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ENDOTRIVIAL MODULES FOR THE SCHUR COVERS OF THE SYMMETRIC AND ALTERNATING GROUPS

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Abstract. We investigate the endotrivial modules for the Schur covers of the symmetric and alternating groups and determine the structure of their group of endotrivial modules in all characteristics. We provide a full description of this group by generators and relations in all cases.

1. Introduction

The endotrivial modules over the group algebra kG of a finite group G of order divisible by the characteristic $p > 0$ of a field k have seen considerable interest over the last forty years, with a surge in the last fifteen years. They have been classified when G is a p-group [CT04] and many contributions towards a general classification have been obtained since for some general classes of finite groups, see e.g. [CMN09, CMN06, CHM10, MT07, CMT11, CMT13, CMT14, LMS, CMN14, LM] and the references therein. The study of endotrivial kG-modules and their group of endotrivial modules $T(G)$ (which is finitely generated) is of particular interest in modular representation theory as it forms an important part of the Picard group of self-equivalences of the stable category of finitely generated kG-modules. In particular the self-equivalences of Morita type are induced by tensoring with endotrivial modules.

At present, notable efforts are made to determine the group $T(G)$ for quasi-simple groups G, as it is hoped that the general question of classifying endotrivial modules for arbitrary finite groups can be reduced to this class of groups (cf. [CMT11]). The results in this article complement those obtained in the aforementioned papers. More precisely, we build on the results of [CMN09, CHM10], where the group of endotrivial modules for the symmetric and alternating groups is fully determined, to describe $T(G)$ for their Schur covers.

For $n \geq 4$ the Schur multiplier of the symmetric and alternating groups is nontrivial. In this case we let $2^{\pm}.\mathfrak{S}_n$ denote the two non-isomorphic double covers of the symmetric group $\mathfrak{S}_n$ and $d.\mathfrak{A}_n$ the d-fold cover of the alternating group $\mathfrak{A}_n$ as in [CCN$^+$85, §6.7]. So $d = 2$ or possibly $d \in \{3, 6\}$ if $n \in \{6, 7\}$.

Our two main results are as follows.

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Theorem A. *Let k be an algebraically closed field of characteristic 2. The following hold:*

$$(1) \qquad T(2.\mathfrak{A}_n) \cong \begin{cases} \mathbb{Z}/2 \oplus \mathbb{Z}/4 & \text{if } 4 \leq n \leq 7, \\ \mathbb{Z} & \text{if } 8 \leq n. \end{cases}$$

$$(2) \qquad T(2^+.\mathfrak{S}_n) \cong \begin{cases} \mathbb{Z}/2 \oplus \mathbb{Z} & \text{if } n \in \{4,5\}, \\ \mathbb{Z}^2 & \text{if } n \in \{8,9\}, \\ \mathbb{Z} & \text{if } n \in \{6,7\} \text{ or } 10 \leq n\,. \end{cases}$$

$$(3) \qquad T(2^-.\mathfrak{S}_n) \cong \begin{cases} \mathbb{Z}/2 \oplus \mathbb{Z}/4 & \text{if } n \in \{4,5\}, \\ \mathbb{Z} & \text{if } 6 \leq n. \end{cases}$$

Theorem B. *Let k be an algebraically closed field of characteristic $p \geq 3$. Let G be one of the double covers $2.\mathfrak{A}_n$ or $2^{\pm}.\mathfrak{S}_n$ with $n \geq 4$. The following hold:*

(1) *If $p = 3$ and $n = 6$, then $T(2.\mathfrak{A}_6) \cong \mathbb{Z}/8 \oplus \mathbb{Z}^2$ and $T(2.\mathfrak{S}_6) = \mathrm{Inf}_{\mathfrak{S}_6}^{2.\mathfrak{S}_6}(T(\mathfrak{S}_6)) \cong (\mathbb{Z}/2)^2 \oplus \mathbb{Z}$.*

(2) *If $n \geq p+4$, then the inflation homomorphism $\mathrm{Inf}_{G/Z(G)}^{G} : T(G/Z(G)) \longrightarrow T(G)$ is an isomorphism.*

(3) *If $p \geq 3$ and $p \leq n \leq min\{2p-1, p+3\}$ then $T(G) \cong \mathbb{Z}/2e \oplus \mathbb{Z}/2$, where e denotes the inertial index of the principal block of kG.*

In addition, when the torsion-free rank of $T(G)$ is greater than 1, we provide generators for the torsion-free part in all cases, and similarly when the torsion subgroup is nontrivial, then we describe its elements explicitly.

The paper is organised as follows. In Section 2, we recall the necessary background on endotrivial modules and in Section 3 that about the Schur covers of alternating and symmetric groups. Theorem A is proved in Section 4 and Theorem B in Section 5. In both these sections we describe the groups $T(G)$ by generators and relations. Finally, in Section 6, we handle the structure of the group of endotrivial modules of the four exceptional Schur covers $3.\mathfrak{A}_6$, $6.\mathfrak{A}_6$, $3.\mathfrak{A}_7$ and $6.\mathfrak{A}_7$.

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2. Preliminaries

Throughout, unless otherwise stated, we let G denote a finite group, p a prime number dividing the order of G, k an algebraically closed field of characteristic p, and $(K, \mathcal{O}, k)$ a splitting p-modular system for G and its subgroups. All kG-modules we consider are assumed to be finitely generated left kG-modules. We denote by $B_0(kG)$ the principal block of kG and by $\mathrm{Irr}(G)$ the set of irreducible complex characters of G. We refer the reader to the standard literature for these concepts (e.g. [Ben98]).

2.1. **Background results on endotrivial modules.**

A kG-module M is called *endotrivial* if its k-endomorphism ring decomposes as a kG-module into a direct sum $\mathrm{End}_k(M) \cong M^* \otimes_k M \cong k \oplus (\mathrm{proj})$, where $M^* = \mathrm{Hom}_k(M,k)$ denotes the k-dual of M, k the trivial kG-module and (proj) some projective module (possibly zero). If M is endotrivial, then M splits as $M \cong M_0 \oplus (\mathrm{proj})$, where M_0 is, up to isomorphism, the unique indecomposable endotrivial direct summand of M. The set $T(G)$ of isomorphism classes of indecomposable endotrivial kG-modules is an abelian group for the binary operation induced by the tensor product $\otimes_k$ of endotrivial modules, that is $[M]+[N] = [(M \otimes_k N)_0]$. The identity element is the class of the trivial module $[k]$, and we have $-[M] = [M^*]$ for any $[M] \in T(G)$. We denote by $X(G)$ the group of isomorphism classes of one-dimensional kG-modules (endowed with $\otimes_k$). Recall that $X(H) \cong (G/[G,G])_{p'}$, the p'-part of the abelianization of G. In particular $X(G)$ is isomorphic to a finite subgroup of $T(G)$.

By [CMN06, Corollary 2.5], the group $T(G)$ is known to be finitely generated (note that this is a consequence to the corresponding result originally obtained by Puig when G is a p-group) so that we may write $T(G) = TT(G) \oplus TF(G)$, where $TT(G)$ is the torsion subgroup of $T(G)$ and $TF(G)$ a torsion-free complement. The $\mathbb{Z}$-rank of $TF(G)$ is called the *torsion-free rank* of $T(G)$. Moreover, if the torsion-free rank is one, then we may choose $TF(G)$ such that $TF(G) = \langle [\Omega_G(k)] \rangle$, where Ω_G denotes the Heller translate. We write $T_0(G)$ for the subgroup of $T(G)$ formed by the isomorphism classes of the indecomposable endotrivial modules which belong to the principal block $B_0(kG)$.

We start with the following general lifting result which allows us to use ordinary character theory in our investigations. Hereafter the term *$\mathcal{O}G$-lattice* means an $\mathcal{O}$-free $\mathcal{O}G$-module.

Theorem 2.1 ([LMS, Theorem 1.3 and Cor. 2.3])**.**
Let $(K,\mathcal{O},k)$ be a splitting p-modular system for a finite group G. Let M be a kG-module.

