Electromagnetic Mode Profile Shaping in Waveguides

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***Abstract*—Electromagnetic mode profile shaping, would be a very useful technique, with applications including in accelerator science and data transmission. Two methods are proposed, one using a negative permittivity, the other using a wire medium.**

I. Introduction

Electromagnetic modes in a waveguide with uniform cross-section have longitudinal profiles that are sinusoidal profiles. However, there are many situation where non sinusoidal behaviour may be particularly useful. In accelerator science, for example, one may wish to have a flatter profile, see Figs. 2-4 which would accelerate a longer bunch, and therefore more electrons, for a given peak power. By contrast a peakier profile, Fig. 1 would accelerate shorter bunches for the same total power. Also due to the steeper gradient, it would enable one to give a shorter bunch more chirp which is necessary for bunch compression. The peakier profiles would also be useful for data transmission where one can have the higher peak when the signal is being observed. The usual method for modifying the profile is by modifying the boundary of the waveguide, essentially converting it into a series of cavities. In this article we examine an alternative approach by modifying the constitutive relations, in particular the permittivity. Thus we set $\epsilon = \epsilon(\omega, z)$. To simplify the analysis we consider $\epsilon(\omega, z)$ to be periodic in z.

In this article we consider two scenarios. The first is for transverse waves, section II. In this case we can make ϵ to be only temporally dispersive. There are many methods for making such a medium. For example one can compose the medium from many thin slices of media, each with a different value of ϵ, thereby approximating the continuous variation. Another method is to set up a standing wave in a non-linear material. The linearised permittivity about this solution would have the spatial dependence as desired. However for the examples given in this article to work would require regions to have negative permittivity, $\epsilon < 0$ which would require some kind of metamaterial. It also does not work for longitudinal waves.

A second scenario, section IV, is to make ϵ spatially dispersive as well as inhomogeneous. That is we set $\epsilon = \epsilon(\omega, k, z)$ where k is the Fourier conjugate variable associated with z. It is natural to ask the meaning of a function depending on both k and z. This is addressed in [1]–[3]. The method, as described below, is to replace the permittivity with a differential equation. Making the corresponding material can be relatively easy – a wire medium oriented along the propagation axis as in fig. 5 is naturally spatially dispersive [4]–[8], and by periodically varying the radius of the wires, will naturally generate the correct spatial inhomogeneity.

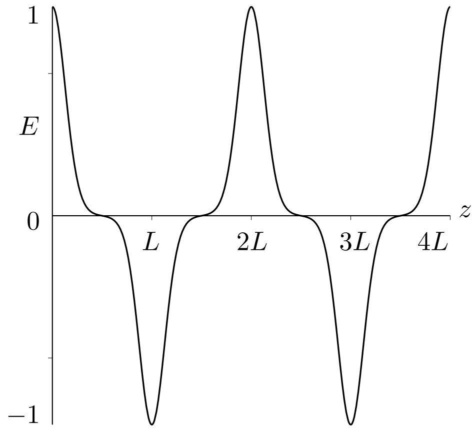

Fig. 1. A peakier solution to Mathieu's equation: In (3) with $q = -10.0$ and $a = \text{MathieuA}(1, -10.0) = -13.937$

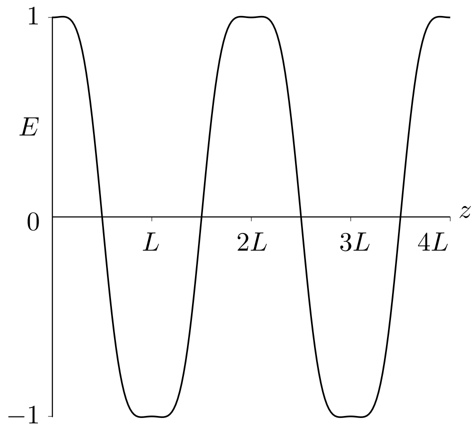

Fig. 2. A flatter solution to Mathieu's equation: In (3) with $q = 1.0$ and $a = \text{MathieuA}(1, 1.0) = 1.8591$

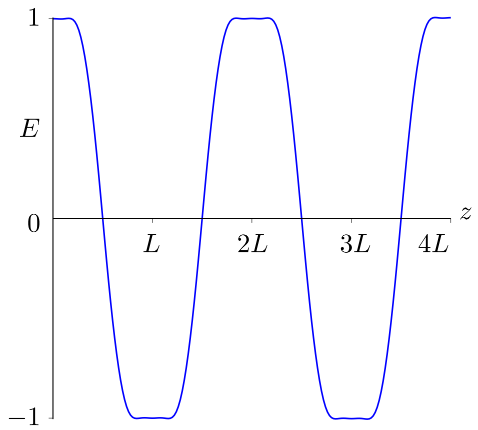

Fig. 3. An even flatter profile: A solution to the equation $\pi^2 L^{-2} E'' + (1.9266 - 2.2\cos(2\pi Z/L) + 0.4\cos(8\pi Z/L))E = 0$.

