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On the performance of overlapping and non-overlapping temporal demand aggregation approaches

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Abstract

Temporal demand aggregation has been shown in the academic literature to be an intuitively appealing and effective approach to deal with demand uncertainty for fast moving and intermittent moving items. There are two different types of temporal aggregation: non-overlapping and overlapping. In the former case, the time series are divided into consecutive non-overlapping buckets of time where the length of the time bucket equals the aggregation level. The latter case is similar to a moving window technique where the window's size is equal to the aggregation level. At each period, the window is moved one step ahead, so the oldest observation is dropped and the newest is included. In a stock-control context, the aggregation level is generally set to equal the lead-time. In this paper, we analytically compare the statistical performance of the two approaches. By means of numerical and empirical investigations, we show that unless the demand history is short, there is a clear advantage of using overlapping blocks instead of the non-overlapping approach. It is also found that the margin of this advantage becomes greater for longer lead-times.

Keywords: Temporal aggregation, overlapping, non-overlapping, empirical investigation.

1. Introduction

Demand forecasting is the starting point for most planning and control organisational activities. Moreover, one of the most important challenges facing modern companies is demand uncertainty (Chen and Blue, 2010; Rostami-Tabar et al., 2013). High variability in demand for both fast moving and slow or intermittent moving items (items with a high proportion of zero observations) pose considerable difficulties in terms of forecasting and stock control (Syntetos and Boylan, 2001; Teunter et al., 2010; Strijbosch et al., 2011). Stock control is particularly challenging in military, aerospace, automotive and other sectors in which there is a volatile demand across a wide variety of stock keeping units (SKUs).

Fast-moving demand may be subject to erratic 'spikes'. Similarly, slow moving or intermittent demand may be 'lumpy' with infrequent high volume of demand. In both cases, the data is not only noisy but, often, its distribution does not confirm to any of the standard distributions, such as the normal or the negative binomial.

There are many approaches that may be used to address the non-standard demand patterns often observed in practice. Non-parametric approaches are promising because, by definition, the standard shapes of parametric distributions do not need to be assumed. An advantage of using parametric methods is that it is usually straightforward to find the distribution of lead-time demand from the distribution of time per period. In the non-parametric case, we also need to find this lead-time distribution. An appealing approach to address this problem is known as temporal aggregation (Nikolopoulos et al., 2011; Babai et al., 2012; Syntetos, 2014; Kourentzes and Petropoulos, 2015).

Temporal aggregation refers to the process by which a low frequency time series (e.g. quarterly) is derived from a high frequency time series (e.g. monthly). Such aggregation may reduce the coefficient of variation of the data, thereby allowing for more accurate forecasts. This approach is particularly appealing when forecasts of total demand over an aggregated time period are needed; this is a common requirement in inventory control. Note that there is another aggregation approach discussed in the literature and often applied in practice that may also lead to the reduction of demand uncertainty and the improvement of stock control performance. This approach is referred to as cross-sectional

aggregation, which involves aggregating different time series in order to improve performance across a group of items (Kefeng and Philip, 2003; Zhang and Burke, 2011; Rostami-Tabar et al., 2015).

With regards to temporal aggregation, there are two different types of aggregation: non-overlapping and overlapping. In the former case, the time series are divided into consecutive non-overlapping buckets of time where the length of the time bucket equals the aggregation level. The latter case is similar to a moving window technique where the window's size equals the aggregation level. At each period, the window is moved one step ahead, and so the oldest observation is dropped and the newest is included.

The potential forecasting benefit of non-overlapping temporal aggregation was recognised by Willemain et al. (1994) for intermittent demand. In this context, Nikolopoulos et al. (2011) and Babai et al. (2012) analysed the empirical forecasting and stock control performance of the non-overlapping temporal aggregation approach. They have shown that an aggregation approach may offer considerable improvements in forecasting and stock control. In the context of fast-moving demand following an Auto-Regressive Moving Average (ARMA) demand processes, Rostami-Tabar et al. (2013, 2014) analysed the effect of non-overlapping temporal aggregation on demand forecasting. They showed that, for high values of positive autocorrelation in the disaggregated demand, the non-overlapping aggregation approach is outperformed by non-aggregation. More recently, Kourentzes et al. (2014) proposed a Multi Aggregation Prediction Algorithm (MAPA) which uses multiple non-overlapping temporal aggregation levels simultaneously and combines the forecasts to produce an overall

forecast. They found that MAPA improved forecasting accuracy, in terms of both error variance and bias.

The overlapping temporal aggregation approach has also been discussed in the literature. Among others, Mohammadipour and Boylan (2012) have analysed the theoretical and empirical forecasting outperformance of overlapping temporal aggregation under Integer ARMA processes. Porras and Dekker (2008) have shown good stock control performance for overlapping temporal aggregation, based on an empirical investigation conducted with a Dutch petrochemical complex.

Although the literature on the overlapping and non-overlapping temporal aggregation approaches has been growing in recent years, these two aggregation approaches have never been compared, to the best of our knowledge. This constitutes the objective of this work in which we attempt to establish some theoretical properties of the approaches. Based on these properties, we compare the performance of the two approaches through numerical and empirical investigations.

The contribution of this paper is threefold:

1. We derive the variance expression of the overlapping blocks estimator of the cumulative demand distribution for independently and identically distributed (i.i.d.) demand;

2. We provide a general condition, for i.i.d. demand, for the variance of the overlapping blocks approach to be lower than the non-overlapping blocks (NOB) approach;

3. We show that there exists situations under which the non-overlapping blocks approach is less variable than the overlapping one.

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The remainder of the paper is structured as follows. In Section 2, we describe the assumptions and notations used in this study, and we present the theoretical comparative analysis of the two aggregation approaches. More results are provided in Section 3 when the demand is Poisson distributed. A numerical evaluation of the theoretical results obtained for Poisson distributed demand is presented in Section 4, followed by an empirical analysis conducted in Section 5. The paper concludes in Section 6 with a summary of the paper's results, along with suggestions for further research in this area.

2. Theoretical analysis

2.1 Assumptions and notations

The aim of this section is to provide some theoretical properties of aggregated demand under both temporal aggregation approaches.

We assume that historical demand values are observed from an independently and identically distributed (i.i.d) time series $\{Y_1, Y_2, ..., Y_n\}$. For the reminder of the paper, we adopt the following notation:

n : length of the demand history (i.e. the number of observed demand values);

m : aggregation level (i.e. the number of demand values to be aggregated);

y : value of the aggregation of m successive demands

$F_m(y)$: Population cumulative distribution function (CDF) of the aggregation of m successive demand values;

$\hat{F}_m^{OB}(y)$: estimated cumulative distribution function of the aggregated demand under the Overlapping Blocks (OB) approach;

$\hat{F}_m^{NOB}(y)$: estimated cumulative distribution function of the aggregated demand under the Non-Overlapping Blocks (NOB) approach;

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$S_m^{NOB}(CSL)$: Order-Up-To-Level (OUTL) calculated with the NOB approach for a given

Cycle Service Level target, CSL;

$S_m^{OB}(CSL)$: OUTL calculated with the OB approach for a given Cycle Service Level target, CSL.

