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Generic singularities of nilpotent orbit closures

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A B S T R A C T

According to a theorem of Brieskorn and Slodowy, the intersection of the nilpotent cone of a simple Lie algebra with a transverse slice to the subregular nilpotent orbit is a simple surface singularity. At the opposite extremity of the poset of nilpotent orbits, the closure of the minimal nilpotent orbit is also an isolated symplectic singularity, called a minimal singularity. For classical Lie algebras, Kraft and Procesi showed that these two types of singularities suffice to describe all generic singularities of nilpotent orbit closures: specifically, any such singularity is either a simple surface singularity, a minimal singularity, or a union of two simple surface singularities of type A_{2k-1}. In the present paper, we complete the picture by determining the generic singularities of all nilpotent orbit closures in *exceptional* Lie algebras (up to normalization in a few cases). We summarize the results in some graphs at the end of the paper.
In most cases, we also obtain simple surface singularities or minimal singularities, though often with more complicated branching than occurs in the classical types. There are, however, six singularities that do not occur in the classical types. Three of these are unibranch non-normal singularities: an $\mathrm{SL}_2(\mathbb{C})$-variety whose normalization is $\mathbb{A}^2$, an $\mathrm{Sp}_4(\mathbb{C})$-vari-

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ety whose normalization is $\mathbb{A}^4$, and a two-dimensional variety whose normalization is the simple surface singularity A_3. In addition, there are three 4-dimensional isolated singularities each appearing once. We also study an intrinsic symmetry action on the singularities, extending Slodowy's work for the singularity of the nilpotent cone at a point in the subregular orbit.

1. Introduction

1.1. Generic singularities of nilpotent orbit closures

Let G be a connected, simple algebraic group of adjoint type over the complex numbers $\mathbb{C}$, with Lie algebra $\mathfrak{g}$. A nilpotent orbit $\mathcal{O}$ in $\mathfrak{g}$ is the orbit of a nilpotent element under the adjoint action of G. Its closure $\overline{\mathcal{O}}$ is a union of finitely many nilpotent orbits. The partial order on nilpotent orbits is defined to be the closure ordering.

We are interested in the singularities of $\overline{\mathcal{O}}$ at points of maximal orbits of its singular locus. Such singularities are known as the *generic singularities* of $\overline{\mathcal{O}}$. Kraft and Procesi determined the generic singularities in the classical types, while Brieskorn and Slodowy determined the generic singularities of the whole nilpotent cone $\mathcal{N}$ for $\mathfrak{g}$ of any type. The goal of this paper is to determine the generic singularities of $\overline{\mathcal{O}}$ when $\mathfrak{g}$ is of exceptional type.

In fact, the singular locus of $\overline{\mathcal{O}}$ coincides with the boundary of $\mathcal{O}$ in $\overline{\mathcal{O}}$, as was shown by Namikawa using results of Kaledin [30,46]. This result also follows from the main theorem in this paper in the exceptional types and from Kraft and Procesi's work in the classical types [32,33]. Therefore to study generic singularities of $\overline{\mathcal{O}}$, it suffices to consider each maximal orbit $\mathcal{O}'$ in the boundary of $\mathcal{O}$ in $\overline{\mathcal{O}}$. We call such an $\mathcal{O}'$ a *minimal degeneration* of $\mathcal{O}$.

The local geometry of $\overline{\mathcal{O}}$ at a point $e \in \mathcal{O}'$ is determined by the intersection of $\overline{\mathcal{O}}$ with a transverse slice in $\mathfrak{g}$ to $\mathcal{O}'$ at e. Such a transverse slice in $\mathfrak{g}$ always exists and is provided by the affine space $\mathcal{S}_e = e + \mathfrak{g}^f$, known as the Slodowy slice. Here, e and f are the nilpotent parts of an $\mathfrak{sl}_2$-triple and $\mathfrak{g}^f$ is the centralizer of f in $\mathfrak{g}$. The local geometry of $\overline{\mathcal{O}}$ at a point e is therefore encoded in $\mathcal{S}_{\mathcal{O},e} = \overline{\mathcal{O}} \cap \mathcal{S}_e$, which we call a nilpotent Slodowy slice. If $\mathcal{O}'$ is a minimal degeneration of $\mathcal{O}$, then $\mathcal{S}_{\mathcal{O},e}$ has an isolated singularity at e. The generic singularities of $\overline{\mathcal{O}}$ can therefore be determined by studying the various $\mathcal{S}_{\mathcal{O},e}$, as $\mathcal{O}'$ runs over all minimal degenerations and $e \in \mathcal{O}'$. The isomorphism type of the variety $\mathcal{S}_{\mathcal{O},e}$ is independent of the choice of e.

The main theorem of this paper is a classification of $\mathcal{S}_{\mathcal{O},e}$ up to algebraic isomorphism for each minimal degeneration $\mathcal{O}'$ of $\mathcal{O}$ in the exceptional types. In a few cases, however, we are only able to determine the normalization of $\mathcal{S}_{\mathcal{O},e}$ and in a few others, we have determined $\mathcal{S}_{\mathcal{O},e}$ only up to local analytic isomorphism.

1.2. Symplectic varieties

Recall from [4] that a symplectic variety is a normal variety W with a holomorphic symplectic form ω on its smooth locus such that for any resolution $\pi : Z \to W$, the pull-back $\pi^*\omega$ extends to a regular 2-form on Z. If this 2-form is symplectic (i.e. if it is non-degenerate everywhere), then π is called a symplectic resolution. By a result of Namikawa [45], a normal variety is symplectic if and only if its singularities are rational Gorenstein and its smooth part carries a holomorphic symplectic form.

The normalization of a nilpotent orbit $\overline{\mathcal{O}}$ is a symplectic variety: it is well-known that $\mathcal{O}$ admits a holomorphic non-degenerate closed 2-form (see [14, Ch. 1.4]) and by work of Hinich [26] and Panyushev [48], the normalization of $\overline{\mathcal{O}}$ has only rational Gorenstein singularities. Hence the normalization of $\overline{\mathcal{O}}$ is a symplectic variety.

Since the normalization of $\overline{\mathcal{O}}$ has rational Gorenstein singularities, the normalization $\widetilde{\mathcal{S}}_{\mathcal{O},e}$ of $\mathcal{S}_{\mathcal{O},e}$ also has rational Gorenstein singularities. The smooth locus of $\mathcal{S}_{\mathcal{O},e}$ admits a symplectic form by restriction of the symplectic form on $\mathcal{O}$ [22, Corollary 7.2], and this yields a symplectic form on the smooth locus of $\widetilde{\mathcal{S}}_{\mathcal{O},e}$ since $\mathcal{S}_{\mathcal{O},e}$ is smooth in codimension one. Thus by the aforementioned result of Namikawa, $\widetilde{\mathcal{S}}_{\mathcal{O},e}$ is also a symplectic variety.

The term *symplectic singularity* refers to a singularity of a symplectic variety. A better understanding of isolated symplectic singularities could shed light on the long-standing conjecture (e.g. [36]) that a Fano contact manifold is homogeneous. The importance of finding new examples of isolated symplectic singularities was stressed in [4]. It is therefore of interest to determine generic singularities of nilpotent orbits, as a means to find new examples of isolated symplectic singularities. Our study of the isolated symplectic singularity $\widetilde{\mathcal{S}}_{\mathcal{O},e}$ contributes to this program.

1.3. Motivation from representation theory

The topology and geometry of the nilpotent cone $\mathcal{N}$ have played an important role in representation theory centered around Springer's construction of Weyl group representations and the resulting Springer correspondence (e.g., [7,28,39,40,55]). The second author of the present paper defined a modular version of Springer's correspondence [29] to the effect that the modular representation theory of the Weyl group of $\mathfrak{g}$ is encoded in the geometry of $\mathcal{N}$. In particular, its decomposition matrix is a part of the decomposition matrix for equivariant perverse sheaves on $\mathcal{N}$. The connection with decomposition numbers makes it desirable to be able to compute the stalks of intersection cohomology complexes with modular coefficients. In this setting the Lusztig–Shoji algorithm to compute Green functions is not available and one has to use other methods, such as Deligne's construction which is general, but hard to use in practice. To actually compute modular stalks it is necessary to have a good understanding of the geometry. The case of a minimal degeneration is the most tractable.

The decomposition matrices of the exceptional Weyl groups are known, so here we are not trying to use the geometry to obtain new information in modular representation theory. However, it is interesting to see how the reappearance of certain singularities in different nilpotent cones leads to equalities (or more complicated relationships) between parts of decomposition matrices. In the GL_n case, the row and column removal rule for nilpotent singularities of [32] gives a geometric explanation for a similar rule for decomposition matrices of symmetric groups [27,29].

It would also be interesting to investigate whether the equivalences of singularities that we obtain in exceptional nilpotent cones have some significance for studying primitive ideals in finite W-algebras (see the survey article [38]).

1.4. Simple surface singularities and their symmetries

1.4.1. Simple surface singularities

Let Γ be a finite subgroup of $\mathrm{SL}_2(\mathbb{C}) \cong \mathrm{Sp}_2(\mathbb{C})$. Then Γ acts on $\mathbb{C}^2$ and the quotient variety $\mathbb{C}^2/\Gamma$ is an affine symplectic variety with an isolated singularity at the image of 0. This variety is known as a simple surface singularity and also as a rational double point, a du Val singularity, or a Kleinian singularity.

Up to conjugacy in $\mathrm{SL}_2(\mathbb{C})$, such Γ are in bijection with the simply-laced, simple Lie algebras over $\mathbb{C}$. The bijection is obtained via the exceptional fiber of a minimal resolution of $\mathbb{C}^2/\Gamma$. The exceptional fiber (that is, the inverse image of 0) is a union of projective lines which intersect transversally. The dual graph of the resolution is given by one vertex for each projective line in the exceptional fiber and an edge joining two vertices when the corresponding projective lines intersect. The dual graph is always a connected, simply-laced Dynkin diagram, which defines the Lie algebra attached to $\mathbb{C}^2/\Gamma$. Hence we denote simple surface singularities using the upper-case letters $A_k, D_k (k \geq 4), E_6, E_7, E_8$, according to the associated simple Lie algebra.

In dimension two, an isolated symplectic singularity is equivalent to a simple surface singularity, that is, it is locally analytically isomorphic to some $\mathbb{C}^2/\Gamma$ (cf. [4, Section 2.1]). An algebraic version of this result is provided by Proposition 5.2. More generally, if $\Gamma \subset \mathrm{Sp}_{2n}(\mathbb{C})$ is a finite subgroup which acts freely on $\mathbb{C}^{2n}\backslash\{0\}$, then the quotient $\mathbb{C}^{2n}/\Gamma$ is an isolated symplectic singularity.

1.4.2. Symmetries of simple surface singularities

Any automorphism of the simple surface singularity $X = \mathbb{C}^2/\Gamma$ fixes $0 \in X$ and induces a permutation of the projective lines in the exceptional fiber of a minimal resolution. Hence it gives rise to a graph automorphism of the dual graph Δ of X. Let $\mathrm{Aut}(\Delta)$ be the group of graph automorphisms of Δ. Then $\mathrm{Aut}(\Delta) = 1$ when $\mathfrak{g}$ is A_1, E_7, or E_8; $\mathrm{Aut}(\Delta) = \mathfrak{S}_3$ when $\mathfrak{g}$ is D_4; and otherwise, $\mathrm{Aut}(\Delta) = \mathfrak{S}_2$.

We now address the question of when the action of $\mathrm{Aut}(\Delta)$ on the dual graph comes from an algebraic action on X (cf. [50, III.6]). When X is of type $A_{2k-1} (k \geq 2)$, D_{k+1} $(k \geq 3)$, or E_6, then $\mathrm{Aut}(\Delta)$ comes from an algebraic action on X. In fact, the action is induced from a subgroup $\Gamma' \subset SL_2(\mathbb{C})$ containing Γ as a normal subgroup. More precisely, there exists such a Γ' with $\Gamma'/\Gamma \cong \mathrm{Aut}(\Delta)$ and the induced action of Γ'/Γ on the dual graph of X coincides with the action of $\mathrm{Aut}(\Delta)$ on Δ via this isomorphism. Such a Γ' is unique. The result also holds for any subgroup of $\mathrm{Aut}(\Delta)$, which is relevant only for the D_4 case.

Slodowy denotes the pair (X, K) consisting of X together with the induced action of $K = \Gamma'/\Gamma$ on X by

$$\begin{aligned}
&B_k, && \text{when } X \text{ is of type } A_{2k-1} \text{ and } K = \mathfrak{S}_2,\\
&C_k, && \text{when } X \text{ is of type } D_{k+1} \text{ and } K = \mathfrak{S}_2,\\
&F_4, && \text{when } X \text{ is of type } E_6 \text{ and } K = \mathfrak{S}_2,\\
&G_2, && \text{when } X \text{ is of type } D_4 \text{ and } K = \mathfrak{S}_3.
\end{aligned}$$

The reasons for this notation will become clear shortly. We also refer to corresponding pairs (Δ, K), where Δ is the dual graph and K is a subgroup of $\mathrm{Aut}(\Delta)$, in the same way. The symmetry of the cyclic group of order 3 when X is of type D_4 is not considered.

When X is of type A_{2k}, the symmetry of X did not arise in Slodowy's work. It does, however, make an appearance in this paper. In this case $\mathrm{Aut}(\Delta) = \mathfrak{S}_2$, but the action on the dual graph does not lift to an action on X. Instead, there is a cyclic group $\langle\sigma\rangle$ of order 4 acting on X, with σ acting by non-trivial involution on Δ, but σ^2 acts non-trivially on X. This cyclic action is induced from a $\Gamma' \subset \mathrm{SL}_2(\mathbb{C})$ corresponding to D_{2k+3}. We define the symmetry of X to be the induced action of Γ' on X and denote it by A_{2k}^+. Only the singularities A_2^+ and A_4^+ will appear in the sequel, and then only when $\mathfrak{g}$ is of type E_7 or E_8.

1.5. The regular nilpotent orbit

1.5.1. Generic singularities of the nilpotent cone

The problem of describing the generic singularities of the nilpotent cone $\mathcal{N}$ of $\mathfrak{g}$ was carried out by Brieskorn [9] and Slodowy [50] in confirming a conjecture of Grothendieck. In their setting $\mathcal{O}$ is the regular nilpotent orbit and so $\overline{\mathcal{O}}$ equals $\mathcal{N}$, and there is only one minimal degeneration, at the subregular nilpotent orbit $\mathcal{O}'$. Slodowy's result from [50, IV.8.3] is that when $e \in \mathcal{O}'$, the slice $\mathcal{S}_{\mathcal{O},e}$ is algebraically isomorphic to a simple surface singularity. Moreover, as in [9], when the Dynkin diagram of $\mathfrak{g}$ is simply-laced, the Lie algebra associated to this simple surface singularity is $\mathfrak{g}$. On the other hand, when $\mathfrak{g}$ is not simply-laced, the singularity $\mathcal{S}_{\mathcal{O},e}$ is determined from the list in §1.4.2. For example, if $\mathfrak{g}$ is of type B_k, then $\mathcal{S}_{\mathcal{O},e}$ is a type A_{2k-1} singularity. This explains the notation in the list in §1.4.2. Next we explain an intrinsic realization of the symmetry of $\mathcal{S}_{\mathcal{O},e}$ when $\mathfrak{g}$ is not simply-laced.

1.5.2. Intrinsic symmetry action on the slice

Let Δ be the Dynkin diagram of $\mathfrak{g}$ and $K \subset \mathrm{Aut}(\Delta)$ be a subgroup. The group $\mathrm{Aut}(\Delta)$ is trivial unless $\mathfrak{g}$ is simply-laced. The action of K on Δ can be lifted to an action on

$\mathfrak{g}$ as in [47, Chapter 4.3]: namely, fix a canonical system of generators of $\mathfrak{g}$. Then there is a subgroup $\tilde{K} \subset \mathrm{Aut}(\mathfrak{g})$, isomorphic to K, which permutes the canonical system of generators, and whose induced action on Δ coincides with K. Any two choices of systems of generators define conjugate subgroups of $\mathrm{Aut}(\mathfrak{g})$. The automorphisms in $\tilde{K}$ are called *diagram automorphisms* of $\mathfrak{g}$.

Now given $\mathfrak{g}$ we can associate a pair $(\mathfrak{g}_s, \tilde{K})$ where $\mathfrak{g}_s$ is a simple, simply-laced Lie algebra with Dynkin diagram Δ_s and $\tilde{K} \subset \mathrm{Aut}(\mathfrak{g}_s)$ is a lifting of some $K \subset \mathrm{Aut}(\Delta_s)$ and $\mathfrak{g} \cong (\mathfrak{g}_s)^{\tilde{K}}$. If $\mathfrak{g}$ is already simply-laced, then $\mathfrak{g} = \mathfrak{g}_s$ and $\tilde{K} = 1$. If $\mathfrak{g}$ is not simply-laced, then the pair (Δ_s, K) appears in the list in §1.4.2, but according to the type of ${}^L\mathfrak{g}$, where ${}^L\mathfrak{g}$ is the Langlands dual Lie algebra of $\mathfrak{g}$.

Recall $\mathfrak{s} \cong \mathfrak{sl}_2(\mathbb{C})$ is the subalgebra of $\mathfrak{g}$ generated by e and f. Let $C(\mathfrak{s})$ be the centralizer of $\mathfrak{s}$ in G. Then $C(\mathfrak{s})$ acts on $\mathcal{S}_{\mathcal{O},e}$ for any nilpotent orbit $\mathcal{O}$, fixing the point e. Also the component group $A(e)$ of the centralizer in G of e is isomorphic to the component group of $C(\mathfrak{s})$ (see §2.2 for more details). When e is in the subregular nilpotent orbit, Slodowy observed that $C(\mathfrak{s})$ is a semidirect product of its connected component $C(\mathfrak{s})^\circ$ and a subgroup $H \cong A(e)$. Moreover H is well-defined up to conjugacy in $C(\mathfrak{s})$. This is immediate except when $\mathfrak{g}$ is of type B_k, since otherwise $C(\mathfrak{s})^\circ$ is trivial. Also, $A(e) \cong K$. In particular, $A(e)$ is trivial if $\mathfrak{g}$ is simply-laced since G is adjoint.

Now let $(({}^L\Delta)_s, {}^LK)$ be the pair attached above to ${}^L\mathfrak{g}$. We have $A(e) \cong H \cong K \cong {}^LK$. Then Slodowy's classification and symmetry result can be summarized as follows: the pair $(\mathcal{S}_{\mathcal{O},e}, H)$, of $\mathcal{S}_{\mathcal{O},e}$ together with the action of H, corresponds to the pair $(({}^L\Delta)_s, {}^LK)$ [50, IV.8.4].

1.6. *The other nilpotent orbits in Lie algebras of classical type*

Kraft and Procesi described the generic singularities of nilpotent orbit closures for all the classical groups, up to smooth equivalence (see §2.1 for the definition of smooth equivalence) [32,33].

1.6.1. *Minimal singularities*

Let $\mathcal{O}_{\min}$ be the minimal (non-zero) nilpotent orbit in a simple Lie algebra $\mathfrak{g}$. Then $\overline{\mathcal{O}}_{\min}$ has an isolated symplectic singularity at $0 \in \overline{\mathcal{O}}_{\min}$. Following Kraft and Procesi [33, 14.3], we refer to $\overline{\mathcal{O}}_{\min}$ by the lower case letters for the ambient simple Lie algebra: $a_k, b_k, c_k, d_k (k \geq 4), g_2, f_4, e_6, e_7, e_8$. The equivalence classes of these singularities, under smooth equivalence, are called *minimal singularities.*

1.6.2. *Generic singularities in the classical types*

The results of Kraft and Procesi for Lie algebras of classical type can be summarized as follows: an irreducible component of a generic singularity is either a simple surface singularity or a minimal singularity, up to smooth equivalence. Moreover, when a generic singularity is not irreducible, then it is smoothly equivalent to a union of two simple surface singularities of type A_{2k-1} meeting transversally in the singular point. This is denoted $2A_{2k-1}$. In more detail:

Theorem 1.1. *[32,33] Assume $\mathcal{O}'$ is a minimal degeneration of $\mathcal{O}$ in a simple complex Lie algebra of classical type. Let $e \in \mathcal{O}'$. Then*

(a) *If the codimension of $\mathcal{O}'$ in $\overline{\mathcal{O}}$ is two, then $\mathcal{S}_{\mathcal{O},e}$ is smoothly equivalent to a simple surface singularity of type A_k, D_k, or $2A_{2k-1}$. The last two singularities do not occur for $\mathfrak{sl}_n(\mathbb{C})$, and the singularity A_k for k even does not occur in the classical Lie algebras besides $\mathfrak{sl}_n(\mathbb{C})$.*
(b) *If the codimension is greater than two, then $\mathcal{S}_{\mathcal{O},e}$ is smoothly equivalent to a_k, b_k, c_k, or d_k. The last three singularities do not occur for $\mathfrak{sl}_n(\mathbb{C})$.*

1.7. The case of type G_2

This case was studied by Levasseur–Smith [37] and Kraft [31]. Let $\mathcal{O}_8$ denote the $\widetilde{A}_1$ orbit and let $\mathcal{O}_6$ denote the minimal orbit. Kraft showed that the closure of the subregular orbit has A_1 singularity along $\mathcal{O}_8$. Levasseur–Smith showed that $\overline{\mathcal{O}}_8$ has non-normal locus equal to $\overline{\mathcal{O}}_6$ and that the natural map from the closure of the minimal orbit in $\mathfrak{so}_7(\mathbb{C})$ to $\overline{\mathcal{O}}_8$ is the normalization map and is bijective. From these results it follows that the singularity of $\overline{\mathcal{O}}_8$ along $\mathcal{O}_6$ is non-normal with smooth normalization. We describe this singularity in §4.4.3 and show that its normalization is $\mathbb{C}^2$.

1.8. Main results

We now summarize the main results of the paper describing the classification of generic singularities in the exceptional Lie algebras. Here and in the sequel, we may write the degeneration $\mathcal{O}'$ of $\mathcal{O}$ as $(\mathcal{O}, \mathcal{O}')$, that is, with the larger orbit appearing first. In this subsection $\mathcal{O}'$ is a minimal degeneration of $\mathcal{O}$ and $e \in \mathcal{O}'$.

1.8.1. Overview

Most generic singularities are like those in the classical types: the irreducible components are either simple surface singularities or minimal singularities. But some new features occur in the exceptional groups. There is more complicated branching and several singularities occur which did not occur in the classical types. Among the latter are three singularities whose irreducible components are not normal (one of these already occurs in G_2 as the singularity of $\widetilde{A}_1$ in the minimal orbit), and three additional singularities of dimension four.

A key observation is that all irreducible components of $\mathcal{S}_{\mathcal{O},e}$ are mutually isomorphic since the action of $C(\mathfrak{s})$ is transitive on irreducible components (§2.4). This result is not true in general when $\mathcal{O}'$ is not a minimal degeneration of $\mathcal{O}$.

For most minimal degenerations we determine the isomorphism type of $\mathcal{S}_{\mathcal{O},e}$, a stronger result than classifying the singularity up to smooth equivalence. In ten of these cases, all in E_8 (§10.2), we can only determine the isomorphism type of $\mathcal{S}_{\mathcal{O},e}$ up to normalization. In the remaining four cases, $\mathcal{S}_{\mathcal{O},e}$ is determined only up to smooth equivalence (§12). It is possible to use the methods here to establish that Kraft and Procesi's results in Theorem 1.1 hold up to algebraic isomorphism (rather than smooth equivalence), but we defer the details to a later paper.

We also calculate the symmetry action on $\mathcal{S}_{\mathcal{O},e}$ induced from $A(e)$, as Slodowy did when $\mathcal{O}$ is the regular nilpotent orbit. This involves extending Slodowy's result on the splitting of $C(\mathfrak{s})$ and introducing the notion of symmetry on a minimal singularity. Again, it is possible to carry out this program for the classical groups, but we also defer the details to a later paper.

1.8.2. Symmetry of a minimal singularity

Let $\mathfrak{g}$ be a simple, simply-laced Lie algebra with Dynkin diagram Δ. As in §1.5.2, let $\tilde{K} \subset \mathrm{Aut}(\mathfrak{g})$ be a subgroup of diagram automorphisms lifting a subgroup $K \subset \mathrm{Aut}(\Delta)$. We call a pair $(\overline{\mathcal{O}}_{\min}, \tilde{K})$, consisting of $\overline{\mathcal{O}}_{\min}$ with the action of $\tilde{K}$, a symmetry of a minimal singularity. We write these pairs as a_k^+, $d_k^+ (k \geq 4)$, d_4^{++} (for the action of the full automorphism group), and e_6^+. As in the surface cases, $|K| = 3$ in D_4 does not arise.

1.8.3. Intrinsic symmetry action on a slice: general case

In §6.1 it is shown that the splitting of $C(\mathfrak{s})$ that Slodowy observed for the subregular orbit holds in general, with four exceptions. More precisely, there exists a subgroup $H \subset C(\mathfrak{s})$ such that $C(\mathfrak{s}) \cong C(\mathfrak{s})^\circ \rtimes H$. So in particular $H \cong C(\mathfrak{s})/C(\mathfrak{s})^\circ \cong A(e)$. The choice of splitting is in general no longer unique up to conjugacy in $C(\mathfrak{s})$, but if we choose H to represent diagram automorphisms of the semisimple part of $\mathfrak{c}(\mathfrak{s})$, then the image of H in $\mathrm{Aut}(\mathfrak{c}(\mathfrak{s}))$ is unique up to conjugacy in $\mathrm{Aut}(\mathfrak{c}(\mathfrak{s}))$. The four exceptions to the splitting of $C(\mathfrak{s})$ have $|A(e)| = 2$, but the best possible result is that there exists $H \subset C(\mathfrak{s})$, cyclic of order 4, with $C(\mathfrak{s}) = C(\mathfrak{s})^\circ \cdot H$ [51, §3.4].

Next, imitating §1.5.2, we describe the action of H on $\mathcal{S}_{\mathcal{O},e}$. The four cases where $C(\mathfrak{s})$ does not split give rise to the symmetries which include A_2^+ and A_4^+ (§1.4.2). Three of these four cases (when $\mathcal{O}'$ has type $A_4 + A_1$ in E_7 and E_8 or type $E_6(a_1) + A_1$ in E_8) are well-known: under the Springer correspondence, their Weyl group representations lead to unexpected phenomena (see, for example, [12, pg. 373]). The phenomena observed here for these three orbits is directly related to the fact that $A(e) = \mathfrak{S}_2$ acts without fixed points on the irreducible components of the Springer fiber over e. It is not clear why the fourth orbit (of type $D_7(a_2)$ in E_8) appears in the same company as these three orbits.

1.8.4. Additional singularities

In the Lie algebras of exceptional type, there are six varieties, arising as components of slice singularities, which are neither simple surface singularities nor minimal singularities.

The ten cases in type E_8 where we know the singularity only up to normalization would give further examples if they turned out to be non-normal.

Non-normal cases.

The variety m. Let $V(i)$ denote the irreducible representation of highest weight $i \in \mathbb{Z}_{\geq 0}$ of $\mathrm{SL}_2(\mathbb{C})$. Consider the linear representation of $\mathrm{SL}_2(\mathbb{C})$ on $V = V(2) \oplus V(3)$. Let $v_2 \in V(2)$ and $v_3 \in V(3)$ be highest weight vectors for a Borel subgroup of $\mathrm{SL}_2(\mathbb{C})$. The variety m is defined to be the closure in V of the $\mathrm{SL}_2(\mathbb{C})$-orbit through $v = v_2 + v_3$, a two-dimensional variety with an isolated singularity at zero. It is not normal, but has smooth normalization, equal to the affine plane $\mathbb{A}^2$. This is an example of an S-variety (i.e., the closure of the orbit of a sum of highest weight vectors) studied in [57], where these properties are proved (see §3.2.1). The first case where m appears is for the minimal degeneration $(\tilde{A}_1, A_1)$ in G_2. This singularity appears at least once in each exceptional Lie algebra, always for two non-special orbits which lie in the same special piece (see §1.9.2).

The variety m'. This is a four-dimensional analogue of m, with $\mathrm{SL}_2(\mathbb{C})$ replaced by $\mathrm{Sp}_4(\mathbb{C})$. It is an S-variety [57] with respect to the $\mathrm{Sp}_4(\mathbb{C})$-representation on $V = V(2\omega_1) \oplus V(3\omega_1)$ where $V(\omega_1)$ is the defining 4-dimensional representation of $\mathrm{Sp}_4(\mathbb{C})$, so that $V(2\omega_1)$ is the adjoint representation. Let $v_2 \in V(2\omega_1)$ and $v_3 \in V(3\omega_1)$ be highest weight vectors for a Borel subgroup of $\mathrm{Sp}_4(\mathbb{C})$. The variety m' is defined to be the closure in V of the $\mathrm{Sp}_4(\mathbb{C})$-orbit through $v = v_2 + v_3$, a four-dimensional variety with an isolated singularity at zero. It is not normal, but has smooth normalization, equal to $\mathbb{A}^4$ (see §3.2.1). The singularity m' occurs exactly once, for the minimal degeneration $(A_3 + 2A_1, 2A_2 + 2A_1)$ in E_8.

The variety μ. The coordinate ring of the simple surface singularity A_3 is $R = \mathbb{C}[st, s^4, t^4]$, as a hypersurface in $\mathbb{C}^3$. We define the variety μ by $\mu = \operatorname{Spec} R'$ where $R' = \mathbb{C}[(st)^2, (st)^3, s^4, t^4, s^5t, st^5]$. This variety is non-normal and its normalization is isomorphic to A_3 via the inclusion of R' in R. Using the methods of §5, the normalization of $\mathcal{S}_{\mathcal{O},e}$ for $(D_7(a_1), E_8(b_6))$ in E_8 is shown to be isomorphic to A_3 with an order two symmetry arising from $A(e)$. In [21] we will show that $\mathcal{S}_{\mathcal{O},e}$ is smoothly equivalent to μ. The closure of $\mathcal{O}$ was known to be non-normal, but our result establishes that it is non-normal in codimension two.

Normal cases. These three singularities are each of dimension four and normal.

The degeneration $(2A_2 + A_1, A_2 + 2A_1)$ in E_6. Let $\zeta = e^{\frac{2\pi i}{3}}$ and let Γ be the cyclic subgroup of $\mathrm{Sp}_4(\mathbb{C})$ of order three generated by the diagonal matrix with eigenvalues ζ, ζ, ζ^{-1}, ζ^{-1}. Then $\mathbb{C}^4/\Gamma$ has an isolated singularity at 0, and we denote this variety by τ. We show in §12.2 that $\mathcal{S}_{\mathcal{O},e}$ is smoothly equivalent to τ for the minimal degeneration $(2A_2 + A_1, A_2 + 2A_1)$ in E_6.

The degeneration $(A_4 + A_1, A_3 + A_2 + A_1)$ in E_7. Let $\mathfrak{S}_2$ be the cyclic group of order two acting on $\mathfrak{sl}_3(\mathbb{C})$ via an outer involution. All such involutions are conjugate in $\mathrm{Aut}(\mathfrak{sl}_3(\mathbb{C}))$. The quotient $a_2/\mathfrak{S}_2$ has an isolated singularity at 0 since there are no minimal nilpotent elements in $\mathfrak{sl}_3(\mathbb{C})$ which are fixed by an outer involution. We will prove in [21] that $\mathcal{S}_{\mathcal{O},e}$ is smoothly equivalent to $a_2/\mathfrak{S}_2$ for the minimal degeneration $(A_4 + A_1, A_3 + A_2 + A_1)$ in E_7.

The degeneration $(A_4 + A_3, A_4 + A_2 + A_1)$ *in* E_8. Let Δ be a dihedral group of order 10, acting on $\mathbb{C}^4$ via the sum of the reflection representation and its dual. Then it turns out that the blow-up of $\mathbb{C}^4/\Delta$ at its singular locus has an isolated singularity at a point lying over 0. We denote this blow-up by χ. We show in §12.3 that $\mathcal{S}_{\mathcal{O},e}$ is smoothly equivalent to χ for the minimal degeneration $(A_4 + A_3, A_4 + A_2 + A_1)$ in E_8.[1]

1.8.5. Statement of the main theorem

In our main theorem we classify the generic singularities of nilpotent orbit closures in a simple Lie algebra of exceptional type. The graphs at the end of the paper list the precise results.

Theorem 1.2. *Let $\mathcal{O}'$ be a minimal degeneration of $\mathcal{O}$ in a simple Lie algebra of exceptional type. Let $e \in \mathcal{O}'$. Taking into consideration the intrinsic symmetry of $A(e)$, we have*

(a) *If the codimension of $\mathcal{O}'$ in $\overline{\mathcal{O}}$ is two, then, with one exception, $\mathcal{S}_{\mathcal{O},e}$ is isomorphic either to a simple surface singularity of type $A - G$ or to one of the following*

$$A_2^+, A_4^+, 2A_1, 3A_1, 3C_2, 3C_3, 3C_5, 4G_2, 5G_2, 10G_2, \text{ or } m,$$

up to normalization for ten cases in E_8. Here, kX_n denotes k copies of X_n meeting pairwise transversally at the common singular point. In the one remaining case, the singularity is smoothly equivalent to μ.

(b) *If the codimension is greater than two, then, with three exceptions, $\mathcal{S}_{\mathcal{O},e}$ is isomorphic either to a minimal singularity of type $a - g$ or to one of the following types:*

$$a_2^+, a_3^+, a_4^+, a_5^+, 2a_2, d_4^{++}, e_6^+, 2g_2, \text{ or } m',$$

where the branched cases $2a_2$ and $2g_2$ denote two minimal singularities meeting transversally at the common singular point. The singularities for the three remaining cases are smoothly equivalent to $\tau, a_2/\mathfrak{S}_2$, and χ, respectively.

In the statement of the theorem, we have recorded the induced symmetry of $A(e)$ relative to the stabilizer in $A(e)$ of an irreducible component of $\mathcal{S}_{\mathcal{O},e}$. See §6.2 for a complete statement of the intrinsic symmetry action.

[1] In private communication with the authors, Bellamy has pointed out that it can be deduced from his work [5] that the symplectic quotient of $\mathbb{C}^4$ by a dihedral group of order $4n + 2$ has a unique $\mathbb{Q}$-factorial terminalization. Since $\mathcal{O}$ is a rigid orbit, the singularity χ is also $\mathbb{Q}$-factorial terminal (see §5.2). Hence χ can be identified with the unique $\mathbb{Q}$-factorial terminalization of $\mathbb{C}^4/\Delta$.

1.8.6. Brief description of methods

The methods in §4 are relevant for cases when $\mathcal{S}_{\mathcal{O},e}$ has a dense $C(\mathfrak{s})$-orbit and are motivated by arguments in [32]. For the higher codimension minimal degenerations (except the three which are normal of codimension four) and the codimension two minimal degenerations where the singularity is kA_1 or m, we show that $\mathcal{S}_{\mathcal{O},e}$ has a dense $C(\mathfrak{s})$-orbit. This allows us to show that the irreducible components of $\mathcal{S}_{\mathcal{O},e}$ are S-varieties for $C(\mathfrak{s})^\circ$ (which are permuted transitively by $C(\mathfrak{s})$), and we determine their isomorphism type. Proposition 3.3 contains precise information about the connection between $C(\mathfrak{s})$ and $\mathcal{S}_{\mathcal{O},e}$ in these cases.

The methods in §5 are applicable to the surface cases. The idea is to use the fact that the normalization of a transverse slice is a simple surface singularity and then obtain a minimal resolution of the singularity by restricting the $\mathbb{Q}$-factorial terminalizations of the nilpotent orbit closure to the transverse slice. Then we can apply a formula of Borho–MacPherson to compute the number of projective lines in the exceptional fiber and the action of $A(e)$ on the projective lines. Proposition 3.1 summarizes the surface cases.

These two methods, as summarized in Propositions 3.1 and 3.3, are sufficient to handle all the cases in the main theorem except when μ, χ, τ, or $a_2/\mathfrak{S}_2$ occur. The two cases where χ or τ occur are dealt with separately in §12. The two cases where μ or $a_2/\mathfrak{S}_2$ occur are deferred to subsequent papers.

The determination of the symmetry action is given in §6, and the calculations supporting Propositions 3.1 and 3.3 are given in §7, §8, §9, §10.

1.9. Some consequences

1.9.1. Isolated symplectic singularities coming from nilpotent orbits

Examples of isolated symplectic singularities include $\overline{\mathcal{O}}_{\min}$ and quotient singularities $\mathbb{C}^{2n}/\Gamma$, where $\Gamma \subset \mathrm{Sp}_{2n}(\mathbb{C})$ is a finite subgroup acting freely on $\mathbb{C}^{2n} \setminus \{0\}$. It was established in [4] that an isolated symplectic singularity with smooth projective tangent cone is locally analytically isomorphic to some $\overline{\mathcal{O}}_{\min}$. It turns out that all of the isolated symplectic singularities we obtain, with one exception, are finite quotients of $\overline{\mathcal{O}}_{\min}$ or $\mathbb{C}^{2n}$. It seems very likely that the singularity χ described above is not equivalent to a singularity of this form.

Another byproduct is the discovery of examples of symplectic contractions to an affine variety whose generic positive-dimensional fiber is of type A_2 and with a non-trivial monodromy action. These examples correspond to minimal degenerations with singularity A_2^+. The orbits $\mathcal{O}$ in these cases have closures which admit a generalized Springer resolution. Examples include the even orbits $A_4 + A_2$ in E_7 and $E_8(b_6)$ in E_8. In [58], a symplectic contraction to a projective variety of the same type is constructed. As far as we know, our examples are the first affine examples that have been constructed. This disproves Conjecture 4.2 in [3].

1.9.2. Special pieces

For a special nilpotent orbit $\mathcal{O}$, the *special piece* $\mathcal{P}(\mathcal{O})$ containing $\mathcal{O}$ is the union of all nilpotent orbits $\mathcal{O}' \subset \overline{\mathcal{O}}$ which are not contained in $\overline{\mathcal{O}}_1$ for any special nilpotent orbit $\mathcal{O}_1$ with $\mathcal{O}_1 \subset \overline{\mathcal{O}}$ and $\mathcal{O}_1 \neq \mathcal{O}$. This is a locally-closed subvariety of $\overline{\mathcal{O}}$ and it is rationally smooth (see [41] and the references therein). To explain rational smoothness geometrically, Lusztig conjectured in [41] that every special piece is a finite quotient of a smooth variety. This conjecture is known for classical types by [34], but for exceptional types it is still open.

Each special piece contains a unique minimal orbit under the closure ordering. Motivated by the aforementioned conjecture of Lusztig, we studied the transverse slice of $\mathcal{P}(\mathcal{O})$ to this minimal orbit. We shall prove in [21] the following:

Theorem 1.3. *Consider a special piece $\mathcal{P}(\mathcal{O})$ in any simple Lie algebra. Then a nilpotent Slodowy slice in $\mathcal{P}(\mathcal{O})$ to the minimal orbit in $\mathcal{P}(\mathcal{O})$ is isomorphic to*

$$(\mathfrak{h}_n \oplus \mathfrak{h}_n^*)^k / \mathfrak{S}_{n+1}$$

where $\mathfrak{h}_n$ is the n-dimensional reflection representation of the symmetric group $\mathfrak{S}_{n+1}$ and k and n are uniquely determined integers.

This theorem also includes the Lie algebras of classical types where $n = 1$, but k can be arbitrarily large. In the exceptional types Theorem 1.2 covers the cases where $\mathcal{P}(\mathcal{O})$ consists of two orbits, in which case $n = k = 1$ (that is, the slice is isomorphic to the A_1 simple surface singularity). This leaves only those special pieces containing more than two orbits. Some of these remaining cases can be handled quickly with the same techniques, but others require more difficult calculations.

1.9.3. Normality of nilpotent orbit closures

By work of Kraft and Procesi [33], together with the remaining cases covered in [53], in classical Lie algebras the failure of $\overline{\mathcal{O}}$ to be normal is explained by branching for a minimal degeneration, and then only with two branches and in codimension two. In exceptional Lie algebras, the question of which nilpotent orbit closures are normal has not been completely solved in E_7 or E_8, but in [10, Section 7.8] a list of non-normal nilpotent orbit closures is given, which is expected to be the complete list.

