# Dispersion in Closed, Off-Axis Orbit Bumps

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## Abstract

In this paper we present a proof to show that there exists no system of linear or nonlinear optics which can simultaneously close multiple local orbit bumps and dispersion through a single beam transport region. The second combiner ring in the CLIC drive beam recombination system, CR2, is used as an example of where such conditions are necessary. We determine the properties of a lattice which is capable of closing the local orbit bumps and dispersion and show that all resulting solutions are either unphysical or trivial.

*Keywords:* Dispersion; Orbit bump; Off-axis; Beam dynamics

## 1. Introduction

Typical local orbit bumps in beam transport systems vary on the timescale of 0.1-100 s and therefore may use conventional dipole magnets. Faster orbit bumps can be achieved with the use of kicker magnets which may operate on timescales from 10 ns up to 100 ms. Such systems can be designed to correct the dispersion function either side of the local orbit bump with relative ease. For some applications, such as the injection into the second combiner ring CR2 for the CLIC drive beam recombination system, multiple local orbit bumps are required on sub-nanosecond timescales; thus RF deflectors are required rather than conventional dipole magnets or kicker magnets.

The CLIC drive beam requires 2×24 pulses, each consisting of 2904 bunches with a bunch spacing of 82 ps. To achieve this, the CLIC drive beam linac produces 24×24 sub-pulses with a bunch spacing of 2 ns. A recombination system is used to interleave bunches over 3 stages to produce the required pulse trains (Figure 1). Further details of this system can be found in [1].

The second combiner ring stores bunch trains for up to 3.5 turns; on each turn an additional bunch train is injected such that the bunches are interleaved with the stored bunches. The

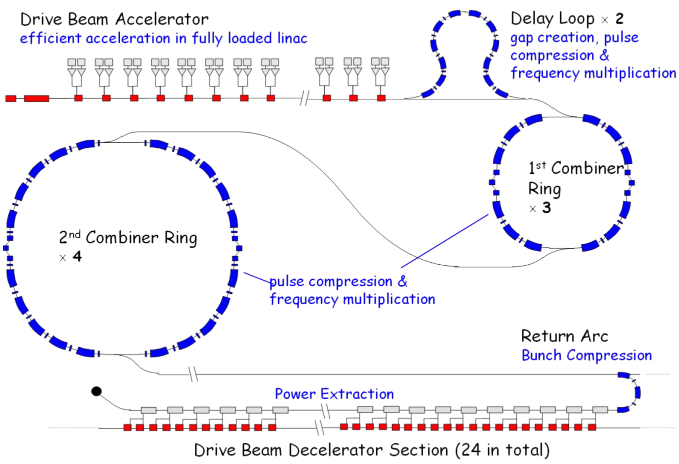


Figure 1: A schematic diagram of the CLIC drive beam recombination system [1].

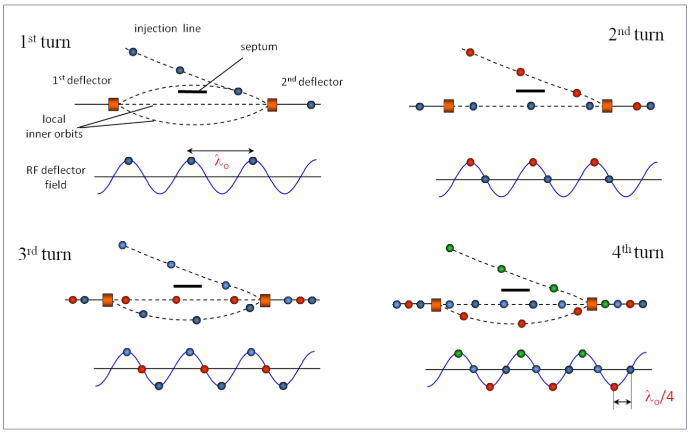


Figure 2: A schematic diagram to show how the combiner ring injection region interleaves bunches over 4 turns [1]. On each turn the stored bunches take different trajectories.

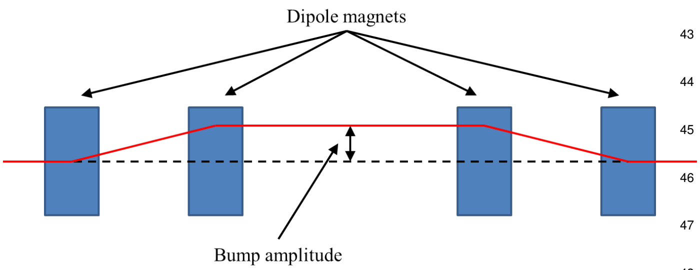


Figure 3: A schematic diagram of a 4-bump.

principle of the injection scheme is depicted in Figure 2; as is shown, there are two stored trajectories and the injection trajectory passing at the same time through the injection region. In order to avoid beam losses at the injection septum magnet a bump amplitude of ~3 cm is required and to interleave bunches with a bunch spacing of 82 ps (12 GHz) a 3 GHz RF deflector is required with the bunches 90° apart in RF phase.

A conventional orbit bump can be achieved with the use of 4 deflectors to create a dispersion-free 4-bump (Figure 3). A deflecting cavity has been designed at SLAC [2] which is similar to the design which would be required by CLIC to achieve a 3 GHz RF 4-bump. The SLAC deflecting cavity has a frequency of 2.815 GHz and an iris radius of 2.2 cm ($\sim 0.2\lambda$); the CLIC CR2 RF deflectors would require a frequency of 3 GHz and an iris radius of ~4 cm ($\sim 0.4\lambda$).

