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The generic rigidity of triangulated spheres with blocks and holes

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Abstract

A simple graph $G = (V, E)$ is 3-rigid if its generic bar-joint frameworks in $\mathbb{R}^3$ are infinitesimally rigid. Block and hole graphs are derived from triangulated spheres by the removal of edges and the addition of minimally rigid subgraphs, known as blocks, in some of the resulting holes. Combinatorial characterisations of minimal 3-rigidity are obtained for these graphs in the case of a single block and finitely many holes or a single hole and finitely many blocks. These results confirm a conjecture of Whiteley from 1988 and special cases of a stronger conjecture of Finbow-Singh and Whiteley from 2013.

Keywords: infinitesimal rigidity, vertex splitting, bar-joint framework, combinatorial rigidity
2010 MSC: 52C25, 05C75

1. Introduction

A classical result of Cauchy [1] asserts that a convex polyhedron in three-dimensional Euclidean space is continuously rigid, when viewed as a bar-joint framework, if and only if the faces are triangles. Dehn [2] subsequently showed that this is also equivalent to the stronger condition of infinitesimal rigidity. If the joints of such a framework are perturbed to generic positions, with the bar lengths correspondingly adjusted, then infinitesimal rigidity may be established more directly by vertex splitting. In this case convexity is not necessary and it follows that the graphs of triangulated spheres are 3-*rigid* in the sense that their generic placements in $\mathbb{R}^3$ provide infinitesimally rigid bar-joint frameworks. This is a theorem of Gluck [5] and in fact these graphs are minimally 3-rigid (isostatic) in view of their flexibility on the removal of any edge. The vertex splitting method was introduced into geometric rigidity theory by Whiteley [9] and it plays a key role in our arguments.

While the general problem of characterising the rigidity or minimal rigidity of generic three-dimensional bar-joint frameworks remains open, an interesting class of graphs which are derived from convex polyhedra has been considered in this regard by Whiteley [8], Finbow-Singh, Ross

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and Whiteley [4] and Finbow-Singh and Whiteley [3]. These graphs arise from surgery on a triangulated sphere involving the excision of the disjoint interiors of some triangulated discs and the insertion of minimally rigid blocks into some of the resulting holes. Even in the case of a single block and a single hole of the same perimeter length $n \geq 4$ the resulting block and hole graph need not be 3-rigid. A necessary and sufficient condition, obtained in [3], for the minimal rigidity of such an n-tower case, with disjoint block and hole boundaries, is that there exist n vertex disjoint paths connecting the vertices of the boundaries.

1.1. The main result

In what follows we introduce some new methods which provide, in particular, characterisations of minimal 3-rigidity for the class of block and hole graphs with a single block and finitely many holes. Such graphs may be viewed as the structure graphs of triangulated domes with windows, where the role of terra firma is played by the single block. In fact, the *girth inequalities*, defined in Sect. 4, provide a computable necessary and sufficient condition for 3-rigidity in terms of lower bounds on the lengths of cycles of edges around sets of windows.

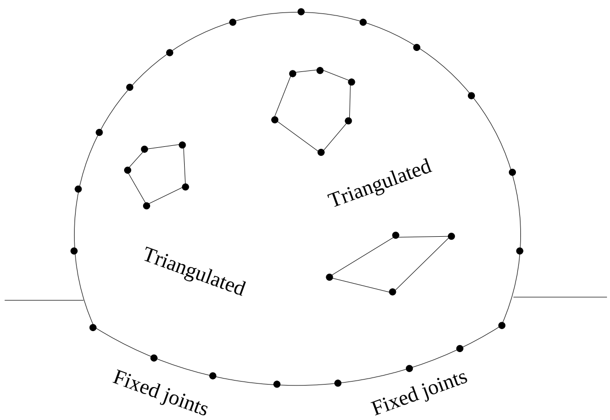

Figure 1: A triangulated dome with windows.

The main result is as follows.

Theorem 1. *Let $\hat{G}$ be a block and hole graph with a single block and finitely many holes, or, single hole and finitely many blocks. Then the following statements are equivalent.*

(i) $\hat{G}$ *is minimally* 3*-rigid.*

(ii) $\hat{G}$ *is* $(3,6)$*-tight.*

(iii) $\hat{G}$ *is constructible from* K_3 *by the moves of vertex splitting and isostatic block substitution.*

(iv) $\hat{G}$ *satisfies the girth inequalities.*

Condition (*ii*) is a well known necessary condition for minimally 3-rigid graphs which requires the Maxwell count $|E| = 3|V| - 6$ together with corresponding sparsity inequalities for subgraphs (see Sect. 2). The construction scheme in (*iii*) involves three phases of reduction for a $(3, 6)$-tight block and hole graph, namely,

1. discrete homotopy reduction by $(3, 6)$-tight preserving edge contractions,
2. graph division over critical separating cycles of edges, and,
3. admissible block-hole boundary contractions.

The girth inequalities in (*iv*) are a reformulation of the *cut cycle inequalities* of [3]. For the single block case, the equivalence of conditions (*i*) − (*iii*) is established in Sect. 3 and the equivalence with condition (*iv*) is established in Sect. 4. The same equivalences are then obtained for the "dual" class of block and hole graphs with a single hole in Corollary 48. In fact, the dual of *any* generically isostatic block and hole graph is generically isostatic (see [4]).

Theorem 1 confirms the single hole case and the single block case of Conjecture 5.1 in [3] (see also Remark 13 below). Example 50 shows that the conjecture is not true in general. A further corollary of Theorem 1 is that the following conjectures, paraphrased from [8], are true.

Conjecture 2 ([8, Conjectures 4.2 and 4.3])**.** *Let $\hat{G}$ be a block and hole graph with one pentagonal block and two quadrilateral holes, or, two quadrilateral blocks and one pentagonal hole. If $\hat{G}$ is* 5*-connected then it is minimally* 3*-rigid.*

The Appendix provides a proof of the preservation of minimal 3-rigidity under vertex splitting (established in [9]) and a simple proof of Gluck's theorem ([5]) on the 3-rigidity of graphs of triangulated spheres.

2. Block and hole graphs

A cycle of edges in a simple graph is a sequence $e_1, e_2, \ldots, e_r$, with $r \geq 3$, for which there exist distinct vertices $v_1, v_2, \ldots, v_r$, such that $e_i = v_i v_{i+1}$ for $i < r$ and $e_r = v_r v_1$.

2.1. Face graphs

Let $S = (V, E)$ be the graph of a triangulated sphere, that is, S is a planar simple 3-connected graph such that each face of S is bounded by a 3-cycle. Let c be a cycle in S of length four or more. Then c determines two complementary planar subgraphs of S, each with a single non-triangular face bordered by the edges of c. Such a subgraph is referred to as a *simplicial disc* of S with boundary cycle c. The boundary cycle of a simplicial disc D is also denoted by ∂D. The *edge interior* of D is the set of edges in D that do not belong to ∂D. A collection of simplicial discs is *internally-disjoint* if their respective edge interiors are pairwise disjoint.

Definition 3. *A* face graph*, G, is obtained from the graph of a triangulated sphere, S, by,*

1. *choosing a collection of internally disjoint simplicial discs in S,*

2. *removing the edge interiors of each of these simplicial discs,*

3. *labelling the non-triangular faces of the resulting planar graph by either B or H.*

A labelling of the triangular faces of G by the letter T would be redundant but nevertheless an edge of G is said to be of type BB, BH, HH, BT, HT or TT according to the labelling of its adjacent faces. A face graph is of type (m, n) if the number of B-labelled faces is m and the number of H-labelled faces is n.

Example 4. *The complete graph K_4 is the graph of a triangulated sphere and may be expressed as the union of two simplicial discs with a common* 4*-cycle boundary. The edge interiors of these simplicial discs each contain a single edge. Remove these edge interiors to obtain a* 4*-cycle and label the two resulting faces by B and H. This is the smallest example of a face graph of type* $(1, 1)$.

If $\mathcal{B}$ and $\mathcal{H}$ are collections of internally-disjoint simplicial discs of S then the notation $G = S(\mathcal{B}, \mathcal{H})$ indicates that the B-labelled faces of the face graph G correspond to the simplicial discs in $\mathcal{B}$ and the H-labelled faces of G correspond to the simplicial discs in $\mathcal{H}$.

2.2. Block and hole graphs

Let $G = S(\mathcal{B}, \mathcal{H})$ be a face graph derived from S and let $\mathcal{B} = \{B_1, B_2, \ldots, B_m\}$ be the simplicial discs in S which determine the B-labelled faces of G.

Definition 5. *A* block and hole graph *on $G = S(\mathcal{B}, \mathcal{H})$ is a graph $\hat{G}$ of the form $\hat{G} = G \cup \hat{B}_1 \cup \cdots \cup \hat{B}_m$ where,*

1. *$\hat{B}_1, \hat{B}_2, \ldots, \hat{B}_m$ are minimally* 3*-rigid graphs which are either pairwise disjoint, or, intersect at vertices and edges of G,*

2. *$G \cap \hat{B}_i = \partial B_i$ for each $i = 1, 2, \ldots, m$.*

As in [3, 4], we refer to the subgraphs $\hat{B}_i$ as the *blocks* or *isostatic blocks* of $\hat{G}$. The following isostatic block substitution principle asserts that one may substitute isostatic blocks without altering the rigidity properties of $\hat{G}$. The proof is an application of [7, Corollary 2.8].

