ENDOTRIVIAL MODULES FOR FINITE GROUPS OF LIE TYPE A IN NONDEFINING CHARACTERISTIC

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ABSTRACT. Let G be a finite group such that $\mathrm{SL}(n,q) \subseteq G \subseteq \mathrm{GL}(n,q)$ and Z be a central subgroup of G. In this paper we determine the group $T(G/Z)$ consisting of the equivalence classes of endotrivial $k(G/Z)$-modules where k is an algebraically closed field of characteristic p such that p does not divide q. The results in this paper complete the classification of endotrivial modules for all finite groups of (untwisted) Lie Type A, initiated earlier by the authors.

1. INTRODUCTION

Let G be a finite group and k be a field of characteristic $p > 0$. The group of endotrivial kG-modules was first introduced for p-groups by Dade [18, 19] nearly forty years ago. He showed that the endotrivial modules for a Sylow p-subgroup S of G are the building blocks of the endo-permutation kS-modules which are the sources of the irreducible kG-modules when the group G is p-nilpotent. For any finite group G, tensoring with an endotrival kG-module induces a self-equivalence on the stable category of kG-modules modulo projectives. Thus the group of endotrivial modules is an important part of the Picard group of self-equivalences of the stable category, namely, the self-equivalences of Morita type. In addition, the endotrivial modules are the modules whose deformation rings are universal and not just versal (see [6]).

The endotrivial modules for an abelian p-group were classified by Dade and a complete classification of endotrivial modules over any p-group was completed several years later by the first author and Thévenaz [14, 15, 7] building on the work of Alperin [2] and others. Since then there has been an effort to compute the group $T(G)$ of endotrivial modules for almost simple and quasi-simple groups G. The proofs of [12] suggest that this might be an important step in the computation of $T(G)$ for an arbitrary finite group G. The group $T(G)$ has been determined for finite groups of Lie type in the defining characteristic in [9], and for symmetric and alternating groups in [10, 8]. Other results can be found in [12, 22, 24, 25].

Every one of the papers in this project has produced important advances for computing and determining endotrivial modules. This paper continues that development, presenting a significant improvement of a method introduced in [11]. The method was inspired by the development by Balmer [5] of "weak H-homomorphisms" which describe the kernel of the restriction map $T(G) \to T(H)$ when H is a subgroup of G that contains a Sylow p-subgroup of G. The new technique with some

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variations allows the computations of the group of endotrivial modules for all finite groups of (untwisted) Lie type A in nondefining characteristic. In all but a few examples of small Lie rank and small characteristic we show that the torsion part of $T(G)$ equals the isomorphism classes of one-dimensional modules. There are a couple of instances in this paper when it is necessary to call upon a somewhat more sophisticated variation of the method developed in [16].

The goal of this paper is to describe $T(G)$ for all finite groups of Lie type A in nondefining characteristic, completing the work started in [11]. The general result is the following. The structure of the groups of endotrivial modules for cases not covered by this theorem are treated in later sections.

Theorem 1.1. *Let k be an algebraically closed field of prime characteristic p and q a prime power with p not dividing q, and let e be the least positive integer such that p divides $q^e - 1$. Let G be a finite group of order divisible by p such that $\mathrm{SL}(n,q) \subseteq G \subseteq \mathrm{GL}(n,q)$, and let Z be a central subgroup of G. Assume that the following conditions hold.*

(a) *In all cases, $n \geq 2e$.*
(b) *If $e = 1$, $n = 2$ and $p \geq 3$, then Z does not contain a Sylow p-subgroup of $Z(G)$.*
(c) *If $e = 1$ and $n = p = 3$, then Z does not contain a Sylow 3-subgroup of $Z(G)$ (which happens if and only if and only if 3 divides $|G/(Z \cdot \mathrm{SL}(3,q))|$).*
(d) *If $p = 2$, then $n > 3$.*

Then

$$T(G/Z) \cong \mathbb{Z} \oplus X(G/Z),$$

where $X(G/Z)$ is the group under tensor product of $k(G/Z)$-modules of dimension one and the torsion free part of $T(G/Z)$ is generated by the class of $\Omega(k)$.

Theorem 1.1 is established in Sections 5 and 6. This follows some preliminaries on endotrivial modules in Section 2, a description of the main method that we use in most proofs in Section 3, and preliminaries on groups of Lie type A in Section 4. The proof of the main theorem stated above is accomplished in two major steps. In Section 5, we treat the case that $G = \mathrm{SL}(n,q)$ and $Z = \{1\}$. In Section 6, the result is extended to any G and Z subject to the assumptions of Theorem 1.1.

