h : X \rightarrow \Re$, where h can now be thought of directly as inputs such as sensor readings and $\Re$ is the set of all reals.[1]

Definition 1. **($\Re$-valued fuzzy measure)** For a finite set X, a FM is a (set-valued) function $g : 2^X \rightarrow [0, 1]$, such that

1. (Boundary Condition) $g(\phi) = 0$;
2. (Monotonicity) If $A, B \subseteq X$, $A \subseteq B$, then $g(A) \leq g(B)$.

Note, if X is an infinite set, a third condition guaranteeing continuity is required. However, this is a moot point for finite X, as considered in this paper and most practical applications. While it is not necessary in general, we often assume $g(X) =$

[1] We do note that this would likely affect the utilization of the SFI, as the measure and integrand would likely reside at different scales, perhaps negatively affecting the results of the max and min operations. However, this does not impact the CFI in the same way, i.e., mathematically.

1 (normality). The following are two well-known FMs.

Definition 2. **(Sugeno λ-measure)** For sets $A, B \subseteq X$, such that $A \cap B = \phi$,

$$g_\lambda(A \cap B) = g_\lambda(A) + g_\lambda(B) + \lambda g_\lambda(A) g_\lambda(B), \quad (1)$$

for some $\lambda > -1$. This measure is built from a set of *densities*, i.e., measure on just the singletons (where $g^i = g(\{x_i\})$). In particular, Sugeno showed that λ can be found by solving

$$\lambda + 1 = \prod_{i=1}^{N} \left(1 + \lambda g^i\right), \lambda > -1, \quad (2)$$

where it can be shown that there exists exactly one real solution such that $\lambda > -1$. The Sugeno λ-measure is appealing since we can automatically construct the lattice from just the densities. This is important as there are $2^N - 2$ parameters in a FM ($2^N - 2 - N$ if we have the densities). Note, when $\lambda = 0$ we obtain the common additive (probability) measure.

Definition 3. **(S-Decomposable Measure)** Let S be a t-conorm. A FM g is called an S-decomposable measure if $g(\phi) = 0$, $g(X) = 1$, and for all A, B such that $A \cap B = \phi$

$$g(A \cup B) = S(g(A), g(B)) \quad (3)$$

One famous example is the possibility measure (a W^*-decomposable measure, where W^* is the Lukasiewicz t-conorm). Other measures, e.g., our measures of agreement, specificity and the combined (meta) measure for crowd sourcing [24, 25], have been put forth to derive FMs from data. In other settings, the FM is learned using genetic algorithms [26], quadratic programming [15] or gradient descent [8]. Next, we review two common and classical FIs.

B. Real-valued FI

Definition 4. **(Sugeno FI)** Given a finite set X, a FM g and a function $h : X \to \Re$, the SFI of h with respect to g is

$$\int_S h \circ g = S_g(h) = \bigvee_{i=1}^{N} \left(h_{\pi(i)} \wedge g(A_{\pi(i)})\right), \quad (4)$$

where $h_{\pi(i)} = h(x_{\pi(i)})$ and π is a permutation on X, $h_{\pi(1)} \geq h_{\pi(2)} \geq ... \geq h_{\pi(N)}$, $A_{\pi(i)} = \{x_{\pi(1)}, ..., x_{\pi(i)}\}$ [1, 15].

Definition 5. (**Choquet FI**) Given a finite set X, a FM g and a function $h : X \to \Re$, the CFI of h with respect to g is

$$\int_C h \circ g = C_g(h) = \sum_{i=1}^{N} h_{\pi(i)} \left(g(A_{\pi(i)}) - g(A_{\pi(i-1)})\right), \quad (5)$$

where $g(A_{\pi(0)}) = 0$. In addition, Sugeno [1] and Grabisch [14] proved the following relevant properties of FIs.

Property 1. **(Continuity of $\int \mathbf{h} \circ \mathbf{g}$)** The Sugeno and Choquet FIs are continuous.

Property 2. **(Boundedness of $\int \mathbf{h} \circ \mathbf{g}$)** The function $\int h \circ g$ is bounded between

$$\bigwedge_{i=1}^{N} h_i \leq \int h \circ g \leq \bigvee_{i=1}^{N} h_i. \quad (6)$$

Property 3. **(Idempotency of $\int \mathbf{h} \circ \mathbf{g}$)** $\int h \circ g$ is idempotent, i.e., $\int h \circ g$ with respect to $h_1 = h_2 = \ldots = h_N = a$ is a.

Property 4. **(Monotonicity of $\int \mathbf{h} \circ \mathbf{g}$)** $\int h \circ g$ is a monotonically non-decreasing function, i.e.,

$$\forall i, h_i \geq z_i \Rightarrow \int h \circ g \geq \int z \circ g.$$

C. Interval-valued FI

The classical FI (S_g and C_g) was extended by Grabisch [15] for the case of *continuous* (i.e., closed) interval-valued integrands. Let $\bar{h}(x_i) \subseteq I$ be the continuous interval-valued evidence from source x_i, where $I = \{\bar{u} \subseteq \Re : \bar{u} = [u^-, u^+], u^- \leq u^+\}$ is the set of all $\Re$-valued continuous intervals.[2] Grabisch's work is based on the interval and fuzzy arithmetic work of Dubois and Prade [27]. A significant finding of theirs, with respect to FIs, is the following.

Theorem 1. [27] If a function φ is continuous and non-decreasing, then, when defined on continuous intervals, it produces the continuous interval $\varphi(\bar{u}) = [\varphi(u^-), \varphi(u^+)]$.

Dubois and Prade extended their interval proofs and formed an α-cut based definition for normal convex FSs, i.e., FNs (adopting a decomposition theorem approach that is a direct result of the EP). The interval approach is of particular benefit as it provides a computationally efficient algorithmic basis for performing fuzzy arithmetic and the FI. Grabisch leveraged the properties of the FI and Dubois and Prade's findings to extend the FI.

Definition 6. **(FI with I-valued integrand)** Let $\bar{h} : X \to I$ denote the I-valued partial support function and let $\bar{h}_i = [[\bar{h}_i]^-, [\bar{h}_i]^+]$ denote the i^{th} interval (where $[\bar{h}_i]^-$ and $[\bar{h}_i]^+$ are the left and right interval endpoints respectively). The interval-valued FI is [14]

$$\int \bar{h} \circ g = \left[\int [\bar{h}]^- \circ g, \int [\bar{h}]^+ \circ g\right]. \quad (7)$$

That is, the FI on a continuous interval is nothing more than a closed interval with the FI applied to the interval endpoints. Further, $\int \bar{h} \circ g$ has the following properties (which are relevant to the current investigation).

Property 5. **(Boundedness of $\int \bar{\mathbf{h}} \circ \mathbf{g}$)** $\int \bar{h} \circ g$ produces an interval $\bar{a} = [a^-, a^+]$ such that

$$\bigwedge_{i=1}^{N} [h_i]^- \leq \left(\int [\bar{h}]^- \circ g\right) = a^- \leq \bigvee_{i=1}^{N} [h_i]^-, \quad (8)$$

and

[2] Again, in most practical circumstances, e.g., confidence level fusion, $\bar{h}$ is constrained to be continuous interval subsets of the unit-interval. However, without loss of generality, we consider the case of $\Re$.

$$\bigwedge_{i=1}^{N}[h_i]^+ \leq \left(\int [\bar{h}]^+ \circ g\right) = a^+ \leq \bigvee_{i=1}^{N}[h_i]^+. \qquad (9)$$

In addition, the I-valued FI also satisfies idempotency, monotonicity and it is continuous.

D. Fuzzy number-valued FI

We begin this section with a quick review of the EP for the case of FS-valued integrands. This is important as it guides our proposed extensions and helps unify notation.

Definition 7. Let $H : X \to FS(\Re)$ be a FS-valued integrand. The EP of the FI of H with respect to $\Re$-valued g is

$$\left(\int H \circ g\right)(a) = \bigvee_{\mathbf{z}\in S_a}\left\{\bigwedge_{i=1}^{N} H_i(z_i)\right\}, \qquad (10)$$

$$S_a = \left\{\mathbf{z} : \mathbf{z} \in \Re^N, \int \mathbf{z} \circ g = a\right\}, \qquad (11)$$

where $\mathbf{z} = (z_1, ..., z_N)$ is a vector of numbers. The set S_a is all admissible N-tuples of values such that the $\Re$-valued FI of $\mathbf{z}$, given FM g, maps to value a.

While theoretically useful, this EP formulation does not lend itself to convenient (namely efficient) calculation. It is also difficult to intuit the behavior and inner workings of the EP in an application setting. Before presenting a new FS-valued FI definition and efficient algorithm for calculation we first review the original extension of the FI for the case of fuzzy-valued information.