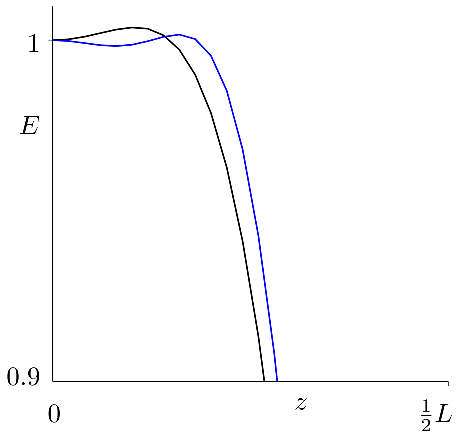

Fig. 4. Close up showing difference between the profiles in Fig. 2 and Fig. 3.

II. Transverse modes periodic inhomogeneous permittivity

Consider the single frequency transverse mode with $\boldsymbol{E} = e^{i\omega t}\tilde{E}(z)\boldsymbol{i}$, $\boldsymbol{P} = e^{i\omega t}\tilde{P}(z)\boldsymbol{i}$ and $\boldsymbol{H} = e^{i\omega t}\tilde{H}(z)\boldsymbol{j}$, together with the permittivity $\epsilon(\omega, z) = \epsilon_0\epsilon_{\mathrm{r}}(\omega, z)$ and vacuum permeability μ_0. Then Maxwell's equations give

$$\tilde{E}'' + \omega^2 c^{-2}\epsilon_{\mathrm{r}}(\omega, z)\tilde{E} = 0 \qquad (1)$$

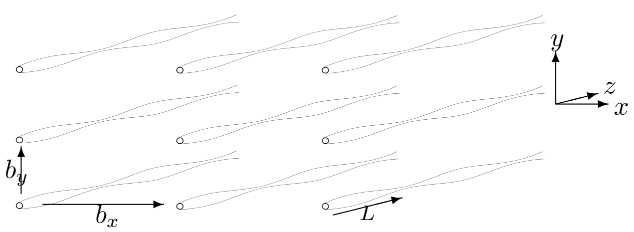

Fig. 5. Wire medium with a periodic variation in the radius of the wires. The inter wire spacing are (b_y, b_z) and the period of the longitudinal variation is L.

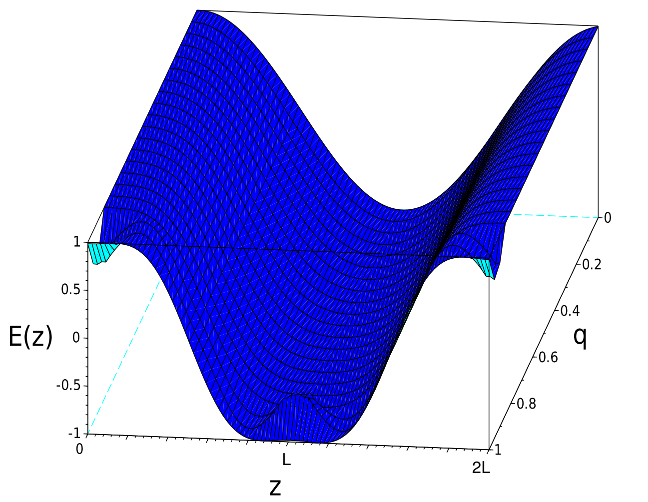

Fig. 6. One period of the set of Mathieu mode profiles $E(z)$, for a range of $q \in [0, 1]$, and using the appropriately matching a parameter value.

where $' = \frac{d}{dz}$. Set

$$\epsilon_{\mathrm{r}}(\omega, z) = \pi^2 c^2 \omega^{-2} L^{-2}\big(a(\omega) - 2q(\omega)\cos(2\pi z/L)\big) \qquad (2)$$

where L is twice the period of the ϵ variation: $\epsilon(\omega, z) = \epsilon(\omega, z + L\pi)$. Then we get the (rescaled) Mathieu's equations

$$\pi^{-2} L^2 \tilde{E}'' + \big(a - 2q\cos(2\pi z/L)\big)\tilde{E} = 0 \qquad (3)$$

The profiles of $E(z)$ are given in Fig. 1 and Fig. 2. By using higher spatial harmonics, we can make the wave even flatter. See Fig. 3 and Fig. 4.

III. Bandstructure and Mathieu Modes

From a physical perspective, obtaining a range of Mathieu Modes from a periodic structure is straightforward: we can specify the required periodic structure, and then solve for the bandstructure and field profiles. This can be done either using commercial software such as CST Studio, or open source solvers such as MPB, both of which offer convenient ways of extracting a bandstructure and field model profiles from a given unit cell. Notably, MPB calculates fully-vectorial eigenmodes of Maxwell's equations with periodic boundary conditions by preconditioned conjugate-gradient minimization of the block Rayleigh quotient in a planewave basis [9].