However, the orbit bumps in the injection region might also be closed with two RF deflectors and a lattice of multipoles (such as quadrupoles) as depicted in Figure 4. CLIC has opted to investigate this design scheme for the CR1 and CR2 injection regions [1] and it is this scheme which is investigated in this report.

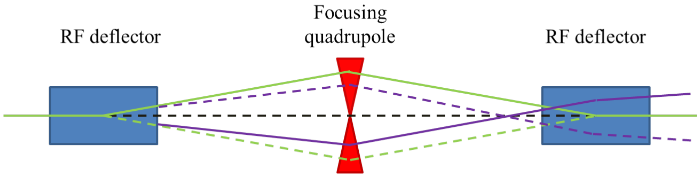


Figure 4: A schematic diagram of a local orbit bump (green) and dispersion (purple) with two RF deflectors and a single focusing quadrupole. The solid and dashed green lines show equal and opposite amplitude orbit bumps through the injection region.

If the beam centroid were to travel on-axis through the quadrupole, this lattice would be a double-bend achromat and would be dispersion-free. However, the dipole term introduced by traveling off-axis through the quadrupole gives a contribution to the dispersion which prevents the dispersion closing through the lattice. As will be shown, there exists no system of linear or nonlinear optics between the RF deflectors which can simultaneously correct both the dispersion and the orbit bump.

If there were only one trajectory through the CR2 injection region, a dispersion suppression region could be placed downstream of the injection region to compensate the residual dispersion from the injection region. However, as there are multiple simultaneous trajectories through the injection region, each trajectory will give rise to different residual dispersion; thus a dispersion suppressor would not be able to simultaneously correct the dispersion for all trajectories.

In Section 2 of this paper, we show that if a lattice exists which can create a dispersion-free closed orbit bump then a symmetric lattice can be designed which can also create a dispersion-free closed orbit bump. By considering the central region of an arbitrary symmetric lattice, we determine the general conditions under which a dispersion-free closed orbit bump can be achieved. In Section 3 we define equations for the residual dispersion from the quadrupoles due to the off-axis trajectory of the beam and use this to define specific conditions on

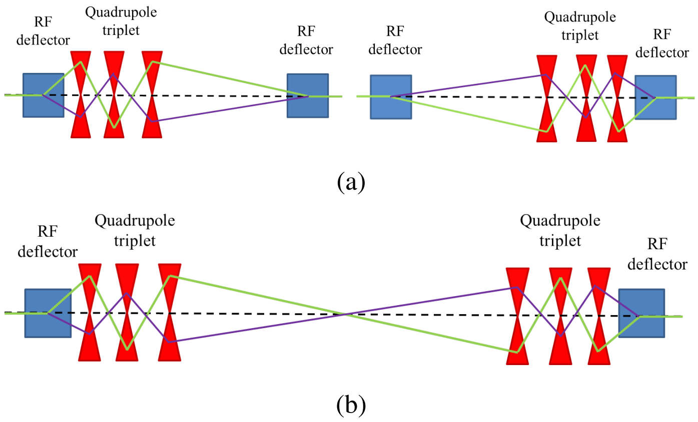


Figure 5: Diagrams illustrating the orbit deviation (green) and dispersion (purple) to show that the reflection of an asymmetric solution is also a solution (a) and that the two can be joined to form a symmetrised solution (b).

the lattice parameters for a dispersion-free closed orbit bump to exist. In Section 4 we investigate the specific conditions under which a dispersion-free closed orbit bump may exist and show that these lead to solutions which are either unphysical or trivial, thus showing that no solution exists for a linear lattice. In Section 5 we consider a non-linear lattice and show that the results from Section 4 imply that no solution exists for a non-linear solution either.

## 2. Requirements for a linear solution

### *2.1. Symmeterisation of an arbitrary lattice*

If we consider an arbitrary lattice which is able to create a dispersion-free closed orbit bump then by symmetry the reflection of this lattice must also create a dispersion-free closed orbit bump (Figure 5a). By combining the original lattice and its reflection and removing the central RF deflectors, a new lattice can be created which is symmetric and able to create a dispersion-free closed orbit bump (Figure 5b). We define this new lattice as the symmeterised lattice [3].

In order to correctly form the symmeterised lattice from the original lattice, there is a transverse offset between the elements in the two halves of the symmeterised lattice, as depicted in Figure 8. To explain the origin of this transverse offset, we should first derive expressions for the trajectory and dispersion functions due to a dipole. From figure 6 it can be shown that the trajectory after the dipole is

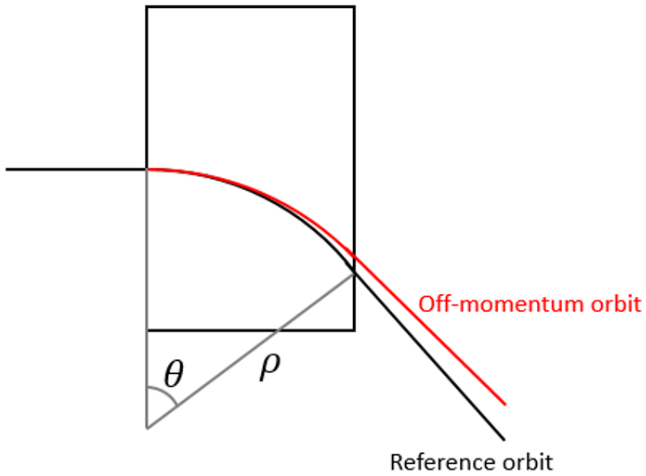


Figure 6: A diagram of the trajectory through a dipole field for a reference particle (black) and an off-momentum particle (red).

