On Spacetime Transformation Optics: Temporal and Spatial Dispersion

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The electromagnetic implementation of cloaking, the hiding of *objects* from sight by diverting and reassembling illuminating electromagnetic fields has now been with us ten years, while the notion of hiding *events* is now five. Both schemes as initially presented neglected the inevitable dispersion that arises when a designed medium replaces vacuum under transformation. Here we define a transformation design protocol that incorporates both spacetime transformations and dispersive material responses in a natural and rigorous way. We show how this methodology is applied to an event cloak designed to appear as a homogeneous and isotropic but *dispersive* medium. The consequences for spacetime transformation design in dispersive materials are discussed, and some parameter and bandwidth constraints identified.

I. INTRODUCTION

Transformation Design – the use of the mathematical transformation of reference materials into those interesting 'device' properties is an area of active research interest – all the way from the most abstract theory and conceptualising [1–5], through to concrete theoretical proposals [6–9] and technological implementations [10–13].

In order to achieve the graduated and controllable modulation of material properties that are a necessary part of any transformation device, we need to understand the underlying behaviour which generates them. From a fundamental (microscopic) perspective, all non-vacuum material properties are dynamic in nature, resulting from the reaction of atoms, molecules, or more complex structures (metamaterials) to the impinging electromagnetic field, and thus changing how that field propagates. It is then an effective - and most likely homogenised [14, 15] – version of this dynamic process which we can often simplify into macroscopic permittivity and permeability functions, or perhaps even just a refractive index [16]. The sole remaining symptom of the original dynamics is then the frequency dependence of these constitutive quantities.

Ordinary spatial-only cloaking relies on a material response to achieve the device properties necessary for their operation, most notably one can consider the split-ring resonators used in the original proposal [17]. Here, the intrinsically dynamic nature of material responses are typically not too much of a problem – we can specify an operating frequency and bandwidth, and hope that our expertise at metamaterial construction allows us to achieve the necessary material properties [18–20].

Spacetime or 'event' cloaks [21], or any other spacetime transformation devices [1, 2, 22], are more subtle constructions, with an intrinsic and carefully calibrated space and time dependence. Event cloaks have been implemented in nonlinear optics using dispersion as an intrinsic feature of their operation [23], in order to engineer an effective controllable speed profile. However, the full details of spacetime cloaking were not addressed in either the original paper, or in the recent experiments. A spacetime transformation not only affects the required material parameters, but also changes the underlying dynamics of the material response. As we show in this paper, even the introduction of a simple one-pole resonance dispersion model in the design medium results in a number of unexpected features, such as the resultant device's medium characteristics being determined by third derivatives of the spacetime coordinates, the generation of spatial dispersion from purely temporal dispersion, and the induction of magneto-electricity, beyond what is known to occur in 'dispersion-free' spacetime cloaking.

This means that the transformation design process requires us to either adjust our material design to compensate for these extra complications, or engineer that extra complication so as to match our design specification. In practise this will probably reduce to an additional and rather pragmatic trade-off of the sort we already make when attempting to build an ordinary spatial-only transformation device – what degree of approximation can we tolerate when attempting to match our desired performance range? The results in this paper aim to show not only the true transformation rules needed for dispersive media, but also to inform us of how those affect the implementation trade-offs that will be needed.

Since electrodynamics is fundamentally a four-dimensional theory, all transformation optics devices should be seen as spacetime ones [24], where purely spatial phase-preserving transformation devices are obtained by restricting the time transformation to the identity. This identity transformation, when used for a cloak designed to look like vacuum, has the effect of forcing the ray trajectories to become spacelike through some regions. The common method for avoiding this, which is to assume the cloak is constructed inside a background medium with refractive index sufficiently greater than one, is the same as assuming a non-identity time transformation. In any case, for any *single* frequency of operation, such faster-than-light propagation is not strictly prohibited by relativity, since the indistinguishability of phase fronts implies that no signal can be transmitted. Nevertheless, the Kramers-Kronig relations concomitantly ensure that any phase-preserving cloak must be inherently and unavoidably dispersive. This unavoidable dispersion has been shown to betray the presence of moving cloaks [25] and could have other consequences. A fuller understanding of dispersion in transformation optics is therefore desirable even from the perspective of purely spatial applications, as well as from the potential for dispersion engineering [26].

This paper is organised as follows. First, in section II we describe a dispersive spacetime cloak in general terms and introduce our mathematical 'morphism' terminology. Section III then describes the handling of the one-pole or Lorentzian resonance which will make up our design medium – that which we want observers to infer exists. We show how this oscillator, whose most direct description is its temporal differential equation, can be represented in either the frequency domain or in terms of an integral kernel. Next, section IV shows how the transformation design process is implemented in this most general dispersive and spacetime case, either as operators, or again as an integral kernel. Section V then applies the general machinery to our chosen case, and calculates what kind of temporally and spatially varying oscillator is needed for a spacetime cloak to perfectly mimic our desired 'one-pole' appearance. We then use these results to estimate some bandwidth/parameter constraints that could be applied to our cloak design if we only had temporal oscillators to build with. Finally, in section VI we present our conclusions.

II. BACKGROUND

In this work we envisage a simple event cloak scheme, but in contrast to the original formulation based on a homogeneous, isotropic, and dispersion-free background, we want our *device* to hide an event inside a dispersive medium. That is, we are going to *design* an event cloak so that the device itself, despite its many complications, seems to an observer to be acting like a simple homogeneous and isotropic material that follows the standard Lorentz model. Naturally, since this is a linear system, the method could be straightforwardly generalised to encompass a sum

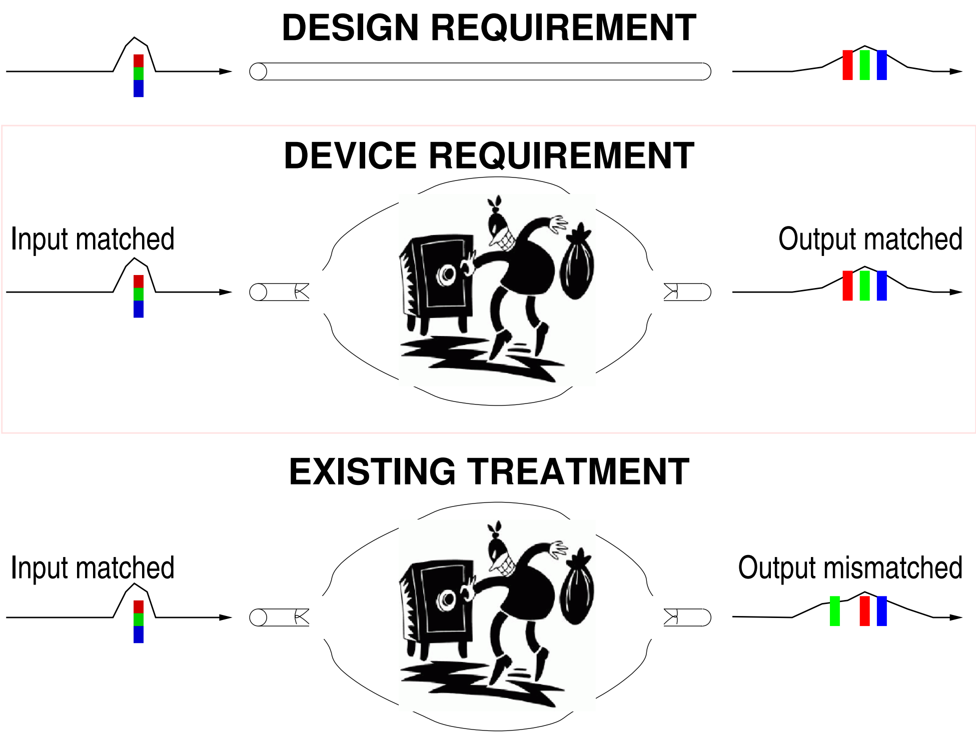

FIG. 1: The requirements of spacetime transformation schemes: whilst the visible behaviour of the device should only be to (top) alter any incident illumination by the expected dispersion properties, the actual transformation device (middle) also must hide a chosen event from any observer. Existing treatments, which ignore the effects of spacetime transformations on the dispersion properties, will not perfectly match the design requirement – even if all the beam steering and scattering suppression is still implemented correctly. The implication is that the observer will suspect that tampering has occurred – depicted here by the output illumination pulse having the wrong frequency chirp – even though the event itself is still perfectly hidden. Figure used with permission from [27].

of Lorentz oscillators as well [28]; and it is worth noting that with careful parameter choice, the Drude model for material response can be encoded within the Lorentz model. We also show how to define material responses as differential equations for polarization 2-forms, and this methodology is general enough to also handle many other (i.e. non-Lorentzian) response models. We also provide an integral kernel approach that is even more general.

Now consider how our cloak needs to work in practise. An optical pulse which started with a fixed phase, but then travelled through an ordinary dispersive medium, will typically emerge with some chirp, simply because its different frequency components experience different phase velocities, as well as generating a group velocity for the pulse as a whole. This situation is depicted at the top of fig. 1, and is how our spacetime cloaking device is designed to appear to an observer. We however, want to hide an event inside a different 'device' medium, whose spatial and temporal properties not only hide our chosen event, but *also* mimic an ordinary dispersive medium of our choice, as seen at the middle of fig. 1. If we do not properly consider *all* aspects of how the medium might need to be adapted to the true spacetime nature of our cloaking transformation, an observer may see evidence of tampering despite the cloaked event remaining hidden, as shown at the bottom of fig. 1.

To be clear, this chirp-induced betrayal of the cloak's presence is an entirely different effect than that of wave scattering by a reduced parameter cloak [29], and can be present even when ray trajectories are perfectly preserved by the cloak. Our interest in this work is not on the effectiveness or scattering reduction achieved by some implementation of a cloaking device. Although an important point, and certainly so in the more mature area of spatial cloaking, its applicability is more relevant to specific implementations than to the fundamentals we address here.

The goal is therefore to construct constitutive relations of a medium such that (a) there is a space time cloak, and (b) that the observer sees frequency dispersion corresponding to a one pole Lorentz resonance. The challenge with incorporating dispersion is that the new constitutive relations are now functions of both frequency ω and time t. However, since ω and t are conjugate Fourier variables, we need somehow to give meaning to constitutive media properties – the permittivity and permeability – which will depend on both. Further, since we perform transformations in space and time, the new constitutive relations will mix both time t and space x with frequency ω and wavevector k. Thus the new constitutive relations are both inhomogeneous in space *and* spatially dispersive [1] [30–32].

In this work we will consider the case of general spacetime transformation design incorporating dispersive effects, for which we will typically use the idea of an event cloak as a proxy. We will use two approaches to describing the constitutive properties: a differential operator approach which is particularly useful for the one-pole resonance that is the main focus of this article; and an integral kernel method valid for more general linear media. Our mathematical underpinning of the physics

[1] As an interesting aside, in this there is an analogy to quantum mechanics, in that when writing down expressions in t and ω they do not commute and we find that the commutator $[t,\omega] = -i$. This implies that a solution to Maxwell's equations cannot be single mode $e^{i\omega t}$. This may give some insight into the nature of quantum mechanics by analogy with electromagnetic modes in dispersive inhomogeneous media.

is now as follows.

The device itself will consist of some complicated arrangement of material properties, but it is designed to appear as if it were simple. The mathematical description of this (simple) design space takes place on a 'design manifold' $\mathscr{M}$, and the description of the device takes place on a 'device manifold' $\tilde{\mathscr{M}}$. As depicted in fig. 2, linking the two is a transformation or morphism φ which expresses how spacetime points on $\tilde{\mathscr{M}}$ (i.e. inside the device) need to be located on $\mathscr{M}$ – so that fields travelling through the device emerge in time and space as if having travelled through our designed $\mathscr{M}$. For example, in the well-known spatial and dispersionless cloak case, we have that origin-avoiding trajectories in $\tilde{\mathscr{M}}$ become straight lines in $\mathscr{M}$; for a dispersionless event cloak the curtain-map ([21], Fig. 6) used converts between a space with a diamond-shaped cloaking region and one with the diamond closed up.

In this paper we will use the coordinate free notation of exterior differential forms[2] [33, 34] – although we could, for example, always use an indexed notation, this would complicate the equations unnecessarily. When we map the manifolds onto charts with coordinate systems (see fig. 2), we are using an 'active transformation', not a *passive* 'coordinate transformation'. It is important to appreciate that the physics of Transformation Optics is independent of this coordinate representation, a point discussed further in [4]. One other crucial point is that on both manifolds, the underlying spacetime metric is taken to be Minkowski with Lorentzian signature $(-,+,+,+)$. Much of the work here applies for general constitutive relations on curved spacetimes, however we do exploit the fact that the there is a timelike killing vector given by ∂_t. Further, although the notion of an effective 'optical metric' [35] can indeed be useful, we do not need or use it

[2] We have used the standard tools of coordinate free differential geometry, i.e. the wedge product, exterior derivative, internal contraction, Lie derivative, Hodge dual and pullback. These are defined as follows:
The exterior derivative d increases the degree of a form by 1. For 0–forms also known as scalar fields, $d\phi = \frac{\partial\phi}{\partial x^a}\, dx^a$ and on the wedge product $\alpha\wedge\beta$ of a p–form α, and a form of arbitrary degree β, via $d\,(\alpha\wedge\beta) = d\alpha\wedge\beta + (-1)^p\alpha\wedge d\beta$.
The internal contraction is defined for the vector field V, i_V acts on a 1–form α as $i_V\alpha = V^\alpha\alpha_\alpha$, and on the wedge product $\alpha\wedge\beta$ of a p–form α, and a form of arbitrary degree β, via $i_V\,(\alpha\wedge\beta) = i_V\alpha\wedge\beta + (-1)^p\alpha\wedge i_V\beta$. The internal contraction operator i_V therefore reduces the degree of a form by 1.
The Lie derivative maps p–forms to p–forms via Cartan's identity $L_V = d\; i_V + i_V\; d$.
The Hodge dual $\star$ takes p forms to $4-p$ forms, can be succinctly and uniquely defined by the requirement that it is tensorial, $\star(\alpha\wedge dx^a) = g^{ab} i_{\partial/\partial x^a}\star\alpha$ and that the 4-volume form $\star 1$ has the correct orientation with $\star\star 1 = -1$.
Given a map $\varphi : \tilde{\mathscr{M}} \to \mathscr{M}$, the pullback φ^* maps p–forms on $\mathscr{M}$ to p–forms on $\tilde{\mathscr{M}}$ and satisfies: For 0–forms $\varphi^*\phi = \phi\circ\varphi$ and on arbitrary forms α, β, $\varphi^*(\alpha\wedge\beta) = \varphi^*(\alpha)\wedge\varphi^*(\beta)$ and $\varphi^*(d\alpha) = d(\varphi^*\alpha)$.
All these operations distribute across addition.