Definition 8. (FI for FN-valued integrand) [15] Let $\hat{H} : X \to FN(\Re)$. Grabisch's representation theorem FI definition of $\hat{H}$ with respect to $\Re$-valued g is

$$\left(\int \hat{H} \circ g\right)(a) = \bigcup_{\alpha\in[0,1]} \alpha\left[{}^{\alpha}\left(\int \hat{H} \circ g\right)(a)\right]$$
$$= \bigcup_{\alpha\in[0,1]} \alpha\left[\left(\int {}^{\alpha}\hat{H} \circ g\right)(a)\right], \qquad (12)$$

where ${}^{\alpha}\hat{H} = [[{}^{\alpha}\hat{H}]^-, [{}^{\alpha}\hat{H}]^+]$ are the closed intervals of the level-cuts of the members of $\hat{H}$ at α. Alternatively, Eq. (12) can be expressed as

$$\left(\int \hat{H} \circ g\right)(a) = \sup\left\{\alpha \in [0,1] : a \in \int {}^{\alpha}\hat{H} \circ g\right\}. \quad (13)$$

Specifically, Grabisch showed that Eqs. (12)(13) has the advantage that they can be efficiently calculated in terms of FIs on intervals (i.e., the α-cuts),

$${}^{\alpha}\left[\int \hat{H} \circ g\right] = \left[\left(\int [{}^{\alpha}\bar{H}]^- \circ g\right), \left(\int [{}^{\alpha}\bar{H}]^+ \circ g\right)\right]. \quad (14)$$

The proof for Eqs. (12)-(14) can be found in [15]. The proof begins with Dubois and Prades analysis of the behavior of functions on continuous intervals. It is followed by a decomposition theorem approach to representing FNs. Dubois and Prade's analysis of α-cuts and interval arithmetic is invoked next. Specifically, the function on the FN is nothing more than the function applied to closed intervals acquired by α-cuts. Grabisch's contribution was the verification of the properties of FIs such that Dubois and Prades fuzzy arithmetic findings could be applied for the FI. Next, we turn to our extension for sub-normal, convex FS-valued information.

E. Sub-Normal, Convex FS-Valued FI

Grabisch's extension for FN-valued integrands is helpful for many cases encountered in practice. However, there exist a number of applications in which the evidence is instead sub-normal but still convex. For example, this can be the case in skeletal age-at-death estimation and multi-*source* (e.g., feature, algorithm, sensor, confidence) fusion. The problem is, the current set of tools (extensions) are not directly applicable. Too often in the fuzzy set community we restrict our analysis to *simple* sets (i.e., normal and convex) as they are mathematically and computationally easy to work with. Without a valid extension we are forced to use another tool or make simplifications, e.g., the inputs are normalized so that their heights equal 1. Herein, we explore a new definition for the case of sub-normal and convex integrands. Furthermore, we show that the algorithm for calculating the result of such information aggregation is no more complex than the case of convex and normal so there is no reason to simplify or pre-process the information. First, we provide two real world examples from our own research to illustrate where this extension is of utility.

Example 1. Consider the application of multi-*source*, specifically multi-aging-method fusion for skeletal age-at-death estimation in forensic anthropology [4–6]. The domain of the input, $\bar{h}$, is the age of an individual at the time of their death ($\Re^+$). Each aging method is defined with respect to a set of age stages (intervals). A FS-valued integrand is built from $\bar{h}$, where the height of each FS is determined by the quality of the skeletal remain used (e.g., skull). These FSs capture the uncertainty in age-at-death estimation, arising from the impossibility to provide an exact age estimate, in particular in the face of complicating factors such as less than perfect skeletal remains. The goal of aggregation is to combine the evidence from the aging methods while taking into account skeletal quality and the worth (reliability) of different sets of aging methods.

Example 2. Consider the scenario of multi-camera, or multi-look in a single camera, fusion. In this case, we want to fuse information, e.g., signals, features, algorithms, or decisions, from an *electro-optical/infrared* (EO/IR) camera and/or *forward looking ground penetrating radar* (FLGPR) system. In [28, 29], such a system is described for ground-based vehicle explosive hazard detection. Imagery is co-registered with each other and/or the world (universal transverse Mercator space). One challenge is that pixels in one image or sensor do not represent the same area on Earth. Objects further down track have fewer pixels on target, viz., each pixel represents a larger physical area. In addition, factors like a camera's field of view, lens, position on the vehicle, focal plane array, etc., all contribute to this problem. An important question is, how do we accurately link (and possibly aggregate) the different signals, features, algorithms and/or decisions? This problem is ultimately one of positional and evidence uncertainty. Keller et al. [7] provided such a sensor-based example that yielded and subsequently operated on FSs. Specifically, positional uncertainty provided the width and shape of the set, while height is based on our uncertainty in the target. However they had to normalize and reduce the original sub-normal and non-convex information into a FN for processing.

In order to address applications like those discussed, we put forth the following FI extension initially introduced in [6].

Definition 9. (**Sub-normal, convex FS-valued integrand**) Let $\hat{\hat{H}}$ be a convex, sub-normal integrand and g be a $\Re$-valued FM. The *sub-normal FI* (SuFI) is

$$\left(\int \hat{\hat{H}} \circ g\right)(a) = \bigcup_{\alpha \in [0,\beta]} \alpha \left[\left(\int {}^{\alpha}\hat{\hat{H}} \circ g\right)(a)\right], \tag{15}$$

$$\beta = \bigwedge_{i=1}^{N} Height\left(\hat{\hat{H}}_i\right), \tag{16}$$

where the height of FS A is

$$Height(A) = \sup_{a \in \Re} \left(\mu_A(a)\right). \tag{17}$$

As shown in [6], Height($\int \hat{\hat{H}} \circ g$) is bounded (according to the EP) by the minimum height set (β), thus

$$\left[\int \hat{\hat{H}} \circ g\right]_{\alpha > \beta} = \phi. \tag{18}$$

Furthermore, Eq. (15) has the benefit that it can also be efficiently calculated via FI interval operations,

$${}^{\alpha}\left[\int \hat{\hat{H}} \circ g\right] = \left[\left(\int [{}^{\alpha}\hat{\hat{H}}]^{-} \circ g\right), \left(\int [{}^{\alpha}\hat{\hat{H}}]^{+} \circ g\right)\right], \tag{19}$$

$\forall \alpha \in [0, \beta]$. The efficient algorithm for calculating the SuFI is presented in Algorithm 1.

Remark 1. Note, in the case of all normal FSs, $\hat{H}$, Eq. (15) is Grabisch's definition for FNs (as $\beta = 1$).

Remark 2. As discussed in [6], we assert that SuFI is an extremely *limiting* generalization. Specifically, the limitation resides with the EP. To see this, consider another related multi-sensor fusion situation in which three different sources, e.g., GPR, IR, and visual spectrum, are being aggregated using the FI. Imagine that one of the sources, say GPR, turns out to be very unreliable. Now, consider that the GPR is assigned a very small density, e.g., 0.1, relative to 1 and 0.8 for the IR and visual spectrum sources. In addition, let a generalized triangular membership function be defined as $[a, b, c, d]$, where $a \leq b \leq c$ are the left, center and right points that define the shape and d is the height of the membership function. Furthermore, let IR and visual spectrum have a high confidence in the FS input of *near* 1 (e.g., $[0.9, 1, 1.1, 1]$) and let GPR have a relatively low confidence in the FS input *near* 0 (e.g., $[-0.1, 0, 0.1, 0.2]$). Regardless of the choice of FM (min, max, average, etc.), a negative impact is observed as a result of GPR having a low confidence (height). Intuitively, we would expect that because the GPR has very little relative worth, i.e., a density value of 0.1, that the GPR decision would influence the decision result very little. However, the height of the resultant set is bounded by β, which is 0.2 in this scenario. The point is, SuFI provides a way to calculate a result; however, this result is not intuitively pleasing in some circumstances. For the provided sensor fusion example, we should intuitively more-or-less ignore the GPR input based on the SuFI algorithm result.

III. Discontinuous interval and IFS-valued FI

The problem with SuFI is that it only addresses sub-normality, but not non-convexity. To this end, we put forth an extension of the FI for the case of discontinuous intervals [16]. First, we review the concept of an *interval*-valued FS (IFS). The extension to the case of discontinuous intervals is simplified if we represent $\bar{h}$ as a convex, normal IFS, which is more formally defined below.

Definition 10. (**Mapping of $\bar{h}$ to IFS $\bar{H}$**) Let $\bar{h}_i$ be an interval. An IFS $\bar{H}_i$ is a convex and normal FS such that

$$\bar{H}_i(z) = \begin{cases} 1, & \forall z \in \bar{h}_i, \\ 0, & \text{else.} \end{cases}$$

Remark 3. For all intents and purposes, $\bar{H}_i$ and $\bar{h}_i$ are equivalent representations of the interval $\bar{h}_i$. $\bar{H}_i$ is simply a FS version of $\bar{h}_i$ that we will use in conjunction with the EP.