Our analysis sheds some new light on the normality question. The occurrence of m, m', and μ at a minimal degeneration of $\mathcal{O}$ gives a new geometric explanation for why $\overline{\mathcal{O}}$ is not normal. Previously the only geometric explanation for the failure of normality was branching (see [6]) and the appearance of the non-normal singularity in the closure of the $\tilde{A}_1$ orbit in G_2, which was known to be unibranch (see [31]).

We also establish: (1) for many $\overline{\mathcal{O}}$ known to be non-normal that $\overline{\mathcal{O}}$ is normal at points in some minimal degeneration; and (2) for many $\overline{\mathcal{O}}$ that are expected to be normal that $\overline{\mathcal{O}}$ is indeed normal at points in all of its minimal degenerations. So we are able to make a contribution to determining the non-normal locus of $\overline{\mathcal{O}}$. Examples of the above phenomena are given starting in §7.2. Along these same lines, we also note that a consequence of Theorem 1.3 is that the special pieces are normal, a question studied by Achar and Sage in [1].

1.9.4. Duality

An intriguing result from [32] for $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$ is the following: a simple surface singularity of type A_k is always interchanged with a minimal singularity of type a_k under the order-reversing involution on the set of nilpotent orbits in $\mathfrak{g}$ given by transposition of partitions.

This result leads to a generalization in the other Lie algebras, both classical and exceptional, by restricting to the set of special nilpotent orbits, which are reversed under the Lusztig–Spaltenstein involution. For a minimal degeneration of one special orbit to another, in most cases a simple surface singularity is interchanged with a singularity corresponding to the closure of the minimal *special* nilpotent orbit of dual type. There are a number of complicating factors outside of $\mathfrak{sl}_n(\mathbb{C})$, related to Lusztig's canonical quotient and the existence of multiple branches. The duality can also be formulated as one from special orbits in $\mathfrak{g}$ to those in ${}^L\mathfrak{g}$, the more natural setting for Lusztig–Spaltenstein duality.

Numerical evidence for such a duality was discovered by Lusztig in the classical groups using the tables in [33]. The duality is already hinted at by Slodowy's result for the regular nilpotent orbit in §1.5.2, which requires passing from $\mathfrak{g}$ to ${}^L\mathfrak{g}$. In a subsequent article [20] we will give a more complete account of the phenomenon of duality for degenerations between special orbits.

1.10. Notation

G will be a connected, simple algebraic group of adjoint type over the complex numbers $\mathbb{C}$ with Lie algebra $\mathfrak{g}$, and $\mathcal{O}$ and $\mathcal{O}'$ will be nilpotent $\operatorname{Ad} G$-orbits in $\mathfrak{g}$. We use the notation in [12, p. 401-407] to refer to nilpotent orbits. For $x \in \mathfrak{g}$, $\mathcal{O}_x$ refers to the orbit $\operatorname{Ad} G(x)$, also written $G \cdot x$. For $x \in G$ or $\mathfrak{g}$ we denote by G^x (resp. $\mathfrak{g}^x$) the centralizer of x in G (resp. $\mathfrak{g}$). Similar notation applies to other algebraic groups which arise, including as subgroups of G. For a subalgebra $\mathfrak{z} \subset \mathfrak{g}$, we denote by $C(\mathfrak{z})$ its centralizer in G and $\mathfrak{c}(\mathfrak{z})$ its centralizer in $\mathfrak{g}$.

Generally, e is a nilpotent element in an $\mathfrak{sl}_2(\mathbb{C})$-subalgebra $\mathfrak{s}$ with standard basis e, h, f. If $e_0 \in \mathfrak{c}(\mathfrak{s})$ is a nilpotent element, we use $\mathfrak{s}_0$ for an $\mathfrak{sl}_2(\mathbb{C})$-subalgebra in $\mathfrak{c}(\mathfrak{s})$ with standard basis e_0, h_0, f_0. Usually $\mathcal{O}'$ is a nilpotent orbit in $\overline{\mathcal{O}}$ with $\mathcal{O}' \neq \mathcal{O}$ and $e \in \mathcal{O}'$. We write $(\mathcal{O}, \mathcal{O}')$ for such a pair of nilpotent orbits. Often, but not always, $\mathcal{O}'$ is a minimal degeneration of $\mathcal{O}$. The nilpotent Slodowy slice $\overline{\mathcal{O}} \cap (e + \mathfrak{g}^f)$ is denoted $\mathcal{S}_{\mathcal{O},e}$.

The field of fractions of an integral domain A will be denoted $\operatorname{Frac}(A)$. The symmetric group on n letters is denoted $\mathfrak{S}_n$. Where we refer to explicit elements of $\mathfrak{g}$, we use the structure constants in GAP [56].

2. Transverse slices

2.1. Smooth equivalence

To study singularities it is useful to introduce the notion of smooth equivalence. Given two varieties X and Y and two points $x \in X$ and $y \in Y$, the singularity of X at x is *smoothly equivalent* to the singularity of Y at y if there exists a variety Z, a point $z \in Z$ and morphisms

$$\varphi : Z \to X \text{ and } \psi : Z \to Y$$

which are smooth at z and such that $\varphi(z) = x$ and $\psi(z) = y$ (see [25, 1.7]). This defines an equivalence relation on pointed varieties (X, x) and the equivalence class of (X, x) will be denoted $\operatorname{Sing}(X, x)$. As in [32, §2.1], two singularities (X, x) and (Y, y) with $\dim Y = \dim X + r$ are equivalent if and only if $(X \times \mathbb{A}^r, (x, 0))$ is locally analytically isomorphic to (Y, y).

Let $\mathcal{O}'$ and $\mathcal{O}$ be nilpotent orbits in $\mathfrak{g}$ with $\mathcal{O}' \subset \overline{\mathcal{O}}$. Let $e \in \mathcal{O}'$. The local geometry of $\overline{\mathcal{O}}$ at e is captured by the equivalence class of $(\overline{\mathcal{O}}, e)$ under smooth equivalence. The equivalence class of the singularity $(\overline{\mathcal{O}}, e)$ will be denoted $\operatorname{Sing}(\mathcal{O}, \mathcal{O}')$ since the equivalence class is independent of the choice of element in $\mathcal{O}' = \mathcal{O}_e$.

2.2. Transverse slices

The main tool in studying $\operatorname{Sing}(\mathcal{O}, \mathcal{O}')$ is the transverse slice. Both [50, III.5.1] and [33, §12] are references for what follows.

Let X be a variety on which G acts, and let $x \in X$. A *transverse slice* in X to $G \cdot x$ at x is a locally closed subvariety S of X with $x \in S$ such that the morphism

$$G \times S \to X, (g, s) \mapsto g \cdot s$$

is smooth at $(1, x)$ and such that the dimension of S is minimal subject to these requirements. It is immediate that $\operatorname{Sing}(X, x) = \operatorname{Sing}(S, x)$. If X is a vector space then it is easy to construct such a transverse slice as $x + \mathfrak{u}$ where $\mathfrak{u}$ is a vector space complement to $T_x(G \cdot x) = [\mathfrak{g}, x]$ in X. More generally, this also suffices to construct a transverse slice to a G-stable subvariety $Y \subset X$ containing x by taking the intersection $(x + \mathfrak{u}) \cap Y$ [50, III.5.1, Lemma 2]. In such a case $\operatorname{codim}_Y(G \cdot x) = \dim_x((x + \mathfrak{u}) \cap Y)$.

These observations are especially helpful for nilpotent orbits in the adjoint representation, where there is a natural choice of transverse slice. As before, pick $e \in \mathcal{O}'$. Then there exists $h, f \in \mathfrak{g}$ so that $\{e, h, f\} \subset \mathfrak{g}$ is an $\mathfrak{sl}_2$-triple. Then by the representation theory of $\mathfrak{sl}_2(\mathbb{C})$, we have $[e, \mathfrak{g}] \oplus \mathfrak{g}^f = \mathfrak{g}$. The affine space

$$\mathcal{S}_e = e + \mathfrak{g}^f$$

is a transverse slice of $\mathfrak{g}$ at e, called the *Slodowy slice*. The variety

$$\mathcal{S}_{\mathcal{O},e} := \mathcal{S}_e \cap \overline{\mathcal{O}}$$

is then a transverse slice of $\overline{\mathcal{O}}$ to $\mathcal{O}'$ at the point e, which we call a nilpotent Slodowy slice. In this setting

$$\operatorname{codim}_{\overline{\mathcal{O}}}(\mathcal{O}') = \dim \mathcal{S}_{\mathcal{O},e}. \tag{2.1}$$

Since any two $\mathfrak{sl}_2$-triples for e are conjugate by an element of G^e, the isomorphism type of $\mathcal{S}_{\mathcal{O},e}$ is independent of the choice of $\mathfrak{sl}_2$-triple. Moreover, the isomorphism type of $\mathcal{S}_{\mathcal{O},e}$ is independent of the choice of $e \in \mathcal{O}'$. By focusing on $\mathcal{S}_{\mathcal{O},e}$ we reduce the study of $\operatorname{Sing}(\mathcal{O}, \mathcal{O}')$ to the study of the singularity of $\mathcal{S}_{\mathcal{O},e}$ at e. In fact, most of our results will be concerned with determining the isomorphism type of $\mathcal{S}_{\mathcal{O},e}$.

2.3. Group actions on $\mathcal{S}_{\mathcal{O},e}$

An important feature of the transverse slice $\mathcal{S}_{\mathcal{O},e}$ is that it carries the action of two commuting algebraic groups, which both fix e. Let $\mathfrak{s}$ be the subalgebra spanned by $\{e, h, f\}$ and $C(\mathfrak{s})$ the centralizer of $\mathfrak{s}$ in G. Then $C(\mathfrak{s})$ is a maximal reductive subgroup of G^e and $C(\mathfrak{s})$ acts on $\mathcal{S}_{\mathcal{O},e}$, fixing e.

The second group which acts is $\mathbb{C}^*$. Since $[h, f] = -2f$, $\operatorname{ad} h$ preserves the subspace $\mathfrak{g}^f$ and by $\mathfrak{sl}_2$-theory all of its eigenvalues are nonpositive integers. Set

$$\mathfrak{g}^f(i) = \{x \in \mathfrak{g}^f : [h, x] = ix\}$$

for $i \leq 0$. The special case $\mathfrak{g}^f(0)$ is simply $\mathfrak{c}(\mathfrak{s})$, the centralizer of $\mathfrak{s}$ in $\mathfrak{g}$, which coincides with $\operatorname{Lie}(C(\mathfrak{s}))$.

Define $\phi : SL_2(\mathbb{C}) \to G$ such that the image of $d\phi$ is equal to $\mathfrak{s}$, with $d\phi \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) = e$ and $d\phi \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = h$. Set $\gamma(t) = \phi \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix}$ for $t \in \mathbb{C}^*$. On the one hand, $\operatorname{Ad}\gamma(t)$ preserves $\overline{\mathcal{O}}$ and so does the scalar action of $\mathbb{C}^*$ on $\mathfrak{g}$ since $\overline{\mathcal{O}}$ is conical in $\mathfrak{g}$. On the other hand, for $x_i \in \mathfrak{g}^f(-i)$ and $t \in \mathbb{C}^*$,

$$\operatorname{Ad}\gamma(t)(e + x_0 + \ldots + x_m) = t^{-2}e + x_0 + tx_1 + \ldots + t^m x_m.$$

Composing this action with the scalar action of t^2 on $\mathfrak{g}$, gives an action of $t \in \mathbb{C}^*$ on $e + \mathfrak{g}^f$ by

$$t \cdot (e + x_0 + x_1 + \ldots) = e + t^2 x_0 + t^3 x_1 + \ldots, \tag{2.2}$$

which preserves $\mathcal{S}_{\mathcal{O},e} = \overline{\mathcal{O}} \cap \mathcal{S}_e$. The $\mathbb{C}^*$-action fixes e and commutes with the action of $C(\mathfrak{s})$, since $C(\mathfrak{s})$ commutes with $\operatorname{ad} h$ and so preserves each $\mathfrak{g}^f(i)$. Thus $C(\mathfrak{s}) \times \mathbb{C}^*$ acts on $\mathcal{S}_{\mathcal{O},e}$.

2.4. Branching and component group action

The $C(\mathfrak{s}) \times \mathbb{C}^*$-action on $\mathcal{S}_{\mathcal{O},e}$ has consequences for the irreducible components of $\mathcal{S}_{\mathcal{O},e}$.

An irreducible variety X is *unibranch* at x if the normalization $\pi : (\tilde{X}, \tilde{x}) \to (X, x)$ of (X, x) is locally a homeomorphism at x. Since the $\mathbb{C}^*$-action on $\mathcal{S}_{\mathcal{O},e}$ in (2.2) is attracting to e, $\mathcal{S}_{\mathcal{O},e}$ is connected and its irreducible components are unibranch at e. Consequently the number of irreducible components of $\mathcal{S}_{\mathcal{O},e}$ is equal to the number of branches of $\overline{\mathcal{O}}$ at e. The latter can be determined from the tables of Green functions in [6,49], as discussed in [6, 5(E)–(F)].

The identity component $C(\mathfrak{s})^\circ$ of $C(\mathfrak{s})$, being connected, preserves each irreducible component of $\mathcal{S}_{\mathcal{O},e}$, hence there is a natural action of $C(\mathfrak{s})/C(\mathfrak{s})^\circ$ on the irreducible components of $\mathcal{S}_{\mathcal{O},e}$. The finite group $C(\mathfrak{s})/C(\mathfrak{s})^\circ$ is isomorphic to the component group $A(e) := G^e/(G^e)^\circ$ of G^e via $C(\mathfrak{s}) \hookrightarrow G^e \twoheadrightarrow G^e/(G^e)^\circ$. Since any two $\mathfrak{sl}_2$-triples containing e are conjugate by an element of $(G^e)^\circ$, this gives a well-defined action of $A(e)$ on the set of irreducible components of $\mathcal{S}_{\mathcal{O},e}$. Moreover, as noted in [6], the permutation representation of $A(e)$ on the branches of $\overline{\mathcal{O}}$ at e, and hence on the irreducible components of $\mathcal{S}_{\mathcal{O},e}$, can be computed. For a minimal degeneration, the situation is particularly nice. We observe by looking at the tables in [6,49] that

Proposition 2.1. *When $\mathcal{O}'$ is a minimal degeneration of $\mathcal{O}$ in an exceptional Lie algebra, the action of $A(e)$ on the set of irreducible components of $\mathcal{S}_{\mathcal{O},e}$ is transitive. In particular, the irreducible components of $S_{\mathcal{O},e}$ are mutually isomorphic.*

The proposition also holds in the classical types, which can be deduced using the results in [33]. In §6.2 we will discuss the full symmetry action on $\mathcal{S}_{\mathcal{O},e}$ induced from $A(e)$.

3. Statement of the key propositions

In this section we state the key propositions which underlie Theorem 1.2. The propositions give more precise information about $\mathcal{S}_{\mathcal{O},e}$. Throughout this section $\mathcal{O}'$ is always a minimal degeneration of $\mathcal{O}$.

3.1. Surface cases

The case of a minimal degeneration of codimension two is summarized by the following proposition.

Proposition 3.1. *Let $\mathcal{O}'$ be a minimal degeneration of $\mathcal{O}$ of codimension 2. Then there exists a finite subgroup $\Gamma \subset SL_2(\mathbb{C})$ such that the normalization $\widetilde{\mathcal{S}}_{\mathcal{O},e}$ of $\mathcal{S}_{\mathcal{O},e}$ is isomorphic to a disjoint union of k copies of X where $X = \mathbb{C}^2/\Gamma$.*

This is proved in §5 where techniques for determining Γ and k are given. For most cases in Proposition 3.1 we can show that the irreducible components of $\mathcal{S}_{\mathcal{O},e}$ are normal either by knowing that $\overline{\mathcal{O}}$ is normal, by using Lemma 4.1 to move to a smaller subalgebra where the slice is known to be normal, or by doing an explicit computation using Lemma 4.3. In the surface case, we found only two ways that an irreducible component of $\mathcal{S}_{\mathcal{O},e}$ fails to be normal:

- When $\Gamma = 1$, we show below that $\mathcal{S}_{\mathcal{O},e} \cong m$ (§1.8.4). This happens for several different minimal degenerations.
- When $(\mathcal{O}, \mathcal{O}') = (D_7(a_1), E_8(b_6))$, we have $\Gamma \cong \mathbb{Z}/4$, but $\mathcal{S}_{\mathcal{O},e}$ is not normal. Instead, $\mathcal{S}_{\mathcal{O},e}$ is smoothly equivalent to μ (§1.8.4).

A handful of cases in E_8 are determined only up to normalization (see §10.2).

Remark 3.2. The isomorphism in Proposition 3.1 is compatible with the natural $\mathbb{C}^*$-actions on both sides. On $\widetilde{\mathcal{S}}_{\mathcal{O},e}$ the $\mathbb{C}^*$-action is the one induced from §2.3; on $\mathbb{C}^2/\Gamma$ it is the one coming from the central torus in $\mathrm{GL}_2(\mathbb{C})$. This follows from Proposition 5.2.

3.2. *Cases with a dense $C(\mathfrak{s})$-orbit*

Next we consider all the cases of codimension 4 or greater other than the three normal cases in §1.8.4. Together with the surface cases where $|\Gamma| = 1$ or 2, these $\mathcal{S}_{\mathcal{O},e}$ have a dense orbit for the action of $C(\mathfrak{s})$. More is true, their irreducible components are examples of S-varieties [57].

3.2.1. *S-varieties*

Let $\{\Lambda_1, \dots \Lambda_r\}$ be a set of dominant weights for a maximal torus in a fixed Borel subgroup of $C(\mathfrak{s})^\circ$. It will also be convenient to view the Λ_i's as weights for the Lie algebra of this maximal torus. Let $V(\Lambda_i)$ be the irreducible representation of $C(\mathfrak{s})^\circ$ of highest weight Λ_i and $v_i \in V(\Lambda_i)$ a non-zero highest weight vector. Then the S-variety $X(\Lambda_1, \dots \Lambda_r)$ is defined to be the closure in $V := V(\Lambda_1) \oplus \cdots \oplus V(\Lambda_r)$ of the $C(\mathfrak{s})^\circ$-orbit through $v := (v_1, \dots, v_r)$. In [57] it is shown that S-varieties are exactly the varieties which carry a dense $C(\mathfrak{s})^\circ$-orbit and every point in this dense orbit has stabilizer containing a maximal unipotent subgroup of $C(\mathfrak{s})^\circ$.

In all the situations encountered in this paper, we find $\Lambda_i = b_i\lambda$ for each i, where $b_i \in \mathbb{N}$ and λ is a fixed dominant weight. In such cases Theorems 6 (and its Corollary), 8 and 10 from [57] reduce to the following, respectively: (1) the complement of $C(\mathfrak{s})^\circ \cdot v$ in X is the origin in V; (2) the determining invariant of the $C(\mathfrak{s})^\circ$-isomorphism type of $X := X(b_1\lambda, \dots, b_r\lambda)$ is the monoid in $\mathbb{N}$ generated by $b_1, \dots, b_r$; and (3) the normalization of X is $X(d\lambda)$, where d is the greatest common divisor of $b_1, \dots, b_r$. More is true in (2): if $b_1, \dots, b_j$ generate the same monoid as $b_1, \dots, b_r$ for $j < r$, then the natural projection from X to $X(b_1\lambda, \dots, b_j\lambda)$ is an isomorphism. We will assume that $b_1 \leq b_2 \leq \cdots \leq$

b_r. If V factors through $Z \subset C(\mathfrak{s})^\circ$ with Lie algebra $\mathfrak{sl}_2(\mathbb{C})$, then we sometimes write $X(b_1, b_2, \dots)$ instead of $X(b_1\lambda, b_2\lambda, \dots)$ where λ is the fundamental weight for $\mathfrak{sl}_2(\mathbb{C})$.

3.2.2. Let $\pi_0 : \mathcal{S}_{\mathcal{O},e} \to \mathfrak{c}(\mathfrak{s})$ be the restriction of the $C(\mathfrak{s}) \times \mathbb{C}^*$-equivariant linear projection of $\mathcal{S}_e$ onto $\mathfrak{c}(\mathfrak{s}) = \mathfrak{g}^f(0)$. Let $\pi_{0,1} : \mathcal{S}_{\mathcal{O},e} \to \mathfrak{g}^f(0) \oplus \mathfrak{g}^f(-1)$ be the restriction of the $C(\mathfrak{s}) \times \mathbb{C}^*$-equivariant linear projection of $\mathcal{S}_e$ onto $\mathfrak{g}^f(0) \oplus \mathfrak{g}^f(-1)$. Recall $v \in \mathfrak{c}(\mathfrak{s})$ belongs to a minimal nilpotent $C(\mathfrak{s})^\circ$-orbit if and only if v is a highest weight vector (relative to a Borel subgroup of $C(\mathfrak{s})$) in a simple summand of $\mathfrak{c}(\mathfrak{s})$. The proof of the next proposition is given in §4.5.

Proposition 3.3. *Let $\mathcal{O}'$ be a minimal degeneration of $\mathcal{O}$ of codimension at least four (other than the three normal cases in §1.8.4) or of codimension two with $|\Gamma| = 1$ or 2.*

Then there exists $J = \{i_1, \dots, i_r\} \subset \mathbb{N}$ so that for each $i \in J$ there exists a highest weight vector $x_i \in \mathfrak{g}^f(-i)$ for the action of $C(\mathfrak{s})^\circ$, and there exists $x_0 \in \mathfrak{c}(\mathfrak{s})$ minimal nilpotent, such that

- $e + x_0 + \sum_{i \in J} x_i \in \mathcal{S}_{\mathcal{O},e}$,
- *if the weight of x_0 is given by λ, then the weight of x_i equals $(\frac{i}{2} + 1)\lambda$,*
- *one of the irreducible components of $\mathcal{S}_{\mathcal{O},e}$ is $e + X$, where X is the corresponding S-variety*

$$X := X\left(\lambda, (\tfrac{i_1}{2} + 1)\lambda, (\tfrac{i_2}{2} + 1)\lambda, \dots, (\tfrac{i_r}{2} + 1)\lambda\right) \subset \mathfrak{g}^f$$

  for $C(\mathfrak{s})^\circ$, through the vector $v := x_0 + \sum_{i \in J} x_i$,
- *the irreducible components of $\mathcal{S}_{\mathcal{O},e}$ are permuted transitively by $C(\mathfrak{s})$, whence $\mathcal{S}_{\mathcal{O},e} = \overline{C(\mathfrak{s}) \cdot (e + v)}$.*

Moreover, there are two cases:

1. *All $i \in J$ are even. Then π_0 induces an isomorphism of each irreducible component of $\mathcal{S}_{\mathcal{O},e}$ with $X(\lambda)$. Furthermore, $X(\lambda)$ is a minimal singularity, being isomorphic to the closure of the $C(\mathfrak{s})^\circ$-orbit through the minimal nilpotent element $x_0 \in V(\lambda) \subset \mathfrak{c}(\mathfrak{s})$. Hence each component is a normal variety.*
2. *We have $1 \in J$. This case occurs only if $\mathfrak{c}(\mathfrak{s})$ contains a simple factor $\mathfrak{z}$ of type $\mathfrak{sl}_2(\mathbb{C})$ or $\mathfrak{sp}_4(\mathbb{C})$ and $\mathfrak{z} = V(\lambda)$. Note that $\lambda = 2\omega$ where $V(\omega)$ is the defining representation of $\mathfrak{z}$. Then $\pi_{0,1}$ gives an isomorphism $\mathcal{S}_{\mathcal{O},e} \cong X(\lambda, \frac{3}{2}\lambda) = X(2\omega, 3\omega)$. In the $\mathfrak{z} = \mathfrak{sl}_2(\mathbb{C})$ case, $\mathcal{S}_{\mathcal{O},e} \cong m$ and in the $\mathfrak{z} = \mathfrak{sp}_4(\mathbb{C})$ case, $\mathcal{S}_{\mathcal{O},e} \cong m'$. In both cases $\mathcal{S}_{\mathcal{O},e}$ is irreducible and non-normal.*

Remark 3.4. In case (1) of Proposition 3.3, when the codimension is at least four, we find that $J = \emptyset$ except for the two minimal degenerations ending in the orbit $D_4(a_1) + A_2$ in E_8, where $J = \{2\}$ (see §10.1.2). On the other hand, when the codimension is two in case (1), then $\mathcal{S}_{\mathcal{O},e} \cong kA_1$. If $k > 1$, then always $J = \emptyset$. If $k = 1$, then J can be either $\emptyset, \{2\}$, or $\{2, 4\}$.

In case (2) of Proposition 3.3, the possibilities for J that arise are $\{1\}, \{1, 2\}$, or $\{1, 2, 3\}$; however, the singularity m' only occurs for $(A_3 + 2A_1, 2A_2 + 2A_1)$ in E_8, where $J = \{1\}$.

Remark 3.5. From the previous remark, we see that $\mathcal{S}_{\mathcal{O},e}$ is not irreducible only when $J = \emptyset$. In that case, each irreducible component of $\mathcal{S}_{\mathcal{O},e}$ corresponds to the minimal orbit closure in a unique simple summand of $\mathfrak{c}(\mathfrak{s})$. The direct sum of these summands is an irreducible representation for $C(\mathfrak{s})$ and the summands are permuted transitively by $C(\mathfrak{s})/C(\mathfrak{s})^\circ$ as expected from Proposition 2.1. The first non-irreducible example covered by the Proposition is $(C_3(a_1), B_2)$, see Table 1.

Remark 3.6. In each case of Proposition 3.3, the map π_0 is surjective onto the closure of a minimal nilpotent $C(\mathfrak{s})$-orbit in $\mathfrak{c}(\mathfrak{s})$ (namely, the one through x_0). It is not true, however, that every such minimal nilpotent orbit will arise in this way. For example, when $e \in \mathcal{O}' = 3A_1$ in E_6, the centralizer $\mathfrak{c}(\mathfrak{s})$ has type $A_2 + A_1$. If e_0 belongs to the minimal orbit in the summand of type A_2 and e_0' belongs to the minimal orbit in the summand of type A_1, then $\mathcal{O}' \subset \overline{\mathcal{O}}_{e+e_o'} \subset \overline{\mathcal{O}}_{e+e_o}$ and $\mathcal{O}'$ is a minimal degeneration of $\mathcal{O}_{e+e_o'}$ and of no other orbit. So the minimal orbit in the A_2 summand is not in the image of π_0 for any minimal degeneration ending in $\mathcal{O}'$.

4. Tools for establishing Proposition 3.3

In this section we give some tools for identifying $\mathcal{S}_{\mathcal{O},e}$, which can often be applied even when the degeneration is not minimal. Therefore we do not in general assume degenerations are minimal in this section. At the end of the section in §4.5, we prove Proposition 3.3. We keep the notation that $\mathcal{O}$ and $\mathcal{O}'$ are nilpotent G-orbits in $\mathfrak{g}$ with $\mathcal{O}' \subset \overline{\mathcal{O}}$ and $e \in \mathcal{O}'$. Unless specified otherwise, $\mathcal{O}'$ is not assumed to be a minimal degeneration of $\mathcal{O}$.

4.1. Some reduction lemmas

The first two lemmas give frameworks to relate $\mathcal{S}_{\mathcal{O},e}$ to a variety attached to a proper subalgebra of $\mathfrak{g}$. Both lemmas are variants of [33, Cor 13.3].

4.1.1. Passing to a reductive subalgebra

Let $M \subset G$ be a closed reductive subgroup and $\mathfrak{m} = \mathrm{Lie}(M)$. Assume that $e \in \mathfrak{m} \cap \mathcal{O}'$. Let $x \in \mathfrak{m} \cap \mathcal{O}$ and suppose that $M \cdot e \subset \overline{M \cdot x}$. Since $\mathfrak{m}$ is reductive, we may assume the $\mathfrak{sl}_2$-subalgebra $\mathfrak{s}$ containing e lies in $\mathfrak{m}$. Let $\mathcal{S}_{M\cdot x,e}$ be the nilpotent Slodowy slice $\overline{M \cdot x} \cap (e + \mathfrak{m}^f)$ in $\mathfrak{m}$. Of course, $\mathcal{S}_{M\cdot x,e} \subset \mathcal{S}_{\mathcal{O},e}$.

Lemma 4.1. *Suppose that* $\operatorname{codim}_{\overline{M\cdot x}}(M\cdot e) = \operatorname{codim}_{\overline{\mathcal{O}}}(\mathcal{O}')$ *and* $\mathcal{S}_{M\cdot x,e}$ *is equidimensional. Then* $\mathcal{S}_{M\cdot x,e}$ *is a union of irreducible components of* $\mathcal{S}_{\mathcal{O},e}$*. Moreover if* $\overline{\mathcal{O}}$ *is unibranch at* e *or if the number of branches of* $\overline{\mathcal{O}}$ *at* e *equals the number of branches of* $\overline{M\cdot x}$ *at* e*, then* $\mathcal{S}_{M\cdot x,e} = \mathcal{S}_{\mathcal{O},e}$*.*

Proof. The first conclusion follows from (2.1) and the fact that $\mathcal{S}_{M\cdot x,e} \subset \mathcal{S}_{\mathcal{O},e}$, together with the hypotheses of the lemma. The second conclusion follows from the fact that the irreducible components of $\mathcal{S}_{\mathcal{O},e}$ and $\mathcal{S}_{M\cdot x,e}$ are unibranch (§2.4). □

Example 4.2. Let $\mathfrak{g}$ be of type E_8, $\mathfrak{m}$ a Levi subalgebra of $\mathfrak{g}$ of type E_6, and $(\mathcal{O},\mathcal{O}')$ of type $(D_5, E_6(a_3))$. Since $\overline{\mathcal{O}} \cap \mathfrak{m}$ is known to be normal in E_6, we are able to conclude (see §8.2) that $\mathcal{S}_{M\cdot x,e}$ is geometrically a simple surface singularity of type D_4. Since $\overline{\mathcal{O}}$ is unibranch at e and the codimension hypothesis of the lemma holds (both sides equal two), it follows that $\mathcal{S}_{\mathcal{O},e} = \mathcal{S}_{M\cdot x,e}$ and so $\mathcal{S}_{\mathcal{O},e}$ has the same singularity. Here, $\overline{\mathcal{O}}$ is conjectured to be normal, but this is still an open question.

Lemma 4.1 is needed for the cases in Table 12, but we also use it to check results that can be obtained by other methods.

4.1.2. The case of a $C(\mathfrak{s})$*-orbit of maximal possible dimension*

Every $x \in \mathcal{S}_{\mathcal{O},e}$ can be written as

$$x = e + x_0 + \sum_{i>0} x_i \tag{4.1}$$

with $x_i \in \mathfrak{g}^f(-i)$. Set

$$x_+ = \sum_{i\geq 0} x_i \quad \text{and} \quad X = \overline{C(\mathfrak{s})\cdot x_+}.$$

Since $C(\mathfrak{s})$ fixes e, we have $C(\mathfrak{s})\cdot x = e + C(\mathfrak{s})\cdot x_+$ and thus $\overline{C(\mathfrak{s})\cdot x} = e + X$. Also, $e + X \cong X$ as $C(\mathfrak{s})$-varieties. By construction X is equidimensional, with irreducible components permuted transitively by $C(\mathfrak{s})/C(\mathfrak{s})^\circ$. Of course $\overline{C(\mathfrak{s})\cdot x} \subset \mathcal{S}_{\mathcal{O},e}$. Hence the same argument as in the previous lemma gives

Lemma 4.3. *Let* $x \in \mathcal{S}_{\mathcal{O},e}$ *be written as in* (4.1)*. Suppose that* $\dim(C(\mathfrak{s})\cdot x) = \operatorname{codim}_{\overline{\mathcal{O}}}(\mathcal{O}')$*. Then* $e + X$ *is a union of irreducible components of* $\mathcal{S}_{\mathcal{O},e}$*. Moreover, if the number of branches of* $\overline{\mathcal{O}}$ *at* e *equals the number of irreducible components of* X*, then* $e + X = \mathcal{S}_{\mathcal{O},e}$*.*

The previous lemma allows us to study $\mathcal{S}_{\mathcal{O},e}$ by studying $\overline{C(\mathfrak{s})\cdot x_+}$, which is often easier to understand. Of course any $x \in \mathcal{S}_{\mathcal{O},e}$ satisfying the dimension hypothesis in the lemma must lie in $\mathcal{S}_{\mathcal{O},e} \cap \mathcal{O}$. Furthermore we have

Lemma 4.4. *Let $x \in \mathcal{S}_{\mathcal{O},e}$ be written as in (4.1). Then* $\dim(C(\mathfrak{s}) \cdot x) = \operatorname{codim}_{\overline{\mathcal{O}}}(\mathcal{O}')$ *if and only if*

$$x_0 \text{ is nilpotent and } \dim(C(\mathfrak{s}) \cdot x_0) = \operatorname{codim}_{\overline{\mathcal{O}}}(\mathcal{O}'). \tag{4.2}$$

Proof. We always have $\dim(C(\mathfrak{s}) \cdot x_0) \leq \dim(C(\mathfrak{s}) \cdot x) \leq \dim \mathcal{S}_{\mathcal{O},e} = \operatorname{codim}_{\overline{\mathcal{O}}}(\mathcal{O}')$, so the reverse direction is straightforward and does not use the hypothesis that x_0 is nilpotent.

For the forward direction, we are given that $\dim(C(\mathfrak{s}) \cdot x) = \dim \mathcal{S}_{\mathcal{O},e}$. Consider the $\mathbb{C}^*$-action (§2.3) on $\mathcal{S}_{\mathcal{O},e}$. We have $C(\mathfrak{s}) \cdot x \subset (C(\mathfrak{s}) \times \mathbb{C}^*) \cdot x \subset \mathcal{S}_{\mathcal{O},e}$ and all have the same dimension, so $C(\mathfrak{s}) \cdot x$ is dense in $(C(\mathfrak{s}) \times \mathbb{C}^*) \cdot x$. Therefore for $\lambda \in \mathbb{C}^*$, we have $\lambda \cdot (C(\mathfrak{s}) \cdot x)$ meets $C(\mathfrak{s}) \cdot x$, from which it follows that $\lambda \cdot x \in C(\mathfrak{s}) \cdot x$. So in fact $\mathbb{C}^*$ preserves $C(\mathfrak{s}) \cdot x$. But if $\mathbb{C}^* \cdot x \subset C(\mathfrak{s}) \cdot x$, then $\mathbb{C}^* \cdot x_0 \subset C(\mathfrak{s}) \cdot x_0$. The $\mathbb{C}^*$-action on $\mathfrak{c}(\mathfrak{s})$ is contracting, hence $0 \in \overline{C(\mathfrak{s}) \cdot x_0}$, and x_0 must be nilpotent in $\mathfrak{c}(\mathfrak{s})$.

Next by [22, Corollary 7.2], $\mathcal{S}_{\mathcal{O},e} \cap \mathcal{O}$ is a symplectic subvariety of $\mathcal{O}$, so the symplectic form on $T_x(\mathcal{O})$ remains non-degenerate on restriction to $T_x(\mathcal{S}_{\mathcal{O},e} \cap \mathcal{O})$. As usual, we identify $T_x(\mathcal{O})$ with $[x, \mathfrak{g}]$ and the symplectic form on $T_x(\mathcal{O})$ is then expressed as $\langle [x,u], [x,v] \rangle := \kappa(x, [u,v])$ for $u, v \in \mathfrak{g}$, where κ is the Killing form on $\mathfrak{g}$. But since $C(\mathfrak{s}) \cdot x$ has dimension equal to $\mathcal{S}_{\mathcal{O},e} \cap \mathcal{O}$, we can identify $T_x(\mathcal{S}_{\mathcal{O},e} \cap \mathcal{O})$ with $[x, \mathfrak{c}(\mathfrak{s})]$.

Now suppose $u \in \mathfrak{c}(\mathfrak{s})$ satisfies $[x_0, u] = 0$. Then for any $v \in \mathfrak{c}(\mathfrak{s})$, we have

$$\langle [x,u], [x,v] \rangle = \kappa(x, [u,v]) = \kappa(x_0, [u,v])$$

since $[u,v] \in \mathfrak{c}(\mathfrak{s}) = \mathfrak{g}^f(0)$ pairs nontrivially only with elements in the 0-eigenspace of $\operatorname{ad} h$. But then $\langle [x,u], [x,v] \rangle = \kappa(x_0, [u,v]) = \kappa([x_0,u], v) = 0$. Hence $[x,u]$ is in the kernel of the symplectic form and thus $[x,u] = 0$ by non-degeneracy of the form. This shows that $\mathfrak{c}(\mathfrak{s})^{x_0} \subset \mathfrak{c}(\mathfrak{s})^x$, which forces $\dim(C(\mathfrak{s}) \cdot x) = \dim(C(\mathfrak{s}) \cdot x_0)$, as desired. $\square$

The case where $x_i = 0$ for all $i \geq 1$ in (4.1), and (4.2) holds, is particularly common and is also easier to handle. In that case, $x = e + x_0$ is a sum of two commuting nilpotent elements and $X = \overline{C(\mathfrak{s}) \cdot x_0}$ is the closure of a nilpotent orbit in $\mathfrak{c}(\mathfrak{s})$, which is a union of irreducible components of $\mathcal{S}_{\mathcal{O},e}$. We discuss this case in detail in §4.3. We next prove a lemma useful for when some x_i is nonzero with $i \geq 1$.

4.1.3. Assume $x \in \mathcal{S}_{\mathcal{O},e} \cap \mathcal{O}$, written as in (4.1), satisfies (4.2). The next lemma uses the $\mathbb{C}^*$-action on $\mathcal{S}_{\mathcal{O},e}$ to say more about the x_i's which appear in (4.1). Since x_0 is nilpotent, we can find an $\mathfrak{sl}_2$-subalgebra $\mathfrak{s}_{x_0}$ in $\mathfrak{c}(\mathfrak{s})$ containing x_0, with standard semisimple basis element h_{x_0}.

Lemma 4.5. *Let $x \in \mathcal{O}$ be written as in (4.1) so that (4.2) holds. Then the following are true.*

(1) $[x_0, x_i] = 0$ *for* $i \geq 0$.
(2) $[h_{x_0}, x_i] = (i+2)x_i$ *for* $i \geq 0$.

(3) *If x_0 lies in a minimal nilpotent $C(\mathfrak{s})^\circ$-orbit of $\mathfrak{c}(\mathfrak{s})$, then each nonzero x_i is a highest weight vector for a Borel subgroup B of $C(\mathfrak{s})^\circ$. In particular, X is a union of S-varieties. If x_0 has weight λ relative to a maximal torus of B, then x_i has weight $(\frac{i}{2}+1)\lambda$.*

Proof. (1) By Lemma 4.4 we know that $\dim(C(\mathfrak{s}){\cdot}x_0) = \dim(C(\mathfrak{s}){\cdot}x)$, which is equivalent to $\mathfrak{c}(\mathfrak{s})^{x_0} = \mathfrak{c}(\mathfrak{s})^x$, which is equivalent to $\mathfrak{c}(\mathfrak{s})^{x_0} \subset \mathfrak{c}(\mathfrak{s})^{x_i}$ for $i \geq 0$. Since x_0 commutes with itself, we get $[x_0, x_i] = 0$ for $i \geq 0$.

(2) From the proof of Lemma 4.4, the dimension condition in (4.2) implies that the $\mathbb{C}^*$-action on $\mathcal{S}_{\mathcal{O},e}$ preserves $C(\mathfrak{s}) \cdot x$. Let $\mathcal{C} := C(\mathfrak{s}) \times \mathbb{C}^*$. The equality $\dim(C(\mathfrak{s}) \cdot x_0) = \dim(C(\mathfrak{s}){\cdot}x)$ therefore implies $\dim(\mathcal{C}{\cdot}x_0) = \dim(\mathcal{C}{\cdot}x)$, which means we have the inclusion of identity components of centralizers

$$(\mathcal{C}^{x_0})^\circ \subset (\mathcal{C}^{x_i})^\circ \text{ for all } i. \tag{4.3}$$

Now write $\chi(t)$ for the element $(1, t) \in C(\mathfrak{s}) \times \mathbb{C}^*$. Let $\phi : \mathbb{C}^* \to C(\mathfrak{s}) \times \mathbb{C}^*$ be the cocharacter coming from h_{x_0}. Of course $\phi(\mathbb{C}^*) \subset C(\mathfrak{s})$ commutes with $\chi(\mathbb{C}^*)$. Consider now the action of the element $\chi(t^{-1})\phi(t)$ on x_0 for $t \in \mathbb{C}^*$. We have $\phi(t).x_0 = t^2 x_0$ since $[h_{x_0}, x_0] = 2$ and $\chi(t).x_0 = t^{0+2}x_0$ since $x_0 \in \mathfrak{g}^f(0)$, so the one-dimensional torus $\{\chi(t^{-1})\phi(t) \mid t \in \mathbb{C}^*\}$ fixes x_0 and hence by (4.3), it also fixes each x_i. The result follows from $\phi(t).x_i = \chi(t).x_i = t^{i+2}x_i$ since $x_i \in \mathfrak{g}^f(-i)$. Combining with part (1), we see that each nonzero x_i is a highest weight vector for the Borel subalgebra of $\mathfrak{s}_{x_0}$ spanned by x_0 and h_{x_0}.