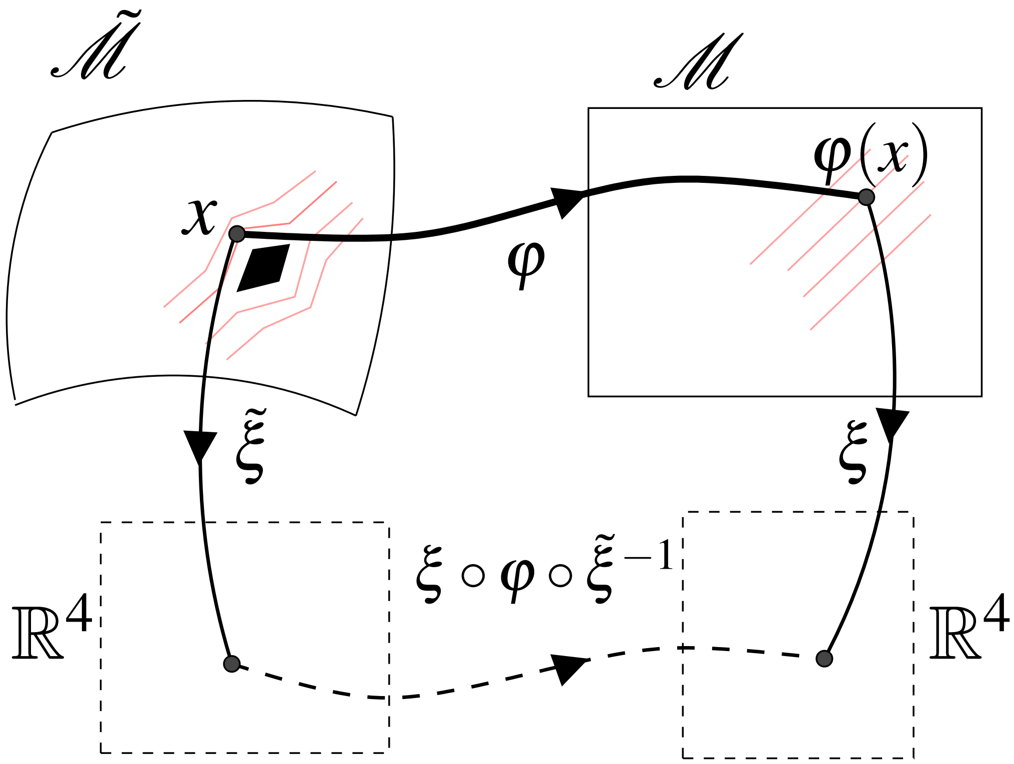

FIG. 2: Diffeomorphism: points x in the device manifold $\tilde{\mathscr{M}}$ are mapped to points $\varphi(x)$ in the design manifold $\mathscr{M}$ by the morphism (mapping) φ. Even though throughout this work we use, where possible, a coordinate independent notation, coordinates can be constructed. For example, with $\tilde{\xi}$ mapping manifold $\tilde{\mathscr{M}}$ to $\mathbb{R}^4$ and ξ mapping $\mathscr{M}$ likewise. Nevertheless, at the implementation stage, specific coordinate systems are invaluable, since φ can be represented by $\xi \circ \varphi \circ \tilde{\xi}^{-1}$ in $\mathbb{R}^4$ which then gives us the 'blueprint' for our device.

here. However an alternative interpretation of transformation optics is to consider two metrics on the device manifold $\tilde{\mathscr{M}}$: the Minkowski $\tilde{g}$ and the optical g_{opt}. Since there is no longer a single preferred metric, knowing which aspects of electromagnetism are independent of the metric is useful, as in the premetric formulation [34].

Maxwell's equations in our chosen notation are

$$dF = 0 \qquad \text{and} \qquad d\star\left(\varepsilon_0 F + \Pi\right) = c^{-2} \star J\,, \tag{1}$$

where $\star$ denotes the Hodge dual, c is the speed of light in vacuum, J is the current density, and ε_0 is the vacuum permittivity[3]. We can also split the electromagnetic 2–form F so that its electric

[3] Note that dimensionally $[dx^a] = \mathrm{m}$ and $[\partial_a] = \mathrm{m}^{-1}$. The metric, which converts vectors to 1–forms, has units

field E and magnetic field B 1–form sub-components are visible, and show how the dielectric polarization P and the magnetisation M appears inside the polarization 2–form Π, i.e.

$$F = dt \wedge E + c\star(dt \wedge B) \qquad \text{and} \qquad \Pi = dt \wedge P - \star(dt \wedge M). \tag{2}$$

where

$$E = i_t F, \quad B = -c^{-1} i_t \star F, \quad P = i_t \Pi \quad \text{and} \quad M = i_t \star \Pi\,. \tag{3}$$

Here $i_t = i_{\frac{\partial}{\partial t}}$ is the internal contraction operator taking the 2–form $dt \wedge E$ to the 1–form E, for example. We use the symbol $\partial_t = \frac{\partial}{\partial t}$ etc. for both the partial derivative and the corresponding vector field.

Since we will have to deal with the dynamics of the material response explicitly, we do not use a susceptibility tensor χ as might normally be expected. Instead we represent the material using constitutive operators Ψ and Φ which take 2–form fields to 2–form fields, so that

$$\Psi\,\Pi = \Phi\,F, \tag{4}$$

or, in components

$$\Psi_{ab}^{cd}\,\Pi_{cd} = \Phi_{ab}^{cd}\,F_{cd}\ , \tag{5}$$

where Ψ_{ab}^{cd} and Φ_{ab}^{cd} are antisymmetric in cd and ab. The reasons for using (4) will become clear later, but for now we note that the structure of this matches that of the differential equation for P as given below in (6). The summation convention is used throughout, and spacetime indices $a,b,\ldots = 0,1,2,3$, space indices $\mu,\nu,\ldots = 1,2,3$. Finally, throughout this paper we use the non-unitary, angular frequency definition of Fourier transform.

III. DESIGN GOAL: THE ONE-POLE RESONANCE

Our goal requires that the constitutive properties of the design medium appear to be a stationary, homogeneous, isotropic material with the behaviour of a single-pole Lorentz oscillator. In the

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of m^2. The Hodge dual has units which depend on the degree: $[\star\alpha] = \mathrm{m}^{4-2\deg(\alpha)}[\alpha]$. Integration and exterior differentiation have no effect on units. The components of the electric field have dimensions $[E^a] = \mathrm{Vm}^{-1}$. Thus the vector $E_{\text{vec}} = E^a\partial_a$ has dimensions $[E_{\text{vec}}] = \mathrm{Vm}^{-2}$, and the dimensions of the 1–form E are $[E] = \mathrm{V}$ (and $[\int E] = \mathrm{V}$). Likewise, since $[E] = [cB]$, then $[B] = \mathrm{Vsm}^{-1}$. Also, $[F] = [dt \wedge E] = [c\star(dt \wedge B)] = [c\,(dt \wedge B)] = \mathrm{sV}$. Thus $[\varepsilon_0 F] = (\mathrm{CV}^{-1}\mathrm{m}^{-1})(\mathrm{sV}) = \mathrm{Cm}^{-1}\mathrm{s}$. This gives $[\star\varepsilon_0 F] = \mathrm{Cm}^{-1}\mathrm{s}$ and hence $[\varepsilon_0 c^2 d\star F] = \mathrm{Cms}^{-1}$. The components of the current density J^a have dimensions $[J^a] = \mathrm{Cs}^{-1}\mathrm{m}^{-2}$. Thus the vector current is $[J_{\text{vec}}] = \mathrm{Cs}^{-1}\mathrm{m}^{-3}$, then 1–form $[J] = \mathrm{Cs}^{-1}\mathrm{m}^{-1}$ and the 3–form $[\star J] = \mathrm{Cs}^{-1}\mathrm{m}$. Thus (1) is dimensionally correct.

simple case of a global Lorentz coordinate system, we could simply write a temporal differential equation (see e.g. [36]) using partial derivatives acting on the relevant components; however our aim demands that we use a more general spacetime form with Lie derivatives, i.e.

$$\left(L_t^2 + \gamma L_t + \omega_{\mathrm{R}}^2\right) P = \omega_{\mathrm{P}}^2 \, E \ , \tag{6}$$

where L_t denotes the Lie derivative $L_t = L_{\partial_t}$ and ω_{R} is the resonance frequency, γ is the damping, ω_{P} is the coupling strength. We choose, as a matter of model construction, Lie transport, and therefore the Lie derivative, as opposed to parallel transport and its corresponding covariant derivative. For static media the two models are indistinguishable. However, the spacetime transformation optics for a covariant derivative formulation will be distinct.

In the simple case of a global Lorentz coordinate system, the Lie derivatives can be replaced by partial derivatives acting on the relevant components. In such a situation, the frequency domain behaviour of the Lorentz oscillator has the form

$$\varepsilon(\omega) = \varepsilon_0 \left(1 + \frac{\omega_{\mathrm{P}}^2}{-\omega^2 + i\gamma\omega + \omega_{\mathrm{R}}^2} \right) \qquad \text{and} \qquad \mu = \mu_0 \ , \tag{7}$$

where μ_0 is the vacuum permeability. The electric and displacement fields E, D are linked by constitutive relations depending on the material polarization P which can be extracted from either (6) or (7) above. Assuming that the magnetic response is that of the vacuum, i.e. $M = 0$, we have

$$H = \mu_0^{-1} B \qquad \text{and} \qquad D = \varepsilon_0 E + P \ . \tag{8}$$

One point of note is that the operator $\left(L_t^2 + \gamma L_t + \omega_{\mathrm{R}}^2\right)$ is not one-to-one. This is because $\left(L_t^2 + \gamma L_t + \omega_{\mathrm{R}}^2\right) e^{i\sigma t} = 0$ where σ is a root of $-\omega^2 + i\gamma\omega + \omega_{\mathrm{R}}^2 = 0$. In this case the 1–form C such that $P = Ce^{i\sigma t}$ satisfies $L_t C = 0$. As a result, we find that ε is infinite, so we denote these 'plasma resonance modes', and exclude them from our analysis. However, if the damping $\gamma > 0$ and we deal only with real frequencies, then the plasma resonance modes are automatically excluded.

We now need to represent this dynamic material response as a constitutive property of Maxwell's equations. We will do this first in a frequency domain picture, then as an integral kernel.

A. Operator representation

When relating the time and frequency versions of the dynamic material response, it is useful to first show explicitly how the constitutive relations may be written as in (4) for the case of the one-pole resonance considered here. We do this by proposing (and proving) the following Lemma.

Lemma 1. *We can write the material response from* (6), (8) *as the constitutive relations in* (4) *by setting*

$$\Psi = L_t^2 + \gamma L_t + \omega_{\mathrm{R}}^2 \qquad \text{and} \qquad \Phi = \omega_{\mathrm{P}}^2 \, dt \wedge i_t \ , \tag{9}$$

i.e.

$$\left(L_t^2 + \gamma L_t + \omega_{\mathrm{R}}^2\right) \Pi = \omega_{\mathrm{P}}^2 \, dt \wedge i_t F \ , \tag{10}$$

as long as we avoid plasma resonance modes.

Proof. We can write (8) and (6) as

$$\left(L_t^2 + \gamma L_t + \omega_{\mathrm{R}}^2\right) i_t \Pi = \omega_{\mathrm{P}}^2 \, i_t F \qquad \text{and} \qquad i_t \star \Pi = 0 \ , \tag{11}$$

However, from (2) we have $\Pi = dt \wedge i_t \Pi$. Then from (6) we have

$$\left(L_t^2 + \gamma L_t + \omega_{\mathrm{R}}^2\right) \Pi = \left(L_t^2 + \gamma L_t + \omega_{\mathrm{R}}^2\right) (dt \wedge i_t \Pi) = dt \wedge \left(L_t^2 + \gamma L_t + \omega_{\mathrm{R}}^2\right) (i_t \Pi) = \omega_{\mathrm{P}}^2 dt \wedge i_t F.$$

Thus (11) implies (10). Clearly (10) implies the first equation in (11). In addition since ∂_t is Killing so that $L_t \star = \star L_t$ we have

$$\left(L_t^2 + \gamma L_t + \omega_{\mathrm{R}}^2\right) i_t \star \Pi = i_t \star \left(L_t^2 + \gamma L_t + \omega_{\mathrm{R}}^2\right) \, \Pi = i_t \star \left(\omega_{\mathrm{P}}^2 dt \wedge i_t F\right) = 0 \ .$$

As discussed above, we have excluded any consideration of plasma resonance modes. As a result, the proof is completed since the last equation above implies the second equation in (11). □

Following this Lemma, we can create frequency dependent constitutive relations which can be specified by the components of Ψ and Φ. By replacing $i\omega \leftrightarrow L_t$ we get

$$\Psi_{ab}^{cd}(\omega) = (-\omega^2 + i\gamma\omega + \omega_{\mathrm{R}}^2)(\delta_a^c \delta_b^d - \delta_a^d \delta_b^c), \quad \Phi_{0\nu}^{0\mu} = \omega_{\mathrm{P}}{}^2 \, \delta_\nu^\mu, \quad \Phi_{\sigma\rho}^{\mu\nu}(\omega, k, t, x) = 0 \ . \tag{12}$$

This means that the usual susceptibility matrix, in Fourier transform space, is defined simply by

$$\chi_{ab}^{cd}(\omega) = (\Psi^{-1})_{ef}^{cd}(\omega) \Phi_{ab}^{ef}(\omega) \ . \tag{13}$$

We prefer to work with Ψ and Φ instead of the more usual χ, as we can then avoid the difficulties in forming the operator inverse of Ψ (cf. (12)), i.e. where we need that

$$\tfrac{1}{2} \left(\Psi^{-1}\right)_{ef}^{cd} \Psi_{cd}^{ab} = \left(\delta_e^a \delta_f^b - \delta_e^b \delta_f^a\right) . \tag{14}$$

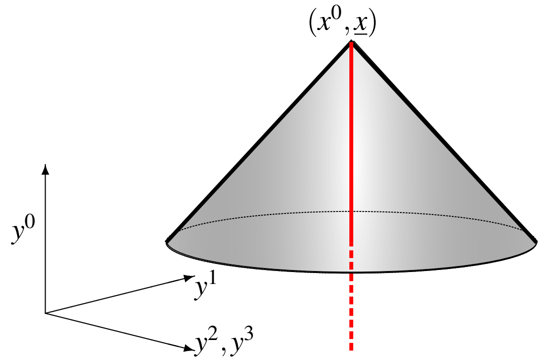

FIG. 3: The support (red) in $\mathscr{M}_{\mathrm{Y}}$ of the kernel κ given in (21), for the point $(x^0, \underline{x}) \in \mathscr{M}_{\mathrm{X}}$ lying in its backward lightcone. This may be contrasted with the support of the transformed kernel, given below in fig. 4

B. Integral kernel representation

It is well-known that the polarization properties of a linear, isotropic, non-magnetic medium with a local, but non-instantaneous response may be written in terms of an integral kernel as [37]

$$P(t) = \int_{-\infty}^{\infty} \kappa_{\text{temp}}(t,\tau) E(\tau) d\tau \,, \tag{15}$$

where causality requires that $\kappa_{\text{temp}}(t,\tau) = 0$ for $t > \tau$. In a usual time translation invariant kernel $\kappa_{\text{temp}}(t,\tau) = \kappa_{\text{temp}}(t-\tau)$ and Fourier techniques are convenient. However, a necessary feature of a spacetime transformation is that the resulting kernel will cease to be time translation invariant, as seen in (49) and (50) below.