We can now apply the EP to express the FI with an $\bar{H}$-valued integrand. Because $\bar{H}_i(z) \in \{0, 1\}$, the EP reduces to

$$\left(\int \bar{H} \circ g\right)(a) = \begin{cases} 1, & \exists \mathbf{z} \in S_a : \bigwedge_{i=1}^{N} \bar{H}_i(z_i) = 1, \\ 0, & \text{else.} \end{cases} \tag{20}$$

This equation can be further reduced (into *support* form) by using the interval notation $\bar{H}_i = [[{}^{\alpha}\bar{H}_i]^{-}, [{}^{\alpha}\bar{H}_i]^{+}]$,

$$\left(\int \bar{H} \circ g\right)(a) = \begin{cases} 1, & \exists \mathbf{z} \in S_a : [{}^{0+}\bar{H}_i]^{-} \leq z_i \leq [{}^{0+}\bar{H}_i]^{+} \\ 0, & \text{else,} \end{cases} \tag{21}$$

where ${}^{0+}$ denotes a strong alpha-cut at 0.

Remark 4. Equation (21) is a direct result of the EP and the mapping of $\bar{h}$ to $\bar{H}$. This equation shows that the FI with an $\bar{H}$-valued integrand is nothing more than an "inclusion check" at each a for the existence of any admissible $\mathbf{z}$ that satisfies $\int \mathbf{z} \circ g = a$, with $[{}^{0+}\bar{H}_i]^{-} \leq z_i \leq [{}^{0+}\bar{H}_i]^{+}$, $i = \{1, \ldots, N\}$.

Proposition 2. For continuous intervals, the FI at (21) leads directly to the interval FI proposed by Grabisch at Eq. (7).

Algorithm 1 Computation of the SuFI algorithm

Input the $\Re$-valued FM g ▷ *e.g., use the Sugeno λ-FM, learn g from data, manually specify g, etc.*
Input partial support function $\hat{\hat{H}}$ ▷ *sub-normal FN*
Calculate $\beta = \bigwedge_{i=1}^{N} Height(\hat{\hat{H}}_i))$ ▷ *minimum height of partial support FSs*
for each $\alpha \in (0, \beta]$ **do** ▷ *note, this step is discretized in practice*
    $[\int \hat{\hat{H}} \circ g]_\alpha = [\int [^\alpha \hat{\hat{H}}]^- \circ g, \int [^\alpha \hat{\hat{H}}]^+ \circ g]$ ▷ *the $\Re$-based (integrand and measure) FI*
end for

Proof: We know $\int \mathbf{z} \circ g$ is monotonic and non-decreasing. Hence, because $\left(\int \bar{H} \circ g\right)(a) = 1$ iff there exists a $\mathbf{z} \in S_a$ such that $[^{0+}\bar{H}_i]^- \leq z_i \leq [^{0+}\bar{H}_i]^+$, $\forall i$, then for any admissible $\mathbf{z}$,

$$a^- = \int [^{0+}\bar{H}]^- \circ g \leq \int \mathbf{z} \circ g \leq \int [^{0+}\bar{H}]^+ \circ g = a^+.$$

Hence, $\left(\int \bar{H} \circ g\right)(a) = 1$ *only* on the interval $[a^-, a^+]$, which implies Eq. (7). ■

A. FI for discontinuous intervals

Definition 11. (Discontinuous interval) [16] Let a $\bar{\bar{h}}_i$ be the discontinuous interval-valued evidence offered by source x_i, defined as

$$\bar{\bar{h}}_i = \bigcup_{j=1}^{M_i} [\bar{h}_i]_j, \tag{22}$$

where each interval is disjoint and M_i is the number of continuous (sub-)intervals $[\bar{h}_i]_j$, which make up the overall discontinuous interval $\bar{\bar{h}}_i$.

This simple representation of a discontinuous interval will be important to our definition of the FI for discontinuous interval integrands (and ultimately, FS-valued integrands). To extend $\bar{h}$ to $\bar{\bar{h}}$ and $\bar{H}$ to $\bar{\bar{H}}$, we first use the following lemma.

Lemma 3. (Extension of $\bar{h}$ to $\bar{\bar{h}}$ and $\bar{H}$ to $\bar{\bar{H}}$) For each $[\bar{h}_i]_j$, let

$$Z_{ji} = \{z \in \Re : z \in [\bar{h}_i]_j\}, \tag{23}$$

is all points, $z \in \Re$, in the j^{th} sub-interval, $[\bar{h}_i]_j$. Furthermore, let

$$Z_i = \bigcup_{j=1}^{M_i} Z_{ji}, \tag{24}$$

that is, all $z \in \Re$ that are in $\bar{\bar{h}}_i$, where $\bar{\bar{h}}_i$ is defined at Eq. (22). Therefore, Z_i makes up the support of $\bar{\bar{H}}_i$. That is,

$$\bar{\bar{H}}_i(z) = \begin{cases} 1, & z \in Z_i, \\ 0, & \text{else.} \end{cases} \tag{25}$$

Because we have written the discontinuous intervals $\bar{\bar{h}}$ as FSs $\bar{\bar{H}}$, it is easy to now apply the EP to define a FI for discontinuous intervals.

Definition 12. (FI for $\bar{\bar{h}}$-valued integrand) The FI for $\bar{\bar{h}}$ is

$$\left(\int \bar{\bar{H}} \circ g\right)(a) = \begin{cases} 1, & \exists \mathbf{z} \in S_a : z_i \in Z_i \ , \forall i, \\ 0, & \text{else.} \end{cases} \tag{26}$$

Thus,

$$\int \bar{\bar{h}} \circ g = \quad \left\{ a \in \Re : \left(\int \bar{\bar{H}} \circ g\right)(a) = 1 \right\}. \tag{27}$$

Remark 5. Our above definition of the FI for discontinuous intervals at Eq. (27) is derived directly from the EP; hence, it is theoretically valid. However, Eq. (27) does not provide a computationally attractive solution as do the FIs at Eq. (7) and Eq. (15), and at our EP-derived interval FI at Eq. (21). But, as will be shown below, we can express Eq. (27) as the union of the FIs of numbers, much in the way that Eq. (7) and Eq. (15) also do.

Theorem 4. The FI of discontinuous interval-valued integrand $\bar{\bar{h}}$ with respect to the $\Re$-valued FM g can be computed as

$$\int \bar{\bar{h}} \circ g = \bigcup_{k=1}^{\mathcal{M}} \int [\bar{h}]_k \circ g = \bigcup_{k=1}^{\mathcal{M}} \left[\int [\bar{h}]_k^- \circ g, \int [\bar{h}]_k^+ \circ g \right], \tag{28}$$

where $[\bar{h}]_k$ is the kth N-tuple of the power set of all sub-intervals in $\bar{\bar{h}}$ and $\mathcal{M} = \prod_{i=1}^{N} M_i$; e.g., $[\bar{h}]_1 = \{[\bar{h}_1]_1, \ldots, [\bar{h}_N]_1\}$ and $[\bar{h}]_{\mathcal{M}} = \{[\bar{h}_1]_{M_1}, \ldots, [\bar{h}_N]_{M_N}\}$.

Proof: Let Z_i be the sets defined in Lemma 3. The set of possible $\mathbf{z} \in S_a : z_i \in Z_i$, $\forall i$, in Eq. (26) can be expressed as

$$\begin{aligned} \mathbf{Z}_a &= \left\{ \mathbf{z} \in \Re^N : z_i \in Z_i, \int \mathbf{z} \circ g = a \right\} \\ &= \left\{ \mathbf{z} \in \Re^N : z_i \in \bigcup_{j=1}^{M_i} Z_{ji}, \int \mathbf{z} \circ g = a \right\}. \end{aligned} \tag{29}$$

By distributing the union, we can reformulate $\mathbf{Z}_a$ as

$$\mathbf{Z}_a = \bigcup_{k=1}^{\mathcal{M}} \left\{ \mathbf{z} \in \Re^N : z_i \in \mathcal{Z}_k, \int \mathbf{z} \circ g = a \right\}. \tag{30}$$

where $\mathcal{Z}_k$ is the kth tuple of the power set of Z_is, viz., $\mathcal{Z}_1 = \{Z_{11}, \ldots, Z_{1N}\}$ and $\mathcal{Z}_{\mathcal{M}} = \{Z_{M_1 1}, \ldots, Z_{M_N N}\}$. Let $\mathbf{Z}_{ak}$ be the kth term in the union in Eq. (30), where $\mathbf{Z}_a = \bigcup_{k=1}^{\mathcal{M}} \mathbf{Z}_{ak}$, then Eq. (26) can be written as

$$\left(\int \bar{\bar{H}} \circ g\right)(a) = \begin{cases} 1, & \exists \mathbf{z} \in \bigcup_{k=1}^{\mathcal{M}} \mathbf{Z}_{ak}, \\ 0, & \text{else.} \end{cases} \tag{31}$$