(1) *If M is endotrivial, then M is liftable to an endotrivial $\mathcal{O}G$-lattice.*
(2) *Suppose that M is endotrivial and lifts to a $\mathbb{C}G$-module affording the character χ. Then $|\chi(g)| = 1$ for every p-singular element $g \in G$.*

The class of endotrivial modules is stable under the operations of restriction and inflation, under the following assumptions.

Lemma 2.2. *Let G be a finite group and H a subgroup of G.*

(1) *If $p \,|\, |H|$, then restriction along the inclusion $H \hookrightarrow G$ induces a group homomorphism $\mathrm{Res}^G_H : T(G) \longrightarrow T(H)$ by mapping $[M] \mapsto [(M{\downarrow}^G_H)_0]$. If, moreover, H contains the normaliser of a Sylow p-subgroup in G, then Res^G_H is injective, with partial inverse induced by Green correspondence.*
(2) *If $Z \trianglelefteq G$ is a p'-subgroup of G, then restriction along the quotient map $G \to G/Z$ induces an injective group homomorphism $\mathrm{Inf}^G_{G/Z} : T(G/Z) \longrightarrow T(G)$ by mapping $[M] \mapsto [\mathrm{Inf}^G_{G/Z}(M)]$, where $\mathrm{Inf}^G_{G/Z}(M)$ denotes the module M regarded as a kG-module with trivial action of Z.*

For part (1) see [CMN06, Prop. 2.6 and Rem. 2.9], and for part (2) see [LM, Lemma 3.2(1)].

We now present the results needed in the sequel in order to investigate the structure of $T(G)$ as a finitely generated abelian group. We start with the torsion-free rank of $T(G)$. Recall that the *p-rank* of a finite group G is the logarithm to base p of the maximum of the orders of the elementary abelian p-subgroups of G. Also, by *maximal elementary abelian p-subgroup* of G, we mean an elementary abelian p-subgroup which is not properly contained in another elementary abelian p-subgroup of G.

Theorem 2.3 ([CMN06, Theorem 3.1])**.** *The torsion-free rank of $T(G)$ is equal to the number of G-conjugacy classes of maximal elementary abelian p-subgroups of rank 2 if G has p-rank at most 2, or that number plus one if G has p-rank greater than 2.*

In contrast $T(G) = TT(G)$ is a finite for groups with Sylow p-subgroups of p-rank 1 (i.e. cyclic or possibly generalised quaternion if $p = 2$).

Moreover if $TF(G)$ is not cyclic, then we do not have means to find generators for it, except in some very special instances, as in the case of central extensions by nontrivial p-subgroups for which Theorem 2.5 below applies. Independently from considerations about generators, if we only regard $TF(G)$ as a torsion-free abelian group, there are two bounds which are relevant to the study of endotrivial modules.

Theorem 2.4. *Let G be a finite group.*

(1) *If G has a maximal elementary abelian p-subgroup of rank 2, then G has p-rank at most p if p is odd, or at most 4 if $p = 2$. Moreover both bounds are optimal.*
(2) *G has at most $p+1$ G-conjugacy classes of maximal elementary abelian p-subgroups of rank 2 if p is odd, or at most 5 if $p = 2$. Moreover both bounds are optimal.*

Consequently if G has p-rank greater than p if p is odd, or greater than 4 if $p = 2$, then $TF(G) = \langle [\Omega_G(k)] \rangle \cong \mathbb{Z}$ is infinite cyclic.

Part (1) was proved by MacWilliams [Mac70, Four Generator Theorem, p. 349] for $p = 2$, and by Glauberman and the second author [GM10] for p odd. Part (2) was proved by Carlson [Car07] for $p = 2$ and by the second author [Maz08] for p odd.

The groups G we handle in this paper are central extensions by cyclic subgroups of order dividing 6. Therefore, in investigating $T(G)$ in characteristic 2 and 3, there are cases in which the following theorem enables us to obtain generators for $TF(G)$. The key result on the p-local structure of a finite p-group which is used in Theorem 2.5 is the following (cf. [CMT14, Section 3]). Suppose that P is a non-abelian finite p-group with $TF(P)$ not cyclic. Thus there is at least one maximal elementary abelian subgroup, say $E = \langle z, s \rangle \cong C_p \times C_p$, with $z \in Z(P)$ and $s \notin Z(P)$. It turns out that $C_P(E) = C_P(s) = \langle s \rangle \times L$ where L has p-rank 1 and its isomorphism type is independent of the choice of a maximal elementary abelian subgroup E. Hence we can define a constant m which only depends on P as follows:

$$m = \begin{cases} 1 & \text{if } |L| \leq 2; \\ 2 & \text{if } L \text{ cyclic of order } \geq 3; \\ 4 & \text{if } L \text{ is generalised quaternion } (p = 2). \end{cases}$$

Theorem 2.5 ([CMT14, Theorem 5.2])**.** *Let G be a group such that the torsion-free rank n_G of $T(G)$ is at least 2. Let $P \in \mathrm{Syl}_p(G)$, let Z be the unique central subgroup of P of order p, and assume that Z is normal in G. For $2 \leq i \leq n_G$ choose a representative E_i*

of a conjugacy class of maximal elementary abelian subgroups of P of rank 2. Let $a = mp$ where m is the integer defined above.

(1) *For $2 \leq i \leq n_G$, there is a subquotient N_i of the kG-module $\Omega^a_G(k)$ which is endotrivial, and such that*

$$N_i{\downarrow}^G_{E_i} \cong \Omega^a_{E_i}(k) \oplus (\mathrm{proj}) \quad \textit{and} \quad N_i{\downarrow}^G_{E_j} \cong k \oplus (\mathrm{proj}) \quad \textit{whenever } j \neq i.$$

(2) *$TF(G)$ is generated by $[\Omega_G(k)]$, $[N_2]$, $\ldots$, $[N_{n_G}]$.*

Next we describe the known results on the torsion subgroup $TT(G)$ of $T(G)$. Given $P \in \mathrm{Syl}_p(G)$, we define

$$K(G) = \ker\left(\mathrm{Res}^G_P : T(G) \longrightarrow T(P)\right).$$

In other words, $K(G)$ is the subgroup of $T(G)$ formed by the equivalence classes of the trivial source endotrivial modules. In particular, $X(G) \leq K(G) \leq TT(G)$.

Lemma 2.6. *Let G be a finite group and $P \in \mathrm{Syl}_p(G)$.*

(1) *$TT(P) = \{[k]\}$ unless P is cyclic, generalised quaternion, or semi-dihedral.*
(2) *$K(G) = TT(G)$ whenever $TT(P) = \{[k]\}$.*
(3) *If ${}^xP \cap P$ is nontrivial for all $x \in G$, then $K(G) = X(G)$.*
(4) *If $1 \longrightarrow Z \longrightarrow G \longrightarrow H \longrightarrow 1$ is a central extension with $Z = Z(G)$ of order divisible by p, then $K(G) = X(G)$.*

Part (1) is a consequence of the classification of endotrivial modules over p-groups (see [CT04]), Part (2) is [CMT11, Lemma 2.3], part (3) is [MT07, Lemma 2.6], while part (4) is a particular case of (3). For the structure of $TT(G)$ when $P \in \mathrm{Syl}_p(G)$ is generalised quaternion or semi-dihedral, we refer the reader directly to [CMT13]. We review here known facts about $T(G)$ when $P \in \mathrm{Syl}_p(G)$ is cyclic.

Theorem 2.7 ([MT07, Theorem 3.2] and [LMS, Lemma 3.2])**.** *Let G be a finite group with a cyclic Sylow p-subgroup P of order at least 3. Let Z be the unique subgroup of P of order p and let $H = N_G(Z)$. Let $e = |N_G(Z) : C_G(Z)|$ denote the inertial index of the principal block $B_0(kG)$ of kG. The following hold.*

(1) *$T(G) = \{\ [\mathrm{Ind}^G_H(M)] \mid [M] \in T(H)\ \} \cong T(H)$.*
(2) *The sequence*

$$0 \longrightarrow X(H) \longrightarrow T(H) \xrightarrow{\mathrm{Res}^H_P} T(P) \longrightarrow 0$$

is exact. Moreover $T(P) = \langle [\Omega_P(k)] \rangle \cong \mathbb{Z}/2$ and the sequence splits if e is odd.