Sections 7 through 11 deal with the cases that are excluded by the hypotheses of Theorem 1.1. In the nontrivial cases excluded by condition (a), namely, $e \leq n < 2e$, the Sylow p-subgroup of G is cyclic and the structure of $T(G)$ was provided in [11, Theorem 1.2]. For the sake of completeness, the theorem is stated in an appendix (cf. Theorem 11.1). The one additional case in which the Sylow p-subgroup of G/Z is cyclic is the case excluded by hypothesis (b) of Theorem 1.1. This case is dealt with in Theorem 7.1.

The case excluded from Theorem 1.1 by condition (c), is treated in Section 8. In Sections 9 and 10 we compute of the groups of endotrivial modules when $p = 2$ and $n = 2$ or 3, excluded from Theorem 1.1 by condition (d). These sections use results of [16], that show the existence of trivial source endotrivial modules of dimension greater than one. Table 1 summarizes the cases when such modules occur. The notation is that of Theorem 1.1 with p^t the highest power of p dividing $q^e - 1$. For conciseness, we have omitted from the first row the details of the conditions for a Sylow p-subgroup S of G/Z to be cyclic (and nontrivial).

n	Z	S	condition(s)	$TT(G/Z)$
n	Z	C_{p^t}		$X(N_{G/Z}(S))$
3	$\lvert Z\rvert_3 = \lvert Z(G)\rvert_3$	$C_3 \times C_3$	$q \equiv 4,7 \pmod 9$ $\frac{q-1}{\lvert \mathrm{Det}(G)\rvert} \equiv 0 \pmod 3$	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus X(G/Z)$
2	$\lvert Z\rvert \equiv 2 \pmod 4$	$C_2 \times C_2$	$q \equiv 3 \pmod 8$ $\lvert G : \mathrm{SL}(2,q)\rvert \not\equiv 0 \pmod 2$	$\mathbb{Z}/3\mathbb{Z} \oplus X(G/Z)$
2	$\lvert Z\rvert_2 = 2\lvert G : \mathrm{SL}(2,q)\rvert_2$	$C_2 \times C_2$	$q \equiv 5 \pmod 8$	$\mathbb{Z}/3\mathbb{Z} \oplus X(G/Z)$

TABLE 1. Trivial source endotrivial modules for finite groups of Lie type A in nondefining characteristic.

Our results for the nondefining characteristic, taken together with the results in [9] (for the defining characteristic), provide a complete description of the group of endotrivial modules for finite groups of (untwisted) Lie type A over algebraically closed fields of arbitrary characteristic.

2. Endotrivial Modules

Throughout the paper, let k be an algebraically closed field of prime characteristic p and G be a finite group with p dividing the order of G. All kG-modules in this paper are assumed to be finitely generated. For kG-modules M and N, let $M^* = \mathrm{Hom}_k(M, k)$ denote the k-dual of M and write $M \otimes N = M \otimes_k N$. The modules M^* and $M \otimes N$ become kG-modules under the the usual Hopf algebra structure on kG.

A kG-module M is *endotrivial* provided its endomorphism algebra $\mathrm{End}_k(M)$ splits as the direct sum of k and a projective kG-module. That is, since $\mathrm{Hom}_k(M, N) \cong M^* \otimes N$, as kG-modules, M is endotrivial if and only if

$$\mathrm{End}_k(M) \cong M^* \otimes M \cong k \ \oplus \ P$$

for some projective kG-module P.

Any endotrivial kG-module M has a unique indecomposable nonprojective endotrivial direct summand M_0 ([9]). This allows us to define an equivalence relation on the class of endotrivial kG-modules; namely, two endotrivial kG-modules are equivalent if they have isomorphic indecomposable nonprojective summands. That is, two endotrivial kG-modules are equivalent if they are isomorphic in the stable category. The set of equivalence classes of endotrivial kG-modules is an abelian group with the operation induced by the tensor of product over k,

$$[M] + [N] = [M \otimes N].$$

The identity element of $T(G)$ is $[k]$, and the inverse of $[M]$ is $[M^*]$. The group $T(G)$ is called the *group of endotrivial kG-modules*.