We seek to generalise the integral kernel approach of (15) to four dimensions so that both temporal and spatial effects may be accounted for simultaneously. This will of course be necessary as a precursor to understanding how the constitutive relations behave under the action of a spacetime cloak.

Both the electromagnetic field F and the polarization field Π are 2–forms. We write

$$\Pi_{ab}(x) = \tfrac{1}{4}\varepsilon^{cdef} \int_{\mathscr{M}} \kappa_{abcd}(x,y) F_{ef}(y) dy^0 \wedge dy^1 \wedge dy^2 \wedge dy^3. \tag{16}$$

where $\kappa_{abcd}(x,y)$ is a generalisation of $\kappa_{\text{temp}}(t,\tau)$. The two parameters t and τ are generalised to two spacetime events x and y. For convenience let $x \in \mathscr{M}_{\mathrm{X}}$ and $y \in \mathscr{M}_{\mathrm{Y}}$ where $\mathscr{M}_{\mathrm{X}}$, $\mathscr{M}_{\mathrm{Y}}$ are copies of the spacetime $\mathscr{M}$. We use coordinates $x = (x^0, x^1, x^2, x^3)$ with $x^0 = t$ and $\underline{x} = (x^1, x^2, x^3)$.

As in standard Green's function theory we lift the linear operators on $\mathscr{M}$, Ψ and Φ given by (9) into linear operators Ψ_{X} and Φ_{X} on the product $\mathscr{M}_{\mathrm{X}} \times \mathscr{M}_{\mathrm{Y}}$ by requiring they act only on the x

coordinates. That is we set

$$\Psi_{\mathrm{X}} = L_{x^0}^2 + \gamma L_{x^0} + \omega_{\mathrm{R}}^2 \qquad \text{and} \qquad \Phi_{\mathrm{X}} = \omega_{\mathrm{P}}^2 dx^0 \wedge i_{x^0} \, . \tag{17}$$

Thus Π given by (16) will satisfy the differential equation (10) if the 4–form κ satisfies

$$\Psi_{\mathrm{X}} \kappa = \Phi_{\mathrm{X}} \Delta, \tag{18}$$

where

$$\Delta = \tfrac{1}{4} \varepsilon_{abcd} \delta^{(4)}(x-y) \; dx^a \wedge dx^b \wedge dy^c \wedge dy^d, \tag{19}$$

and

$$\kappa = \tfrac{1}{4} \kappa_{abcd}(x,y) \; dx^a \wedge dx^b \wedge dy^c \wedge dy^d, \tag{20}$$

are 4–forms on the product of two copies of spacetime, $\mathscr{M}_{\mathrm{X}} \times \mathscr{M}_{\mathrm{Y}}$.

We may write a solution of (18) as

$$\kappa = \omega_{\mathrm{P}}^2 \frac{\theta(x^0-y^0)\delta^{(3)}(\underline{x}-\underline{y})}{2(\sigma_+ - \sigma_-)} \left[e^{\sigma_+(x^0-y^0)} - e^{\sigma_-(x^0-y^0)} \right] \varepsilon_{\lambda\nu\rho} \;\; dx^0 \wedge dx^\lambda \wedge dy^\nu \wedge dy^\rho , \tag{21}$$

where $\theta(x^0-y^0)$ is the Heaviside function, $\underline{x} = (x^1, x^2, x^3)$ and σ_+, σ_- are the two roots

$$\sigma_\pm = -\tfrac{1}{2}\gamma \pm \sqrt{\tfrac{1}{4}\gamma^2 - \omega_{\mathrm{R}}^2} \, . \tag{22}$$

The proof that (21) is a solution to (18) is given in lemma 3 in the Appendix. The support of the kernel κ, that is the set of points (x,y) such that $\kappa(x,y) \neq 0$, is given in fig. 3. Indeed the support of any causal temporally dispersive kernel is given by (a subset of) fig. 3.

We see that the coefficients in κ in (21) are actually functions of the difference $x-y$, using the affine structure of Minkowski spacetime. This is because it is the Green's function for a spacetime-homogeneous differential equation, i.e. the differential operator is independent of position. However in general, either for general relativity or for inhomogeneous media, κ will *not* have this structure. Thus we will see that $\tilde{\kappa}$, which is the kernel to generate $\tilde{\Pi}$, the spacetime deformation of Π, will not correspond to any difference.

IV. TRANSFORMATION DESIGN

As motivated in section II, we are now going to invoke some suitable diffeomorphism φ as a part of a transformation design process. At first, the steps that need to be taken might seem relatively straightforward. Under the diffeomorphism $\varphi : \tilde{\mathscr{M}} \to \mathscr{M}$ the operators Ψ and Φ become $\tilde{\Psi}$ and $\tilde{\Phi}$ respectively.

Although the goal for this article is to find the explicit form of the operators $\tilde{\Psi}$ and $\tilde{\Phi}$ for the diffeomorphism corresponding to a spacetime cloak with a single pole resonance, in fact all the results in this section apply to any constitutive relations which can be represented by the differential equation (4), for example a sum of Lorentz operators. In this general case the transformed polarisation 2–form $\tilde{\Pi}$, satisfies the operator equation

$$\tilde{\Psi}\,\tilde{\Pi} = \tilde{\Phi}\,\tilde{F}. \tag{23}$$

Here $\tilde{\Psi}$ and $\tilde{\Phi}$ will be constructed out of the (morphed) Lie derivatives $L_{\tilde{t}}$ and $L_{\tilde{x}}$ as well as internal contractions. Of course we will again stay away from resonances so that $\tilde{\Psi}\,\tilde{\Pi} = 0$ if and only if $\tilde{\Pi} = 0$. It is now tempting to immediately apply the identities $-i\tilde{\omega} \leftrightarrow L_{\tilde{t}}$ and $i\tilde{k} \leftrightarrow L_{\tilde{x}}$ in order to obtain matrix entries $\tilde{\Psi}_{ab}^{cd}(\tilde{\omega},\tilde{k},\tilde{t},\tilde{x})$ and $\tilde{\Phi}_{ab}^{cd}(\tilde{\omega},\tilde{k},\tilde{t},\tilde{x})$ and then apply the inverse of $\tilde{\Psi}_{ab}^{cd}$ in order to construct a single susceptibility matrix $\tilde{\chi}_{ab}^{cd}(\tilde{\omega},\tilde{k},\tilde{t},\tilde{x})$. However, we must be careful when we write down $\tilde{\Psi}(\tilde{\omega},\tilde{k},\tilde{t},\tilde{x})$ as neither $\tilde{\omega}$ and $\tilde{t}$, nor $\tilde{k}$ and $\tilde{x}$, commute – just as happens in a similar manner in quantum mechanics. Therefore, in writing down $\tilde{\Psi}(\tilde{\omega},\tilde{k},\tilde{t},\tilde{x})$ we must be sure to retain the proper ordering of these variables; and this consideration makes computation of the inverse even more problematic. Thus although we might *formally* write down $\tilde{\Psi}^{-1}$, calculating it in practise would be very difficult.

It is helpful to identify two distinct scales of spatio-temporal variation: the scale on which the spacetime cloak varies, and the scale on which the envelope of the optical field varies. If the spacetime cloak varies sufficiently slowly that we can neglect second order and higher derivatives of φ, then we refer to this situation as the *Gradual Transformation Approximation* (GTA). A similar approximation, albeit more restrictive, applied to the envelope of the optical field results in the so-called *Slowly Varying Envelope Approximation* (SVEA), i.e. the fields are of the form

$$E(\tilde{t},\tilde{x}) = \mathscr{A}(\tilde{t},\tilde{x}) \exp\left(-i\tilde{\omega}_0\tilde{t} + i\tilde{k}_0\tilde{x}\right), \tag{24}$$

where $\mathscr{A}(\tilde{t},\tilde{x})$ is a pulse envelope whose variation can be considered negligible [38]. Assuming the SVEA, and writing the Lie derivatives $L_{\tilde{t}}$ and $L_{\tilde{x}}$ to the right of $\tilde{t}$ and $\tilde{x}$, then the role of $\tilde{\omega}_0$ and $\tilde{k}_0$ can be identified with $\tilde{\omega}$ and $\tilde{k}$ in $\tilde{\chi}_{ab}^{cd}(\tilde{\omega},\tilde{k},\tilde{t},\tilde{x})$. If we assume both the GTA and the SVEA, the transformed constitutive relations are much simpler, but still necessarily contain both spatial and temporal dispersion, as given in (61) below.

Now let φ be the diffeomorphism between the device manifold $\tilde{\mathscr{M}}$ and the design manifold $\mathscr{M}$,

$$\varphi : \tilde{\mathscr{M}} \to \mathscr{M} \ . \tag{25}$$

See fig. 2. Let $\tilde{\star}$ be the Hodge dual defined with respect to the Minkowski spacetime metric on $\tilde{\mathscr{M}}$. We emphasise that $\tilde{\star}$ is distinct from the induced Hodge dual with respect to the optical metric given by $\tilde{\star}_{\text{optical}} = \varphi^* \star (\varphi^{-1})^*$. We recall that Maxwell's equations on $\mathscr{M}$ are given by (1). Now setting

$$\tilde{F} = \varphi^* F \ . \tag{26}$$

Maxwell's equations on $\tilde{\mathscr{M}}$ read

$$d\tilde{F} = 0 \qquad \text{and} \qquad d\,\tilde{\star}\,\left(\varepsilon_0 \tilde{F} + \tilde{\Pi}\right) = c^{-2} \tilde{\star} \tilde{J} \ , \tag{27}$$

where the first of these follows immediately from (26), on account of the exterior derivative d commuting with the pullback φ^*. Note that in the following we will define $\varphi^{-*} = (\varphi^{-1})^*$.

The derivation of $\tilde{\Psi}$ and $\tilde{\Phi}$ is given for a generic transformation in section IV A below and the specific example of the one pole Lorentz oscillator in section V.

The alternative method of representing the constitutive relations is in terms of the integral kernel as described in section III B. The goal here is two calculate the 4–form $\tilde{\kappa}$ on the product manifold $\tilde{\mathscr{M}}_{\text{X}} \times \tilde{\mathscr{M}}_{\text{Y}}$

$$\tilde{\kappa} = \tfrac{1}{4} \tilde{\kappa}_{abcd}(x,y) d\tilde{x}^a \wedge d\tilde{x}^b \wedge d\tilde{y}^c \wedge d\tilde{y}^d , \tag{28}$$

so that

$$\tilde{\Pi}_{ab}(x) = \tfrac{1}{4} \varepsilon^{cdef} \int_{\tilde{\mathscr{M}}_{\text{Y}}} \tilde{\kappa}_{abcd}(x,y)\ \tilde{F}_{ef}(y)\, d\tilde{y}^0 \wedge d\tilde{y}^1 \wedge d\tilde{y}^2 \wedge d\tilde{y}^3 \ , \tag{29}$$

where $\tilde{\mathscr{M}}_{\text{X}}$ and $\tilde{\mathscr{M}}_{\text{Y}}$ are two copies of $\tilde{\mathscr{M}}$ corresponding to the $\tilde{x}$ and $\tilde{y}$ coordinates respectively. The calculation of $\tilde{\kappa}$ for a generic transformation is given in section IV B, and for the one pole Lorentz oscillator in section V B.

A. Operator representation

In this section we show how the operators Ψ and Φ are transformed under diffeomorphism (to $\tilde{\Psi}$ and $\tilde{\Phi}$ respectively), and demonstrate the invariance of Maxwell's equations. We start with the operator equation for the polarisation 2–forms Π and $\tilde{\Pi}$ on $\mathscr{M}$ and $\tilde{\mathscr{M}}$ respectively given by (4) and (23). The only assumption we make is that Ψ is *Killing* and *Closed*, that is that it commutes with the Hodge dual and the exterior derivative respectively, i.e.

$$\Psi \star = \star\, \Psi \qquad \text{and} \qquad \Psi\, d = d\, \Psi. \tag{30}$$

A sufficient condition so that (30) holds is that Ψ is created out of lie derivatives with respect to Killing vectors. Since $L_t = L_{\frac{\partial}{\partial t}}$ is the Lie derivative with respect to the Killing vector $\frac{\partial}{\partial t}$ then $d\,L_t = L_t\, d$ and $\star\, L_t = L_t \star$. Hence Ψ given by (9) satisfies (30).

Theorem 2. *Let*

$$\tilde{\Psi} = \varphi^*\, \Psi \star^{-1} \varphi^{-*}\, \tilde{\star}, \quad \tilde{\Phi} = \varepsilon_0 \varphi^* \Psi \varphi^{-*} - \varepsilon_0\, \tilde{\Psi} + \varphi^* \Phi \varphi^{-*} \quad \text{and} \quad \tilde{J} = \tilde{\star}^{-1} \varphi^* \star J\ , \tag{31}$$

then

$$\Psi \varphi^{-*} \left[d\tilde{\star} \left(\varepsilon_0 \tilde{F} + \tilde{\Pi} \right) - c^{-2} \tilde{\star} \tilde{J} \right] = 0\ . \tag{32}$$

Again assuming we avoid the plasma resonances, then (32) is equivalent to (27).