(3) Since x_0 lies in a minimal nilpotent $C(\mathfrak{s})$-orbit of $\mathfrak{c}(\mathfrak{s})$, the stabilizer of the line through x_0 is a parabolic subgroup P of $C(\mathfrak{s})$, containing $\phi(\mathbb{C}^*)$. Let B be a Borel subgroup of P containing $\phi(\mathbb{C}^*)$ and let T be a maximal torus in B with $\phi(\mathbb{C}^*) \subset T$. Let U be the unipotent radical of B, which acts trivially on x_0. Now x_0 is a root vector relative to T, so T^{x_0} is codimension one in T. Thus T is generated by T^{x_0} and $\phi(\mathbb{C}^*)$, so each element of B can be written as $ut_0\phi(t)$ for $u \in U$, $t_0 \in T^{x_0}$, and $t \in \mathbb{C}^*$. It follows that the connected subgroup $\{ut_0\phi(t)\chi(t^{-1})\}$ of $C(\mathfrak{s}) \times \mathbb{C}^*$ centralizes x_0 and hence centralizes each x_i by (4.3). Hence B (and indeed also P) preserves the line through x_i since $\chi(\mathbb{C}^*)$ does; in other words, x_i is a highest weight vector relative to (B, T). Moreover, the weight of x_i must equal $r\lambda$, with r rational, since x_i and x_0 are both acted upon trivially by T^{x_0}. On the one hand, $\phi(t).x_i = t^{i+2}x_i$ from part (2), and on the other hand, $\phi(t).x_i = (r\lambda)(\phi(t))x_i = (\lambda(\phi(t)))^r x_i = (t^2)^r x_i$. We conclude that $r = (\frac{i}{2}+1)$. $\square$

Another way to phrase part (2) of Lemma 4.5 is that x must be in the 2-eigenspace for $\operatorname{ad}(h+h_{x_0})$. This fact can be used to help locate an x written as in (4.1) so that (4.2) holds, see Lemma 4.10.

Part (3) of Lemma 4.5 will be used in the proof of Proposition 3.3 in §4.5, since for each minimal degeneration covered by the proposition, it turns out that there exists $x \in \mathcal{S}_{\mathcal{O},e}$ satisfying (4.2) with x_0 in a minimal nilpotent orbit of $\mathfrak{c}(\mathfrak{s})$.

4.1.4. When the degeneration is codimension two and $\mathfrak{c}(\mathfrak{s})$ has a simple summand isomorphic to $\mathfrak{sl}_2(\mathbb{C})$, the next lemma gives a criterion which guarantees that $\mathcal{S}_{\mathcal{O},e}$ has a dense $C(\mathfrak{s})$-orbit, allowing us to apply the previous lemmas.

Lemma 4.6. *Let $\mathcal{O}'$ be a degeneration of $\mathcal{O}$ of codimension two (necessarily minimal). Suppose $C(\mathfrak{s})^\circ$ contains a simple factor Z with Lie algebra $\mathfrak{z} \cong \mathfrak{sl}_2(\mathbb{C})$. Let $C(\mathfrak{z})$ be the centralizer of $\mathfrak{z}$ in G, with Lie algebra $\mathfrak{c}(\mathfrak{z})$.*

(1) *If Z acts non-trivially on $\mathcal{S}_{\mathcal{O},e}$, then there exists $x \in \mathcal{S}_{\mathcal{O},e} \cap \mathcal{O}$, written as in (4.1), satisfying (4.2) with $x_0 \in \mathfrak{z}$.*
(2) *If $x \notin \mathfrak{c}(\mathfrak{z})$, or if $x \in \mathfrak{c}(\mathfrak{z})$ but $e \notin \overline{C(\mathfrak{z}) \cdot x}$, then Z acts non-trivially on $\mathcal{S}_{\mathcal{O},e}$.*

Proof. (1) Consider the decomposition of $\mathfrak{g}^f$ under Z. Since the action of Z is non-trivial on $\mathcal{S}_{\mathcal{O},e}$, there exists $x = e + x_+ \in \mathcal{S}_{\mathcal{O},e}$ and a non-trivial irreducible Z-submodule V of some $\mathfrak{g}^f(-j)$ so that x_+ has nonzero image x_V under the $C(\mathfrak{s}) \times \mathbb{C}^*$-equivariant projection π_V of $\mathfrak{g}^f$ onto V. The equivariance of π_V ensures that the image $\pi_V(\mathcal{S}_{\mathcal{O},e})$ is conical for the scalar action on V. Let Y be the projectivization of $\pi_V(\mathcal{S}_{\mathcal{O},e})$, which is in $\mathbb{P}(V)$. Since $\dim \mathcal{S}_{\mathcal{O},e} = 2$ by hypothesis, we know $\dim Y \leq 1$. Now Y has a closed Z-orbit, necessarily irreducible. If Y contains a closed orbit which is a point, then Z preserves a line in V, contradicting that V is non-trivial irreducible. Thus, since $\dim Y \leq 1$, each irreducible component of Y must be a one-dimensional (closed) Z-orbit. The stabilizer of any point in Y is therefore a proper parabolic subgroup in Z, namely a Borel subgroup B. Since then $\dim(Z \cdot x_V) = 2 = \dim \mathcal{S}_{\mathcal{O},e}$, we have also $\dim(Z \cdot x) = \dim \mathcal{S}_{\mathcal{O},e}$. Then Lemma 4.4 ensures that x satisfies (4.2) with $x_0 \in \mathfrak{c}(\mathfrak{s})$, when x is expressed as in (4.1). But then for dimension reasons the other connected simple factors of $C(\mathfrak{s})$ must preserve $Z \cdot x_0$, hence act trivially, which implies that $x_0 \in \mathfrak{z}$.

(2) If Z acts trivially on $\mathcal{S}_{\mathcal{O},e}$, then $\mathcal{S}_{\mathcal{O},e} \subset \mathfrak{c}(\mathfrak{z})$. Hence $x \in \mathfrak{c}(\mathfrak{z})$. Since $\mathfrak{s} \subset \mathfrak{c}(\mathfrak{z})$, it follows that the $\mathbb{C}^*$-action on $\mathcal{S}_{\mathcal{O},e}$ preserves nilpotent $C(\mathfrak{z})$-orbits. Hence $e \in \overline{C(\mathfrak{z}) \cdot x}$. (This part did not use the assumption on the codimension of $\mathcal{O}'$ in $\mathcal{O}$.) $\square$

Corollary 4.7. *Let Z act non-trivially on $\mathcal{S}_{\mathcal{O},e}$. Then there exist $b_i \in \mathbb{N}$ satisfying $2 < b_2 < b_3 < \dots$ so that each irreducible component of $\mathcal{S}_{\mathcal{O},e}$ is an S-variety for Z of the form $X(2, b_2, b_3, \dots)$.*

Consequently, the components of $\mathcal{S}_{\mathcal{O},e}$ are normal if and only if all the b_i are even, in which case each component is isomorphic to the nilcone in $\mathfrak{sl}_2(\mathbb{C})$. In the non-normal case, the normalization of an irreducible component is $\mathbb{A}^2$.

Proof. By part (1) of the lemma, there exists $x \in \mathcal{S}_{\mathcal{O},e} \cap \mathcal{O}$, when written as in (4.1), satisfying (4.2) with $x_0 \in \mathfrak{z}$. Hence x_0 must belong to the minimal nilpotent orbit in $\mathfrak{z}$. By Lemma 4.5 the x_i's are highest weight vectors, of weight $i+2$, for a Borel subalgebra of $\mathfrak{z}$. Thus the irreducible component $\overline{Z \cdot x}$ of $\mathcal{S}_{\mathcal{O},e}$ is an S-variety for Z of the form $X(2, b_2, b_3, \dots)$ with $2 < b_2 < b_3 < \dots$. But since the degeneration is minimal, $C(\mathfrak{s})$ acts transitively on the irreducible components of $\mathcal{S}_{\mathcal{O},e}$ by Proposition 2.1, and thus each irreducible component of $\mathcal{S}_{\mathcal{O},e}$ takes this same form.

By §3.2.1, $X(2, b_2, b_3, \dots)$ is normal if and only if the b_i's are all even, in which case it is isomorphic to $X(2)$, the nilcone in $\mathfrak{sl}_2(\mathbb{C})$, which is the A_1-singularity. Otherwise, its normalization is $X(1) \cong \mathbb{A}^2$. $\square$

4.1.5. An example in C_3

Let $\mathfrak{g} = \mathfrak{sp}_6(\mathbf{C})$. Nilpotent orbits in $\mathfrak{g}$ can be parametrized by the Jordan partition for any element in the orbit, viewed as a 6×6-matrix. Pick $e \in \mathfrak{g}$ with partition $[2^3]$. Then $\mathfrak{c}(\mathfrak{s}) \cong \mathfrak{sl}_2(\mathbb{C})$. Set $\mathfrak{z} = \mathfrak{c}(\mathfrak{s})$. A nonzero nilpotent $e_0 \in \mathfrak{z}$ has type $[3^2]$ in $\mathfrak{g}$, so $\mathfrak{c}(\mathfrak{z}) = \mathfrak{s}$ and thus the only non-zero nilpotent G-orbit that meets $\mathfrak{c}(\mathfrak{z})$ is the one through e. Consequently, part (2) of Lemma 4.6 applies to any $\mathcal{O}$ of which $\mathcal{O}' = \mathcal{O}_e$ is a degeneration of codimension two. Let $\mathcal{O}_{[4,1^2]}$ and $\mathcal{O}_{[3^2]}$ be the nilpotent orbits with given partition type. Then $\mathcal{O}'$ is a minimal degeneration of both orbits, in each case of codimension two. So in both cases Corollary 4.7 applies. Now $C(\mathfrak{s})$ acts on $\mathfrak{g}^f$ with $\mathfrak{g}^f(0) = \mathfrak{c}(\mathfrak{s}) = V(2)$ and $\mathfrak{g}^f(-2) \cong V(4) \oplus V(0)$. Also $e + e_0$ belongs to $\mathcal{O}_{[4,2]}$ (see §4.2), so we can say that for both $\mathcal{O} = \mathcal{O}_{[4,1^2]}$ and $\mathcal{O} = \mathcal{O}_{[3^2]}$, each irreducible component of $\mathcal{S}_{\mathcal{O},e}$ is of the form $X(2, 4) \cong X(2)$, which is the nilcone in $\mathfrak{sl}_2(\mathbb{C})$. But since $\overline{\mathcal{O}}$ is normal for both orbits $\mathcal{O}$ by [33], it follows that $\mathcal{S}_{\mathcal{O},e}$ is irreducible. In particular the singularity of $\overline{\mathcal{O}}$ at e is an A_1-singularity in both cases, as was already known from [33].

4.2. Locating nilpotent elements in $\mathfrak{c}(\mathfrak{s})$

In order to make use of Lemma 4.4 or part (2) of Lemma 4.6, we will need to describe nilpotent elements in $c(\mathfrak{s})$ relative to the embedding of $c(\mathfrak{s})$ in $\mathfrak{g}$. We will also need to be able to start with nilpotent $e_0 \in \mathfrak{c}(\mathfrak{s})$ and then compute the G-orbit to which $e + e_0$ belongs.

First, if $e_0 \in \mathfrak{c}(\mathfrak{s})$, then e_0 centralizes the semisimple element $h \in \mathfrak{s}$. Hence $e_0 \in \mathfrak{g}^h$, which is a Levi subalgebra of $\mathfrak{g}$. Assume h lies in a chosen Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and is dominant for a chosen Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$ containing $\mathfrak{h}$. The type of the Levi subalgebra $\mathfrak{g}^h$ can then be read off from the weighted Dynkin diagram for h: the Dynkin diagram for the semisimple part of $\mathfrak{g}^h$ corresponds to the zeros of the diagram. Therefore in order to locate a nilpotent element in $\mathfrak{c}(\mathfrak{s})$, we first choose a nilpotent element $e_0 \in \mathfrak{g}^h$; the G^h-orbits of such elements are known by Dynkin's and Bala and Carter's results [12]. In particular we can compute the semisimple element $h_0 \in \mathfrak{g}^h \cap \mathfrak{h}$ of an $\mathfrak{sl}_2$-subalgebra $\mathfrak{s}_0$ through e_0 in $\mathfrak{g}^h$.

Next, we compute $h + h_0$ and see whether it corresponds to a nilpotent orbit in $\mathfrak{g}$: for if e and e_0 (or some conjugate of e_0 under G^h) commute, then $h + h_0$ will be the semisimple element in an $\mathfrak{sl}_2$-subalgebra through the nilpotent element $e + e_0$. Together with knowledge of the Cartan–Killing type of the reductive Lie algebra $\mathfrak{c}(\mathfrak{s}) \subset \mathfrak{g}^h$ (see [12]), this search usually suffices to locate the nilpotent orbit through e_0 in $\mathfrak{g}$ for nilpotent elements $e_0 \in \mathfrak{c}(\mathfrak{s})$ and the resulting nilpotent orbit through $e + e_0$. In particular we carried out this approach for all the minimal nilpotent $C(\mathfrak{s})$-orbits in $\mathfrak{c}(\mathfrak{s})$. Two special situations are worth mentioning.

4.2.1. One special situation is when e_0 is minimal in $\mathfrak{g}$, that is, of type A_1. Then the semisimple part of $\mathfrak{c}(\mathfrak{s}_0)$ is the semisimple part of a Levi subalgebra of $\mathfrak{g}$, the one corresponding to the nodes in the Dynkin diagram which are not adjacent to the affine node in the extended Dynkin diagram. Of course $e \in \mathfrak{c}(\mathfrak{s}_0)$. Consequently it is easy to locate all e which have $e_0 \in \mathfrak{c}(\mathfrak{s})$ when e_0 is of type A_1 in $\mathfrak{g}$.

We will see in Corollary 4.9 that Lemma 4.3 always applies in this setting with $x = e + e_0$. Moreover the type of x in $\mathfrak{g}$ is easy to determine: if we know the type of e in $\mathfrak{c}(\mathfrak{s}_0)$, call it X, then x has generalized Bala–Carter type $X + A_1$. Then the usual type can be looked up in [51] or in Dynkin's seminal paper [16].

For example, in E_8 when e_0 is of type A_1, then $\mathfrak{c}(\mathfrak{s})$ is of Cartan–Killing type E_7. Any nilpotent element e in a Levi subalgebra of type E_7 will have a conjugate of e_0 in $c(\mathfrak{s})$. If, for instance, e is a regular nilpotent element, then $e + e_0$ has generalized Bala–Carter type $E_7 + A_1$, which is the same as $E_8(a_3)$.

There is another way to determine $e+e_0$ when e_0 is minimal in $\mathfrak{g}$. It has the advantage of locating the simple summand of $\mathfrak{c}(\mathfrak{s})$ in which e_0 lies. As above, assume h is dominant relative to $\mathfrak{b}$. Since $e_0 \in \mathfrak{g}^h$ has type A_1, the semisimple element $h_0 \in \mathfrak{h}$ is equal to the coroot of a long root θ for $\mathfrak{g}^h$. Therefore, $\alpha(h_0) \geq -2$ for any root of $\mathfrak{g}$ and equality holds if and only $\alpha = -\theta$. Now choose h_0 dominant in $\mathfrak{g}^h$ (relative to $\mathfrak{b} \cap \mathfrak{g}^h$). Then $\alpha(h_0) \geq -1$ for all simple roots α of $\mathfrak{g}$ since $-\theta$ is a negative root. Moreover $\alpha(h_0) = -1$ only if α is not a simple root for $\mathfrak{g}^h$. In that case $\alpha(h) \geq 1$ since the simple roots of $\mathfrak{g}^h$ correspond to the zeros of the weighted Dynkin diagram for h. This shows that $\alpha(h + h_0) \geq 0$ for all simple roots α of $\mathfrak{g}$ and thus $h + h_0$ yields the weighted Dynkin diagram for $e + e_0$ without having to conjugate by an element of the Weyl group.

For example, let e belong to the orbit $E_7(a_3)$ in E_8, which has weighted Dynkin diagram

$$\begin{smallmatrix} 2 & 0 & 1 & 0 & 1 & 0 & 2 \\ & & 0 & & & & \end{smallmatrix}.$$

Then $\mathfrak{g}^h$ has type $4A_1$ and $\mathfrak{c}(\mathfrak{s})$ is isomorphic to $\mathfrak{sl}_2(\mathbb{C})$ since $\mathfrak{c}(\mathfrak{s})$ has rank one (because e is distinguished in a Levi subalgebra of rank 7) and $\mathfrak{c}(\mathfrak{s})$ contains e_0, a nonzero nilpotent element. We want to know in which summand of $\mathfrak{g}^h$ the element e_0 lies and what is $e + e_0$. The diagram for h_0 relative to $\mathfrak{g}$, and dominant for $\mathfrak{g}^h$, is either:

$$\begin{smallmatrix} 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ & & 0 & & & & \end{smallmatrix} \text{ or } \begin{smallmatrix} 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ & & 0 & & & & \end{smallmatrix} \text{ or } \begin{smallmatrix} -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ & & 0 & & & & \end{smallmatrix} \text{ or } \begin{smallmatrix} 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ & & 2 & & & & \end{smallmatrix}.$$

Only the second choice leads to a weighted Dynkin diagram for $h+h_0$, namely for $D_7(a_1)$. Hence we know the type of $e + e_0$ and the embedding of $\mathfrak{c}(\mathfrak{s})$ in $\mathfrak{g}^h$.

4.2.2. The other special situation occurs when $\mathfrak{c}(\mathfrak{s})$ has rank 1. Let $\mathfrak{l}$ be a minimal Levi subalgebra containing e. Then $\mathfrak{l}$ has semisimple rank equal to the rank of $\mathfrak{g}$ minus one. Assume that $\mathfrak{l}$ is a standard Levi subalgebra. Let α_i be the simple root of $\mathfrak{g}$ which is not a simple root of $\mathfrak{l}$. For nonzero $e_0 \in \mathfrak{c}(\mathfrak{s})$, the corresponding h_0 centralizes $\mathfrak{l}$ and hence lies in the one-dimensional subalgebra of $\mathfrak{h}$ spanned by the coweight $\omega_i^\vee$ for α_i. Since the values in any weighted Dynkin diagram are $0, 1$, or 2, if h_0 is dominant, then h_0 must be either $\omega_i^\vee$ or $2\omega_i^\vee$.

For example, let e be of type A_7 in E_8, which has weighted Dynkin diagram $\begin{smallmatrix} 1&0&1&0&1&1&0\\ &&0&&&& \end{smallmatrix}$. Then $\mathfrak{c}(\mathfrak{s})$ has type A_1 and the weighted Dynkin diagram of a nonzero $h_0 \in \mathfrak{c}(\mathfrak{s})$ must either be

$$\begin{smallmatrix} 0&0&0&0&0&0&0\\ &&1&&&& \end{smallmatrix} \text{ or } \begin{smallmatrix} 0&0&0&0&0&0&0\\ &&2&&&& \end{smallmatrix}.$$

Both of these are actual weighted Dynkin diagrams in E_8, the first is $4A_1$ and the second is $D_4(a_1) + A_2$. Only the orbit $4A_1$ meets $\mathfrak{g}^h$ (which has semisimple type exactly $4A_1$). Therefore a nonzero nilpotent element $e_0 \in \mathfrak{c}(\mathfrak{s}) \subset \mathfrak{g}^h$ has type $4A_1$ in $\mathfrak{g}$.

4.3. The case where $x_i = 0$ for $i \geq 1$ in (4.1), and (4.2) holds

Once a nilpotent $e_0 \in \mathfrak{c}(\mathfrak{s})$ is located, as in the previous section, with corresponding semisimple element $h_0 \in \mathfrak{c}(\mathfrak{s})$, we can compute $h + h_0$ and check by hand whether the dimension condition

$$\dim C(\mathfrak{s}) \cdot e_0 = \operatorname{codim}_{\overline{\mathcal{O}}}(\mathcal{O}_e) \tag{4.4}$$

holds for the orbit $\mathcal{O}$ through $e + e_0$. If it does, then certainly $x := e + e_0$ satisfies (4.1) with $x_0 = e_0$ and $x_i = 0$ for $i \geq 1$, and the dimension condition in (4.2) just becomes (4.4). By Lemmas 4.3 and 4.4, the union of some of the irreducible components of $\mathcal{S}_{\mathcal{O},e}$ is thus isomorphic to $\overline{C(\mathfrak{s}) \cdot e_0}$. Next we give a condition for (4.4) to hold for the orbit $\mathcal{O} = \mathcal{O}_{e+e_0}$ and show that this condition is always true when e_0 belongs to the minimal orbit in $\mathfrak{g}$.

As before, let $\mathfrak{s}_0$ be an $\mathfrak{sl}_2$-subalgebra in $\mathfrak{c}(\mathfrak{s})$ with standard basis e_0, h_0, f_0. Clearly, $\mathfrak{s}$ and $\mathfrak{s}_0$ commute. We will now establish an equivalent condition to the dimension condition (4.4) in terms of the decomposition of $\mathfrak{g}$ into irreducible subrepresentations for $\mathfrak{s} \oplus \mathfrak{s}_0 \cong \mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C})$.

Let $V_{m,n}$ denote an irreducible representation of $\mathfrak{s} \oplus \mathfrak{s}_0$ with $h \in \mathfrak{s}$ acting by m and $h_0 \in \mathfrak{s}_0$ acting by n on a highest weight vector $u \in V_{m,n}$ annihilated by both e and e_0. The eigenvalues of $h + h_0$ on $V_{m,n}$ are either all even if m and n have the same parity or all odd if m and n have opposite parities. In the former case the quantity

$$\min(m, n) + 1$$

is equal to the dimension of the 0-eigenspace of $h + h_0$; in the latter case, it is equal to the dimension of the 1-eigenspace of $h + h_0$. This is analogous to what occurs in the proof of the Clebsch–Gordan formula.

Let

$$\mathfrak{g} = \bigoplus_{i=1}^{N} V^{(i)}_{m_i,n_i} \tag{4.5}$$

be a decomposition into irreducible subrepresentations $V^{(i)}_{m_i,n_i} \cong V_{m_i,n_i}$ for the action of $\mathfrak{s} \oplus \mathfrak{s}_0$. The relationship between (4.4) and this decomposition in (4.5) is the following:

Proposition 4.8. *Let $\mathcal{O}$ be the orbit through $e + e_0$. The dimension condition (4.4) holds if and only if*

$$m_i \geq n_i \textit{ whenever } m_i > 0. \tag{4.6}$$

Proof. By $\mathfrak{sl}_2(\mathbb{C})$-theory, the sum of the dimensions of the 0-eigenspace and the 1-eigenspace for $\operatorname{ad}(h + h_0)$ on $\mathfrak{g}$ equals the dimension of the centralizer of $x = e + e_0$ in $\mathfrak{g}$. It therefore follows that

$$\dim \mathfrak{g}^x = \sum_{i=1}^{N} (\min(m_i, n_i) + 1)\,.$$

At the same time

$$\dim \mathfrak{g}^e = \sum_{i=1}^{N} (n_i + 1)$$

since the kernel of $\operatorname{ad}(e)$ on $V^{(i)}_{m_i,n_i}$ is isomorphic to $V(n_i)$. Here, $V(n_i)$ is an irreducible representation of $\mathfrak{s}_0 \cong \mathfrak{sl}_2(\mathbb{C})$ of highest weight n_i, hence of dimension $n_i + 1$. Putting the two formulas together, the codimension of $\mathcal{O}_e$ in $\overline{\mathcal{O}}_x$ is equal to

$$\sum_{i=1}^{N} (n_i - \min(m_i, n_i))\,.$$

It is also necessary to compute $\dim \mathfrak{c}(\mathfrak{s})^{e_0}$. Since $\mathfrak{s}_0 \subset \mathfrak{c}(\mathfrak{s})$ and $\mathfrak{c}(\mathfrak{s})$ is exactly $\ker \operatorname{ad} e \cap \ker \operatorname{ad} h$, it follows that $\mathfrak{c}(\mathfrak{s})$ coincides with the sum of all $V^{(i)}_{m_i,n_i}$ where $m_i = 0$. The centralizer $\mathfrak{c}(\mathfrak{s})^{e_0}$ is then the span of the highest weight vectors of these $V^{(i)}_{0,n_i}$ and hence its dimension is given by the number of these subrepresentations. That is,

$$\dim \mathfrak{c}(\mathfrak{s})^{e_0} = \#\{1 \leq i \leq N \mid m_i = 0\}.$$

Thus

$$\dim C(\mathfrak{s}) \cdot e_0 = \dim \mathfrak{c}(\mathfrak{s}) - \dim \mathfrak{c}(\mathfrak{s})^{e_0} = \sum_{m_i=0} (n_i + 1) - \sum_{m_i=0} 1 = \sum_{m_i=0} n_i.$$

The equality of $\dim C(\mathfrak{s}) \cdot e_0$ and the codimension of $\mathcal{O}_e$ in $\overline{\mathcal{O}}_x$ is therefore equivalent to $\min(m_i, n_i) = n_i$ for all i with $m_i \neq 0$. $\square$

It follows from the above proof that if $\mathcal{J} = \{i \mid n_i > m_i > 0\}$, then

$$\dim \mathcal{S}_{\mathcal{O},e} - \dim C(\mathfrak{s}) \cdot e_0 = \sum_{i \in \mathcal{J}} (n_i - m_i). \tag{4.7}$$

The element $e_0 \in \mathfrak{g}$ is called height 2 if all the eigenvalues of $\operatorname{ad} h_0$ on $\mathfrak{g}$ are at most 2, and e is called even if all the eigenvalues of $\operatorname{ad} h$ on $\mathfrak{g}$ are even.

Corollary 4.9. *Suppose that either (1) e_0 belongs to the minimal nilpotent orbit in $\mathfrak{g}$, or (2) e_0 is of height 2 in $\mathfrak{g}$ and e is even. Then the dimension condition (4.4) holds.*

Proof. If e_0 belongs to the minimal nilpotent orbit of $\mathfrak{g}$, then e_0 is of height two and the 2-eigenspace of $\operatorname{ad} h_0$ is spanned by e_0. This is the case since h_0 is conjugate to the coroot of the highest root. But since $\mathfrak{s}_0 \subset \mathfrak{c}(\mathfrak{s})$, it follows that $\mathfrak{s}_0 \cong V_{0,2}$ is the unique subrepresentation of $\mathfrak{g}$ isomorphic to $V_{m,n}$ with $n \geq 2$. Therefore all other $V^{(i)}_{m_i,n_i}$ must have $n_i = 0$ or $n_i = 1$ and so condition (4.6) holds.

Next assume the second hypothesis. Since e is even, all $V^{(i)}_{m_i,n_i}$ with $m_i > 0$ satisfy $m_i \geq 2$. Since e_0 is of height two, $n_i \leq 2$ and thus condition (4.6) is true. $\square$

4.4. The case where $x_i \neq 0$ for some $i \geq 1$ in (4.1), and (4.2) holds

Let $e_0 \in \mathfrak{c}(\mathfrak{s})$ be nilpotent and suppose that the dimension condition (4.4) does not hold for $\mathcal{O} = \mathcal{O}_{e+e_0}$. It may happen instead that Lemma 4.3 applies for a different nilpotent orbit $\mathcal{O}$ with $\mathcal{O}_e \subset \overline{\mathcal{O}} \subset \overline{\mathcal{O}}_{e+e_\circ}$. More precisely, it may be possible to find $x \in \mathcal{S}_{\mathcal{O},e}$, written as in (4.1), so that $x_0 = e_0$ and (4.4) does hold for this $\mathcal{O}$. Then Lemma 4.4 ensures that Lemma 4.3 applies to $\mathcal{S}_{\mathcal{O},e}$. Now in such a situation, Lemma 4.5 implies that x must lie in the 2-eigenspace for $\operatorname{ad}(h + h_0)$. We now use this information to give one way to help locate such an x when it exists.

4.4.1. A smaller slice result

Let $y = e + e_0$, which is nilpotent with corresponding semisimple element $h_y = h + h_0$. Write $\mathfrak{g}_j$ for the j-eigenspace of $\operatorname{ad} h_y$ on $\mathfrak{g}$. The centralizer $G_0 := G^{h_y}$ has Lie algebra $\mathfrak{g}_0$ and G_0 acts on each $\mathfrak{g}_j$. Then $y \in \mathfrak{g}_2$ and the G_0-orbit through y is the unique dense orbit in $\mathfrak{g}_2$. Now $e \in \mathfrak{g}_2$ since

$$[h_y, e] = [h + h_0, e] = 2e + 0 = 2e. \tag{4.8}$$

We want to find a transverse slice in $\mathfrak{g}_2$ to the G_0-orbit through e. In fact, since $\mathfrak{g}_2$ is a direct sum of $\operatorname{ad} h$-eigenspaces, the decomposition $\mathfrak{g} = \operatorname{Im} \operatorname{ad} e \oplus \ker \operatorname{ad} f$ restricts to a decomposition

$$\mathfrak{g}_2 = [e, \mathfrak{g}_0] \oplus (\mathfrak{g}_2 \cap \ker \operatorname{ad} f).$$

Therefore, setting $\mathcal{S}_e^{(2)} = e + (\mathfrak{g}_2 \cap \ker \operatorname{ad} f)$, it follows that the affine space $\mathcal{S}_e^{(2)}$ is a transverse slice of $\mathfrak{g}_2$ at e with respect to the G_0-action. Consequently, every G_0-orbit in $\mathfrak{g}_2$ containing e in its closure meets $\mathcal{S}_e^{(2)}$.

Let $\mathfrak{g}(r, s)$ denote the subspace of $\mathfrak{g}$ where $\operatorname{ad} h$ has eigenvalue r and $\operatorname{ad} h_0$ has eigenvalue s. Define $\mathfrak{g}^f(r, s) = \mathfrak{g}(r, s) \cap \ker \operatorname{ad} f$. Then

$$\mathfrak{g}_2 \cap \ker \operatorname{ad} f = \bigoplus_{r \geq 0} \mathfrak{g}^f(-r, r + 2).$$

Next, we relate this decomposition to the decomposition (4.5) of $\mathfrak{g}$ under $\mathfrak{s} \oplus \mathfrak{s}_0$. Let

$$\mathcal{E} = \{i \mid n_i > m_i > 0 \text{ and } n_i - m_i \text{ even}\}$$

where (m_i, n_i) are defined in (4.5). Then $\mathcal{E} \subset \mathcal{J}$. For each $i \in \mathcal{E}$, let w_i be a nonzero vector in the one-dimensional space $V^{(i)}_{m_i,n_i} \cap \mathfrak{g}(-m_i, m_i + 2)$. Then w_i is a lowest weight vector for $\mathfrak{s}$, but not in general a highest weight vector for $\mathfrak{s}_0$. The set $\{w_i \mid i \in \mathcal{E}\}$ is then a basis for

$$\bigoplus_{r \geq 1} \mathfrak{g}^f(-r, r + 2)$$

since each vector in $\mathfrak{g}^f(-r, r + 2)$ lies in a sum of subrepresentations of type $V_{r,s}$ with $r + 2 \leq s$ and $s - r$ even. The subspace $\mathfrak{g}^f(0, 2)$ is just the 2-eigenspace of $\operatorname{ad} h_0$ in $\mathfrak{c}(\mathfrak{s})$, which coincides with $\mathfrak{c}(\mathfrak{s}) \cap \mathfrak{g}(0, 2)$. It contains e_0. A consequence of the above observations is the following

Lemma 4.10. *Let $x \in \mathfrak{g}_2$. If $e \in \overline{G_0 \cdot x}$, then some G_0-conjugate of x can be expressed as*

$$e + w + \sum_{i \in \mathcal{E}} d_i w_i \tag{4.9}$$

where $w \in \mathfrak{c}(\mathfrak{s}) \cap \mathfrak{g}(0, 2)$ and $d_i \in \mathbb{C}$.

Given a nilpotent orbit $\mathcal{O}$, Lemma 4.10 gives a way to show the existence of some $x \in \mathcal{O}$ that can be written as in (4.1) with $x_0 = w$ nilpotent. But it does not guarantee that w is equal to the prescribed e_0 or that (4.4) holds. When $w = e_0$ and (4.4) holds, which is the case we are interested in, then we also know by Lemma 4.5 that the w_i which appear in (4.9) with $d_i \neq 0$ must satisfy $n_i - m_i = 2$, so only the terms in $\mathcal{E}$ with $n_i - m_i = 2$ will contribute in this case.

4.4.2. Applying Lemma 4.10

In order to apply Lemma 4.10 for some $x \in \mathfrak{g}$ with $\overline{\mathcal{O}}_e \subset \overline{\mathcal{O}}_x \subset \overline{\mathcal{O}}_y$, we need to check two things, after possibly replacing x by a conjugate:

(1) $x \in \mathfrak{g}_2$
(2) $e \in \overline{G_0 \cdot x}$

The first condition can often be shown as follows. Let $\mathfrak{s}_x$ be an $\mathfrak{sl}_2$-subalgebra through some conjugate of x with standard semisimple element $h^x \in \mathfrak{h}$. In all cases we are interested in, there exists nilpotent $e_0^x \in \mathfrak{c}(\mathfrak{s}_x)$ with semisimple element $h_0^x \in \mathfrak{h}$, such that $h^x + h_0^x = h_y$, after possibly replacing x again by a conjugate. Then just as in (4.8), $x \in \mathfrak{g}_2$ and the first condition holds.

We may further assume that h_y is dominant with respect to the Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$ and h^x is dominant for the corresponding Borel subalgebra $\mathfrak{b}_y$ of $\mathfrak{g}^{h_y}$. Then since $[h^x, x] = 2x$ and $[h_y, x] = 2x$, it follows that x belongs to

$$I_x := \mathfrak{g}_2 \cap \bigoplus_{i \geq 2} \mathfrak{g}(h^x; i),$$

where $\mathfrak{g}(h^x; i)$ are the eigenspaces for $\operatorname{ad} h^x$. This subspace is preserved by the action of $\mathfrak{b}_y$. Thus $G_0 \cdot I_x = \overline{G_0 \cdot x}$. We can carry out a similar process for e and obtain a subspace $I_e \subset \mathfrak{g}_2$, with $G_0 \cdot I_e = \overline{G_0 \cdot e}$. Then if $I_e \subset I_x$, it necessarily follows that

$$G_0 \cdot I_e \subset G_0 \cdot I_x$$

and the second condition holds. For the cases we are interested in, this approach will suffice to check the hypothesis in Lemma 4.10.

4.4.3. Example: $(\tilde{A}_1, A_1)$ in type G_2

Let $\mathfrak{g}$ be of type G_2 and let $e \in \mathfrak{g}$ be minimal nilpotent. Then $\mathfrak{c}(\mathfrak{s}) \cong \mathfrak{sl}_2(\mathbb{C})$. Let $e_0 \in \mathfrak{c}(\mathfrak{s})$ be minimal nilpotent, which has type $\tilde{A}_1$ in $\mathfrak{g}$. The decomposition in (4.5) is $\mathfrak{g} = V(0,2) \oplus V(2,0) \oplus V(1,3)$. Therefore $(m_i, n_i) = (1,3)$ for the unique $i \in \mathcal{E}$ and (4.4) fails for $\mathcal{O} = \mathcal{O}_{e+e_0}$ by Proposition 4.8. Indeed, $e + e_0$ has type $G_2(a_1)$ and thus if $\mathcal{O}$ is the orbit of type $\tilde{A}_1$, then $\mathcal{O}_e \subset \overline{\mathcal{O}} \subset \overline{\mathcal{O}}_{e+e_0}$. Since $\mathfrak{s}$ and $\mathfrak{c}(\mathfrak{s})$ are mutual centralizers, and $\overline{\mathcal{O}}$ is unibranch at e, the argument in Example 4.1.5 gives that $\mathcal{S}_{\mathcal{O},e}$ takes the form of the S-variety $X(2,3)$ for $SL_2(\mathbb{C})$, which is isomorphic to m.

We can show also that Lemma 4.10 holds by checking the two conditions in §4.4.2. Fix nonzero $w_i \in V(1,3)$ satisfying $[e_0, w_i] = 0$ and $[f, w_i] = 0$. Choose h_y so that its weighted diagram is the usual weighted Dynkin diagram $2\ 0$ of y and choose h^x and h_0^x to have weighted diagrams $0\ 1$ and $2\ {-1}$, respectively. Then $h_y = h^x + h_0^x$ and thus by the above discussion we may replace x by a conjugate and assume $x \in \mathfrak{g}_2$. Similarly, let h and h_0 have weighted diagrams ${-1}\ 1$ and $3\ {-1}$, respectively. Then I_e is one-dimensional and $I_e \subset I_x$. The two conditions in §4.4.2 are met, so by Lemma 4.10 there exists $x \in \mathcal{O}$

with $x = e + ae_0 + dw_i$ for $a, d \in \mathbb{C}$. Now $d \neq 0$ since x and $e + ae_0$ are not in the same G-orbit for any value of a, and $a \neq 0$ by Lemma 4.4, and thus we get another proof that $\mathcal{S}_{\mathcal{O},e}$ takes the form $X(2,3)$.

4.4.4. Finding w_i for $i \in \mathcal{E}$

We sometimes need to do explicit computations to verify (4.9) or to show that $w = e_0$, especially for degenerations which are not minimal (e.g., §4.4.5) or where Lemma 4.6 does not apply. In these cases there arises the need for an analogue of the result describing isomorphisms between S-varieties (§3.2.1). Here we describe a way that is often helpful in finding w_i for $i \in \mathcal{E}$, which frequently leads to an isomorphism of $\overline{C(\mathfrak{s}) \cdot x}$ with $\overline{C(\mathfrak{s}) \cdot e_0}$ in Lemma 4.3, when such an isomorphism exists.

Write $\mathfrak{g}(h;j)$ for the j-eigenspace of $\operatorname{ad} h$. Since $\mathfrak{c}(\mathfrak{s}) \subset \mathfrak{g}^h = \mathfrak{g}(h;0)$, the $\mathfrak{g}(h;j)$ are $\mathfrak{c}(\mathfrak{s})$-modules. Also $\mathfrak{g}^h \oplus \mathfrak{g}(h;1)$ is isomorphic to $\mathfrak{g}^f$, as $\mathfrak{c}(\mathfrak{s})$-modules. Indeed, for $j \geq 0$,

$$\mathfrak{g}^f(-2j) \cong (\operatorname{ad} f)^j(\mathfrak{g}^h) \cap \mathfrak{g}^f \text{ and}$$
$$\mathfrak{g}^f(-2j-1) \cong (\operatorname{ad} f)^{j+1}(\mathfrak{g}(h;1)) \cap \mathfrak{g}^f,$$

as $\mathfrak{c}(\mathfrak{s})$-modules.