2.2 Analytical results

In this subsection, we first recall some known statistical properties of the NOB estimator,

namely the expressions of the expected value and the variance of the estimator, $\hat{F}_m^{NOB}(y)$.

Then, we derive the same statistical properties of the OB estimator, $\hat{F}_m^{OB}(y)$.

Under the NOB approach, it is straightforward to establish the bias and variance

properties of the estimated Cumulative Distribution Function (CDF). The estimator

$\hat{F}_m^{NOB}(y)$ is an unbiased estimator of the population CDF, i.e.

$$E\left[\hat{F}_m^{NOB}(y)\right]= F_m(y) \tag{1}$$

In addition, the variance of $\hat{F}_m^{NOB}(y)$ is given by:

$$Var\left[\hat{F}_m^{NOB}(y)\right]= \frac{F_m(y)}{k} - \frac{F_m(y)^2}{k} \tag{2}$$

where: $k = \dfrac{n}{m}$ (this result is based on the assumption that n is an integer multiple of m).

Under the OB approach, the expectation and the variance of the estimator $\hat{F}_m^{OB}(y)$ are

given in Proposition 1.

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Proposition 1.

The mean and variance of the estimator $\hat{F}_m^{OB}(y)$ *are given by i) and ii) respectively.*

i) $E\left[\hat{F}_m^{OB}(y)\right] = F_m(y)$

ii) $Var\left[\hat{F}_m^{OB}(y)\right] = \dfrac{F_m(y)}{k} + \left[\dfrac{(k-m)(k-m+1)}{k^2} - 1\right]F_m(y)^2 + \dfrac{1}{k^2}\sum_{s=1}^{m-1}2(k-m+s)\Theta_s(y)$

where

$k = n - m + 1$ *and* $\Theta_s(y) = \sum_{Y_1=0}^{y}\ldots\ldots\sum_{Y_m=0}^{y-Y_1-\ldots-Y_{m-1}}\sum_{Y_{m+1}=0}^{y-Y_{m-s+1}-\ldots-Y_m}\ldots\ldots\sum_{Y_{2m-s}=0}^{y-Y_{m-s+1}-\ldots-Y_{2m-s-1}}\prod_{j=1}^{2m-s}p(Y_j)$

The proof of Proposition 1 is given in *Appendix* A. The first term on the right-hand-side of the variance expression represents the cross-terms, in the calculation of expected squares, with complete commonality of time periods. The second term represents the cross-terms with no commonality. The final term represents cross-terms with partial commonality of time periods, with the variable *s* signifying the number of common periods. The variable $\Theta_s(y)$ represents the probability that the sum of the observations in each block of a pair, with *s* common periods between the two blocks, does not exceed the value y.

Both approaches lead to unbiased estimators for i.i.d. demand. Moreover, the variances of the estimator under both approaches are functions of $F_m(y)$ and $F_m(y)^2$. Under the overlapping approach, there is in addition a new term $\Theta_s(y)$ that results from taking into account the correlation between the blocks, when estimating the cumulative probability

of the aggregated demand. Furthermore, it is obvious that when $m = 1$, $Var\left[\hat{F}_m^{OB}(y)\right]$

reduces to $Var\left[\hat{F}_m^{NOB}(y)\right]$, as the two approaches are the same.

In the remainder of the paper, the performance of the two approaches will be compared

based on the analytical results presented above. In the theoretical analysis, the

outperformance of an approach will be assessed by the reduction in the variance of the

CDF (Cumulative Distribution Function) estimates. In the empirical analysis, stock

control performance will also be examined.

2.3 Comparative analysis

In this subsection, we find the condition under which the NOB approach gives a lower

variance of the estimate of the cumulative distribution, for a given value of y, than the

OB approach. The condition follows from Proposition 1 and is expressed below as

Proposition 2.

Proposition 2
The NOB approach has a lower variance of the estimate of the cumulative distribution,
for a given value of y, than the OB approach, for aggregation level m, and length of
history n (a multiple of m) if and only if:

$$\sum_{s=1}^{m-1}\frac{2(n-2m+s+1)}{(n-m+1)^2}\Theta_s(y) > \left[\frac{m}{n} - \frac{1}{(n-m+1)}\right]F_m(y) + \left[1 - \frac{m}{n} - \frac{(n-2m+1)(n-2m+2)}{(n-m+1)^2}\right]F_m(y)^2$$

Proof of Proposition 2.

The relative reduction in the variance of the estimate of the CDF stemming from the use

of the OB approach instead of NOB, for given values of block size (m), length of history

(n) and aggregate demand over the block (y), $\Delta(m,n,y)$, may be defined as:

$$\Delta(m, n, y) = \frac{Var\left[\hat{F}_m^{OB}(y)\right] - Var\left[\hat{F}_m^{NOB}(y)\right]}{Var\left[\hat{F}_m^{NOB}(y)\right]} \tag{3}$$

The non-overlapping blocks approach outperforms the overlapping one if and only if $\Delta(m, n, y) > 0$, which occurs when:

$$\frac{F_m(y)}{(n-m+1)} + \left[\frac{(n-2m+1)(n-2m+2)}{(n-m+1)^2} - 1\right] F_m(y)^2$$
$$+ \frac{1}{(n-m+1)^2} \sum_{s=1}^{m-1} 2(n-2m+s+1)\Theta_s(y) - \frac{m}{n}\left(F_m(y) - F_m(y)^2\right) > 0 \tag{4}$$

Or equivalently:

$$\frac{1}{(n-m+1)^2} \sum_{s=1}^{m-1} 2(n-2m+s+1)\Theta_s(y) > \left[\frac{m}{n} - \frac{1}{(n-m+1)}\right] F_m(y)$$
$$+ \left[1 - \frac{m}{n} - \frac{(n-2m+1)(n-2m+2)}{(n-m+1)^2}\right] F_m(y)^2 \tag{5}$$

which ends the proof of Proposition 2. □

Proposition 2 shows that for the simplest non-degenerate case, $m = 2$, the general condition for the outperformance of the NOB approach is:

$$\Theta_1(y) > \frac{(n-1)}{2n} F_2(y) + \frac{(n+1)}{2n} F_2(y)^2 \tag{6}$$

This inequality is not highly sensitive to the value of n, as will be demonstrated in Section 4 of this paper. However, there are some sensitivities to low values of n. For example, if $F_2(y) = 0.8$, then the right-hand side of the inequality takes the value 0.6933 for $n = 3$, but a value of 0.7133 for $n = 12$. Therefore, we may expect short demand histories to favour NOB, all other things being equal.