(3) *$T_0(G) = \langle [\Omega_G(k)] \rangle \cong \mathbb{Z}/2e$.*
(4) *The number of p-blocks of kG containing indecomposable endotrivial kG-modules is equal to $\frac{|X(H)|}{e}$, and each of these has inertial index equal to e and contains a simple endotrivial kG-module.*

An obvious consequence of Theorem 2.7(4) is that in the case of cyclic Sylow p-subgroups, we have $T(G) = \left\langle [\Omega_G(k)], [S_2], \ldots, [S_{|X(H)|/e}] \right\rangle$, where the modules S_i for $2 \leq i \leq |X(H)|/e$ are simple endotrivial modules in pairwise distinct non-principal blocks of kG.

3. The Schur covers of the alternating and symmetric groups

For a detailed construction of the Schur covers of the alternating and symmetric groups, we refer the reader to [HH92, Chap. 2] and [Wil09, §2.7.2]. We recall that for $n \geq 4$ the Schur multiplier of the symmetric and alternating groups is nontrivial of order 2, except in the cases of $\mathfrak{A}_6$ and $\mathfrak{A}_7$ in which case it has order 6. We let $2^{\pm}.\mathfrak{S}_n$ denote the two isoclinic double covers of the symmetric group $\mathfrak{S}_n$, and $d.\mathfrak{A}_n$ denote the d-fold Schur cover of the alternating group $\mathfrak{A}_n$. As in [CCN+85, § 6.7], the group $2^{+}.\mathfrak{S}_n$ is the double cover of $\mathfrak{S}_n$ in which transpositions of $\mathfrak{S}_n$ lift to involutions, while in $2^{-}.\mathfrak{S}_n$ transpositions lift to elements of order 4. We recall that $2^{+}.\mathfrak{S}_n \cong 2^{-}.\mathfrak{S}_n$ if and only if $n = 6$.

We use the presentation by generators and relations of $2^{\pm}.\mathfrak{S}_n$ given in [HH92, CCN+85], and the convention $[g, h] = g^{-1}h^{-1}gh$ for commutators. So

$$2^{\pm}.\mathfrak{S}_n = \langle z, t_1, \ldots, t_{n-1}\rangle / \langle \mathcal{R}\rangle$$

where $\mathcal{R}$ is generated by the relations

$$\begin{aligned} z^2 &= 1; \\ t_j^2 &= z^{\alpha} \quad , \quad 1 \leq j \leq n-1; \\ (t_j t_{j+1})^3 &= z^{\alpha} \quad , \quad 1 \leq j \leq n-2; \\ [z, t_j] &= 1 \quad , \quad 1 \leq j \leq n-1; \\ [t_j, t_k] &= z \quad \text{if} \quad |j-k| > 1, 1 \leq j, k \leq n-1; \end{aligned}$$

where $\alpha = 0$ for $G = 2^{+}.\mathfrak{S}_n$ and $\alpha = 1$ for $G = 2^{-}.\mathfrak{S}_n$. Then $2.\mathfrak{A}_n$ is the preimage of $\mathfrak{A}_n$ in either double cover of $\mathfrak{S}_n$ via the natural quotient map $G \mapsto G/Z(G)$, where $Z(G) = \langle z\rangle$. It follows that

$$2.\mathfrak{A}_n = \langle z, x_1, \ldots, x_{n-2}\rangle / \big(\langle \mathcal{R}\rangle \cap \langle z, x_1, \ldots, x_{n-2}\rangle\big)$$

where $x_j = t_j t_{j+1}$ for $1 \leq j \leq n-2$.

For later use, we record a few combinatorial equalities from [Wil09, § 2.7.2]. Write $t_j = [j, j+1]$ for an element of $2^{\pm}.\mathfrak{S}_n$ which lifts the transposition $(j, j+1)$. Then, the lifts of transpositions are of the form $\pm[i, j]$ and have order 2 in $2^{+}.\mathfrak{S}_n$, respectively 4 in $2^{-}.\mathfrak{S}_n$. In particular we calculate $[1, 2]^{[1,2]} = -[2, 1]$ and $[1, 2]^{[3,4]} = -[1, 2]$, so that $([1, 2][3, 4])^2 = -1$ and so, independently of the order of the lift of a transposition, we conclude that in $2.\mathfrak{A}_n$ the noncentral involutions are the lifts of l-fold transpositions for $l \equiv 0 \pmod 4$. For convenience, we call a permutation g an *l-fold transposition*, for some positive integer l, if g is the product of l disjoint transpositions. In particular transpositions are 1-fold transpositions and $(1, 2)(3, 4)$ is a 2-fold transposition. The noncentral involutions of $2^{-}.\mathfrak{S}_n$ are the lifts of l-fold transpositions with $l \equiv 3 \pmod 4$, whereas the latter lift to elements of order 4 in $2^{+}.\mathfrak{S}_n$.

Next we recall some facts about the ordinary character theory of $2^{\pm}.\mathfrak{S}_n$ (see [HH92, §6 and §8]). Let $\mathcal{D}(n)$ denote the set of partitions of n into distinct parts, i.e. $\lambda = (\lambda_1 > \lambda_2 > \cdots > \lambda_t)$ with $\sum_j \lambda_j = n$. A partition λ is called *even*, respectively *odd*, if its number of even parts is even, respectively odd. The faithful complex irreducible characters of $2^{\pm}.\mathfrak{S}_n$ are parametrised by the partitions $\lambda \in \mathcal{D}(n)$ as follows (see [HH92, Theorem 8.6]):

(1) If λ is even, then there is one irreducible character $\psi_\lambda \in \mathrm{Irr}(2^{\pm}.\mathfrak{S}_n)$ which splits upon restriction to $2.\mathfrak{A}_n$ into two distinct irreducible constituents $\psi_\lambda^{\pm}$.
(2) If λ is odd, then there are two irreducible characters $\psi_\lambda^{\pm} \in \mathrm{Irr}(2^{\pm}.\mathfrak{S}_n)$ which have the same irreducible restriction to $2.\mathfrak{A}_n$.

Moreover, the necessary information for our investigation on the values of the faithful elements in $\mathrm{Irr}(2^{\pm}.\mathfrak{S}_n)$ and $\mathrm{Irr}(2^{\pm}.\mathfrak{A}_n)$ is provided by [HH92, Theorem 8.7]. In particular, for $g \in 2^{\pm}.\mathfrak{S}_n$ and $\lambda \in \mathcal{D}(n)$, the following hold.

(1) If λ is odd, then $\psi_\lambda(g) = 0$ if g lifts a cycle type of $\mathfrak{S}_n$ which is not in $\mathcal{P}^0(n) \cup \{\lambda\}$, where $\mathcal{P}^0(n)$ denotes the set of partitions of n with only odd parts.
(2) If λ is even, then $\psi_\lambda(g) = 0$ if g lifts a cycle type of $\mathfrak{S}_n$ which is not in $\mathcal{P}^0(n)$.
(3) If λ is even, then the two irreducible constituents $\psi_\lambda^{\pm}$ of the restriction of ψ_λ to $2.\mathfrak{A}_n$ are such that $\psi_\lambda^{+}(g) = \psi_\lambda^{-}(g)$ for all $g \in 2.\mathfrak{A}_n$ such that g does not project to an element of cycle type $\lambda \in \mathfrak{A}_n$.

Finally, motivated by Theorem 2.3 and Lemma 2.6, we summarise in Table 1 below the relevant information on the Sylow 2-subgroups of the Schur covers of the alternating and symmetric groups and their 2-rank.