Lemma 6. *Let $G = S(\mathcal{B}, \mathcal{H})$ be a face graph and suppose there exists a block and hole graph on G which is simple and minimally* 3*-rigid. Then every simple block and hole graph on G is minimally* 3*-rigid.*

The graph of a triangulated sphere is minimally 3-rigid ([5]) and so such graphs provide a natural choice for the isostatic blocks in a block and hole graph.

Example 7. *Let $G = S(\mathcal{B}, \mathcal{H})$ be a face graph and for each $B_i \in \mathcal{B}$ construct an isostatic block $B_i^\dagger$ with,*

$$V(B_i^\dagger) = V(\partial B_i) \cup \{x_i, y_i\}, \quad E(B_i^\dagger) = E(\partial B_i) \cup \{(v, x_i), (v, y_i) : v \in V(\partial B_i)\}$$

The graph $B_i^\dagger$ is referred to as a simplicial discus *with* poles *at x_i and y_i. The resulting block and hole graph $G \cup B_1^\dagger \cup \cdots \cup B_m^\dagger$, denoted by $G^\dagger$, is referred to as the* discus and hole graph *for G. Note that $G^\dagger$ is a simple graph which is uniquely determined by G. The discus and hole graphs will be used in Sect. 3 to establish a construction scheme for $(3,6)$-tight block and hole graphs with a single block.*

In general, a block and hole graph may not be simple. This can occur if two B-labelled faces of G share a pair of non-adjacent vertices.

Example 8. *Let $G = S(\mathcal{B}, \mathcal{H})$ be a face graph and for each $B_i \in \mathcal{B}$ construct an isostatic block B_i° as follows: Define B_i° to be the graph of a triangulated sphere which is obtained from the boundary cycle ∂B_i by adjoining $2(|\partial B_i| - 3)$ edges so that B_i° is the union of two internally-disjoint simplicial discs with common boundary cycle ∂B_i. The resulting block and hole graph $G \cup B_1^\circ \cup \cdots \cup B_m^\circ$ will be denoted G°. Note that G° is not uniquely determined and may not be simple. However, G° has the convenient property that its vertex set is that of G. This construction will be applied in Sect. 4 to characterise isostatic block and hole graphs in terms of girth inequalities.*

There is a simple relationship between a face graph G and its associated block and hole graphs. It is convenient therefore to focus the reduction analysis at the level of face graphs. This perspective also underlines a duality principle of the theory under B, H transposition, a feature exposed in [4] and discussed in Sect. 4.4.

2.3. Freedom numbers

Let $f(J)$ denote the *freedom number* $3|V(J)| - |E(J)|$ of a graph J. A simple graph J *satisfies the Maxwell count* if $f(J) = 6$.

Lemma 9. *Let G, K and K' be graphs with the following properties,*

(i) K and K' both satisfy the Maxwell count, and,

(ii) $G \cap K = G \cap K'$.

If $G \cup K$ satisfies the Maxwell count then $G \cup K'$ satisfies the Maxwell count.

Proof. The result follows on considering the freedom numbers,

$$f(G \cup K') = f(G) + f(K') - f(G \cap K') = f(G) + f(K) - f(G \cap K) = f(G \cup K) = 6.$$

□

A simple graph G is said to be $(3,6)$-*sparse* if $f(J) \geq 6$ for any subgraph J containing at least two edges. The graph G is $(3,6)$-*tight* if it is $(3,6)$-sparse and satisfies the Maxwell count.

Lemma 10. *Let G, K and K' be simple graphs with the following properties,*

(i) K and K' are both $(3,6)$-tight,

(ii) $G \cap K = G \cap K'$,

(iii) if $v, w \in V(G \cap K')$ and $vw \in E(K')$ then $vw \in E(G)$.

If $G \cup K$ is $(3,6)$-sparse (respectively, $(3,6)$-tight) then $G \cup K'$ is $(3,6)$-sparse (respectively, $(3,6)$-tight).

Proof. Suppose that $G \cup K$ is $(3,6)$-sparse and let J be a subgraph of $G \cup K'$ which contains at least two edges. It is sufficient to consider the case where J is connected. If J is a subgraph of G then $f(J) \geq 6$ since $G \cup K$ is $(3,6)$-sparse. If J is not a subgraph of G then there are two possible cases.

Case 1) Suppose that $J \cap K'$ contains exactly one edge vw and that this edge is not in G. Then, by condition (iii), either $v \notin V(G)$ or $w \notin V(G)$. It follows that,

$$f(J) = f(J \cap G) + (f(J \cap K') - f(J \cap (G \cap K'))) \geq 5 + 2 = 7.$$

Case 2) Suppose that $J \cap K'$ contains two or more edges. Since K satisfies the Maxwell count, $f(J \cap K') \geq 6 = f(K)$ and, since $G \cup K$ is $(3,6)$-sparse,

$$\begin{aligned} f(J) &= f(J \cap G) + f(J \cap K') - f(J \cap (G \cap K')) \\ &\geq f(J \cap G) + f(K) - f(J \cap (G \cap K)) \\ &= f((J \cap G) \cup K) \geq 6. \end{aligned}$$

In each case, $f(J) \geq 6$ and so $G \cup K'$ is $(3,6)$-sparse. If $G \cup K$ is $(3,6)$-tight then by the above argument, and Lemma 9, $G \cup K'$ is also $(3,6)$-tight. □

It is well-known that minimally 3-rigid graphs, and hence the isostatic blocks of a block and hole graph, are necessarily $(3,6)$-tight (see for example [6]). The following corollary refers to the discus and hole graph described in Example 7.

Corollary 11. *Let $G = S(\mathcal{B}, \mathcal{H})$ be a face graph of type (m, n).*

(i) Suppose there exists a block and hole graph on G which satisfies the Maxwell count. Then every block and hole graph on G satisfies the Maxwell count.

(ii) Suppose there exists a block and hole graph on G which is simple and $(3,6)$-sparse (respectively, simple and $(3,6)$-tight). Then the discus and hole graph $G^\dagger$ is $(3,6)$-sparse (respectively, $(3,6)$-tight).

Proof. The statements follow by applying Lemmas 9 and 10 respectively with K and K' representing two different choices of isostatic block for a given B-labelled face of G. Note that in the case of (ii), if $B_i \in \mathcal{B}$ then there are no edges vw of the simplicial discus $B_i^\dagger$ with $v, w \in \partial B_i$ other than the edges of the boundary cycle ∂B_i. Thus condition (iii) of Lemma 10 is satisfied. □

2.4. 3-connectedness

Recall that a graph is 3-connected if there exists no pair of vertices $\{x, y\}$ with the property that there are two other vertices which cannot be connected by an edge path avoiding x and y. Such a pair is referred to here as a separation pair. The block and hole graphs $\hat{G}$ which are derived from face graphs G need not be 3-connected. However, it is shown below that in the single block case 3-connectedness is a consequence of $(3, 6)$-tightness.

Lemma 12. *Every* $(3, 6)$*-tight block and hole graph with a single block is* 3*-connected.*

Proof. Let $\hat{G}$ be a $(3, 6)$-tight block and hole graph with a single block and suppose that $\hat{G}$ is not 3-connected. Then there exists a separation pair $\{x, y\}$ with associated connected components $K_1, K_2, \ldots, K_r$. That is each K_j is a maximal connected subgraph in which every pair of vertices may be connected by a path of edges whose internal vertices do not include x or y.

Let K_1 be the component which contains an edge of $\hat{B}_1$ and hence all of $\hat{B}_1$. The graph K_1 and its complementary graph K_1' with $E(K') = E(\hat{G})\backslash E(K)$ each have more than one edge and their intersection is $\{x, y\}$. Thus $f(K_1 \cap K_1') = 6$ and

$$f(K_1) + f(K_1') = f(\hat{G}) + f(K_1 \cap K_1') = 12$$

It follows that the $(3, 6)$-sparse graphs K_1 and K_1' are both $(3, 6)$-tight. In particular, K_1' must be the graph of a triangulated sphere and it follows that K_1' contains the edge xy. Now $K_1 \cup \{xy\}$ is a subgraph of $\hat{G}$ which fails the $(3, 6)$-sparsity count, which is a contradiction. □

Remark 13. *The definition of a block and hole graph $\hat{G}$ is somewhat more liberal than the block and hole graphs $\hat{\mathcal{P}}$ of Finbow-Singh and Whiteley [3]. A graph $\hat{\mathcal{P}}$ is defined by considering a planar* 3*-connected graph $\mathcal{P}$ whose faces are labelled with the letters B, H and D. The B-labelled faces are replaced with isostatic block graphs and the D-labelled faces are triangulated. The resulting graph $\hat{\mathcal{P}}$ is called a* base polyhedron *reflecting the fact that it is the starting point for an "expanded" graph $\hat{\mathcal{P}}^E$. This is obtained by a further triangulation process involving adding vertices on edges of DD type, and vertices interior to triangles. In particular $\hat{\mathcal{P}}$ and $\hat{\mathcal{P}}^E$ are also* 3*-connected.*

3. Edge contraction and critical cycle division

For m, n nonnegative integers let $\mathcal{G}(m, n)$ be the set of all face graphs of type (m, n) for which the discus and hole graph $G^\dagger$ is $(3, 6)$-tight. In particular, the graphs of $\mathcal{G}(0, 0)$ are triangulations of a triangle and the sets $\mathcal{G}(0, n)$ and $\mathcal{G}(m, 0)$ are empty for $n, m \geq 1$.