Proof. Let $G = \varepsilon_0 F + \Pi$ and $\tilde{G} = \varepsilon_0 \tilde{F} + \tilde{\Pi}$. Recall that the exterior derivative commutes with the pull-backs φ^* and φ^{-*}. Thus using (26), (4), (23) and (30) in turn we have

$$\begin{aligned}
\Psi \varphi^{-*} d\tilde{\star}\ \left(\varepsilon_0 \tilde{F} + \tilde{\Pi} \right) &= \Psi d \varphi^{-*} \tilde{\star}\ \tilde{G} = d \Psi \varphi^{-*} \tilde{\star}\ \tilde{G} = d \Psi \star \star^{-1} \varphi^{-*} \tilde{\star}\ \tilde{G} = d \star \Psi \star^{-1} \varphi^{-*} \tilde{\star}\ \tilde{G} \\
&= d \star \varphi^{-*} \left(\varphi^* \star^{-1} \Psi \varphi^{-*} \tilde{\star} \right) \tilde{G} = d \star \varphi^{-*} \tilde{\Psi} \left(\varepsilon_0 \tilde{F} + \tilde{\Pi} \right) = d \star \varphi^{-*} \left(\varepsilon_0 \tilde{\Psi} \tilde{F} + \tilde{\Psi} \tilde{\Pi} \right) \\
&= d \star \varphi^{-*} \left(\varepsilon_0 \tilde{\Psi} \tilde{F} + \tilde{\Phi} \tilde{F} \right) = d \star \varphi^{-*} \left(\varepsilon_0 \tilde{\Psi} \tilde{F} + \left(\varepsilon_0 \varphi^* \Psi \varphi^{-*} - \varepsilon_0 \tilde{\Psi} + \varphi^* \Phi \varphi^{-*} \right) \tilde{F} \right) \\
&= d \star \varphi^{-*} \left(\varepsilon_0 \varphi^* \Psi \varphi^{-*} \tilde{F} + \varphi^* \Phi \varphi^{-*} \tilde{F} \right) = d \star \left(\varepsilon_0 \Psi \varphi^{-*} \tilde{F} + \Phi \varphi^{-*} \tilde{F} \right) \\
&= d \star \left(\varepsilon_0 \Psi F + \Phi F \right) = d \star \left(\varepsilon_0 \Psi F + \Psi \Pi \right) = d \star \Psi \left(\varepsilon_0 F + \Pi \right) = d \Psi \star \left(\varepsilon_0 F + \Pi \right) \\
&= \Psi d \star \left(\varepsilon_0 F + \Pi \right) = c^{-2} \Psi \star J = c^{-2} \Psi \varphi^{-*} \varphi^* \star J = c^{-2} \Psi \varphi^{-*} \tilde{\star}\, \tilde{J}.
\end{aligned}$$

□

If we have the inverse of Ψ (for non-resonant modes) then, by comparison with (13), we can write down an expression for induced susceptibility $\tilde{\chi}(\tilde{\omega}, \tilde{k}, \tilde{t}, \tilde{x}) = \tilde{\Psi}^{-1} \tilde{\Phi}$, where the polarisation $\tilde{\Pi} = \tilde{\chi} \tilde{F}$. Again preserving the order of $(\tilde{\omega}, \tilde{k}, \tilde{t}, \tilde{x})$ we then have

$$\tilde{\chi} = \varepsilon_0 \left(\tilde{Z}^{-1} - 1 \right) + \tilde{\Psi}^{-1} \varphi^* \Phi \varphi^{-*} \quad \text{where} \quad \tilde{Z} = \varphi^* \star^{-1} \varphi^{-*} \tilde{\star} \quad \text{so that} \quad \tilde{\Psi} = \varphi^* \Psi \varphi^{-*} \tilde{Z}\ . \tag{33}$$

However, as stated previously, the operator $\tilde{\Psi}^{-1}$ is not a simple matrix inverse of $\tilde{\Psi}$ since the entries in $\tilde{\Psi}$ depend on $(\tilde{\omega},\tilde{k},\tilde{t},\tilde{x})$ which do not commute. For this article we calculate the operators $\tilde{\Psi}$ and $\tilde{\Phi}$, which are the most useful, and do not consider the operator $\tilde{\Psi}^{-1}$.

Given φ, one can calculate the map $\tilde{Z}$ which takes 2–forms on $\tilde{\mathscr{M}}$ to 2–forms on $\tilde{\mathscr{M}}$ and may be written with respect to components $\tilde{Z}^{ab}_{cd}$ so that for any 2–form α,

$$\tilde{Z}\alpha = \tfrac{1}{4}\tilde{Z}^{ab}_{cd}\, d\tilde{x}^c \wedge d\tilde{x}^d \wedge (i_{\tilde{x}^a}\, i_{\tilde{x}^b}\, \alpha) = \varphi^* \star^{-1} \varphi^{-*} \tilde{\star}\, \alpha. \tag{34}$$

The map $\tilde{Z}$ is sufficient to encode any spatial transformation device, whether or not it has temporal dispersion, and all spacetime transformation devices without dispersion. The components of the constitutive tensor, $\tilde{Z}^{ab}_{cd}$, may be written as a block 2×2 tableau representing the permittivity, the (inverse) permeability and the magneto-electric tensors (see [21], Eq. (10)).

Using $\tilde{\Phi} = \varepsilon_0 \varphi^* \Psi \varphi^{-*} - \varepsilon_0 \tilde{\Psi} + \varphi^* \Phi \varphi^{-*}$ and $\tilde{\Psi} = \varphi^*\ \Psi\ \varphi^{-*}\ \tilde{Z}$ it is clear that the key step remaining in order to calculate $\tilde{\Psi}$ and $\tilde{\Phi}$ is to calculate the operators $\varphi^* \Psi \varphi^{-*}$ and $\varphi^* \Phi \varphi^{-*}$. This is achieved in section V A.

B. Integral kernel representation

In this section we show how the integral kernel κ is transformed into $\tilde{\kappa}$ under the diffeomorphism φ. The map $\varphi : \tilde{\mathscr{M}} \to \mathscr{M}$ generalises to the maps $\varphi_{\mathrm{X}} : \tilde{\mathscr{M}}_{\mathrm{X}} \to \mathscr{M}_{\mathrm{X}}$ and $\varphi_{\mathrm{Y}} : \tilde{\mathscr{M}}_{\mathrm{X}} \to \mathscr{M}_{\mathrm{Y}}$. Likewise for π_{X} and π_{Y} to give the following commutative diagram:

$$\begin{tikzcd}
& \tilde{\mathscr{M}}_{\mathrm{X}} \times \tilde{\mathscr{M}}_{\mathrm{Y}} \arrow[dl, "\tilde{\pi}_{\mathrm{X}}"'] \arrow[dr, "\tilde{\pi}_{\mathrm{Y}}"] \arrow[rrr, bend left, "\varphi_{\mathrm{XY}}"] & & & \mathscr{M}_{\mathrm{X}} \times \mathscr{M}_{\mathrm{Y}} \arrow[dl, "\pi_{\mathrm{X}}"'] \arrow[dr, "\pi_{\mathrm{Y}}"] & \\
\tilde{\mathscr{M}}_{\mathrm{X}} \arrow[rrr, bend right, "\varphi_{\mathrm{X}}"'] & & \tilde{\mathscr{M}}_{\mathrm{Y}} \arrow[rrr, bend right, "\varphi_{\mathrm{Y}}"'] & \mathscr{M}_{\mathrm{X}} & & \mathscr{M}_{\mathrm{Y}}
\end{tikzcd} \tag{35}$$

where $\varphi_{\mathrm{XY}} = \varphi_{\mathrm{X}} \times \varphi_{\mathrm{Y}}$. Let

$$\tilde{\kappa} = \tilde{Z}_{\mathrm{X}}^{-1}\, \varphi_{\mathrm{XY}}^* \left(\varepsilon_0 \Delta + \kappa\right) - \varepsilon_0 \tilde{\Delta}, \tag{36}$$

where

$$\tilde{Z}_{\mathrm{X}}^{-1} = \left(\tilde{\star}_X^{-1}\ \varphi_{\mathrm{XY}}^*\ \star_X\ \varphi_{\mathrm{XY}}^{-*}\right), \tag{37}$$

and $\tilde{\Delta} = \frac{1}{4}\varepsilon_{abcd}\delta^{(4)}(x-y)d\tilde{x}^a \wedge d\tilde{x}^b \wedge d\tilde{y}^c \wedge d\tilde{y}^d$. We show in lemma 4 in the Appendix that $\tilde{\Delta} = \varphi^*_{\mathrm{XY}}\Delta$. Here $\star_X$ means the Hodge dual applies only to the dx^a components. Similar to (34) we can calculate the components of $\tilde{Z}_{\mathrm{X}}^{-1}$ so that for any 4–form α, which has degree 2 with respect to both $\tilde{x}^a$ and $\tilde{y}^a$, i.e. $\alpha = \frac{1}{4}\alpha_{abcd}\, d\tilde{x}^a \wedge d\tilde{x}^b \wedge d\tilde{y}^c \wedge d\tilde{y}^d$. We have

$$\begin{aligned}(\tilde{Z}_{\mathrm{X}}^{-1})\alpha &= \tilde{\star}_X^{-1}\, \varphi^*_{\mathrm{X}} \star_X \varphi_{\mathrm{X}}^{-*}\, \alpha = \tfrac{1}{4}\left(\tilde{Z}_{\mathrm{X}}^{-1}\right)^{ab}_{cd}\, d\tilde{x}^c \wedge d\tilde{x}^d \wedge (i_{\tilde{x}^c}\, i_{\tilde{x}^d}\, \alpha) \\ &= \frac{1}{8}\left(\tilde{Z}_{\mathrm{X}}^{-1}\right)^{ab}_{cd}\, \alpha_{abef}\, d\tilde{x}^c \wedge d\tilde{x}^d \wedge d\tilde{y}^e \wedge d\tilde{y}^f .\end{aligned} \tag{38}$$

Here the coefficients $(\tilde{Z}_{\mathrm{X}}^{-1})^{ab}_{cd}$ are the same as $(\tilde{Z}^{-1})^{ab}_{cd}$, i.e. the inverse of the $\tilde{Z}^{ab}_{cd}$ given in (34). The proof that $\tilde{\kappa}$ given by (36) satisfies Maxwell's equation (27) is given lemma 5 in the Appendix. From $\tilde{\Delta} = \varphi^*_{\mathrm{XY}}\Delta$ we can write (36) as

$$\tilde{\kappa} = \tilde{Z}_{\mathrm{X}}^{-1}\varphi^*_{\mathrm{XY}}\ \kappa + \varepsilon_0\tilde{Z}_{\mathrm{X}}^{-1}\tilde{\Delta} - \varepsilon_0\tilde{\Delta}. \tag{39}$$

Applying (38) we can calculate the second term in (39) as

$$\tilde{Z}_{\mathrm{X}}^{-1}\,\tilde{\Delta} = \frac{1}{8}\left(\tilde{Z}_{\mathrm{X}}^{-1}\right)^{ab}_{cd}\ \delta^{(4)}(x-y)\varepsilon_{abef}\, d\tilde{x}^c \wedge d\tilde{x}^d \wedge d\tilde{y}^e \wedge d\tilde{y}^f\ . \tag{40}$$

Thus the challenge is to calculate the 4–form $\varphi^*_{\mathrm{XY}}\ \kappa$. This is achieved in section V B.

V. DEVICE PROPERTIES

Here we consider how a one-pole resonance is transformed under the diffeomorphism φ. In the operator representation it is convenient to make the restriction that t is transformed under φ to $\tilde{t}$. This restriction is relaxed in the integral kernel. Finally, in Sec. V C, we consider the task of implementing a transformation in terms of the required material properties – i.e. the material dynamics, and its associated dispersion. Because of the complexity of a situation involving a general transformation, we utilize some simplifying assumptions to clarify which parts of the transformed dynamics are most important.

A. Operator representation

As stated above, our goal is to calculate $\tilde{\Psi}$ and $\tilde{\Phi}$, so that the polarisation 2–form $\tilde{\Pi}$ satisfies (23). Let $\mathscr{M}$ have coordinates (t,x,y,z) and $\tilde{\mathscr{M}}$ have coordinates $(\tilde{t},\tilde{x},\tilde{y},\tilde{z})$ with the diffeomorphism (25) given by

$$\begin{aligned} &\varphi^* t = \tilde{t}, \quad \varphi^* x = x(\tilde{t},\tilde{x}), \quad \varphi^* y = \tilde{y} \quad \text{and} \quad \varphi^* z = \tilde{z}, \\ \text{and inverse relations} \quad &\varphi^{-*}\tilde{t} = t\,, \quad \varphi^{-*}\tilde{x} = \tilde{x}(t,x)\,, \quad \varphi^{-*}\tilde{y} = y \quad \text{and} \quad \varphi^{-*}\tilde{z} = z\,. \end{aligned} \tag{41}$$

Thus we have chosen to transform only between (t,x) and $(\tilde{t},\tilde{x})$ and furthermore we have set $t = \tilde{t}$ so that we are transforming only in space, albeit in a time dependent manner. Equation (41) implies $\partial_x \tilde{t} = 0$, $\partial_t \tilde{t} = 1$, $\partial_{\tilde{x}} t = 0$ and $\partial_{\tilde{t}} t = 1$. Thus

$$\begin{aligned} &\partial_x = (\partial_x \tilde{x})\,\partial_{\tilde{x}}\,, \qquad \partial_t = (\partial_t \tilde{x})\,\partial_{\tilde{x}} + \partial_{\tilde{t}}\,, \\ \text{and} \qquad &\partial_{\tilde{x}} = (\partial_{\tilde{x}} x)\,\partial_x\,, \qquad \partial_{\tilde{t}} = (\partial_{\tilde{t}} x)\,\partial_x + \partial_t. \end{aligned} \tag{42}$$