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} \rho\,(1 - \cos\theta) \\ \tan\theta \end{pmatrix}, \tag{1}$$

where $\rho$ is the radius of curvature of a particle through the dipole field and $\theta$ is the deflection angle.

The dispersion through a sector bend dipole is often expressed as [4]

$$\begin{pmatrix} D_x \\ D'_x \end{pmatrix} = -\begin{pmatrix} \rho\,(1 - \cos\theta) \\ \sin\theta \end{pmatrix}, \tag{2}$$

however, this expression provides the dispersion in terms of the local (curvilinear) coordinate system where longitudinal axis, $S$, is the tangent of the reference orbit at some point. We need to determine the dispersion in terms of a global coordinate system where the longitudinal axis, $z$, is fixed because this is the coordinate system that the trajectory is determined in for Eq. 1. Figure 7 shows the difference between the local and global coordinate systems.

In the global coordinate system, the length of the dipole can be expressed in terms of $\rho$ and $\theta$ as

$$L = \rho \sin\theta \tag{3}$$

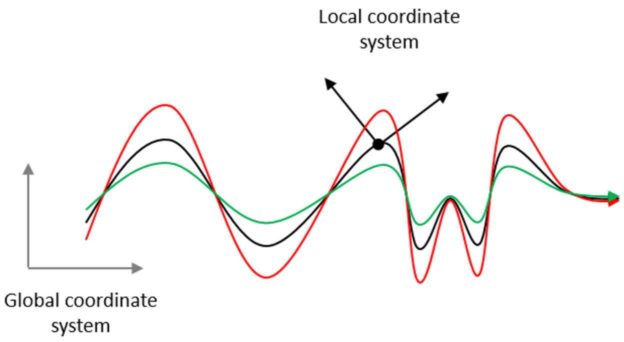


Figure 7: A diagram showing the global and local coordinate systems.

and to determine the dispersion in the global coordinate system, we need to define the trajectory in terms of a small momentum offset, $\frac{\delta p}{p}$. First we will define the bending radius of the dipole magnet in terms of $\frac{\delta p}{p}$ as

$$\rho = \frac{\rho_0}{1 + \frac{\delta p}{p}} \tag{4}$$

and from Eq. 3 we can also define the momentum dependence of deflection angle as

$$\sin\theta = \frac{L}{\rho_0}\left(1 + \frac{\delta p}{p}\right). \tag{5}$$

Therefore

$$\cos\theta = \sqrt{1 - \sin^2\theta} = \sqrt{1 - \frac{L^2}{\rho_0^2}\left(1 + \frac{\delta p}{p}\right)^2} \tag{6}$$

and we can substitute Eqs. 4 and 6 into Eq. 1 to obtain

$$x = \frac{\rho_0}{1 + \frac{\delta p}{p}}\left(1 - \sqrt{1 - \frac{L^2}{\rho_0^2}\left(1 + \frac{\delta p}{p}\right)^2}\right). \tag{7}$$

By expanding the terms inside the square root and neglecting all terms of $\frac{\delta p}{p}$ which are second order or higher, we obtain

$$x = \frac{\rho_0}{1 + \frac{\delta p}{p}}\left(1 - \sqrt{1 - \frac{L^2}{\rho_0^2} - \frac{2L^2}{\rho_0^2}\frac{\delta p}{p}}\right). \tag{8}$$

and from Eqs. 5 and 6, this can be simplified as

$$x = \frac{\rho_0}{1 + \frac{\delta p}{p}}\left(1 - \cos\theta\sqrt{1 - 2\tan^2\theta\frac{\delta p}{p}}\right). \tag{9}$$

By using the expansions $(1 + x)^{-1} = 1 - x + \ldots$ and $\sqrt{1 - x} = 1 - \frac{x}{2} + \ldots$ as well as neglecting all terms of $\frac{\delta p}{p}$ which are second

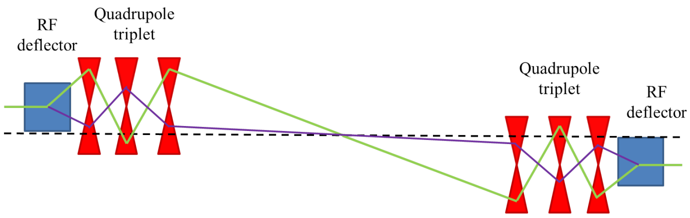


Figure 8: A schematic diagram of the orbit deviation (green) and dispersion (purple) through a symmeterised lattice depicting the transverse offset in quadrupoles required to symmeterise an asymmetric lattice.

order or higher, we can obtain the expression for position as a function of momentum offset as

$$x = \rho_0\left(1 - \cos\theta - (1 - \cos\theta)\frac{\delta p}{p} + \frac{\sin^2\theta}{\cos\theta}\frac{\delta p}{p}\right). \tag{10}$$

The dispersion is defined as $D = p\frac{\partial x}{\partial \delta p}$, therefore

$$D_x(z) = -\rho_0\left(1 - \cos\theta - \frac{\sin^2\theta}{\cos\theta}\right) = \frac{\rho_0}{\cos\theta}(1 - \cos\theta) \tag{11}$$

and

$$D'_x(z) = \frac{\sin\theta}{\cos^3\theta} = \frac{\tan\theta}{\cos^2\theta}. \tag{12}$$

Hence in a global coordinate system, the trajectory and dispersion functions after a dipole can be expressed as.

$$\begin{gathered}\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} \rho(1 - \cos\theta) \\ \tan\theta \end{pmatrix} \\ \begin{pmatrix} D_x \\ D'_x \end{pmatrix} = \begin{pmatrix} \frac{\rho}{\cos\theta}(1 - \cos\theta) \\ \frac{\tan\theta}{\cos^2\theta} \end{pmatrix}\end{gathered} \tag{13}$$