3.1. TT edge contractions

The first reduction move for block and hole graphs is based on an edge contraction move for face graphs. A TT edge in a face graph G is said to be *contractible* if it belongs to two triangular faces and to no other 3-cycle of G. In this case the deletion of the edge and the identification of its vertices determines a graph move $G \to G'$ on the class of face graphs, called a *TT edge contraction*, which preserves the boundary cycles of the labelled faces of G.

Definition 14. *A* terminal face graph *G in $\mathcal{G}(m,n)$ is one for which there exist no TT edge contractions $G \to G'$ with $G' \in \mathcal{G}(m,n)$.*

Example 15. *A cycle graph with length at least* 4*, with exterior face labelled B and interior face labelled H is evidently a terminal graph in $\mathcal{G}(1,1)$.*

Example 16. *Fig. 2 shows a face graph G with a contractible TT edge which is nevertheless a terminal face graph of $\mathcal{G}(1,5)$. The discus and hole graph for the contracted graph G' fails to be $(3,6)$-tight since there is an extra edge added to the simplicial discus $B^\dagger$. Each block and hole graph $\hat{G}$ is evidently reducible by inverse Henneberg moves to a single block (i.e. by successively removing degree* 3 *vertices, see for example [6]). However, there is a systematic method of reduction described below in which each move is a form of edge contraction or cycle division.*

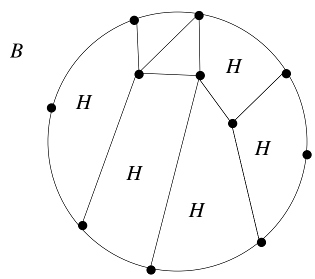

Figure 2: A terminal face graph in $\mathcal{G}(1,5)$.

Example 17. *The* 6*-vertex graph of Fig. 3 is a terminal face graph in $\mathcal{G}(2,2)$ whose block and hole graphs (variants of the double banana graph) are not* 3*-rigid. The graph G° (see Ex. 8), which in this case is unique, is not a simple graph.*

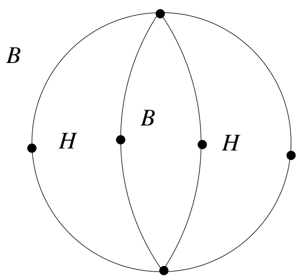

Figure 3: A terminal face graph in $\mathcal{G}(2,2)$.

Remark 18. *The contraction of a TT edge in a graph which is both $(3,6)$-tight and* 3*-connected may remove either one of these properties while maintaining the other. However, for a block and hole graph with a single block the situation is more straightforward since, by Lemma 12,* 3*-connectedness is a consequence of $(3,6)$-tightness. In particular, if G is a terminal face graph in $\mathcal{G}(1,n)$, for some $n \geq 1$, then the discus and hole graph $G^\dagger$ is both $(3,6)$-tight and* 3*-connected.*

3.2. Critical separating cycles

Let c be a cycle of edges in a face graph G and fix a planar realisation of G. Then c determines two new face graphs G_1 and G_2 which consist of the edges of c together with the edges and labelled faces of G which lie outside (resp. inside) c. If c is not a 3-cycle then the unlabelled face in G_1 (and in G_2) which is bounded by c is assigned the label H. The discus and hole graph for G_1 (resp. G_2) will be denoted $Ext(c)$ (resp. $Int(c)$). Note that $G^\dagger = Ext(c) \cup Int(c)$ and $Ext(c) \cap Int(c) = c$.

Definition 19. *A* critical separating cycle *for a face graph G is a cycle c with the property that either $Ext(c)$ or $Int(c)$ is $(3,6)$-tight.*

The boundary of a B-labelled face is always a critical separating cycle. Moreover, if $G^\dagger$ is $(3,6)$-tight then the boundary of every face of G is a critical separating cycle.

Lemma 20. *Let G be a face graph in $\mathcal{G}(m,n)$. If c is a 3-cycle in G then c is a critical separating cycle for G and both $Ext(c)$ and $Int(c)$ are $(3,6)$-tight.*

Proof. Since $G^\dagger$ is $(3,6)$-sparse, both $Ext(c)$ and $Int(c)$ are $(3,6)$-sparse. Note that $f(G^\dagger) = f(c) = 6$, $f(Ext(c)) \geq 6$ and $f(Int(c)) \geq 6$. Thus applying the formula,

$$f(G^\dagger) = f(Ext(c)) + f(Int(c)) - f(c),$$

it follows that both $Ext(c)$ and $Int(c)$ are $(3,6)$-tight. □

For face graphs of type $(1,n)$ a planar depiction may be chosen for which the unbounded face is B-labelled. Thus for any cycle c, it may be assumed that $Ext(c)$ contains the isostatic block and $Int(c)$ is a subgraph of a triangulated sphere.

Lemma 21. *Let G be a face graph in $\mathcal{G}(1,n)$. Then a cycle c is a critical separating cycle for G if and only if $Ext(c)$ is $(3,6)$-tight.*

Proof. If c is a 3-cycle then apply Lemma 20. If c is not a 3-cycle then $Int(c)$ is a subgraph of a triangulated sphere with $f(Int(c)) \geq 6 + (|c| - 3) > 6$. □

Proposition 22. *Let G be a face graph of type $(1,n)$ and suppose that there are no TT or BH edges in G.*

(i) If $G^\dagger$ satisfies the Maxwell count then G contains a cycle π, which is not the boundary of a face, such that $Ext(\pi)$ satisfies the Maxwell count.

(ii) If $G \in \mathcal{G}(1,n)$ then G contains a critical separating cycle for G which is not the boundary of a face.

Proof. Since G contains no edges of type TT or BH, every edge in the boundary cycle ∂B is of type BT (see Fig. 4) and so each vertex v in ∂B must be contained in an H-labelled face H_v. If each vertex v in ∂B is contained in a distinct H-labelled face H_v then let $r = |\partial B|$ and let $v_1, \ldots, v_r$ be the vertices of ∂B. Let $H_1, \ldots, H_n$ be the H-labelled faces of G, indexed so that $H_i = H_{v_i}$ for each $i = 1, 2, \ldots, r$. Note that $r \leq n$. Since the block and hole graphs G° satisfy the Maxwell count it follows that,

$$r - 3 = |\partial B| - 3 = \sum_{i=1}^{n}(|\partial H_i| - 3) \geq \sum_{i=1}^{r}(|\partial H_i| - 3) \geq r.$$

This is a contradiction and so $H_v = H_w$ for some distinct vertices $v, w \in \partial B$. The boundary of this common H-labelled face is composed of two edge-disjoint paths c_1 and c_2 joining v to w. The boundary cycle ∂B is also composed of two edge-disjoint paths joining v to w. Let π_1 be the path in Fig. 4 which moves anti-clockwise along ∂B from v to w and then along c_1 from w to v. Similarly, let π_2 be the path which moves clockwise along ∂B from v to w and then along c_2 from w to v. Note that π_1 and π_2 are cycles in G with $Ext(\pi_1) \cap Ext(\pi_2) = B^\dagger$. Thus,

$$f(G^\dagger) = f(Ext(\pi_1)) + f(Ext(\pi_2)) - f(B^\dagger),$$

and so, since $f(G^\dagger) = f(B^\dagger) = 6$, it follows that $f(Ext(\pi_1)) = f(Ext(\pi_2)) = 6$. Hence $Ext(\pi_1)$ and $Ext(\pi_1)$ both satisfy the Maxwell count. This proves (i) and now (ii) follows immediately. □

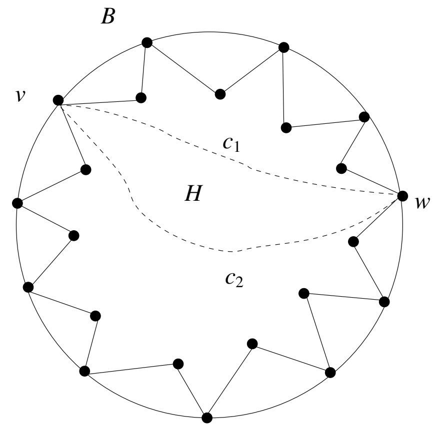

Figure 4: $H = H_v = H_w$.

3.3. Separating cycle division

The next reduction move for block and hole graphs is based on a division of the face graph with respect to a critical separating cycle of edges. The usefulness of this arises from the fact that critical separating cycles arise when there are obstructions to TT edge contraction.

Definition 23. *Let G be a face graph with a single B-labelled face and consider a planar realisation in which the unbounded face is labelled by B. Let c be a cycle in G.*

Define G_1 to be the face graph obtained from G and c by,

(i) removing all edges and vertices interior to c, and,

(ii) if $|c| \geq 4$, viewing the edges of c as the boundary of a new face with label H.

Define G_2 to be the face graph obtained from G and c by,

(i) removing all edges and vertices which are exterior to c, and,

(ii) if $|c| \geq 4$, viewing the edges of c as the boundary of a new face with label B.

This division process $G \to \{G_1, G_2\}$ is referred to as a separating cycle division *for the face graph G and cycle c.*

Note that, under this separating cycle division, $G_1^\dagger = Ext(c)$. If $|c| = 3$ then $G_2^\dagger = Int(c)$ while if $|c| \geq 4$ then $G_2^\dagger = Int(c) \cup B^\dagger$ where $B^\dagger$ is the simplicial discus with perimeter vertices in c.