As stated in section IV A the key step to calculate $\tilde{\Psi}$ and $\tilde{\Phi}$ is $\varphi^*\,\Psi\,\varphi^{-*}$ and $\varphi^*\,\Phi\,\varphi^{-*}$. These are given by

$$\varphi^*\Psi\varphi^{-*} = \varphi^*(L_t^2 + \gamma L_t + \omega_{\mathrm{P}}^2)\varphi^{-*} = \varphi^* L_t^2 \varphi^{-*} + \gamma\varphi^* L_t \varphi^{-*} + \omega_{\mathrm{P}}^2\,, \tag{43}$$

where

$$\varphi^* L_t \varphi^{-*} = L_{\tilde{t}} + (\partial_t \tilde{x}) L_{\tilde{x}} + (\partial_{\tilde{t}}\partial_t \tilde{x}) d\tilde{t} \wedge i_{\tilde{x}} + (\partial_{\tilde{x}}\partial_t \tilde{x}) d\tilde{x} \wedge i_{\tilde{x}}\,, \tag{44}$$

and

$$\begin{aligned} \varphi^* L_t^2 \varphi^{-*} = {} & (\partial_t^2 \tilde{x}) L_{\tilde{x}} + L_{\tilde{t}}^2 + 2(\partial_t \tilde{x}) L_{\tilde{t}} L_{\tilde{x}} + (\partial_t \tilde{x})^2 L_{\tilde{x}}^2 + 2(\partial_{\tilde{t}}\partial_t \tilde{x}) d\tilde{t} \wedge i_{\tilde{x}} L_{\tilde{t}} + 2(\partial_{\tilde{x}}\partial_t \tilde{x}) d\tilde{x} \wedge i_{\tilde{x}} L_{\tilde{t}} \\ & + 2(\partial_t \tilde{x})(\partial_{\tilde{t}}\partial_t \tilde{x}) d\tilde{t} \wedge i_{\tilde{x}} L_{\tilde{x}} + 2(\partial_t \tilde{x})(\partial_{\tilde{x}}\partial_t \tilde{x}) d\tilde{x} \wedge i_{\tilde{x}} L_{\tilde{x}} \\ & + \left((\partial_t \partial_{\tilde{t}}\partial_t \tilde{x}) + (\partial_{\tilde{t}}\partial_t \tilde{x})(\partial_{\tilde{x}}\partial_t \tilde{x})\right) d\tilde{t} \wedge i_{\tilde{x}} + \left((\partial_t \partial_{\tilde{x}}\partial_t \tilde{x}) + (\partial_{\tilde{x}}\partial_t \tilde{x})^2\right) d\tilde{x} \wedge i_{\tilde{x}}\,. \end{aligned} \tag{45}$$

Likewise

$$\varphi^*\,\Phi\,\varphi^{-*} = \omega_{\mathrm{P}}^2\,(d\tilde{t} \wedge i_{\tilde{t}} + (\partial_t \tilde{x}) d\tilde{t} \wedge i_{\tilde{x}})\,. \tag{46}$$

In (44)-46, we can interpret the action of '$d\tilde{x} \wedge i_{\tilde{t}}$' as taking the polarization-like parts of what it operates on (i.e. Z) and applies them to the magnetization-like parts; conversely '$d\tilde{t} \wedge i_{\tilde{x}}$' takes the magnetization parts and applies them to the polarization. We have also taken expressions such as $\partial_{\tilde{t}}\partial_t \tilde{x}$, which means first differentiate $\tilde{x}(t,x)$ with respect to t, then consider the resulting expression as a function of $(\tilde{t},\tilde{x})$ and differentiate those with respect to $\tilde{t}$. Expanding this out, we obtain[4]

$$\partial_{\tilde{t}}\partial_t \tilde{x} = \frac{\partial^2 \tilde{x}}{\partial t^2} + \frac{\partial x}{\partial \tilde{t}}\,\frac{\partial^2 \tilde{x}}{\partial x \partial t}\,, \tag{47}$$

[4] Note that the partial derivatives do not commute, $[\partial_{\tilde{t}}, \partial_t] \neq 0$. This is because t and $\tilde{t}$ belong to different coordinate systems.

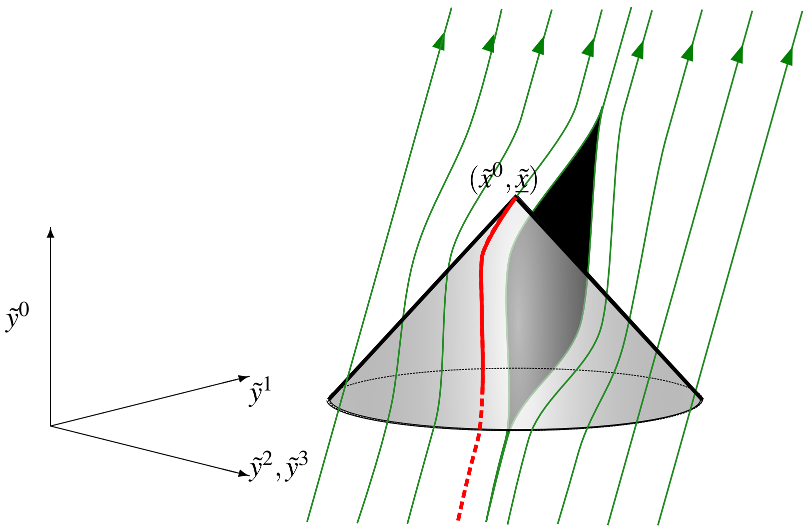

FIG. 4: The support (red) in $\tilde{\mathscr{M}}_{\mathrm{Y}}$ of the kernel $\varphi^*_{\mathrm{XY}}\kappa$ and $\tilde{\kappa}$ (49), for the point $(\tilde{x}^0, \tilde{\underline{x}}) \in \mathscr{M}_{\mathrm{X}}$. Here the green lines correspond to the lines $\left\{\phi(y^0, \underline{y}) | y^0 \in \mathbb{R}\right\}$ for fixed $\underline{y}$.

In (45) we have third derivatives. An example expanded out becomes

$$\partial_t \partial_{\tilde{t}} \partial_t \tilde{x} = \frac{\partial^3 \tilde{x}}{\partial t^3} + \frac{\partial^2 x}{\partial \tilde{t}^2}\frac{\partial^2 \tilde{x}}{\partial x \partial t} + \frac{\partial \tilde{x}}{\partial t}\frac{\partial^2 x}{\partial \tilde{x} \partial \tilde{t}}\frac{\partial^2 \tilde{x}}{\partial x \partial t} + \frac{\partial x}{\partial \tilde{t}}\frac{\partial^3 \tilde{x}}{\partial x \partial t^2}\ , \tag{48}$$

The reason for the third derivative of the coordinate transformation is that since F is a 2–form, we need one derivative of the coordinate transformation which needs to be differentiated twice more since Ψ contains L_t^2. The proof of (43)-(46) is given in the Appendix, where we no longer impose $t = \tilde{t}$. The general transformation (t,x) to $(\tilde{t},\tilde{x})$ includes, for example, the curtain map introduced in the original proposal for the spacetime cloak [21], which utilized a Lorentz boost.

B. Integral kernel representation

In the integral kernel representation, our goal is to calculate $\tilde{\kappa}$, so that the polarisation 2–form $\tilde{\Pi}$ given by (29) satisfies Maxwell's equation (27).

The transformation of the kernel is much easier. We simply set

$$\varphi^*_{\mathrm{XY}}\kappa = \chi_0 \frac{\theta\left(x^0 - y^0\right)\delta^{(3)}(\underline{x} - \underline{y})}{2(\sigma_+ - \sigma_-)}\left(e^{\sigma_+(x^0-y^0)} - e^{\sigma_-(x^0-y^0)}\right)\varepsilon_{\lambda\nu\rho}\, dx^0 \wedge dx^\lambda \wedge dy^\nu \wedge dy^\rho\ , \tag{49}$$

where we regard

$$x_a = x_a(\tilde{x})\,, \quad y_a = y_a(\tilde{y})\,, \quad dx^a = \frac{\partial x^a}{\partial \tilde{x}^b} d\tilde{x}^b \quad \text{and} \quad dy^a = \frac{\partial y^a}{\partial \tilde{y}^b} d\tilde{y}^b\ . \tag{50}$$

By substituting (50) into (49) we can generate $\varphi^*_{\mathrm{XY}}\,\kappa$ and hence using (39) we can generate $\tilde{\kappa}$. It is clear that having done this, the components of $\tilde{\kappa}$ will not be functions of $\tilde{x}-\tilde{y}$. This is as expected since $\tilde{\kappa}$ is not the Fourier transform of a function with arguments $(\tilde{\omega},\tilde{k})$. The support of $\tilde{\kappa}$ in (49) is given in fig. 4.

In this representation it is easy to see that we are effectively doing transformation optics intended to mimic a dispersive medium. As described in section IV B, we have restricted the design medium to be homogeneous and isotropic, with the causal and light cone structure of Minkowski spacetime. The integral kernel representation for more general dispersive design media entails complications that are beyond the scope of this initial work, and will form the basis of further study.

C. Practical Constraints

We expect it to be extremely challenging to construct the spatio-temporal dispersive medium required for perfect event cloaking, and indeed this will likely even be true of simpler spacetime transformation devices [2]. However, in practise we will likely require only that its performance is better than some chosen benchmark. The question then to be answered here is a more practical one – if we only have ordinary temporally dispersive materials available, and we wish to keep the cloaking within some performance range, what constraints does that put on the device design and operating parameters? Here we will consider how the transformation gradients induced by a design morphism, as indicated by basic scales and sizes of the device, affect the magnitude of the additional terms that appear for the transformed medium, as calculated in V A.

We assume a morphism involving a time-dependent spatial distortion, but with matched time coordinates (i.e. $t=\tilde{t}$). We assume a plane-polarized light, the light (of course) being purely transverse in nature; it is a quasi continuous wave, centered about a frequency ω, and with wavelength in the design medium of λ and wavevector $k=\lambda/2\pi$. Specifically, we assume the electric field has the carrier-envelope form given in (24); and although it is now common to allow for a wideband field envelope [39], here we make the more stringent SVEA discussed above.

Before assuming that the transformation is gradual (i.e. making the GTA), let us first consider the likely scope and deformation of spacetime points caused by a cloaking morphism, taking the experiment of Fridman et al. [23] as a guide. For a 1D waveguide based event cloak, there are only three important parameters – the distance L over which the cloak opens and closes, the maximum shift Δ in position of any point, and the average speed of light c_o within the cloak. Using [23], we therefore set $L = 200$m and $\Delta = 5$mm. These distances are proportional to the time scales over which the cloak operates: it has a total duration of $T = L/c_o \sim 1\mu$s, and opens a 'time gap' of $\tau = \Delta/c_o \sim 25$ps.

Subject to these assumptions, we can write out the morphed/device polarization response/operator using (44) and (45) in (10). Before then, however, we will consider the prefactors of each term, each of which relates to derivatives of the morphism, and estimate their sizes. This will make it clear which terms we might need to keep, and which to discard: obviuosly we expect that the first order terms will be most significant, we also calculate the second order ones for completeness. These prefactors, with $\tilde{t} \equiv t$, are

$$a_1 = \partial_t \tilde{x} \qquad \sim \Delta/T \sim 5 \times 10^{-3}/10^{-6} \qquad \simeq \; 5 \times 10^3 \;\; [\text{m/s}] \,, \tag{51}$$

$$b_1 = \partial_t^2 \tilde{x} \qquad \sim \Delta/T^2 \sim 5 \times 10^{-3}/10^{-12} \qquad \simeq \; 5 \times 10^9 \;, \;\; [\text{m/s}^2] \tag{52}$$

$$b_2 = \partial_x \partial_t \tilde{x} \qquad \sim \Delta/LT \sim 5 \times 10^{-3}/200 \times 10^{-6} \qquad \simeq \; 25 \;\; [1/\text{s}] \,. \tag{53}$$

We now need to combine these with the properties of the design medium and the electromagnetic field, and compare the results to the benchmark frequency scale. Consequently, let us now assume that our cloak is also trying to mimic a medium with resonant frequency of about 300THz (i.e. $\omega_{\mathrm{R}} \sim 2 \times 10^{15}$rad/s), and a decay rate of nanoseconds ($\gamma \sim 10^9\text{s}^{-1}$). We further assume that the illumination frequency is comparable to this resonant frequency, at $200THz$ (1432nm) (i.e. $\omega_0 \sim 1.4 \times 10^{15}\text{rad.s}^{-1}$), we therefore also know that allowing for the refactive index of silica, that $k_0 = \omega_0/c_{\text{silica}} \sim 0.7 \times 10^7$. Thus

$$ka_1/\omega \sim 4 \times 10^{10} \text{ rad/s}/1.4 \times 10^{15} \text{ rad/s} \qquad \simeq \; 3 \times 10^{-5}, \tag{54}$$

$$2\pi b_1/c\omega \sim 2\pi \times 5 \times 10^9 \text{ m/s}^2/4 \times 10^{23} \text{ m.rad/s}^2 \qquad \simeq \; 7 \times 10^{-14}, \tag{55}$$

$$b_2/\omega \sim 25 \text{ rad/s}/1.4 \times 10^{15} \text{ rad/s} \qquad \simeq \; 2 \times 10^{-14}. \tag{56}$$

Clearly, by far the most significant correction here is the first order a_1 term, which appears in (44) and (45) in concert with the spatial derivative $L_{\tilde{x}}$; keeping only these is consistent with the GTA. The very small size of the second order terms also assures us that ignoring the third order terms was reasonable. This means that the a_1-dependent additional dynamics of the polarization field is dependent on spatial gradients, that is, that each (originally independent) Lorentz oscillator now is affected by its neighbourhood. However, since this dominant correction is of order 10^{-5}, we can see that the Fridman et al. experiment [23] was safely in the regime where the spacetime effects discussed in this paper are negligible. However, if we hoped to extend the cloaking interval to beyond 25ns instead of ps ($\Delta = 5$m) without also extending the total cloaking length L, then we could be pushing the limitations of the GTA, and more care would be required.

Nevertheless, retaining the a_1 parts as being the primary (GTA) corrections, we can now write a corrected (transformed) differential operator for the polarization dynamics, which is

$$\tilde{\Psi}_{\mathrm{GTA}} = (L_{\tilde{t}} + a_1 L_{\tilde{x}})^2 + \gamma (L_{\tilde{t}} + a_1 L_{\tilde{x}}) + \omega_{\mathrm{R}}^2. \tag{57}$$

Under this level of approximation, we can see that although the device medium has spatial dispersion in addition to the required temporal dispersion, there is no dynamic magnetoelectric cross-coupling between the polarization part of $\tilde{Z}$ and its magnetization part (and vice versa). However, the transformed driving term $\tilde{\Phi}$ in (46) does have a magnetic driving contribution of order a_1; the polarization and magnetization thus evolve independently, but with matching dynamics.