Suppose that $\mathfrak{g}^h$ is a direct sum of classical Lie algebras. Then for $M \in \mathfrak{g}^h$, the matrix power M^r is in $\mathfrak{g}^h$ for r odd, or if all the factors of $\mathfrak{g}^h$ are type A, then for any r. Of course $[M, M^r] = 0$ in $\mathfrak{g}^h$, and hence in $\mathfrak{g}$. Set $M := e_0$, where as before $e_0 \in \mathfrak{c}(\mathfrak{s})$ is nilpotent. The identity $[h_0, M^r] = 2rM^r$ holds in $\mathfrak{g}^h$ because $[h_0, M] = h_0M - Mh_0 = 2M$, where matrix multiplication takes place in $\mathfrak{g}^h$; hence this identity also holds in $\mathfrak{g}$. Thus M^r, if nonzero, is a highest weight vector for $\mathfrak{s}_0$ relative to e_0 and h_0. Now assume M^r is not zero. Then for some largest j,

$$(\operatorname{ad} f)^j M^r \in \mathfrak{g}^f(-2j)$$

is nonzero. Since $\mathfrak{s}$ and $\mathfrak{s}_0$ commute, $(\operatorname{ad} f)^j M^r$ is both a highest weight vector for $\mathfrak{s}_0$ and a lowest weight vector for $\mathfrak{s}$ (relative to f and h).

Now suppose $\mathcal{E} \neq \emptyset$ and consider (m_i, n_i) for $i \in \mathcal{E}$. Suppose m_i is even. In the cases of interest (see Lemma 4.5 and the paragraph after Lemma 4.10), we have $n_i - m_i = 2$. In such cases we often find that w_i can be taken to be

$$(\operatorname{ad} f)^{\frac{m_i}{2}}(M^{\frac{m_i}{2}+1}).$$

Moreover, if x in (4.9) is a linear combination of such w_i's and $w = e_0$, then it follows that $\overline{C(\mathfrak{s}) \cdot x} \cong \overline{C(\mathfrak{s}) \cdot e_0}$ via the projection π_0 (§3.2.2) since the G^h-action, and thus the $C(\mathfrak{s})$-action, commutes with taking matrix powers.

4.4.5. Example: the non-minimal degeneration $(C_3, \tilde{A}_2)$ in F_4

We illustrate the previous discussion in F_4 in proving that $\mathcal{S}_{\mathcal{O},e}$ contains an irreducible component isomorphic to the nilpotent cone $\mathcal{N}_{G_2}$ in G_2, when $\mathcal{O}$ is of type C_3 and e

lies in the $\tilde{A}_2$ orbit. For this choice of e, the centralizer $C(\mathfrak{s})$ is connected, simple of type G_2. Let $e_0 \in \mathfrak{c}(\mathfrak{s})$ be regular nilpotent. Then $e + e_0$ lies in the orbit $F_4(a_2)$ and the decomposition of $\mathfrak{g}$ in (4.5) is

$$V(0,2) \oplus V(0,10) \oplus V(2,0) \oplus V(4,6),$$

so $\mathcal{E}$ has a single element, with $(m_i, n_i) = (4,6)$. Then $\mathcal{O}_e \subset \overline{\mathcal{O}} \subset \overline{\mathcal{O}}_{e+e_0}$ and we could use §4.4.2 to show that there exists $x \in \mathcal{O}$ satisfying (4.9) with some additional work. Instead, we report on a direct computation using GAP. Let

$$e = e_{0010} + e_{0001}, \quad f = 2f_{0010} + 2f_{0001}, \quad h = [e,f],$$

and

$$e_0 = e_{0111} - e_{0120} + e_{1000}.$$

The space $\mathfrak{g}(-4,6)$ is one-dimensional, spanned by $w_i := e_{1220}$. This is also a highest weight vector for the full action of $C(\mathfrak{s})$ on $\mathfrak{g}^f(-4) \cong V(\omega_2)$, the 7-dimensional irreducible representation of G_2. We computed in GAP that there is an $x \in \mathcal{O}$ with $x = e + e_0 - \frac{1}{4}w_i$, which establishes (4.1) with $x_0 = e_0$ and $x_4 = w_i$. Since $\dim \mathcal{S}_{\mathcal{O},e} = \dim C(\mathfrak{s}) \cdot e_0$, Lemma 4.3 applies and thus $\mathcal{S}_{\mathcal{O},e}$ contains $e + X$ as an irreducible component, where $X := \overline{C(\mathfrak{s}) \cdot (e_0 - \frac{1}{4}w_i)}$.

We now show that X is isomorphic to $\overline{C(\mathfrak{s}) \cdot e_0}$, which is the nilcone of $\mathfrak{c}(\mathfrak{s})$, by relating the choice of w_i to the discussion in §4.4.4. We have $\mathfrak{g}^h \cong \mathfrak{so}_7(\mathbb{C}) \oplus \mathbb{C}$ and the $\mathfrak{so}_7(\mathbb{C})$ component contains $\mathfrak{c}(\mathfrak{s})$ and decomposes under $\mathfrak{c}(\mathfrak{s})$ into $\mathfrak{c}(\mathfrak{s}) \oplus V(\omega_2)$. Now $\operatorname{ad} f$ annihilates $\mathfrak{c}(\mathfrak{s})$, while $(\operatorname{ad} f)^2$ carries the $V(\omega_2)$ summand isomorphically onto $\mathfrak{g}^f(-4)$. Let $M = e_0 \in \mathfrak{c}(\mathfrak{s}) \subset \mathfrak{so}_7(\mathbb{C})$. Then $M^3 \in \mathfrak{so}_7(\mathbb{C})$ and $M^3 \neq 0$ since e_0 has type B_3 in $\mathfrak{g}$ (i.e., the embedding of $\mathfrak{c}(\mathfrak{s})$ of type G_2 in $\mathfrak{so}_7(\mathbb{C})$ is the expected one). Since M^3 is centralized by e_0 and is an eigenvector for $\operatorname{ad} h_0$ with eigenvalue 6, we have $M^3 \in V(\omega_2)$ since only the eigenvalues 2 and 10 are possible for the $\mathfrak{c}(\mathfrak{s})$ summand. Moreover, $(\operatorname{ad} f)^2(M^3)$ is a nonzero vector in $\mathfrak{g}(-4,6)$ and so must be a multiple of w_i. Although X is not an S-variety (since e_0 is not minimal in $\mathfrak{c}(\mathfrak{s})$), it is the closure of the $C(\mathfrak{s})$-orbit through $(e_0, w_i) \in \mathfrak{c}(\mathfrak{s}) \oplus \mathfrak{g}^f(-4)$, which can now be described as the set of elements $(M, M^3) \in \mathfrak{c}(\mathfrak{s}) \oplus V(\omega_2) = \mathfrak{so}_7(\mathbb{C})$ with $M \in \mathfrak{c}(\mathfrak{s})$ nilpotent. Hence, there is a $C(\mathfrak{s})$-equivariant isomorphism of X with $\overline{C(\mathfrak{s}) \cdot e_0} \cong \mathcal{N}_{G_2}$ coming from π_0.

Remark 4.11. There are two branches of $\overline{\mathcal{O}}$ in a neighborhood of e. These two branches are not conjugate under the action of G^e, which shows that Proposition 2.1 does not generally hold for degenerations which are not minimal. The other branch of $\overline{\mathcal{O}}$ at e splits into three separate branches in a neighborhood of a point in the orbit $F_4(a_3)$ (see §7.3).

4.5. Proof of Proposition 3.3

The proof is case-by-case until we exhaust all minimal degenerations covered by the Proposition. First, we consider those e for which there exists $e_0 \in \mathfrak{c}(\mathfrak{s})$ that is minimal nilpotent in $\mathfrak{g}$, and then compute the G-orbit $\mathcal{O}$ to which $e + e_0$ belongs (§4.2.1). Corollary 4.9 ensures that (4.4) holds for this $\mathcal{O}$, and then applying Lemma 4.3 to $x := e + e_0$, we conclude that $e + \overline{C(\mathfrak{s}) \cdot e_0}$ is a union of irreducible components of $\mathcal{S}_{\mathcal{O},e}$. Such degenerations turn out always to be minimal degenerations, and so $C(\mathfrak{s})$ acts transitively on the irreducible components of $\mathcal{S}_{\mathcal{O},e}$ by Proposition 2.1. Hence $\mathcal{S}_{\mathcal{O},e} = e + \overline{C(\mathfrak{s}) \cdot e_0}$. The results are recorded in Tables 1, 3, 6, and 9 for each of the exceptional groups F_4, E_6, E_7, and E_8, respectively. Next, we consider all other cases where e_0 belongs to a minimal nilpotent $C(\mathfrak{s})$-orbit in $\mathfrak{c}(\mathfrak{s})$ and check whether or not (4.4) holds for $\mathcal{O} = \mathcal{O}_{e+e_0}$. In the cases where it does hold, the degeneration $(\mathcal{O}, \mathcal{O}_e)$ turns out to be a minimal degeneration, and thus $\mathcal{S}_{\mathcal{O},e} = e + \overline{C(\mathfrak{s}) \cdot e_0}$ as in the first step. The results are recorded in the first lines of Tables 2, 4, 7, and 10. These two sets of calculations cover all the minimal degenerations in Proposition 3.3 where $J = \emptyset$.

For the remaining cases, we study those e_0 which are minimal in $\mathfrak{c}(\mathfrak{s})$, but where (4.4) does not hold for the orbit through $e + e_0$. For such e and $e + e_0$, we look for nilpotent orbits $\mathcal{O}$ with $\mathcal{O}_e \subset \overline{\mathcal{O}} \subset \overline{\mathcal{O}}_{e+e_\circ}$ such that $\mathcal{O}_e$ is a minimal degeneration of $\mathcal{O}$ and $\dim C(\mathfrak{s}) \cdot e_0 = \operatorname{codim}_{\overline{\mathcal{O}}}(\mathcal{O}_e)$. Then $\mathcal{O}' = \mathcal{O}_e$ and $\mathcal{O}$ are candidate orbits to apply Lemma 4.3. For the cases where the degeneration is dimension two, which is the vast majority, we can show that Lemma 4.6 (and hence Corollary 4.7) applies. Sometimes, though, we have to restrict to a subalgebra as in Lemma 4.1 or carry out a computer calculation to determine for which $i \in \mathbb{N}$ the corresponding x_i is nonzero in (4.1). There are just three others cases, all of dimension four, and for these we can show that there exists $x \in \mathcal{O}$ satisfying (4.2) by restricting to a subalgebra as in Lemma 4.1 (§10.1.1, §10.1.2). Thus for all the remaining cases, which are the ones in the Proposition where $J \neq \emptyset$, we find that Lemma 4.3 applies and Lemma 4.5 ensures that the x_i's are highest weight vectors for $C(\mathfrak{s})$ with weights as prescribed in the Proposition. The possibilities for J turn out to be $\{2\}, \{2, 4\}, \{1\}, \{1, 2\}$, and $\{1, 2, 3\}$, as noted in Remark 3.4. By §3.2.1, the first two possibilities give the isomorphism under π_0 in (1) of the Proposition, and the last three possibilities give the isomorphism under $\pi_{0,1}$ in (2) of the Proposition.

Comparing with the surface cases treated in §5, in order to know which surface cases have $|\Gamma| = 1$ or 2, we find that all the cases in Proposition 3.3 have been addressed. The results are recorded in Tables 2, 4, 7, and 10, where $\mathcal{E} \neq \emptyset$. The set J consists of those m_i with $i \in \mathcal{E}$ and $d_i \neq 0$ in (4.9), or equivalently, $x_{m_i} \neq 0$ in (4.1). Such m_i are the ones in the boldface pairs (m_i, n_i) in these tables. They all must satisfy $n_i - m_i = 2$ by Lemma 4.5.

5. Geometric method for surface singularities

In this section we consider a minimal degeneration $\mathcal{O}'$ of $\mathcal{O}$ such that $\mathcal{O}'$ is of codimension 2 in $\overline{\mathcal{O}}$. Let $e \in \mathcal{O}'$. We show that the normalization of each irreducible component of

$\mathcal{S}_{\mathcal{O},e}$ is isomorphic to $\mathbb{C}^2/\Gamma$ for some finite subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{C})$. Our method allows us to determine the group Γ, hence we determine $\mathcal{S}_{\mathcal{O},e}$ up to normalization. As mentioned in §3.1, we can often use results on normality of nilpotent orbit closures or other methods (e.g. Lemma 4.1) to decide whether the irreducible components of $\mathcal{S}_{\mathcal{O},e}$ are normal. Sometimes we have to state our results up to normalization.

5.1. Two-dimensional Slodowy slices

Recall that a contracting $\mathbb{C}^*$-action on a variety X is a $\mathbb{C}^*$-action on X with a unique fixed point $o \in X$ such that for any $x \in X$, we have $\lim_{\lambda \to 0} \lambda \cdot x = o$. Recall from [4] that a symplectic variety is a normal variety W with a holomorphic symplectic form ω on its smooth locus such that for any resolution $\pi : Z \to W$, the pull-back $\pi^*\omega$ extends to a regular 2-form on Z. For a nilpotent orbit, we write $\widetilde{\mathcal{O}}$ for the normalization of $\overline{\mathcal{O}}$.

Lemma 5.1. *The normalization $\widetilde{\mathcal{S}}_{\mathcal{O},e}$ of $\mathcal{S}_{\mathcal{O},e}$ is an affine normal variety with each irreducible component having at most an isolated symplectic singularity and endowed with a contracting $\mathbb{C}^*$-action.*

Proof. As $\widetilde{\mathcal{O}}$ has rational Gorenstein singularities by [26] and [48], $\widetilde{\mathcal{S}}_{\mathcal{O},e}$ has only rational Gorenstein singularities. On the other hand, there exists a symplectic form on its smooth locus, hence $\widetilde{\mathcal{S}}_{\mathcal{O},e}$ has only symplectic singularities by [45] (Theorem 6). By construction, the contracting $\mathbb{C}^*$-action on $\mathcal{S}_{\mathcal{O},e}$ in §2.3 has positive weights, hence it lifts to a contracting $\mathbb{C}^*$-action on $\widetilde{\mathcal{S}}_{\mathcal{O},e}$. □

The two-dimensional symplectic singularities are exactly rational double points (cf. [4, Section 2.1]). The following is immediate from [17, Lemma 2.6].

Proposition 5.2. *Let X be an affine irreducible surface with an isolated rational double point at o. If there exists a contracting $\mathbb{C}^*$-action on X, then X is isomorphic to $\mathbb{C}^2/\Gamma$ for some finite subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{C})$.*

Note that by Proposition 2.1, the irreducible components of $\mathcal{S}_{\mathcal{O},e}$ are mutually isomorphic. As an immediate corollary, we get

Corollary 5.3. *Let $\mathcal{S}_{\mathcal{O},e}$ be a two-dimensional nilpotent Slodowy slice. Then there exists a finite subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{C})$ such that each irreducible component of the normalization $\widetilde{\mathcal{S}}_{\mathcal{O},e}$ is isomorphic to $\mathbb{C}^2/\Gamma$.*

Hence to determine $\widetilde{\mathcal{S}}_{\mathcal{O},e}$, we only need to determine the subgroup Γ. In the following, we shall describe a way to construct the minimal resolution of $\widetilde{\mathcal{S}}_{\mathcal{O},e}$. Then the configuration of exceptional $\mathbb{P}^1$'s in the minimal resolution will determine Γ.

5.2. $\mathbb{Q}$-factorial terminalization for nilpotent orbit closures

A general reference for the minimal model program in algebraic geometry is [44]. Here we recall some basic definitions.

Let X be a normal variety. A Weil divisor D on X is called $\mathbb{Q}$-*Cartier* if ND is a Cartier divisor for some non-zero integer N. We say that X is $\mathbb{Q}$-*Gorenstein* if its canonical divisor K_X is $\mathbb{Q}$-Cartier. The variety X is called $\mathbb{Q}$-*factorial* if every Weil divisor on X is $\mathbb{Q}$-Cartier. A $\mathbb{Q}$-Gorenstein variety X is said to have *terminal singularities* if there exists a resolution $\pi : Z \to X$ such that $K_Z = \pi^* K_X + \sum_{i=1}^{k} a_i E_i$ with $a_i > 0$ for all i, where $E_i, i = 1, \cdots, k$ are the irreducible components of the exceptional divisor of π. A $\mathbb{Q}$-*factorial terminalization* of a $\mathbb{Q}$-Gorenstein variety X is a projective birational morphism $\pi : Z \to X$ such that $K_Z = \pi^* K_X$ and Z is $\mathbb{Q}$-factorial with only terminal singularities.

It is well-known that two-dimensional terminal singularities are necessarily smooth (cf. Theorem 4-6-5 [44]), hence a normal variety X with only terminal singularities is smooth in codimension 2, that is, $\operatorname{codim}_X \operatorname{Sing}(X) \geq 3$.

For the normalization of the closure of a nilpotent orbit, one way to obtain its $\mathbb{Q}$-factorial terminalization is by the following method. Consider a parabolic subgroup Q in G. Let L be a Levi subgroup of Q. For a nilpotent element $t \in \operatorname{Lie}(L)$, we denote by $\mathcal{O}_t^L$ its orbit under L in $\operatorname{Lie}(L)$. Let $\mathfrak{n}(\mathfrak{q})$ be the nilradical of $\operatorname{Lie}(Q)$. Then the natural map $p : G \times^Q (\mathfrak{n}(\mathfrak{q}) + \overline{\mathcal{O}}_t^L) \to \mathfrak{g}$ has image equal to $\overline{\mathcal{O}}$ for some nilpotent orbit $\mathcal{O}$ and p is called a *generalized Springer map* for $\mathcal{O}$. Then $\mathcal{O}$ is said to be induced from $(L, \mathcal{O}_t^L)$ [42]. When $t = 0$, then $\mathcal{O}$ is called the Richardson orbit for Q and $G \times^Q \mathfrak{n}(\mathfrak{q})$ identifies with the cotangent bundle $T^*(G/Q)$; if in addition p is birational, then we call p a generalized Springer resolution. By [18], those are the only symplectic resolutions of nilpotent orbit closures. More generally, if p is birational and the normalization of $\overline{\mathcal{O}}_t^L$ is $\mathbb{Q}$-factorial terminal, then the normalization of p gives a $\mathbb{Q}$-factorial terminalization of $\widetilde{\mathcal{O}}$, the normalization of $\overline{\mathcal{O}}$. In [19], it was proved in confirming a conjecture of Namikawa that for a nilpotent orbit $\mathcal{O}$ in an exceptional Lie algebra, either $\widetilde{\mathcal{O}}$ is $\mathbb{Q}$-factorial terminal or every $\mathbb{Q}$-factorial terminalization of $\overline{\mathcal{O}}$ is given by a generalized Springer map.

5.3. Minimal resolutions of two-dimensional nilpotent Slodowy slices

We now use the generalized Springer maps to construct a minimal resolution of $\widetilde{\mathcal{S}}_{\mathcal{O},e}$ when $\mathcal{S}_{\mathcal{O},e}$ is two-dimensional.

Recall from [19] that in a simple Lie algebra of exceptional type, $\widetilde{\mathcal{O}}$ has only terminal singularities if and only if $\mathcal{O}$ is either a rigid orbit or it belongs to the following list: $2A_1, A_2 + A_1, A_2 + 2A_1$ in E_6; $A_2 + A_1$, $A_4 + A_1$ in E_7; $A_4 + A_1$, $A_4 + 2A_1$ in E_8.

First consider the case where $\widetilde{\mathcal{O}}$ has only terminal singularities. Then $\widetilde{\mathcal{O}}$ is smooth in codimension two by the previous subsection. This implies that the singularities of $\overline{\mathcal{O}}$ along $\mathcal{O}_e$ are smoothable by its normalization. In other words, $\widetilde{\mathcal{S}}_{\mathcal{O},e}$ is smooth, which is then isomorphic to $\mathbb{C}^2$ by Proposition 5.2 and we are done.

Example 5.4. Consider again the minimal degeneration $(\widetilde{A}_1, A_1)$ in G_2 from §4.4.3. As $\mathcal{O} = \mathcal{O}_{\widetilde{A}_1}$ is a rigid orbit, its normalization has $\mathbb{Q}$-factorial terminal singularities by [19]. In particular, the singular locus of $\widetilde{\mathcal{O}}$ has codimension at least 4. Since the orbit A_1 is of codimension two in $\overline{\mathcal{O}}$, this implies that $\overline{\mathcal{O}}$ is non-normal and $\widetilde{\mathcal{S}}_{\mathcal{O},e} \cong \mathbb{C}^2$ for $e \in \mathcal{O}_{A_1}$, which is consistent with the description $\mathcal{S}_{\mathcal{O},e} \cong m$ in §4.4.3.

Next, assume that the normalization $\widetilde{\mathcal{O}}$ is not terminal. Then by [19], $\mathcal{O}$ is an induced orbit and $\widetilde{\mathcal{O}}$ admits a $\mathbb{Q}$-factorial terminalization $\pi : Z \to \widetilde{\mathcal{O}}$ given by the normalization of a generalized Springer map. We denote by U the open subset $\mathcal{O} \cup \mathcal{O}_e$ of $\overline{\mathcal{O}}$ and $\nu : \widetilde{U} \to U$ the normalization map. As Z has only terminal singularities, it is smooth in codimension two. As π is G-equivariant and $\mathcal{O}_e \subset \overline{\mathcal{O}}$ is of codimension two, we get that $\pi(\mathrm{Sing}(Z)) \cap \nu^{-1}(\mathcal{O}_e) = \emptyset$. We deduce that $V := \pi^{-1}(\widetilde{U})$ is smooth. In particular, we obtain a symplectic resolution $\pi|_V : V \to \widetilde{U}$. By restriction, we get a resolution $\pi : \pi^{-1}(\widetilde{\mathcal{S}}_{\mathcal{O},e}) \to \widetilde{\mathcal{S}}_{\mathcal{O},e}$, which is a symplectic, hence minimal, resolution.

Let $y \in \nu^{-1}(e)$. If we know: (1) the number of $\mathbb{P}^1$'s in $\pi^{-1}(y)$ and in $\pi^{-1}(\nu^{-1}(e))$; and (2) the action of $A(e)$ on the $\mathbb{P}^1$'s in $\pi^{-1}(\nu^{-1}(e))$, then in most cases we can determine the configuration of $\mathbb{P}^1$'s in $\pi^{-1}(\nu^{-1}(e))$, and hence in $\pi^{-1}(y)$, and therefore determine $\widetilde{\mathcal{S}}_{\mathcal{O},e}$. We next introduce some methods to compute this information.

5.4. The method of Borho–MacPherson

Let W be the Weyl group of G. The Springer correspondence assigns to any irreducible W-module a unique pair $(\mathcal{O}, \phi)$ consisting of a nilpotent orbit $\mathcal{O}$ in $\mathfrak{g}$ and an irreducible representation ϕ of the component group $A(x)$ where $x \in \mathcal{O}$. The corresponding irreducible W-module will be denoted by $\rho_{(x,\phi)}$.

Let W_L denote the Weyl group of L, viewed as a subgroup of W. Let $\mathcal{B}_x$ denote the Springer fiber over x for the resolution of the nilpotent cone $\mathcal{N}$ in $\mathfrak{g}$ and let $\mathcal{B}_t^L$ be the Springer fiber of t for the group L. If $\mathcal{O}_t^L$ is the orbit of L through the nilpotent element $t \in \mathrm{Lie}(L)$, we denote by $\rho^L_{(t,1)}$ the W_L-module corresponding to the pair $(\mathcal{O}_t^L, 1)$ via the Springer correspondence for L.

Lemma 5.5. *Let $Z = G \times^Q (\mathfrak{n}(\mathfrak{q}) + \overline{\mathcal{O}}_t^L)$. Let $p : Z \to \overline{\mathcal{O}}$ be the generalized Springer map. Let $\mathcal{O}' \subset \overline{\mathcal{O}}$ be a nilpotent orbit of codimension $2d$. Assume that Z is rationally smooth at all points of $p^{-1}(e)$ for $e \in \mathcal{O}'$. Then the number of irreducible components of $p^{-1}(e)$ of dimension d is given by the formula*

$$\frac{\deg \rho^L_{(t,1)}}{\dim H^{top}(\mathcal{B}_t^L)} \bigoplus_{\phi \in \mathrm{Irr}\, A(e)} \deg \phi \cdot [\mathrm{Res}^W_{W_L} \rho_{(e,\phi)} : \rho^L_{(t,1)}],$$

where the sum is over the irreducible representations ϕ of $A(e)$ appearing in the Springer correspondence for G.

Proof. By [8, Thm. 3.3], we have $H^{top}(p^{-1}(e)) \otimes H^{top}(\mathcal{B}_t^L) \cong H^{top}(\mathcal{B}_e)^{\rho^L_{(t,1)}}$, where the right hand side denotes the $\rho^L_{(t,1)}$-isotypical component of the restriction of $H^{top}(\mathcal{B}_e)$ to W_L. Recall that $H^{top}(\mathcal{B}_e) = \oplus_\phi \rho_{(e,\phi)} \otimes \phi$, which gives

$$h^{top}(p^{-1}(e)) \cdot h^{top}(\mathcal{B}_t^L) = \deg \rho^L_{(t,1)} \sum_\phi \deg \phi \cdot [\mathrm{Res}^W_{W_L} \rho_{(e,\phi)} : \rho^L_{(t,1)}],$$

where $h^{top}(X)$ denotes the dimension of $H^{top}(X)$. □

Now the component group $A(e)$ acts on the left-hand side of

$$H^{top}(p^{-1}(e)) \otimes H^{top}(\mathcal{B}_t^L) \cong H^{top}(\mathcal{B}_e)^{\rho^L_{(t,1)}}$$

where it acts trivially on $H^{top}(\mathcal{B}_t^L)$. It also acts on the right-hand side since the $A(e)$-action commutes with the W-action, and hence the W_L-action. Note that the action of $A(e)$ is compatible with the isomorphism (see Corollary 3.5 [8]). This gives the following

Corollary 5.6. *The permutation action of $A(e)$ on the irreducible components of dimension d of $p^{-1}(e)$ gives rise to the linear representation*

$$\bigoplus_{\phi \in \mathrm{Irr}\, A(e)} \deg \rho^L_{(t,1)} [\mathrm{Res}^W_{W_L} \rho_{(e,\phi)} : \rho^L_{(t,1)}] \phi \tag{5.1}$$

In particular the number of orbits of $A(e)$ on the irreducible components of $p^{-1}(e)$ of dimension d equals the multiplicity of the trivial representation of $A(e)$ in (5.1). The number of $A(e)$-orbits is therefore equal to $\deg \rho^L_{(t,1)} [\mathrm{Res}^W_{W_L} \rho_{(e,1)} : \rho^L_{(t,1)}]$.

Example 5.7. Let $\mathfrak{g}$ be of type F_4. Let $\mathcal{O}$ be the nilpotent orbit of type B_3 and $\mathcal{O}'$ of type $F_4(a_3)$. Then $\mathcal{O}' \subset \overline{\mathcal{O}}$ is codimension two. Since $\mathcal{O}$ is even, its weighted Dynkin diagram shows that $\mathcal{O}$ is Richardson for the parabolic subgroup Q with Levi subgroup L of semisimple type $\widetilde{A}_2$. This gives rise to the generalized Springer map $p : G \times^Q \mathfrak{n}(\mathfrak{q}) \to \overline{\mathcal{O}}$ as in Lemma 5.5, with $t = 0$. The map p is birational because e is even. Since $\overline{\mathcal{O}}$ is normal and p is birational, the restriction of p gives a minimal resolution of $\widetilde{\mathcal{S}}_{\mathcal{O},e} = \mathcal{S}_{\mathcal{O},e}$ where $e \in \mathcal{O}'$ as in §5.3.

Now $A(e) = \mathfrak{S}_4$. Since $t = 0$, the representation $\rho^L_{(t,1)}$ is the sign representation of W_L. By the Springer correspondence for F_4, $\rho_{(e,[21^2])} = \phi'_{1,12}$, $\rho_{(e,[2^2])} = \phi''_{6,6}$, $\rho_{(e,[31])} = \phi'_{9,6}$ and $\rho_{(e,[4])} = \phi_{12,4}$ (see [12, pg. 428]). The multiplicity of the sign representation in the restriction of $\rho_{(e,[2^2])}$ to W_L is 1 and in the restriction of $\rho_{(e,[4])}$ is 2 and it is zero otherwise. By Lemma 5.5, the number of $\mathbb{P}^1$'s in $p^{-1}(e)$ is $1 \cdot 2 + 2 \cdot 1 = 4$ and by Corollary 5.6, the group $A(e)$ fixes one component and permutes the remaining three components transitively. Consequently the dual graph of $\mathcal{S}_{\mathcal{O},e} = \widetilde{\mathcal{S}}_{\mathcal{O},e}$ is the Dynkin diagram of type D_4 and $A(e)$ acts on the dual graph via the unique quotient of $A(e)$ isomorphic to $\mathfrak{S}_3$. Hence the singularity is G_2.

The fact that the dual graph is D_4 could also be obtained by restricting to a maximal subalgebra of type B_4 (§4.1.1). In this way we would only need to know that the degeneration in F_4 is unibranch, instead of the stronger statement that $\overline{\mathcal{O}}$ is normal.

5.5. Orbital varieties and the exceptional divisor of π

The next lemma (see [19, Lemma 4.3]) can sometimes be used to simplify computations.

Lemma 5.8. *Let $\mathcal{O}$ be a nilpotent orbit with* $\mathrm{Pic}(\mathcal{O})$ *finite and such that there is a generalized Springer resolution $\pi : G \times^Q \mathfrak{n}(\mathfrak{q}) \to \overline{\mathcal{O}}$ (see §5.2). Then the number of irreducible exceptional divisors of π equals $b_2(G/Q)$, the second Betti number of G/Q, which is equal to the rank of G minus the semisimple rank of a Levi subgroup of Q.*

From [19, Prop 4.4] it follows that $\mathrm{Pic}(\mathcal{O}_x)$ is finite whenever the character group of G^x is finite. Picking an $\mathfrak{sl}_2$-subalgebra $\mathfrak{s}_x$ containing x, the latter is equivalent to the finiteness of the character group of $C(\mathfrak{s}_x)$, or equivalently, to the finiteness of the center of $C(\mathfrak{s}_x)$. The latter can be read off from the tables in [2] or deduced from the tables in [51]. Such calculations are closely related to those in §6. In the exceptional groups, $\mathrm{Pic}(\mathcal{O})$ is finite unless $\mathcal{O}$ is one of the following orbits in E_6: $2A_1, A_2 + A_1, A_2 + 2A_1, A_3, A_3 + A_1, A_4, A_4 + A_1, D_5(a_1), D_5$. For these orbits in E_6, the number of irreducible exceptional divisors of a generalized Springer resolution or a $\mathbb{Q}$-factorial terminalization has been explicitly computed in the proof of [19, Prop 4.4].

Let $\mathcal{O}_1, \dots \mathcal{O}_s$ be the maximal orbits in the complement of $\mathcal{O}$ in $\overline{\mathcal{O}}$. We restrict to the case where all $\mathcal{O}_i$'s are codimension two in $\overline{\mathcal{O}}$. Then the irreducible exceptional divisors of π have a description in terms of the orbital varieties for the $\mathcal{O}_i$'s. Recall that an orbital variety for $\mathcal{O}_i$ is an irreducible component of $\overline{\mathcal{O}}_i \cap \mathfrak{n}$ where $\mathfrak{n} := \mathfrak{n}(\mathfrak{b})$ is the nilradical of the Borel subalgebra $\mathfrak{b}$. It is known that each orbital variety has dimension $\frac{1}{2} \dim \mathcal{O}_i$. Let X be an orbital variety for $\mathcal{O}_i$ which is contained in $\mathfrak{n}(\mathfrak{q})$. Then X is of codimension one in $\mathfrak{n}(\mathfrak{q})$ since $\mathcal{O}_i$ is of codimension two in $\overline{\mathcal{O}}$ and $\dim \mathfrak{n}(\mathfrak{q}) = \frac{1}{2} \dim \mathcal{O}$. Moreover X is stable under the action of the connected group Q since $X \subset Q \cdot X \subset \overline{\mathcal{O}}_i \cap \mathfrak{n}$ and X is maximal irreducible in $\overline{\mathcal{O}}_i \cap \mathfrak{n}$.

Let π_X be the restriction of π to $G \times^Q X$. The image of π_X is $\overline{\mathcal{O}}_i$ since X is irreducible and Q is a parabolic. By dimension considerations, $\pi_X^{-1}(\overline{\mathcal{O}}_i) = G \times^Q X$ is an irreducible exceptional divisor of π. Conversely, any irreducible exceptional divisor of π equals $G \times^Q Y$ for some irreducible component Y of $\overline{\mathcal{O}}_i \cap \mathfrak{n}(\mathfrak{q})$. Now $\dim Y$ can only equal $\dim \mathfrak{n}(\mathfrak{q}) - 1$ or $\dim \mathfrak{n}(\mathfrak{q}) - 2$ since $\mathrm{Im}\, \pi_Y = \overline{\mathcal{O}}_i$. In the former case, Y is an orbital variety of X contained in $\mathfrak{n}(\mathfrak{q})$. In the latter case, $\pi_Y^{-1}(e_i)$ is finite where $e_i \in \mathcal{O}_i$, contradicting the fact, from above, that the irreducible components of $\pi^{-1}(e_i)$ are $\mathbb{P}^1$'s. This shows that the irreducible exceptional divisors of π are exactly the $G \times^Q X$ where X is an orbital variety of some $\mathcal{O}_i$ lying in $\mathfrak{n}(\mathfrak{q})$.

Next, the map $G \times^B X \to G \times^Q X$ has connected fibers isomorphic to Q/B. It follows from [54] that the $\mathbb{P}^1$'s in $\pi_X^{-1}(e_i)$ are permuted transitively under the induced action of $A(e_i)$ since the analogous statement holds for the irreducible components of $p_X^{-1}(e_i)$ where $p_X : G \times^B X \to \mathcal{N}$. Consequently, if $\operatorname{Pic}(\mathcal{O})$ is finite and r_i equals the number of $A(e_i)$-orbits on $\pi^{-1}(e_i)$, then $\sum r_i = b_2(G/Q)$ by Lemma 5.8. See, for example, [58, Thm 1.3] for a more general setting where this phenomenon occurs.

Example 5.9. Consider the minimal degeneration where $\mathcal{O}$ has type $\widetilde{A}_2$ and $\mathcal{O}'$ has type $A_1 + \widetilde{A_1}$ in F_4. The codimension of $\mathcal{O}'$ in $\overline{\mathcal{O}}$ is two. The orbit $\mathcal{O}$ is Richardson for the parabolic subgroup Q whose Levi subgroup has type B_3. Moreover the map $\pi : Z := G \times^Q \mathfrak{n}(\mathfrak{q}) \to \overline{\mathcal{O}}$ is birational, hence a generalized Springer resolution. The hypotheses of Lemma 5.8 hold. Since $b_2(Z) = 1$ and there is no other minimal degeneration of $\mathcal{O}$, there must be exactly one irreducible component in $\pi^{-1}(\overline{\mathcal{O}'})$. Since $A(e) = 1$ for $e \in \mathcal{O}'$, there is only one irreducible component in $\pi^{-1}(e)$. Since $\overline{\mathcal{O}}$ is normal, the singularity of $\overline{\mathcal{O}}$ at e is of type A_1.

5.6. Three remaining cases

There are three cases where the information in Lemma 5.5 and Corollary 5.6 is not sufficient to determine a minimal surface degeneration, even up to normalization. They are $(E_6(a_1), D_5)$ in E_6, $(E_7(a_1), E_7(a_2))$ in E_7, and $(E_8(a_1), E_8(a_2))$ in E_8. In this section we give an ad hoc way to determine the singularity.

In each of the three cases, the larger orbit $\mathcal{O}$ is the subregular nilpotent orbit and so $\overline{\mathcal{O}}$ is normal. Since $\mathfrak{g}$ is simply-laced, $A(x)$ is trivial for $x \in \mathcal{O}$. Hence for any parabolic subgroup Q with Levi factor A_1 the map $\pi : G \times^Q \mathfrak{n}(\mathfrak{q}) \to \overline{\mathcal{O}}$ is birational. Moreover in each case the smaller orbit $\mathcal{O}'$ is the unique maximal orbit in $\overline{\mathcal{O}} \backslash \mathcal{O}$. Since $A(e) = 1$ for $e \in \mathcal{O}'$, there are $\operatorname{rank}(\mathfrak{g}) - 1$ $\mathbb{P}^1$'s in $\pi^{-1}(e)$ by §5.5. At the same time, this uniqueness means that $\mathcal{O}'$ is the Richardson orbit for any parabolic Q' with Levi factor of semisimple type $A_1 \times A_1$, so if Q' is such a parabolic, then $\mathfrak{n}(\mathfrak{q}')$ is an orbital variety for $\mathcal{O}'$. Hence if we fix Q corresponding to a simple root α, then we find an orbital variety $\mathfrak{n}(\mathfrak{q}') \subset \mathfrak{n}(\mathfrak{q})$ for $\mathcal{O}'$ for each simple root β not connected to α in the Dynkin diagram. Since $A(e)$ is trivial, each of these $\mathfrak{n}(\mathfrak{q}')$ gives rise to a unique $\mathbb{P}^1$ in $\pi^{-1}(e)$. By looking in the Levi subalgebra corresponding to the simple roots not connected to α, it is possible to determine the intersection pattern of these $\mathbb{P}^1$'s.

5.6.1. The case of $(E_6(a_1), D_5)$ in E_6

There are 5 $\mathbb{P}^1$'s in $\pi^{-1}(e)$. The singularity could only be A_5 or D_5 since $\overline{\mathcal{O}}$ is normal. If we choose α so that the remaining simple roots form a root system of type A_5, then there are 4 orbital varieties of the form $\mathfrak{n}(\mathfrak{q}')$ in $\mathfrak{n}(\mathfrak{q})$. The 4 $\mathbb{P}^1$'s have intersection diagram of type $A_2 + A_2$. This could only happen for a dual graph of type A_5, so $\mathcal{S}_{\mathcal{O},e} \cong A_5$.

5.6.2. The case of $(E_7(a_1), E_7(a_2))$ *in* E_7

There are 6 $\mathbb{P}^1$'s in $\pi^{-1}(e)$. The singularity could only be A_6, D_6, or E_6 since $\overline{\mathcal{O}}$ is normal. Choosing α so that the remaining simple roots form a system of type E_6, there are 5 orbital varieties of the form $\mathfrak{n}(\mathfrak{q}')$ in $\mathfrak{n}(\mathfrak{q})$. Then the 5 $\mathbb{P}^1$'s have intersection diagram of type D_5. This eliminates A_6 as a possibility. If we choose α so that the remaining simple roots form a system of type A_6, then there are 5 orbital varieties of the form $\mathfrak{n}(\mathfrak{q}')$ in $\mathfrak{n}(\mathfrak{q})$ and the corresponding 5 $\mathbb{P}^1$'s have intersection diagram of type $A_2 + A_3$. This eliminates E_6, hence $\mathcal{S}_{\mathcal{O},e} \cong D_6$.

5.6.3. The case of $(E_8(a_1), E_8(a_2))$ *in* E_8

There are 7 $\mathbb{P}^1$'s in $\pi^{-1}(e)$. The singularity could only be A_7, D_7, or E_7 since $\overline{\mathcal{O}}$ is normal. If we choose α so that the remaining simple roots form a system of type E_7, then there are 6 orbital varieties of the form $\mathfrak{n}(\mathfrak{q}')$ in $\mathfrak{n}(\mathfrak{q})$. The corresponding 6 $\mathbb{P}^1$'s have intersection diagram of type E_6. Hence $\mathcal{S}_{\mathcal{O},e} \cong E_7$.

Remark 5.10. Ben Johnson and the fourth author have also confirmed these three results using Broer's description of the ideal defining the closure of the subregular nilpotent orbit and the Magma algebra system.

6. On the splitting of $C(\mathfrak{s})$ and intrinsic symmetry action

6.1. The splitting of $C(\mathfrak{s})$

In this section we establish the splitting on $C(\mathfrak{s})$ discussed in §1.8.3. That is, we determine when

$$C(\mathfrak{s}) \cong C(\mathfrak{s})^\circ \rtimes H$$

for some $H \subset C(\mathfrak{s})$. Necessarily $H \cong A(e)$. We continue to assume that G is of adjoint type.

In the classical groups, $C(\mathfrak{s})$ is a product of orthogonal groups and a connected group, possibly up to a quotient by a central subgroup of order two. Since the result holds for any orthogonal group, it holds for $C(\mathfrak{s})$.