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If we keep n fixed, and consider the effect of the distribution on the inequality, it is clear that distributions which have higher values of $\Theta_1(y)$, with the same cumulative distributions, $F_2(y)$, will also favour NOB. These are often associated with more highly intermittent series. Please note that demand intermittence has been characterised in the academic literature by various criteria (Williams, 1984; Johnston and Boylan, 1996; Syntetos et al., 2005). In this paper, the probability of zero demand is the considered criterion, i.e. we assume that a higher intermittence of the demand is characterised by a higher probability of the demand being equal to zero. For example, a series with $p(0) = 0.8$ and $p(1) = 0.1$ is more intermittent than a series with $p(0) = 0.6$ and $p(1) = 0.3666$ but has the same cumulative value $F_2(1) = 0.8$. However, it has a higher value of $\Theta_1(1)$ (= 0.7120) than the less intermittent series, which has a lower value of $\Theta_1(1)$ (=0.6926). Thus, the less intermittent series favours the OB approach, for all values of $n \geq 3$, whereas the more highly intermittent series favours the NOB approach for $n = 3$ and, on further analysis of the right-hand side of the inequality, for $n \leq 9$. Thus, more intermittent demand histories would seem to favour the NOB approach, all other things being equal. It should be noted that these insights have arisen from consideration of the case of $m = 2$. Examples for $m = 3$ will be examined later in the paper.

We now analyse the asymptotic comparative performance of both approaches, i.e. the performance for large values of the length of the demand history n.

Proposition 3

Asymptotically (i.e. as $n \to \infty$), the NOB approach has a lower variance of the estimate of the cumulative distribution than the OB approach if and only if :

$$\frac{1}{(m-1)} \sum_{s=1}^{m-1} \Theta_s(y) > \frac{1}{2}\left[F_m(y) + F_m(y)^2 \right]$$

where the notation is unchanged from Proposition 2.

Proof of Proposition 3.

For large values of n, the above condition can be written as:

$$\sum_{s=1}^{m-1} \Theta_s(y) > \frac{(n-m+1)^2}{2n}\left[\frac{m}{n} - \frac{1}{(n-m+1)} \right] F_m(y)$$
$$+ \frac{(n-m+1)^2}{2n}\left[1 - \frac{m}{n} - \frac{(n-2m+1)(n-2m+2)}{(n-m+1)^2} \right] F_m(y)^2 \qquad (7)$$

which is equivalent to:

$$\sum_{s=1}^{m-1} \Theta_s(y) > \frac{(n-m+1)^2}{2n^2}\left[\frac{(m-1)n - m^2 + m}{(n-m+1)} \right] F_m(y)$$
$$+ \frac{1}{2n^2}\left[(m-1)n^2 - nm^2 + m - m^3 + 2m^2 + 2mn - n \right] F_m(y)^2 \qquad (8)$$

Hence, asymptotically, when $n \to \infty$, $\Delta(m.n.y)$ (i.e. NOB outperforms OB) if and only

if $\frac{1}{(m-1)} \sum_{s=1}^{m-1} \Theta_s(y) > \frac{1}{2}\left[F_m(y) + F_m(y)^2 \right]$

This leads to Proposition 3. □

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Proposition 3 shows that, asymptotically, the comparative performance of the two approaches is determined by the relation between the variables $\sum_{s=1}^{m-1} \Theta_s(y)$ and $F_m(y)$. The outperformance conditions presented in Proposition 2 and Proposition 3 will be analysed numerically in Section 4.

3. Analytical results under Poisson distributed demand

We now investigate the comparative performance of the two approaches when the demand is distributed according to a Poisson distribution with parameter λ, i.e. when the probability of having k demands per time unit is given by:

$$p(k) = e^{-\lambda} \frac{\lambda^k}{k!} \text{ for any integer } k \geq 0.$$

Before proceeding to the special case of Poisson demand, we recall that $\Theta_s(y)$ and $F_m(y)$ can be written as follows:

$$\Theta_s(y) = \sum_{Y_1=0}^{y} \cdots \sum_{Y_m=0}^{y-Y_1-\ldots-Y_{m-1}} \sum_{Y_{m+1}=0}^{y-Y_{m-s+1}-\ldots-Y_m} \cdots \cdots \sum_{Y_{2m-s}=0}^{y-Y_{m-s+1}-\ldots-Y_{2m-s-1}} \prod_{j=1}^{2m-s} p(Y_j) \tag{9}$$

and $$F_m(y) = \sum_{Y_1=0}^{y} \sum_{Y_2=0}^{y-Y_1} \sum_{Y_3=0}^{y-Y_1-Y_2} \cdots \cdots \sum_{Y_m=0}^{y-Y_1-\ldots-Y_{m-1}} \prod_{j=1}^{m} p(Y_j) \tag{10}$$

The relative benefit stemming from the use of OB instead of NOB, denoted by $\Delta(m,n,y)$, can be written as following for the general case:

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$$\Delta(m,n,y) = \frac{\dfrac{F_m(y)}{(n-m+1)} + \left[\dfrac{(n-2m+1)(n-2m+2)}{(n-m+1)^2} - 1\right] F_m(y)^2 + \displaystyle\sum_{s=1}^{m-1} \dfrac{2(n-2m+s+1)}{(n-m+1)^2} \Theta_s(y)}{\dfrac{m}{n}\left[F_m(y) - F_m(y)^2\right]} - 1 \quad (11)$$

which is equivalent to:

$$\Delta(\mathrm{m},n,y) = \frac{\dfrac{1}{(n-m+1)} + \left[\dfrac{(n-2m+1)(n-2m+2)}{(n-m+1)^2} - 1\right] F_m(y) + \displaystyle\sum_{s=1}^{m-1} \dfrac{2(n-2m+s+1)}{(n-m+1)^2} \dfrac{\Theta_s(y)}{F_m(y)}}{\dfrac{m}{n}\left[1 - F_m(y)\right]} - 1 \quad (12)$$

Now, for Poisson demand, we may establish the asymptotic result given in Proposition 4.

Proposition 4

When λ tends towards infinity, the relative benefit stemming from the use of the OB approach tends towards $\dfrac{n}{m(n-m+1)} - 1$. In addition, when both λ and n tend towards infinity, the relative benefit stemming from the use of the OB approach tends to $\dfrac{1-m}{m}$.

The proof of Proposition 4 is given in Appendix B. The significance of the result lies in the large reductions in variance that may be accrued by using the OB approach, instead of NOB. For an aggregation level of $m = 2$, reductions of up to 50% may be attained for very long demand histories and high mean demands. For higher values of m, the reduction in variance is even higher.

4. Numerical analysis

For the purpose of the numerical analysis, we consider the case of Poisson distributed demand. We first numerically compare the outperformance conditions presented in Proposition 2 and Proposition 3 for $m = 2$. Then we numerically analyse the comparative performance of the two approaches for different values of n, m and λ.

Let $\Theta_1^{*}(2,n,y)$ denotes the value of $\Theta_1(2,n,y)$ at which the NOB approach has the same variance as the OB approach for $m=2$, i.e.

$$\Theta_1^{*}(2,n,y) = \frac{(n-1)}{2n} F_2(y) + \frac{(n+1)}{2n} F_2(y)^2 \tag{13}$$

Then, the asymptotic condition of the outperformance of the NOB approach is reached at the following limiting value:

$$\lim_{n\to\infty} \Theta_1^{*}(2,n,y) = \frac{1}{2}\left[F_2(y) + F_2(y)^2\right] \tag{14}$$

In Table 1, we show some numerical values of $\Theta_1^{*}(2,n,y)$ and $\lim_{n\to\infty} \Theta_1^{*}(2,\infty,y)$ for different values of n.