Although an analysis of the differential equation governing the material response is perhaps the most physical approach [36], it is rather more common to look at the material response in the frequency/wavevector domain, and consider the dispersion relations. To get these we use the SVEA, where the nearly CW nature of the illuminating field means that its effect on the polarization is dominated by the exponential carrier oscillations, i.e. by ω_0 and k_0. It can now be easily seen that

$$\varphi^* \tilde{\Psi} \varphi^{-*} = -\left(\tilde{\omega} - a_1 \tilde{k}\right)^2 + i\gamma \left(\tilde{\omega} - a_1 \tilde{k}\right) + \omega_{\mathrm{R}}^2, \tag{58}$$

so that

$$\tilde{\chi} = \varepsilon_0 \left(\tilde{Z}^{-1} - 1\right) + \frac{\omega_{\mathrm{P}}^2}{-\left(\tilde{\omega} - a_1 \tilde{k}\right)^2 + i\gamma \left(\tilde{\omega} - a_1 \tilde{k}\right) + \omega_{\mathrm{R}}^2} \, . \, \tilde{Z}^{-1} \left(d\tilde{t} \wedge i_{\tilde{t}} + a_1 d\tilde{t} \wedge i_{\tilde{x}}\right). \tag{59}$$

The components of the inverse map $\tilde{Z}^{-1}$ can easily be calculated. For example, from the first term in (59), we see the electric field in the $\tilde{y}$-direction gives rise to a magnetisation in the $\tilde{z}$-direction as well as a polarisation in the $\tilde{y}$-direction since

$$\tilde{Z}^{-1}(d\tilde{t} \wedge d\tilde{y}) = (\partial_{\tilde{t}} x)(d\tilde{x} \wedge d\tilde{y}) + (\partial_{\tilde{x}} x)(d\tilde{t} \wedge d\tilde{y}), \tag{60}$$

where we should note that the derivatives here are of x, and not of $\tilde{x}$ as previously (i.e. as for a_1, b_1, b_2). Thus

$$\begin{aligned} \varepsilon_0 \, d\tilde{t} \wedge d\tilde{y} + \tilde{\chi}(d\tilde{t} \wedge d\tilde{y}) = &\left[\varepsilon_0 + \frac{\omega_{\mathrm{P}}^2}{-\left(\tilde{\omega} - a_1 \tilde{k}\right)^2 + i\gamma \left(\tilde{\omega} - a_1 \tilde{k}\right) + \omega_{\mathrm{R}}^2}\right] \\ &\times \left[(\partial_{\tilde{t}} x)(d\tilde{x} \wedge d\tilde{y}) + (\partial_{\tilde{x}} x)(d\tilde{t} \wedge d\tilde{y})\right] \end{aligned} \tag{61}$$

We emphasize again, that despite the application of both the GTA and the SVEA, the resultant medium is still spatially and temporally dispersive and magneto-electric.

As we would expect, this is more complicated than that of the simple one pole resonance specified by the design requirement in (7). We can see that the morphism induces an effective frequency shift $\propto a_1 k_0$, and the loss term is augmented by an induced evanescence $\propto \gamma a_1$; we now have spatial effects which can be considered as spatial dispersion. For a symmetric cloak, we would expect that the a_i coefficient would change sign at the half-way point, for zero net effect. In a perfect cloak we will expect the effect of the modified medium to cancel out – indeed that is the point of the transformation design process – but in any imperfect implementation that may not be the case.

Although it is easy enough to calculate the sizes of these morphism-induced changes to the medium, and discuss the significance of their magnitudes, it is less easy to infer their integrated effect as some illuminating field passes through the device medium. In fig. 1 we indicated that a spacetime cloak built without the additional material dynamics would hide the event, and not return any evidence of what it was, but that – for pictorial purposes, at least – a diverted illuminating pulse would end up with the wrong chirp. This mismatch could raise suspicions that all was not as it should seem.

However, what the signature would be in practise is less clear, as the effect on the illuminating field at any given position depends on the current local state of that medium. Further, that state is also an integral over its past, a past that involves its internal dynamics, the propagating and changing field, *and* an evolving morphism. Since a detailed simulation of the polarization and field dynamics is beyond the scope of this paper, we instead have focussed on the magnitudes of the non-Lorentzian a_1 terms, and whether or not the effects of neglecting them will or will not be significant. This is because from a practical perspective, they mean we have the challenging task of constructing a metamaterial that naturally follows such a non-Lorentzian dynamics, but we would prefer to keep our device simpler and more achievable.

VI. CONCLUSION

In this paper we have for the first time accounted for dispersion in spacetime cloaking. Since dispersion is common to all media through which electromagnetic waves propagate, this advance must be regarded as significant. In particular, we have given the design recipe for fooling an observer into thinking that the electromagnetic signal she receives has travelled through a uniform one-pole resonance medium, whereas in reality, the electromagnetic signal has been distorted and reformed so as to permit events to occur undetected for a brief period.

As we have seen, the problem presented a considerable mathematical challenge, one that we have addressed using both an operator and an integral kernel method, applied within a framework which treats the electromagnetic field as a differential form. Some simplifying assumptions enabled us to make further progress. We identified the GTA and SVEA as being approximations that allow us to ignore several of the terms generated under spacetime transformation. Nevertheless, even when both these approximations are operative, the inherent nature of the transformation is such that space and time become mixed, so that the medium prescribed by the transformation is one that must have defined inhomogeneous temporal and inhomogeneous spatial dispersion. The metamaterial design will still be a considerable technological challenge. Rather than attempt to address how these challenges can be met, we instead focussed on using our results to estimate the likely impact of imperfections on practical spacetime cloaks. We did this by estimating the size of various terms using the experimental parameters of Fridman et al's spacetime cloak [23] as a guide. We found that the dominant dynamics of the induced polarization field depend on spatial gradients, and that at this level of approximation magneto-electric effects can be ignored.

Our rather complete analysis of the problem of achieving spacetime cloaking using dispersive media will likely assist experimentalists in their design of spacetime cloaks. A key step achieved here is that in each case the limitations of any practical design can be quantitatively assessed.

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Appendix A: Additional Lemmas

In this appendix we demonstrate the results used in this paper. We use some additional notation. We use multi-indices on forms so that $dx^{ab} = dx^a \wedge dx^b$.

Lemma 3. *The kernel given by* (21) *satisfies the operator equation* (18).

Proof. From (9) the operators

$$\Psi_{\mathrm{X}} = L_{x^0}^2 + \gamma L_{x^0} + \omega_{\mathrm{R}}^2 \qquad \text{and} \qquad \Phi_{\mathrm{X}} = \omega_{\mathrm{P}}^2\, dx^0 \wedge i_{x^0}$$

Acting Ψ_{X} on κ gives

$$\begin{aligned}
&\Psi_{\mathrm{X}}\, \kappa/\omega_{\mathrm{P}}^2 \\
&= \left(L_{x^0}^2 + \gamma L_{x^0} + \omega_{\mathrm{R}}^2\right) \\
&\quad \times \left(\frac{\theta(x^0-y^0)\, \delta^{(3)}(\underline{x}-\underline{y})}{2(\sigma_+ - \sigma_-)} \left(e^{\sigma_+(x^0-y^0)} - e^{\sigma_-(x^0-y^0)}\right) \varepsilon_{\mu\nu\rho}\, dx^{0\mu} \wedge dy^{\nu\rho} \right) \\
&= \frac{\delta^{(3)}(\underline{x}-\underline{y})}{2(\sigma_+ - \sigma_-)} \left(L_{x^0}^2 + \gamma L_{x^0} + \omega_{\mathrm{R}}^2\right) \left(\theta(x^0-y^0) \left(e^{\sigma_+(x^0-y^0)} - e^{\sigma_-(x^0-y^0)}\right)\right) \varepsilon_{\mu\nu\rho}\, dx^{0\mu} \wedge dy^{\nu\rho} \\
&= \frac{\delta^{(3)}(\underline{x}-\underline{y})}{2(\sigma_+ - \sigma_-)} \Big[\theta(x^0-y^0) \left(L_{x^0}^2 + \gamma L_{x^0} + \omega_{\mathrm{R}}^2\right) \left(e^{\sigma_+(x^0-y^0)} - e^{\sigma_-(x^0-y^0)}\right) \\
&\qquad + \left(\left(L_{x^0}^2 + \gamma L_{x^0}\right) \theta(x^0-y^0)\right) \left(e^{\sigma_+(x^0-y^0)} - e^{\sigma_-(x^0-y^0)}\right) \\
&\qquad + 2\left(L_{x^0} \theta(x^0-y^0)\right) \left(L_{x^0}\left(e^{\sigma_+(x^0-y^0)} - e^{\sigma_-(x^0-y^0)}\right)\right) \Big] \varepsilon_{\mu\nu\rho}\, dx^{0\mu} \wedge dy^{\nu\rho} \\
&= \frac{\delta^{(3)}(\underline{x}-\underline{y})}{2(\sigma_+ - \sigma_-)} \Big[\left(\delta'(x^0-y^0) + \gamma\delta(x^0-y^0)\right) \left(e^{\sigma_+(x^0-y^0)} - e^{\sigma_-(x^0-y^0)}\right) \\
&\qquad + 2\left(\delta(x^0-y^0)\right) \left(L_{x^0}\left(e^{\sigma_+(x^0-y^0)} - e^{\sigma_-(x^0-y^0)}\right)\right) \Big] \varepsilon_{\mu\nu\rho}\, dx^{0\mu} \wedge dy^{\nu\rho} \\
&= \frac{\delta^{(3)}(\underline{x}-\underline{y})}{2(\sigma_+ - \sigma_-)} \Big[\left(\delta(x^0-y^0)\right) \left(L_{x^0}\left(e^{\sigma_+(x^0-y^0)} - e^{\sigma_-(x^0-y^0)}\right)\right) \Big] \varepsilon_{\mu\nu\rho}\, dx^{0\mu} \wedge dy^{\nu\rho} \\
&= \frac{\delta^{(3)}(\underline{x}-\underline{y})}{2}\, \delta(x^0-y^0) \varepsilon_{\mu\nu\rho}\, dx^{0\mu} \wedge dy^{\nu\rho} = \tfrac{1}{2} \varepsilon_{\mu\nu\rho} \delta^{(4)}(x-y) dx^{0\mu} \wedge dy^{\nu\rho}
\end{aligned}$$

where we have used the identity that $\delta(z) f(z) = 0$ and $\delta'(z) f(z) = -f'(0)$ for any functions $f(z)$ such that $f(0) = 0$. Acting Φ_{X} on Δ gives

$$\begin{aligned}
\Phi_{\mathrm{X}}\, \Delta &= \tfrac{1}{4} \varepsilon_{abcd} \Phi_{\mathrm{X}}\, \delta^{(4)}(x-y) dx^{ab} \wedge dy^{cd} = \tfrac{1}{4} \varepsilon_{abcd}\, \omega_{\mathrm{P}}^2\, \delta^{(4)}(x-y) dx^0 \wedge i_{x^0}(dx^{ab} \wedge dy^{cd}) \\
&= \tfrac{1}{4} \varepsilon_{abcd}\, \omega_{\mathrm{P}}^2\, \delta^{(4)}(x-y) dx^0 \wedge (\delta_0^a dx^b - \delta_0^b dx^a) \wedge dy^{cd} \\
&= \tfrac{1}{2} \varepsilon_{0bcd}\, \omega_{\mathrm{P}}^2\, \delta^{(4)}(x-y) dx^{0b} \wedge dy^{cd} = \tfrac{1}{2} \varepsilon_{\mu\nu\rho}\, \omega_{\mathrm{P}}^2\, \delta^{(4)}(x-y) dx^{0\mu} \wedge dy^{\nu\rho}
\end{aligned}$$

Hence (18).

□

Lemma 4. *The 4–form distribution* Δ *is preserved under the map* $\varphi_{\mathrm{XY}} : \tilde{\mathscr{M}}_{\mathrm{X}} \times \tilde{\mathscr{M}}_{\mathrm{Y}} \to \mathscr{M}_{\mathrm{X}} \times \mathscr{M}_{\mathrm{Y}}$*, that is*

$$\varphi_{\mathrm{XY}}^{*}\Delta = \tilde{\Delta} \tag{A1}$$

Proof. We show (A1) is true by acting on an arbitrary test form α in $\mathscr{M}_{\mathrm{X}} \times \mathscr{M}_{\mathrm{Y}}$ to form the integral $\int_{\mathscr{M}_{\mathrm{X}} \times \mathscr{M}_{\mathrm{Y}}} \alpha \wedge \Delta$. Note that if α is not of the degree (2,2), i.e. $\alpha \neq \frac{1}{4}\alpha_{abcd}dx^{ab} \wedge dy^{cd}$ then $\int_{\mathscr{M}_{\mathrm{X}} \times \mathscr{M}_{\mathrm{Y}}} \alpha \wedge \Delta = 0$.