If we assume that Eq. 13 describes the change in trajectory and dispersion from the second RF deflector of the original lattice in Figure 5, the incident trajectory and dispersion, respectively, of a particle beam in order to create a dispersion-free closed orbit bump after the RF deflector can be expressed as

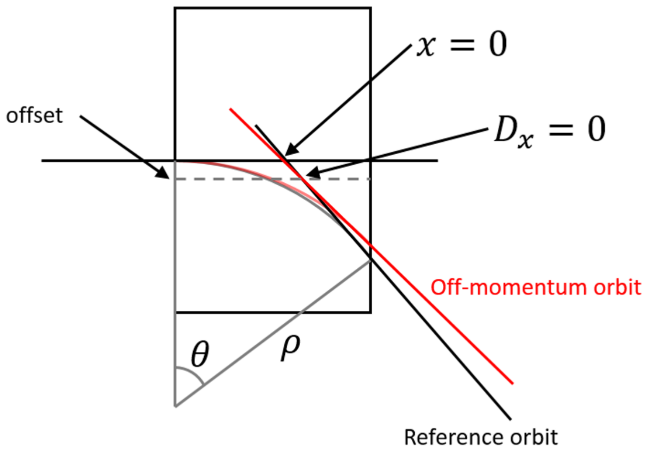


Figure 9: A schematic diagram of the orbit deviation (green) and dispersion (purple) through a symmeterised lattice depicting the transverse offset in quadrupoles required to symmeterise an asymmetric lattice.

$$\begin{pmatrix} x_0 \\ x'_0 \end{pmatrix} = \begin{pmatrix} \rho\,(1-\cos\theta) \\ -\tan\theta \end{pmatrix} \qquad \begin{pmatrix} D_x \\ D'_x \end{pmatrix} = \begin{pmatrix} \frac{\rho}{\cos\theta}\,(1-\cos\theta) \\ -\frac{\tan\theta}{\cos^2\theta} \end{pmatrix} . \tag{14}$$

Figure 9 shows that if the dipole is replaced with a drift space, $x = 0$ and $D_x = 0$ occur at different locations. We can determine the lengths, $L_x$ and $L_D$, for $x = 0$ and $D_x = 0$ respectively as

$$\begin{aligned} L_x &= -\frac{x_0}{x'_0} = \frac{\rho(1-\cos\theta)}{\tan\theta} \\ L_D &= -\frac{D_{x,0}}{D'_{x,0}} = \frac{\rho\cos^2\theta(1-\cos\theta)}{\tan\theta\cos\theta} = L_x\cos\theta \end{aligned} . \tag{15}$$

If we replace the central RF deflectors in the symmeterised lattice (Figure 5b) with drift lengths $L_D$ then the dispersion $D_x = 0$ in the centre of the lattice as required, but the trajectory has an offset $x = x_0\,(1-\cos\theta) = \rho\,(1-\cos\theta)^2$. Therefore if the elements in the first half of the lattice are given a transverse offset of $-\rho\,(1-\cos\theta)^2$ relative to the longitudinal axis of symmetry and the elements in the second half of the lattice are given a transverse offset of $\rho\,(1-\cos\theta)^2$ then $x = D_x = 0$ at the midpoint of the lattice as required.

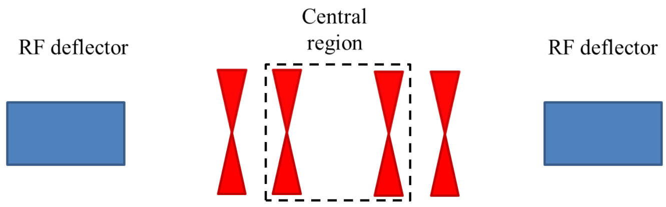


Figure 10: Diagrams to show the central region for a symmeterised lattice.

### 2.2. Central region

Having shown that a symmetric lattice must exist if any solution exists, we are able to greatly simplify the problem. By applying the symmeterisation technique described above to an arbitrary lattice, the trajectory and dispersion must form odd functions about the midpoint of the symmeterised lattice. We shall define the 'central region' as the symmetric doublet at the centre of the symmeterised lattice as depicted in Figure 10.

In order to obtain an odd function through the central region for the trajectory and dispersion, we require the following boundary conditions to be satisfied, where the subscripts 0 and 1 represents the initial and final states respectively for the trajectory and dispersion functions.

$$\begin{pmatrix} x_1 \\ x'_1 \end{pmatrix} = \begin{pmatrix} -x_0 \\ x'_0 \end{pmatrix} \qquad \begin{pmatrix} D_{x,1} \\ D'_{x,1} \end{pmatrix} = \begin{pmatrix} -D_{x,0} \\ D'_{x,0} \end{pmatrix} . \tag{16}$$

We define the transfer matrix through the central region as $\mathbf{N}$, where $\det\mathbf{N} = 1$. Therefore

$$\begin{aligned} \begin{pmatrix} x_1 \\ x'_1 \end{pmatrix} &= \begin{pmatrix} \mathbf{N}_{11} & \mathbf{N}_{12} \\ \mathbf{N}_{21} & \mathbf{N}_{22} \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix} \\ \begin{pmatrix} D_{x,1} \\ D'_{x,1} \end{pmatrix} &= \begin{pmatrix} \mathbf{N}_{11} & \mathbf{N}_{12} \\ \mathbf{N}_{21} & \mathbf{N}_{22} \end{pmatrix} \begin{pmatrix} D_{x,0} \\ D'_{x,0} \end{pmatrix} + \begin{pmatrix} D_{doub} \\ D'_{doub} \end{pmatrix} \end{aligned} \tag{17}$$

where $D_{doub}$ and $D'_{doub}$ are the residual dispersion and derivative introduced by the off-axis trajectory of the beam through the quadrupoles; these terms will be defined explicitely in Section 4. From Eqs. 16 and 17, we can define simultaneous equations which must be satisfied for the trajectory to form an odd function through the central region.