It is known that the 2-rank of $2.\mathfrak{A}_n$ is equal to $3\lfloor \frac{n}{8} \rfloor + 1$, see e.g. [GLS98, Prop. 5.2.10]. To work out the isomorphism types of the Sylow 2-subgroups, we used [GLS98, CCN+85, BCP97]. For $n \geq 3$, we denote by D_{2^n} and Q_{2^n} a dihedral and a generalised quaternion group of order 2^n respectively, and for $n \geq 4$, we write SD_{2^n} for a semi-dihedral group of order 2^n. In addition, $R_{m,r}$ denotes the group $\mathsf{SmallGroup}(m, r)$, of order m, in the MAGMA Database of Small Groups (see [BCP97]).

Table 1. Schur covers of the symmetric and alternating groups with 2-rank less than 5 and their Sylow 2-subgroups.

G	2-rank	$P \in \mathrm{Syl}_2(G)$	G	2-rank	$P \in \mathrm{Syl}_2(G)$
$2.\mathfrak{A}_4, 2.\mathfrak{A}_5$	1	Q_8	$2^-.\mathfrak{S}_4, 2^-.\mathfrak{S}_5$	1	Q_{16}
$2.\mathfrak{A}_6,\ 2.\mathfrak{A}_7,\ 6.\mathfrak{A}_6,\ 6.\mathfrak{A}_7$	1	Q_{16}	$2^+.\mathfrak{S}_4, 2^+.\mathfrak{S}_5$	2	SD_{16}
$3.\mathfrak{A}_6,\ 3.\mathfrak{A}_7$	2	D_8	$2.\mathfrak{S}_6,\ 2^{\pm}.\mathfrak{S}_7$	2	$R_{32,44}$
$2.\mathfrak{A}_n, 8 \leq n \leq 15$	4		$2^{\pm}.\mathfrak{S}_n, 8 \leq n \leq 15$	≥ 4	

4. Double covers in characteristic 2

Throughout this section $p = 2$ and k denotes an algebraically closed field of characteristic 2. We prove Theorem A.

4.1. The structure of $T(G)$ when the 2-rank is one.

From Table 1, the Schur covers of the alternating and symmetric groups have 2-rank one if and only if they have generalised quaternion Sylow 2-subgroups.

Proposition 4.1. *Let G be one of the groups $2.\mathfrak{A}_n$ for $4 \leq n \leq 7$, or $2^-.\mathfrak{S}_n$ for $n \in \{4, 5\}$. Then $T(G) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/4$.*

Proof. A Sylow 2-subgroup P of G is generalised quaternion of order 8 for $2.\mathfrak{A}_4$ and $2.\mathfrak{A}_5$, and of order 16 otherwise, see Table 1. Therefore in all cases $T(P) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/4$ by [CT06, Theorem 6.3]. Now $Z(G)$ is a nontrivial normal 2-subgroup of G and $X(G) \cong (G/[G,G])_{2'}$ is trivial in all cases. By [CMT13, Theorem 4.5] we conclude that $T(G) \cong T(P)$ as required. $\square$

Note that the $\mathbb{Z}/4$ summand is generated by the class of $\Omega_G(k)$, since a projective resolution of the trivial module for G is periodic of period 4. The construction of the generator of the $\mathbb{Z}/2$ summand is detailed in the proof of [CMT13, Theorem 4.5].

4.2. **The structure of $T(G)$ when the 2-rank is at least 2.**

As pointed out in Section 2, in order to determine the torsion subgroup $TT(G)$ of $T(G)$ it suffices to find the trivial source endotrivial modules, i.e. the group $K(G)$. Our objective is further simplified, because it is well known that Sylow 2-subgroups of symmetric and alternating groups are selfnormalising except in few small groups (cf. [Wei25, Corollary 2 and Theorem p. 124]). So the same holds in a double cover of an alternating or symmetric group.

Proposition 4.2. *Let G be a double cover of an alternating or symmetric group, whose 2-rank is at least 2. Then $TT(G) = \{[k]\}$, unless $G = 2^+.\mathfrak{S}_4$ or $2^+.\mathfrak{S}_5$ in which case $TT(G) \cong \mathbb{Z}/2$.*

Proof. Let G be a double cover of an alternating or symmetric group of degree at least 6. Then a Sylow 2-subgroup P of G is selfnormalising and $TT(P) = \{[k]\}$ by Lemma 2.6. So $X(N_G(P)) = \{[k]\}$ and a fortiori $TT(G) = \{[k]\}$.

Now suppose that $G = 2^+.\mathfrak{S}_4$ or $G = 2^+.\mathfrak{S}_5$, and let $P \in \mathrm{Syl}_2(G)$. Then $P \cong SD_{16}$ is semi-dihedral of order 16 and $P = N_G(P)$. Thus [CMT13, Prop. 6.5] and [CT04, Theorem 7.1] yield $T(G) \cong \mathrm{Res}^G_P(T(G)) \cong T(P) \cong \mathbb{Z}/2 \oplus \mathbb{Z}$. The claim follows. $\square$

We refer the reader to [CMT13, Theorem 6.4(2)] for a more precise description of the nontrivial selfdual endotrivial kG-module of a group G with semi-dihedral Sylow 2-subgroups.

We now handle the group $TF(G)$. From Theorem 2.3 and Theorem 2.4, we gather that $TF(G) = \langle [\Omega_G(k)] \rangle \cong \mathbb{Z}$ whenever G is a double cover of an alternating or symmetric group of degree at least 16, because then the 2-rank is greater than 4 (see Table 1). Thus we are left with the cases when the 2-rank of G is between 2 and 4 (see Table 1), and we want to find which of these, if any, have maximal Klein-four subgroups.

Proposition 4.3. *Let G be one of the groups $2.\mathfrak{A}_n$ or $2^{\pm}.\mathfrak{S}_n$ with $8 \leq n \leq 15$. Then G has no maximal Klein-four subgroups unless $G = 2^+.\mathfrak{S}_n$ with $n \in \{8, 9\}$, in which case there is a unique G-conjugacy class of maximal Klein-four subgroups.*

Proof. We first consider $G = 2.\mathfrak{A}_n$ for $8 \leq n \leq 15$ and use the presentation of G given in Section 3. It suffices to show that the centraliser of any involution contains at least three commuting involutions. We use the notation introduced in Section 3, and the explicit calculations in [Wil09, §2.7.2], briefly recalled in Section 3.

The involutions of G are the lifts of l-fold transpositions for $l \equiv 0 \pmod 4$, and any involution of G is G-conjugate to the lift $x = [1,2][3,4][5,6][7,8]$ of $(1,2)(3,4)(5,6)(7,8)$.

So $y = [1,3][2,4][5,7][6,8]$ centralises x and $xy = yx = [1,4][2,3][5,8][6,7]$ is an involution too. Therefore $C_G(x) \geq \langle x, y, z\rangle \cong C_2^3$ as required.

Let us now take $G = 2^{\pm}.\mathfrak{S}_n$ for $8 \leq n \leq 15$. Proceeding as for the alternating group, we calculate the 2-ranks of the centralisers of any involution of G. The involutions of G are those of $2.\mathfrak{A}_n$ together with the l-fold transpositions with

$$l \equiv \begin{cases} 1 \pmod 4 & \text{if } G = 2^{+}.\mathfrak{S}_n\,, \text{ or} \\ 3 \pmod 4 & \text{if } G = 2^{-}.\mathfrak{S}_n\,, \end{cases}$$

as noted in Section 3.

Building on the first part of the proof, it suffices to consider involutions lifting odd permutations and find their centralisers. Now $C_{2.\mathfrak{A}_n}(x)$ has index 2 in $C_G(x)$, which gives $C_G(x) = \langle x\rangle \times C_{2.\mathfrak{A}_n}(x)$. Because $z \in C_{2.\mathfrak{A}_n}(x)$, the 2-rank of $C_G(x)$ is at least 3 if and only if there exists a noncentral involution in $2.\mathfrak{A}_n$ which centralises x. If this holds, then G cannot have any maximal Klein-four subgroup.