Let $\mathcal{C} \subset A(e)$ be a conjugacy class. There exists $s \in C(\mathfrak{s})$ whose image $\bar{s}$ in $A(e)$ lies in $\mathcal{C}$ such that the order of s equals the order of $\bar{s}$, except when e belongs to one the following four orbits:

$$A_4 + A_1 \text{ in } E_7; \qquad A_4 + A_1, D_7(a_2) \text{ and } E_6(a_1) + A_1 \text{ in } E_8. \tag{6.1}$$

For these four orbits, which all have $A(e) = \mathfrak{S}_2$, the best result is an s of order 4 to represent the non-trivial $\mathcal{C}$ in $A(e)$ [51, §3.4]. Hence the splitting holds for all other orbits where $A(e) = \mathfrak{S}_2$, with $H = \{1, s\}$.

This leaves the cases where $A(e) = \mathfrak{S}_3, \mathfrak{S}_4$, or $\mathfrak{S}_5$. If e is distinguished, meaning $C(\mathfrak{s})^\circ = 1$, there is nothing to check. This leaves a handful of cases where $A(e) = \mathfrak{S}_3$ and e is not distinguished. The first such case is $e = D_4(a_1)$ in E_6, which we now explain.

6.1.1. $\mathfrak{S}_3$ cases

Let G be of type E_6 and $s \in G$ be an involution with G^s of semisimple type $A_5 + A_1$. Then there exist $\tilde{e} \in \mathfrak{g}^s$ nilpotent of type $2A_2$. Let $\tilde{\mathfrak{s}} \subset \mathfrak{g}^s$ be an $\mathfrak{sl}_2$-triple through $\tilde{e}$. Then $\mathfrak{c}(\tilde{\mathfrak{s}})$ has type G_2. It is easy to compute $\mathfrak{g}^s \cap \mathfrak{c}(\tilde{\mathfrak{s}})$ inside of $A_5 + A_1$; it is a semisimple subalgebra of type $A_1 + A_1$. Let $\tilde{e}_0$ be regular nilpotent in $\mathfrak{g}^s \cap \mathfrak{c}(\tilde{\mathfrak{s}})$. Then $\tilde{e}_0$ is in the subregular nilpotent orbit in $\mathfrak{c}(\tilde{\mathfrak{s}})$. Clearly s belongs to the centralizer of $\tilde{e}_0$ in $C(\tilde{\mathfrak{s}})$, which is a finite group $H \cong \mathfrak{S}_3$, from the case of the subregular orbit in G_2. Next, a calculation in $A_5 + A_1$ shows that $\tilde{e} + \tilde{e}_0$ has generalized Bala–Carter type $A_3 + 2A_1$. From this we conclude that $e = \tilde{e} + \tilde{e}_0$ belongs to the nilpotent orbit $D_4(a_1)$ in E_6 and s represents an involution in $A(e)$ [51, §4].

A similar argument works if $s \in G$ is an element of order 3 with G^s of semisimple type $3A_2$. Therefore the centralizer $H \cong \mathfrak{S}_3$ of $\tilde{e}_0$ in $C(\tilde{\mathfrak{s}})$ also centralizes $\tilde{e} + \tilde{e}_0$ and the image of H in $A(e)$ is all of $A(e)$. This proves the splitting for $e = D_4(a_1)$ in E_6. The same procedure works for the other $\mathfrak{S}_3$ cases.

6.1.2. We have shown

Proposition 6.1. *There exists $H \subset C(\mathfrak{s})$ such that*

$$C(\mathfrak{s}) \cong C(\mathfrak{s})^\circ \rtimes H,$$

except when e belongs to one of the four orbits in (6.1). For those four cases, $A(e) = \mathfrak{S}_2$ and

$$C(\mathfrak{s}) = C(\mathfrak{s})^\circ \cdot H$$

where $H \subset C(\mathfrak{s})$ is cyclic of order 4.

While the above splitting is unique up to conjugacy in $C(\mathfrak{s})$ in the subregular case (§1.5.2), this is not the case in general, as the next example shows.

Example 6.2. Let e be in the A_2 orbit in $\mathfrak{g}$ of type E_8. Then $\mathfrak{c}(\mathfrak{s})$ has type E_6 and $A(e) = \mathfrak{S}_2$. The generalized Bala–Carter notation for the non-trivial class $\mathcal{C}$ in $A(e)$ is $(4A_1)''$. From this it follows that both conjugacy classes of involution in G can represent $\mathcal{C}$. For one choice of involution $s_1 \in C(\mathfrak{s})$ lifting $\mathcal{C}$, $\mathfrak{g}^{s_1}$ has type D_8. The partition of e in $\mathfrak{g}^{s_1}$ is $[2^8]$, so the reductive centralizer of e in $\mathfrak{g}^{s_1}$ is $\mathfrak{sp}_8$. For the other choice $s_2 \in C(\mathfrak{s})$ lifting $\mathcal{C}$, $\mathfrak{g}^{s_2}$ has type $E_7 + A_1$ and e corresponds to $(3A_1)'' + A_1$. Hence the reductive centralizer of e in $\mathfrak{g}^{s_2}$ is of type F_4. Consequently, there are two choices of splitting in Proposition 6.1 that are not only non-conjugate under $C(\mathfrak{s})$, but also in $\mathrm{Aut}(\mathfrak{c}(\mathfrak{s}))$.

Although the choice of splitting in Proposition 6.1 is not unique up to conjugacy in $C(\mathfrak{s})$ or even $\mathrm{Aut}(\mathfrak{c}(\mathfrak{s}))$, we can restrict the choice of H further so that the image of H in $\mathrm{Aut}(\mathfrak{c}(\mathfrak{s}))$ will be well-defined up to conjugacy in $\mathrm{Aut}(\mathfrak{c}(\mathfrak{s}))$. Let $\mathfrak{c}(\mathfrak{s})^{\mathrm{ss}}$ be the semisimple summand of $\mathfrak{c}(\mathfrak{s})$. Let

$$a : C(\mathfrak{s}) \to \mathrm{Aut}(\mathfrak{c}(\mathfrak{s})^{\mathrm{ss}})$$

be the natural map. Then $\mathrm{Im}\, a = \mathrm{Int}(\mathfrak{c}(\mathfrak{s})^{\mathrm{ss}}) \rtimes K$ for some subgroup of diagram automorphisms (§1.5.2). By a case-by-case check, H in the Proposition 6.1 can be chosen so that H maps onto K via a. Then the image of H in $\mathrm{Aut}(\mathfrak{c}(\mathfrak{s}))$ is well-defined up to conjugacy in $\mathrm{Aut}(\mathfrak{c}(\mathfrak{s}))$. In the above example, $H = \langle s_2 \rangle$ has the desired property, since F_4 is the fixed subalgebra under the non-trivial diagram automorphism of E_6. We note that [2] is the original source for computing the image of the map a.

6.2. *Computing the intrinsic symmetry*

Having chosen H with $a(H) = K$ as above, we can determine the action of H on $\mathcal{S}_{\mathcal{O},e}$. Here, we restrict to the exceptional groups and to a minimal degeneration $\mathcal{O}'$ of $\mathcal{O}$, with $e \in \mathcal{O}'$. We summarize the possibilities and record the action of H on $\mathcal{S}_{\mathcal{O},e}$ in the graphs at the end of the paper.

6.2.1. *Minimal singularities: $A(e) = \mathfrak{S}_2$ cases*

Let $\mathcal{S}_{\mathcal{O},e}$ be an irreducible minimal singularity admitting an involution as in §1.8.2. If $|H| = 2$, then it turns out that H realizes this involution. There is one case of this kind when $|H| = 4$, when $e = A_4 + A_1$ in E_8 and $\mathcal{S}_{\mathcal{O},e} \cong a_2$. Let $H = \langle s \rangle$. Then $s \in H$ realizes the involution on $\mathcal{S}_{\mathcal{O},e}$ and s^2 acts trivially on $\mathcal{S}_{\mathcal{O},e}$. We will still refer to this singularity with induced symmetry by a_2^+.

If $\mathcal{S}_{\mathcal{O},e}$ is a reducible minimal singularity, then it is turns out that $\mathcal{S}_{\mathcal{O},e}$ has exactly two irreducible components and H interchanges the two components. The only three cases which occur are the singularities with symmetry action $[2A_1]^+, [2a_2]^+$, and $[2g_2]^+$.

6.2.2. *Minimal singularities: $A(e) = \mathfrak{S}_3$ cases*

If $\mathcal{S}_{\mathcal{O},e}$ is the unique irreducible minimal singularity admitting an action of $\mathfrak{S}_3$ as in §1.8.2, then H realizes the full symmetry d_4^{++}. This only occurs once, in E_8.

If $\mathcal{S}_{\mathcal{O},e}$ is a reducible minimal singularity, then $\mathcal{S}_{\mathcal{O},e}$ turns out to have 3 irreducible components and H acts by permuting transitively the three components. In other words, the stabilizer of a component acts trivially on the component. All of these cases are of the form $3A_1$ and the singularity with symmetry action is denoted $[3A_1]^{++}$.

6.2.3. *Simple surface singularities: $A(e) = \mathfrak{S}_2$ cases*

If $\mathcal{S}_{\mathcal{O},e}$ is an irreducible simple surface singularity admitting an involution as in §1.4.2 (or in the case of A_2 and A_4, admitting the appropriate cyclic action of order 4), then H realizes this symmetry. To show this, we first checked that $A(e)$ has the appropriate action on the dual graph of a minimal resolution in Corollary 5.6. Then since $C(\mathfrak{s})$ acts symplectically on $\mathcal{S}_{\mathcal{O},e}$, Corollary 1.1 and Theorem 1.2 in [13] imply that H corresponds to the $\Gamma' \subset \mathrm{SL}_2(\mathbb{C})$ which defines the symmetry involution.

The only reducible surface singularities with $A(e) = \mathfrak{S}_2$ are those with $\mathcal{S}_{\mathcal{O},e} \cong 2A_1$, hence covered previously.

6.2.4. Simple surface singularities: $A(e) = \mathfrak{S}_3$ cases

If $\mathcal{S}_{\mathcal{O},e}$ is an irreducible simple surface singularity admitting an $\mathfrak{S}_3$ action as in §1.4.2, then H realizes the symmetry action and so $\mathcal{S}_{\mathcal{O},e} \cong G_2$.

An unusual situation occurs for the minimal degeneration $(D_7(a_1), E_8(b_6))$. Here, $A(e) = \mathfrak{S}_3$, but $\mathcal{S}_{\mathcal{O},e}$ only admits a two-fold symmetry, compatible with its normalization $\widetilde{\mathcal{S}}_{\mathcal{O},e}$ which is A_3. Here, $\Gamma \subset \mathrm{SL}_2(\mathbb{C})$ corresponding to $\widetilde{\mathcal{S}}_{\mathcal{O},e}$ is cyclic of order 4. The normal cyclic subgroup of $H \cong \mathfrak{S}_3$ is generated by an element s with $\mathfrak{g}^s$ of type $E_6 + A_2$ and hence s acts without fixed point on the orbit $D_7(a_1)$ since the latter orbit does not meet the subalgebra $E_6 + A_2$. On the other hand, using Corollary 5.6, we see that $A(e)$ induces the involution on the dual graph of a minimal resolution of $\widetilde{\mathcal{S}}_{\mathcal{O},e}$. Since $C(\mathfrak{s})$ acts symplectically on $\mathcal{S}_{\mathcal{O},e}$ and $\widetilde{\mathcal{S}}_{\mathcal{O},e}$, Corollary 1.1 and Theorem 1.2 in [13] imply that H acts on $\widetilde{\mathcal{S}}_{\mathcal{O},e} = \mathbb{C}^2/\Gamma$ via the action of $\Gamma' \subset \mathrm{SL}_2(\mathbb{C})$, the binary dihedral group of order 24 containing Γ as normal subgroup.

If $\mathcal{S}_{\mathcal{O},e}$ is a reducible surface singularity, then $\mathcal{S}_{\mathcal{O},e}$ is isomorphic to $3C_2, 3C_3, 3(C_5)$, or the previously covered $[3A_1]^{++}$. We have omitted the superscript in $3C_2$, etc. The notation means that H permutes the three components transitively and the stabilizer of any component is order 2, which acts by the indicated symmetry. The notation (C_5) refers to the fact that we do not know whether an irreducible component is normal.

6.2.5. Simple surface singularities: $A(e) = \mathfrak{S}_4$ case

This only occurs in F_4. One degeneration has $\mathcal{S}_{\mathcal{O},e} \cong G_2$ (see §5.7). Here, the Klein 4-group in H acts trivially on $\mathcal{S}_{\mathcal{O},e}$ and the quotient action realizes the full symmetry of $\mathfrak{S}_3$ on $\mathcal{S}_{\mathcal{O},e}$. This follows either from the list of possible symplectic automorphisms of $\mathcal{S}_{\mathcal{O},e}$ or from a direct calculation that the Klein 4-group in H fixes $\mathcal{S}_{\mathcal{O},e}$ pointwise.

The other degeneration has $\mathcal{S}_{\mathcal{O},e} \cong 4G_2$ (see §7.2). Here, H permutes the four components transitively and the stabilizer of any component is an $\mathfrak{S}_3$, which acts by the indicated symmetry.

6.2.6. Simple surface singularities: $A(e) = \mathfrak{S}_5$ case

This only occurs in E_8. One degeneration has $\mathcal{S}_{\mathcal{O},e} \cong 10G_2$. Here, H permutes the ten components transitively and the stabilizer of any component is a Young subgroup $\mathfrak{S}_3 \times \mathfrak{S}_2$. The $\mathfrak{S}_2$ factor acts trivially on the given component and the $\mathfrak{S}_3$ factor acts by the indicated symmetry.

The other degeneration has $\mathcal{S}_{\mathcal{O},e} \cong 5G_2$. Here, H permutes the five components transitively and the stabilizer of any component is a $\mathfrak{S}_4$. The $\mathfrak{S}_4$ factor acts on the given component as in the F_4 case above.

Table 1
$\mathbf{F_4}$: cases with $e_0 \in \mathfrak{c}(\mathfrak{s})$ of type A_1 in $\mathfrak{g}$.

e	$e + e_0 \in \mathcal{O}$	$\mathfrak{c}(\mathfrak{s})$	Isomorphism type of $\mathcal{S}_{\mathcal{O},e}$
A_1	$2A_1 = \tilde{A}_1$	C_3	c_3
$\tilde{A}_1$	$\tilde{A}_1 + A_1$	A_3	a_3^+
$A_1 + \tilde{A}_1$	$2A_1 + \tilde{A}_1 = A_2$	$A_1 + \mathbf{A_1}$	A_1
$\tilde{A}_2$	$\tilde{A}_2 + A_1$	G_2	g_2
B_2	$B_2 + A_1 = C_3(a_1)$	$\mathbf{2A_1}$	$[2A_1]^+$
$C_3(a_1)$	$C_3(a_1) + A_1 = F_4(a_3)$	A_1	A_1
C_3	$C_3 + A_1 = F_4(a_2)$	A_1	A_1

Remark 6.3. Even when $A(e)$ is non-trivial, it might not induce a non-trivial symmetry on any $\mathcal{S}_{\mathcal{O},e}$. For example, when $e = C_3(a_1)$ in F_4, the only degeneration above $\mathcal{O}_e$ has $\mathcal{S}_{\mathcal{O},e} \cong A_1$. Here, H acts trivially on $\mathcal{S}_{\mathcal{O},e}$, reflecting the fact that $\mathrm{SL}_2(\mathbb{C})$ has no outer automorphisms. Indeed, $C(\mathfrak{s})$ is just the direct product $C(\mathfrak{s})^\circ \times H$.

7. Results for F_4

7.1. Details in the proof of Proposition 3.3

Here we record the details for establishing Proposition 3.3 for $\mathfrak{g}$ of type F_4, as outlined in §4.5. First, we enumerate the G-orbits of those e such that $\mathfrak{c}(\mathfrak{s})$ has non-trivial intersection with the minimal nilpotent orbit in $\mathfrak{g}$. To that end, let $e_0 \in \mathfrak{g}$ be minimal nilpotent and recall that $\mathfrak{s}_0$ is an $\mathfrak{sl}_2(\mathbb{C})$-subalgebra through e_0. The centralizer $\mathfrak{c}(\mathfrak{s}_0)$ is a simple subalgebra of type C_3, equal to the semisimple part of a Levi subalgebra of $\mathfrak{g}$. The relevant nonzero nilpotent elements $e \in \mathfrak{c}(\mathfrak{s}_0)$ are therefore those in the G-orbits

$$A_1, \tilde{A}_1, A_1 + \tilde{A}_1, \tilde{A}_2, B_2, C_3(a_1) \text{ and } C_3$$

and hence Corollary 4.9 applies to these elements. The computation of $e + e_0 \in \mathcal{O}$ proceeds as in §4.2. The results are in Table 1. We use boldface font in Table 1 to locate the simple factors whose minimal nilpotent orbit is of type A_1 in $\mathfrak{g}$. Where more than one such simple factor is in boldface, this indicates that the factors are conjugate under the action of $C(\mathfrak{s})$. The first two lines of Table 2 have the remaining cases where Lemma 4.3 applies with $x = e + e_0$ for an element e_0 in a minimal nilpotent orbit of $\mathfrak{c}(\mathfrak{s})$. This now exhausts all minimal degenerations covered by Proposition 3.3 with $J = \emptyset$.

The remaining minimal degenerations in the proposition, of which there are four, are all codimension two and unibranch. We use §7.2 to determine that these exhaust the remaining codimension two cases with $|\Gamma| = 1$ or 2. We now show that all four cases are S-varieties for $SL_2(\mathbb{C})$ of the form $X(2, i_1 + 2, i_2 + 2, \dots)$, with $J = \{i_1, i_2, \dots\}$ among those listed in Remark 3.4. All cases can be handled with Lemma 4.6 and Corollary 4.7, but in one case we need to pass to a subalgebra (as in Lemma 4.1) and in another, do an explicit computation to find the exact form of $\mathcal{S}_{\mathcal{O},e}$. The values of (m_j, n_j) for $j \in \mathcal{E}$ are

Table 2
$\mathbf{F_4}$: Remaining relevant cases with e_0 minimal in $\mathfrak{c}(\mathfrak{s})$.

e	e_0	$e+e_0$	$\mathcal{O}$	$\mathfrak{c}(\mathfrak{s})$	(m_i,n_i) for $i\in\mathcal{E}$	Isomorphism type of $\mathcal{S}_{\mathcal{O},e}$
A_2	$\tilde{A}_1$	$A_2+\tilde{A}_1$	$A_2+\tilde{A}_1$	A_2	$\emptyset$	a_2^+
B_3	$\tilde{A}_2$	$F_4(a_2)$	$F_4(a_2)$	A_1	$\emptyset$	A_1
$A_1+\tilde{A}_1$	$\tilde{A}_2$	$C_3(a_1)$	$\tilde{A}_2$	$\mathbf{A_1}+A_1$	$(\mathbf{2},\mathbf{4})$	A_1
$A_2+\tilde{A}_1$	$A_2+\tilde{A}_1$	$F_4(a_3)$	$\tilde{A}_2+A_1$	A_1	$(\mathbf{1},\mathbf{3}),(\mathbf{2},\mathbf{4})$	m
			B_2		$(1,3),(\mathbf{2},\mathbf{4})$	A_1
$\tilde{A}_2+A_1$	$A_1+\tilde{A}_1$	$F_4(a_3)$	$C_3(a_1)$	A_1	$(\mathbf{1},\mathbf{3})$	m

listed in Table 2. Boldface is used for those (m_j,n_j) where $x_{m_j}\neq 0$ in (4.1). Equivalently, the set J consists of the m_j's in boldface.

7.1.1. The degeneration $(\tilde{A}_2, A_1+\tilde{A}_1)$

For e of type $A_1+\tilde{A}_1$, $\mathfrak{c}(\mathfrak{s})\cong\mathfrak{sl}_2(\mathbb{C})\oplus\mathfrak{sl}_2(\mathbb{C})$. The nonzero nilpotent elements in one simple factor of $\mathfrak{c}(\mathfrak{s})$ are minimal in $\mathfrak{g}$ and this case was handled earlier. The nonzero nilpotent elements in the other simple factor $\mathfrak{z}$ are of type $\tilde{A}_2$ in $\mathfrak{g}$. Let $e_0\in\mathfrak{z}$ be such an element. The centralizer $\mathfrak{c}(\mathfrak{z})$ is contained in a Levi subalgebra of $\mathfrak{g}$ whose semisimple type is B_3, and thus $\mathfrak{c}(\mathfrak{z})$ does not meet the G-orbit $\mathcal{O}$ of type $\tilde{A}_2$. Hence Lemma 4.6 applies to $\mathcal{S}_{\mathcal{O},e}$. Now $e+e_0$ is of type $C_3(a_1)$ and $(m_i,n_i)=(2,4)$ for the unique element $i\in\mathcal{E}$. The argument in Example 4.1.5 then gives that $\mathcal{S}_{\mathcal{O},e}=e+X(2,4)\cong X(2)$ is an A_1-singularity.

In fact Example 4.1.5 can be used more directly. This also illustrates the process of passing to a subalgebra to establish that $\mathcal{S}_{\mathcal{O},e}$ has the desired form as an S-variety. Let $\mathfrak{l}$ be a Levi subalgebra of $\mathfrak{g}$ whose semisimple type is C_3. The G-orbit through e meets $\mathfrak{l}$ in the orbit $[2^3]$, so we may assume $e\in\mathfrak{s}\subset\mathfrak{l}$. Then $\mathfrak{c}(\mathfrak{s})\cap\mathfrak{l}$ coincides with $\mathfrak{z}$ and $\mathcal{O}\cap\mathfrak{l}$ coincides with the orbit in $\mathfrak{l}$ of type $[3^2]$. For dimension reasons it follows that $\mathcal{S}_{\mathcal{O}\cap\mathfrak{l},e}\subset\mathfrak{l}$ equals $\mathcal{S}_{\mathcal{O},e}$. Thus Example 4.1.5 directly gives $\mathcal{S}_{\mathcal{O},e}=e+X(2,4)$.

7.1.2. The degeneration $(C_3(a_1),\tilde{A}_2+A_1)$

For e of type $\tilde{A}_2+A_1$, $\mathfrak{c}(\mathfrak{s})\cong\mathfrak{sl}_2(\mathbb{C})$. Let $e_0\in\mathfrak{c}(\mathfrak{s})$ be a nonzero element in $\mathfrak{z}=\mathfrak{c}(\mathfrak{s})$, which has type $A_1+\tilde{A}_1$ in $\mathfrak{g}$. The orbit $\mathcal{O}$ of type $C_3(a_1)$ does not meet $\mathfrak{c}(\mathfrak{z})$, so Lemma 4.6 applies. The sum $e+e_0$ is of type $F_4(a_3)$ and $(m_i,n_i)=(1,3)$ for the unique element $i\in\mathcal{E}$. Hence as in Example 4.4.3, we have $\mathcal{S}_{\mathcal{O},e}=e+X(2,3)\cong m$. The result can also be obtained by reducing to the subalgebra $\mathfrak{s}'\oplus\mathfrak{c}(\mathfrak{s}')$, where $\mathfrak{s}'$ is the $\mathfrak{sl}_2$-subalgebra through an element e' of type $\tilde{A}_2$. A key factor making this work is that $\mathfrak{c}(\mathfrak{s}')$ has type G_2 and we can directly use Example 4.4.3. We omit the details.

7.1.3. The degenerations $(B_2, A_2+\tilde{A}_1)$ *and* $(\tilde{A}_2+A_1, A_2+\tilde{A}_1)$

For e of type $A_2+\tilde{A}_1$, $\mathfrak{c}(\mathfrak{s})\cong\mathfrak{sl}_2(\mathbb{C})$. The nonzero nilpotent orbit in $\mathfrak{c}(\mathfrak{s})$ also has type $A_2+\tilde{A}_1$ in $\mathfrak{g}$. Hence for $\mathfrak{z}:=\mathfrak{c}(\mathfrak{s})$, we have $\mathfrak{c}(\mathfrak{z})=\mathfrak{s}$ and so Lemma 4.6 applies for $\mathcal{O}$ both of type B_2 and of type $\tilde{A}_2+A_1$. Let $e_0\in\mathfrak{z}$ be nonzero nilpotent. The sum $e+e_0$

is of type $F_4(a_3)$ and $\{(1,3),(2,4)\}$ are the values for the two elements in $\mathcal{E}$. Indeed the decomposition of $\mathfrak{g}$ in (4.5) is

$$V(0,2) \oplus V(1,3) \oplus V(2,0) \oplus V(2,4) \oplus V(3,1) \oplus V(4,2).$$

So the only remaining question to determine the isomorphism type of $\mathcal{S}_{\mathcal{O},e}$ is whether x_1 is nonzero when expressing $x \in \mathcal{O}$ as in (4.1).

For $\mathcal{O}$ of type B_2 in $\mathfrak{g}$, we see that $\mathcal{O}$ meets the maximal simple subalgebra $\mathfrak{l} \cong \mathfrak{so}_9(\mathbb{C})$ in F_4 in the orbit with partition $[4^2,1]$, while the orbit $\mathcal{O}_e$ meets $\mathfrak{l}$ in the orbit with partition $[3^3]$. So we may assume $\mathfrak{s} \subset \mathfrak{l}$ and consider $\mathcal{S}_{\mathcal{O}\cap\mathfrak{l},e}$. The centralizer of $\mathfrak{s}$ in $\mathfrak{l}$ remains $\mathfrak{sl}_2(\mathbb{C})$, so we may also assume that $\mathfrak{s}_0 = \mathfrak{c}(\mathfrak{s}) \subset \mathfrak{l}$. Calculating $\mathcal{E}$ for $\mathfrak{s}$ and $\mathfrak{s}_0$ relative to $\mathfrak{l}$, we find that only $(2,4)$ occurs. Hence we can identify $\mathcal{S}_{\mathcal{O}\cap\mathfrak{l},e}$ with $\mathcal{S}_{\mathcal{O},e}$ since both are dimension two. We conclude that $x_1 = 0$ in (4.1). It follows that $\mathcal{S}_{\mathcal{O},e} = e + X(2,4) \cong X(2)$.

On the other hand, for the orbit $\mathcal{O}$ of type $\tilde{A}_2 + A_1$, we have to carry out an explicit computation in GAP. We find that both x_1 and x_2 are nonzero in (4.1) and thus $\mathcal{S}_{\mathcal{O},e} = e + X(2,3,4)$, which is isomorphic to m by §3.2.1.

7.2. *Remaining surface singularities*

This section summarizes the calculations of the singularities of the minimal degenerations of codimension two, using the methods in §5.

For the cases in Proposition 3.3, we did not need to know whether a nilpotent orbit has closure which is normal to determine the singularity type of a minimal degeneration. Knowing the branching was sufficient. Indeed, the closure of the orbit B_2 is non-normal, but it was shown above that it is normal at points in the orbit $A_2+\tilde{A}_1$ since the singularity of that degeneration is of type A_1. Similarly for the orbit $\tilde{A}_2$. The remaining non-normal orbit closures, of which there are three [11], are detected through a minimal degeneration: either the closure is branched at a minimal degeneration (as for C_3) or is isomorphic to m at a minimal degeneration (as for $C_3(a_1)$ and for $\tilde{A}_2 + A_1$). In what follows we use the fact that the orbit $F_4(a_1)$ has closure which is normal [11] to classify the type of its minimal degeneration. This is the only case where we need to know whether the closure is normal in order to resolve the type of a minimal degeneration in F_4.

(1) $(\mathcal{O}, \mathcal{O}') = (F_4(a_1), F_4(a_2))$. The even orbit $F_4(a_1)$ is Richardson for the parabolic subgroup Q with Levi factor of type $\tilde{A}_1$ and the resulting map $p : G \times^Q \mathfrak{n}(\mathfrak{q}) \to \overline{\mathcal{O}}$ is birational, hence a generalized Springer resolution. The hypotheses of Lemma 5.8 hold and $b_2(G/Q) = 3$. Since $\mathcal{O}'$ is the unique orbit of codimension two in $\overline{\mathcal{O}}$, it follows from §5.5 that there are 3 orbits of $A(e) = \mathfrak{S}_2$ on the irreducible components of $p^{-1}(e)$. On the other hand, there are a total of four irreducible components of $p^{-1}(e)$ by §5.4. Thus the singularity must be C_3, given that $\overline{\mathcal{O}}$ is normal.

Table 3
$\mathbf{E_6}$: cases with $e_0 \in \mathfrak{c}(\mathfrak{s})$ of type A_1 in $\mathfrak{g}$.

e	$e + e_0 \in \mathcal{O}$	$\mathfrak{c}(\mathfrak{s})$	Isomorphism type of $\mathcal{S}_{\mathcal{O},e}$
A_1	$2A_1$	A_5	a_5
$2A_1$	$3A_1$	$B_3 + T_1$	b_3
$3A_1$	$4A_1 = A_2$	$A_2 + \mathbf{A_1}$	A_1
A_2	$A_2 + A_1$	$\mathbf{2A_2}$	$[2a_2]^+$
$A_2 + A_1$	$A_2 + 2A_1$	$A_2 + T_1$	a_2
$2A_2$	$2A_2 + A_1$	G_2	g_2
A_3	$A_3 + A_1$	$B_2 + T_1$	b_2
$A_3 + A_1$	$A_3 + 2A_1 = D_4(a_1)$	$A_1 + T_1$	A_1
A_4	$A_4 + A_1$	$A_1 + T_1$	A_1
A_5	$A_5 + A_1 = E_6(a_3)$	A_1	A_1

(2) $(\mathcal{O}, \mathcal{O}') = (C_3, F_4(a_3))$. The orbit $\mathcal{O}$ is Richardson for the parabolic subgroup Q with Levi factor of type A_2. The map $p : G \times^Q \mathfrak{n}(\mathfrak{q}) \to \overline{\mathcal{O}}$ is birational, hence a generalized Springer resolution, since $A(x) = 1$ for $x \in \mathcal{O}$. If $e \in \mathcal{O}'$, then $A(e) = \mathfrak{S}_4$. By Lemma 5.5 and Corollary 5.6, there are 16 irreducible components in $p^{-1}(e)$ with two orbits under $A(e)$. The number of orbits can also be deduced from §5.5. Looking at the possibilities for the dual graph, it is clear that $\overline{\mathcal{O}}$ is non-normal and the normalization map $\nu : \widetilde{\mathcal{O}} \to \overline{\mathcal{O}}$ restricts to a degree 4 map over $\mathcal{O}'$. This also follows from [49] (§2.4). By Corollary 5.6, there is a fixed component of $\pi^{-1}(y)$ under the $A(e)$-action for $y = \nu^{-1}(e)$. This implies that the singularity of $\widetilde{\mathcal{O}}$ at y is G_2. We show in §7.3 that $\mathcal{S}_{\mathcal{O},e}$ is isomorphic to $4G_2$. In other words, the irreducible components of $\mathcal{S}_{\mathcal{O},e}$ are normal and hence each is isomorphic to G_2.

(3) $(\mathcal{O}, \mathcal{O}') = (B_3, F_4(a_3))$. The singularity is G_2 by §5.7.

7.3. *The degeneration* $(C_3, F_4(a_3))$ *is* $4G_2$

We now show each irreducible component of this slice is normal. By §4.4.5 the nilpotent Slodowy slice $\mathcal{S}$ of C_3 at $\tilde{A}_2$ contains an irreducible component isomorphic to the nilpotent cone $\mathcal{N}_{G_2}$. Recall e belongs to the $\tilde{A}_2$ orbit, with corresponding $\mathfrak{sl}_2$-subalgebra $\mathfrak{s}$. Let $e_0 \in \mathfrak{c}(\mathfrak{s})$ be subregular nilpotent. Then a calculation shows that $e' := e + e_0$ lies in the $F_4(a_3)$ orbit in $\mathfrak{g}$ and also that e' lies in the component of $\mathcal{S}$ isomorphic to $\mathcal{N}_{G_2}$. Hence the nilpotent Slodowy slice of C_3 at $F_4(a_3)$ contains a component that is smoothly equivalent to the nilpotent Slodowy slice in $\mathfrak{c}(\mathfrak{s})$ of G_2 at $G_2(a_1)$. But then this component must be isomorphic to the simple surface singularity D_4 by Lemma 5.3. Incorporating the symmetry of $A(e') = \mathfrak{S}_4$, the nilpotent Slodowy slice of C_3 at $F_4(a_3)$ is isomorphic to $4G_2$.

8. Results for E_6

8.1. *Details in the proof of Proposition 3.3*

In Table 3 we list the cases where Corollary 4.9 holds for e_0 in the minimal orbit of E_6. Here $\mathfrak{c}(\mathfrak{s}_0)$ is the semisimple part of a Levi subalgebra and has type A_5. The relevant

Table 4
$\mathbf{E_6}$: Remaining relevant cases with e_0 minimal in $\mathfrak{c}(\mathfrak{s})$.

e	e_0	$e+e_0$	$\mathcal{O}$	$\mathfrak{c}(\mathfrak{s})$	(m_i, n_i) for $i \in \mathcal{E}$	Isomorphism type of $\mathcal{S}_{\mathcal{O},e}$
D_4	$2A_1$	$D_5(a_1)$	$D_5(a_1)$	A_2	$\emptyset$	a_2
A_2+2A_1	A_2+2A_1	$D_4(a_1)$	A_3	A_1+T_1	$(1,3), (1,3), (\mathbf{2}, \mathbf{4})$	A_1
$2A_2+A_1$	$3A_1$	$D_4(a_1)$	A_3+A_1	A_1	$(\mathbf{1}, \mathbf{3})$	m

Table 5
Surface singularities using §5: E_6.

Degeneration	Induced from	$\sharp\mathbb{P}^1$'s	$A(e)$	$\sharp$ orbits of $A(e)$	$\mathcal{S}_{\mathcal{O},e}$
$(E_6(a_1), D_5)$	$(A_1, 0)$	5	1		A_5
$(D_5, E_6(a_3))$	$(2A_1, 0)$	4	S_2	3	C_3
$(D_5(a_1), A_4+A_1)$	$(A_2+A_1, 0)$	2	1		A_2
(A_5, A_4+A_1)	$(D_4, 32^21)$	2	1		A_2
$(D_4, D_4(a_1))$	$(2A_2, 0)$	4	S_3	2	G_2
$(A_4, D_4(a_1))$	$(A_3, 0)$	9	S_3	2	$3C_2$

nonzero nilpotent G-orbits are those that have non-trivial intersection with $\mathfrak{c}(\mathfrak{s}_0)$. In the first line of Table 4 is the remaining case where Lemma 4.3 applies with $x = e + e_0$ for an element e_0 in a minimal nilpotent orbit of $\mathfrak{c}(\mathfrak{s})$. This exhausts all minimal degenerations covered by the proposition with $J = \emptyset$. There are only two cases where $J \neq \emptyset$, both of codimension two. The degeneration $(A_3, A_2 + 2A_1)$ follows from working in the Levi subalgebra of semisimple type D_5, similar to §7.1.1. The degeneration $(A_3+A_1, 2A_2+A_1)$ is similar to §7.1.2. Details are given in Table 4.

8.2. Remaining surface singularities, and an exceptional degeneration

The results are listed in Table 5. In the first four entries of the table, we use the fact that the larger orbit has closure which is normal [52]. The entry for $(E_6(a_1), D_5)$ is from §5.6.1. The entry for $(A_4, D_4(a_1))$ is $3C_2$ since the irreducible components are isomorphic and one of them is isomorphic to C_2 from Table 13. Alternatively, it follows from working in the Levi subalgebra of semisimple type D_5 and using Lemma 4.1 and [33]. The entry for $(D_4, D_4(a_1))$ is also clear from working in the Levi subalgebra of semisimple type D_4. The degenerations $(E_6(a_3), D_5(a_1))$ and $(2A_2, A_2 + A_1)$ are both A_2 using larger slices (see Table 13). Note that the $2A_2$ orbit is unibranch at $A_2 + A_1$, but its closure is not normal.

The exceptional degeneration $(2A_2 + A_1, A_2 + 2A_1)$ of codimension four is treated in §12.

9. Results for E_7

9.1. Details in the proof of Proposition 3.3

In Table 6 we list the cases where Corollary 4.9 applies. Here $\mathfrak{c}(\mathfrak{s}_0)$ is the semisimple part of a Levi subalgebra and has type D_6. The relevant nonzero nilpotent G-orbits are

Table 6
$\mathbf{E_7}$: cases with $e_0 \in \mathfrak{c}(\mathfrak{s})$ of type A_1 in $\mathfrak{g}$.

e	$e + e_0 \in \mathcal{O}$	$\mathfrak{c}(\mathfrak{s})$	Isomorphism type of $\mathcal{S}_{\mathcal{O},e}$
A_1	$2A_1$	D_6	d_6
$2A_1$	$(3A_1)''$	$B_4 + \mathbf{A_1}$	A_1
	$(3A_1)'$	$\mathbf{B_4} + A_1$	b_4
$(3A_1)''$	$4A_1$	F_4	f_4
$(3A_1)'$	$4A_1$	$\mathbf{C_3} + A_1$	c_3
	A_2	$C_3 + \mathbf{A_1}$	A_1
A_2	$A_2 + A_1$	A_5	a_5^+
$4A_1$	$5A_1 = A_2 + A_1$	C_3	c_3
$A_2 + A_1$	$A_2 + 2A_1$	$A_3 + T_1$	a_3^+
$A_2 + 2A_1$	$A_2 + 3A_1$	$A_1 + A_1 + \mathbf{A_1}$	A_1
A_3	$(A_3 + A_1)''$	$B_3 + \mathbf{A_1}$	A_1
	$(A_3 + A_1)'$	$\mathbf{B_3} + A_1$	b_3
$2A_2$	$2A_2 + A_1$	$\mathbf{G_2} + A_1$	g_2
$(A_3 + A_1)'$	$A_3 + 2A_1$	$A_1 + A_1 + \mathbf{A_1}$	A_1
	$(A_3 + 2A_1)' = D_4(a_1)$	$A_1 + \mathbf{A_1} + A_1$	A_1
$(A_3 + A_1)''$	$A_3 + 2A_1$	B_3	b_3
$D_4(a_1)$	$D_4(a_1) + A_1$	$\mathbf{3A_1}$	$[3A_1]^{++}$
$A_3 + 2A_1$	$A_3 + 3A_1 = D_4(a_1) + A_1$	$A_1 + \mathbf{A_1}$	A_1
D_4	$D_4 + A_1$	C_3	c_3
$D_4(a_1) + A_1$	$D_4(a_1) + 2A_1 = A_3 + A_2$	$\mathbf{2A_1}$	$[2A_1]^+$
$A_3 + A_2$	$A_3 + A_2 + A_1$	$A_1 + T_1$	A_1
A_4	$A_4 + A_1$	$A_2 + T_1$	a_2^+
$D_4 + A_1$	$D_4 + 2A_1 = D_5(a_1)$	B_2	b_2
$(A_5)''$	$A_5 + A_1$	G_2	g_2
$D_5(a_1)$	$D_5(a_1) + A_1$	$A_1 + T_1$	A_1
$(A_5)'$	$(A_5 + A_1)' = E_6(a_3)$	$A_1 + \mathbf{A_1}$	A_1
$D_6(a_2)$	$D_6(a_2) + A_1 = E_7(a_5)$	A_1	A_1
D_5	$D_5 + A_1$	$A_1 + \mathbf{A_1}$	A_1
$D_6(a_1)$	$D_6(a_1) + A_1 = E_7(a_4)$	A_1	A_1
D_6	$D_6 + A_1 = E_7(a_3)$	A_1	A_1

those that have non-trivial intersection with $\mathfrak{c}(\mathfrak{s}_0)$. In the first several lines of Table 7 are the remaining cases where Lemma 4.3 applies with $x = e + e_0$ for an element e_0 in a minimal nilpotent orbit of $\mathfrak{c}(\mathfrak{s})$.

The nine remaining cases (all of codimension two), involving e from six different G-orbits, are listed in Table 7. The cases where e is type $A_2 + 2A_1$ or $2A_2 + A_1$ follow by restricting to a subalgebra of type E_6. The case where e is type $A_5 + A_1$ proceeds as in Example 4.4.3. The two cases where e is type $D_5(a_1) + A_1$ are similar to Example 4.1.5. The three minimal degenerations lying above the orbit $A_4 + A_2$ and the one above the orbit $A_3 + A_2 + A_1$ satisfy part (2) of Lemma 4.6. Since all the m_i are even for $i \in \mathcal{E}$, Corollary 4.7 gives that these four degenerations are A_1-singularities and satisfy the proposition. Still, we carry out an explicit computer calculation in GAP to show that both x_2 and x_4 are nonzero for these degenerations, so that in each of these cases, $\mathcal{S}_{\mathcal{O},e}$ takes the form $e + X(2, 4, 6)$. The details are omitted.