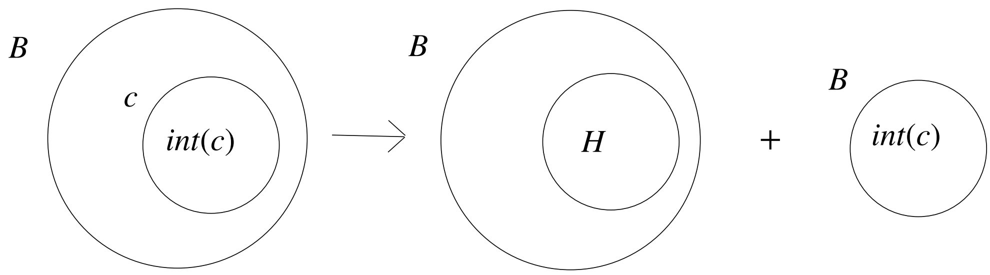

Figure 5: Separating cycle division in a face graph.

Lemma 24. *Let G be a face graph in $\mathcal{G}(1, n)$ with a separating cycle division $G \to \{G_1, G_2\}$ for a critical separating cycle c in G.*

(i) If $|c| = 3$ then $G_1 \in \mathcal{G}(1, n)$ and $G_2 \in \mathcal{G}(0, 0)$.

(ii) If $|c| \geq 4$ then $G_1 \in \mathcal{G}(1, n - l + 1)$ and $G_2 \in \mathcal{G}(1, l)$, where l is the number of H-labelled faces interior to c.

Proof. (i) By Lemma 20, G_1 and G_2 both have $(3, 6)$-tight discus and hole graphs. Since G_2 has no B-labelled faces it must be the graph of a triangulated sphere.

(ii) By Lemma 21, $G_1^\dagger = Ext(c)$ is $(3, 6)$-tight. That $G_2^\dagger$ is $(3, 6)$-tight follows from Lemma 10 since $G^\dagger = Ext(c) \cup Int(c)$ is $(3, 6)$-tight and $Ext(c)$ (which intersects $Int(c)$ in c) may be substituted by the simplicial discus $B^\dagger$ with vertices in c to obtain $G_2^\dagger$. $\square$

It can happen that the only critical separating cycles in a face graph $G \in \mathcal{G}(m, n)$ are the trivial ones, that is, the boundary cycles of the faces of G.

Definition 25. *A face graph G in $\mathcal{G}(m, n)$ is* indivisible *if every critical separating cycle for G is the boundary cycle of a face of G.*

In the next section it is shown how repetition of $(3, 6)$-tight-preserving TT edge contractions may lead to the appearance of critical separating cycles. Through a repeated edge contraction and cycle division process a set of terminal and indivisible face graphs may be obtained. Such a face graph is illustrated in Fig. 6.

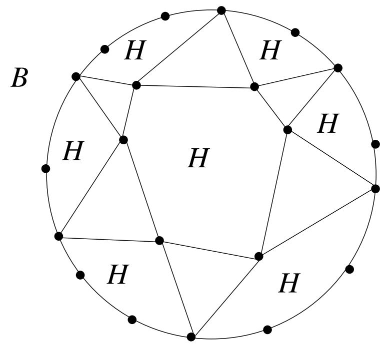

Figure 6: A face graph in $\mathcal{G}(1,7)$ which is both terminal and indivisible.

3.4. Key Lemmas

If c is a cycle in a face graph G, which is not the boundary of a face, then $int(c)$ denotes the subgraph of $G^\dagger$ obtained from $Int(c)$ by the removal of the edges of c. The following result will be referred to as the "hole-filling" lemma.

Lemma 26. *Let G be a face graph in $\mathcal{G}(1,n)$. Let K be a subgraph of $G^\dagger$ and suppose that c is a cycle in $G \cap K$ with $E(K \cap int(c)) = \emptyset$.*

(i) $f(K \cup int(c)) \leq f(K)$.

(ii) If K is $(3,6)$-tight then $K \cup int(c)$ is $(3,6)$-tight.

Proof. Since $G^\dagger$ is $(3,6)$-sparse, $f(K \cup int(c)) \geq 6$ and $f(Ext(c)) \geq 6$. Note that,

$$6 = f(G^\dagger) = f(Ext(c)) + f(int(c)) - 3|c|,$$

and so $f(int(c)) - 3|c| \leq 0$. It follows that,

$$f(K \cup int(c)) = f(K) + f(int(c)) - 3|c| \leq f(K).$$

This proves (i). To prove (ii) apply the above argument with $f(K) = 6$. □

The following lemma plays a key role in the proof of the main result.

Lemma 27. *Let G be a face graph in $\mathcal{G}(1,n)$ with $n \geq 1$. Let e be a contractible TT edge in G with contracted face graph G'. Then the following statements are equivalent.*

(i) $G' \notin \mathcal{G}(1,n)$.

(ii) The edge e lies on a critical separating cycle of G.

Proof. Suppose that $G' \notin \mathcal{G}(1,n)$ and let $e = uv$. Then the discus and hole graph $(G')^\dagger$ is not $(3,6)$-tight and so there exists a subgraph K' in $(G')^\dagger$ with $f(K') \leq 5$. Let v' be the vertex in G' obtained by the identification of u and v. Evidently, $v' \in V(K')$ since, otherwise, $G^\dagger$ must contain a copy of K' and this contradicts the $(3,6)$-sparsity count for $G^\dagger$. There are two pairs of edges xu, xv and yu, yv in G which are identified with xv' and yv' in G' on contraction of e (see Fig. 7).

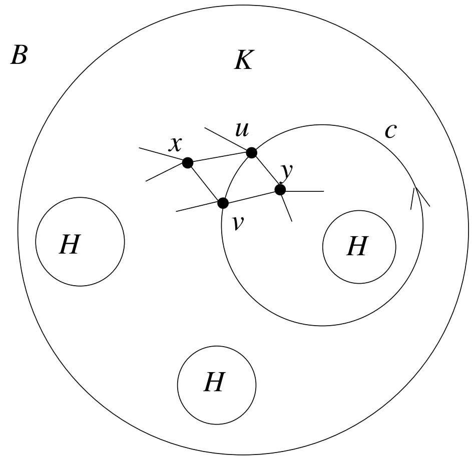

Figure 7: Locating a critical separating cycle for e.

Case (a). Suppose that $x, y \in V(K')$. Let K be the subgraph obtained from K' by first adjoining the edges xv' and yv' to K' (if necessary) and then reversing the TT edge contraction on e. Then $f(K') \geq f(K) \geq 6$ which is a contradiction.

Case (b). Suppose that $x \in V(K')$ and $y \notin V(K')$. Let K be the subgraph of $G^\dagger$ obtained from K' by first adjoining the edge xv' to K' (if necessary) and then reversing the TT edge contraction on e. Then $f(K) \leq f(K') + 1 \leq 6$ and so $f(K) = 6$. In particular, K is $(3,6)$-tight. Rechoose K, if necessary, to be a maximal $(3,6)$-tight graph in $G^\dagger$ which contains the edge e and does not contain the vertex y. Note that K must be connected and must contain the isostatic block in $G^\dagger$. Since K is maximal, by the hole-filling lemma (Lemma 26), $K = Ext(c)$ for some cycle c in G. This cycle is a critical separating cycle for G, and so (i) implies (ii) in this case.

Case (c). Suppose that $x \notin V(K')$ and $y \notin V(K')$. Let K be the subgraph of $G^\dagger$ obtained from K' by reversing the TT edge contraction on e. Then $f(K) = f(K') + 2 \leq 7$ and so $f(K) \in \{6, 7\}$. Once again assume that K is a maximal subgraph with this property. Then K must be connected and must contain the isostatic block in $G^\dagger$. By the planarity of G there are two cycles c, d of G, passing through e, with $int(c)$ and $int(d)$ disjoint from K and containing x and y respectively. Since K is maximal, by the hole-filling lemma (Lemma 26), $K = Ext(c) \cap Ext(d)$. Note that $f(Ext(c)) \geq 6$, $f(Ext(d)) \geq 6$ and

$$6 = f(G^\dagger) = f(Ext(c)) + f(Ext(d)) - f(K).$$

Thus, since $f(K) \in \{6, 7\}$, at least one of c and d is a critical separating cycle and so (i) implies (ii).

For the converse, suppose that the contractible edge e lies on a critical separating cycle c. Then c is a separating cycle for a division $G \to \{G_1, G_2\}$ and $G_1^\dagger$ is a $(3,6)$-tight subgraph of $G^\dagger$. Since the edge e lies in exactly one triangular face of $G_1^\dagger$, the graph obtained from $G_1^\dagger$ by contracting e is a subgraph of $(G')^\dagger$ with freedom number 5 and so (i) does not hold. $\square$

Corollary 28. *Let G be a face graph in $\mathcal{G}(1, n)$ which is both terminal and indivisible. Then G contains no TT edges.*

Proof. Suppose there exists a TT edge e in G. Since G is terminal, either e is not contractible or e is contractible but the graph obtained by contracting e is not in $\mathcal{G}(1,n)$. If e is not contractible then it must be contained in a non-facial 3-cycle c. By Lemma 20, c is a critical separating cycle for G. However, this contradicts the indivisibility of G. If e is contractible then by Lemma 27, e lies on a critical separating cycle. Again this contradicts the indivisibility of G and so the result follows. □

3.5. *Contracting edges of BH type*

A BH edge e of a face graph G is *contractible* if it does not belong to any 3-cycle in G. A *BH edge contraction* is a graph move $G \to G'$ on the class of face graphs under which the vertices of a contractible BH edge of G are identified. At the level of the discus and hole graph $G^\dagger$, a contractible BH edge e is contained in a simplicial discus $B^\dagger$ and is an edge of exactly two 3-cycles of $G^\dagger$. The contraction of e preserves the freedom number of $G^\dagger$ and can be reversed by vertex splitting. Thus, prima facie, there is the possibility of reducing an indivisible terminal face graph with a $(3,6)$-tight discus and hole graph to a smaller face graph which also has a $(3,6)$-tight discus and hole graph. In the case of a block and hole graph with a single block this is always the case.