Therefore,

$$\left(\int [\bar{H}]_k \circ g\right)(a) = \begin{cases} 1, & \exists \mathbf{z} \in \mathbf{Z}_{ak}, \\ 0, & \text{else}, \end{cases} \tag{32}$$

where this is the standard FS-integrand FI of the kth tuple of subset components of $\bar{\bar{H}}$, i.e., $\bar{\bar{H}} = \bigcup_{k=1}^{\mathcal{M}} [\bar{H}]_k$, where the support of $[\bar{H}]_k$ is $\mathcal{Z}_k$. Combining Eq. (31) and Eq. (32) gives

$$\left(\int \bar{\bar{H}} \circ g\right)(a) = \bigcup_{k=1}^{\mathcal{M}} \left(\int [\bar{H}]_k \circ g\right)(a), \tag{33}$$

which combined with the result of Proposition 2 proves the theorem. ∎

Remark 6. The advantage of our formulation of the FI for discontinuous intervals at Eq. (28) is that it is simply the union of all the combinations of continuous I-valued results. Moreover, since $(\int [\bar{h}]_k \circ g) = [\int [\bar{h}]_k^- \circ g, \int [\bar{h}]_k^+ \circ g]$, each continuous I-interval FI is (a) characterized by the FI on the interval endpoints and (b) continuous on the interval (albeit, on the continuous I-interval sub-parts of $\bar{\bar{h}}$). This allows for the efficient calculation of $\bar{\bar{h}} \circ g$ in terms of just the union of the resulting continuous closed intervals, which only require the $\Re$-valued FI to be calculated on the interval endpoints. This concept holds for the corresponding case of $\int \bar{\bar{H}} \circ g$.

Remark 7. Our definition of the discontinuous-interval FI reduces to the existing form of the FI for continuous I-valued intervals and FNs as the union-based decomposition results in a set $\mathcal{Z}$ of size one, viz. $\mathcal{M} = 1$.

IV. Non-convex, sub-normal FS-valued FI

The previous sections provide a foundation upon which we can understand and establish a definition and corresponding computationally efficient algorithm to calculate the sub-normal and non-convex FS-valued integrand FI (gFI). First, we review a few relevant properties of FSs and α-cuts on FSs.

Remark 8. Let A be a FS defined on domain $\Re$. By definition,

- ${}^{\alpha}A$ may be a discontinuous interval as a FS can be non-convex
- The level sets of A are monotonically non-increasing.

Remark 9. An EP-based formulation of $\left(\int H \circ g\right)$ implies

- $\left(\int H \circ g\right)$ is a FS
- ${}^{\alpha}\left(\int H \circ g\right) = \phi$ if $\alpha > \beta$, ${}^{\alpha}\left(\int H \circ g\right) \neq \phi$ otherwise.

Lemma 5. For $\Delta \geq 0$ and $\alpha + \Delta \leq 1$,

$$\left(\int {}^{\alpha+\Delta}H \circ g\right) \subseteq \left(\int {}^{\alpha}H \circ g\right). \tag{34}$$

That is, the FI defined on α-cuts is monotonically decreasing (set-wise) with respect to Δ.

Proof: In order to prove Eq. (34), let

$$\bar{\bar{z}}_i^1 = \bigcup_{j_1=1}^{M_i^1} \left[\bar{z}_i^1\right]_{j_1}, \tag{35}$$

$$\bar{\bar{z}}_i^2 = \bigcup_{j_2=1}^{M_i^2} \left[\bar{z}_i^2\right]_{j_2}, \tag{36}$$

be the discontinous interval-valued partial support functions at $(\alpha + \Delta)$ and α respectively. By definition (of a FS), each continuous interval at $(\alpha + \Delta)$, $\left[\bar{z}_i^1\right]_{j_1}$, is a subset of a corresponding interval at α, $\left[\bar{z}_i^2\right]_{j_2}$. In the following, we use k_1 and k_2 to denote two combinations of continuous-valued sub-intervals such that each interval at $(\alpha + \Delta)$ is a subset of its corresponding interval at α. Thanks to the I-valued work of Dubois, Prade and Grabisch, we know

$$\left(\int \left[\bar{z}^2\right]_{k_2}^- \circ g\right) \leq \left(\int \left[\bar{z}^1\right]_{k_1}^- \circ g\right) \leq \left(\int \left[\bar{z}^1\right]_{k_1}^+ \circ g\right) \leq \left(\int \left[\bar{z}^2\right]_{k_2}^+ \circ g\right), \tag{37}$$

as

$$\left[\bar{z}_i^2\right]_{k_2}^- \leq \left[\bar{z}_i^1\right]_{k_1}^- \leq \left[\bar{z}_i^1\right]_{k_1}^+ \leq \left[\bar{z}_i^2\right]_{k_2}^+.$$

Therefore,

$$\bigcup_{k_1=1}^{\mathcal{M}_1} \left(\int \left[\bar{z}^1\right]_{k_1} \circ g\right) \subseteq \bigcup_{k_2=1}^{\mathcal{M}_2} \left(\int \left[\bar{z}^2\right]_{k_2} \circ g\right), \tag{38}$$

which completes this lemma. ∎

The final property that we must show is that the level cuts of $\left(\int H \circ g\right)$ are equal to our FI for discontinious intervals. Specifically, the proof is based on the EP, as that is what we consider as "truth" herein.

Proposition 6. The following sets are equal,

$${}^{\alpha}\left(\int H \circ g\right) = \bigcup_{k=1}^{\mathcal{M}_\alpha} \left(\int \left[{}^{\alpha}H\right]_k \circ g\right). \tag{39}$$

Proof: This proof is trivial (given the definitions put forth thus far). According to the EP, the LHS of Eq. (39) is all admissible $\mathbf{z} \in S_a$ (for $\forall a \in \Re$) such that $\bigwedge_{i=1}^{N} H_i(z_i) \geq \alpha$. Similarly, we showed (Theorem 4) that our discontinious interval-valued FI is all $\mathbf{z} \in S_a$ such that each $H_i(z_i) \geq \alpha$. ∎

Definition 13. (FI for FS-valued integrand) The representation theorem and EP-based definition of the FI of FS-valued integrand H and $\Re$-valued g is

$$\left(\int H \circ g\right)(a) = \bigcup_{\alpha \in [0,\beta]} \alpha \left[\left(\int {}^{\alpha}H \circ g\right)(a)\right], \tag{40}$$

or alternatively

$$\left(\int H \circ g\right)(a) = \sup\left\{\alpha \in [0,\beta] : a \in \int {}^{\alpha}H \circ g\right\}. \tag{41}$$

Furthermore, based on Lemma 5, Proposition 6, and Theorem 4, the FI can be (efficiently) calculated in terms of discontinuous interval FIs at α using Eq. (28). That is, the extension of the FI for the general case of FS-valued inputs is simply an α-cut decomposition and union of all possible continuous (closed interval) I-valued integral, and ultimately $\Re$-valued, calculations. Algorithm 2 is a formal description of a computationally efficient method for calculating the gFI.

Algorithm 2 takes as input the $\Re$-valued FM g and partial support function H. The gFI does not place additional constraints on the FM (beyond boundedness and monotonicity). We can learn g from data, an expert can specify it, etc. Next, the minimum height of the different $H(\{x_i\})$ sets is calculated. In practice, the integral is computed (approximated) on a computer by discretizing the range ($[0,1]$, specifically $[0,\beta]$ for the gFI). The quality, in terms of approximation error, of the result depends on at least two factors. The first factor is the number of samples used. Too high of a sampling rate will result in excessive computational complexity versus the increase in precision achieved. Conversely, too few samples results in poor result resolution and greater approximation error. To the best of our knowledge, there has not been any investigation into characterizing the resulting approximation error in terms of the sampling rate for the FI. In practice, users typically select the sampling rate based on the profile of a specific computing device or application. Second, approximation error depends largely on the shape of the fuzzy sets. If all the inputs are triangular or trapezoidal membership functions versus Gaussian or some other non-piecewise linear function, the approximation is simpler and will likely require fewer samples. As steps $5-7$ in Algorithm 2 show, the gFI breaks down into a series of interval-valued FI calculations on the different alpha cuts. Specifically, step 5 is the first continuous interval integral calculation and step 7 is the repeated calculation, and union across those calculations, of the resulting continuous interval integrals for a specific α.

V. Non Extension Principle-Based FI

As already stated, the SuFI is a *harsh* way to aggregate multiple sub-normal, convex FS-valued inputs. Namely, it is extreme with respect to the resultant height restriction, β, which is ultimately due to Zadeh's EP. Furthermore, gFI suffers from the same problem as it is also a valid extension of the EP. In [4, 5], we show an alternative non-direct (i.e., non-EP based) method, called NDFI, to generate FS-valued results from sub-normal, convex FS-valued inputs based on the $\Re$-valued SFI. In Alg. 3, we put forth an extended version of NDFI for FS-valued inputs. In this respect, NDFI can be compared to the EP-based gFI.