9.2. *Remaining surface singularities*

The results using §5 are collected in Table 8. We have used the fact that $E_7(a_1)$, $E_7(a_2)$, $E_7(a_3)$, E_6, $E_6(a_1)$ have closure which is normal [10, Section 7.8]. The method

Table 7
$\mathbf{E_7}$: Remaining relevant cases with e_0 minimal in $\mathfrak{c}(\mathfrak{s})$.

e	e_0	$e+e_0$	$\mathcal{O}$	$\mathfrak{c}(\mathfrak{s})$	(m_i,n_i) for $i\in\mathcal{E}$	Isomorphism type of $\mathcal{S}_{\mathcal{O},e}$
A_2+2A_1	$2A_1$	A_2	A_2	$A_1+\mathbf{A_1}+A_1$	$\emptyset$	A_1
A_2+3A_1	$2A_1$	$2A_2+A_1$	$2A_2+A_1$	G_2	$\emptyset$	g_2
$2A_2$	$(3A_1)''$	$(A_3+A_1)''$	$(A_3+A_1)''$	$G_2+\mathbf{A_1}$	$\emptyset$	A_1
$(A_5)'$	$(3A_1)''$	$D_6(a_2)$	$D_6(a_2)$	$\mathbf{A_1}+A_1$	$\emptyset$	A_1
D_5	$2A_1$	$D_6(a_1)$	$D_6(a_1)$	$\mathbf{A_1}+A_1$	$\emptyset$	A_1
D_5+A_1	$2A_1$	$E_7(a_4)$	$E_7(a_4)$	A_1	$\emptyset$	A_1
A_6	A_2+3A_1	$E_7(a_4)$	$E_7(a_4)$	A_1	$\emptyset$	A_1
$E_6(a_3)$	$(3A_1)''$	$E_7(a_5)$	$E_7(a_5)$	A_1	$\emptyset$	A_1
E_6	$(3A_1)''$	$E_7(a_2)$	$E_7(a_2)$	A_1	$\emptyset$	A_1
A_2+2A_1	A_2+2A_1	$D_4(a_1)$	A_3	$\mathbf{A_1}+A_1+A_1$	$(1,3)^4$, $(\mathbf{2},\mathbf{4})$	A_1
$2A_2+A_1$	$(3A_1)'$	$D_4(a_1)$	$(A_3+A_1)'$	$A_1+\mathbf{A_1}$	$(\mathbf{1},\mathbf{3})$	m
$A_3+A_2+A_1$	A_4+A_2	$E_7(a_5)$	D_4+A_1	A_1	$(\mathbf{2},\mathbf{4})$, $(2,8)$, $(\mathbf{4},\mathbf{6})$	A_1
A_4+A_2	$A_3+A_2+A_1$	$E_7(a_5)$	A_5+A_1	A_1	$(\mathbf{2},\mathbf{4})$, $(\mathbf{4},\mathbf{6})$	A_1
			$(A_5)'$		$(\mathbf{2},\mathbf{4})$, $(\mathbf{4},\mathbf{6})$	A_1
			$D_5(a_1)+A_1$		$(\mathbf{2},\mathbf{4})$, $(\mathbf{4},\mathbf{6})$	A_1
A_5+A_1	$(3A_1)'$	$E_7(a_5)$	$D_6(a_2)$	A_1	$(\mathbf{1},\mathbf{3})$	m
$D_5(a_1)+A_1$	$2A_2$	$E_7(a_5)$	$E_6(a_3)$	A_1	$(\mathbf{2},\mathbf{4})$	A_1
			$D_6(a_2)$		$(\mathbf{2},\mathbf{4})$	A_1

Table 8
Surface singularities using §5: E_7.

Degeneration	Induced from	$\sharp\mathbb{P}^1$'s	$A(e)$	$\sharp$ orbits of $A(e)$	$\mathcal{S}_{\mathcal{O},e}$
$(E_7(a_1),E_7(a_2))$	$(A_1,0)$	6	1		D_6
$(E_7(a_2),E_7(a_3))$	$(2A_1,0)$	5	S_2	4	C_4
$(E_7(a_3),E_6(a_1))$	$((3A_1)',0)$	5	S_2	3	B_3
$(E_6,E_6(a_1))$	$((3A_1)'',0)$	6	S_2	4	F_4
$(E_6(a_1),E_7(a_4))$	$(4A_1,0)$	4	S_2	3	C_3
$(D_6,E_7(a_4))$	$(D_4,32^21)$	4	S_2	3	C_3
$(A_6,E_7(a_5))$	$(A_2+3A_1,0)$	4	S_3	2	G_2
$(D_5+A_1,E_7(a_5))$	$(2A_2,0)$	4	S_3	2	G_2
$(D_6(a_1),E_7(a_5))$	$(A_3,0)$	12	S_3	3	$3C_3$

from [52] can be used to show D_6 has closure which is normal. The entry for $(E_7(a_1),E_7(a_2))$ is from §5.6.2. For the three degenerations above $E_7(a_5)$, the irreducible components of $\mathcal{S}_{\mathcal{O},e}$ are normal (see §9.3).

The remaining six minimal degenerations are unibranch, but either the larger orbit has non-normal closure or it is not known whether the larger orbit has closure which is normal. In all cases we are able to determine that the slice is normal and hence fully determine the singularity. The corresponding action of $A(e)$ is determined using §5. The degeneration $(D_5,E_6(a_3))$ is C_3 and $(D_5(a_1),A_4+A_1)$ is A_2^+ by restriction to E_6, see Table 12. The other four degenerations follow from Table 13.

9.3. *Additional calculations: three degenerations above $E_7(a_5)$*

The proofs are similar to the one in §7.3 and proceed by first showing that a larger slice is isomorphic to the whole nilcone of a smaller Lie algebra.

For $(A_6, E_7(a_5))$ and $(D_5+A_1, E_7(a_5))$, we first show that the degenerations (A_6, A_5'') and $(D_5 + A_1, A_5'')$ are both isomorphic to $\mathcal{N}_{G_2}$. Then we use the fact that $E_7(a_5)$ corresponds to the subregular orbit in G_2. The result follows, as in §7.3, since these singularities are unibranch. In more detail: let e be in the orbit A_5''. Then $\mathfrak{c}(\mathfrak{s})$ is of type G_2. Let e_0 be a regular nilpotent element in $\mathfrak{c}(\mathfrak{s})$. Then $e + e_0$ lies in the orbit $E_7(a_4)$ and $(m_i, n_i) = (4, 6)$ for the unique element in $\mathcal{E}$. The simple part of $\mathfrak{g}^h$ is $\mathfrak{so}_8(\mathbb{C})$. Let $w_i = (\operatorname{ad} f)^2(M^3)$ with $M = e_0 \in \mathfrak{c}(\mathfrak{s}) \subset \mathfrak{so}_7 \subset \mathfrak{so}_8$ (§4.4.4). Using GAP we showed that there is a unique scalar $b \neq 0$ such that $e + e_0 + bw_i$ is in the orbit A_6, and similarly for $D_5 + A_1$. The rest of the proof in §4.4.5 applies to give the result.

For the case of $(D_6(a_1), E_7(a_5))$, we first show that the degeneration $(D_6(a_1), D_4)$ has one branch which is isomorphic to $\mathcal{N}_{C_3}$. (There are two branches of $D_6(a_1)$ above D_4.) Let e be in the orbit D_4. Then $\mathfrak{c}(\mathfrak{s}) \cong \mathfrak{sp}_6$. Let e_0 be a regular nilpotent element in $\mathfrak{c}(\mathfrak{s})$. Then $e + e_0$ lies in the orbit $E_7(a_4)$ and $\mathfrak{g}$ decomposes in (4.5) as

$$V(0,10) \oplus V(0,6) \oplus V(0,2) \oplus V(2,0) \oplus V(6,4) \oplus V(6,8) \oplus V(10,0),$$

reflecting that $\mathfrak{c}(\mathfrak{s})$ decomposes under $\mathfrak{s}_0$ as $V(10) \oplus V(6) \oplus V(2)$ and $\mathfrak{g}^f(-6)$ decomposes under $\mathfrak{s}_0$ as $V(4) \oplus V(8)$, which is 14-dimensional and as a representation of $\mathfrak{c}(\mathfrak{s})$ is $V(\omega_2)$. Also $(m_i, n_i) = (6, 8)$ for the unique element in $\mathcal{E}$.

The semisimple part of $\mathfrak{g}^h$ is isomorphic to $\mathfrak{sl}_6(\mathbb{C})$. If we take $M = e_0$, then $M^4 \in \mathfrak{sl}_6$ is nonzero since M is regular in $\mathfrak{sl}_6$. It cannot be in $\mathfrak{sp}_6$ since only odd powers of M are. It satisfies $[h_0, M^4] = 8M^4$ and so it must be a highest weight vector in $V(8)$ for $\mathfrak{s}_0$ with respect to e_0. Hence we can choose $w_i = (\operatorname{ad} f)^3(M^4)$ (§4.4.4). We checked using GAP that there is an x in the orbit $D_6(a_1)$ with

$$x = e + e_0 + bw_i$$

with $b \neq 0$. Since the elements $\overline{C(\mathfrak{s}) \cdot (e_0 + bw_i)}$ consist of pairs $(M, M^4) \in \mathfrak{sp}_6 \oplus V(\omega_2) \cong \mathfrak{sl}_6$ with $M \in \mathfrak{sp}_6$ nilpotent, the slice for $(D_6(a_1), D_4)$ contains an irreducible component isomorphic to $\mathcal{N}_{C_3}$ (the dimensions match). Since elements in the slice belonging to the $E_7(a_5)$-orbit correspond to the subregular elements in $\mathcal{N}_{C_3}$, one branch of $(D_6(a_1), E_7(a_5))$ is isomorphic to C_3, hence the singularity is $3C_3$.

10. Results for E_8

10.1. Details in the proof of Proposition 3.3

In Table 9 we list the cases where Corollary 4.9 applies. The centralizer $\mathfrak{c}(\mathfrak{s}_0)$ is the semisimple part of a Levi subalgebra of type E_7. The nonzero nilpotent G-orbits meeting $\mathfrak{c}(\mathfrak{s}_0)$ are those which appear in the table. The first several lines of Table 10 contain the remaining cases where Lemma 4.3 applies with $x = e + e_0$ for an element e_0 in a minimal nilpotent orbit of $\mathfrak{c}(\mathfrak{s})$.

Table 9
$\mathbf{E_8}$: cases with $e_0 \in \mathfrak{c}(\mathfrak{s})$ of type A_1 in $\mathfrak{g}$.

e	$e + e_0 \in \mathcal{O}$	$\mathfrak{c}(\mathfrak{s})$	Isomorphism type of $\mathcal{S}_{\mathcal{O},e}$
A_1	$2A_1$	E_7	e_7
$2A_1$	$3A_1$	B_6	b_6
$3A_1$	$4A_1$	$\mathbf{F_4} + A_1$	f_4
	A_2	$F_4 + \mathbf{A_1}$	A_1
A_2	$A_2 + A_1$	E_6	e_6^+
$4A_1$	$5A_1 = A_2 + A_1$	C_4	c_4
$A_2 + A_1$	$A_2 + 2A_1$	A_5	a_5^+
$A_2 + 2A_1$	$A_2 + 3A_1$	$\mathbf{B_3} + A_1$	b_3
$A_2 + 3A_1$	$A_2 + 4A_1 = 2A_2$	$G_2 + \mathbf{A_1}$	A_1
A_3	$A_3 + A_1$	B_5	b_5
$2A_2$	$2A_2 + A_1$	$\mathbf{2G_2}$	$[2g_2]^+$
$2A_2 + A_1$	$2A_2 + 2A_1$	$\mathbf{G_2} + A_1$	g_2
$A_3 + A_1$	$A_3 + 2A_1$	$\mathbf{B_3} + A_1$	b_3
	$(A_3 + 2A_1)'' = D_4(a_1)$	$B_3 + \mathbf{A_1}$	A_1
$A_3 + 2A_1$	$A_3 + 3A_1 = D_4(a_1) + A_1$	$\mathbf{B_2} + A_1$	b_2
$D_4(a_1)$	$D_4(a_1) + A_1$	D_4	d_4^{++}
$D_4(a_1) + A_1$	$D_4(a_1) + 2A_1 = A_3 + A_2$	$\mathbf{3A_1}$	$[3A_1]^{++}$
$A_3 + A_2$	$A_3 + A_2 + A_1$	$B_2 + T_1$	b_2
$A_3 + A_2 + A_1$	$A_3 + A_2 + 2A_1 = D_4(a_1) + A_2$	$A_1 + \mathbf{A_1}$	A_1
A_4	$A_4 + A_1$	A_4	a_4^+
D_4	$D_4 + A_1$	F_4	f_4
$D_4 + A_1$	$D_4 + 2A_1 = D_5(a_1)$	C_3	c_3
$A_4 + A_1$	$A_4 + 2A_1$	$A_2 + T_1$	a_2^+
$D_5(a_1)$	$D_5(a_1) + A_1$	A_3	a_3^+
$A_4 + A_2$	$A_4 + A_2 + A_1$	$A_1 + \mathbf{A_1}$	A_1
$D_5(a_1) + A_1$	$D_5(a_1) + 2A_1 = D_4 + A_2$	$A_1 + \mathbf{A_1}$	A_1
A_5	$A_5 + A_1$	$\mathbf{G_2} + A_1$	g_2
	$A_5 + A_1 = E_6(a_3)$	$G_2 + \mathbf{A_1}$	A_1
$A_5 + A_1$	$A_5 + 2A_1 = E_6(a_3) + A_1$	$A_1 + \mathbf{A_1}$	A_1
$E_6(a_3)$	$E_6(a_3) + A_1$	G_2	g_2
$D_6(a_2)$	$D_6(a_2) + A_1 = E_7(a_5)$	$\mathbf{2A_1}$	$[2A_1]^+$
D_5	$D_5 + A_1$	B_3	b_3
$E_7(a_5)$	$E_7(a_5) + A_1 = E_8(a_7)$	A_1	A_1
$D_5 + A_1$	$D_5 + 2A_1 = D_6(a_1)$	$A_1 + \mathbf{A_1}$	A_1
$D_6(a_1)$	$D_6(a_1) + A_1 = E_7(a_4)$	$\mathbf{2A_1}$	$[2A_1]^+$
A_6	$A_6 + A_1$	$A_1 + \mathbf{A_1}$	A_1
$E_7(a_4)$	$E_7(a_4) + A_1 = D_5 + A_2$	A_1	A_1
$E_6(a_1)$	$E_6(a_1) + A_1$	A_2	a_2^+
D_6	$D_6 + A_1 = E_7(a_3)$	B_2	b_2
E_6	$E_6 + A_1$	G_2	g_2
$E_7(a_3)$	$E_7(a_3) + A_1 = D_7(a_1)$	A_1	A_1
$E_7(a_2)$	$E_7(a_2) + A_1 = E_8(b_5)$	A_1	A_1
$E_7(a_1)$	$E_7(a_1) + A_1 = E_8(b_4)$	A_1	A_1
E_7	$E_7 + A_1 = E_8(a_3)$	A_1	A_1

The remaining cases of the proposition, where $J \neq \emptyset$, are listed in Table 10 and include two non-surface cases. All cases follow by restriction to a subalgebra or by using Lemma 4.6 and Corollary 4.7, except for the two degenerations above $A_4 + A_3$. We now

Table 10
$\mathbf{E_8}$: Remaining relevant cases with e_0 minimal in $\mathfrak{c}(\mathfrak{s})$.

e	e_0	$e+e_0$	$\mathcal{O}$	$\mathfrak{c}(\mathfrak{s})$	(m_i,n_i) for $i\in\mathcal{E}$	Isomorphism type of $\mathcal{S}_{\mathcal{O},e}$
$2A_3$	$2A_1$	A_4+2A_1	A_4+2A_1	B_2	$\emptyset$	b_2
D_4+A_2	$2A_1$	$D_5(a_1)+A_2$	$D_5(a_1)+A_2$	A_2	$\emptyset$	a_2^+
A_6	A_2+3A_1	$E_7(a_4)$	$E_7(a_4)$	$\mathbf{A_1}+A_1$	$\emptyset$	A_1
A_4+2A_1	$2A_1$	A_4+A_2	A_4+A_2	A_1+T_1	$\emptyset$	A_1
A_6+A_1	A_2+3A_1	D_5+A_2	D_5+A_2	A_1	$\emptyset$	A_1
A_7	$4A_1$	$E_8(b_6)$	$E_8(b_6)$	A_1	$\emptyset$	A_1
D_7	$2A_1$	$E_8(a_5)$	$E_8(a_5)$	A_1	$\emptyset$	A_1
A_2+2A_1	A_2+2A_1	$D_4(a_1)$	A_3	$B_3+\mathbf{A_1}$	$(1,3)^8$, $(\mathbf{2},\mathbf{4})$	A_1
$2A_2+A_1$	$3A_1$	$D_4(a_1)$	A_3+A_1	$G_2+\mathbf{A_1}$	$(\mathbf{1},\mathbf{3})$	m
$2A_2+2A_1$	$3A_1$	$D_4(a_1)+A_1$	A_3+2A_1	B_2	$(\mathbf{1},\mathbf{3})$	m'
$A_3+A_2+A_1$	A_4+A_2	$E_7(a_5)$	D_4+A_1	$\mathbf{A_1}+A_1$	$(\mathbf{2},\mathbf{4})$, $(2,8)$, $(\mathbf{4},\mathbf{6})$	A_1
$D_4(a_1)+A_2$	A_2+2A_1	A_4+2A_1	$2A_3$	A_2	$(\mathbf{2},\mathbf{4})$	a_2^+
			A_4+A_1		$(\mathbf{2},\mathbf{4})$	a_2^+
A_4+A_2	$A_3+A_2+A_1$	$E_7(a_5)$	A_5	$\mathbf{A_1}+A_1$	$(\mathbf{2},\mathbf{4})$, $(\mathbf{4},\mathbf{6})$	A_1
			$D_5(a_1)+A_1$		$(\mathbf{2},\mathbf{4})$, $(\mathbf{4},\mathbf{6})$	A_1
$D_5(a_1)+A_1$	$2A_2$	$E_7(a_5)$	$E_6(a_3)$	$\mathbf{A_1}+A_1$	$(\mathbf{2},\mathbf{4})$	A_1
$A_4+A_2+A_1$	$A_3+A_2+A_1$	$E_8(a_7)$	D_4+A_2	A_1	$(1,5)$, $(\mathbf{2},\mathbf{4})$, $(3,5)$, $(\mathbf{4},\mathbf{6})$	A_1
A_4+A_3	$2A_2+2A_1$	$E_8(a_7)$	A_5+A_1	A_1	$(\mathbf{1},\mathbf{3})$, $(\mathbf{2},\mathbf{4})$, $(\mathbf{3},\mathbf{5})$	m
			$D_5(a_1)+A_2$		$(\mathbf{1},\mathbf{3})$, $(\mathbf{2},\mathbf{4})$, $(\mathbf{3},\mathbf{5})$	m
A_5+A_1	$3A_1$	$E_7(a_5)$	$D_6(a_2)$	$\mathbf{A_1}+A_1$	$(\mathbf{1},\mathbf{3})$	m
$D_5(a_1)+A_2$	A_2+2A_1	$E_8(a_7)$	$E_6(a_3)+A_1$	A_1	$(\mathbf{1},\mathbf{3})$, $(\mathbf{2},\mathbf{4})$	m
			$D_6(a_2)$		$(1,3)$, $(\mathbf{2},\mathbf{4})$	A_1
$E_6(a_3)+A_1$	$3A_1$	$E_8(a_7)$	$E_7(a_5)$	A_1	$(\mathbf{1},\mathbf{3})$	m
E_6+A_1	$3A_1$	$E_8(b_5)$	$E_7(a_2)$	A_1	$(\mathbf{1},\mathbf{3})$	m

discuss those two cases and the two non-surface cases, but omit the details for the other degenerations.

10.1.1. The degeneration $(A_3+2A_1, 2A_2+2A_1)$

Here e is in the orbit $2A_2+2A_1$ and $\mathfrak{c}(\mathfrak{s}) \cong \mathfrak{sp}_4(\mathbb{C})$. Let e_0 be in the minimal nilpotent orbit of $\mathfrak{c}(\mathfrak{s})$. In this case $\mathcal{E}$ has one element corresponding to $(m_i,n_i)=(1,3)$. Consider the Levi subalgebra $\mathfrak{l}$ of type E_6+A_1. Then without loss of generality $e\in\mathfrak{l}$ (with nonzero component on the A_1 factor) and $e_0\in\mathfrak{l}$ (contained in the E_6 factor). By the results for E_6, there is an x in the orbit $\mathcal{O}$ of type A_3+2A_1 (in E_8) with $x=e+e_0+e_1$ for a choice of $e_1\in\mathfrak{g}^f(-1)$ corresponding to $(1,3)$. Moreover, writing $\mathfrak{c}(\mathfrak{s})=V(2\omega_1)$, then e_1 is a highest weight vector for a $\mathfrak{c}(\mathfrak{s})$-module $V(3\omega_1)\subset\mathfrak{g}^f(-1)$. Hence $\mathcal{S}_{\mathcal{O},e}=e+X(2\omega_1,3\omega_1)\cong m'$ since (4.4) holds and the singularity is unibranch.

10.1.2. The degenerations $(A_4+A_1, D_4(a_1)+A_2)$ *and* $(2A_3, D_4(a_1)+A_2)$

Here $\mathfrak{c}(\mathfrak{s}) \cong \mathfrak{sl}_3(\mathbb{C})$. All the orbits meet the semisimple subalgebra $\mathfrak{l}$ of $\mathfrak{g}$ of type D_5+A_3: $\mathcal{O}_e$ meets $\mathfrak{l}$ in the orbit $[3^3,1]\cup[4]$; A_4+A_1 meets $\mathfrak{l}$ in the orbit $[5,2^2,1]\cup[4]$; and $2A_3$ meets $\mathfrak{l}$ in the orbit $[4^2,1^2]\cup[4]$. Then just as in the case $(B_2, A_2+\tilde{A}_1)$ in §7.1.3, there exists $x\in\mathcal{O}$ with $x=e+e_0+e_2$ for some $e_2\in\mathfrak{g}^f(-2)$ corresponding to the pair $(2,4)$, for $\mathcal{O}$ either of type A_4+A_1 or type $2A_3$. Identifying $\mathfrak{c}(\mathfrak{s})$ with $V(\theta)$ where θ is a highest root of $\mathfrak{c}(\mathfrak{s})$, we have e_2 is a highest weight vector for a $\mathfrak{c}(\mathfrak{s})$-module

$V(2\theta) \subset \mathfrak{g}^f(-2)$. Hence for both orbits $\mathcal{S}_{\mathcal{O},e} = e + X(\theta, 2\theta) \cong X(\theta)$, as desired (for two different choices of e_2, related by a scalar).

10.1.3. The degenerations $(A_5 + A_1, A_4 + A_3)$ *and* $(D_5(a_1) + A_2, A_4 + A_3)$

Corollary 4.7 applies, but is not sufficient to pin down the singularity, so we carry out an explicit computation. In both of these cases, e lies in the orbit $A_4 + A_3$, for which $\mathfrak{c}(\mathfrak{s}) \cong \mathfrak{sl}_2(\mathbb{C})$. This case is a more complicated version of §7.1.3 in F_4. Using the information in [35, p. 146] (adjusted for sign differences in GAP), let

$$e = -(4f_{\alpha_1} + 6f_{\alpha_3} + 6f_{\alpha_4} + 4f_{\alpha_2} + 3f_{\alpha_6} + 4f_{\alpha_7} + 3f_{\alpha_8}), \quad f = \sum_{i \neq 5} e_{\alpha_i} \quad \text{and} \quad h = [e, f].$$

A nilpositive element in $\mathfrak{c}(\mathfrak{s})$ is

$$e_0 = 2e_{\substack{1122211\\1}} - e_{\substack{1232110\\1}} + 2e_{\substack{0122221\\1}} + e_{\substack{1232100\\2}} + e_{\substack{1222210\\1}} + e_{\substack{1222111\\1}},$$

embedded in an $\mathfrak{sl}_2$-triple $\{e_0, h_0, f_0\}$ for $\mathfrak{c}(\mathfrak{s})$. Then the three elements in $\mathcal{E}$ correspond to $\{(1, 3), (2, 4), (3, 5)\}$. The spaces $\mathfrak{g}^f(-1)$, $\mathfrak{g}^f(-2)$ and $\mathfrak{g}^f(-3)$ contain highest weight modules for $\mathfrak{c}(\mathfrak{s})$ with respective highest weights 3, 4 and 5, and highest weight vectors:

$$e_1 = e_{\substack{2443210\\2}} - e_{\substack{1343211\\2}} + e_{\substack{1243221\\2}} - e_{\substack{1233321\\2}}, \quad e_2 = e_{\substack{2454321\\2}} + e_{\substack{2354321\\3}}, \quad e_3 = e_{\substack{2465432\\3}}.$$

We checked in GAP that

$$e + e_0 + 3e_2 \pm (2e_1 + 4e_3) \in D_5(a_1) + A_2, \quad \text{and}$$
$$e + e_0 - \tfrac{7}{6}e_2 \pm \sqrt{\tfrac{8}{27}}(e_1 - \tfrac{19}{3}e_3) \in A_5 + A_1.$$

Hence in both cases $\mathcal{S}_{\mathcal{O},e} = e + X(2, 3, 4, 5) \cong X(2, 3) = m$.

10.2. Remaining surface singularities, and an exceptional degeneration

The results using §5 are collected in Table 11. We use the fact that E_8, $E_8(a_1)$, $E_8(a_2)$, $E_8(a_3)$, $E_8(a_4)$ have closure which is normal [10, Section 7.8]. The method from [52] can be used to show $E_7, E_8(b_4)$, and $E_8(a_5)$ have closure which is normal. The entry for $(E_8(a_1), E_8(a_2))$ is from §5.6.3. That each irreducible component of $(D_6(a_1), E_8(a_7))$ and $(A_6, E_8(a_7))$ is G_2 follows from the fact that the degeneration (E_6, D_4) contains a branch isomorphic to the nilpotent cone in F_4, and then from the results in F_4.

There are 19 other cases. For nine of them, the degenerations are unibranch, but either the larger orbit has non-normal closure or it is not known whether the larger orbit has closure which is normal. Nevertheless, in these cases we are able to show that the slice is normal and hence fully determine the singularity. The action of $A(e)$ is determined using §5. The degeneration $(D_5, E_6(a_3))$ is C_3 and the degeneration

Table 11
Surface singularities using §5: E_8.

Degeneration	Induced from	$\sharp\mathbb{P}^1$'s	$A(e)$	$\sharp$ orbits of $A(e)$	$\mathcal{S}_{\mathcal{O},e}$
$(E_8(a_1), E_8(a_2))$	$(A_1, 0)$	7	1		E_7
$(E_8(a_2), E_8(a_3))$	$(2A_1, 0)$	7	S_2	6	C_6
$(E_8(a_3), E_8(a_4))$	$(3A_1, 0)$	6	S_2	4	F_4
$(E_8(a_4), E_8(b_4))$	$(4A_1, 0)$	5	S_2	4	C_4
$(E_8(a_5), E_8(b_5))$	$(A_2 + 3A_1, 0)$	4	S_3	2	G_2
$(E_7(a_1), E_8(b_5))$	$(A_3, 0)$	18	S_3	5	$3(C_5)$
$(E_8(b_5), E_8(a_6))$	$(2A_2 + A_1, 0)$	4	S_3	2	(G_2)
$(E_7(a_3), E_6(a_1) + A_1)$	$(D_6, 3^2 2^2 1^2)$	4	S_2	2	(A_4^+)
$(D_7(a_2), D_5 + A_2)$	$(2A_3, 0)$	3	S_2	2	(C_2)
$(E_7, E_8(b_4))$	$(D_4, 32^2 1)$	6	S_2	4	F_4
$(D_7, E_8(a_6))$	$(D_4 + A_2, 32^2 1 + 0)$	4	S_3	2	(G_2)
$(E_8(b_4), E_8(a_5))$	$(A_2 + 2A_1, 0)$	4	S_2	3	C_3
$(E_7(a_2), D_7(a_1))$	$(D_5, 32^2 1^3)$	5	S_2	3	(B_3)
$(D_7(a_1), E_8(b_6))$	$(A_3 + A_2, 0)$	3	S_3	2	$(C_2) = \mu$
$(E_6 + A_1, E_8(b_6))$	$(E_6, 2A_2 + A_1)$	4	S_3	2	(G_2)
$(A_7, D_7(a_2))$	$(D_5 + A_2, 32^2 1^3 + 0)$	2	S_2	1	(A_2^+)
$(E_6(a_1) + A_1, D_7(a_2))$	$(E_7, A_4 + A_1)$	2	S_2	1	(A_2^+)
$(D_6, D_5 + A_2)$	$(D_6, 32^4 1)$	3	S_2	2	(C_2)
$(D_6(a_1), E_8(a_7))$	$(A_5, 0)$	40	S_5	2	$10G_2$
$(A_6, E_8(a_7))$	$(D_4 + A_2, 0)$	20	S_5	2	$5G_2$

Table 12
Some surface cases where Lemma 4.1 can be applied.

$\mathfrak{g}$	e	$x \in \mathcal{O}$	Subalgebra	$\mathcal{S}_{\mathcal{O},e}$
E_7, E_8	$E_6(a_3)$	D_5	E_6	C_3
E_7, E_8	$A_4 + A_1$	$D_5(a_1)$	E_6	A_2^+
E_7, E_8	$A_3 + A_2$	A_4	D_6	C_2

$(D_5(a_1), A_4 + A_1)$ is A_2^+, both by restriction to E_6 (see Table 12). The other degenerations follow from Table 13. For the other ten cases, the result is determined up to normalization. In four of these cases, the orbit closure is known to be non-normal: $(E_7(a_1), E_8(b_5))$, $(E_7(a_3), E_6(a_1)+A_1)$, $(D_7(a_2), D_5+A_1)$, (D_6, D_5+A_2). The latter three are unibranched. The orbit closures, and hence the slices, for the other six are expected to be normal. We use (Y) to denote a singularity with normalization Y.

The exceptional degeneration $(A_4 + A_3, A_4 + A_2 + A_1)$ of codimension four is treated in §12.

11. Slices related to entire nilcones

The main goal of the paper was to study $\mathcal{S}_{\mathcal{O},e}$ for a minimal degeneration. Many of the same ideas can be used to show that $\mathcal{S}_{\mathcal{O},e}$ has a familiar description when the degeneration is not minimal. In particular, there are many cases where $\mathcal{S}_{\mathcal{O},e}$ is isomorphic to the closure of a non-minimal orbit in a nilcone for a subalgebra of $\mathfrak{g}$ or is isomorphic to a slice between two orbits in such a nilcone. Rather than listing all these cases here,

Table 13
Slices containing a smaller nilcone.

$\mathfrak{g}$	Degeneration and nilcone
F_4	$(B_3, \tilde{A}_2) = \mathcal{N}_{G_2}$
	$(C_3, \tilde{A}_2) \supset \mathcal{N}_{G_2}$
E_6	$(E_6(a_3), D_4) = \mathcal{N}_{A_2}$
	$(A_4, A_3) \supset \mathcal{N}_{C_2}$
	$(2A_2, A_2) = [2\mathcal{N}_{A_2}]^+$
E_7	$(E_7(a_4), D_5) = \mathcal{N}_{2A_1}$
	$(D_6(a_1), D_4) \supset \mathcal{N}_{C_3}$
	$(E_7(a_5), A_5') = \mathcal{N}_{2A_1}$
	$(A_6, A_5'') = \mathcal{N}_{G_2}$
	$(D_5 + A_1, A_5'') = \mathcal{N}_{G_2}$
	$(A_4 + A_2, A_4) = \mathcal{N}_{A_2}^+$
	$(D_4, A_2 + 3A_1) = \mathcal{N}_{G_2}$
	$(D_4, 2A_2) = \mathcal{N}_{G_2}$
	$(D_4(a_1) + A_1, (A_3 + A_1)') = \mathcal{N}_{2A_1}$
	$(A_5'', A_3) \supset \mathcal{N}_{B_3}$
E_8	$(E_8(a_5), E_6) = \mathcal{N}_{G_2}$
	$(E_8(a_6), D_6) = \mathcal{N}_{C_2}$
	$(E_6, D_4) \supset \mathcal{N}_{F_4}$
	$(D_5 + A_1, A_5) \supset \mathcal{N}_{G_2}$
	$(A_6, E_6(a_3)) \supset \mathcal{N}_{G_2}$
	$(D_6(a_1), E_6(a_3)) \supset \mathcal{N}_{G_2}$
	$(E_8(b_6), E_6(a_1)) = \mathcal{N}_{A_2}^+$
	$(A_4, A_3 + 2A_1) \supset \mathcal{N}_{C_2}$
	$(D_4, 2A_2) = 2\mathcal{N}_{G_2}$

we write down some cases where $\mathcal{S}_{\mathcal{O},e}$, or one of its irreducible components, is isomorphic to an entire nilcone. Some of these were used to show in the surface case that $\mathcal{S}_{\mathcal{O},e}$, or an irreducible component of $\mathcal{S}_{\mathcal{O},e}$, is normal (e.g., starting with §7.3). These examples are relevant for the duality discussed in §1.9.4, to be explored in future work. They are also examples where $C(\mathfrak{s})$ acts with a dense orbit.

11.1. Exceptional groups

The results are listed in Table 13. The notation $\mathcal{N}_X$ refers to the nilcone in the Lie algebra of type X. The proofs use Lemma 4.3, usually for $x \neq e + e_0$, and often require a computer calculation.

11.2. Slices isomorphic to entire nilcones: two slices in $\mathfrak{sl}_N$

These two examples are special cases of isomorphisms discovered by Henderson [24] using Maffei's work on quiver varieties [43]. Here we give direct proofs that fit into the framework of Lemma 4.3 and §4.4. We are grateful to Henderson for bringing these examples to our attention.

11.2.1. First slice

It is slightly more convenient to work in $\mathfrak{g} = \mathfrak{gl}_{nk}$. Assume $n \geq 2$ and $k \geq 1$. Consider the nilpotent orbit $\mathcal{O}'$ with partition $[n^k]$. Write $k = p(n+1) + q$ with $0 \leq q < n+1$, which gives $kn = (pn + q - 1)(n + 1) + (n + 1 - q)$ for maximally dividing kn by $n + 1$. Let $\mathcal{O}$ be the nilpotent orbit with partition $[(n+1)^{pn+q-1}, n+1-q]$, which is a partition of kn. Then $\mathcal{O}' \subset \overline{\mathcal{O}}$ by the dominance order for partitions. Moreover, $X \in \overline{\mathcal{O}}$ implies $X^{n+1} = 0$ and $\mathcal{O}$ is maximal for nilpotent orbits in $\mathfrak{gl}_{nk}$ with this property.

Proposition 11.1. *[24, Corollary 9.5] Let $e \in \mathcal{O}'$. The variety $\mathcal{S}_{\mathcal{O},e}$ is isomorphic to*

$$\mathcal{Y} := \{Y \in \mathfrak{gl}_k \mid Y^{n+1} = 0\}.$$

In particular, $\mathcal{S}_{\mathcal{O},e}$ is isomorphic to the closure of the nilpotent orbit in $\mathfrak{gl}_k$ with partition $[(n+1)^p, q]$, which is the whole nilcone when $k \leq n + 1$.

Proof. Let I_k be the $k \times k$ identity matrix. Define $e = (e_{ij})$, $h = (h_{ij})$, and $f = (f_{ij})$ to be $n \times n$-block matrices, with blocks of size $k \times k$, as follows:

$$e_{ij} = \begin{cases} j(n-j)I_k & i = j+1 \\ 0 & \text{else} \end{cases}, \; h_{ij} = \begin{cases} (2i-n-1)I_k & i = j \\ 0 & \text{else} \end{cases}, \; f_{ij} = \begin{cases} I_k & j = i+1 \\ 0 & \text{else} \end{cases}$$

The Jordan type of e and f is $[n^k]$, and so $e, f \in \mathcal{O}'$. The elements $\{e, h, f\}$ are a standard basis of an $\mathfrak{sl}_2$-subalgebra $\mathfrak{s}$, as in the $k = 1$ case. Also, as in the $k = 1$ case, the centralizer $\mathfrak{g}^f$ consists of $n \times n$-block matrices $Z = (z_{ij})$ of the form

$$z_{ij} = \begin{cases} Y_{j-i} & j \geq i \\ 0 & \text{otherwise} \end{cases}$$

for any choice of $Y_0, Y_1, \ldots, Y_{n-1} \in \mathfrak{gl}_k$. We abbreviate this matrix by $Z(\{Y_i\})$. In particular, $\mathfrak{c}(\mathfrak{s}) \cong \mathfrak{gl}_k$ consists of the matrices of the form $Z(\{Y, 0, \ldots, 0\})$.

We are interested in

$$\mathcal{S}_{\mathcal{O},e} := \mathcal{S}_e \cap \overline{\mathcal{O}} = \mathcal{S}_e \cap \{X \in \mathfrak{g} \mid X^{n+1} = 0\},$$

where as before $\mathcal{S}_e = e + \mathfrak{g}^f$. Let $M = e + Z(\{Y_i\}) \in \mathcal{S}_e$. Set $Y_0 = -\frac{1}{n}Y$ for a fixed matrix Y for reasons that will become clear shortly. Since $M^{n+1} = 0$, we can find constraints on the entries of M^{n+1}. The $(n, 1)$-entry of M^{n+1} is equal to $rY_1 + sY_0^2$ where r is a sum of products of the coefficient in e, hence nonzero. Thus $rY_1 + sY_0^2 = 0$ and Y_1 is proportional to Y^2. Given this fact, the $(n, 2)$-entry of M^{n+1} is equal to $r'Y_2 + s'Y_0^3$ where r' is nonzero. Hence Y_2 is proportional to Y^3, and so on. In this way, we conclude that $Y_i = c_iY^{i+1}$ for all $i = 0, 1, 2, \ldots, n - 1$, where the $c_i \in \mathbb{C}$ are uniquely determined constants (which depend on n, but not k). Consequently $M \in \mathcal{S}_{\mathcal{O},e}$ takes the form $e + Z(\{c_iY^{i+1}\})$ for some Y. We were not able to find a general formula for the c_i's, but in all cases that we computed, the c_i's were nonzero, which we expect to be true in general.

Now let $T^n + \sum_{i=1}^{n-1} a_iT^{n-i} \in \mathbb{C}[T]$ be the characteristic polynomial for the $n \times n$-matrix $e + Z(\{c_iI_1\})$ in the $k = 1$ case. A direct computation with block matrices then shows that $p(T) := T^n + \sum_{i=1}^{n-1} a_i Y^i T^{n-i}$ is the characteristic polynomial of M, viewing M as an $n \times n$-matrix over the commutative ring $\mathbb{C}[Y]$, where Y acts by simultaneous multiplication on each of the block entries of M. By the Cayley–Hamilton Theorem over $\mathbb{C}[Y]$, it follows that $p(M) = 0$. In fact, $p(T)$ is the minimal polynomial of M over $\mathbb{C}[Y]$. Indeed, for $1 \leq i \leq n-1$, the i-th block lower diagonal of M^i consists of non-zero scalar matrices while everything below that diagonal is zero. Thus M cannot satisfy a polynomial of degree less than n over $\mathbb{C}[Y]$.