It is well-known that the group of endotrivial modules is a finitely generated abelian group. Therefore,

$$T(G) \cong TF(G) \ \oplus \ TT(G)$$

where $TT(G)$ is the torsion subgroup of $T(G)$ and $TF(G)$ is a torsion free complement. The rank of $TF(G)$ depends only on the p-local structure of $T(G)$, as described in the next theorem. Recall that the p-rank of a group is the maximum of the ranks of elementary abelian p-subgroups of G, and a maximal elementary abelian p-subgroup is an elementary abelian p-subgroup which is not properly contained in any other elementary abelian p-subgroup. Let n_G be the number of conjugacy classes of maximal elementary abelian p-subgroups of G of order p^2.

Theorem 2.1. [9, Theorem 3.1] *Let G be a finite group. The rank of $TF(G)$ is equal to the number n_G defined above if G has p-rank at most 2, and is equal to $n_G + 1$ if G has rank at least 3.*

We say that a kG-module has *trivial Sylow restriction* if its restriction to a Sylow p-subgroup S of G is isomorphic to the direct sum of k with some projective module. Equivalently, a kG-module with trivial Sylow restriction is the direct sum of a trivial source endotrivial kG-module and some projective module. In particular, its equivalence class is in the kernel of the restriction map $T(G) \to T(S)$. The next result was proved in [9, Proposition 2.6 (d)] and is very important to our development. Its proof is based on the fact that an indecomposable module with trivial Sylow restriction is a direct summand of $k_S^{\uparrow G}$ where S is a Sylow p-subgroup of G.

Proposition 2.2. *If G has a nontrivial normal p-subgroup, then every indecomposable kG-module with trivial Sylow restriction has dimension one.*

Another easy result that we find useful is the following.

Proposition 2.3. *Suppose that a Sylow p-subgroup S of G is self-normalizing (i.e. $N_G(S) = S$). Then the only indecomposable kG-module with trivial Sylow restriction is the trivial module.*

Proof. The Green correspondent of any indecomposable kG-module M with trivial Sylow restriction must have dimension one by the above proposition. Hence the Green correspondent is the trivial module and $M \cong k$. □

The following theorem has several applications to finite groups of Lie type. Note that the first condition in the statement is equivalent to saying that G contains the derived subgroup $[H, H] \times [J, J]$ of $H \times J$.

Theorem 2.4. *Suppose that H and J are finite groups and that G is a normal subgroup of the direct product $H \times J$ such that the orders of both $G \cap H$ and $G \cap J$ are divisible by p (here we are identifying H with $H \times \{1\}$ and J with $\{1\} \times J$ in $H \times J$). Then any indecomposable kG-module with trivial Sylow restriction has dimension one.*

Proof. Let $\widehat{H} = H \cap G$ and $\widehat{J} = J \cap G$. Let Q and T denote Sylow p-subgroups of $\widehat{H}$ and $\widehat{J}$, respectively, and let S be a Sylow p-subgroup of G that contains $Q \times T$.

Note that $T, Q \trianglelefteq S$. Let $W = (\widehat{H} \times \widehat{J})S$. By hypothesis G is normal in $H \times J$. This implies that H and J centralize $G/(\widehat{H} \times \widehat{J})$. Therefore, this quotient is abelian, and W is normal in G.

Suppose that M is an indecomposable kW-module with trivial Sylow restriction. Then $M_{\downarrow \widehat{H}S} \cong \chi \oplus (\text{proj})$ for some indecomposable $k(\widehat{H}S)$-module χ. We know that χ has dimension one because $\widehat{H}S$ has a nontrivial normal p-subgroup, namely T. Moreover, T is centralized by every element of $\widehat{H}$.

It follows that M is a direct summand of $\chi^{\uparrow W} \cong kW \otimes_{k(\widehat{H}S)} \chi$. Observe that all of the left coset representatives of $\widehat{H}S$ in W can be taken to be elements of $\widehat{J}$. Because these elements centralize Q and because the p-group Q acts trivially on a one-dimensional module, it must be that Q acts trivially on $\chi^{\uparrow W}$ and hence also on M. Therefore, the restriction of M to S can have no nonzero projective summands and M must have dimension one.

Suppose that N is a kG-module with trivial Sylow restriction. Then $N_{\downarrow W} \cong \Theta \oplus (\text{proj})$, where Θ has dimension one. This means that N is a direct summand of $\Theta^{\uparrow G}$ and because W is normal in G, $(\Theta^{\uparrow G})_{\downarrow W}$ is a direct sum of conjugates of Θ. It follows that N must have dimension one. □

3. The Main Method

In this section we introduce conditions that imply the triviality of any indecomposable kG-module with trivial Sylow restriction. The method was suggested by the work of Balmer [5], though none of the results of [5] are directly required in this paper. It is worth pointing out that the method works for perfect groups (i.e., $[G, G] = G$), and, with some effort, it can be adapted to other cases to prove that indecomposable kG-modules with trivial Sylow restriction have dimension one. The statement proved in Theorem 3.1 below is sufficient for this paper. A somewhat different version of the method is contained in the paper [16].