Table 1. Numerical comparison of the outperformance conditions

n	$F_2(y)$	$\Theta_1^{*}(2,n,y)$	$\lim_{n\to\infty} \Theta_1^{*}(2,n,y)$
	0.90	0.844	0.855
4	0.95	0.920	0.926
	0.99	0.984	0.985
	0.90	0.849	0.855
8	0.95	0.923	0.926
	0.99	0.984	0.985
	0.90	0.851	0.855
12	0.95	0.924	0.926
	0.99	0.985	0.985
	0.90	0.853	0.855
24	0.95	0.925	0.926
	0.99	0.985	0.985

The results of Table 1 show that the value of $\Theta_1^{*}(2,n,y)$ is not highly sensitive to the value of n. Consequently, the gap between $\Theta_1^{*}(2,n,y)$ and its asymptotic value is low

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for low values of n and becomes very low for $n \geq 12$, with almost no difference (to the third decimal place) for $F_2(y) = 0.99$. This shows that for such demand histories, the asymptotic condition is a very good approximation for assessing the comparative performance of the two approaches.

In Figure 1, we plot the variation of $\Delta(m, n, y)$ with respect to the length of the demand history n and the mean demand λ for $y = 1$ and for $m = 2$ and $m = 3$. To simplify the presentation of the results, we do not show findings for $m \geq 4$ but it is straightforward to extend to these cases.
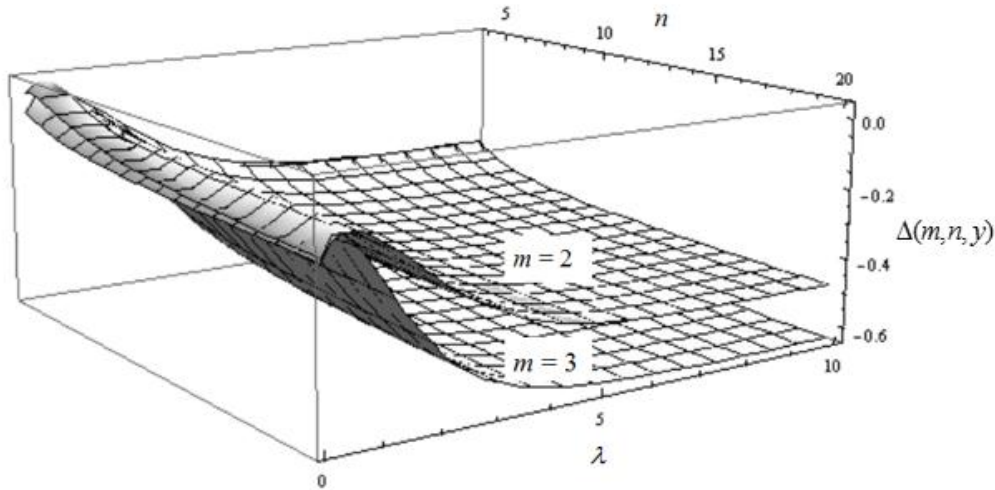

Figure 1. The relative benefit of the OB approach for Poisson distributed demand

The results in Figure 1 show that the ratio $\Delta(m, n, y)$ is negative in almost all cases, which means that, under the Poisson assumption, the OB approach leads to a variance reduction, which shows the outperformance of this approach. The results also show that, for very low values of λ, i.e. slow moving demand, the relative benefit from using OB instead of NOB is low when n is low. Moreover, there are very few values where the

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relative benefit is positive, which correspond to cases where both λ and *n* are very low. Figure 1 illustrates that the relative benefit stemming from the use of OB instead of NOB increases substantially as the aggregation level *m* increases from 2 periods to 3 periods. Further increases will follow as *m* increases to 4 periods or more.

In Table 2, we show the benefit of using the OB approach as compared to the NOB for an infinite value of λ (i.e. as an approximation for a very fast moving item) and different values of *n*. Again the results are reported for *m* = 2 and *m* = 3.

Table 2. Asymptotic benefit under Poisson distributed demand

n		6	12	18	24	30	∞
% benefit	*m* = 2	-40.00	-45.45	-47.06	-47.83	-48.28	-50.00
	m = 3	-50.00	-60.00	-62.50	-63.64	-64.29	-66.67

The numerical results in Table 2 show that for a very fast moving item, characterised by Poisson distributed demand, the benefit of using the OB approach is very high varying from 40% to 50% for *m* = 2 and from 50% to 66.7% for *m* = 3. Obviously, this benefit is relatively lower for slow moving items, i.e. for lower values of λ. To illustrate this effect, we report in Figure 2 the relative benefit as a function of *n* for $\lambda = 0.5$ and *y* = 1.
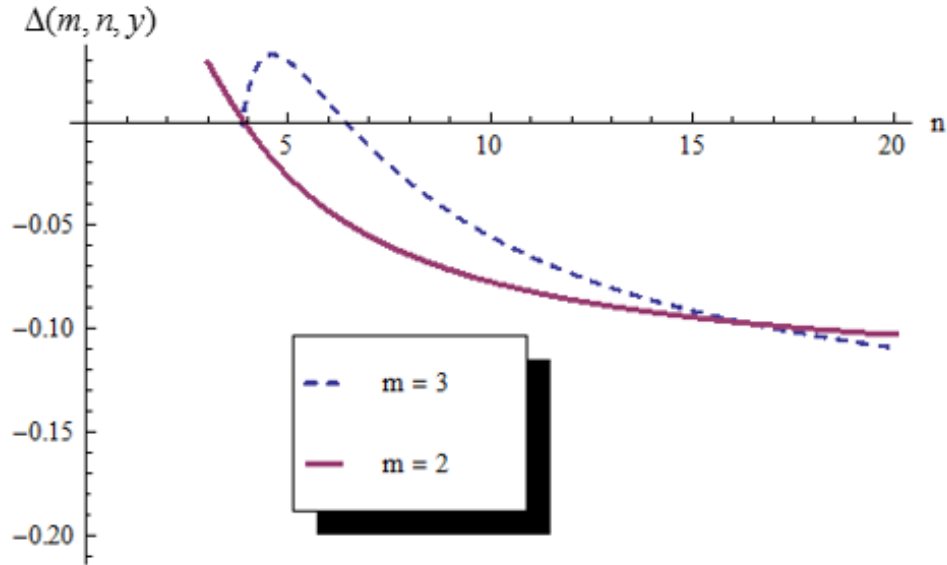
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Figure 2. Relative benefit of using OB with respect to *n* when $\lambda = 0.5$ and $y = 1$

Figure 2 shows that for $m = 2$ and values of $n \leq 4$ (i.e. a very short demand history), the NOB approach performs better than OB with a benefit varying between 0% and 3%. Moreover, when *m* increases, (e.g. $m = 3$), the NOB approach performs better than OB for values of *n* that can go up to 6. However, for higher values of *n*, OB outperforms NOB and the benefit of using the former approach can go up to 11%.