Lemma 29. *Let $G \in \mathcal{G}(1,n)$, $n \geq 1$, and let G' be derived from G by a BH edge contraction. Then G' is a face graph in either $\mathcal{G}(1,n)$, $\mathcal{G}(1,n-1)$ or $\mathcal{G}(0,0)$.*

Proof. Let $e = uv$ be the contractible BH edge in G with B_1 and H_1 the adjacent labelled faces of G and v' the vertex in G' obtained on identifying of u and v. Then e is contained in exactly two 3-cycles of $G^\dagger$ which lie in the simplicial discus $B_1^\dagger$. Clearly, $(G')^\dagger$ satisfies the Maxwell count since $f((G')^\dagger) = f(G^\dagger) = 6$. The BH edge contraction on e reduces the length of the boundary cycle ∂B_1 by one. If this reduction of the boundary cycle results in a 3-cycle then G' has no B-labelled face. Moreover, the Maxwell count for G' ensures that there are no H-labelled faces in G'. Thus $G' \in \mathcal{G}(0,0)$. If G' has one B-labelled face then it must have either n or $n-1$ H-labelled faces, depending on whether or not the BH edge contraction on e reduces the boundary cycle ∂H_1 to a 3-cycle. It remains to show that $(G')^\dagger$ is $(3,6)$-sparse in this case.

If K' is a subgraph of $(G')^\dagger$ then K' may be obtained from a subgraph K of $G^\dagger$ by the contraction of e. Let x and y be the polar vertices of the simplicial discus $B_1^\dagger$. If K' contains neither of the vertices x,y then K is a subgraph of G with $f(K) \geq 6 + (|\partial B_1| - 3) + (|\partial H_1| - 3) \geq 8$. Thus $f(K') = f(K) - 2 \geq 6$. Suppose that K' contains exactly one of the polar vertices x,y. Then, assuming it is the vertex x, it follows that K is a subgraph of the triangulated sphere obtained from G by substituting the simplicial disc B_1 with the discus hemisphere for the vertex x and by inserting simplicial discs in the H-labelled faces of G. It follows that K' is also a subgraph of a triangulated sphere and so $f(K') \geq 6$. Now suppose that K' contains both of the polar vertices x,y. It is sufficient to consider the case when K' contains the edges xv' and yv' and to assume that $xu, xv, yu, yv \in K$. Then $f(K') = f(K) \geq 6$. It follows that $(G')^\dagger$ is $(3,6)$-sparse. □

For multiblock graphs a BH edge contraction need not preserve $(3,6)$-tightness.

Example 30. *Let $G \in \mathcal{G}(2,3)$ be the face graph illustrated in Fig. 8. Contraction of the edge e leads to a vertex which is adjacent to four vertices in ∂B_1 and so the associated discus and hole graph is not* $(3,6)$*-tight.*

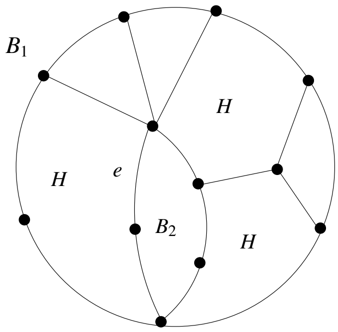

Figure 8: A contractible BH edge e in a face graph $G \in \mathcal{G}(2,3)$ with inadmissible contraction.

The following analogue of Lemma 29 applies to multi-block graphs.

Lemma 31. *Let G be a face graph in $\mathcal{G}(m,n)$ with $m,n \geq 1$. Let e be an edge of a path P in $\partial B_i \cap \partial H_j$ which has length* 3 *or more and let G' be the face graph, of type (m',n') obtained by the contraction of e. Then $G' \in \mathcal{G}(m',n')$.*

Proof. The proof follows by applying the $(3,6)$-tight graph substitution principle of Lemma 10. Consider the graph obtained from $G^\dagger$ by removing the poles of the simplicial discus $B_i^\dagger$ and the interior vertices of P. This graph plays the role of G in Lemma 10. Let $(B_i')^\dagger$ denote the simplicial discus obtained from $B_i^\dagger$ on contracting e. Now $B_i^\dagger$ and $(B_i')^\dagger$ play the roles of K and K' respectively in Lemma 10. Note that since the path containing e has length at least 3, condition (*iii*) of Lemma 10 is satisfied. Thus, since $G^\dagger$ is $(3,6)$-tight, $(G')^\dagger$ is also $(3,6)$-tight. □

In the light of Lemma 29, the indivisible terminal face graph of Fig. 6 may be reduced by BH edge contractions and further edge contraction reductions become possible in view of the emerging edges of type TT. One can continue such reductions until termination at the terminal graph of $\mathcal{G}(0,0)$ which is K_3. In fact this kind of reduction is possible in general and forms a key part of the proof of Theorem 36.

Definition 32. *A face graph G is BH-*reduced *if it contains no contractible BH edges.*

Corollary 33. *For each $n \geq 1$, there is no face graph in $\mathcal{G}(1,n)$ which is terminal, indivisible and BH-reduced.*

Proof. Suppose there exists $G \in \mathcal{G}(1,n)$ which is terminal, indivisible and BH-reduced. By Corollary 28, G contains no TT edges. If an edge e in G is of type BH then, since G is BH-reduced, e is not contractible and so must be contained in a non-facial 3-cycle c of G. By Lemma 20, c is a critical separating cycle for G. However, this contradicts the assumption that G is indivisible and so G contains no BH edges. By Proposition 22, G contains a critical separating cycle for G which is not the boundary of a face. However, this contradicts the indivisibility of G and so there can be no face graph in $\mathcal{G}(1,n)$ which is terminal, indivisible and BH-reduced. $\square$

Corollary 34. *Let G be a face graph in $\mathcal{G}(1,n)$. Then there exists a rooted tree in which each node is labelled by a face graph such that,*

(i) the root node is labelled G,

(ii) every node has either one child which is obtained from its parent node by a TT or BH edge contraction, or, two children which are obtained from their parent node by a critical separating cycle division,

(iii) each node is either contained in $\mathcal{G}(1,m)$ for some $m \leq n$ and is not a leaf, or, is contained in $\mathcal{G}(0,0)$ (in which case it is a leaf).

Proof. The statement follows by applying Corollary 33 together with Lemma 24 and Lemma 29. $\square$

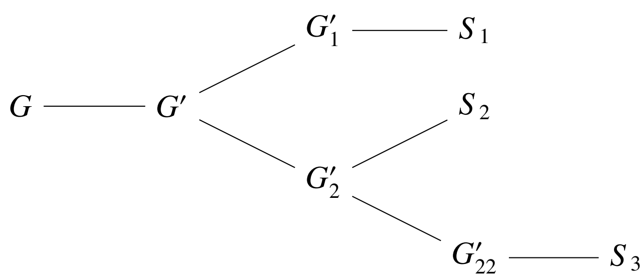

Figure 9: Deconstructing a face graph $G \in \mathcal{G}(1,n)$. Each node is obtained from its parent by a TT or BH edge contraction, or, by a critical separating cycle division. Each leaf is contained in $\mathcal{G}(0,0)$.

In the case of general block and hole graphs one can also perform division at critical cycles, and there are counterparts to Lemma 27 and Corollary 28. However, as the following example shows, there are face graphs in $\mathcal{G}(m,n)$, $m \geq 2$, which are terminal, indivisible and BH-reduced.

Example 35. *Fig. 10 shows a face graph $G \in \mathcal{G}(2,6)$ which is terminal, indivisible and BH-reduced. Note that the associated block and hole graphs $\hat{G}$ are* 3*-rigid. This follows from the fact that they are constructible from K_3 by vertex splitting together with Henneberg degree* 3 *and degree* 4 *vertex extension moves.*

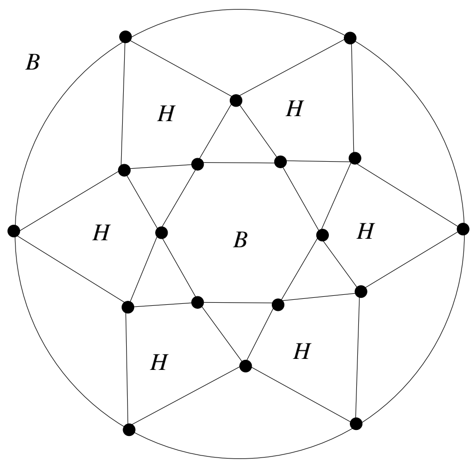

Figure 10: A face graph in $\mathcal{G}(2,6)$ which is terminal, indivisible and BH-reduced.