Whereas the gFI decomposes the FI into a sequence of interval-based FI calculations across the membership domain, the NDFI decomposes the FI into a sequence of $\Re$-valued FI calculations across the input domain. In addition, the gFI is an EP-based generalization of the FI for fuzzy inputs, whereas the NDFI is an aggregation "in-place" of FSs *using* the FI. It is trivial to verify that the sets generated by the NDFI are valid FSs as they passes the vertical line test. In addition, the NDFI typically produces sub-normal and non-convex results, whereas Grabisch's prior extension yields FN results and the gFI yields FSs. In addition, the gFI produces results between the minimum and maximum. The NDFI also generates FSs between the minimum and maximum, however only in regions (in the input domain) between the minimum and maximum that is covered by at least one input. The difference between the NDFI and the gFI is apparent with respect to $\left(\int H \circ g\right)(a)$. At a, the gFI calculation is governed by the EP, whereas the NDFI is

$$\left(\int H \circ g\right)(a) = \int \mathbf{z}_a \circ g, \tag{42}$$

where $\mathbf{z}_a = (H_1(a), ..., H_N(a))$. The EP formulation uses all $\Re$-based FIs whose result is a and a t-norm of the membership degrees of the FS inputs at those locations. The NDFI is a $\Re$-based FI at a. The NDFI and gFI *fuse* information in very different ways. The NDFI integrates *vertically* while gFI integrates *horizontally*. In the next sub-sections we present the NDFI for age-at-death estimation and different examples are given for NDFI versus gFI.

A. Age-at-death estimation using NDFI

The following sub-section shows why the NDFI was created and it helps demonstrate its utility. Age-at-death estimation of an individual skeleton is important to forensic and biological anthropologists for identification and demographic analysis. It has been shown that current individual aging methods are often unreliable because of skeletal variation and taphonomic factors [4]. Previously, we introduced the NDFI as an way to estimate adult skeletal age-at-death [4]. In particular, focus was placed on the production of numeric [4], graphical [4, 5] and linguistic descriptions of age-at-death [5]. The NDFI algorithm takes as input multiple age-range intervals representing age-at-death estimations from different methods. It also takes into account the accuracies of these methods as well as the condition of the bones being examined. Advantages of NDFI, relative to related work in forensic anthropology, are that it does not require a skeletal population for training and it produces additional information (numeric, graphical and linguistic) that can assist an investigator.

Our age-at-death NDFI approach takes I-valued inputs, e.g., "method 1 says the skeleton is between the ages of 20 to 35 at the time of death". We also have information, namely correlation coefficients, representing the *reliability* of each aging method. Last, we have a $[0,1]$ value indicating the quality of each bone found. Each aging method is based on, and ultimately bounded by, the quality of the remains. The membership function for method i with respect to its interval-valued input and corresponding bone quality value, q_i, is

$$\mu_{A_i}(x) = \begin{cases} q_i, & \text{if } v_i^- \leq x \leq v_i^+ \\ 0, & \text{otherwise,} \end{cases} \tag{43}$$

where μ_{A_i} is the membership function and $[v_i^-, v_i^+]$ are the extreme interval endpoints in the age interval for aging method i (e.g., the interval $[10,15]$ years). At the moment, the FSs have only 0 and q_i membership values. The NDFI algorithm is formally described in Alg. 4. Figure 1 is a result of the NDFI algorithm for skeleton 208 from the Terry Anatomical Collection [4, 5].

Algorithm 2 Algorithm to calculate the *generalized FI* (gFI)

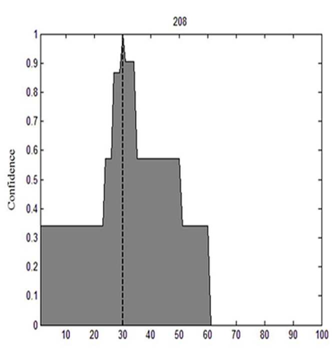
Input the $\Re$-valued FM g ▷ *e.g., use the Sugeno λ-FM, learn g from data or manually specify g, etc.*
Input FS-valued partial support function H ▷ *i.e., H_i for $i = \{1, ..., N\}$*
Calculate $\beta = \bigwedge_{i=1}^{N} Height(H_i)$ ▷ *minimum height of partial support FSs*
for each $\alpha \in (0, \beta]$ **do** ▷ *note, this step is discretized in practice*
  $[\int H \circ g]_\alpha = \left(\int [^\alpha H]_1 \circ g\right)$ ▷ *first gFI calculation, I-based integrand and $\Re$-based FM FI*
  for k = 2 to $\mathcal{M}_\alpha$ **do** ▷ *all discontinuous interval combinations at α*
    $[\int H \circ g]_\alpha = [\int H \circ g]_\alpha \bigcup \left(\int [^\alpha H]_k \circ g\right)$ ▷ *the I-based integrand and $\Re$-based FM FI*
  end for
end for

Algorithm 3 Algorithm to calculate the *non-direct FI* (NDFI)

Input the $\Re$-valued FM g ▷ *e.g., use the Sugeno λ-FM, learn g from data, manually specify g, etc.*
Input the FS-valued partial support function H ▷ *i.e., H_i for $i = \{1, ..., N\}$*
Discretize the output domain, $D = \{d_1, ..., d_{|D|}\}$ ▷ *e.g., $D = \{0, 0.01, ..., 1\}$*
Initialize the (FS) result to $R[d_k] = 0$
for each $d_k \in D$ **do** ▷ *for each output domain location*
  for each $i \in \{1, ..., N\}$ **do** ▷ *for each input*
    Let $z_i = H(d_k)$ ▷ *i.e., $\mathbf{z}$ is the vector of memberships at d_k*
    $R[d_k] = \int \mathbf{z} \circ g$ ▷ *calculated using the $\Re$-valued (integrand and measure) FI of $\mathbf{z}$ with g*
  end for
end for

Algorithm 4 NDFI algorithm for skeletal age-at-death estimation for forensic anthropology [4, 5]

Input fuzzy measure g ▷ *e.g., use the Sugeno λ-FM, learn g from data, manually specify g, etc.*
Input bone quality weathering values, $\{q_1, ..., q_N\}$ ▷ *Where $q_i \in [0, 1]$*
Input age-at-death intervals for each aging method, $\{\bar{v}_1, ..., \bar{v}_N\}$ ▷ *Where $\bar{v}_i$ is an age interval, e.g., $\bar{v}_i = [5, 20]$ years*
Discretized the output domain, $D = \{d_1, ..., d_{|D|}\}$ ▷ *e.g., $D = \{1, 2, ..., 110\}$*
Initialize the (FS-valued) result to $R[d_k] = 0$
for each $d_k \in D$ **do** ▷ *i.e., each discrete age*
  for $i = 1$ to N **do** ▷ *Calculate the partial support function at d_k*
    if $d_k \geq v_i^-$ and $d_k \leq v_i^+$ **then** ▷ *Where $-$ and $+$ are the left and right endpoints, e.g., $[v_i^-, v_i^+]$*
      $z_i = q_i$ ▷ *Age method i indicates possible age-at-death, use bone quality q_i*
    else
      $z_i = 0$ ▷ *Age method i indicates not a possible age-at-death, so no support in the hypothesis*
    end if
  end for
  $R[d_k] = \int \mathbf{z} \circ g$ ▷ *Fuzzy membership at d_k is the $\Re$-valued (integrand and measure) FI of $\mathbf{z}$ with g*
end for



Skeleton	208
Chronological Age	30 yrs
Sex	Female
Sugeno Age Range	30 yrs
Pubic Symphysis	27-30 yrs
Auricular Surface	30-34 yrs
Ectocranial Suture – Vault	24-60 yrs
Ectocranial Suture - Lateral-Anterior	< 50 yrs

Fig. 1. Example age-at-death skeletal estimation fusion result (skeleton 208 from the Terry Anatomical Collection) for the NDFI algorithm [4, 5]. The true age-at-death is 30 years. The sex is female. Four different aging methods were used. Information about the FM, anthropological details and a wider range of rich examples can be found in [4, 5].

In summary, the NDFI is based on the idea of multiple hypothesis testing. A single hypothesis is: 'the skeleton was at age *k* at death (a specific age, not range).' The (classical) SFI is repeatedly applied, once for each possible age using the respective accuracy, range and quality information. Every age, in discrete one year increments from 1 to 110 is tested. The age indicators are based on whether or not the age tested is in their respective interval. The h values are a function (t-norm) of the quality, a $[0, 1]$ value, and the age-quality membership function. Again, the result of this procedure is a collection of (age tested, FI result) pairs, which is a FS defined over the age domain. In this respect, we were able to address sub-normal FSs. Refer to [4, 5] for more details regarding the application of NDFI to skeletal age-at-death estimation.