In summary, the most interesting insights that could be provided based on the analysis of the Poisson case are the following. With the exception of the context of short demand histories' length, overall the OB approach outperforms the NOB approach and the relative benefit stemming from the use of OB increases substantially as the aggregation level or the rate of the demand increases.

5. Empirical analysis

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In this section, we discuss details related to our empirical investigation. First, we provide information on the empirical data used in this research. Then, we present the experimental structure employed, some simulation related details and the empirical results.

5.1. Empirical data

For the purpose of our investigation, we use a dataset that comes from the US jewellery industry. It contains the weekly demand history of 4,076 intermittent demand SKUs over a period of one year (52 weeks). The lead time is fixed for all SKUs to one week. Descriptive statistics are provided in Table 3. 'Demand sizes' refer to the sizes of demand, when demand occurs, i.e. the relevant statistics refer only to demand occurring periods. 'Demand intervals' refer to the number of periods with zero demand plus one (for example, if there are no zero demands between two periods with occurring demands, the demand interval is one). 'Demand per period' refers to all periods, including both the demand occurring and non-demand occurring ones. In order to illustrate the calculation of demand sizes, demand intervals and demand per period, we consider the following example with a demand history of 24 periods:

| 0 | 0 | 0 | 3 | 0 | 0 | 2 | 0 | 0 | 0 | 0 | 2 | 0 | 0 | 0 | 4 | 0 | 0 | 0 | 0 | 0 | 6 | 0 | 1 |

In this example, the demand sizes are 3, 2, 2, 4, 6 and 1 which provides an average demand size of 3. Let us suppose that the period immediately preceding this history contained a non-zero demand value. Then the demand intervals are equal to 4, 3, 5, 4, 6 and 2, which leads to an average demand interval of 4 and a demand per period equal to 0.75.

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These calculations are done per SKU in the dataset then averaged across SKUs to result in the statistics shown in Table 3.

Table 3. Descriptive statistics of the empirical data

4,076 SKUs	Demand Sizes		Demand Intervals		Demand per period	
	Mean	St. Deviation	Mean	St. Deviation	Mean	St. Deviation
Min	1.000	0.000	1.275	0.564	0.115	0.323
25%ile	1.100	0.316	3.286	2.629	0.192	0.445
Median	1.200	0.426	4.364	3.711	0.250	0.530
75%ile	1.353	0.651	5.625	5.046	0.365	0.673
Max	3.194	3.682	8.667	12.987	2.212	2.539

The descriptive statistics reported in Table 3 show that most of the SKUs are slow moving since the distribution of demand intervals is mainly concentrated around a median of 4.4 weeks, and most demands are for one or two units. It should also be noted that the average minimum demand interval across SKUs is 1.3 (i.e. a value that is strictly higher than 1) which means that for each SKU there are some zero demands between demand occurrences. In the following subsections, we analyse the empirical performance of both approaches. Firstly, variance reductions are assessed, and then the stock control results are examined.

5.2. Statistical performance

We present the summary statistics of the variance ratio of the NOB approach to the OB approach, $Var\left[\hat{F}_m^{NOB}(y)\right]/Var\left[\hat{F}_m^{OB}(y)\right]$. Note that the calculation of the variances is performed by using the analytical results given in (2) and in Proposition 1 and by considering the entire demand history composed of 52 periods, which means that a relatively long demand history is considered.

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Table 4. Empirical comparative performance for long demand histories

	$m = 2$			$m = 3$		
	Variance ratio for y=0	Variance ratio for y=1	Variance ratio for y=2	Variance ratio for y=0	Variance ratio for y=1	Variance ratio for y=2
Min	1.0214	1.0196	0.9953	1.0249	1.0289	0.9970
25% percentile	1.0392	1.1016	1.1244	1.0691	1.1225	1.1660
Median	1.0521	1.1332	1.4948	1.0941	1.1524	1.3429
Average	1.0662	1.1842	1.3853	1.1220	1.1721	1.3061
75% percentile	1.0806	1.1827	1.6292	1.1511	1.1854	1.4620
Max	1.4342	1.6308	1.7311	1.9506	1.5562	1.5274

The results in Table 4 show that, for a long demand history, the OB approach is expected to outperform NOB for almost all the SKUs, with a median benefit, when $y = 1$, of 13% for $m = 2$ and 15% for $m = 3$. Furthermore, when $y = 2$, the median benefit increases and can go up to 49% for $m = 2$ and 34% for $m = 3$.

We now present the summary statistics of the variance ratio, $Var\left[\hat{F}_m^{NOB}(y)\right]/Var\left[\hat{F}_m^{OB}(y)\right]$, for a length of the history $n = 9$. The calculation of the variances is no longer based on the analytical results given in Proposition 1. Instead, it is performed by splitting the demand history in five blocks composed of 10 periods (or 9 periods) when $m = 2$ (or $m = 3$), i.e. the values of $\hat{F}_m^{NOB}(y)$ and $\hat{F}_m^{OB}(y)$ are first calculated for each block (by first calculating the aggregated demands over the block and then deducing the probability that the aggregated demand over the block is less or equal to y) and then their variances are calculated by considering the 5 blocks. Note that in the former case, we consider a demand history composed of 50 periods whereas in the latter, only 45 periods are considered. The number of 5 blocks is chosen to ensure a trade-off between the length of the blocks and the number of blocks. A short block length leads to poor estimation of the

probabilities whereas a long length of blocks implies a low number of blocks which makes it difficult to calculate the variance of the estimates over the blocks.

Table 5. Empirical comparative performance for short demand histories

	$m = 2$			$m = 3$		
	Variance ratio for y=0	Variance ratio for y=1	Variance ratio for y=2	Variance ratio for y=0	Variance ratio for y=1	Variance ratio for y=2
Min	0	0	0	0	0	0
25% percentile	0.7650	0.7088	0.7069	0.7259	0.7424	0.8167
Median	0.9504	0.9927	1.1270	0.9722	1.0889	1.3611
Average	1.0195	1.49284	1.52613	1.1534	1.6431	2.2713
75% percentile	1.1719	1.7357	1.7446	1.3199	1.8148	5.4444
Max	7.5600	16.2000	16.2000	9.0741	13.6111	13.6111

The results in Table 5 show that a high proportion (i.e. approximately 50%) of the SKUs has a variance ratio less than 1, i.e. the NOB approach outperforms OB. These results, showing performance superiority of the NOB approach for a short demand history, are consistent with the theoretical results established earlier in this paper, which demonstrated that such outperformance may arise in these circumstances. A more detailed analysis was conducted but no link was established between the variance ratio and the degree of intermittence (as characterised by the length of demand intervals). This is not consistent with the analytical findings in Section 2.3, and the discrepancy would seem to merit further empirical research on other datasets.