First assume that $n \geq 10$ and $G = 2^{-}.\mathfrak{S}_n$. Any involution not in $2.\mathfrak{A}_n$ is conjugate to the lift of the 3-fold transposition $x = [1,2][3,4][5,6]$, or for $n \in \{14, 15\}$ possibly the 7-fold transposition $x' = [1,2][3,4]\cdots[13,14]$. In the first case, we take $y = [1,3][2,4][5,6][7,8]$, which gives $xy = yx = [1,4][2,3][7,8]$. Thus $C_G(x) \geq \langle x, y, z\rangle \cong C_2^3$ as required. For x', we take $y = [1,2][3,4][5,6][7,8]$, so that $x'y = yx' = [9,10][11,12][13,14]$ and we are done in this case.

Suppose $n \geq 10$ and $G = 2^{+}.\mathfrak{S}_n$. Any involution not in $2.\mathfrak{A}_n$ is conjugate to either $x = [1,2]$, or $x' = [1,2][3,4]\cdots[9,10]$. Now $xx' = x'x = [3,4][5,6][7,8][9,10]$ so that $C_G(x) \geq \langle x, x', z\rangle \cong C_2^3$ and we are done for these groups too.

Suppose now $G = 2^{-}.\mathfrak{S}_n$ with $n \in \{8, 9\}$. By the above, any involution of G not in $2.\mathfrak{A}_n$ is conjugate to $x = [1,2][3,4][5,6]$. Let $y = [1,2][3,5][4,6][7,8]$, so that $xy = yx = [3,6][4,5][7,8]$, and we get $C_G(x) \geq \langle x, y, z\rangle \cong C_2^3$ as required.

We are left with $2^{+}.\mathfrak{S}_n$ and $n \in \{8, 9\}$. We claim that G has a maximal Klein-four subgroup, unique up to conjugacy, namely $\langle x, z\rangle$ where $x = [1,2]$. As noted above, it is enough to prove that there is no noncentral involution in $2.\mathfrak{A}_n$ which centralises x. Any such involution must be the lift of a 4-fold transposition. The centraliser in $\mathfrak{S}_n$ of the transposition $(1,2)$ is the direct product $\langle(1,2)\rangle \times \mathfrak{S}_{n-2}$ where the latter symmetric group permutes the set $\{3, \ldots, n\}$ of cardinality 6 or 7, and so does not contain any 4-fold transposition whose lift in $2.\mathfrak{A}_n$ is an involution. Therefore $C_{2.\mathfrak{A}_n}(x)$ has Sylow 2-subgroup Q of rank 1. Indeed, by the results for $2.\mathfrak{A}_6$ and $2.\mathfrak{A}_7$ in Table 1, we deduce that Q is quaternion of order 16. $\square$

As a consequence of Proposition 4.3, $TF(G) \cong \mathbb{Z}$ in characteristic 2 for G a double cover of an alternating or symmetric group of degree at least 8, except for $2^{+}.\mathfrak{S}_8$ and $2^{+}.\mathfrak{S}_9$, in which case the torsion-free rank is equal to 2. Actually, we can be more thorough and give generators for $TF(G)$ when $G = 2^{+}.\mathfrak{S}_n$ with $n \in \{8, 9\}$.

Theorem 4.4. *Let G be a double cover of an alternating or symmetric group of 2-rank at least 2. That is, G is one of $2.\mathfrak{A}_n$ with $n \geq 8$, or $2^{\pm}.\mathfrak{S}_n$ with $n \geq 6$, or $2^{+}.\mathfrak{S}_n$ with*

$n \in \{4,5\}$. *Then:*

$$TF(G) \cong \begin{cases} \mathbb{Z}^2 & \text{if } G \text{ is one of } 2^+.\mathfrak{S}_8, 2^+.\mathfrak{S}_9, \\ \mathbb{Z} & \text{otherwise.} \end{cases}$$

Furthermore, if $G = 2^+.\mathfrak{S}_n$ with $n \in \{8,9\}$, let F be a maximal Klein-four subgroup and let E be any non-maximal Klein-four subgroup of G. Then

$$TF(G) = \langle [\Omega_G(k)], [M] \rangle \cong \mathbb{Z}^2 \,,$$

where

$$\mathrm{Res}^G_F(M) \cong \Omega^8_F(k) \oplus (\mathrm{proj}) \quad \textit{and} \quad \mathrm{Res}^G_E(M) \cong k \oplus (\mathrm{proj}) \;.$$

Proof. By Theorem 2.3, we need to find the conjugacy classes of maximal Klein-four subgroups of G. By [GLS98, Prop. 5.2.10], if G is one of $2.\mathfrak{A}_n$ or $2^\pm.\mathfrak{S}_n$ with $n \geq 16$, then G has 2-rank greater than 4, so that Theorem 2.4 shows that G cannot have maximal Klein-four subgroups. If $G = 2^+.\mathfrak{S}_4$ or $G = 2^+.\mathfrak{S}_5$, then a Sylow 2-subgroup of G is semi-dihedral (of order 16), and so has rank 2 and a unique conjugacy class of (maximal) Klein-four subgroups. If $G = 2.\mathfrak{S}_6$ or $G = 2^\pm.\mathfrak{S}_7$, then a Sylow 2-subgroup of G is of the form $P \cong R_{32,44}$ where $R_{32,44}$ has 2-rank 2 (see Table 1). A direct computation with MAGMA [BCP97] shows that P has two conjugacy classes of Klein-four subgroups which fuse in G. Finally, Proposition 4.3 and [GLS98, Prop. 5.2.10] show that G has rank 4 and no maximal Klein-four subgroup for $G = 2.\mathfrak{A}_n$, or $2^\pm.\mathfrak{S}_n$ with $8 \leq n \leq 16$, except $G = 2^+.\mathfrak{S}_n$ with $n \in \{8,9\}$. In this case, G has 2-rank 4 and one conjugacy class of maximal Klein-four subgroups.

Therefore Theorem 2.3 shows that $TF(G) = \langle [\Omega_G(k)] \rangle \cong \mathbb{Z}$ in all these groups except $G = 2^+.\mathfrak{S}_n$ with $n \in \{8,9\}$, in which case $TF(G) \cong \mathbb{Z}^2$.

To complete the theorem, we need to find one more generator for $TF(G)$ in these latter two cases, as we can pick $[\Omega_G(k)]$ by "default". This is an immediate application of Theorem 2.5. In our case the proof of Proposition 4.3 shows that a noncentral involution x of a maximal Klein-four subgroup F of G has centraliser whose Sylow 2-subgroup has the form $\langle x \rangle \times Q_{16}$. So the integer a in Theorem 2.5 is equal to 8, while the $\mathbb{Z}$-rank of $TF(G)$ is $n_G = 2$. So there is a subquotient M of the kG-module $\Omega^8_G(k)$ which is endotrivial and subject to the conditions:

$$\mathrm{Res}^G_F(M) \cong \Omega^8_F(k) \oplus (\mathrm{proj}) \quad \text{and} \quad \mathrm{Res}^G_E(M) \cong k \oplus (\mathrm{proj})$$

for any non-maximal Klein-four subgroup E of G. Then Theorem 2.5 says that $TF(G) = \langle [\Omega_G(k)], [M] \rangle \cong \mathbb{Z}^2$, as asserted. □

Proposition 4.1 together with Proposition 4.2 and Theorem 4.4 complete the proof of Theorem A, and the classification of endotrivial modules for the double covers of alternating and symmetric groups in characteristic 2.

5. Double covers in odd characteristic

Throughout this section we let p denote an odd prime and k an algebraically closed field of characteristic p. The objective of this section is to prove Theorem B. Building on [CMN09, CHM10], the question comes down to whether there are faithful endotrivial

kG-modules. We start by showing that the answer is negative in most cases, by adapting the proof of [LMS, Thm. 4.5] from simple modules to indecomposable modules in general.