However, just setting up the structural index and periodic variation as a function of position, as compatible with eqn. (3), is not sufficient to uniquely identify a chosen field profile from the complete bandstructure and its modes. It is also necessary to correctly specify either the mode energy or wavevector to get the point on the bandstructure which matches the selected mathematical waveform. Alternatively, the material properties need to be set up with the correct frequency response, namely going as $1/\omega^2$.

This done, we can e.g. investigate the progression towards the interesting flat-top profile seen in fig. 2. We generate a series of structures with q increasing from zero to one, and with the a chosen to match q, starting at $a = 1$ and $q = 0$,

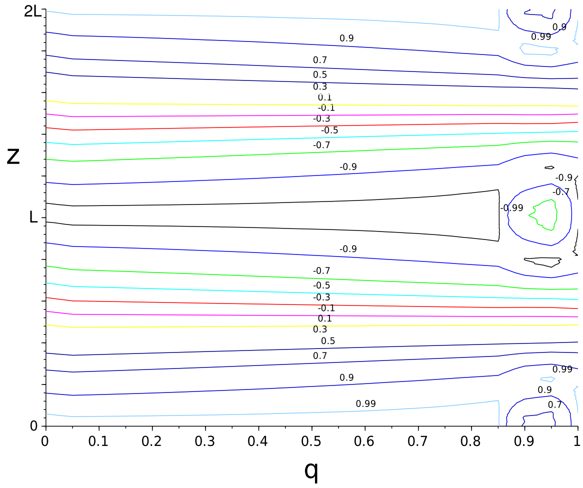

Fig. 7. Contour plot based on the stack of Mathieu mode profiles in $E(z)$ as shown in the previous figure. The profiles vary from sinusoidal for $q = 0$ on the left side of this figure, through to the flat topped and/or more complicated forms on the right as q increases. The flat topped profiles are characterised by the increasing separation of the high-value contours (at e.g. $|E| = 0.90$ or 0.99), as is most easily seen around $z \simeq L$.

and extract the correct Mathieu mode from the bandstructure information. The most simple situation is for a 1D structure, which gives us the progression or waveforms seen in fig. 6. However, despite the clear non-sinusoidal behavior for $q \geq 0.85$, more subtle features are less evident. Therefore fig. 7 provides a contour plot of the same data, which demonstrates clearly the broadening of the peaks – as we would expect from fig. 2 – and the attendant steepening of the transitions.

An extension of these numerical results to 2D and 3D Mathieu-like structures, such as the wire medium discussed in this paper, will also be shown in the conference presentation.

IV. Wire Media: Longitudinal modes with spatially dispersive inhomogeneous permittivity

Consider longitudinal modes so that the electric and polarization fields are longitudinal and the magnetic field vanishes, i.e. $\boldsymbol{E} = e^{i\omega t}\tilde{E}(z)\boldsymbol{k}$, $\boldsymbol{P} = e^{i\omega t}\tilde{P}(z)\boldsymbol{k}$ and $\boldsymbol{B} = \boldsymbol{0}$, then Maxwell's equations are automatically satisfied if

$$\epsilon_0\tilde{E} + \tilde{P} = 0 \tag{4}$$

I.e. $\boldsymbol{D} = \boldsymbol{0}$, thus we are looking for epsilon near zero (ENZ) media. When the medium is homogeneous we will use an empirical model of the permittivity via

$$\epsilon(\omega, k) = \epsilon_0 - \frac{\epsilon_0 k_{\rm p}^2}{L(\omega) - \beta^2 k^2} \tag{5}$$

Combining (4) and (5) we obtain the dispersion relation

$$L(\omega) - \beta^2 k^2 = k_{\rm p}^2 \tag{6}$$

Taking the Fourier transform of $\hat{P}(k) = \epsilon(\omega, k)\hat{E}(k)$ with respect to k using $\tilde{P}(z) = \int_{-\infty}^{\infty} e^{ikz}\hat{P}(k)dk$ one obtains the differential equation

$$\beta^2\tilde{E}'' + L(\omega)\tilde{E} = k_{\rm p}(z)^2\tilde{E} \tag{7}$$

The simplest method to include an inhomogeneity in the permittivity is to let the plasma frequency $k_{\rm p}$ depend on position z, that is $k_{\rm p} = k_{\rm p}(z)$. Thus (7) becomes

$$\beta^2\tilde{E}'' + \big(L(\omega) + k_{\rm p}(z)\big)\tilde{E} = 0 \tag{8}$$

Again by choosing the appropriate periodic function for $k_{\rm p}(z)$ one can replace (8) by the Mathieu equation. This can be achieved with a wire medium, as detailed in [6].

V. Conclusion

Two methods of mode profile shaping are suggested: one using transverse waves and a varying permittivity, the second is by using a wire medium with periodic variation. We have observed that for the non spatially dispersive scenario in section II, the example waves given in this article require $\epsilon < 0$. Although this is easy to implement numerically, it would be much harder to construct the medium. As a result, numerical investigations of the type shown in section III are invaluable in working out how to minimize the technological demands, especially when extended into a full 3D model. We are currently implementing the wire medium, and optimising the shapes of the wires to form the desired field profiles.

Acknowledgement

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