5.3. Stock control performance

In this subsection, we conduct an empirical comparison of the two approaches when an Order-Up-To-Level (OUTL) inventory control policy is used. Under each approach, the OUTL is calculated at the end of each period as the minimum number y such that $\hat{F}_m(y)$ is higher than a target cycle service level (*CSL*, the probability of no stock-outs during a

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replenishment cycle), as shown in (15) and (16). We assume that unmet demand is backordered.

$$S_m^{NOB}(CSL) = \arg\min_y \left(\hat{F}_m^{NOB}(y) \geq CSL \right) \tag{15}$$

$$S_m^{OB}(CSL) = \arg\min_y \left(\hat{F}_m^{OB}(y) \geq CSL \right) \tag{16}$$

In order to simulate the performance of each approach, we split the demand history available (i.e. 52 periods) for each SKU into two parts. The 1st part (26 periods) is used in order to initialise the inventory level (that is assumed to be equal to the initial OUTL). The 2nd part (26 periods) is used for the out-of-sample generation of results and evaluation of performance. In each part, n historical periods are considered to aggregate the demand using one of the two approaches and then $\hat{F}_m^{NOB}(y)$ and $\hat{F}_m^{OB}(y)$ are calculated in order to determine the OUTLs $S_m^{NOB}(CSL)$ and $S_m^{OB}(CSL)$. The initial values of $S_m^{NOB}(CSL)$ and $S_m^{OB}(CSL)$ are calculated at the end of period 26 and then they are updated in each period of the out-of-sample as described above. The order of events in a period (i.e. a week) is as follows: the demand is first observed, the order (placed one week ago) is received, and a new order is then placed. This dynamic simulation of the inventory system uses an OUTL method calculated differently with the two aggregation approaches. This enables us to calculate at the end of the demand history, the average inventory holding volumes and backordering volumes, which allows a direct comparison of the performance of the two approaches. We refer the reader to Syntetos and Boylan (2006) for more details on this methodology of inventory performance evaluation of estimation methods.

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We analyse the stock control results by assessing the trade-offs between the average (across all SKUs) inventory holding volumes and the average backordering volumes. This analysis is conducted for three target *CSLs*, namely *CSL* = 80%, 90%, 95%, three demand history lengths, $n = 6, 12, 24$, and two aggregation levels, $m = 2, 3$.

We begin by presenting in Figures 3 and 4 the results for $n = 24$ in the form of efficiency curves for inventory holding and backordering volumes.
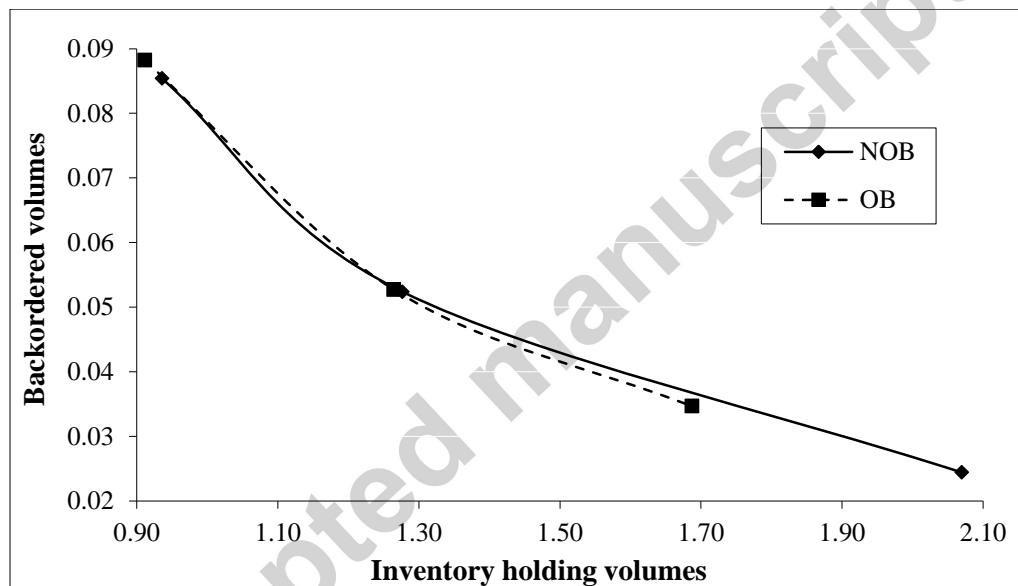

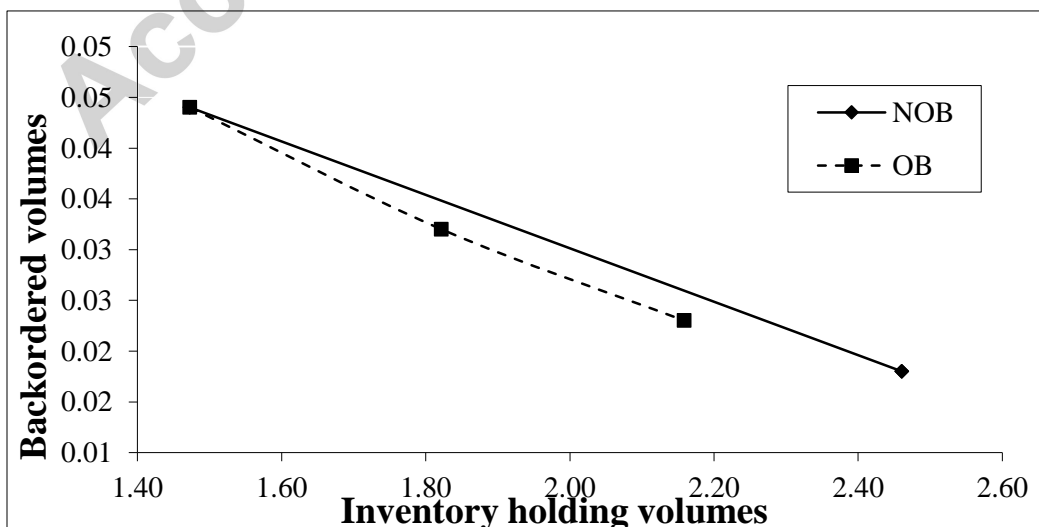Figure 3. Efficiency curves for $m = 2$ and $n = 24$

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Figure 4. Efficiency curves for $m = 3$ and $n = 24$

Figures 3 and 4 demonstrate overall better inventory performance of OB for the case of $n = 24$. Moreover, the improvement in inventory performance is greater for $m = 3$ than $m = 2$. This is consistent with expectations from our theoretical results on variance reductions.

For the cases of $n = 6$ and $n = 12$, no benefits in inventory performance have been identified. In some cases, it is not even possible to draw efficiency curves because of insensitivity of the inventory holding volumes to the target service level. Insensitivity of inventory holding volumes is due to all target service levels (80%, 90% and 95%) being achieved at the same OUTL. Such a situation would not arise for faster moving SKUs but is more prevalent for slow moving items. This problem is exacerbated for shorter demand histories as there is less opportunity to smooth the CDF with fewer blocks.