Theorem 5.1. *Let p be an odd prime and let G be a double cover of an alternating or symmetric group. Then the inflation homomorphism $\mathrm{Inf}_{G/Z(G)}^{G}: T(G/Z(G)) \longrightarrow T(G)$ is an isomorphism in the following cases.*

(1) *$G = 2.\mathfrak{A}_n$ or $2^{\pm}.\mathfrak{S}_n$ with $n \geq p+4$; or*
(2) *$G = 2.\mathfrak{S}_6$ and $p = 3$.*

Proof. By Lemma 2.2(2), the inflation homomorphism $\mathrm{Inf}_{G/Z(G)}^{G}$ is injective, hence it suffices to show that no faithful block of kG contains an indecomposable endotrivial module. Assume that M is an indecomposable endotrivial kG-module belonging to a faithful block of kG. By Theorem 2.1, M is liftable to an endotrivial $\mathbb{C}G$-module. Let χ_M denote the corresponding complex character. Then χ_M is a sum of faithful characters in $\mathrm{Irr}(G)$. By Theorem 2.1(2), it suffices to find a p-singular element $g \in G$ such that $|\chi_M(g)| \neq 1$. We use the notation for characters and their parametrisation introduced in Section 3.

First assume G is one of $2.\mathfrak{A}_n$, or $2^{\pm}.\mathfrak{S}_n$ with $n \geq p+4$ and let $g \in G$ be a p-singular element whose projection to $G/Z(G)$ has cycle type $\mu = (p)(2)^2(1)^{n-p-4}$. Let $\lambda \in \mathcal{D}_n$. If λ is odd, then for both characters $\psi_\lambda^{\pm} \in \mathrm{Irr}(2^{\pm}.\mathfrak{S}_n)$, we have $\psi_\lambda^{\pm}(g) = 0$ by [HH92, Theorem 8.7(ii)]. A fortiori, their restrictions to $2.\mathfrak{A}_n$ must take value zero on g as well. Note that $\psi_\lambda^{+}|_{2.\mathfrak{A}_n} = \psi_\lambda^{-}|_{2.\mathfrak{A}_n} \in \mathrm{Irr}(2.\mathfrak{A}_n)$. If λ is even, then for $\psi_\lambda \in \mathrm{Irr}(2^{\pm}.\mathfrak{S}_n)$, again by [HH92, Theorem 8.7(iii) and (iv)], we obtain $\psi_\lambda(g) = 0$. Moreover, the two irreducible constituents $\psi_\lambda^{\pm}$ of the restriction of ψ_λ to $2.\mathfrak{A}_n$ are such that $\psi_\lambda^{+}(g) = \psi_\lambda^{-}(g)$, so that in fact $\psi_\lambda^{\pm}(g) = 0$. Consequently, any sum of faithful irreducible characters of G takes value zero on g and it follows that $\chi_M(g) = 0$. Whence M cannot be endotrivial by Theorem 2.1(2).

Suppose now that $G = 2.\mathfrak{S}_6$ and that $p = 3$. In this case consider an element $g' \in G$ which projects to an element of cycle type $\mu' = (3)(2)(1)$. Then again by [HH92, Theorem 8.7(i),(ii) and (iii)], we have $\psi_\lambda(g') = 0$ if $\lambda \in \mathcal{D}(6)$ is even. Otherwise, $\psi_\lambda^{\pm}(g') = 0$ if $\lambda = (6)$, and we calculate $\psi_{\mu'}^{\pm}(g') = \pm i^{(\frac{6-3+1}{2})}(\frac{3\cdot 2\cdot 1}{2})^{\frac{1}{2}} = \pm\sqrt{3}$. This forces $\chi_M(g') \in \sqrt{3}\mathbb{Z}$ and thus M cannot be endotrivial by Theorem 2.1(2). □

The argument above does not apply to the group $G = 2.\mathfrak{A}_6$ in characteristic 3. However in this case $T(G)$ is already known since G is in fact a finite group of Lie type in defining characteristic.

Lemma 5.2. *In characteristic $p = 3$, we have $T(2.\mathfrak{A}_6) \cong \mathbb{Z}/8 \oplus \mathbb{Z}$. Moreover the four faithful indecomposable torsion endotrivial $k2.\mathfrak{A}_6$-modules are uniserial modules of dimension 10 with two composition factors of dimension 2 and one composition factor of dimension 6.*

Proof. Let $G = 2.\mathfrak{A}_6$ and $p = 3$. Then $P \in \mathrm{Syl}_3(G)$ is elementary abelian of order 9 and it is well known ([CCN+85]) that $G \cong \mathrm{SL}_2(9)$. Thus [CMN06, Corollary 5.3] says that $T(G) \cong X(T) \oplus T(P) \cong \mathbb{Z}/8 \oplus \mathbb{Z}$ where $P \in \mathrm{Syl}_3(G)$ and T denotes the torus of $\mathrm{SL}_2(9)$ of diagonal matrices with determinant 1 (see also [Bon11, §2.1.1]). Hence we have $X(T) \cong \mathbb{F}_9^{\times} \cong \mathbb{Z}/8$. The assertion about the structure of the faithful endotrivial modules is proven either by a direct computation with MAGMA [BCP97], or by inducing the linear characters of $kN_G(P)$ and using the decomposition matrix of $2.\mathfrak{A}_6$ (cf. [Bre12]).

$\square$

Hence we are left with investigating endotrivial modules for the double covers of alternating and symmetric groups of degree $p \leq n \leq p+3$ and when a Sylow p-subgroup is cyclic.

Theorem 5.3. *Let p be an odd prime and let G be one of the double covers $2.\mathfrak{A}_n$ or $2^{\pm}.\mathfrak{S}_n$ with $p \leq n \leq min\{2p-1, p+3\}$. Let e be the inertial index of $B_0(kG)$. Then $T(G) \cong \mathbb{Z}/2e \oplus \mathbb{Z}/2$.*

Proof. Let $P \in \mathrm{Syl}_p(G)$. Note that $P \cong C_p$. For convenience, set $N := N_G(P)$, $X := X(N)$ and $Z := Z(G)$. By Theorem 2.7, we have $T(G) \cong T(N)$ and $T(N)$ is an extension of $\mathbb{Z}/2$ by X. Moreover, Theorem 2.7(3) shows that $T(G)$ has a subgroup $T_0(G) = \langle[\Omega_G(k)]\rangle \cong \mathbb{Z}/2e$, which consists of the indecomposable endotrivial modules in $B_0(kG)$.

First assume that $G = 2.\mathfrak{A}_n$. If $p > 3$, we know from [LMS, Proof of Prop. 4.6] that

$$e = \begin{cases} \frac{p-1}{2} & \text{if } n \in \{p, p+1\}, \text{ and} \\ p-1 & \text{if } n \in \{p+2, p+3\}, \end{cases}$$

and that in both cases $|X|/e = 2$. The same holds if $p = 3$ and $n \in \{4, 5\}$. So by Theorem 2.7(4) there are exactly two kN-blocks containing indecomposable endotrivial modules, namely the principal block $B_0(kN)$ and a faithful block B_1. Now it follows from Theorem 2.7(3) and [Bes91, Theorem 2.3] that any indecomposable endotrivial kN-module in B_1 can be written as $M \otimes_k \theta$, where $M \in B_0(kN)$ and $\theta \in X$ is a B_1-module. Consequently $T(N)$ has exponent $2e$ and we obtain $T(G) \cong \mathbb{Z}/2e \oplus \mathbb{Z}/2$.

Next assume $G = 2^{\pm}.S_n$. We need to determine whether G has faithful endotrivial modules. If $p > 3$ and $n \in \{p, p+1\}$, then we know from [LMS, Prop. 4.6(1),(2)] that $2.\mathfrak{A}_n$ has faithful simple endotrivial modules which are the restrictions of faithful simple endotrivial $k2^{\pm}.\mathfrak{S}_n$-modules, labelled by odd partitions. Thus as G is a double cover of $\mathfrak{S}_n$, we must have $|T(G)| = 2|T(\mathfrak{S}_n)|$. A direct computation shows that the same holds if $p = 3$ and $n = 4$. In addition, $T(\mathfrak{S}_n) \cong \mathbb{Z}/2e$ by [CMN09, Theorem A(b)] and the claim follows by the same argument as in the case of $2.\mathfrak{A}_n$ showing that the elements of $T(N)$ have order at most $2e$, since any indecomposable endotrivial kN-module V can be written as $V \cong M \otimes_k \theta$, where $M \in B_0(kN)$ and $\theta \in X$ belongs to the same block as V. This forces $T(G) \cong \mathbb{Z}/2e \oplus \mathbb{Z}/2$. (Notice that in this case $e = p-1$ as $N_{\mathfrak{S}_n}(P) \cong C_p \rtimes C_{p-1}$.)