3.6. Generic rigidity of block and hole graphs

Let J be a simple graph and let v be a vertex of J with adjacent vertices $v_1, v_2, \dots, v_n$, $n \geq 2$. Construct a new graph $\tilde{J}$ from J by,

1. removing the vertex v and its incident edges from J,
2. adjoining two new vertices w_1, w_2,
3. adjoining the edge w_1v_j or the edge w_2v_j for each $j = 3, 4, \dots, n$.
4. adjoining the five edges $v_1w_1, v_2w_1, v_1w_2, v_2w_2$ and w_1w_2.

The graph move $J \to \tilde{J}$ is called *vertex splitting*. It is shown in [9] that if J is minimally 3-rigid then so too is $\tilde{J}$. (See also the Appendix).

Theorem 36. *Let $\hat{G}$ be a block and hole graph with a single block. Then the following statements are equivalent.*

(i) *$\hat{G}$ is minimally* 3*-rigid.*

(ii) *$\hat{G}$ is* $(3,6)$*-tight.*

(iii) *$\hat{G}$ is constructible from K_3 by the moves of vertex splitting and isostatic block substitution.*

Proof. The implication $(i) \Rightarrow (ii)$ is well known for general minimally 3-rigid graphs. The implication $(iii) \Rightarrow (i)$ follows from the isostatic block substitution principle (Lemma 6) and the fact that vertex splitting preserves minimal 3-rigidity (see Appendix).

To prove $(ii) \Rightarrow (iii)$, apply the following induction argument based on the number of vertices of the underlying face graph. Let $P(k)$ be the statement that every $(3,6)$-tight block and hole graph $\hat{G}$ with a single block and $|V(G)| = k$ is constructible from K_3 by the moves of vertex splitting and isostatic block substitution. Note that if $|V(G)| = 4$ then G is a 4-cycle with one B-labelled face and one H-labelled face. In this case, every block and hole graph $\hat{G}$ is clearly constructible from K_3 by applying a single vertex splitting move to obtain the minimally 3-rigid graph K_4 and then substituting this K_4 with the required isostatic block for $\hat{G}$. Thus the statement $P(4)$ is true and this establishes the base of the induction.

Now assume that the statement $P(k)$ holds for all $k = 4, 5, \dots, l-1$ where $l \geq 5$. Let $\hat{G}$ be a $(3, 6)$-tight block and hole graph with a single block and $|V(G)| = l$. By Corollary 11, the discus and hole graph $G^\dagger$ is also $(3, 6)$-tight and so $G \in \mathcal{G}(1, n)$ for some n. Thus G admits a TT edge contraction, a BH edge contraction or a critical separating cycle division as described in the reduction scheme for face graphs in $\mathcal{G}(1, n)$ (Corollary 34). In the case of a TT or BH edge contraction $G \to G'$, the contracted face graph G' has fewer vertices than G and is contained in either $\mathcal{G}(1, m)$ for some $m \leq n$, or, in $\mathcal{G}(0, 0)$. In the former case, the induction hypothesis implies that $(G')^\dagger$ is constructible from K_3 by the moves of vertex splitting and isostatic block substitution. In the latter case, $(G')^\dagger$ is the graph of a triangulated sphere and so is constructible from K_3 by vertex splitting alone (see Appendix). It follows that $G^\dagger$ is itself constructible from K_3 by vertex splitting and isostatic block substitution. In the case of a critical separating cycle division $G \to \{G_1, G_2\}$, G is obtained from two face graphs G_1 and G_2, each with fewer vertices than G. Moreover, for each $j = 1, 2$ either $G_j \in \mathcal{G}(1, m_j)$ for some $m_j \leq n$, or, $G_j \in \mathcal{G}(0, 0)$. Thus it again follows that both $G_1^\dagger$ and $G_2^\dagger$ are constructible from K_3 by vertex splitting and isostatic block substitution. Note that $G_1^\dagger$ is minimally 3-rigid and so may be used as a substitute for the isostatic block of $G_2^\dagger$. In this way $G^\dagger$ is shown to be constructible from K_3 in the required manner. This establishes the inductive step and so the proof of the implication $(ii) \Rightarrow (iii)$ is complete. □

4. Girth inequalities

We now examine certain cycle length inequalities for block and hole graphs that were considered in Finbow-Singh and Whiteley [3]. Recall from Ex. 7 that G° denotes the block and hole graph obtained from a face graph G by adjoining $2(|\partial B| - 3)$ edges to each B-labelled face so that each isostatic block B° is the graph of a triangulated sphere.

4.1. Index of a collection of labelled faces

Let $\mathcal{B}'$ and $\mathcal{H}'$ respectively be collections of B-labelled and H-labelled faces of a face graph G. The *index* of the collection $\mathcal{B}' \cup \mathcal{H}'$ is defined as,

$$\operatorname{ind}(\mathcal{B}' \cup \mathcal{H}') = \sum_{B \in \mathcal{B}'} (|\partial B| - 3) - \sum_{H \in \mathcal{H}'} (|\partial H| - 3).$$

Lemma 37. *Let $G = S(\mathcal{B}, \mathcal{H})$ be a face graph of type (m, n).*

(i) If C and C' are two collections of labelled faces of G then,

$$\operatorname{ind}(C \cup C') = \operatorname{ind}(C) + \operatorname{ind}(C') - \operatorname{ind}(C \cap C').$$

(ii) $f(G^\circ) = 6 - \operatorname{ind}(\mathcal{B} \cup \mathcal{H})$.

(iii) If G° satisfies the Maxwell count then,

$$\mathrm{ind}((\mathcal{B} \cup \mathcal{H})\backslash C) = -\,\mathrm{ind}(C)$$

for each collection C of labelled faces of G.

Proof. (i) This follows by simple counting.

(ii) The face graph G is obtained from the graph of a triangulated sphere S. By construction,

$$|E(G^\circ)| = |E(S)| + \mathrm{ind}(\mathcal{B} \cup \mathcal{H}).$$

Moreover, S and G° have the same vertex set and so,

$$f(G^\circ) = 3|V(G^\circ)| - |E(G^\circ)| = f(S) - \mathrm{ind}(\mathcal{B} \cup \mathcal{H}).$$

The graph of a triangulated sphere S must satisfy the Maxwell count and so the result follows.

(iii) Let C be a collection of labelled faces of G. By (i),

$$\mathrm{ind}(\mathcal{B} \cup \mathcal{H}) = \mathrm{ind}(C) + \mathrm{ind}((\mathcal{B} \cup \mathcal{H})\backslash C).$$

If G° satisfies the Maxwell count then, by (ii), $\mathrm{ind}(\mathcal{B} \cup \mathcal{H}) = 0$ and so the result follows. □

Definition 38. *A face graph G is said to satisfy the* girth inequalities *if, for every cycle c in G, and every planar realisation of G,*

$$|c| \geq |\,\mathrm{ind}(C)| + 3$$

where C is the collection of B-labelled and H-labelled faces of G which lie inside c.

A block and hole graph $\hat{G}$ is said to satisfy the girth inequalities if it is derived from a face graph G which satisfies the girth inequalities.

Example 39. *Let G be a face graph of type $(1, 1)$, so that G has exactly one B-labelled face and exactly one H-labelled face. Then G satisfies the girth inequalities if and only if the lengths of the boundaries of the B-labelled face and the H-labelled face are equal and, letting r denote this common boundary length, every cycle in G which winds around H has length at least r.*

Lemma 40. *Let G be a face graph of type (m, n). If G satisfies the girth inequalities then G° satisfies the Maxwell count.*

Proof. By Lemma 37(ii) it is sufficient to show that $\mathrm{ind}(\mathcal{B} \cup \mathcal{H}) = 0$. Choose any H-labelled face H_1 in G and let $C = (\mathcal{B} \cup \mathcal{H})\backslash\{H_1\}$. Applying the girth inequalities,

$$\mathrm{ind}(\mathcal{B} \cup \mathcal{H}) = \mathrm{ind}(C) - (|\partial H_1| - 3) \leq |\,\mathrm{ind}(C)| - (|\partial H_1| - 3) \leq 0.$$

To obtain the reverse inequality, choose any B-labelled face B_1 in G and let $C' = (\mathcal{B} \cup \mathcal{H})\backslash\{B_1\}$. By the girth inequalities,

$$\mathrm{ind}(\mathcal{B} \cup \mathcal{H}) = (|\partial B_1| - 3) + \mathrm{ind}(C') \geq |\,\mathrm{ind}(C')| + \mathrm{ind}(C') \geq 0.$$

□

Proposition 41. *Let c be a cycle in a face graph G of type (m, n) and let C be a collection of labelled faces of G which lie inside c for some planar realisation of G.*

(i) If G° is simple and $(3, 6)$-sparse then $|c| \geq \operatorname{ind}(C) + 3$.

(ii) If G° is simple and $(3, 6)$-tight then $|c| \geq |\operatorname{ind}(C)| + 3$.

In particular, if G° is simple and $(3, 6)$-tight then G satisfies the girth inequalities.

Proof. Let S be the graph of a triangulated sphere and let c be a cycle of edges of length greater than 3. Then c determines two simplicial discs D_1 and D_2 with intersection equal to c. Since each simplicial disc may be completed to the graph of a triangulated sphere by the addition of $|c| - 3$ edges it follows that,

$$f(D_1) = f(D_2) = 6 + (|c| - 3).$$

Suppose a graph K_1 is derived from D_1 by keeping the same vertex set and subtracting and adding various edges. Then K_1 will fail the sparsity count $f(K_1) \geq 6$ if the total change in the number of edges is an increase by more than $|c| - 3$ edges.