VI. Application: Exploration of gFI and NDFI

In this section, we begin with a demonstration of the behavior of the gFI for FS-valued integrands and different FMs. Next, we compare and contrast the inner-workings of the gFI and NDFI in the context of a multi-sensor data fusion

TABLE III

FS-VALUED PARTIAL SUPPORT FUNCTION H^1. EACH H_i^1 IS THE UNION OF MULTIPLE TRAPEZOIDAL MEMBERSHIP FUNCTIONS. A TRAPEZOID IS SPECIFIED BY FOUR ORDERED VALUES, $a \leq b \leq c \leq d$, AND A HEIGHT. $H_{i,j}^1$ DENOTES THE j^{th} TRAPEZOIDAL MEMBERSHIP FUNCTION FOR THE i^{th} SOURCE. FIGURE (2) IS AN ILLUSTRATION OF THESE FSS.

	Height	a	b	c	d
$H_{1,1}^1$	1	$\frac{3}{30}$	$\frac{5}{30}$	$\frac{7}{30}$	$\frac{10}{30}$
$H_{1,2}^1$	0.8	$\frac{5}{30}$	$\frac{11}{30}$	$\frac{11}{30}$	$\frac{12}{30}$
$H_{1,3}^1$	0.7	$\frac{12}{30}$	$\frac{16}{30}$	$\frac{16}{30}$	$\frac{17}{30}$
$H_{1,4}^1$	0.9	$\frac{16}{30}$	$\frac{17}{30}$	$\frac{17}{30}$	$\frac{25}{30}$
$H_{2,1}^1$	1	$\frac{1}{30}$	$\frac{2}{30}$	$\frac{3}{30}$	$\frac{12}{30}$
$H_{3,1}^1$	0.8	$\frac{20}{30}$	$\frac{22}{30}$	$\frac{22}{30}$	$\frac{24}{30}$

scenario. Last, we explore the utility of the gFI and NDFI for skeletal age-at-death in anthropology.

A. *gFI for different fuzzy measures*

In this sub-section we demonstrate the calculation of the gFI for a few different common FMs. Specifically, as shown in [2], the CFI acts like a number of known aggregation operators based on the selection of FM. For example, when sets of equal size (cardinality) in the FM have equal measure value, the CFI produces an *ordered weighted average* (OWA), which encompasses all linear combinations of order statistics including minimum, maximum, average, etc. The following three examples demonstrate common OWAs and the result of the gFI. The reason for showing these examples is to illustrate the impact of the gFI, both in terms of sub-normality but also non-convexity. Graphically, they help us gain some insight into the inner workings of the gFI. Each sub-section makes use of the following FS-valued partial support function, H^1 (reported in Table III and shown in Fig. 2(a). This sub-normal and non-convex partial support function is generated by taking the union of different trapezoidal membership functions. This is a simple and tractable way to produce a FS that can be easily reproduced by the reader.

1) Example 1 (Max FM): Let g_1 be a FM that is of value 1 at all points in the lattice. Therefore, $\big(g_1(x_{\pi(1)}) - 0\big)$ for $\pi(1)$ in the CFI and all other FM differences are 0. Thus, the CFI simply selects the largest h value, $h_{\pi(1)}$. Figure (2b) shows the use of g_1 with respect to H^1.

2) Example 2 (Min FM): Let g_2 be a FM that is of value 0 at all points in the lattice except for 1 at $g_2(X)$. Therefore, the FM difference weightings in the CFI are all 0 except for 1 at $i = N$. Thus, the CI selects the smallest h value, $h_{\pi(N)}$. Figure (2b) shows the use of g_2 with respect to H^1.

3) Example 3 (Mean FM): Let g_3 be a FM that is of value $\frac{k}{N}$ at *layer k* in the lattice. Thus, at layer 1, i.e., the densities, each measure has value $\frac{1}{N}$. At layer 2, the value is $\frac{2}{N}$ and so on (yielding value 1 at $g_3(X)$). Therefore, each FM difference weighting in the CI is of value $\frac{1}{N}$, yielding the expected value. Figure (2b) shows the use of $g1$ with respect to H^1.

These three examples tell the following story. First, one can clearly see that the height of each gFI result is equal to that of the sub-normal FS H_3^1. One can also see that the convexity of the result depends entirely on the shape of the input FSs and the FM. The min result is a trapezoid and the max result is very similar to a triangle. However, it is clear that the average FM result is non-convex. Moreover, one can see that the convexity of the result at a given α is also very much dictated by both the input and the specific FM. Again, these examples are provided as graphical illustrations of the gFI to more clearly illustrate the inner workings of the gFI definition and approximation algorithm.

B. *Comparison of gFI and NDFI*

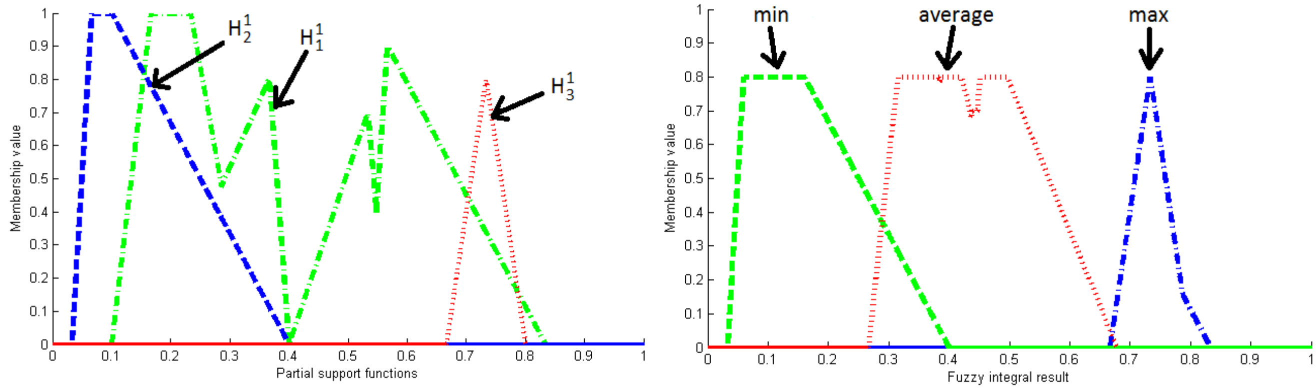
Upon beginning this investigation, the underlying questions were: "What is the direct method of extending the FI for FS-valued integrands?", and "Does it produce a better or the same result as NDFI?". The short answer is no, the gFI does not produce the same result as NDFI. In fact, the two approaches aggregate information in very different ways. It is unfortunately not simple to declare one approach as definitively better than the other. Each approach has its own respective advantages and disadvantages and the appropriate choice depend in part on the application and what one is trying to accomplish. These pros and cons are illustrated through the following numeric examples and a high-level comparative summary is provided at the end in Table V.

1) Example 1 ($\hat{H}$): Consider the example in Fig. 3. This scenario contains two inputs $X = \{x_1, x_2\}$ with partial support function $\hat{H}^2$. The two FS inputs are characterized by the triangular membership functions $\mu_{\hat{H}_1^2} = [0, 0.2, 0.4]$ and $\mu_{\hat{H}_2^2} = [0.6, 0.8, 1]$. The reliability of these sources is given by the FM, $g_4(x_1) = 0.5, g_4(x_2) = 0.5, g_4(X) = 1$.

The gFI, specifically the gSFI, produces a result which, although technically a FS, is the singleton 0.5, with a membership of 1. If the FM is changed to $g_5(x_1) = 1, g_5(x_2) = 1, g_5(X) = 1$, the gFI produces the triangular FS $[0.6, 0.8, 1]$ with height of 1. Note, this is exactly equal to μ_{H_2}.

NDFI produces very different results, shown in Fig. 4. View (a) shows the NDFI algorithm result for FM g_4. The result is two triangles, $[0, 0.2, 0.4]$ and $[0.6, 0.8, 1]$, both with heights of 0.5. For g_5, shown in view (b), the result is the same; however, each triangle has a height of 1. A possible downside of NDFI is that for this very straight-forward example, the result is a non-convex (and for g_4, sub-normal) FS.

This example could be considered as the combination (e.g., average and maximum) of two FNs, with linguistic representations of 'about 0.2' and 'about 0.8.' Intuitively, we expect the output to look like the inputs: in this case, a triangular FS with the linguistic interpretation of something like 'about 0.5' or 'about 0.8' (depending on the FM). The NDFI algorithm, again depending on the selection of FM, produces a result that is differently shaped from each of the inputs. In contrast, gFI produces outputs that look very much like the inputs, namely triangular FSs, which would be easily interpreted. However, a limitation of gFI is that if *any* of the inputs are sub-normal FSs then the output will have a maximum membership of



(a)

(b)

Fig. 2. (a) Partial support function H^1 and (b) gFI result. In (a), green is H_1^1, blue is H_2^1 and red is H_3^1. In (b), green is min, blue is max and red is average. The gFI used is the *generalized Choquet fuzzy integral* (gCFI).