The next step is to show that Y^{n+1} must be the zero matrix. Since $p(M) = 0$,

$$0 = Mp(M) - b_1 Y p(M) = \sum_{i=2}^{n} (a_i - a_1 a_{i-1}) Y^i M^{n-i+1} - a_1 a_n Y^{n+1}.$$

Since the minimal polynomial of M over $\mathbb{C}[Y]$ has degree n, it follows that $(a_i - a_1 a_{i-1})Y^i = 0$ for $i = 2, \dots, n$ and $a_1 a_n Y^{n+1} = 0$. Note that $a_1 = 1$ by taking the trace of M since $c_0 = -\frac{1}{n}$. Now if $Y^{n+1} \neq 0$, then recursively $a_i = a_1^i = 1$, but also $a_1 a_n = a_n = 0$, a contradiction. Similarly, if $Y^\ell = 0$ and $Y^{\ell-1} \neq 0$ for some $\ell \leq n+1$, then $a_i = a_1^i = 1$ for $i = 1, 2, \dots, \ell - 1$. We conclude that all elements in $\mathcal{S}_{\mathcal{O},e}$ take the form $e + Z(\{c_i Y^i\})$ where $Y^{n+1} = 0$. Hence $\mathcal{S}_{\mathcal{O},e}$ is isomorphic to a subvariety of $\mathcal{Y}$ via restriction π_0 of the natural projection $\mathcal{S}_e \to \mathfrak{c}(\mathfrak{s})$ by the argument in §4.4.4. Now $\mathcal{S}_{\mathcal{O},e}$ and $\mathcal{Y}$ both have dimension $p^2n^2 + 2pqn + p^2n + q^2 - q$, and the latter variety is irreducible; hence π_0 gives an isomorphism of $\mathcal{S}_{\mathcal{O},e}$ onto $\mathcal{Y}$.

One consequence is the following: since the c_i's, and hence the a_i's, are independent of k, choosing $k > n$, we deduce that all $a_i = 1$, an interesting fact in its own right. $\square$

Remark 11.2. Fix $Y_0 = e_0 \in \mathfrak{c}(\mathfrak{s})$ in the orbit $[(n+1)^p, q]$ and $Y = -nY_0$. In the notation of §4.4, the vector $Z(\{0, \dots, 0, Y^{i+1}, 0, \dots, 0\})$ corresponds to the pair $(i, i+2)$, which lies in $\mathcal{E}$ when $1 \leq i \leq \min(n, k-1)$. The proof shows that there is an $x \in \mathcal{O}$ that can be written as in (4.1) with $x_i := c_i Z(\{0, \dots, 0, Y^{i+1}, 0, \dots, 0\})$ where $0 \leq i \leq \min(n, k-1)$ and such that (4.2) holds.

11.2.2. Second slice

Next, let $\mathcal{O}$ be the orbit in $\mathfrak{gl}_{nk}$ with partition $[(n+k-1, (n-1)^{k-1}]$. Then again $e \in \overline{\mathcal{O}}$. The elements in $\overline{\mathcal{O}}$ correspond to matrices which are nilpotent and which have $\operatorname{rank}(M^i) = k(n-i)$ for $i = 1, 2, \dots, n-1$.

Proposition 11.3. *[24, Corollary 9.3] The variety $\mathcal{S}_{\mathcal{O},e}$ is isomorphic to the nilcone in $\mathfrak{gl}_k$.*

Proof. Up to smooth equivalence, this result is a consequence of [32], by cancellation of the first $n-1$ columns of the partitions for $\mathcal{O}$ and $\mathcal{O}'$. Here, we show that, in fact, $\mathcal{S}_{\mathcal{O},e} \cong \mathcal{N}_{A_{k-1}}$, which also follows from [24, Corollary 9.3].

Keep the notation from the proof of the previous proposition. Let $M \in \mathcal{S}_e$ satisfying the rank conditions $\operatorname{rank}(M^i) = k(n-i)$ for $i = 1, 2, \dots, n-1$. The last rank condition is

$\mathrm{rank}(M^{n-1}) = k$. The bottom, left 2×2-submatrix of M^{n-1} consists of $\begin{pmatrix} rY_0 & sY_1 \\ tI_k & rY_0 \end{pmatrix}$, with each of r, s, t positive, since the coefficients of e are positive. Multiply the last row by $\frac{r}{t}Y_0$ and substract it from the second-to-last row to zero out the $(n-1, 1)$-entry. Then since $\mathrm{rank}(tI_k) = k$, it follows that for $\mathrm{rank}(M^{n-1}) = k$ to hold, necessarily the second-to-last row must be identically zero. In particular, the $(n-1, 2)$-entry is zero, that is, Y_1 is a scalar multiple of Y_0^2. Continuing in this way for the smaller powers of M, we conclude that $Y_i = d_i Y_0^i$ for some $d_i \in \mathbb{C}$, as in the previous proposition.

Next a direct computation shows that M^{n+k-1} has entry $(n, 1)$ which is a scalar multiple of Y_0^k and all other entries are scalar multiples of Y_0^m for $m > k$. If any of these scalar multiples are nonzero, then since $M^{n+k-1} = 0$, it follows that Y_0 is nilpotent, whence $Y_0^k = 0$ since $Y_0 \in \mathfrak{gl}_k$. These multiples are independent of k. The $k = 1$ case implies that the entries in M^{n+k-1} cannot all be zero unless all $d_i = 0$ since e is the only nilpotent element in $\mathcal{S}_e$. We have therefore shown that $\mathcal{S}_{\mathcal{O},e}$ is contained in a variety isomorphic to the nilcone of $\mathfrak{gl}_k$. By dimension reasons, this must be an equality as in the previous proof. □

11.2.3. Example

An example of the first proposition is the degeneration $[2^3] < [3^2]$ and of the second proposition is the degeneration $[2^3] < [4, 1^2]$, both in $\mathfrak{sl}_6$. Both slices are isomorphic to the nilcone of $\mathfrak{sl}_3$. In this setting, the common intermediate orbit $[3, 2, 1]$ corresponds to the minimal nilpotent orbit in $\mathfrak{sl}_3$. Upon restriction to $\mathfrak{sp}_6$, the slice becomes isomorphic to the nilcone in $\mathfrak{so}_3$, which is of type A_1. This gives another proof of §4.1.5, one which does not require knowing that either $[3^2]$ or $[4, 1^2]$ have closures which are unibranch at $[2^3]$.

12. The remaining additional singularities

The singularities μ and $\mathfrak{a}_2/\mathfrak{S}_2$ will be discussed in subsequent work. Here we discuss the minimal degenerations $(2A_2 + A_1, A_2 + 2A_1)$ in E_6 and $(A_4 + A_3, A_4 + A_2 + A_1)$ in E_8 and show that they are singularities of type τ and χ, respectively. Both cases are related to showing that a larger slice is the *cover* of the nilcone in a smaller Lie algebra (compare this with the cases in §11). For the case in E_6, we show for the degeneration $(2A_2 + A_1, A_2)$ that the slice is isomorphic to the affinization of a 3-fold cover of the regular nilpotent orbit in $\mathfrak{sl}_3(\mathbb{C}) \oplus \mathfrak{sl}_3(\mathbb{C})$. For the case in E_8, we show for the degeneration $(A_4 + A_3, A_4)$ that the slice is isomorphic to the affinization of the universal cover of the regular nilpotent orbit in $\mathfrak{sl}_5(\mathbb{C})$.

12.1. Preliminaries

We start with a lemma that extends the results in §4.3. The lemma introduces an alternative transverse slice to some orbits, slightly different from the Slodowy slice. This alternative slice will facilitate the determination of the singularities of the two degenerations in this section. It will also be used in subsequent work for other, non-minimal degenerations. Since this slice is different from the nilpotent Slodowy slice, we are not able to determine the isomorphism type of the nilpotent Slodowy slice, and thus the results in Theorem 1.2 are stated only up to smooth equivalence.

Lemma 12.1. *Let e be a nilpotent element in $\mathfrak{g}$, and let $\mathfrak{s} := \langle e, h, f\rangle$ be an $\mathfrak{sl}_2$-subalgebra containing it. Next, let e_0 be a nilpotent element in $\mathfrak{c}(\mathfrak{s})$, and let $\mathfrak{s}_0 := \langle e_0, h_0, f_0\rangle$ be an $\mathfrak{sl}_2$-subalgebra of $\mathfrak{c}(\mathfrak{s})$. Suppose condition (4.4) is satisfied:*

$$\dim C(\mathfrak{s}) \cdot e_0 = \operatorname{codim}_{\overline{\mathcal{O}}_{e+e_0}} \mathcal{O}_e.$$

Then

$$\mathcal{S}'_{e+e_0} := e + e_0 + \mathfrak{c}(\mathfrak{s})^{f_0} \oplus \sum_{i<0} \mathfrak{g}^f(i)$$

is a transverse slice in $\mathfrak{g}$ to $\mathcal{O}_{e+e_0}$ at $e + e_0$, where $\mathfrak{g}^f(i)$ denotes the $\operatorname{ad} h$-eigenspace for the eigenvalue i in $\mathfrak{g}^f$.

Proof. Decompose $\mathfrak{g}$ under $\mathfrak{s} \oplus \mathfrak{s}_0$ as in (4.5). Then by Proposition 4.8, the dimension hypothesis ensures that the summands $V^{(i)}_{m_i,n_i}$ satisfy $m_i \geq n_i$ whenever $m_i > 0$.

Let $V(m, n)$ be such a summand with $m \geq n$ and $m > 0$, and consider the action of $\mathfrak{s} \oplus \mathfrak{s}_0$ on $V(m, n)$. Then $\dim\ker f = n + 1$ and $\dim\ker f_0 = m + 1$. As discussed in §4.3, $V(m, n)$ decomposes into $n + 1$ irreducible representations under the action of the $\mathfrak{sl}_2$-subalgebra $\langle e + e_0, h + h_0, f + f_0\rangle$. Therefore $\dim\ker(f + f_0) = n + 1$ and so $\dim\ker f = \dim\ker(f + f_0)$. Now $\ker f \cap \operatorname{Im}(e + e_0) = \{0\}$ on $V(m, n)$. Indeed, if $[e + e_0, y] \in \ker f$, then write $y = \sum_{i,j} y_{i,j}$ in the common eigenbasis for h and h_0, where $i, j \in \mathbb{Z}$. If $y_{-m,-n} \neq 0$, then $[e + e_0, y]$ has nonzero component on the $(-m + 2, -n)$-eigenspace since $m > 0$. This contradicts $[e + e_0, y] \in \ker f$, since $\ker f$ coincides with the $(-m)$-eigenspace of h; hence $y_{-m,-n} = 0$. Repeating this argument for $y_{-m+2,-n}$ and then $y_{-m,-n+2}$ shows that they are both zero. Continuing inductively along the diagonals, we get $y_{-m,i} = 0$ for all i. Thus $[e+e_0, y] \in \ker f$ only if $[e+e_0, y] = 0$, as desired. It follows that $\operatorname{Im}(e + e_0) \oplus \ker f$ is a direct sum decomposition of $V(m, n)$ since $\dim\ker f = \dim\ker(f + f_0)$.

On $\mathfrak{c}(\mathfrak{s})$, which is the direct sum of those $V^{(i)}_{m_i,n_i}$ with $m_i = 0$, we clearly have $\mathfrak{c}(\mathfrak{s}) = \operatorname{Im}(e+e_0)\oplus\mathfrak{c}(\mathfrak{s})^{f_0}$ since $\mathfrak{s}$ acts trivially. Therefore, $\mathfrak{c}(\mathfrak{s})^{f_0} \oplus \sum_{i<0} \mathfrak{g}^f(i)$ is a complementary subspace to $[e + e_0, \mathfrak{g}]$ in $\mathfrak{g}$, and we are done. □

Let $(\mathcal{O}, \mathcal{O}')$ be either $(2A_2 + A_1, A_2 + 2A_1)$ in type E_6 or $(A_4 + A_3, A_4 + A_2 + A_1)$ in type E_8. Let $\mathcal{O}''$ be the A_2 orbit in the E_6 case and the A_4 orbit in the E_8 case. Let $e \in \mathcal{O}''$.

Our strategy to study the singularity of $\overline{\mathcal{O}}$ along $\mathcal{O}'$ is to first describe $\mathcal{S}_{\mathcal{O},e}$. In both cases, there exists $x \in \mathcal{O}$ of the form in (4.1) such that (4.2) holds with $x_0 \in \mathfrak{c}(\mathfrak{s})$ regular nilpotent. Hence, $\mathcal{S}_{\mathcal{O},e}$ has a dense $C(\mathfrak{s})$-orbit, and this allows us to describe $\mathcal{S}_{\mathcal{O},e}$ in a concrete way. Of independent interest, $\mathcal{S}_{\mathcal{O},e}$ is the affinization of a cover of the $C(\mathfrak{s})$-orbit through x_0, so unlike in §11, the projection to $\mathfrak{c}(\mathfrak{s})$ of a branch of $\mathcal{S}_{\mathcal{O},e}$ is not an isomorphism. The next step is to show for e_0 in the unique $C(\mathfrak{s})$-orbit of codimension four in the nilcone of $\mathfrak{c}(\mathfrak{s})$ that Lemma 12.1 applies. This allows for the singularity in question to be studied by studying $\overline{\mathcal{O}} \cap \mathcal{S}'_{e+e_0}$, which is manageable since $\mathcal{S}'_{e+e_0} \subset \mathcal{S}_e = e + \mathfrak{g}^f$, and therefore $\overline{\mathcal{O}} \cap \mathcal{S}'_{e+e_0} = \mathcal{S}_{\mathcal{O},e} \cap \mathcal{S}'_{e+e_0}$, so it is enough to work completely inside the concrete $\mathcal{S}_{\mathcal{O},e}$.

Set $Z = C(\mathfrak{s})$ and $\mathfrak{z} = \mathfrak{c}(\mathfrak{s})$. Having found $x \in \mathcal{O}$ of the form $e + x_0 + x_1 + \ldots + x_m$ as above, our approach then consists of the following series of steps:

1. *Describe the (closure of the) set of elements in $Z \cdot x_0$ which are in $e_0 + \mathfrak{z}^{f_0}$.*
2. *For each $y_0 \in Z \cdot x_0$ found in step 1, find an element $z \in Z$ such that $z \cdot x_0 = y_0$.*
3. *With z as in step 2, determine the values of $z \cdot x_1$, $z \cdot x_2$ etc.*

Then since $Z \cdot x$ is dense in $\mathcal{S}_{\mathcal{O},e}$, we arrive at a parametrization of $\overline{\mathcal{O}} \cap \mathcal{S}'_{e+e_0}$.

12.2. $(\mathbf{2A_2 + A_1}, \mathbf{A_2 + 2A_1})$ *in* $\mathbf{E_6}$

Recall $\mathcal{O}''$ is of type A_2. We choose $e \in \mathcal{O}''$ and the rest of $\mathfrak{s}$ as follows:

$$e = e_{\alpha_2} + e_{\substack{12321\\1}}, \quad f = 2f_{\alpha_2} + 2f_{\substack{12321\\1}}, \quad h = [e, f].$$

Then $\mathfrak{z} \cong \mathfrak{sl}_3 \oplus \mathfrak{sl}_3$, with basis of simple roots $\{\alpha_1, \alpha_3, \alpha_5, \alpha_6\}$. Let $\mathfrak{l}_1$ be the subalgebra of $\mathfrak{z}$ with simple roots $\{\alpha_1, \alpha_3\}$ and let $\mathfrak{l}_2$, with simple roots $\{\alpha_5, \alpha_6\}$, so that $\mathfrak{z} = \mathfrak{l}_1 \oplus \mathfrak{l}_2$. Similarly, $Z^\circ = L_1 \times L_2 \cong \mathrm{SL}_3 \times \mathrm{SL}_3$, where $\mathrm{Lie}(L_1) = \mathfrak{l}_1$ and $\mathrm{Lie}(L_2) = \mathfrak{l}_2$, and Z/Z° is cyclic of order 2, generated by an element which interchanges L_1 and L_2.

The Z-orbit structure of $\mathcal{N}(\mathfrak{z})$ is therefore as follows: there is a unique open orbit, which is also connected. We call this the regular orbit. Its complement in $\mathcal{N}(\mathfrak{z})$ has two irreducible components permuted transitively by Z/Z°, and a unique open Z-orbit, which we call the subregular orbit, consisting of pairs (x, y) where one of x, y is regular nilpotent, and the other is subregular, in $\mathfrak{sl}_3$. The closure of this orbit contains the Z-orbit of all pairs (x, y) where both x and y are subregular nilpotent elements of $\mathfrak{sl}_3$. There are three further Z-orbits with representatives $(x, 0)$, as x ranges over all Jordan types in $\mathfrak{sl}_3$.

We recall [35, p. 81] that $\mathfrak{g}^f(-2) = \mathbb{C}f \oplus V \oplus W$ where V is isomorphic to the tensor product of the natural representation of L_1 with the dual of the natural representation of L_2, and $W \cong V^*$. The only other non-trivial space $\mathfrak{g}^f(-i)$ is $\mathfrak{g}^f(-4)$, which is one-dimensional. Moreover, $v_1 := 3f_\beta$ where $\beta = {\substack{01210\\1}}$ is a highest weight vector in V and $w_1 := 3f_{\alpha_2+\alpha_4}$ is a highest weight vector in W, relative to the choice of simple roots above. With respect to the Z°-action, we identify V (respectively, W) with the space of 3×3 matrices, on which $(g, h) \in L_1 \times L_2$ acts via

$$(g, h) \cdot M = gMh^{-1} \text{ (respectively, } (g, h) \cdot M = hMg^{-1}),$$

and we identify v_1 and w_1 with the matrix with 1 in the top right entry, and zero everywhere else.

Let $e_1 = e_{\alpha_1+\alpha_3}$, $e_2 = e_{\alpha_5+\alpha_6}$, $\tilde{e}_1 = e_{\alpha_1} + e_{\alpha_3}$, $\tilde{e}_2 = e_{\alpha_5} + e_{\alpha_6}$. Let $x_0 := \tilde{e}_1 + \tilde{e}_2$, which is a regular nilpotent element in $\mathfrak{z}$ and let $e_0 := e_1 + e_2$. Then e_0 satisfies the dimension hypothesis (4.4) and so we can apply Lemma 12.1 to it. On the other hand, for x_0 the situation in §4.4 applies:

Lemma 12.2. *The element*

$$x := e + x_0 + v_1 + w_1$$

lies in $\mathcal{S}_{\mathcal{O},e} \cap \mathcal{O}$. Thus $S_{\mathcal{O},e} = \overline{Z^\circ \cdot x}$.

Proof. We verified by computer that $x \in \mathcal{O}$. The last part follows, as in §4.1.2, since both $\mathcal{S}_{\mathcal{O},e}$ and $Z^\circ \cdot x_0$ have dimension 12, and $\mathcal{S}_{\mathcal{O},e}$ is irreducible since $\overline{\mathcal{O}}$ is unibranch at e. $\square$

We note that x_0 is in the regular nilpotent Z-orbit in $\mathfrak{z}$ and $e_1 + \tilde{e}_2$ and $\tilde{e}_1 + e_2$ both lie in the subregular nilpotent Z-orbit so that $\overline{Z \cdot x_0} \supset \overline{Z \cdot (e_1 + \tilde{e_2})} \supset \overline{Z \cdot e_0}$. Moreover, we observe that $e + e_1 + \tilde{e}_2$ and $e + \tilde{e}_1 + e_2$ both belong to $\mathcal{O} \cap \mathcal{S}_{\mathcal{O},e}$. This fact can be used to give a conceptual proof of the previous lemma. It is also useful for the next proposition.

The centralizer $(Z^\circ)^{x_0}$ of x_0 in Z° is generated by its identity component, a unipotent group of dimension four, and the nine scalar matrices in the center of $Z^\circ \cong \mathrm{SL}_3 \times \mathrm{SL}_3$. Let U be the index 3 subgroup of this centralizer containing the central cyclic group $\{(\omega^i I, \omega^i I) \mid i \in \{0, 1, 2\}\}$, where $\omega = e^{2\pi i/3}$. Let $p : \mathcal{S}_{\mathcal{O},e} \to \mathcal{N}(\mathfrak{z})$ be the restriction of the Z-equivariant projection of $e + \mathfrak{g}^f$ onto $\mathfrak{z}$. By the previous lemma, p is surjective onto the nilpotent cone $\mathcal{N}(\mathfrak{z})$ in $\mathfrak{z}$. The next proposition is not needed in the proof of the main result, but is of independent interest.

Proposition 12.3. *The slice $\mathcal{S}_{\mathcal{O},e}$ is isomorphic to the affinization of the 3-fold cover Z°/U of the regular nilpotent orbit $Z^\circ \cdot x_0$ in $\mathfrak{sl}_3(\mathbb{C}) \oplus \mathfrak{sl}_3(\mathbb{C})$, and hence is a normal variety. Moreover, p is finite and is an isomorphism when restricted to the complement of $Z^\circ \cdot x$. Finally, $\mathcal{S}_{\mathcal{O},e}$ (and hence the affinization) is smooth at points over the subregular Z-orbit in $\mathcal{N}(\mathfrak{z})$.*

Proof. For dimension reasons the identity component of U acts trivially on v_1 and w_1. A pair of scalar matrices $(\omega^i I, \omega^j I)$ acts on V and W by the scalars ω^{i-j} and ω^{j-i}, respectively. Hence the subgroup of $(Z^\circ)^{x_0}$ that acts trivially on x is exactly U. This shows that $\widetilde{Y} := Z^\circ \cdot x$ identifies with the 3-fold cover Z°/U of the regular orbit $Y := Z^\circ \cdot x_0$ in $\mathfrak{z}$.

Now the regular functions $\mathbb{C}[\mathcal{S}_{\mathcal{O},e}]$ on $\mathcal{S}_{\mathcal{O},e} = \overline{Z^\circ \cdot x}$ embed in $\mathbb{C}[\widetilde{Y}]$, since $\widetilde{Y}$ is dense in $\mathcal{S}_{\mathcal{O},e}$. Since p is surjective onto $\overline{Y} = \mathcal{N}(\mathfrak{z})$, we then have the inclusions $\mathbb{C}[\overline{Y}] \subset \mathbb{C}[\mathcal{S}_{\mathcal{O},e}] \subset \mathbb{C}[\widetilde{Y}]$. Also $\mathbb{C}[Y] \cong \mathbb{C}[\mathcal{N}(\mathfrak{z})]$ since $\mathcal{N}(\mathfrak{z})$ is normal. Now from [23], the ring $\mathbb{C}[\widetilde{Y}]$ is generated as a module over $\mathbb{C}[Y]$ by the unique copies of V and W in $\mathbb{C}[\widetilde{Y}]$. But $\mathbb{C}[\mathcal{S}_{\mathcal{O},e}]$ contains a copy of both V and W, via the Z°-equivariant projection of $\mathcal{S}_{\mathcal{O},e}$ onto the V and W factors in $\mathfrak{g}^f$, respectively. Hence $\mathbb{C}[\mathcal{S}_{\mathcal{O},e}] = \mathbb{C}[\widetilde{Y}]$. This shows in particular that $\mathcal{S}_{\mathcal{O},e}$ is normal and p is finite.

For any non-regular element in $\mathcal{N}(\mathfrak{z})$, its centralizer in Z° will contain a torus that acts non-trivially on any line in V and W. Thus, since p is finite, $\mathcal{S}_{\mathcal{O},e}$ must be zero on the V and W components over such elements. It follows that p is an isomorphism over such elements, that is, when restricted to the complement of $Z^\circ \cdot x$ in $\mathcal{S}_{\mathcal{O},e}$.

Moreover, $\mathcal{O} \cap \mathcal{S}_{\mathcal{O},e}$ consists of exactly two Z-orbits, corresponding to points over the regular and subregular Z-orbits in $\mathfrak{z}$. Since $\mathcal{O} \cap \mathcal{S}_{\mathcal{O},e}$ is smooth, it follows that $\mathcal{S}_{\mathcal{O},e}$ is smooth at points over the subregular orbit. Alternatively, this follows from the fact that transverse slice of $\mathcal{N}(\mathfrak{z})$ at a subregular element is $\mathbb{C}^2/\Gamma'$, where $\Gamma' \subset \mathrm{SL}_2$ is cyclic of order three. The preimage under p of this transverse slice must then be $\mathbb{C}^2$. $\square$

Before continuing, we make some observations about transverse slices in $\mathfrak{sl}_3$. Following up on our identification of V and W with 3×3 matrices, we identify $\mathfrak{l}_1$ and $\mathfrak{l}_2$ with $\mathfrak{sl}_3$ so that e_1 and e_2 correspond to $\begin{pmatrix} 0&0&1\\ 0&0&0\\ 0&0&0 \end{pmatrix}$.

Lemma 12.4. *With the above identification of $\mathfrak{l}_1$ with $\mathfrak{sl}_3$, we have that*

$$e_1 = \begin{pmatrix} 0&0&1\\ 0&0&0\\ 0&0&0 \end{pmatrix}, \; h_1 = \begin{pmatrix} 1&0&0\\ 0&0&0\\ 0&0&-1 \end{pmatrix}, \; f_1 = \begin{pmatrix} 0&0&0\\ 0&0&0\\ 1&0&0 \end{pmatrix},$$

is an $\mathfrak{sl}_2$-triple through e_1. The intersection of $e_1 + \mathfrak{sl}_3{}^{f_1}$ with the nilpotent cone in $\mathfrak{sl}_3$ is the set of elements of the form

$$X_{st} := \begin{pmatrix} \frac{1}{2}st & 0 & 1\\ s^3 & -st & 0\\ -\frac{3}{4}s^2t^2 & t^3 & \frac{1}{2}st \end{pmatrix}$$

for $s, t \in \mathbb{C}$.

Proof. The ideal of the nilpotent cone in $\mathfrak{sl}_3$ is generated by the determinant and the sum of the three diagonal 2×2 minors. The zero set in $e_1 + \mathfrak{sl}_3{}^{f_1}$ of these two functions is exactly the elements X_{st} for $s, t \in \mathbb{C}$. $\square$

Continuing the identification of $\mathfrak{l}_1$ and $\mathfrak{l}_2$ with $\mathfrak{sl}_3$, we have $\tilde{e}_1$ and $\tilde{e}_2$ correspond to $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$.

Lemma 12.5. *If $s \neq 0$ then $g_{st}\tilde{e}_1 g_{st}^{-1} = X_{st}$ where*

$$g_{st} := \begin{pmatrix} -t & -1/s & 0 \\ -s^2 & 0 & 0 \\ st^2/2 & -t/2 & -1/s \end{pmatrix}$$

Moreover, $g_{st} \in L_1$ and $g_{st}^{-1} = \begin{pmatrix} 0 & -1/s^2 & 0 \\ -s & t/s & 0 \\ s^2t/2 & -t^2 & -s \end{pmatrix}$.

Proof. It is easy to check that $\det g_{st} = 1$ (hence lies in L_1) and that g_{st}^{-1} is as described. The columns c_1, c_2, c_3 of g_{st} satisfy $X_{st}c_1 = 0$, $X_{st}c_2 = c_1$, and $X_{st}c_3 = c_2$, from which it follows that $g_{st}\tilde{e}_1 g_{st}^{-1} = X_{st}$. $\square$

As noted above, Lemma 12.1 applies to $e_0 = e_1 + e_2$. Furthermore, $e + e_0 \in \mathcal{O}'$. Thus the affine linear space $\mathcal{S}'_{e+e_0} = e + e_0 + \mathfrak{l}_1^{f_1} + \mathfrak{l}_2^{f_2} + \mathfrak{g}^f(-2) + \mathfrak{g}^f(-4)$ is transverse to $\mathcal{O}'$, and hence $\mathrm{Sing}(\mathcal{O}, \mathcal{O}')$ can be determined by describing the intersection $\overline{\mathcal{O}} \cap \mathcal{S}'_{e+e_0}$.

Theorem 12.6. *The intersection $\overline{\mathcal{O}} \cap \mathcal{S}'_{e+e_0}$ consists of all elements of the form:*

$$e + \left(X_{st}, X_{uv}, \begin{pmatrix} -\frac{1}{2}tu^2v & tv^2 & tu \\ -\frac{1}{2}s^2u^2v & s^2v^2 & s^2u \\ \frac{1}{4}st^2u^2v & -\frac{1}{2}st^2v^2 & -\frac{1}{2}st^2u \end{pmatrix}, \begin{pmatrix} -\frac{1}{2}s^2tv & t^2v & sv \\ -\frac{1}{2}s^2tu^2 & t^2u^2 & su^2 \\ \frac{1}{4}s^2tuv^2 & -\frac{1}{2}t^2uv^2 & -\frac{1}{2}suv^2 \end{pmatrix} \right)$$
$$\in e + \mathfrak{l}_1 \oplus \mathfrak{l}_2 \oplus V \oplus W$$

where $s, t, u, v \in \mathbb{C}$.

Proof. Suppose $s, u \neq 0$. Consider the action of the element $(g_{st}, g_{uv}) \in Z^\circ$ on x. From Lemmas 12.4 and 12.5 (also for the $\mathfrak{l}_2$ version), we have

$$\begin{aligned}(g_{st}, g_{uv}) \cdot x &= e + (g_{st}, g_{uv}).(\tilde{e}_1, \tilde{e}_2, v_1, w_1) \\ &= e + \left(g_{st}\tilde{e}_1 g_{st}^{-1}, g_{uv}\tilde{e}_2 g_{uv}^{-1}, g_{st} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} g_{uv}^{-1}, g_{uv} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} g_{st}^{-1} \right) \\ &= e + \left(X_{st}, X_{uv}, \begin{pmatrix} -\frac{1}{2}tu^2v & tv^2 & tu \\ -\frac{1}{2}s^2u^2v & s^2v^2 & s^2u \\ \frac{1}{4}st^2u^2v & -\frac{1}{2}st^2v^2 & -\frac{1}{2}st^2u \end{pmatrix}, \right. \\ &\qquad \left. \begin{pmatrix} -\frac{1}{2}s^2tv & t^2v & sv \\ -\frac{1}{2}s^2tu^2 & t^2u^2 & su^2 \\ \frac{1}{4}s^2tuv^2 & -\frac{1}{2}t^2uv^2 & -\frac{1}{2}suv^2 \end{pmatrix} \right). \end{aligned} \tag{12.1}$$

By Lemma 12.2, we have $x \in \mathcal{O}$, so the elements in (12.1) lie in $\mathcal{O}$. They also clearly are in $\mathcal{S}'_{e+e_0}$, and so this set of elements, of dimension four, lies in $\overline{\mathcal{O}} \cap \mathcal{S}'_{e+e_0}$. But the latter is of dimension four since this is the codimension of $\mathcal{O}'$ in $\overline{\mathcal{O}}$. Moreover, $\overline{\mathcal{O}}$ is unibranch at points in $\mathcal{O}'$. Hence $\overline{\mathcal{O}} \cap \mathcal{S}'_{e+e_0}$ is irreducible of dimension four, and must be the closure of the set of elements in (12.1) with $s, u \neq 0$. The closure of this latter set is evidently those in (12.1) where s, t, u, v are unrestricted. $\square$

Let Γ be the subgroup of $\mathrm{Sp}_4(\mathbb{C})$ generated by $\mathrm{diag}(\omega, \omega^{-1}, \omega, \omega^{-1})$.

Corollary 12.7. *The singularity* $\mathrm{Sing}(\mathbb{C}^4/\Gamma, 0)$ *is equal to* $\mathrm{Sing}(2A_2 + A_1, A_2 + 2A_1)$.

Proof. By the theorem, the variety $\overline{\mathcal{O}} \cap \mathcal{S}'_{e+e_0}$ is isomorphic to the variety with coordinate ring $\mathbb{C}[st, s^3, t^3, uv, u^3, v^3, sv, tu, s^2u, su^2, t^2v, tv^2]$. It is straightforward to see that this is the invariant subring of $\mathbb{C}[s, t, u, v]$ for the induced action of Γ. Also $e + e_0$ corresponds to the point $s = t = u = v = 0$. Since $\mathrm{Sing}(\overline{\mathcal{O}} \cap \mathcal{S}'_{e+e_0}, e + e_0) = \mathrm{Sing}(2A_2 + A_1, A_2 + 2A_1)$, the result follows. $\square$

We note an interesting consequence of the above description. The closed subset given by setting $s = v$, $t = u$ has coordinate ring $\mathbb{C}[s^3, t^3, st, st^2, s^2t, s^2, t^2]$, which is exactly the coordinate ring of the singularity m. This amounts to taking fixed points in $\overline{\mathcal{O}} \cap \mathcal{S}'_{e+e_0}$ under an appropriate outer involution of $\mathfrak{g}$, giving us another proof that the singularity $(\tilde{A}_2 + A_1, A_2 + \tilde{A}_1)$ in F_4 is smoothly equivalent to m.

12.3. $(\mathbf{A_4} + \mathbf{A_3}, \mathbf{A_4} + \mathbf{A_2} + \mathbf{A_1})$ *in* $\mathbf{E}_8$

12.3.1. We begin by describing a concrete model for the singularity.

Let $\Delta = \langle \sigma, \tau : \sigma^5 = \tau^2 = (\sigma\tau)^2 = 1 \rangle$ be a dihedral group of order 10, acting on $V = \mathbb{C}^4$ by: $\tau(u, v) = (v, u)$ and $\sigma(u, v) = (\zeta u, \zeta^{-1} v)$, where $\zeta = e^{\frac{2\pi i}{5}}$ and $(u, v) \in \mathbb{C}^2 \oplus \mathbb{C}^2 = \mathbb{C}^4$.

Denote by p, q (resp. s, t) the coordinate functions on the first (resp. second) copy of $\mathbb{C}^2$. In particular, $\mathbb{C}[V] = \mathbb{C}[p, q, s, t]$. It is easy to show that the ring of invariants $\mathbb{C}[V]^\Delta$ is generated by $A = pt + qs$, $B = -2ps$, $C = 2qt$ and the functions $F_i = p^{5-i}q^i + s^{5-i}t^i$ for $0 \leq i \leq 5$. We note that $A^2 + BC = (pt - qs)^2$. Since none of the elements of Δ act as complex reflections on V, it follows that the singular points of the quotient V/Δ are the Δ-orbits of points with non-trivial centralizer, hence are the images in V/Δ of the points of the form (u, u) (or equivalently, $(u, \zeta^i u)$) for $u \in \mathbb{C}^2$. Thus the singular locus is properly contained in the zero set of $(A^2 + BC)$ in V/Δ. Let $D = A^2 + BC$ and for $0 \leq i \leq 5$ let $G_i = (p^{5-i}q^i - s^{5-i}t^i)/(pt - qs) \in \mathrm{Frac}(\mathbb{C}[V]^\Delta) = \mathbb{C}(V)^\Delta$. It is easy to see that for $0 \leq i \leq 5$, $DG_i \in \mathbb{C}[V]^\Delta$ vanishes on the singular locus of V/Δ, and that $F_i = AG_i + BG_{i+1}$ for $i \leq 4$ (resp. $F_i = CG_{i-1} - AG_i$ for $i \geq 1$), whence the G_i satisfy: $2AG_i - CG_{i-1} + BG_{i+1} = 0$ for $1 \leq i \leq 4$. (These equations are also satisfied by the F_i's.)

Let $Y = \mathrm{Spec}(\mathbb{C}[A, B, C, G_0, \ldots, G_5])$.

Remark 12.8. a) The singularity Y can be obtained by blowing up V/Δ at its singular locus, as follows. It is not hard to show that the ideal of elements of $\mathbb{C}[V]^\Delta$ which vanish at the singular points is generated by D and $DG_0, \dots, DG_5$. Thus the blowup of V/Δ can be described as the subset of $\mathbb{A}^9 \times \mathbb{P}^6$ which is the closure of the set of elements of the form $(A, B, C, F_0, \dots, F_5, [D : DG_0 : \dots : DG_5])$ with at least one of $D, DG_0, \dots, DG_5 \neq 0$. Clearly, the affine open subset given by $D \neq 0$ has affine coordinates A, B, C, F_i, G_i, and hence is isomorphic to Y. An immediate consequence of this description is that Y is birational to V/Δ.

b) It can be shown that the ideal of relations satisfied by $A, B, C, G_0, \dots, G_5$ is generated by the expressions $2AG_i + BG_{i+1} - CG_{i-1} = 0$ together with ten identities of the form $G_iG_j - G_{i+1}G_{j-1} - p(A, B, C) = 0$, where p is a cubic polynomial. For example, $G_iG_{i+2} - G_{i+1}^2 = \frac{(-1)^i}{8} B^{3-i}C^i$ for $i \leq 3$ and $G_iG_{i+3} - G_{i+1}G_{i+2} = \frac{(-1)^{i+1}}{4} AB^{2-i}C^i$ for $i \leq 2$.

c) It can be shown that all of the remaining affine open subsets of the blow-up given by $DG_i \neq 0$ are smooth, in fact are isomorphic to $\mathbb{A}^4$. For example, the open subset given by $DG_0 \neq 0$ is the affine variety with coordinate ring $R = \mathbb{C}[A, B, C, F_0, \dots, F_5, 1/G_0, G_1/G_0, \dots, G_5/G_0]$. It is an easy calculation (using the identities for the G_i mentioned above) to check that this ring is generated by $B, F_0, 1/G_0$ and G_1/G_0, hence by dimensions is a polynomial ring of rank four. Thus the point of Y corresponding to the maximal ideal (A, B, C, G_i) is the unique singular point of the blow-up of $\mathbb{C}^4/\Delta$. This justifies the more succinct description of $\mathrm{Sing}(\mathcal{O}, \mathcal{O}')$ given in the introduction.

d) In general, a blow-up of a symplectic singularity is not a symplectic singularity. In our case, $\overline{\mathcal{O}}$ inherits a symplectic structure from that of $\mathfrak{g}$, and so (subject to our claim) Y is a symplectic singularity. More generally, it can be shown that the blow-up (at the singular locus) of the quotient of $\mathbb{C}^4$ by any dihedral group (with $\mathbb{C}^4$ identified with two copies of its reflection representation) is a symplectic singularity.

We will next show that $\mathrm{Sing}(\overline{\mathcal{O}}, \mathcal{O}')$ is equivalent to Y.

12.3.2. Let $e = e_{\alpha_1} + e_{\alpha_3} + e_{\alpha_4} + e_{\alpha_2}$, $f = 4f_{\alpha_1} + 6f_{\alpha_3} + 6f_{\alpha_4} + 4f_{\alpha_2}$, $h = [e, f]$. Then $e \in \mathcal{O}''$, the orbit of type A_4, and $\mathfrak{z} \cong \mathfrak{sl}_5(\mathbb{C})$ with basis of simple roots $\{\beta_1, \beta_2, \beta_3, \beta_4\} := \{\alpha_8, \alpha_7, \alpha_6, {}^{2465321}_{\quad 3}\}$, and Z is isomorphic to the semidirect product of $\mathrm{SL}_5(\mathbb{C})$ by an outer involution. Let x_0 belong to the regular nilpotent orbit in $\mathfrak{z}$.

For the purposes of calculation we identify Z° with $\mathrm{SL}_5(\mathbb{C})$ by identifying the basis of simple roots $\{\beta_1, \beta_2, \beta_3, \beta_4\}$ with the basis of simple roots of $\mathrm{SL}_5(\mathbb{C})$ coming from the choice of diagonal maximal torus and upper triangular Borel subgroup, with the usual ordering of simple roots. Let W be the natural module for Z°, corresponding to the defining representation of SL_5. The Z°-module structure of $\mathfrak{g}^f$ includes the following spaces:

$$\mathfrak{g}^f(-2) \cong W \oplus W^* \oplus \mathbb{C}, \quad \mathfrak{g}^f(-4) \cong \Lambda^2(W) \oplus \Lambda^2(W^*) \oplus \mathbb{C}$$

The following vectors are highest weight vectors, relative to the simple roots $\{\beta_i\}$:

$$w_1 = 3e_{\substack{0011111\\1}} - 2e_{\substack{0111111\\0}} \in W, \quad u_1 = 2e_{\substack{1354321\\2}} - 3e_{\substack{2344321\\2}} \in W^*,$$
$$y_1 = e_{\substack{1233321\\1}} \in \Lambda^2(W), \quad z_1 = e_{\substack{0122221\\1}} \in \Lambda^2(W^*).$$

Let $x_0 := e_{\beta_1} + e_{\beta_2} + e_{\beta_3} + e_{\beta_4}$, a regular nilpotent element in $\mathfrak{z}$. Then we verified that

$$x = e + x_0 - w_1 + u_1 + 10y_1 - 10z_1 \in \mathcal{O},$$

where recall $\mathcal{O}$ is of type A_4+A_3, and so it follows that $\mathcal{S}_{\mathcal{O},e} = \overline{Z^\circ \cdot x}$ since both sides are dimension 20 and $\overline{\mathcal{O}}$ is unibranch at e. This leads to a description of $\mathcal{S}_{\mathcal{O},e}$, whose details, which we omit, are similar to those in Proposition 12.3. Recall that $p : \mathcal{S}_{\mathcal{O},e} \to \mathcal{N}(\mathfrak{z})$ is the Z-equivariant projection.