$$\begin{aligned} x_0 &= -\frac{\mathbf{N_{12}}}{1+\mathbf{N_{11}}}x_0' \\ x_0 &= \frac{1-\mathbf{N_{22}}}{\mathbf{N_{21}}}x_0' \end{aligned} \tag{18}$$

The simultaneous equations in Eq. 18 describe two lines which intersect at the point $x_0 = x_0' = 0$, which is a trivial solution. In order for non-trivial solutions to exist, we require that the two lines are equivalent, thus we obtain

$$\det \mathbf{N} = 1 + \mathbf{N_{11}} - \mathbf{N_{22}}. \tag{19}$$

Since $\det \mathbf{N} = 1$, we obtain the constraint $\mathbf{N_{11}} = \mathbf{N_{22}}$ in order to satify the condition on trajectory from Eq. 16. For the dispersion function, from Eqs. 16 and 18 we obtain the simultaneous equations

$$\begin{aligned} D_{x,0} &= \frac{x_0}{x_0'}D_{x,0}' - \frac{D_{doub}}{1+\mathbf{N_{11}}} \\ D_{x,0} &= \frac{x_0}{x_0'}D_{x,0}' - \frac{D_{doub}'}{\mathbf{N_{21}}} \end{aligned}. \tag{20}$$

Therefore in order to create a dispersion-free closed orbit bump the following condition must be satified.

$$\mathbf{N_{21}}D_{doub} - \left(1+\mathbf{N_{11}}\right)D_{doub}' = 0. \tag{21}$$

## 3. Calculation of residual dispersion

In order to determine whether a closed orbit bump through a symmeterised lattice can be designed to be dispersion-free we must evaluate Eq. 21 in terms of lattice parameters and determine under which conditions such a solution may exist. We define the transfer matrix for the quadrupoles as $\mathbf{M}$ and remind the reader that the transfer matrices for a focussing and defocussing quadrupole respectively can be expressed as

$$\begin{aligned} \mathbf{M}_f &= \begin{pmatrix} \cos\left(\sqrt{k_f}l_q\right) & \frac{\sin\left(\sqrt{k_f}l_q\right)}{\sqrt{k_f}} \\ -\sqrt{k_f}\sin\left(\sqrt{k_f}l_q\right) & \cos\left(\sqrt{k_f}l_q\right) \end{pmatrix} \\ \mathbf{M}_d &= \begin{pmatrix} \cosh\left(\sqrt{k_d}l_q\right) & \frac{\sinh\left(\sqrt{k_d}l_q\right)}{\sqrt{k_d}} \\ \sqrt{k_d}\sinh\left(\sqrt{k_d}l_q\right) & \cosh\left(\sqrt{k_d}l_q\right) \end{pmatrix} \end{aligned} \tag{22}$$

and the transfer matrix, $\mathbf{P}$, for a drift length, $L_{dr}$, as

$$\mathbf{P} = \begin{pmatrix} 1 & L_{dr} \\ 0 & 1 \end{pmatrix}. \tag{23}$$

The transfer matrix, $\mathbf{N}$, for the quadrupole doublet can be expressed as

$$\mathbf{N} = \begin{pmatrix} 2\mathbf{M}_{11}^2+L_{dr}\mathbf{M}_{11}\mathbf{M}_{21}-1 & \left(L_{dr}\mathbf{M}_{11}+2\mathbf{M}_{12}\right)\mathbf{M}_{11} \\ \left(L_{dr}\mathbf{M}_{21}+2\mathbf{M}_{11}\right)\mathbf{M}_{21} & 2\mathbf{M}_{11}^2+L_{dr}\mathbf{M}_{11}\mathbf{M}_{21}-1 \end{pmatrix}. \tag{24}$$

The residual dispersion and derivative due to the off-axis beam trajectory through a quadrupole can be expressed as [4]

$$\begin{aligned} D_q &= \mathbf{M}_{12}\left(l_q\right)\int_0^{l_q}\frac{\mathbf{M}_{11}\left(s\right)}{\rho\left(s\right)}ds - \mathbf{M}_{11}\left(l_q\right)\int_0^{l_q}\frac{\mathbf{M}_{12}\left(s\right)}{\rho\left(s\right)}ds \\ D_q' &= \mathbf{M}_{22}\left(l_q\right)\int_0^{l_q}\frac{\mathbf{M}_{11}\left(s\right)}{\rho\left(s\right)}ds - \mathbf{M}_{21}\left(l_q\right)\int_0^{l_q}\frac{\mathbf{M}_{12}\left(s\right)}{\rho\left(s\right)}ds \end{aligned}. \tag{25}$$

Where $\rho\left(s\right)$ can be expressed as[3]

$$\rho\left(s\right) = \mp\frac{\left(1+\left(\mathbf{M}_{21}x_0+\mathbf{M}_{22}x_0'\right)^2\right)^{\frac{3}{2}}}{k\left(\mathbf{M}_{11}x_0+\mathbf{M}_{12}x_0'\right)}. \tag{26}$$