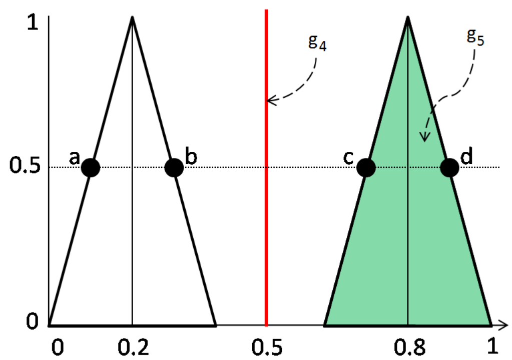
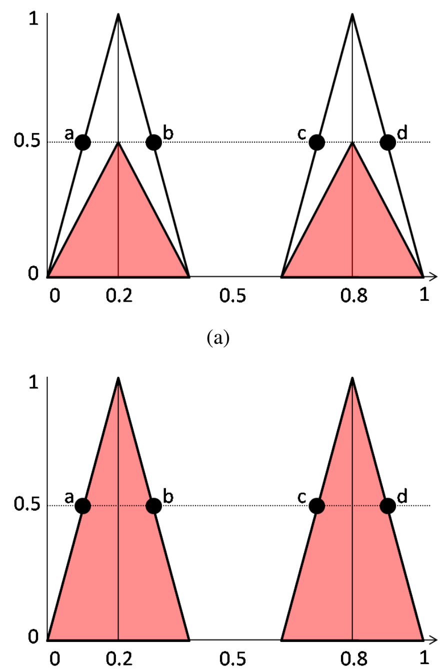


Fig. 3. Illustration of a FS integrand and interval endpoints used to compute gFI at $\alpha = 0.5$. The results for two FMs are provided: red is $g_4(x_1) = 0.5, g_4(x_2) = 0.5, g_4(X) = 1$; and green is $g_5(x_1) = 1, g_5(x_2) = 1, g_5(X) = 1$. The two FSs are characterized by the triangular membership functions $\mu_{\hat{H}_1^2} = [0, 0.2, 0.4]$ and $\mu_{\hat{H}_2^2} = [0.6, 0.8, 1]$.



(a)

(b)

Fig. 4. Illustration of a FS integrand and interval endpoints used to compute NDFI at $\alpha = 0.5$. Case (a) is for FM $g_4(x_1) = 0.5, g_4(x_2) = 0.5, g_4(X) = 1$, while case (b) is for FM $g_5(x_1) = 1, g_5(x_2) = 1, g_5(X) = 1$. The results for these FMs is shown in red. The two FSs are characterized by the triangular membership functions $\mu_{\hat{H}_1^2} = [0, 0.2, 0.4]$ and $\mu_{\hat{H}_2^2} = [0.6, 0.8, 1]$.

the minimum-height sub-normal FS, even if the respective reliability (g) of that sub-normal input is 0-valued (which intuitively means that we should ignore that input as it has no worth in the solution to the FI). Hence, both have their respective drawbacks.

In contrast, for age-at-death estimation in anthropology, we desire a *restricted* result. That is, anthropologists indicate that we should be careful not to produce ages *outside* of intervals indicated by the individual aging methods. For example, if one method reports $[10, 20]$ and another method reports $[60, 100]$ (which, for most practical cases is unlikely), we do not want to produce an age interval such as $[40, 50]$. In addition to fusing the inputs, we would like to have a way to discover that there is disagreement among the sources and we would like to find the age(s) in which we can be most confident. That is, we would like to take into account the agreement between sources, the method's confidences and our confidences in the sources. If one input has a low height, we do not want the FI result to be ultimately limited by this amount. In [4], our objective was to find a way to fuse the various information (FS inputs, bone quality values and numeric values representing the 'worth' of the information sources) and then analyze the result. The result was the introduction of NDFI. In [4], we calculated a single age-at-death number (e.g., died at age 20). We identified FS features and created fuzzy class definitions to assist with interpreting the FS results [5]. We also measured the confidence and specificity of the resultant FSs. The four anthropological FS categories are shown in Fig. 5. These categories represent: specific age (aging methods come together and agree on a single age-at-death), age range (agreement between the sources but no single definitive age),

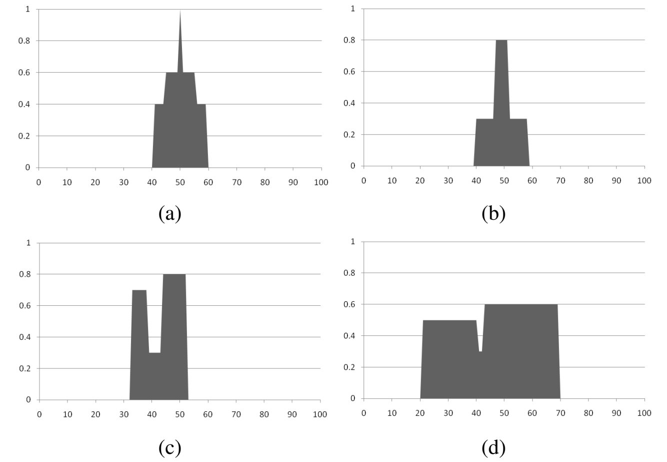


Fig. 5. Interpretation of resultant FS in age-at-death estimation using NDFI [4, 5]. Categories identified by anthropologists include: (a) specific age (aging method have come together and agree on a single age-at-death), (b) age interval (there is agreement between the sources but no single definitive age), (c) disagreement (there is disagreement between the methods, thus multiple plateaus) and (d) inconclusive (so much disagreement or general lack of confidence that it is difficult to conclude anything).

disagreement (there is disagreement between the methods, thus multiple plateaus) and inconclusive (so much disagreement or lack of confidence that it is difficult to conclude anything).

2) Example 2 ($\hat{\hat{H}}$): Consider the example in Fig. 6(a). This scenario contains two inputs $X = \{x_1, x_2\}$ with partial support function $\hat{\hat{H}}^3$. The two FS inputs are characterized by the triangular membership functions $\mu_{\hat{H}_1^3} = [0, 0.2, 0.4]$ and $\mu_{\hat{H}_2^3} = [0.6, 0.8, 1]$, and the FM is $g_6(x_1) = 1, g_6(x_2) = 0, g(X) = 1$ (i.e., no worth is assigned to the second information source). However, in this example let the height of $\mu_{\hat{H}_2^3}$ be 0.01 (sub-normal FS).

The gFI algorithm results in the trapezoidal membership function $[0, 0.002, 0.398, 0.4]$ with height 0.01. Note, this result is different in *shape* from the input. That is, the inputs are triangular while the result is a trapezoid. While the second source is completely untrustworthy ($g_6(x_2) = 0$), it has substantially impacted the result. The resultant height is so low that intuitively we should ignore the result. However, for this second experiment NDFI produces a more pleasing result. That is, a single triangle of height 1 at $[0, 0.2, 0.4]$ and quasi no support (height 0.01) in $[0.6, 0.8, 1]$ (shown in Fig. 6(b)).

3) Example 3: Age-at-Death Estimation: Next, we consider a case from our prior skeletal age-at-death estimation work [4]. This example, presented in Table V, consists of eight aging methods. Each skeletal remain (bone) is associated with a skeletal quality value of less than one, i.e., a $Height(H_i) \leq 1$. By looking at the agreement between these aging methods from an anthropological standpoint, we would expect a result close to the true age-at-death (which is 38). Specifically, we expect a *narrow* interval (not a single age-at-death because the inputs are all interval-valued with width greater than 1) that includes the age 38. The input FSs have heights (their confidence) equal to their respective quality of bone. Additionally, the fusion procedure (gFI or NDFI) is expected

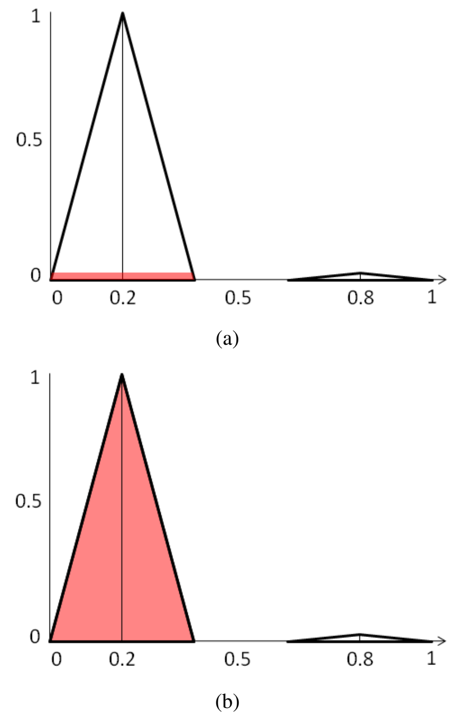


Fig. 6. Results for the FM $g_6(x_1) = 1, g_6(x_2) = 0, g_6(X) = 1$. Case (a) is for gFI and (b) is for NDFI. The two FS are characterized by the triangular membership functions $\mu_{\hat{H}_1^3} = [0, 0.2, 0.4]$ and $\mu_{\hat{H}_2^3} = [0.6, 0.8, 1]$, with $Height(\hat{\hat{H}}_1^3) = 1$ and $Height(\hat{\hat{H}}_2^3) = 0.01$.

to fuse this information with respect to the reliability of the aging methods. In this work, as well as in our previous work, the Sugeno λ-FM is used to build the entire FM from the densities. Figure 7 shows the results of gFI and NDFI. Note, with respect to the gFI, the inputs are first scaled from $[0, 110]$ to $[0, 1]$ (division by 110), the gFI algorithm is run, and the results are then scaled back to $[0, 110]$ (multiplication by 110).