6. Conclusions and future work

In this paper, based on a theoretical analysis, we have shown that the overlapping aggregation approach produces unbiased estimates of the cumulative probability of demand and we have established an expression for its variance, assuming identically and independently distributed demand. Based on this analysis, we have established conditions for the variance outperformance of the two approaches. Our analysis reveals that the overlapping approach outperforms the non-overlapping one in most cases. The exceptional cases occur with short demand histories and very slow demand.

Through an empirical investigation, we have confirmed that the overlapping approach is expected to produce lower variance estimates, for all SKUs, for longer demand histories.

We have also shown that the non-overlapping approach is better than the overlapping one for 50% of the SKUs for a short demand history of length 9 or 10 periods.

When the stock control performance is analysed, the empirical investigation shows that for a demand history of 24 periods, the overlapping approach reduces the inventory backordering volumes, whilst maintaining the same inventory holding volumes. For shorter demand histories, no benefit in inventory performance has been identified due to the lack of opportunity to smooth the CDF with a small number of blocks.

The managerial implications of this research are clear if standard parametric distributions are not appropriate and a non-parametric 'temporal aggregation' approach is adopted. Unless the demand history is short, there is a clear advantage of using overlapping blocks instead of the non-overlapping approach. The margin of this advantage becomes greater for longer lead-times (equal to the length of the block). There are strong analytical arguments to support these managerial implications, in terms of the variance of the estimates of the cumulative distribution function. There is also empirical evidence to demonstrate that these can be translated into stock-control benefits too. However, these benefits should be checked, for example by a simulation exercise, as the evidence to support stock-control improvements needs to be verified in a broader range of inventory environments.

In the light of the above discussion, an interesting avenue for further research would be to broaden the empirical analysis by evaluating more varied datasets, including faster moving demands and lumpier demands. Other approaches need to be identified to address

the issue of smoothing the CDF such as bootstrapping methods which randomly sample non-contiguous time periods. Such methods have been suggested in the literature (e.g. Willemain et al. (2000)) and it would be beneficial to analyse their theoretical properties and empirical performance.

Appendix A. Proof of Proposition 1

We suppose that the observations are split into overlapping blocks of length m, namely $\left\{B_1^{OB}, B_2^{OB}, ..., B_k^{OB}\right\}$ where $k = n - m + 1$ and $B_i^{OB} = \{Y_i, Y_{i+1}, ..., Y_{i+m-1}\}$ for $i = 1, 2, ..., k$.

We define Z_i as the sum of demand values over a block of length m, i.e.

$$Z_i = \sum_{j=1}^{m} Y_{i+j-1}$$

(A1)

We also define for any value of y the indicator function:

$$I_{(-\infty, y]}(Z_i) = \begin{cases} 1 & \text{for } Z_i \leq y \\ 0 & \text{for } Z_i > y \end{cases} \tag{A2}$$

An estimate of $F_m(y)$ using the overlapping approach is then:

$$\hat{F}_m^{OB}(y) = \frac{1}{k} \sum_{i=1}^{k} I_{(-\infty, y]}(Z_i) \tag{A3}$$

We now calculate the expectation and the variance of this estimator $\hat{F}_m^{OB}(y)$.

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The expectation of $\hat{F}_m^{OB}(y)$ is given by:

$$E\left[\hat{F}_m^{OB}(y)\right] = E\left[\frac{1}{k}\sum_{i=1}^{k} I_{(-\infty,y]}(Z_i)\right] = \left[\frac{1}{k}\sum_{i=1}^{k} F_m(y)\right] = F_m(y) \tag{A4}$$

By definition, the variance of the estimator $\hat{F}_m^{OB}(y)$ can be written as:

$$V\left[\hat{F}_m^{OB}(y)\right] = E\left[\left(\hat{F}_m^{OB}(y)\right)^2\right] - \left(E\left[\hat{F}_m^{OB}(y)\right]\right)^2 \tag{A5}$$

The first term may be written:

$$E\left[\left(\hat{F}_m^{OB}(y)\right)^2\right] = \frac{1}{k^2}\sum_{i=j} E\left[\left(I_{(-\infty,y]}(Z_i)\right)^2\right] + \frac{1}{k^2}\sum_{i\neq j} E\left[I_{(-\infty,y]}(Z_i)I_{(-\infty,y]}(Z_j)\right] \tag{A6}$$

By definition of the indicator function, we have:

$$E\left[\left(I_{(-\infty,y]}(Z_i)\right)^2\right] = E\left[I_{(-\infty,y]}(Z_i)\right] = F_m(y), \tag{A7}$$

which implies that:

$$\frac{1}{k^2}\sum_{i=j} E\left[\left(I_{(-\infty,y]}(Z_i)\right)^2\right] = \frac{1}{k^2}\sum_{i=j} F_m(y) = \frac{1}{k^2}\sum_{i=1}^{k} F_m(y) = \frac{1}{k}F_m(y). \tag{A8}$$

We now calculate the term $\dfrac{1}{k^2}\sum_{i\neq j} E\left[I_{(-\infty,y]}(Z_i)I_{(-\infty,y]}(Z_j)\right]$. In this term, there are some

pairs of blocks that do not overlap and some that do overlap. Hence:

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$$\sum_{i \neq j} E\left[I_{(-\infty, y]}(Z_i) I_{(-\infty, y]}(Z_j)\right] = 2\left[\sum_{i=1}^{k} \sum_{j=1}^{i-m} F_m(y)^2 + \sum_{i=1}^{k} \sum_{j=i-m+1}^{i-1} E\left[I_{(-\infty, y]}(Z_i) I_{(-\infty, y]}(Z_j)\right]\right]$$

(A9)

$$\sum_{i=1}^{k} \sum_{j=1}^{i-m} F_m(y)^2 = F_m(y)^2 \sum_{i=1}^{k}(i-m) = \frac{(k-m)(k-m+1)}{2} F_m(y)^2$$

(A10)

The term $\displaystyle\sum_{i=1}^{k} \sum_{j=i-m+1}^{i-1} E\left[I_{(-\infty, y]}(Z_i) I_{(-\infty, y]}(Z_j)\right]$ may be re-expressed by a single summation

(across the diagonals, from the diagonal which is the one just next to the 'main diagonal' to the one which is (m-1) away from the 'main diagonal'). Hence:

$$\sum_{i=1}^{k} \sum_{j=i-m+1}^{i-1} E\left[I_{(-\infty, y]}(Z_i) I_{(-\infty, y]}(Z_j)\right] = \sum_{s=1}^{m-1}(k-m+s)\Theta_s(y)$$

(A11)

where $\displaystyle\Theta_s(y) = \sum_{Y_1=0}^{y} \cdots \sum_{Y_m=0}^{y-Y_1-\ldots-Y_{m-1}} \sum_{Y_{m+1}=0}^{Y-Y_{m-s+1}-\ldots Y_m} \cdots \sum_{Y_{2m-s}=0}^{y-Y_{m-s+1}-\ldots-Y_{2m-s-1}} \prod_{j=1}^{2m-s} p(Y_j)$ (A12)