Thus we set $\alpha = \frac{1}{4}\alpha_{abcd}dx^{ab} \wedge dy^{cd}$. Since α is a test form it has compact support, so that $\int_{\mathscr{M}_{\mathrm{X}} \times \mathscr{M}_{\mathrm{Y}}} \alpha \wedge \Delta$ is finite.

$$\begin{aligned}
\int_{\mathscr{M}_{\mathrm{X}} \times \mathscr{M}_{\mathrm{Y}}} \alpha \wedge \Delta &= \frac{1}{16}\int_{\mathscr{M}_{\mathrm{X}} \times \mathscr{M}_{\mathrm{Y}}} \alpha_{abcd}(x,y)\, \delta^{(4)}(x-y)\, dx^{ab} \wedge dy^{cd} \wedge dx^{ef} \wedge dy^{gh} \varepsilon_{efgh} \\
&= \frac{1}{16}\int_{\mathscr{M}_{\mathrm{X}} \times \mathscr{M}_{\mathrm{Y}}} \alpha_{abcd}(x,y)\, \delta^{(4)}(x-y)\, dx^{abef} \wedge dy^{0123}\, \varepsilon_{efgh}\, \varepsilon^{cdgh} \\
&= \frac{1}{16}\int_{\mathscr{M}_{\mathrm{X}}} \alpha_{abcd}(x,x) dx^{abef}\, \varepsilon_{efgh}\, \varepsilon^{cdgh} \\
&= \frac{1}{8}\int_{\mathscr{M}_{\mathrm{X}}} \alpha_{abcd}(x,x) dx^{abef} (\delta_e^c \delta_f^d - \delta_f^c \delta_e^d) \\
&= \frac{1}{4}\int_{\mathscr{M}_{\mathrm{X}}} \alpha_{abcd}(x,x) dx^{abcd} = \int_{\mathscr{M}_{\mathrm{X}}} \gamma_{\mathrm{X}}^{*}\alpha
\end{aligned}$$

where $\gamma_{\mathrm{X}} : \mathscr{M}_{\mathrm{X}} \to (\mathscr{M}_{\mathrm{X}} \times \mathscr{M}_{\mathrm{Y}})$ with $\gamma_{\mathrm{X}}(x) = (x,x)$. Likewise setting $\tilde{\gamma}_{\mathrm{X}} : \tilde{\mathscr{M}}_{\mathrm{X}} \to (\tilde{\mathscr{M}}_{\mathrm{X}} \times \tilde{\mathscr{M}}_{\mathrm{Y}})$, $\tilde{\gamma}_{\mathrm{X}}(\tilde{x}) = (\tilde{x},\tilde{x})$ then $\gamma_{\mathrm{X}} \circ \varphi_{\mathrm{X}} = \varphi_{\mathrm{XY}} \circ \tilde{\gamma}_{\mathrm{X}}$. Thus

$$\begin{aligned}
\int_{\tilde{\mathscr{M}}_{\mathrm{X}} \times \tilde{\mathscr{M}}_{\mathrm{Y}}} (\varphi_{\mathrm{XY}}^{*}\alpha) \wedge \tilde{\Delta} &= \int_{\tilde{\mathscr{M}}_{\mathrm{X}}} \tilde{\gamma}_{\mathrm{X}}^{*}\, \varphi_{\mathrm{XY}}^{*}\, \alpha = \int_{\tilde{\mathscr{M}}_{\mathrm{X}}} \varphi_{\mathrm{X}}^{*}\, \gamma_{\mathrm{X}}^{*}\alpha = \int_{\mathscr{M}_{\mathrm{X}}} \gamma_{\mathrm{X}}^{*}\alpha = \int_{\mathscr{M}_{\mathrm{X}} \times \mathscr{M}_{\mathrm{Y}}} \alpha \wedge \Delta \\
&= \int_{\tilde{\mathscr{M}}_{\mathrm{X}} \times \tilde{\mathscr{M}}_{\mathrm{Y}}} \varphi_{\mathrm{XY}}^{*}(\alpha \wedge \Delta) = \int_{\tilde{\mathscr{M}}_{\mathrm{X}} \times \tilde{\mathscr{M}}_{\mathrm{Y}}} (\varphi_{\mathrm{XY}}^{*}\, \alpha) \wedge (\varphi_{\mathrm{XY}}^{*}\, \Delta)
\end{aligned}$$

Since this is true for all α we have (A1). □

Lemma 5. *The transformed kernel* $\tilde{\kappa}$ *given by* (36) *satisfies Maxwell's equation* (27).

Proof. There is a slight subtlety with regard to the manifold that F lies in. Let $F_{\mathrm{X}} = \frac{1}{2}F_{ab}(x)dx^{ab}$ and $F_{\mathrm{Y}} = \frac{1}{2}F_{ab}(y)dy^{ab}$ be the same electromagnetic 2–form on $\mathscr{M}_{\mathrm{X}}$ and $\mathscr{M}_{\mathrm{Y}}$ respectively. This gives $F_{\mathrm{X}} = \int_{\mathscr{M}_{\mathrm{Y}}} \Delta \wedge \pi_{\mathrm{Y}}^{*}F_{\mathrm{Y}}$. Likewise with $\tilde{F}_{\mathrm{X}}$ and $\tilde{F}_{\mathrm{Y}}$.

If α and β are forms on $\mathscr{M}_{\mathrm{X}}\times\mathscr{M}_{\mathrm{Y}}$, but β only contains dy, then from (37)

$$\begin{aligned}
\tilde{Z}_{\mathrm{X}}^{-1}(\alpha\wedge\beta) &= \tilde{\star}_X^{-1}\,\varphi_{\mathrm{XY}}^{*}\star_X\,\varphi_{\mathrm{XY}}^{-*}(\alpha\wedge\beta) = \tilde{\star}_X^{-1}\,\varphi_{\mathrm{XY}}^{*}\star_X(\varphi_{\mathrm{XY}}^{-*}\alpha\wedge\varphi_{\mathrm{XY}}^{-*}\beta)\\
&= \tilde{\star}_X^{-1}\,\varphi_{\mathrm{XY}}^{*}(\star_X\,\varphi_{\mathrm{XY}}^{-*}\alpha\wedge\varphi_{\mathrm{XY}}^{-*}\beta) = \tilde{\star}_X^{-1}(\varphi_{\mathrm{XY}}^{*}\star_X\,\varphi_{\mathrm{XY}}^{-*}\alpha\wedge\varphi_{\mathrm{XY}}^{*}\,\varphi_{\mathrm{XY}}^{-*}\beta)\\
&= \tilde{\star}_X^{-1}(\varphi_{\mathrm{XY}}^{*}\star_X\,\varphi_{\mathrm{XY}}^{-*}\alpha\wedge\beta) = \tilde{\star}_X^{-1}\varphi_{\mathrm{XY}}^{*}\star_X\,\varphi_{\mathrm{XY}}^{-*}\alpha\wedge\beta = \tilde{Z}_{\mathrm{X}}^{-1}\alpha\wedge\beta
\end{aligned}$$

Hence we have

$$\begin{aligned}
c^{-2}\,\tilde{\star}\,\tilde{J} &= c^{-2}\,\varphi_{\mathrm{X}}^{*}\star J = \varphi_{\mathrm{X}}^{*}\,d\star_X\,(\varepsilon_0 F_{\mathrm{X}}+\Pi) = d\;\varphi_{\mathrm{X}}^{*}\star_X\int_{\mathscr{M}_{\mathrm{Y}}}(\varepsilon_0\Delta+\kappa)\wedge\pi_{\mathrm{Y}}^{*}F_{\mathrm{Y}}\\
&= d\;\varphi_{\mathrm{X}}^{*}\star_X\varphi_{\mathrm{X}}^{-*}\varphi_{\mathrm{X}}^{*}\int_{\mathscr{M}_{\mathrm{Y}}}(\varepsilon_0\Delta+\kappa)\wedge\pi_{\mathrm{Y}}^{*}F_{\mathrm{Y}}\\
&= d\;\varphi_{\mathrm{X}}^{*}\star_X\varphi_{\mathrm{X}}^{-*}\int_{\tilde{\mathscr{M}}_{\mathrm{Y}}}\varphi_{\mathrm{XY}}^{*}\,((\varepsilon_0\Delta+\kappa)\wedge\pi_{\mathrm{Y}}^{*}FY)\\
&= d\;\tilde{\star}_X\;\tilde{\star}_X^{-1}\varphi_{\mathrm{X}}^{*}\star_X\;\varphi_{\mathrm{X}}^{-*}\int_{\tilde{\mathscr{M}}_{\mathrm{Y}}}\varphi_{\mathrm{XY}}^{*}\;((\varepsilon_0\,\Delta+\kappa)\wedge\pi_{\mathrm{Y}}^{*}F_{\mathrm{Y}})\\
&= d\;\tilde{\star}_X\;Z^{-1}\int_{\tilde{\mathscr{M}}_{\mathrm{Y}}}\varphi_{\mathrm{XY}}^{*}\;((\varepsilon_0\,\Delta+\kappa)\wedge\pi_{\mathrm{Y}}^{*}F_{\mathrm{Y}})\\
&= d\;\tilde{\star}_X\int_{\tilde{\mathscr{M}}_{\mathrm{Y}}}\tilde{Z}_{\mathrm{X}}^{-1}\varphi_{\mathrm{XY}}^{*}\;((\varepsilon_0\,\Delta+\kappa)\wedge\pi_{\mathrm{Y}}^{*}F_{\mathrm{Y}})\\
&= d\;\tilde{\star}_X\int_{\tilde{\mathscr{M}}_{\mathrm{Y}}}\big(\tilde{Z}_{\mathrm{X}}^{-1}\varphi_{\mathrm{XY}}^{*}\,(\varepsilon_0\Delta+\kappa)\big)\wedge(\varphi_{\mathrm{XY}}^{*}\;\pi_{\mathrm{Y}}^{*}F_{\mathrm{Y}})\\
&= d\;\tilde{\star}_X\int_{\tilde{\mathscr{M}}_{\mathrm{Y}}}(\varepsilon_0\tilde{\Delta}+\tilde{\kappa})\wedge\;(\tilde{\pi}_{\mathrm{Y}}^{*}\;\varphi_{\mathrm{Y}}^{*}\;F_{\mathrm{Y}})\\
&= d\;\tilde{\star}_X\int_{\tilde{\mathscr{M}}_{\mathrm{Y}}}\big(\varepsilon_0\,\tilde{\Delta}+\tilde{\kappa}\big)\wedge\;\big(\tilde{\pi}_{\mathrm{Y}}^{*}\;\tilde{F}_{\mathrm{Y}}\big) = d\;\tilde{\star}_X\;\big(\varepsilon_0\tilde{F}_{\mathrm{X}}+\tilde{\Pi}\big)
\end{aligned}$$

□

Appendix B: Derivation of the transformation $(t,x)\to(\tilde{t},\tilde{x})$

Here we demonstrate the formulae (44) and (45). Generalise the transformation so that

$$\begin{aligned}
&\varphi^{*}t=t(\tilde{t},\tilde{x}),\quad \varphi^{*}x=x(\tilde{t},\tilde{x}),\quad \varphi^{*}y=\tilde{y}\quad\text{and}\quad \varphi^{*}z=\tilde{z}\\
\text{and inverse relations}\quad &\varphi^{-*}\tilde{t}=\tilde{t}(t,x),\quad \varphi^{-*}\tilde{x}=\tilde{x}(t,x),\quad \varphi^{-*}\tilde{y}=y\quad\text{and}\quad \varphi^{-*}\tilde{z}=z
\end{aligned}\tag{B1}$$

This implies

$$\varphi_{*}^{-1}\partial_x=(\partial_x\tilde{x})\,\partial_{\tilde{x}}+(\partial_x\tilde{t})\,\partial_{\tilde{t}}\qquad\text{and}\qquad \varphi_{*}^{-1}\partial_t=(\partial_t\tilde{x})\,\partial_{\tilde{x}}+(\partial_t\tilde{t})\,\partial_{\tilde{t}}\tag{B2}$$

and

$$\varphi_{*}\partial_{\tilde{x}}=(\partial_{\tilde{x}}x)\,\partial_x+(\partial_{\tilde{x}}t)\,\partial_t\qquad\text{and}\qquad \varphi_{*}\partial_{\tilde{t}}=(\partial_{\tilde{t}}x)\,\partial_x+(\partial_{\tilde{t}}t)\,\partial_t\tag{B3}$$

This gives the following

Lemma 6. *$\varphi^* L_t \varphi^{-*}$ and $\varphi^* L_t^2 \varphi^{-*}$ are given by*

$$\varphi^* L_t \varphi^{-*} = (\partial_t \tilde{t}) L_{\tilde{t}} + (\partial_t \tilde{x}) L_{\tilde{x}} + (\partial_{\tilde{t}} \partial_t \tilde{t}) d\tilde{t} \wedge i_{\tilde{t}} + (\partial_{\tilde{t}} \partial_t \tilde{x}) d\tilde{t} \wedge i_{\tilde{x}} + (\partial_{\tilde{x}} \partial_t \tilde{t}) d\tilde{x} \wedge i_{\tilde{t}} + (\partial_{\tilde{x}} \partial_t \tilde{x}) d\tilde{x} \wedge i_{\tilde{x}} \tag{B4}$$

and

$$\begin{aligned}
&\varphi^* L_t^2 \varphi^{-*} = \\
&\quad \left(\partial_t^2 \tilde{t}\right) L_{\tilde{t}} + \left(\partial_t^2 \tilde{x}\right) L_{\tilde{x}} \quad + (\partial_t \tilde{t})^2 L_{\tilde{t}}^2 \quad + 2\,(\partial_t \tilde{t})\,(\partial_t \tilde{x})\, L_{\tilde{t}} L_{\tilde{x}} \quad + (\partial_t \tilde{x})^2 L_{\tilde{x}}^2 \\
&\quad + 2\,(\partial_t \tilde{t})\,(\partial_{\tilde{t}} \partial_t \tilde{t})\, d\tilde{t} \wedge i_{\tilde{t}} L_{\tilde{t}} \quad + 2\,(\partial_t \tilde{t})\,(\partial_{\tilde{t}} \partial_t \tilde{x})\, d\tilde{t} \wedge i_{\tilde{x}} L_{\tilde{t}} \quad + 2\,(\partial_t \tilde{t})\,(\partial_{\tilde{x}} \partial_t \tilde{t})\, d\tilde{x} \wedge i_{\tilde{t}} L_{\tilde{t}} \\
&\quad + 2\,(\partial_t \tilde{t})\,(\partial_{\tilde{x}} \partial_t \tilde{x})\, d\tilde{x} \wedge i_{\tilde{x}} L_{\tilde{t}} \quad + 2\,(\partial_t \tilde{x})\,(\partial_{\tilde{t}} \partial_t \tilde{t})\, d\tilde{t} \wedge i_{\tilde{t}} L_{\tilde{x}} \quad + 2\,(\partial_t \tilde{x})\,(\partial_{\tilde{t}} \partial_t \tilde{x})\, d\tilde{t} \wedge i_{\tilde{x}} L_{\tilde{x}} \\
&\quad + 2\,(\partial_t \tilde{x})\,(\partial_{\tilde{x}} \partial_t \tilde{t})\, d\tilde{x} \wedge i_{\tilde{t}} L_{\tilde{x}} + 2\,(\partial_t \tilde{x})\,(\partial_{\tilde{x}} \partial_t \tilde{x})\, d\tilde{x} \wedge i_{\tilde{x}} L_{\tilde{x}} \\
&\quad + \Big((\partial_t \partial_{\tilde{t}} \partial_t \tilde{t}) + (\partial_{\tilde{t}} \partial_t \tilde{t})^2 + (\partial_{\tilde{t}} \partial_t \tilde{x})\,(\partial_{\tilde{x}} \partial_t \tilde{t}) \Big)\, d\tilde{t} \wedge i_{\tilde{t}} \\
&\quad + \left((\partial_t \partial_{\tilde{t}} \partial_t \tilde{x}) + (\partial_{\tilde{t}} \partial_t \tilde{t})\,(\partial_{\tilde{t}} \partial_t \tilde{x}) + (\partial_{\tilde{t}} \partial_t \tilde{x})\,(\partial_{\tilde{x}} \partial_t \tilde{x}) \right) d\tilde{t} \wedge i_{\tilde{x}} \\
&\quad + \left((\partial_t \partial_{\tilde{x}} \partial_t \tilde{t}) + (\partial_{\tilde{x}} \partial_t \tilde{t})\,(\partial_{\tilde{t}} \partial_t \tilde{t}) + (\partial_{\tilde{x}} \partial_t \tilde{x})\,(\partial_{\tilde{x}} \partial_t \tilde{t}) \right) d\tilde{x} \wedge i_{\tilde{t}} \\
&\quad + \Big((\partial_t \partial_{\tilde{x}} \partial_t \tilde{x}) + (\partial_{\tilde{x}} \partial_t \tilde{t})\,(\partial_{\tilde{t}} \partial_t \tilde{x}) + (\partial_{\tilde{x}} \partial_t \tilde{x})^2 \Big)\, d\tilde{x} \wedge i_{\tilde{x}} \\
&\quad + 2\left((\partial_{\tilde{t}} \partial_t \tilde{t}) + (\partial_{\tilde{x}} \partial_t \tilde{x}) - (\partial_{\tilde{t}} \partial_t \tilde{x}) + (\partial_{\tilde{x}} \partial_t \tilde{t}) \right) d\tilde{t} \wedge d\tilde{x} \wedge i_{\tilde{x}}\, i_{\tilde{t}}
\end{aligned} \tag{B5}$$