Consider now the face graph G and suppose it is derived from the graph of a triangulated sphere S. Fix a planar representation of G and let c be a cycle in G. As in the previous paragraph, c determines two simplicial discs D_1 and D_2 in S. Without loss of generality, assume that D_1 contains the edges of S which lie inside c and D_2 contains the edges which lie outside c. Let K_1 and K_2 be the corresponding subgraphs of the block and hole graph G°. Thus K_1 and K_2 are derived from D_1 and D_2 respectively by removing edges which correspond to H-labelled faces in G and adjoining the edges of each isostatic block.

(i) If G° is $(3, 6)$-sparse then $f(K_1) \geq 6$. Thus the total change in the number of edges in deriving K_1 from D_1 does not exceed $|c| - 3$ in magnitude. This implies the inequality $|c| - 3 \geq \operatorname{ind}(C)$.

(ii) Applying the argument for (i) to K_2, $f(K_2) \geq 6$ and so the total change in the number of edges in deriving K_2 from D_2 does not exceed $|c| - 3$. Thus,

$$|c| - 3 \geq \operatorname{ind}((\mathcal{B} \cup \mathcal{H}) \backslash C).$$

By Lemma 37, $\operatorname{ind}((\mathcal{B} \cup \mathcal{H}) \backslash C)) = -\operatorname{ind}(C)$ and so $|c| - 3 \geq |\operatorname{ind}(C)|$. □

4.2. Critical girth cycles

Definition 42. *A cycle c in a face graph G is called a* critical girth cycle *for G if, for some planar realisation of G,*

$$|c| = |\operatorname{ind}(C)| + 3$$

where C is the collection of B-labelled and H-labelled faces of G which lie inside c.

Recall from Def. 19 the definition of a critical separating cycle for a face graph.

Lemma 43. *Let G be a face graph of type (m, n) and suppose the block and hole graphs for G satisfy the Maxwell count. If c is a cycle in G then the following statements are equivalent.*

(i) c is a critical girth cycle for G.

(ii) Either $Ext(c)$ or $Int(c)$ satisfies the Maxwell count.

In particular, if $G \in \mathcal{G}(m,n)$ then c is a critical girth cycle if and only if it is a critical separating cycle.

Proof. Fix a planar realisation for G and let $\mathcal{B}' \cup \mathcal{H}'$ be the labelled faces of G which lie inside c. Let G_1 be the face graph obtained from G by removing edges and vertices which are interior to c and, if $|c| \geq 4$, labelling the face with boundary c by H. Then $f(G_1^\circ) = f(G^\circ) - \text{ind}(\mathcal{B}' \cup \mathcal{H}') + (|c|-3)$. It follows that G_1° satisfies the Maxwell count if and only if $|c| = \text{ind}(\mathcal{B}' \cup \mathcal{H}') + 3$. Similarly, let G_2 be the face graph obtained from G by removing edges and vertices which are exterior to c and, if $|c| \geq 4$, labelling the face with boundary c by H. Then, by Lemma 37(*iii*), G_2° satisfies the Maxwell count if and only if $|c| = -\text{ind}(\mathcal{B}' \cup \mathcal{H}') + 3$. Thus, c is a critical girth cycle if and only if either G_1° or G_2° satisfies the Maxwell count. The result now follows from Corollary 11. □

4.3. One block and n holes

From the arguments of [3] it follows that a block and hole graph with a single block and a single hole is $(3,6)$-tight if and only if the underlying face graph satisfies the girth inequalities. In Theorem 46 this equivalence is extended to the case of block and hole graphs with a single block and n holes for any $n \geq 1$.

Lemma 44. *Let $G \to G'$ be a TT edge contraction or a BH edge contraction on a face graph G of type $(1,n)$. If G satisfies the girth inequalities and contains no critical girth cycles, other than boundary cycles, then G' satisfies the girth inequalities.*

Proof. If G' is obtained from G by contracting a TT edge e then this contraction does not alter the boundary of any labelled face of G. If G' is obtained from G by contracting a BH edge e then this contraction reduces by one the boundary lengths of the B-labelled face and some H-labelled face H_1. All other labelled faces of G are unchanged. Let c' be a cycle in G'. Then there is a cycle c in G such that either $c = c'$, or, c' is obtained from c by contracting the edge e. If e is an edge of c then B_1 and H_1 must lie in complementary regions of the complement of c. Thus the index of the exterior and interior labelled faces for c are, respectively, reduced and increased by one. If e is not an edge of c then the B and H labelled faces both lie either inside or outside c. Thus the index of the exterior and interior labelled faces for c are unchanged. Since c is not a critical girth cycle in G, in each of these cases the girth inequality is satisfied by c'. □

Lemma 45. *Let G be a face graph of type $(1,n)$ and let $G \to \{G_1, G_2\}$ be a separating cycle division on a critical girth cycle c in G. If G satisfies the girth inequalities then G_1 and G_2 both satisfy the girth inequalities.*

Proof. Let C denote the collection of labelled faces of G which lie inside c. Evidently, $\text{ind}(C) \leq 0$ and so, since c is a critical girth cycle in G, $|c|-3 = -\text{ind}(C)$. Moreover, by Lemma 40, G° satisfies the Maxwell count and so, by Lemma 37, $|c|-3 = \text{ind}((\mathcal{B} \cup \mathcal{H}) \backslash C)$. If c_1 is a cycle in G_1 then c_1 is also a cycle in G. Let $\mathcal{D}$ denote the collection of labelled faces of G which lie inside c_1 and let C_1

denote the collection of labelled faces of G_1 which lie inside c_1. Since $|c|-3 = -\operatorname{ind}(C)$, it follows that $\operatorname{ind}(\mathcal{D}) = \operatorname{ind}(C_1)$. Since G satisfies the girth inequalities, $|c_1| \geq |\operatorname{ind}(\mathcal{D})| + 3 = |\operatorname{ind}(C_1)| + 3$. If C_1' denotes the labelled faces of G_1 which lie outside c_1 then, again since $|c| - 3 = -\operatorname{ind}(C)$, it follows that $\operatorname{ind}(C_1') = -\operatorname{ind}(\mathcal{D})$. Thus, $|c_1| \geq |\operatorname{ind}(C_1')|+3$ and so G_1 satisfies the girth inequalities. Similarly, if c_2 is a cycle in G_2 then c_2 is also a cycle in G and, since $|c| - 3 = \operatorname{ind}((\mathcal{B} \cup \mathcal{H})\backslash C)$, it follows that G_2 satisfies the girth inequalities.

□

The following theorem completes the proof of Theorem 1 in the single block case.

Theorem 46. *Let $\hat{G}$ be a block and hole graph with a single block. Then the following are equivalent.*

(i) $\hat{G}$ is minimally 3-rigid.

(ii) G satisfies the girth inequalities.

Proof. If $\hat{G}$ is minimally 3-rigid then, by the isostatic block substitution principle, Lemma 6, G° is minimally 3-rigid for any choice of triangulated sphere B°. In particular, G° is $(3, 6)$-tight and so, by Proposition 41, G satisfies the girth inequalities.

To prove the converse, apply the following induction argument. Let $P(k)$ be the statement that every block and hole graph $\hat{G}$ with a single block which satisfies the girth inequalities and has $|V(G)| = k$, is minimally 3-rigid. The statement $P(4)$ is true since in this case there exists only one face graph G, namely a 4-cycle with one B-labelled face and one H-labelled face. Clearly, G satisfies the girth inequalities and has minimally 3-rigid block and hole graphs. This establishes the base of the induction.

Suppose that $P(k)$ is true for all $k = 4, 5, \ldots, l-1$ and let $\hat{G}$ be a block and hole graph with a single block which satisfies the girth inequalities and has $|V(G)| = l$. Note that, by Lemma 40, each block and hole graph G° satisfies the Maxwell count. If G contains a critical girth cycle c, which is not the boundary of a face, then by Lemma 45 the face graphs G_1 and G_2 obtained by separating cycle division on c both satisfy the girth inequalities. Note that G_1 and G_2 are each either face graphs with a single B-labelled face and fewer vertices than G, or, are triangulations of a triangle. It follows that both G_1 and G_2 have minimally 3-rigid block and hole graphs. By the block substitution principle (Lemma 6) the isostatic block of $G_2^\dagger$ may be substituted with $G_1^\dagger$ to obtain $G^\dagger$. Thus G has minimally 3-rigid block and hole graphs.

Now suppose that there are no critical girth cycles in G, other than the boundary cycles of faces of G. If G contains no edges of type TT or BH then, by Proposition 22, G contains a cycle π, which is not the boundary of a face, such that $Ext(\pi)$ satisfies the Maxwell count. By Lemma 43, π is a critical girth cycle for G. This is a contradiction and so G must contain an edge of type TT or BH. Moreover, such an edge must be contractible since any non-facial 3-cycle would be a critical girth cycle for G.

Suppose a face graph G' is obtained from G by contracting a TT or a BH edge e. Then G' is either a face graph with a single B-labelled face and fewer vertices than G, or, is a triangulation of a triangle. By Lemma 44, G' satisfies the girth inequalities and so G' must have minimally 3-rigid block and hole graphs. Now $G^\dagger$ may be obtained from $(G')^\dagger$ by vertex splitting and so G also has minimally 3-rigid block and hole graphs. This establishes that the statement $P(l)$ is true and so, by the principle of induction, the theorem is proved. □

In [3] the following theorem is obtained.