Where the $\mp$ symbol represents the sign due to a focusing or defocusing quadrupole. By substituting the matrix elements from Eq. 24 and using the angle sum identities for trigonometric and hyperbolic functions, Eq. 25 can be simplified and used to describe the residual dispersion and its derivative through the first quadrupole in the central region as

$$\begin{aligned} D_{q1} &= \int_0^{l_q}\frac{\mathbf{M}_{12}\left(l_q-s\right)}{\rho\left(s\right)}ds \\ D_{q1}' &= \int_0^{l_q}\frac{\mathbf{M}_{11}\left(l_q-s\right)}{\rho\left(s\right)}ds \end{aligned}. \tag{27}$$

As the trajectory through the central region is an odd function about the midpoint of the lattice, the trajectory through the second quadrupole has the opposite sign to the first quadrupole thus the transverse deflection to the beam will have the opposite sign and the dispersion and its derivative through the second quadrupole can be expressed as

$$D_{q2} = -\int_0^{l_q} \frac{\mathbf{M}_{12}\left(l_q - s'\right)}{\rho\left(l_q - s'\right)} ds'$$
$$D'_{q2} = -\int_0^{l_q} \frac{\mathbf{M}_{11}\left(l_q - s'\right)}{\rho\left(l_q - s'\right)} ds' \quad (28)$$

By using the change of variable $s' = l_q - s$, the integrals in Eq. 28 can be expressed as

$$-\int_0^{l_q} \frac{\mathbf{M}_{ij}\left(l_q - s'\right)}{\rho\left(l_q - s'\right)} ds' = \int_{l_q}^{0} \frac{\mathbf{M}_{ij}(s)}{\rho(s)} ds = -\int_0^{l_q} \frac{\mathbf{M}_{ij}(s)}{\rho(s)} ds. \quad (29)$$

Therefore we can express $D_{q1}$ and $D'_{q1}$ respectively as

$$D_{q1} = \mathbf{M}_{11}\left(l_q\right) D_{q2} - \mathbf{M}_{12}\left(l_q\right) D'_{q2}$$
$$D'_{q1} = \mathbf{M}_{21}\left(l_q\right) D_{q2} - \mathbf{M}_{22}\left(l_q\right) D'_{q2} \quad (30)$$

Eq. 30 can be rearranged to express $D_{q2}$ and $D'_{q2}$ in terms of $D_{q1}$ and $D'_{q1}$ as

$$D_{q2} = -\mathbf{M}_{11}\left(l_q\right) D_{q1} + \mathbf{M}_{12}\left(l_q\right) D'_{q1}$$
$$D'_{q2} = -\mathbf{M}_{21}\left(l_q\right) D_{q1} + \mathbf{M}_{22}\left(l_q\right) D'_{q1} \quad (31)$$

The total residual dispersion and its derivative from the central region is

$$\begin{pmatrix} D_{doub} \\ D'_{doub} \end{pmatrix} = \begin{pmatrix} \mathbf{M}_{11} & \mathbf{M}_{12} \\ \mathbf{M}_{21} & \mathbf{M}_{22} \end{pmatrix} \begin{pmatrix} 1 & L_{dr} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} D_{q1} \\ D'_{q1} \end{pmatrix} + \begin{pmatrix} D_{q2} \\ D'_{q2} \end{pmatrix}. \quad (32)$$

Substituting Eq. 31 into Eq. 32 and using the fact that $\mathbf{M}_{11} = \mathbf{M}_{22}$, we can express the total residual dispersion from the central region in terms of the residual dispersion from the first quadrupole as

$$D_{doub} = \left(L_{dr}\mathbf{M}_{11} + 2\mathbf{M}_{12}\right) D'_{q1}$$
$$D'_{doub} = \left(L_{dr}\mathbf{M}_{21} + 2\mathbf{M}_{11}\right) D'_{q1} \quad (33)$$

Substituting Eqs. 24 and 33 into Eq. 21, we obtain the condition for a dispersion-free closed orbit bump in terms of lattice parameters.

$$-2\left(L_{dr}\mathbf{M}_{21} + 2\mathbf{M}_{11}\right) D'_{q1} = 0 \quad (34)$$

Therefore in order to create a dispersion-free closed orbit bump, we require that either $L_{dr}\mathbf{M}_{21} + 2\mathbf{M}_{11} = 0$ or $D'_{q1} = 0$.

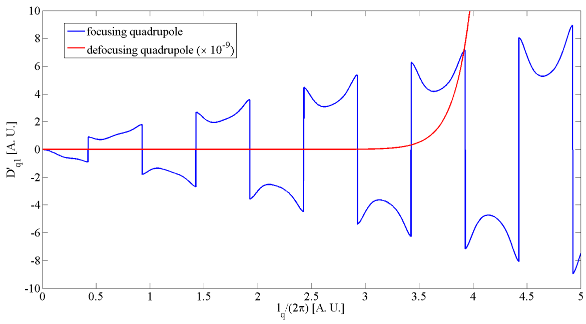


Figure 11: $D'_{q1}$ vs. $l_q$ for a focusing (blue) and defocusing (red) quadrupole.

## 4. Results

### 4.1. $D'_{q1} = 0$

In order to determine under what conditions $D'_{q1} = 0$ occurs, Eqs. 18 and 26 are substituted into Eq. 27, the function was integrated numerically in Matlab and the results shown in Figure 11. For the defocusing quadrupole $D'_{q1}$ increases monotonically and $D'_{q1} = 0$ only occurs at $l_q = 0$ or $x_0 = x'_0 = 0$, which are trivial solutions. For the focusing quadrupole, non-trivial solutions for $D'_{q1} = 0$ occur when $x_0 = 0$, which implies $L_{dr}\mathbf{M}_{11} + 2\mathbf{M}_{12} = 0$ from Eq. 18.