The following observations are made with respect to gFI and NDFI. First, the inputs are trapezoids and the output of gFI is a trapezoid. Specifically, the output is subnormal and convex and its shape is that of the inputs (a trapezoid). In comparison, the output of NDFI is sub-normal and non-convex and its shape does not resemble that of the individual inputs. Second, the interval $[37, 39]$ has the most agreement among the inputs. That is, each age method reports these ages. However, we do not desire an overly simple procedure that just counts the number of times that an age is agreed upon by the aging methods followed by a selection of an interval that has a maximum score. It is very likely that multiple intervals could exist. Additionally, we would like to take the reliability of each aging method into consideration. This is the motivation for taking a generalized SFI approach. That said, gFI returns a single (and very wide or non-specific at that) interval, $[37, 76]$. While the gFI algorithm output does include the true age-at-death, it includes to many other ages as well. In comparison, the NDFI algorithm result indicates a single maximum *plateau* of $[35, 39]$, which for Example 3 is a single interval associated with the highest membership degree (see [4] and [5] for a

TABLE IV
INPUT FOR EXAMPLE 3 FROM OUR PRIOR AGE-AT-DEATH WORK [4]

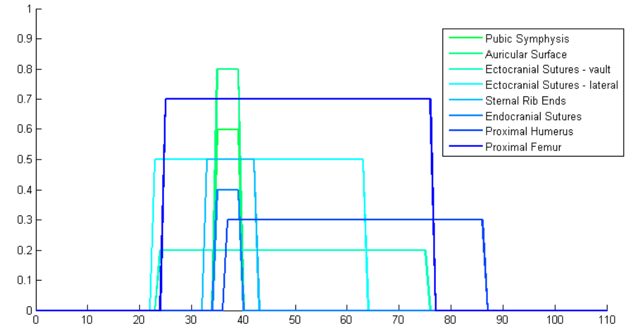
Aging Method	**Quality**	**Age Range**	g^i
Pubic Symphysis	0.6	35-39	.57
Auricular Surface	0.8	35-39	.72
Ectocranial Sutures - vault	0.2	24-75	.59
Ectocranial Sutures - lateral	0.5	23-63	.59
Sternal Rib Ends	0.5	33-42	.75
Endocranial Sutures	0.4	35-39	.51
Proximal Humerus	0.3	37-86	.44
Proximal Femur	0.7	25-76	.56

formal definition of maximum *plateau*). However, in some cases, such as those discussed in [4, 5], multiple plateaus can exist. To summarize Example 3, both NDFI and gFI include the true age of death in their result, however NDFI indicates smaller number of possible ages. The gFI result is a wide (that is, non-specific) interval that is of little-to-no use for age-at-death estimation. It reports that the true age-at-death is one of 40 years. However, the NDFI algorithm result is more specific, i.e., the true age-at-death is one of 5 possible ages (according to the maximum plateau).

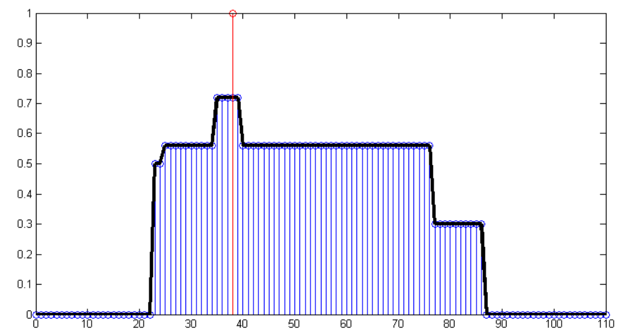
As discussed in our prior work [4, 5], NDFI provides a wealth of additional information. In [5] we put forth a technique to linguistically describe a gFI FS. From that work, the following can be concluded (which is not available in the SuFI algorithm output). First, the *shape* of the resultant FS informs us about the nature of the agreement. For Example 3, the result is of type *interval* (one of many possible ages), however it is not very wide and could potentially be considered as type *specific* (a single age-at-death). Additionally, in [5] we defined a linguistic variable to interpret the confidence of the output decision. For example 3, NDFI reports that the fused result is of *moderate* confidence (the maximum plateau has a height of 0.72), while SuFI (of height 0.2) is of *very low* confidence (and most likely should be ignored).

C. High-level Comparison of the gFI and the NDFI

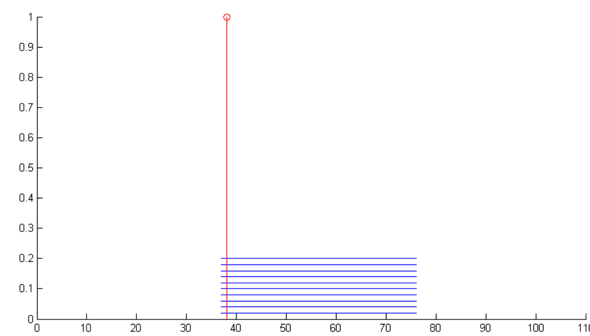
Table V is a summary of the major differences between the gFI and the NDFI. Specifically, Table V tells the following story. The gFI is a direct (i.e., Extension Principle based) generalization of the FI for FS-valued integrands. However, the NDFI is also an extension, be it indirect, of the FI for FS-valued inputs. The gFI and the NDFI are very different in each of the reported categories: height, range, approach and shape of the resultant FS. The "correct" or indeed more appropriate FI extension appears to depend on the application. The NDFI appears to be of utility when the goal is to aggregate the input FSs "in place" with respect to the FI. As described earlier, this is reflected by the fact that the NDFI is the repeated application of the FI for multiple hypotheses. Specifically, there is one hypothesis for each discretized domain location. Furthermore, the NDFI is restricted to the region (range) corresponding to the union of the input FS supports. In this respect, the NDFI cannot draw conclusions in regions where no input support exists. Conversely, the gFI is appropriate when one wants to compute a function with respect to a FS-valued integrand. The output can be anywhere between the



(a)



(b)



(c)

Fig. 7. Results found for the inputs, bone quality values, and Sugeno λ-FM for the densities reported in Table V. In (a), the input FSs are shown. In (b), the NDFI algorithm result is shown. The x-axis is the range $[0, 110]$. In (c), the gFI algorithm result is shown for 11 α-cuts. In (b) and (c), a vertical dashed line is drawn at the true age-at-death (38).

minimum and maximum. However, gFI is not perfect. Zadeh's definition of the Extension Principle *restricts* the gFI result in some applications (such as the discussed case of multi-sensor fusion).

To illustrate, consider the case of skeletal age-at-death (example in Figure 7). We saw that when the input FSs are sub-normal it is possible that the gFI yields a more-or-less unusable result. Not only is the result restricted to a maximum value of 0.2, but the approximated result is a flat membership function stretching from 30 to 70 years. This result is of little-to-no use in skeletal age-at-death estimation. However, the NDFI result clearly indicates an age range subset. Namely, an age range in which there is the greatest agreement across the different input FSs (taking into account the quality, or reliability, of the different inputs/sources).

VII. Conclusion

Herein, a number of FIs for different types of integrand information was reviewed. In addition, a definition and efficient algorithm for the *generalized FI* (gFI) for FS-valued integrands (non-convex and sub-normal) was put forth. This extension was compared to a *non-direct* FI extension, called the NDFI, theoretically as well as empirically for the cases of multi-source (sensor) data fusion and skeletal age-at-death estimation in forensic anthropology. It was demonstrated that both the gFI and the NDFI have their individual benefits and they are indeed different extensions. The overall benefit of this article is the comparison of the definition, calculation and application of different FI extensions.

Acknowledgments

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TABLE V
COMPARISON TABLE FOR THE GFI AND THE NDFI.

Property	**gFI**	**NDFI**
Height	Height of lowest FS (i.e., minimum of Height(H_1), ..., Height(H_N))	Depends on the FM. Anywhere between 0 and maximum FS height
Range	$\int H \circ g$ can be anywhere between the minimum and maximum of input FSs (i.e., the integrand)	Range extremes are similar to the gFI (minimum to maximum). However, the support of the NDFI FS result is restricted to the union of the support of the input FSs
Approach to $(\int H \circ g)(a)$	Extension Principle. Thus, it is the extension of a function for FS-valued integrands	FI calculated at a. Thus, aggregation is being performed across the FSs
Shape of the FS $\int H \circ g$	Can be (and likely is) different from that of the inputs, e.g., for triangular shaped sub-normal FS inputs we can obtain a trapezoidal shaped output. In general, sub-normal (if any input is sub-normal) and convex	In general, will be sub-normal and non-convex

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