If $n \in \{p+2, p+3\}$, we claim that there cannot be any faithful indecomposable endotrivial modules. First if $p = 3$ and $n = 5$, it is easily checked from GAP [Bre12] that there is no faithful block with inertial index e, so that the claim follows from Theorem 2.7(4). Suppose now that $p > 3$ and that there is some faithful block containing an endotrivial kG-module. Then by Theorems 2.7(4) and 2.1(1), we may assume that there is a faithful simple endotrivial kG-module S which lifts to a $\mathbb{C}G$-module with complex character $\psi \in \mathrm{Irr}(G)$ labelled by a partition $\lambda \in \mathcal{D}(n)$ (see Section 3). Then $\mathrm{Res}^G_{2.\mathfrak{A}_n}(S)$ is endotrivial by Lemma 2.2. If λ is odd, then ψ is of the form $\psi^{\pm}_{\lambda} \in \mathrm{Irr}(G)$ and $\mathrm{Res}^G_{2.\mathfrak{A}_n}(S)$ is simple endotrivial and affords $\psi^{+}_{\lambda}|_{2.\mathfrak{A}_n} = \psi^{-}_{\lambda}|_{2.\mathfrak{A}_n} \in \mathrm{Irr}(2.\mathfrak{A}_n)$. This contradicts [LMS, Prop. 4.6(3),(4)] since the faithful simple endotrivial $k2.\mathfrak{A}_n$-modules are parametrised by even partitions. If λ is even, then ψ is of the form ψ_λ and since $2.\mathfrak{A}_n \trianglelefteq G$ we must have $\mathrm{Res}^G_{2.\mathfrak{A}_n}(S) = S_1 \oplus S_2$, where S_1, S_2 are simple of the same dimension affording the two irreducible constituents $\psi_\lambda^\pm \in \mathrm{Irr}(2.\mathfrak{A}_n)$ of ψ_λ by [HH92, Theorem 8.6]. So S cannot be endotrivial because its restriction to $2.\mathfrak{A}_n$ does not split as $M \oplus$ (proj) for some indecomposable endotrivial $k2.\mathfrak{A}_n$-module M. Therefore $T(G) \cong \mathrm{Inf}^G_{G/Z}(T(G/Z)) \cong T(\mathfrak{S}_n) \cong \mathbb{Z}/2e \oplus \mathbb{Z}/2$. (Notice that in this case $e = p-1$ as $N_{\mathfrak{S}_n}(P) \cong (C_p \rtimes C_{p-1}) \times \mathfrak{S}_{n-p}$.) □

6. The exceptional covering groups

In this final section we handle the exceptional Schur covers of the alternating groups of degrees 6 and 7. The only characteristics that we need to discuss are $p \in \{2, 3, 5, 7\}$. We proceed along the same lines as for the double covers, separating the cases $p = 2$ and $p \geq 3$.

From Table 1 the Sylow 2-subgroups of $3.\mathfrak{A}_6$ and $3.\mathfrak{A}_7$ are dihedral of order 8 and those of $6.\mathfrak{A}_6$ and $6.\mathfrak{A}_7$ are generalised quaternion of order 16. (Note that the 2-local structure is the same in degree 6 and in degree 7.)

Proposition 6.1. *Let k be an algebraically closed field of characteristic* 2.

(1) *Let $G = 3.\mathfrak{A}_6$. Then $T(G) \cong \mathbb{Z}/3 \oplus \mathbb{Z}^2$.*
(2) *Let $G = 3.\mathfrak{A}_7$. Then $T(G) \cong \mathbb{Z}^2$.*
(3) *Let $G = 6.\mathfrak{A}_n$ with $n \in \{6, 7\}$. Then $T(G) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/4$.*

Proof. First assume that $G = 6.\mathfrak{A}_n$ with $n \in \{6, 7\}$. Then the arguments of the proof of Proposition 4.1 apply to G and give the required result.

Suppose now that $G = 3.\mathfrak{A}_n$ with $n \in \{6, 7\}$. Let $P \in \mathrm{Syl}_2(G)$ and put $N := N_G(P)$. Then $N \cong C_3 \times D_8$, and by Lemma 2.2 the restriction induces an injective group homomorphism $TT(G) \hookrightarrow X(N) \cong \mathbb{Z}/3$. Thus, by Lemma 2.6, it suffices to determine whether the kG-Green correspondents of the nontrivial modules in $X(N)$ are endotrivial. A direct MAGMA computation shows the following. For $G = 3.\mathfrak{A}_6$, the kG-Green correspondents of the two nontrivial one-dimensional kN-modules are two faithful simple endotrivial $k3.\mathfrak{A}_6$-modules of dimension 9 (note that it was proven in [LMS, Theorem 4.9(5)] that these modules are endotrivial, but not that they are torsion elements in $T(G)$). Therefore, we have $TT(G) \cong \mathbb{Z}/3$. For $G = 3.\mathfrak{A}_7$, the kG-Green correspondents of the two nontrivial one-dimensional kN-modules are indecomposable modules of dimension 15, which cannot be endotrivial because a trivial source endotrivial kG-module must have dimension congruent to 1 modulo 8, whence $TT(G) \cong \{[k]\}$.

Finally, for $TF(3.\mathfrak{A}_n)$ with $n \in \{6, 7\}$, we recall that generators of $TF(G)$ are known when G has a dihedral Sylow 2-subgroup. Namely they can be chosen to be $[\Omega_G(k)]$ and the class of one of the two non-isomorphic indecomposable direct summands of the heart of the projective cover of the trivial kG-module (see [AC86, §6]). □

Let us now turn to the endotrivial modules in odd characteristic.

Proposition 6.2. *Let k be an algebraically closed field of odd characteristic p. Let G be one of the groups $3.\mathfrak{A}_6$, $6.\mathfrak{A}_6$, $3.\mathfrak{A}_7$ or $6.\mathfrak{A}_7$.*

(1) *If* $p = 3$*, then* $T(G) = \langle [\Omega_G(k)], [M] \rangle \cong \mathbb{Z}^2$*, where* M *is a* kG*-module such that*

$$\operatorname{Res}^G_F(M) \cong \Omega^6_F(k) \oplus (\text{proj}) \quad \text{and} \quad \operatorname{Res}^G_E(M) \cong k \oplus (\text{proj})$$

for some maximal elementary abelian 3*-subgroup* F *of* G *of rank* 2 *and for any elementary abelian* 3*-subgroup* E *of rank* 2 *of* G *not conjugate to* F*.*

(2) *If* $p \in \{5, 7\}$*, then* $T(G)$ *is as given in Table 2.*

Table 2. Cyclic Sylow cases for $6.\mathfrak{A}_6$ and $6.\mathfrak{A}_7$

G	p	$X(N)$	e	$T(G)$
$3.\mathfrak{A}_6$	5	$\mathbb{Z}/3 \oplus \mathbb{Z}/2$	2	$\mathbb{Z}/3 \oplus \mathbb{Z}/4$
$6.\mathfrak{A}_6$	5	$\mathbb{Z}/3 \oplus \mathbb{Z}/4$	2	$\mathbb{Z}/3 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/4$
$3.\mathfrak{A}_7$	5	$\mathbb{Z}/3 \oplus \mathbb{Z}/4$	4	$\mathbb{Z}/3 \oplus \mathbb{Z}/8$
$6.\mathfrak{A}_7$	5	$\mathbb{Z}/3 \oplus \mathbb{Z}/8$	4	$\mathbb{Z}/3 \oplus \mathbb{Z}/8 \oplus \mathbb{Z}/2$
$3.\mathfrak{A}_7$	7	$\mathbb{Z}/3 \oplus \mathbb{Z}/3$	3	$\mathbb{Z}/3 \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/2$
$6.\mathfrak{A}_7$	7	$\mathbb{Z}/3 \oplus \mathbb{Z}/6$	3	$\mathbb{Z}/3 \oplus \mathbb{Z}/6 \oplus \mathbb{Z}/2$

Proof. Assume that $p = 3$. Then, $TT(G)$ is trivial since G is a perfect group with a nontrivial normal 3-subgroup (see Lemma 2.6).