Theorem 47. *Let $\hat{G}$ be a block and hole graph with one block and one hole such that $|\partial B| = |\partial H| = r$. If there exist r vertex disjoint paths in G which include the vertices of the labelled faces then $\hat{G}$ is 3-rigid.*

We note that this also follows from Theorem 46. Indeed if the disjoint path condition holds then it is evident that every cycle c associated with the single hole has length at least r since it must cross each of the r paths. Thus the girth inequalities hold. Similarly, Conjecture 2, in our introduction, follows on verifying that the 5-connectedness condition ensures that the girth inequalities hold.

4.4. Block-hole transposition

We next observe that the characterisation of minimally 3-rigid block and hole graphs with a single block also provides a characterisation in the single hole case. Let G_t be the face graph obtained from graph G by replacing B labels by H labels and H labels by B labels.

Corollary 48. *Let $\hat{G}$ be a block and hole graph with a single hole. Then the following are equivalent.*

(i) *$\hat{G}$ is minimally 3-rigid.*

(ii) *$\hat{G}$ is $(3,6)$-tight.*

(iii) *$\hat{G}$ is constructible from K_3 by vertex splitting and isostatic block substitution.*

(iv) *G satisfies the girth inequalities.*

In particular, $\hat{G}$ is minimally 3-rigid if and only if $\hat{G}_t$ is minimally 3-rigid.

Proof. The implications $(iii) \implies (i) \Rightarrow (ii) \Rightarrow (iv)$ have already been established more generally for face graphs of type (m,n). If G satisfies the girth inequalities then G_t also satisfies the girth inequalities and so there exists a reduction scheme for G_t as described in Corollary 34. This same reduction scheme may be applied to show that the block and hole graphs for G are minimally 3-rigid. Thus the equivalence of $(i) - (iv)$ is established. The final statement follows since G satisfies the girth inequalities if and only if G_t satisfies the girth inequalities. □

4.5. Separation conditions

The following separation conditions for block and hole graphs $\hat{G}$ were indicated in [3] (see Conjecture 5.1 and Proposition 5.4) and are necessary conditions for minimal 3-rigidity.

Corollary 49. *Let $\hat{G}$ be a minimally 3-rigid block and hole graph with face graph G of type (m,n).*

(i) *There are no edges in G between nonadjacent vertices in the boundary of a labelled face of G.*

(ii) Each pair of labelled faces in G with the same label share at most two vertices and these vertices must be adjacent.

Proof. (i) If there exists an edge between two nonadjacent vertices in the boundary of a labelled face of G then there exists a cycle in G which violates the girth inequalities.

(ii) If two H-labelled faces in G share more than two vertices then by the girth inequalities there exists a B-labelled face within their joint perimeter cycle. However, this implies that the block and hole graphs for G fail to be 3-connected. Similarly, if two H-labelled faces in G share two nonadjacent vertices then the block and hole graphs for G fail to be 3-connected. By block-hole transposition the result also holds for B-labelled faces.

□

The following example shows that Conjectures 5.1 and 5.2 of [3] are not true in general.

Example 50. *Let G be the face graph of type* $(2,2)$ *with planar realisation illustrated in Fig. 11. The block and hole graph* G° *satisfies the separation conditions of Corollary 49 (and of [3]). Also,* G° *is* $(3,6)$*-tight and, by Proposition 41, G satisfies the girth inequalities. However,* G° *is not minimally* 3*-rigid since it may be reduced to a graph which is not* 3*-connected by inverse Henneberg moves on vertices of degree* 3*.*

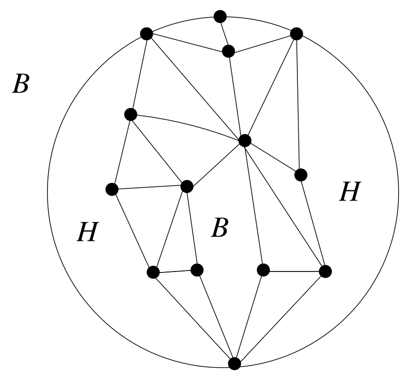

Figure 11: A face graph of type (2, 2) which satisfies the girth inequalities and separation conditions but does not have a 3-rigid block and hole graph.

5. Appendix

A bar-joint framework in $\mathbb{R}^3$ consists of a simple graph $G = (V, E)$ and a placement $p : V \to \mathbb{R}^3$, such that $p(v) \neq p(w)$ for each edge $vw \in E$. An *infinitesimal flex* of (G, p) is an assignment $u : V \to \mathbb{R}^3$ which satisfies the infinitesimal flex condition $(u(v) - u(w)) \cdot (p(v) - p(w)) = 0$ for every edge $vw \in E$. A *trivial infinitesimal flex* of (G, p) is one which extends to an infinitesimal flex of any containing framework, which is to say that it is a linear combination of a translation infinitesimal flex and a rotation infinitesimal flex. The framework (G, p) is *infinitesimally rigid* if the only infinitesimal flexes are trivial and the graph G is 3-rigid if every generic framework (G, p) is infinitesimally rigid. See [6].

5.1. Vertex splitting

The proof of rigidity preservation under vertex splitting indicated in Whiteley [9] is based on static self-stresses and 3-frames. For completeness we give an infinitesimal flex proof of this important result.

Let $G = (V, E)$ with $v_1, v_2, \dots, v_r$ the vertices of V and v_1v_2, v_1v_3, v_1v_4 edges in E. Let $G' = (V', E')$ arise from a vertex splitting move on v_1 which introduces the new vertex v_0 and the new edges v_0v_1, v_0v_2, v_0v_3. Some of the remaining edges v_1v_t may be replaced by the edges v_0v_t. Let $p : V \to \mathbb{R}^3$ be a generic realisation with $p(v_i) = p_i$ and for $n = 1, 2, \dots$ let $q^{(n)} : V' \to \mathbb{R}^3$ be nongeneric realisations which extend p, where $q^{(n)}(v_0) = q_0^n, n = 1, 2, \dots$ is a sequence of points on the line segment from p_1 to p_4 which converges to p_1.

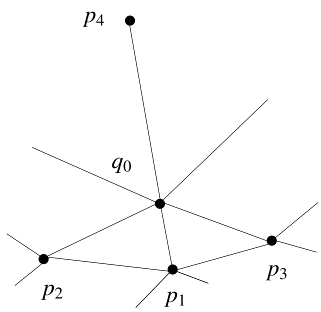

Figure 12: The vertex $q_0 = q^{(1)}(v_0)$.

Let $u^{(n)}, n = 1, 2, \dots,$ be infinitesimal flexes of $(G', q^{(n)}), n = 1, 2, \dots,$ which are of unit norm in $\mathbb{R}^{3(r+1)}$. By taking a subsequence we may assume that $u^{(n)}$ converges to an infinitesimal flex $u^{(\infty)}$ of the degenerate realisation of G' with $q(v_0) = q(v_1) = q_1$. In view of the line segment condition we have,

$$u_0^{(n)} \cdot (p_0^{(n)} - p_4) = u_1^{(n)} \cdot (p_1 - p_4).$$

for each n. Also we have,

$$u_0^{(n)} \cdot (p_0^{(n)} - p_2) = u_1^{(n)} \cdot (p_1 - p_2), \quad u_0^{(n)} \cdot (p_0^{(n)} - p_3) = u_1^{(n)} \cdot (p_1 - p_3),$$

and it follows from the generic position of p_2, p_3 and p_4 that $u_0^{(\infty)} = u_1^{(\infty)}$. Thus $u^{(\infty)}$ restricts to an infinitesimal flex u of (G, p). Note that the norm of u is nonzero.

We now use the general construction of the limit flex in the previous paragraph to show that if G' is not 3-rigid then neither is G. Indeed if G' is not 3-rigid then there exists a sequence as above in which each flex $u^{(n)}$ is orthogonal in $\mathbb{R}^{3(r+1)}$ to the space of trivial infinitesimal flexes. It follows that $u^{(\infty)}$ is similarly orthogonal and that the restriction flex u of (G, p) is orthogonal in $\mathbb{R}^{3r}$ to the space of trivial infinitesimal flexes. Since u is nonzero G is not 3-rigid, as desired.

5.2. A proof of Gluck's theorem

In our terminology Gluck's theorem ([5]) asserts that the (unlabelled) face graphs G of type $(0,0)$ are 3-rigid. For convenience we give a direct proof here. In view of 3-rigidity preservation under vertex splitting it will be enough to show that G derives from K_3 by a sequence of vertex splitting moves. To see this let $P(k)$ be the statement that every plane representation of a face graph G of type $(0,0)$ with $|V(G)| = k$ contains a contractible edge which is not in the topological boundary (of the unbounded component of the complement) of G. The statement clearly holds when $k = 4$. Assume $P(k)$ holds for all $4 \leq k \leq n$ and let G be a face graph of type $(0,0)$ with $|V(G)| = n + 1$. Consider an interior edge of G, $e = uv$ say, with associated edges xu, xv and yu, yv for its adjacent faces. If e is not contractible then there is a nonfacial triangle in G with edges zu, zv and uv. The subgraph consisting of the 3-cycle zu, zv, uv and its interior is a face graph G' of type $(0,0)$ with fewer vertices than G. It contains at least 4 vertices, since it contains x or y, and so by the induction hypothesis G' contains a contractible interior edge. This edge is also a contractible interior edge in G. Thus the statement $P(n+1)$ holds and so by induction $P(k)$ holds for all $k \geq 4$.

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