As we require $x_0 = 0$ in order for $D'_{q1} = 0$ for a focusing quadrupole, from Eq. 26, $D'_{q1}$ can be expressed as

$$D'_{q1} = -\sqrt{k_f}x'_0 \int_0^{l_q} \frac{\cos\left(\sqrt{k_f}\left(l_q - s\right)\right)\sin\left(\sqrt{k_f}s\right)}{\left(1 + x_0'^2\cos^2\left(\sqrt{k_f}s\right)\right)^{\frac{3}{2}}} ds \quad (35)$$

where $l_q = \frac{1}{\sqrt{k_f}}\left(\pi - \tan^{-1}\left(\frac{\sqrt{k_f}L_{dr}}{2}\right)\right)$. Eq. 35 is an elliptic integral of the third kind and $D'_{q1} = 0$ only occurs when $x_0 = x'_0 = 0$ (Figure 12) or when $\sqrt{k}l_q = 0$ (Figure 11) for a focusing quadrupole, both of which are trivial solutions. Therefore there are no non-trivial solutions for $D'_{q1} = 0$ for any linear lattice which can simultaneously close an off-axis orbit and dispersion bump.

### 4.2. $L_{dr}\mathbf{M}_{21} + 2\mathbf{M}_{11} = 0$

From Eq. 22 we can state that $\det\mathbf{M} = 1$ and $\mathbf{M}_{11} = \mathbf{M}_{22}$, from which we obtain

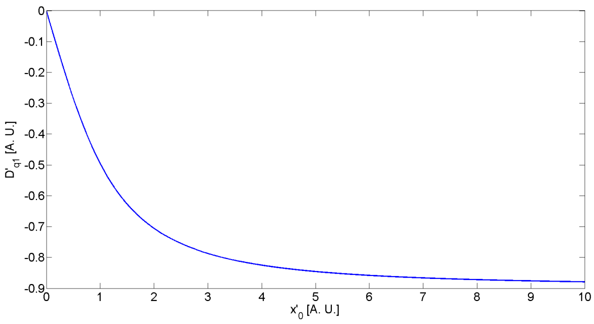


Figure 12: $D'_{q1}$ vs. $x'_0$ for a focusing quadrupole when $x_0 = 0$.

$$\mathbf{M}_{21} = \frac{\mathbf{M}_{11}^2 - 1}{\mathbf{M}_{12}}. \tag{36}$$

Substituting Eq. 36 into $L_{dr}\mathbf{M}_{21} + 2\mathbf{M}_{11} = 0$, we obtain

$$L_{dr}\mathbf{M}_{11} + 2\mathbf{M}_{12} = \frac{L_{dr}}{\mathbf{M}_{11}}. \tag{37}$$

From Eq. 33 we know that $D'_{doub} = 0$ if $L_{dr}\mathbf{M}_{21} + 2\mathbf{M}_{11} = 0$. From Eq. 20 we also require that $D_{doub} = 0$ for $D_{x,0}$ to be finite as $\mathbf{N}_{11} + 1 = 0$. As we have shown that $D'_{q1} = 0$ only leads to trivial solutions, from Eq. 33 we require that $L_{dr}\mathbf{M}_{11} + 2\mathbf{M}_{12} = 0$, and from Eq. 37 this implies that $L_{dr} = 0$ as $\mathbf{M}_{11}$ must be finite for a physical solution to exist.

By substituting $L_{dr} = 0$ into the two conditions which must be satified for a dispersion-free solution to exist we obtain

$$\begin{aligned} L_{dr}\mathbf{M}_{21} + 2\mathbf{M}_{11} = 0 \Rightarrow \mathbf{M}_{11} = 0 \\ L_{dr}\mathbf{M}_{11} + 2\mathbf{M}_{12} = 0 \Rightarrow \mathbf{M}_{12} = 0 \end{aligned}. \tag{38}$$

Therefore if $L_{dr}\mathbf{M}_{21} + 2\mathbf{M}_{11} = 0$, we require that $\det \mathbf{M} = 0$ for a solution to exist, which is unphysical. Therefore the condition $L_{dr}\mathbf{M}_{11} + 2\mathbf{M}_{12} = 0$ only leads to unphysical solutions and the condition $D'_{q1} = 0$ only leads to the trivial solutions $x_0 = x'_0 = 0$ or $\sqrt{k}l_q = 0$; hence there are no non-trivial solutions for dispersion-free closed orbit bump using a lattice comprised entirely of linear elements.

## 5. Nonlinear Extension

Having proven that no linear solution exists which can simultaneously close multiple orbit bumps and dispersion functions, we can consider the case where higher-order multipoles are used to create a nonlinear optical system. If we first start by considering a beam traveling on-axis through a nonlinear lattice, the horizontal position of a particle at the end of the lattice can be expressed as

$$x_1 = \sum_i \mathbf{R}_{1,i}X_{i,0} + \sum_{i,j} \mathbf{T}_{1,i,j}X_{i,0}X_{j,0} + \sum_{i,j,k} \mathbf{U}_{1,i,j,k}X_{i,0}X_{j,0}X_{k,0} + \ldots, \tag{39}$$

where $\mathbf{R}$ is the linear transfer matrix, $\mathbf{T}$ is the second order tensor, $\mathbf{U}$ is the third order tensor and $X_0$ is the phase space coordinate of a particle at the start of the lattice. Dispersion is the momentum dependence of the transverse position and can be defined as $D_x = p\frac{\partial x_1}{\partial \delta p}$, where $p$ is the nominal momentum of the beam and $\delta p$ is the momentum deviation of a particle from the nominal momentum. Using this definition of dispersion with Eq. 39, we obtain

$$D_x = \mathbf{R}_{1,6} + 2\sum_i \mathbf{T}_{1,6,i}X_{i,0} + 3\sum_{i,j} \mathbf{U}_{1,6,i,j}X_{i,0}X_{j,0} + \ldots \tag{40}$$