Proposition 12.9. *The slice $\mathcal{S}_{\mathcal{O},e}$ is isomorphic to the affinization of the universal cover of the regular nilpotent orbit in $\mathfrak{sl}_5(\mathbb{C})$, and hence is a normal variety. Moreover, p is finite and is an isomorphism when restricted to the complement of $Z^\circ \cdot x$. Finally, $\mathcal{S}_{\mathcal{O},e}$ (and hence the affinization) is smooth at points over the subregular Z°-orbit in $\mathcal{N}(\mathfrak{z})$.*

We also note that $\mathcal{O} \cap \mathcal{S}_{\mathcal{O},e}$ is the union of two Z°-orbits, one of which projects under p to the regular orbit and the other, to the subregular orbit in $\mathcal{N}(\mathfrak{z})$.

Let $e_0 \in \mathfrak{z}$ be an element in the orbit with partition type $[3, 2]$, which is codimension four in $\mathcal{N}(\mathfrak{z})$. Then $e + e_0 \in \mathcal{O}'$. Moreover, e_0 satisfies the condition in Lemma 12.1 and so we can study the singularity $(\mathcal{O}, \mathcal{O}')$ by studying $\overline{\mathcal{O}} \cap \mathcal{S}'_{e+e_0}$.

Lemma 12.10. *The intersection $\overline{\mathcal{O}} \cap \mathcal{S}'_{e+e_0}$ is isomorphic to the closure of the set of all $(M, w'_1, w'_1 \wedge w'_2, w'_1 \wedge w'_2 \wedge w'_3, w'_1 \wedge w'_2 \wedge w'_3 \wedge w'_4) \in (e_0 + \mathfrak{z}^{f_0}) \times (W \oplus \Lambda^2(W) \oplus \Lambda^3(W) \oplus \Lambda^4(W))$ such that there is a basis $\{w'_1, w'_2, \ldots, w'_5\}$ for W with $w'_1 \wedge \ldots \wedge w'_5 = 1$ and $Mw'_i = w'_{i-1}$ $(i \geq 2)$, $Mw'_1 = 0$.*

Proof. We can describe $\mathcal{S}_{\mathcal{O},e}$ in the following way: let $\{w_1, w_2, w_3, w_4, w_5\}$ be the standard basis for $\mathbb{C}^5$ and let

$$M_0 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

so that $M_0 w_1 = 0$ and $M_0 w_i = w_{i-1}$ for $2 \leq i \leq 5$. Then $\mathcal{S}_{\mathcal{O},e}$ is isomorphic to the closure in $\mathfrak{sl}_5 \oplus W \oplus \Lambda^2(W) \oplus \Lambda^3(W) \oplus \Lambda^4(W)$ of the SL_5-orbit of $\tilde{M}_0 := (M_0, w_1, w_1 \wedge w_2, w_1 \wedge w_2 \wedge w_3, w_1 \wedge w_2 \wedge w_3 \wedge w_4)$.

To describe the subvariety $\overline{\mathcal{O}} \cap \mathcal{S}'_{e+e_0} = \mathcal{S}_{\mathcal{O},e} \cap \mathcal{S}'_{e+e_0}$, we note that if $M \in e_0 + \mathfrak{z}^{f_0}$ is nilpotent then generically M is regular and therefore there exists a basis $\mathcal{B} = \{w'_1, w'_2, w'_3, w'_4, w'_5\}$ of $\mathbb{C}^5$ such that $Mw'_1 = 0$ and $Mw'_i = w'_{i-1}$ for $i \geq 2$. After scaling, we may assume that $g_{\mathcal{B}} := (w'_1\ w'_2\ w'_3\ w'_4\ w'_5)$ has determinant one. Then the tuple $(M, w'_1, w'_1 \wedge w'_2, w'_1 \wedge w'_2 \wedge w'_3, w'_1 \wedge w'_2 \wedge w'_3 \wedge w'_4) = g_{\mathcal{B}} \cdot \tilde{M}_0$. $\square$

Next, we concretely describe the variety $\mathcal{N}(\mathfrak{z}) \cap (e_0 + \mathfrak{z}^{f_0})$. Let

$$e_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \end{pmatrix}, \quad f_0 = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and then $e_0 + \mathfrak{z}^{f_0}$ consists of all matrices of the form

$$\begin{pmatrix} 2a & b & c & d & g \\ 0 & -3a & h & k & l \\ 2 & 0 & 2a & b & c \\ 0 & 1 & 0 & -3a & h \\ 0 & 0 & 2 & 0 & 2a \end{pmatrix},$$

where $a, b, c, d, g, h, k, l \in \mathbb{C}$. For the purposes of our calculation, we consider the matrices in $e_0 + \mathfrak{z}^{f_0}$ of the form:

$$M = \begin{pmatrix} 2a & b & c - 6a^2 & d - 2ab & 40a^3 - 10ac - \frac{5}{4}bh \\ 0 & -3a & h & 9a^2 - 4c & l - 2ah \\ 2 & 0 & 2a & b & c - 6a^2 \\ 0 & 1 & 0 & -3a & h \\ 0 & 0 & 2 & 0 & 2a \end{pmatrix}$$

A calculation confirms that any such matrix satisfies $\operatorname{Tr} M^2 = \operatorname{Tr} M^3 = 0$, and that the conditions $\operatorname{Tr} M^4 = 0$ and $\operatorname{Tr} M^5 = 0$ are expressed in terms of the coordinates a, b, c, d, h, l as:

$$dh + bl + \frac{8}{3}c^2 = a(9bh - 216a^3 + 72ac), \quad dl = c(9bh - 216a^3 + 48ac). \tag{12.2}$$

Since every irreducible component of the set of (a, b, c, d, h, l) satisfying these two equations has dimension at least four, it follows that the set of matrices given by the coordinates satisfying (12.2) is equal to the set of nilpotent elements of $e_0 + \mathfrak{z}^{f_0}$ (and is therefore irreducible).

It is easy to verify that the rational functions $a = A/6$, $b = -G_0/3$, $c = -BC/16$, $d = BG_1/4$, $h = G_5/3$, $l = CG_4/4$ in $\mathbb{C}(p, q, s, t)$ satisfy (12.2). Since $A, BC, G_0, G_5, BG_1, CG_4$ are regular functions on Y, we have therefore constructed a morphism from Y to $\mathcal{N}(\mathfrak{z}) \cap (e_0 + \mathfrak{z}^{f_0})$, corresponding to the inclusion $\mathbb{C}[A, BC, G_0, G_5, BG_1, CG_4] \subset \mathbb{C}[Y]$. In fact, this morphism corresponds to quotienting Y by the action of a group of order five, as follows: let ρ be the automorphism of order five of V which sends (p, q, s, t) to $(\zeta p, \zeta^{-1} q, \zeta s, \zeta^{-1} t)$. Then ρ normalizes Γ and has an induced action on Y satisfying $\mathbb{C}[Y]^\rho = \mathbb{C}[A, BC, G_0, G_5, BG_1, CG_4]$. (The invariants B^2G_2 and C^2G_3

are contained in this ring, since $BG_2 = CG_0 - 2AG_1$ and $CG_3 = 2AG_4 + BG_5$.) It follows that the coordinates $a = A/6$ etc. given above define an isomorphism from $Y/\langle\rho\rangle$ to $\mathcal{N}(\mathfrak{sl}_5) \cap (e_0 + \mathfrak{z}^{f_0})$.

Remark 12.11. The above discussion indicates an interesting way to view the singularity $([5],[3,2])$ in $\mathfrak{sl}_5$, as an affine open subset of the blow up of the quotient of $\mathbb{C}^4$ by a group of order 50. Indeed, the group generated by Γ and ρ is isomorphic to the complex reflection group $G(5,1,2)$, acting on $\mathbb{C}^4 = U \oplus U^*$ where U is the defining representation for $G(5,1,2)$. Blowing up the quotient at the set of orbits of points of the form (u,u), and restricting to the affine open subset given by $D \neq 0$, one obtains the variety $Y/\langle\rho\rangle$.

We will first give an ad hoc justification that $\mathcal{S}'_{\mathcal{O},e+e_0} := \overline{\mathcal{O}} \cap \mathcal{S}'_{e+e_0}$ is isomorphic to Y, and then a more rigorous proof. Fix a matrix M as above with coordinates $a = A/6$, etc., which we think of as depending on the point $(A,B,C,G_0,\dots,G_5) \in Y$. The space of (column) vectors in W which are annihilated by M is generically of dimension one, spanned by

$$w_1' = \begin{pmatrix} -\frac{1}{6}G_0G_4 + \frac{1}{9}A^2B + \frac{1}{32}B^2C \\ -\frac{1}{4}CG_3 - \frac{1}{12}BG_5 \\ -\frac{1}{6}AB \\ -G_4 \\ B \end{pmatrix}$$

Similarly, the space of (row) vectors in W^* which are annihilated by M is also generically of dimension one, spanned by $u_1' = (C,\ -G_1,\ -\frac{1}{6}AC,\ \frac{1}{3}CG_0 - \frac{1}{2}AG_1,\ \frac{1}{9}A^2C + \frac{1}{32}BC^2 + \frac{1}{6}G_1G_5)$. It follows that if $z \in Z^\circ = \mathrm{SL}_5$ is such that $\operatorname{Ad} z(M_0) = M$, then zw_1 is a scalar multiple of w_1', and $u_1 z$ is a scalar multiple of u_1'. Our more rigorous argument below will (essentially) consist of showing that these scalars, up to multiplication by a fifth root of unity, are independent of p,q,s,t. Thus the ring of regular functions on $\mathcal{S}'_{\mathcal{O},e+e_0}$ also contains elements which naturally correspond to B, C, G_1 and G_4. To continue along this line, we would have to find a vector $w_2' \in W$ such that $Mw_2' = w_1'$, and similarly for u_1'. Then it turns out that the coordinates of $w_1' \wedge w_2'$ and $u_1' \wedge u_2'$ are contained in $\mathbb{C}[Y]$, and include scalar multiples of G_2 and G_3. Thus one obtains a morphism $\varphi : Y \to \mathcal{S}'_{\mathcal{O},e+e_0}$, which (since all of the generators A,B,C,G_i appear somewhere in the coordinates describing φ) is evidently a closed immersion, hence an isomorphism by equality of dimensions and reducedness.

For a more careful analysis, we note that finding a basis $\{w_1',\dots,w_5'\}$ for $\mathbb{C}^5$ such that $Mw_i' = w_{i-1}'$ for $i \geq 2$ and $Mw_1' = 0$ is essentially equivalent to finding an element $w_5' \in \mathbb{C}^5$ such that $M^4 w_5' \neq 0$. Moreover, any transformation of the form $w_5' \mapsto w_5' + \alpha w_4' + \beta w_3' + \gamma w_2' + \delta w_1'$ preserves the elements w_1', $w_1' \wedge w_2'$, $w_1' \wedge w_2' \wedge w_3'$, $w_1' \wedge w_2' \wedge w_3' \wedge w_4'$. Thus, to find $z \cdot \tilde{M}_0$ where $\operatorname{Ad} z(M_0) = M$, it suffices to choose an element w_5' such that $M^4 w_5' \neq 0$, and then to multiply w_5' by an appropriate scalar such that $\det(w_1'\ w_2'\ w_3'\ w_4'\ w_5') = 1$. For this purpose, we first choose

$$w'_5 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \text{ ; then } w'_4 = \begin{pmatrix} \frac{1}{3}A \\ 0 \\ 2 \\ 0 \\ 0 \end{pmatrix},\ w'_3 = \begin{pmatrix} -\frac{2}{9}A^2 - \frac{1}{8}BC \\ \frac{2}{3}G_5 \\ \frac{4}{3}A \\ 0 \\ 4 \end{pmatrix},$$

$$w'_2 = \begin{pmatrix} \frac{4}{9}A^3 + \frac{7}{24}ABC + \frac{1}{3}G_0G_5 \\ CG_4 - \frac{1}{3}AG_5 \\ -\frac{2}{3}A^2 - \frac{1}{2}BC \\ 2G_5 \\ 4A \end{pmatrix}$$

and finally

$$\begin{aligned} w'_1 &= \begin{pmatrix} A^4 + \frac{23}{36}A^2BC + \frac{1}{32}B^2C^2 + AG_0G_5 + \frac{1}{2}BG_1G_5 - \frac{1}{3}CG_0G_4 \\ \frac{1}{2}ACG_4 + \frac{1}{3}BCG_5 \\ \frac{1}{6}ABC \\ CG_4 \\ -BC \end{pmatrix} \\ &= -C \begin{pmatrix} -\frac{1}{6}G_0G_4 + \frac{1}{9}A^2B + \frac{1}{32}B^2C \\ -\frac{1}{4}CG_3 - \frac{1}{12}BG_5 \\ -\frac{1}{6}AB \\ -G_4 \\ B \end{pmatrix} \end{aligned}$$

Then one can show that the determinant of the matrix $(w'_1\ w'_2\ w'_3\ w'_4\ w'_5)$ is $-C^5$. Thus we replace each of w'_i, $1 \le i \le 5$ by $w''_i = -w'_i/C$, which is well-defined whenever $C \neq 0$. In other words, whenever $C \neq 0$ we can construct a matrix $g_{\mathcal{B}} = (w''_1\ w''_2\ w''_3\ w''_4\ w''_5)$ of determinant 1 such that $g_{\mathcal{B}} M_0 g_{\mathcal{B}}^{-1} = M$.

It is now a routine computer calculation to verify that, relative to the obvious basis for $\Lambda^2(W)$, we have

$$w''_1 \wedge w''_2 = \begin{pmatrix} \frac{1}{18}G_0G_4^2 - \frac{5}{288}C^3G_0 + \frac{11}{288}AC^2G_1 - \frac{5}{144}A^2CG_2 + \frac{1}{18}A^3G_3 \\ \frac{1}{36}A^2B^2 + \frac{1}{64}B^3C - \frac{1}{18}AG_0G_3 - \frac{1}{12}BG_0G_4 \\ -\frac{2}{9}A^2G_3 - \frac{7}{24}ABG_4 - \frac{1}{16}B^2G_5 \\ \frac{1}{6}AB^2 + \frac{1}{3}G_0G_3 \\ -\frac{1}{12}C(AG_2 + 2BG_3) \\ \frac{2}{3}G_3G_5 - G_4^2 \\ AG_3 + BG_4 \\ \frac{1}{3}AG_3 + \frac{1}{2}BG_4 \\ -\frac{1}{2}B^2 \\ 2G_3 \end{pmatrix}$$

and similarly

$$w_1'' \wedge w_2'' \wedge w_3'' = \begin{pmatrix} -\frac{1}{18}A^3G_2 + \frac{1}{36}A^2CG_1 + \frac{5}{288}ABCG_2 - \frac{1}{96}BC^2G_1 - \frac{1}{18}G_0G_3G_4 \\ -\frac{1}{9}AG_3G_4 + \frac{1}{24}BG_3G_5 - \frac{1}{8}BG_4^2 \\ \frac{1}{3}A^2G_2 + \frac{1}{12}ABG_3 + \frac{1}{4}BCG_2 - \frac{1}{12}C^2G_0 \\ -\frac{1}{24}ABG_3 + \frac{1}{16}B^2G_4 - \frac{1}{9}A^2G_2 \\ -\frac{1}{8}B^3 + \frac{1}{3}G_0G_2 \\ \frac{2}{3}AG_2 + \frac{1}{2}BG_3 \\ -\frac{1}{3}G_3G_4 - \frac{1}{12}AC^2 \\ AG_2 + BG_3 \\ \frac{1}{2}C^2 \\ -2G_2 \end{pmatrix}.$$

Finally, it is straightforward to show using the identification of $\Lambda^4(W)$ with W^* that

$$w_1'' \wedge w_2'' \wedge w_3'' \wedge w_4'' = \begin{pmatrix} -C & G_1 & \frac{1}{6}AC & \frac{1}{3}CG_0 - \frac{1}{2}AG_1 & \frac{1}{9}A^2C + \frac{1}{32}BC^2 + \frac{1}{6}G_1G_5 \end{pmatrix}.$$

What these computations amount to is the following:

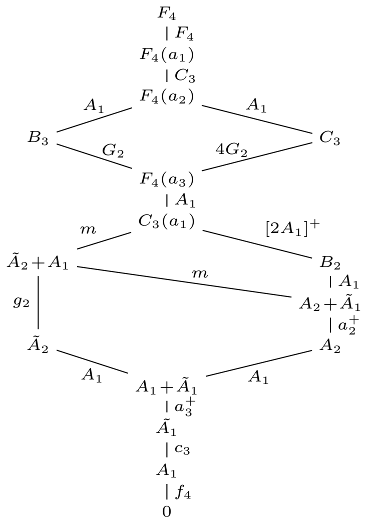
Theorem 12.12. *There is a morphism from the open subset of Y given by $C \neq 0$ to $\mathcal{O} \cap \mathcal{S}'_{e+e_0}$, given by letting the matrix $g_{\mathcal{B}}$ act on $\tilde{M}_0$. Moreover, this morphism extends to an isomorphism from Y to $\mathcal{S}'_{\mathcal{O},e+e_0}$.*

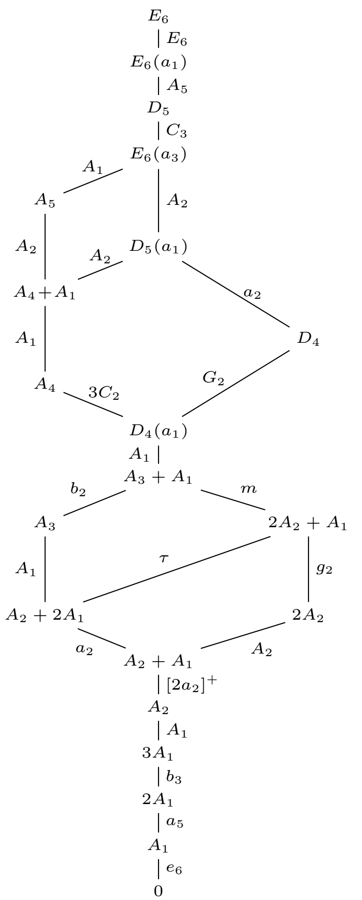
Proof. The first part follows from the above discussion, since $g_{\mathcal{B}}$ has coordinates in the localized ring $\mathbb{C}[Y]_C$. But we can see by our calculations that in fact, the coordinates of M and $w_1'', \ldots, w_1'' \wedge w_2'' \wedge w_3'' \wedge w_4''$ all lie in $\mathbb{C}[Y]$. Thus we can extend the morphism to a morphism φ from Y to $\mathcal{S}'_{\mathcal{O},e+e_0}$. Since each of the generators $A, B, C, G_0, \ldots, G_5$ appears (up to multiplication by a scalar) as a coordinate of the map φ, it follows that φ is a closed immersion, and hence by dimensions, irreducibility and reducedness, is an isomorphism onto $\mathcal{S}'_{\mathcal{O},e+e_0}$. □

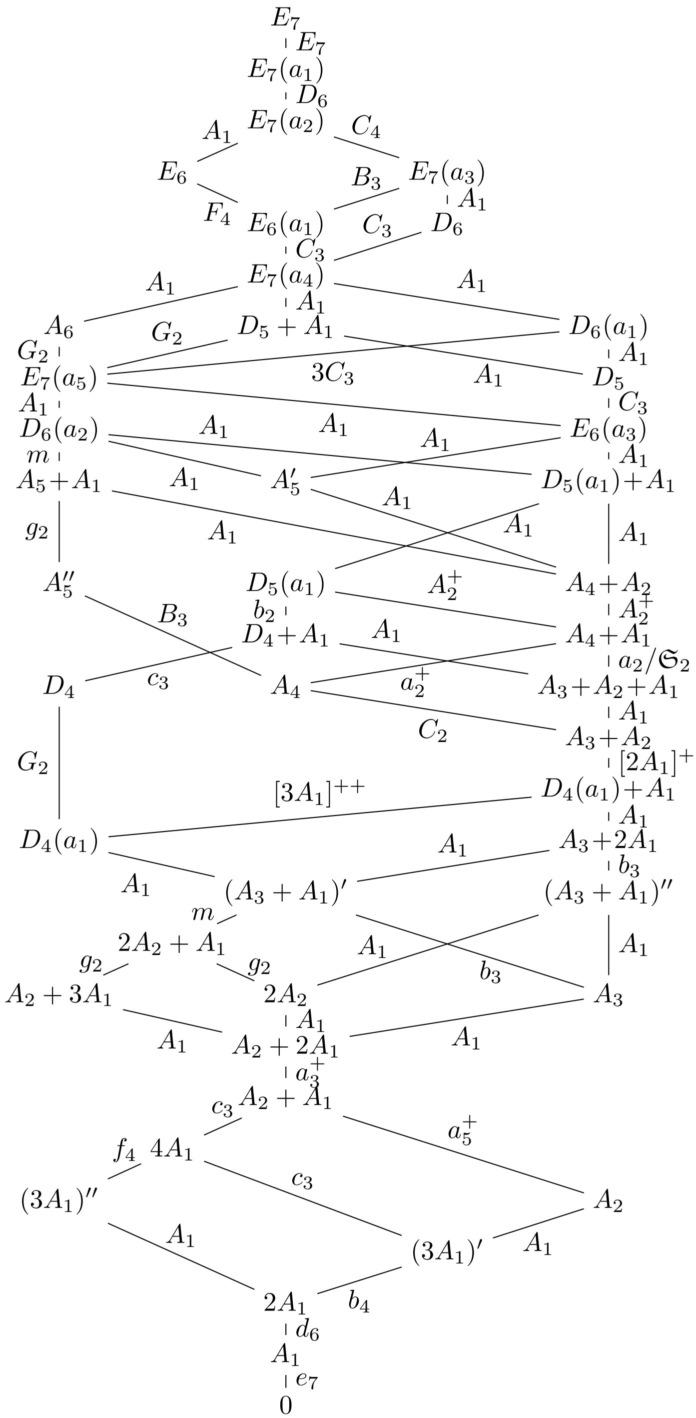
13. Graphs

Capital letters are used to denote simple singularities and lower-case letters to denote singularities of closures of minimal nilpotent orbits. The notation m, m', μ, χ, $a_2/\mathfrak{S}_2$ and τ are explained in §1.8.4. The intrinsic symmetry action induced from $A(e)$ is explained in §6 and the notation is explained in §6.2. We use (Y) to denote a singularity with normalization Y.

$$\begin{array}{c} G_2 \\ \mid G_2 \\ G_2(a_1) \\ \mid A_1 \\ \tilde{A}_1 \\ \mid m \\ A_1 \\ \mid g_2 \\ 0 \end{array}$$


F_4
F_4
$F_4(a_1)$
C_3
$F_4(a_2)$
A_1
A_1
B_3
C_3
G_2
$4G_2$
$F_4(a_3)$
A_1
$C_3(a_1)$
m
$[2A_1]^+$
$\tilde{A}_2 + A_1$
B_2
m
A_1
g_2
$A_2 + \tilde{A}_1$
a_2^+
$\tilde{A}_2$
A_2
A_1
A_1
$A_1 + \tilde{A}_1$
a_3^+
$\tilde{A}_1$
c_3
A_1
f_4
0


E_6
E_6
$E_6(a_1)$
A_5
D_5
C_3
$E_6(a_3)$
A_1
A_5
A_2
A_2
$D_5(a_1)$
A_2
$A_4 + A_1$
a_2
A_1
D_4
A_4
$3C_2$
G_2
$D_4(a_1)$
A_1
$A_3 + A_1$
b_2
m
A_3
$2A_2 + A_1$
τ
A_1
g_2
$A_2 + 2A_1$
$2A_2$
a_2
A_2
$A_2 + A_1$
$[2a_2]^+$
A_2
A_1
$3A_1$
b_3
$2A_1$
a_5
A_1
e_6
0



E_7
E_7
$E_7(a_1)$
D_6
$E_7(a_2)$
A_1
C_4
E_6
$E_7(a_3)$
B_3
A_1
F_4
$E_6(a_1)$
C_3
D_6
C_3
$E_7(a_4)$
A_1
A_1
A_1
A_6
D_5+A_1
$D_6(a_1)$
G_2
G_2
A_1
$E_7(a_5)$
$3C_3$
A_1
D_5
A_1
C_3
$D_6(a_2)$
A_1
A_1
$E_6(a_3)$
A_1
m
A_1
A_5+A_1
A_1
A'_5
$D_5(a_1)+A_1$
A_1
g_2
A_1
A_1
A_1
A''_5
$D_5(a_1)$
A_2^+
A_4+A_2
b_2
A_2^+
B_3
D_4+A_1
A_1
A_4+A_1
$a_2/\mathfrak{S}_2$
D_4
c_3
A_4
a_2^+
$A_3+A_2+A_1$
A_1
C_2
A_3+A_2
G_2
$[2A_1]^+$
$[3A_1]^{++}$
$D_4(a_1)+A_1$
A_1
$D_4(a_1)$
A_1
A_3+2A_1
b_3
A_1
$(A_3+A_1)'$
$(A_3+A_1)''$
m
A_1
$2A_2+A_1$
A_1
g_2
g_2
b_3
A_2+3A_1
$2A_2$
A_3
A_1
A_1
A_2+2A_1
A_1
a_3^+
c_3
A_2+A_1
a_5^+
f_4
$4A_1$
c_3
$(3A_1)''$
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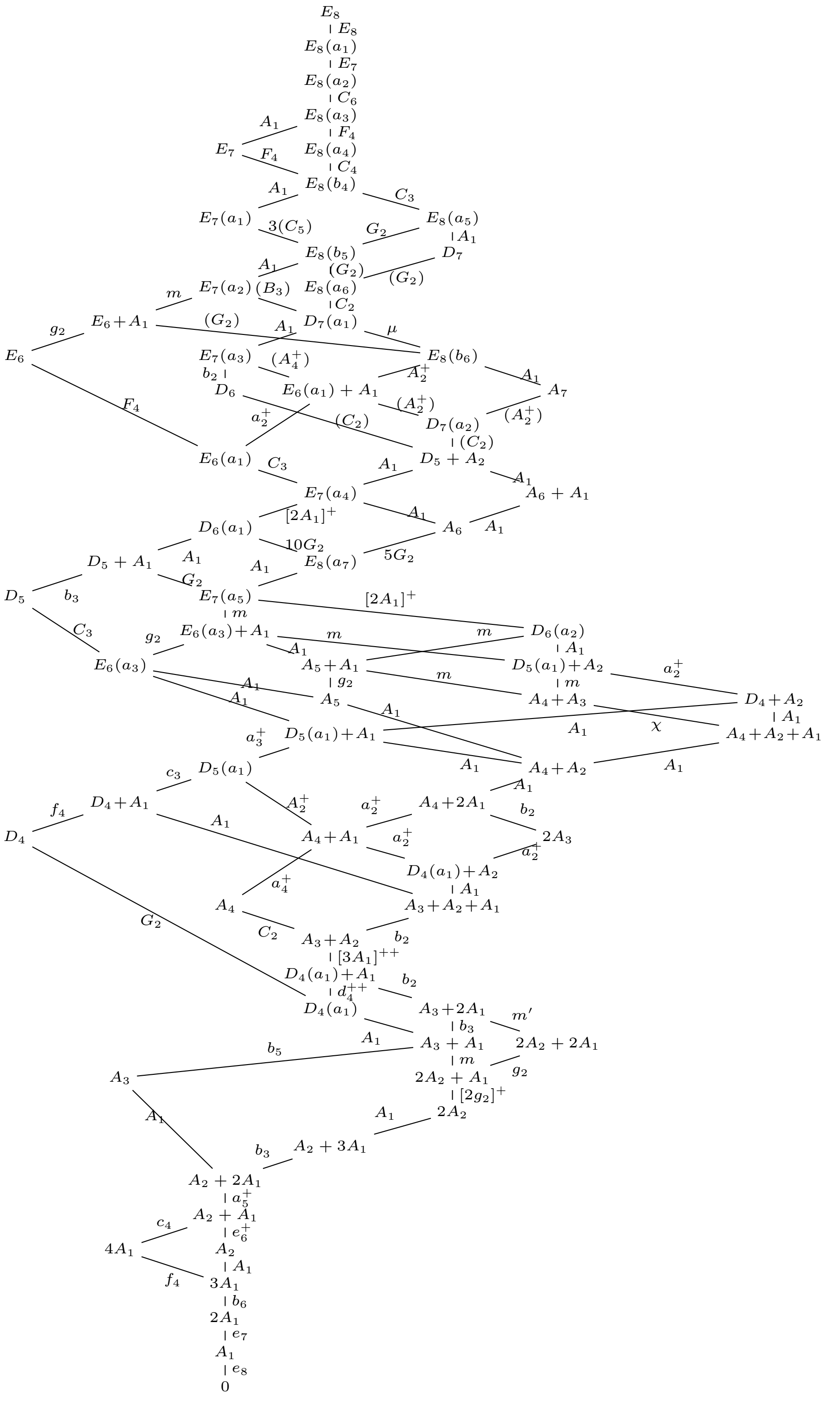
A_1
e_7
0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E_8
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E_7
$E_8(a_2)$
C_6
A_1
$E_8(a_3)$
F_4
E_7
F_4
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C_4
A_1
$E_8(b_4)$
C_3
$E_7(a_1)$
$3(C_5)$
G_2
$E_8(a_5)$
A_1
$E_8(b_5)$
D_7
A_1
(G_2)
m
$E_7(a_2)$
(B_3)
$E_8(a_6)$
(G_2)
C_2
g_2
E_6+A_1
(G_2)
$D_7(a_1)$
A_1
μ
E_6
$E_7(a_3)$
(A_4^+)
$E_8(b_6)$
b_2
A_2^+
A_1
D_6
$E_6(a_1)+A_1$
A_7
F_4
a_2^+
(C_2)
(A_2^+)
(A_2^+)
$D_7(a_2)$
(C_2)
$E_6(a_1)$
C_3
A_1
D_5+A_2
A_1
$E_7(a_4)$
A_6+A_1
$[2A_1]^+$
A_1
$D_6(a_1)$
A_6
A_1
$10G_2$
A_1
$E_8(a_7)$
$5G_2$
D_5+A_1
A_1
G_2
D_5
b_3
$E_7(a_5)$
$[2A_1]^+$
m
C_3
g_2
$E_6(a_3)+A_1$
m
m
$D_6(a_2)$
A_1
$E_6(a_3)$
A_1
A_5+A_1
$D_5(a_1)+A_2$
a_2^+
g_2
m
m
A_1
A_5
A_4+A_3
D_4+A_2
A_1
A_1
A_1
A_1
χ
a_3^+
$D_5(a_1)+A_1$
$A_4+A_2+A_1$
A_1
A_4+A_2
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c_3
$D_5(a_1)$
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D_4+A_1
A_2^+
a_2^+
A_4+2A_1
b_2
D_4
A_1
A_4+A_1
a_2^+
$2A_3$
a_2^+
$D_4(a_1)+A_2$
a_4^+
A_1
A_4
$A_3+A_2+A_1$
G_2
C_2
A_3+A_2
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$[3A_1]^{++}$
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b_2
d_4^{++}
$D_4(a_1)$
A_3+2A_1
m'
b_3
A_1
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m
A_3
g_2
$2A_2+A_1$
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A_1
$2A_2$
b_3
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A_2+2A_1
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A_1
f_4
$3A_1$
b_6
$2A_1$
e_7
A_1
e_8
0

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References

[1] P.N. Achar, D.S. Sage, Perverse coherent sheaves and the geometry of special pieces in the unipotent variety, Adv. Math. 220 (4) (2009) 1265–1296.

[2] A.V. Alekseevskiĭ, Component groups of centralizers of unipotent elements in semisimple algebraic groups, Akad. Nauk Gruzin. SSR Trudy Tbiliss. Mat. Inst. Razmadze 62 (1979) 5–27, collection of articles on algebra, 2.

[3] M. Andreatta, J.A. Wiśniewski, 4-dimensional symplectic contractions, Geom. Dedicata 168 (2014) 311–337.

[4] A. Beauville, Symplectic singularities, Invent. Math. 139 (3) (2000) 541–549.

[5] G. Bellamy, Counting resolutions of symplectic quotient singularities, preprint, arXiv:1405.6925, 2014.

[6] W.M. Beynon, N. Spaltenstein, Green functions of finite Chevalley groups of type E_n $(n = 6, 7, 8)$, J. Algebra 88 (2) (1984) 584–614.

[7] W. Borho, R. MacPherson, Représentations des groupes de Weyl et homologie d'intersection pour les variétés nilpotentes, C. R. Acad. Sci. Paris Sér. I Math. 292 (15) (1981) 707–710.

[8] W. Borho, R. MacPherson, Partial resolutions of nilpotent varieties, in: Analysis and topology on singular spaces, II, III, Luminy, 1981, in: Astérisque, vol. 101, Soc. Math. France, Paris, 1983, pp. 23–74.

[9] E. Brieskorn, Singular elements of semi-simple algebraic groups, in: Actes du Congrès International des Mathématiciens, Tome 2, Nice, 1970, Gauthier-Villars, Paris, 1971, pp. 279–284.

[10] A. Broer, Decomposition varieties in semisimple Lie algebras, Canad. J. Math. 50 (5) (1998) 929–971.

[11] A. Broer, Normal nilpotent varieties in F_4, J. Algebra 207 (2) (1998) 427–448.

[12] R.W. Carter, Finite Groups of Lie Type, Wiley Classics Library, John Wiley & Sons Ltd., Chichester, 1993, Conjugacy Classes and Complex Characters, reprint of the 1985 original, a Wiley-Interscience Publication.

[13] F. Catanese, Automorphisms of rational double points and moduli spaces of surfaces of general type, Compos. Math. 61 (1) (1987) 81–102.

[14] D.H. Collingwood, W.M. McGovern, Nilpotent Orbits in Semisimple Lie Algebras, Van Nostrand Reinhold Co., New York, 1993.

[15] W.A. de Graaf, Computing with nilpotent orbits in simple Lie algebras of exceptional type, LMS J. Comput. Math. 11 (2008) 280–297.

[16] E.B. Dynkin, Semisimple subalgebras of semisimple Lie algebras, Mat. Sb. 30 (72) (1952) 349–462 (3 plates).

[17] H. Flenner, M. Zaidenberg, Rational curves and rational singularities, Math. Z. 244 (3) (2003) 549–575.

[18] B. Fu, Symplectic resolutions for nilpotent orbits, Invent. Math. 151 (1) (2003) 167–186.

[19] B. Fu, On $\mathbb{Q}$-factorial terminalizations of nilpotent orbits, J. Math. Pures Appl. (9) 93 (6) (2010) 623–635.

[20] B. Fu, D. Juteau, P. Levy, E. Sommers, Duality for generic singularities of special slices in simple lie algebras, in preparation.
[21] B. Fu, D. Juteau, P. Levy, E. Sommers, Geometry of special pieces in nilpotent orbits, in preparation.
[22] W.L. Gan, V. Ginzburg, Quantization of Slodowy slices, Int. Math. Res. Not. IMRN 2002 (5) (2002) 243–255.
[23] W.A. Graham, Functions on the universal cover of the principal nilpotent orbit, Invent. Math. 108 (1) (1992) 15–27.
[24] A. Henderson, Singularities of nilpotent orbit closures, Rev. Roum. Math. Pures Appl. 60 (4) (2015) 441–469.
[25] W. Hesselink, Singularities in the nilpotent scheme of a classical group, Trans. Amer. Math. Soc. 222 (1976) 1–32.
[26] V. Hinich, On the singularities of nilpotent orbits, Israel J. Math. 73 (3) (1991) 297–308.
[27] G.D. James, On the decomposition matrices of the symmetric groups III, J. Algebra 71 (1981) 115–122.
[28] A. Joseph, On the variety of a highest weight module, J. Algebra 88 (1) (1984) 238–278.
[29] D. Juteau, Modular Springer correspondence and decomposition numbers, PhD thesis, Université Paris 7 – Denis-Diderot, 2007.
[30] D. Kaledin, Symplectic singularities from the Poisson point of view, J. Reine Angew. Math. 600 (2006) 135–156.
[31] H. Kraft, Closures of conjugacy classes in G_2, J. Algebra 126 (2) (1989) 454–465.
[32] H. Kraft, C. Procesi, Minimal singularities in GL_n, Invent. Math. 62 (3) (1981) 503–515.
[33] H. Kraft, C. Procesi, On the geometry of conjugacy classes in classical groups, Comment. Math. Helv. 57 (4) (1982) 539–602.
[34] H. Kraft, C. Procesi, A special decomposition of the nilpotent cone of a classical lie algebra, Astérisque 173–174 (1989) 271–279.
[35] R. Lawther, D.M. Testerman, Centres of centralizers of unipotent elements in simple algebraic groups, Mem. Amer. Math. Soc. 210 (988) (2011) 188.
[36] C. LeBrun, Fano manifolds, contact structures, and quaternionic geometry, Internat. J. Math. 6 (3) (1995) 419–437.
[37] T. Levasseur, S.P. Smith, Primitive ideals and nilpotent orbits in type G_2, J. Algebra 114 (1) (1988) 81–105.
[38] I. Losev, Finite W-algebras, in: Proceedings of the International Congress of Mathematicians, vol. III, Hindustan Book Agency, New Delhi, 2010, pp. 1281–1307.
[39] G. Lusztig, Green polynomials and singularities of unipotent classes, Adv. Math. 42 (2) (1981) 169–178.
[40] G. Lusztig, Green functions and character sheaves, Ann. of Math. (2) 131 (2) (1990) 355–408.
[41] G. Lusztig, Notes on unipotent classes, Asian J. Math. 1 (1) (1997) 194–207.
[42] G. Lusztig, N. Spaltenstein, Induced unipotent classes, J. Lond. Math. Soc. (2) 19 (1) (1979) 41–52.
[43] A. Maffei, Quiver varieties of type A, Comment. Math. Helv. 80 (1) (2005) 1–27.
[44] K. Matsuki, Introduction to the Mori Program, Universitext, Springer-Verlag, New York, 2002.
[45] Y. Namikawa, Extension of 2-forms and symplectic varieties, J. Reine Angew. Math. 539 (2) (2001) 123–137.
[46] Y. Namikawa, Birational geometry of symplectic resolutions of nilpotent orbits 2 (version 1), arXiv:math/0408274.
[47] A.L. Onishchik, È.B. Vinberg, Lie Groups and Algebraic Groups, Springer Series in Soviet Mathematics, Springer-Verlag, Berlin, 1990, translated from the Russian and with a preface by D.A. Leites.
[48] D.I. Panyushev, Rationality of singularities and the Gorenstein property of nilpotent orbits, Funktsional. Anal. i Prilozhen. 25 (3) (1991) 76–78.
[49] T. Shoji, On the Springer representations of Chevalley groups of type F_4, Comm. Algebra 8 (5) (1980) 409–440.
[50] P. Slodowy, Simple Singularities and Simple Algebraic Groups, Lecture Notes in Mathematics, vol. 815, Springer, Berlin, 1980.
[51] E. Sommers, A generalization of the Bala–Carter theorem for nilpotent orbits, Int. Math. Res. Not. IMRN 1998 (11) (1998) 539–562.
[52] E. Sommers, Normality of nilpotent varieties in E_6, J. Algebra 270 (1) (2003) 288–306.
[53] E. Sommers, Normality of very even nilpotent varieties in D_{2l}, Bull. Lond. Math. Soc. 37 (3) (2005) 351–360.
[54] N. Spaltenstein, Classes unipotentes et sous-groupes de Borel, Lecture Notes in Mathematics, vol. 946, Springer-Verlag, Berlin, 1982.

[55] T.A. Springer, Trigonometric sums, Green functions of finite groups and representations of Weyl groups, Invent. Math. 36 (1976) 173–207.
[56] The GAP Group, GAP – Groups, Algorithms, and Programming, version 4.7.8, 2015.
[57] È.B. Vinberg, V.L. Popov, A certain class of quasihomogeneous affine varieties, Izv. Akad. Nauk Ser. Mat. 36 (1972) 749–764.
[58] J. Wierzba, Contractions of symplectic varieties, J. Algebraic Geom. 12 (3) (2003) 507–534.