Now, a Sylow 3-subgroup P of G is extraspecial of order 27 and exponent 3. We calculate with MAGMA ([BCP97]) that P has two G-conjugacy classes of elementary abelian subgroups of rank 2 and that a representative of each of them is selfcentralising. Therefore Theorem 2.3 yields $TF(G) \cong \mathbb{Z}^2$. More precisely, let F be an elementary abelian subgroup of P of rank 2. The integer a in Theorem 2.5 is equal to 6 and there is a subquotient M of the kG-module $\Omega^6_G(k)$ which is endotrivial and subject to the following conditions:

$$\operatorname{Res}^G_F(M) \cong \Omega^6_F(k) \oplus (\text{proj}) \quad \text{and} \quad \operatorname{Res}^G_E(M) \cong k \oplus (\text{proj})$$

for any elementary abelian subgroup E of G of rank 2 not conjugate to F. Then Theorem 2.5 says that $TF(G) = \langle [\Omega_G(k)], [M] \rangle \cong \mathbb{Z}^2$, as asserted.

Next assume that $p \in \{5, 7\}$. Then a Sylow p-subgroup P of G is cyclic of order p. Let e denote the inertial index of the principal block $B_0(kG)$ (recall that $e = |N_G(P) : C_G(P)|$). By Theorem 2.7, $T(G) \cong T(N)$, where $N = N_G(P)$, and $T(N)$ is an extension of $X(N)$ by $T(P) \cong \mathbb{Z}/2$, which splits when e is odd. Now, both $X(N)$ and e can be calculated directly for both characteristics. This is enough to work out the structure of $T(G)$ when $p = 7$ since e is odd in this case. In addition for $p = 5$ and $G \in \{3.\mathfrak{A}_6, 3.\mathfrak{A}_7\}$ we use the fact that the module $\Omega_G(k)$ generates a cyclic subgroup of order $2e$ of $T(G)$, see Theorem 2.7(3).

Finally for $p = 5$ and $G \in \{6.\mathfrak{A}_6, 6.\mathfrak{A}_7\}$, by Theorem 2.7(4) the number of blocks of kN containing indecomposable endotrivial modules is $|X(N)|/e$, each of which contains precisely $2e$ of them. Furthermore, as in the proof of Theorem 5.3, any indecomposable endotrivial module in a non-principal block of kN can be obtained by tensoring an indecomposable endotrivial module in the principal block $B_0(kN)$ with a one-dimensional module in $X(N)$, which allows us to deduce the exponent of $T(N)$. The results of these computations are detailed in Table 2 and the claims follow. □

References

[AC86] M. Auslander and J. F. Carlson. Almost-split sequences and group rings. *J. Algebra*, 103(1):122–140, 1986.

[BCP97] W. Bosma, J. Cannon, and C. Playoust. The Magma algebra system. I. The user language. *J. Symbolic Comput.*, 24(3-4):235–265, 1997.

[Ben98] D. J. Benson. *Representations and cohomology. I*, volume 30 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 1998.

[Bes91] C. Bessenrodt. Endotrivial modules and the Auslander-Reiten quiver. In *Representation theory of finite groups and finite-dimensional algebras (Bielefeld, 1991)*, volume 95, pages 317–326. Birkhäuser, Basel, 1991.

[Bon11] C. Bonnafé. *Representations of* $\mathrm{SL}_2(\mathbb{F}_q)$, volume 13 of *Algebra and Applications*. Springer-Verlag London, Ltd., London, 2011.

[Bre12] T. Breuer. The GAP Character Table Library, Version 1.2.1. http://www.math.rwth-aachen.de/~Thomas.Breuer/ctbllib, May 2012.

[Car07] J. F. Carlson. Maximal elementary abelian subgroups of rank 2. *J. Group Theory*, 10(1):5–13, 2007.

[CCN+85] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, and R. A. Wilson. *Atlas of finite groups*. Oxford University Press, Eynsham, 1985.

[CHM10] J. F. Carlson, D. J. Hemmer, and N. Mazza. The group of endotrivial modules for the symmetric and alternating groups. *Proc. Edinb. Math. Soc. (2)*, 53(1):83–95, 2010.

[CMN06] J. F. Carlson, N. Mazza, and D. K. Nakano. Endotrivial modules for finite groups of Lie type. *J. Reine Angew. Math.*, 595:93–119, 2006.

[CMN09] J. F. Carlson, N. Mazza, and D. K. Nakano. Endotrivial modules for the symmetric and alternating groups. *Proc. Edinb. Math. Soc. (2)*, 52(1):45–66, 2009.

[CMN14] J. F. Carlson, N. Mazza, and D. K. Nakano. Endotrivial modules for the general linear group in a nondefining characteristic. *Math. Z.*, 278(3-4):901–925, 2014.

[CMT11] J. F. Carlson, N. Mazza, and J. Thévenaz. Endotrivial modules for p-solvable groups. *Trans. Amer. Math. Soc.*, 363(9):4979–4996, 2011.

[CMT13] J. F. Carlson, N. Mazza, and J. Thévenaz. Endotrivial modules over groups with quaternion or semi-dihedral Sylow 2-subgroup. *J. Eur. Math. Soc. (JEMS)*, 15(1):157–177, 2013.

[CMT14] J. F. Carlson, N. Mazza, and J. Thévenaz. Torsion-free endotrivial modules. *J. Algebra*, 398:413–433, 2014.

[CT04] J. F. Carlson and J. Thévenaz. The classification of endo-trivial modules. *Invent. Math.*, 158(2):389–411, 2004.

[CT06] J. F. Carlson and J. Thévenaz. Torsion endo-trivial modules. *Invent. Math.*, 3(4):303–335, 2006.

[GLS98] D. Gorenstein, R. Lyons, and R. Solomon. *The classification of the finite simple groups. Number 3. Part I. Chapter A*, volume 40 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1998.

[GM10] G. Glauberman and N. Mazza. p-groups with maximal elementary abelian subgroups of rank 2. *J. Algebra*, 323(6):1729–1737, 2010.

[HH92] P. N. Hoffman and J. F. Humphreys. *Projective representations of the symmetric groups*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 1992. Q-functions and shifted tableaux, Oxford Science Publications.

[LM] C. Lassueur and N. Mazza. Endotrivial modules for the sporadic groups and their covers. *J. Pure Appl. Algebra*. To appear, DOI:10.1016/j.jpaa.2012.04.003.

[LMS] C. Lassueur, G. Malle, and E. Schulte. Simple endotrivial modules for quasi-simple groups. *J. Reine Angew. Math.* To appear, DOI: 10.1515/crelle-2013-0100.

[Mac70] A. R. MacWilliams. On 2-groups with no normal abelian subgroups of rank 3, and their occurrence as Sylow 2-subgroups of finite simple groups. *Trans. Amer. Math. Soc.*, 150:345–408, 1970.

[Maz08] N. Mazza. Connected components of the category of elementary abelian p-subgroups. *J. Algebra*, 320(12):4242–4248, 2008.

[MT07] N. Mazza and J. Thévenaz. Endotrivial modules in the cyclic case. *Arch. Math. (Basel)*, 89(6):497–503, 2007.

[Wei25] L. Weisner. On the Sylow subgroups of the symmetric and alternating groups. *Amer. J. Math.*, 47(2):121–124, 1925.

[Wil09] R. A. Wilson. *The finite simple groups*, volume 251 of *Graduate Texts in Mathematics*. Springer-Verlag London, Ltd., London, 2009.

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