If we consider only terms dependent on $x_0$ and $x'_0$ in Eq. 40 and consider the start of the lattice to be at the end of the first RF deflector in the bump region, then from Eq. 13, we can express $x' = \frac{\tan\theta}{\rho(1-\cos\theta)}x = \alpha x$ and the dispersion as

$$D_x = \mathbf{R}_{1,6} + 2tx_0 + 3ux_0^2 + \ldots \tag{41}$$

where

$$\begin{aligned} t &= \mathbf{T}_{1,1,6} + \alpha\mathbf{T}_{1,2,6} \\ u &= \mathbf{U}_{1,1,1,6} + 2\alpha\mathbf{U}_{1,1,2,6} + \alpha^2\mathbf{U}_{1,2,2,6} \end{aligned}. \tag{42}$$

As $x_0$ is the horizontal displacement of a particle within the bunch and we assume the bunch size is small, we can assume that $x_0 \ll 1$ and therefore in the on-axis case, the dispersion can be approximated as $D_x = \mathbf{R}_{1,6}$.

However, as we need to consider off-axis trajectories through a nonlinear lattice, we can include a displacement of the bunch centroid, $x_b$, in Eq. 41 and we obtain

$$D_x = \mathbf{R}_{1,6} + 2t\,(x_0 + x_b) + 3u\,(x_0 + x_b)^2 + \ldots$$
$$= \left(\mathbf{R}_{1,6} + 2tx_b + 3ux_b^2 + \ldots\right) + (2t + 6ux_b + \ldots)\,x_0 + 3ux_0^2 + \ldots \quad (43)$$

As previously stated, we assume that $x_0 \ll 1$, hence we can neglect any terms dependent on $x_0$ and the dispersion from an off-axis trajectory becomes

$$D_x = \mathbf{R}_{1,6} + 2tx_b + 3ux_b^2 + \ldots \quad (44)$$

where each higher order term is a perturbation to the linear transfer matrix due to traveling off-axis through a higher order multipole. Hence traveling off-axis through a multipole introduces lower order multipole terms and to determine the dispersion through a nonlinear lattice, we only need to consider dipole and quadrupole terms introduced. If we now consider the magnetic field experienced by a particle traveling on-axis through a multipole we obtain

$$\mathbf{B}_{y,n} = \frac{pc}{e} k_n x_0^n \quad (45)$$

where $n$ is the order of the multipole where $n = 0$ is a dipole, $n = 1$ is a quadrupole and so forth. The magnetic field experienced by a particle traveling off-axis through the multipole is

$$\mathbf{B}_{y,n} = \tfrac{pc}{e} k_n\,(x_0 + x_b)^n$$
$$= \tfrac{pc}{e} k_n \left(x_0^n + nx_b x_0^{n-1} + \ldots + nx_b^{n-1} x_0 + x_b^n\right) . \quad (46)$$

By only considering the dipole and quadrupole terms from Eq. 46 we obtain

$$\tilde{\mathbf{B}}_{y,n} = \frac{pc}{e} n x_b^{n-1} k_n \left(x_0 + \frac{x_b}{n}\right) = \frac{pc}{e} \tilde{k}_n \left(x_0 + \frac{x_b}{n}\right) \quad (47)$$

If we now consider the magnetic field experienced by the beam due to traveling off-axis through a quadrupole, we obtain

$$\mathbf{B}_{y,2} = \frac{pc}{e} k_2\,(x_0 + x_b)\,. \quad (48)$$

By comparing Eqs. 47 and 48, we can see that traveling off-axis through a higher order multipole with a horizontal displacement $x_b$ it is equivalent to traveling off-axis through a quadrupole with a horizontal displacement $\frac{x_b}{n}$. As we have previously shown that there exists no linear lattice in which a dispersion free closed off-axis orbit bump exists, hence no such solution can exist in the nonlinear case either; although as the order of the multipole, $n$, increases, the trajectory asymptotically converges with a closed solution.

As the order of the multipole increases, the dispersion becomes increasingly more strongly dependent on the beam position. In a real machine it is likely that the residual dispersion introduced by beam jitter would become a limiting factor on the maximum order multipole that could be used in such a system.

## 6. Summary

In this paper we have shown that there is no possible linear solution to simultaneously close orbit and dispersion functions. We showed that if a solution exists then it must be possible to create a symmetric lattice which is also a solution. For a symmetric lattice, both the orbit and dispersion must be either symmetric or anti-symmetric about the midpoint of the lattice. This allows us to investigate just the central region of the lattice to determine whether a solution is possible. By considering a quadrupole singlet at the centre of the injection region, we are able to draw conclusions about any lattice consisting of an odd number of quadrupoles. Similarly by considering a doublet at the centre, we are able to draw conclusions about any lattice consisting of an even number of quadrupoles. By considering a quadrupole singlet as the special case of a quadrupole doublet with a drift length $L_{dr} = 0$, we are able to investigate all cases and show that no non-trivial linear solutions exist.

After proving that no linear solution exists, we were able to extend the proof to nonlinear optical systems. By considering the multipole terms experienced by an off-axis beam and by neglecting small terms, we were able to show that the lattice is equivalent to traveling off-axis through a quadrupole with a smaller offset. By relating this to the proof for linear optics, we were able to show that no non-trivial nonlinear optics exist; thus completing the proof that no solution exists to simultaneously correct multiple local orbit bumps and dispersion functions with linear or nonlinear optics.

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