Proof. Dropping the φ^* and φ^{-*} then (B4) follows from

$$L_t = L_{\partial_t} = L_{(\partial_t \tilde{t})\, \partial_{\tilde{t}}} + L_{(\partial_t \tilde{x})\, \partial_{\tilde{x}}} = (\partial_t \tilde{t})\, L_{\partial_{\tilde{t}}} + d(\partial_t \tilde{t}) \wedge i_{\partial_{\tilde{t}}} + (\partial_t \tilde{x})\, L_{\partial_{\tilde{x}}} + d(\partial_t \tilde{x}) \wedge i_{\partial_{\tilde{x}}}$$

Taking L_t of the first and second terms of (B4) we have

$$\begin{aligned}
L_t\left((\partial_t \tilde{t}) L_{\tilde{t}}\right) &= (\partial_t^2 \tilde{t}) L_{\tilde{t}} \ + (\partial_t \tilde{t}) L_t L_{\tilde{t}} \\
&= (\partial_t^2 \tilde{t}) L_{\tilde{t}} \\
&\qquad + (\partial_t \tilde{t}) \Big[(\partial_t \tilde{t}) L_{\tilde{t}} \ + (\partial_t \tilde{x}) L_{\tilde{x}} \ + (\partial_{\tilde{t}} \partial_t \tilde{t}) d\tilde{t} \wedge i_{\tilde{t}} \\
&\qquad\quad + (\partial_{\tilde{t}} \partial_t \tilde{x}) d\tilde{t} \wedge i_{\tilde{x}} \ + (\partial_{\tilde{x}} \partial_t \tilde{t}) d\tilde{x} \wedge i_{\tilde{t}} \ + (\partial_{\tilde{x}} \partial_t \tilde{x}) d\tilde{x} \wedge i_{\tilde{x}} \Big] L_{\tilde{t}} \\
&= (\partial_t^2 \tilde{t}) L_{\tilde{t}} \ + (\partial_t \tilde{t})^2 L_{\tilde{t}}^2 \ + (\partial_t \tilde{t})(\partial_t \tilde{x}) L_{\tilde{t}} L_{\tilde{x}} \ + (\partial_t \tilde{t})(\partial_{\tilde{t}} \partial_t \tilde{t}) d\tilde{t} \wedge i_{\tilde{t}} L_{\tilde{t}} \\
&\qquad\quad + (\partial_t \tilde{t})(\partial_{\tilde{t}} \partial_t \tilde{x}) d\tilde{t} \wedge i_{\tilde{x}} L_{\tilde{t}} \ + (\partial_t \tilde{t})(\partial_{\tilde{x}} \partial_t \tilde{t}) d\tilde{x} \wedge i_{\tilde{t}} L_{\tilde{t}} \ + (\partial_t \tilde{t})(\partial_{\tilde{x}} \partial_t \tilde{x}) d\tilde{x} \wedge i_{\tilde{x}} L_{\tilde{t}}
\end{aligned}$$

and

$$
\begin{aligned}
L_t\left((\partial_t\tilde{x})L_{\tilde{x}}\right) &= (\partial_t^2\tilde{x})L_{\tilde{x}} + (\partial_t\tilde{x})L_tL_{\tilde{x}} \\
&= (\partial_t^2\tilde{x})L_{\tilde{x}} + (\partial_t\tilde{x})\Big[(\partial_t\tilde{t})L_{\tilde{t}} \; + (\partial_t\tilde{x})L_{\tilde{x}} \; + (\partial_{\tilde{t}}\partial_t\tilde{t})d\tilde{t}\wedge i_{\tilde{t}} \\
&\qquad\qquad + (\partial_{\tilde{t}}\partial_t\tilde{x})d\tilde{t}\wedge i_{\tilde{x}} \; + (\partial_{\tilde{x}}\partial_t\tilde{t})d\tilde{x}\wedge i_{\tilde{t}} \; + (\partial_{\tilde{x}}\partial_t\tilde{x})d\tilde{x}\wedge i_{\tilde{x}}\Big]L_{\tilde{x}} \\
&= (\partial_t^2\tilde{x})L_{\tilde{x}} + (\partial_t\tilde{t})(\partial_t\tilde{x})L_{\tilde{t}}L_{\tilde{x}} + (\partial_t\tilde{x})^2L_{\tilde{x}}^2 + (\partial_t\tilde{x})(\partial_{\tilde{t}}\partial_t\tilde{t})d\tilde{t}\wedge i_{\tilde{t}}L_{\tilde{x}} \\
&\quad + (\partial_t\tilde{x})(\partial_{\tilde{t}}\partial_t\tilde{x})d\tilde{t}\wedge i_{\tilde{x}}L_{\tilde{x}} + (\partial_t\tilde{x})(\partial_{\tilde{x}}\partial_t\tilde{t})d\tilde{x}\wedge i_{\tilde{t}}L_{\tilde{x}} + (\partial_t\tilde{x})(\partial_{\tilde{x}}\partial_t\tilde{x})d\tilde{x}\wedge i_{\tilde{x}}L_{\tilde{x}}
\end{aligned}
$$

Now

$$
\begin{aligned}
&L_t\left((\partial_{\tilde{t}}\partial_t\tilde{t})d\tilde{t}\wedge i_{\tilde{t}}\right) = (\partial_t\partial_{\tilde{t}}\partial_t\tilde{t})d\tilde{t}\wedge i_{\tilde{t}} \; + (\partial_{\tilde{t}}\partial_t\tilde{t})L_t\left(d\tilde{t}\wedge i_{\tilde{t}}\right) \\
&= (\partial_t\partial_{\tilde{t}}\partial_t\tilde{t})d\tilde{t}\wedge i_{\tilde{t}} \\
&\quad + (\partial_{\tilde{t}}\partial_t\tilde{t})\Big[(\partial_t\tilde{t})L_{\tilde{t}} + (\partial_t\tilde{x})L_{\tilde{x}} + (\partial_{\tilde{t}}\partial_t\tilde{t})d\tilde{t}\wedge i_{\tilde{t}} \\
&\qquad\qquad + (\partial_{\tilde{t}}\partial_t\tilde{x})d\tilde{t}\wedge i_{\tilde{x}} + (\partial_{\tilde{x}}\partial_t\tilde{t})d\tilde{x}\wedge i_{\tilde{t}} + (\partial_{\tilde{x}}\partial_t\tilde{x})d\tilde{x}\wedge i_{\tilde{x}}\Big]\left(d\tilde{t}\wedge i_{\tilde{t}}\right)
\end{aligned}
$$

The first term on the right hand side of the above is simply included in (B5). The next two terms double the appropriate cross terms since $L_{\tilde{t}}d\tilde{x}\wedge i_{\tilde{x}} = d\tilde{x}\wedge i_{\tilde{x}}L_{\tilde{t}}$. The last for terms can be calculated according to the following table.

		Second			
		$d\tilde{t}\wedge i_{\tilde{t}}$	$d\tilde{t}\wedge i_{\tilde{x}}$	$d\tilde{x}\wedge i_{\tilde{t}}$	$d\tilde{x}\wedge i_{\tilde{x}}$
First	$d\tilde{t}\wedge i_{\tilde{t}}$	$d\tilde{t}\wedge i_{\tilde{t}}$	$d\tilde{t}\wedge i_{\tilde{x}}$	0	$d\tilde{t}\wedge d\tilde{x}\wedge i_{\tilde{x}}\, i_{\tilde{t}}$
	$d\tilde{t}\wedge i_{\tilde{x}}$	0	0	$d\tilde{t}\wedge i_{\tilde{t}} - d\tilde{t}\wedge d\tilde{x}\wedge i_{\tilde{x}}\, i_{\tilde{t}}$	$d\tilde{t}\wedge i_{\tilde{x}}$
	$d\tilde{x}\wedge i_{\tilde{t}}$	$d\tilde{x}\wedge i_{\tilde{t}}$	$d\tilde{x}\wedge i_{\tilde{x}} - d\tilde{t}\wedge d\tilde{x}\wedge i_{\tilde{x}}\, i_{\tilde{t}}$	0	0
	$d\tilde{x}\wedge i_{\tilde{x}}$	$d\tilde{t}\wedge d\tilde{x}\wedge i_{\tilde{x}}\, i_{\tilde{t}}$	0	$d\tilde{x}\wedge i_{\tilde{t}}$	$d\tilde{x}\wedge i_{\tilde{x}}$

which gives

$$
\begin{aligned}
&\Big((\partial_{\tilde t}\partial_t\tilde t)d\tilde t\wedge i_{\tilde t}+(\partial_{\tilde t}\partial_t\tilde x)d\tilde t\wedge i_{\tilde x}+(\partial_{\tilde x}\partial_t\tilde t)d\tilde x\wedge i_{\tilde t}+(\partial_{\tilde x}\partial_t\tilde x)d\tilde x\wedge i_{\tilde x}\Big)^2\\
&\quad=(\partial_{\tilde t}\partial_t\tilde t)^2 d\tilde t\wedge i_{\tilde t}+(\partial_{\tilde t}\partial_t\tilde t)(\partial_{\tilde t}\partial_t\tilde x)d\tilde t\wedge i_{\tilde x}+(\partial_{\tilde t}\partial_t\tilde t)(\partial_{\tilde x}\partial_t\tilde x)d\tilde t\wedge d\tilde x\wedge i_{\tilde x}\,i_{\tilde t}\\
&\qquad+(\partial_{\tilde t}\partial_t\tilde x)(\partial_{\tilde x}\partial_t\tilde t)\,(d\tilde t\wedge i_{\tilde t}-d\tilde t\wedge d\tilde x\wedge i_{\tilde x}\,i_{\tilde t})+(\partial_{\tilde t}\partial_t\tilde x)(\partial_{\tilde x}\partial_t\tilde x)d\tilde t\wedge i_{\tilde x}\\
&\qquad+(\partial_{\tilde x}\partial_t\tilde t)(\partial_{\tilde t}\partial_t\tilde t)d\tilde x\wedge i_{\tilde t}+(\partial_{\tilde x}\partial_t\tilde t)(\partial_{\tilde t}\partial_t\tilde x)\,(d\tilde x\wedge i_{\tilde x}-d\tilde t\wedge d\tilde x\wedge i_{\tilde x}\,i_{\tilde t})\\
&\qquad+(\partial_{\tilde x}\partial_t\tilde x)(\partial_{\tilde t}\partial_t\tilde t)d\tilde t\wedge d\tilde x\wedge i_{\tilde x}\,i_{\tilde t}+(\partial_{\tilde x}\partial_t\tilde x)(\partial_{\tilde x}\partial_t\tilde t)d\tilde x\wedge i_{\tilde t}+(\partial_{\tilde x}\partial_t\tilde x)^2 d\tilde x\wedge i_{\tilde x}\\
&\quad=\big((\partial_{\tilde t}\partial_t\tilde t)^2+(\partial_{\tilde t}\partial_t\tilde x)(\partial_{\tilde x}\partial_t\tilde t)\big)\,d\tilde t\wedge i_{\tilde t}+((\partial_{\tilde t}\partial_t\tilde t)(\partial_{\tilde t}\partial_t\tilde x)+(\partial_{\tilde t}\partial_t\tilde x)(\partial_{\tilde x}\partial_t\tilde x))\,d\tilde t\wedge i_{\tilde x}\\
&\qquad+((\partial_{\tilde x}\partial_t\tilde t)(\partial_{\tilde t}\partial_t\tilde t)+(\partial_{\tilde x}\partial_t\tilde x)(\partial_{\tilde x}\partial_t\tilde t))\,d\tilde x\wedge i_{\tilde t}+\big((\partial_{\tilde x}\partial_t\tilde t)(\partial_{\tilde t}\partial_t\tilde x)+(\partial_{\tilde x}\partial_t\tilde x)^2\big)\,d\tilde x\wedge i_{\tilde x}\\
&\qquad+2\,((\partial_{\tilde t}\partial_t\tilde t)(\partial_{\tilde x}\partial_t\tilde x)-(\partial_{\tilde t}\partial_t\tilde x)(\partial_{\tilde x}\partial_t\tilde t))\,d\tilde t\wedge d\tilde x\wedge i_{\tilde x}\,i_{\tilde t}
\end{aligned}
$$

□

Equations (44)-45 follow by setting $(\partial_t\tilde t)=1$ and $(\partial_x\tilde t)=0$.

For the more general transformation (B1) then Φ is transformed using (31) with

$$
\varphi^*\,\Phi\,\varphi^{-*}=\omega_{\mathrm{P}}^2\Big((\partial_{\tilde t}t)(\partial_t\tilde t)d\tilde t\wedge i_{\tilde t}+(\partial_{\tilde t}t)(\partial_t\tilde x)d\tilde t\wedge i_{\tilde x}+(\partial_{\tilde x}t)(\partial_t\tilde t)d\tilde x\wedge i_{\tilde t}+(\partial_{\tilde x}t)(\partial_t\tilde x)d\tilde x\wedge i_{\tilde x}\Big)\;. \tag{B6}
$$