Assembling the terms in (A8), (A10) and (A11) leads to:

$$Var[\hat{F}_m^{OB}(y)] = \frac{1}{k} F_m(y) + \left[\frac{(k-m)(k-m+1)}{k^2} - 1\right] F_m(y)^2$$

$$+ \frac{2}{k^2} \sum_{s=1}^{m-1}(k-m+s) \sum_{Y_1=0}^{y} \cdots \sum_{Y_m=0}^{y-Y_1-\ldots-Y_{m-1}} \sum_{Y_{m+1}=0}^{Y-Y_{m-s+1}-\ldots Y_m} \cdots \sum_{Y_{2m-s}=0}^{y-Y_{m-s+1}-\ldots-Y_{2m-s-1}} \prod_{j=1}^{2m-s} p(Y_j)$$

(A13)

which ends the proof of Proposition 1. □

Appendix B. Proof of Proposition 4

The relative benefit from using the *OB* approach instead of NOB, $\Delta(m,n,y)$ is given by:

$$\Delta(m,n,y) = \frac{\dfrac{1}{(n-m+1)} + \left[\dfrac{(n-2m+1)(n-2m+2)}{(n-m+1)^2} - 1\right]F_m(y) + \displaystyle\sum_{s=1}^{m-1}\dfrac{2(n-2m+s+1)}{(n-m+1)^2}\dfrac{\Theta_s(y)}{F_m(y)}}{\dfrac{m}{n}\left[1-F_m(y)\right]} - 1 \qquad (A14)$$

It is clear that when λ tends towards infinity, $F_m(y)$ tends towards 0. This means that when λ tends towards infinity, $\Delta(m,n,y)$ tends towards:

$$\Delta(m,n,y) = \frac{\dfrac{1}{(n-m+1)} + \displaystyle\sum_{s=1}^{m-1}\dfrac{2(n-2m+s+1)}{(n-m+1)^2}\lim_{\lambda\to\infty}\dfrac{\Theta_s(y)}{F_m(y)}}{\dfrac{m}{n}} - 1 \qquad (A15)$$

We now calculate $\lim_{\lambda\to\infty}\dfrac{\Theta_s(y)}{F_m(y)}$ where $1\le s\le m-1$.

$$\frac{\Theta_s(y)}{F_m(y)} = \frac{\displaystyle\sum_{Y_1=0}^{y}\cdots\cdots\sum_{Y_m=0}^{y-Y_1-\ldots-Y_{m-1}}\sum_{Y_{m+1}=0}^{y-Y_{m-s+1}-\ldots-Y_m}\cdots\cdots\sum_{Y_{2m-s}=0}^{y-Y_{m-s+1}-\ldots-Y_{2m-s-1}}\prod_{j=1}^{2m-s}p(Y_j)}{\displaystyle\sum_{Y_1=0}^{y}\sum_{Y_2=0}^{y-Y_1}\sum_{Y_3=0}^{y-Y_1-Y_2}\cdots\cdots\cdots\sum_{Y_m=0}^{y-Y_1-\ldots-Y_{m-1}}\prod_{j=1}^{m}p(Y_j)} \qquad (A16)$$

which can be written as:

$$\frac{\Theta_s(y)}{F_m(y)} = \frac{e^{-(2m-s)\lambda}\displaystyle\sum_{Y_1=0}^{y}\cdots\cdots\sum_{Y_m=0}^{y-Y_1-\ldots-Y_{m-1}}\sum_{Y_{m+1}=0}^{y-Y_{m-s+1}-\ldots-Y_m}\cdots\cdots\sum_{Y_{2m-s}=0}^{y-Y_{m-s+1}-\ldots-Y_{2m-s-1}}\prod_{j=1}^{2m-s}\dfrac{\lambda^{Y_j}}{Y_j!}}{e^{-m\lambda}\displaystyle\sum_{Y_1=0}^{y}\sum_{Y_2=0}^{y-Y_1}\sum_{Y_3=0}^{y-Y_1-Y_2}\cdots\cdots\cdots\sum_{Y_m=0}^{y-Y_1-\ldots-Y_{m-1}}\prod_{j=1}^{m}\dfrac{\lambda^{Y_j}}{Y_j!}} \qquad (A17)$$

This means that:

$$\frac{\Theta_s(y)}{F_m(y)} = \frac{e^{-(m-s)\lambda}\sum_{Y_1=0}^{y}.....\sum_{Y_m=0}^{y-Y_1-...-Y_{m-1}}\sum_{Y_{m+1}=0}^{y-Y_{m-s+1}-...-Y_m}......\sum_{Y_{2m-s}=0}^{y-Y_{m-s+1}-...-Y_{2m-s-1}}\prod_{j=1}^{2m-s}\frac{\lambda^{Y_j}}{Y_j!}}{\sum_{Y_1=0}^{y}\sum_{Y_2=0}^{y-Y_1}\sum_{Y_3=0}^{y-Y_1-Y_2}...........\sum_{Y_m=0}^{y-Y_1-...-Y_{m-1}}\prod_{j=1}^{m}\frac{\lambda^{Y_j}}{Y_j!}} \qquad (A18)$$

Since $\dfrac{\sum_{Y_1=0}^{y}.....\sum_{Y_m=0}^{y-Y_1-...-Y_{m-1}}\sum_{Y_{m+1}=0}^{y-Y_{m-s+1}-...-Y_m}......\sum_{Y_{2m-s}=0}^{y-Y_{m-s+1}-...-Y_{2m-s-1}}\prod_{j=1}^{2m-s}\dfrac{\lambda^{Y_j}}{Y_j!}}{\sum_{Y_1=0}^{y}\sum_{Y_2=0}^{y-Y_1}\sum_{Y_3=0}^{y-Y_1-Y_2}...........\sum_{Y_m=0}^{y-Y_1-...-Y_{m-1}}\prod_{j=1}^{m}\dfrac{\lambda^{Y_j}}{Y_j!}}$ is bounded and

$1 \le s \le m-1$ it is clear that: $\lim_{\lambda\to\infty}\dfrac{\Theta_s(y)}{F_m(y)} = 0$ \qquad (A19)

Hence: $\lim_{\lambda\to\infty}\Delta(m,n,y) = \dfrac{\dfrac{1}{(n-m+1)}}{\dfrac{m}{n}} - 1 = \dfrac{n}{m(n-m+1)} - 1$ \qquad (A20)

In addition, since: $\lim_{\lambda\to\infty}\Delta(m,n,y) = \dfrac{\dfrac{1}{(n-m+1)}}{\dfrac{m}{n}} - 1 = \dfrac{n}{m(n-m+1)} - 1$ \qquad (A21)

Then $\lim_{n\to\infty}\lim_{\lambda\to\infty}\Delta(m,n,y) = \lim_{n\to\infty}\dfrac{n}{m(n-m+1)} - 1 = \lim_{n\to\infty}\dfrac{1}{m(1-\dfrac{m}{n}+\dfrac{1}{n})} - 1 = \dfrac{1-m}{m}$ \qquad (A22)

which ends the proof of Proposition 4.   . $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

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