For each nontrivial p-subgroup Q of a given Sylow p-subgroup S of G, we construct a chain of subgroups:

$$\rho^1(Q) \subseteq \rho^2(Q) \subseteq \dots .$$

These were written $\rho_{i-1}(Q)$ in [11] where they were first introduced. The subgroups are defined inductively by the following rule:

$$\rho^1(Q) = [N_G(Q), N_G(Q)] \qquad \text{and}$$
$$\rho^i(Q) = \langle N_G(Q) \cap \rho^{i-1}(R) \mid \{1\} \neq R \subseteq S \rangle \quad \text{for } i > 1.$$

In [16], it is shown that if $\rho^i(S) = N_G(S)$ for some i (or more generally if $\rho^i(Q) = N_G(Q)$ for some nontrivial subgroup $Q \subseteq S$ with $N_G(S) \subseteq N_G(Q)$), then the trivial kG-module is the only indecomposable module with trivial Sylow restriction. The following theorem (Theorem 3.1) is the simplified version of that result needed for most of this paper.

Theorem 3.1. *Let S be a Sylow p-subgroup of G, and let H be a subgroup of G such that $N_G(S) \leq H$. Suppose that the following conditions hold.*

(A) *Every indecomposable kH-module with trivial Sylow restriction has dimension one.*

(B) $H = \langle g_1, \dots, g_m \rangle$ *such that for each* i*, either*

  (1) $g_i \in [H, H]S$*, or*

  (2) *there exists a subgroup* H_i *of* G *such that*

    (a) *every indecomposable* kH_i*-module with trivial Sylow restriction has dimension one,*

    (b) p *divides the order of* $H_i \cap H$*, and*

    (c) $g_i \in [H_i, H_i]$.

Then the trivial module k *is the only indecomposable* kG*-module with trivial Sylow restriction.*

Proof. Suppose that M is a kG-module with trivial Sylow restriction. Then $M_{\downarrow H} \cong \chi \oplus (\text{proj})$ for some kH-module χ having dimension one. So $[H, H]$ and S are in the kernel of χ and any generator g_i of H that satisfies condition (1) must act trivially on χ. Our next objective is to prove that the same holds for any generator g_i of H satisfying condition (2).

Suppose that g_i satisfies condition (2) for some subgroup H_i of G. By (2)(b), we can pick a nontrivial p-subgroup $Q_i \subseteq H_i \cap H$ for each i. By condition (2)(a), $M_{\downarrow H_i} \cong \mu \oplus (\text{proj})$ for some one-dimensional kH_i-module μ. Since g_i is in $[H_i, H_i]$ by (2)(c), g_i acts trivially on μ. As p divides the order of $H_i \cap H$ by (2)(b), any projective $k(H_i \cap H)$-module has dimension divisible by p. So consider the restriction

$$M_{\downarrow(H_i \cap H)} \cong \chi_{\downarrow(H_i \cap H)} \oplus (\text{proj}) \cong \mu_{\downarrow(H_i \cap H)} \oplus (\text{proj}) .$$

By the Krull-Schmidt Theorem $\mu_{\downarrow(H_i \cap H)} \cong \chi_{\downarrow(H_i \cap H)}$, and hence g_i acts trivially on χ.

Since every generator of H acts trivially on χ, it follows that $\chi \cong k_H$, the trivial kH-module. Now, M is indecomposable and H contains the normalizer of S. So M must be the Green correspondent of k_H. That is, $M \cong k$, as asserted. □

Remark 3.2. In most of the applications of Theorem 3.1 in this paper, the group G is a special linear group and the subgroup H is a parabolic or Levi subgroup that contains the normalizer of a Sylow p-subgroup of G. For such subgroups, condition (A) in the hypothesis of the theorem is established using an argument similar to that of Theorem 2.4.

In the case that $H = N_G(Q)$ where Q is a nontrivial characteristic subgroup of the Sylow p-subgroup S of G, the hypotheses of Theorem 3.1 basically say that $\rho^2(Q) = N_G(Q)$, which guarantees that the trivial kG-module is the only indecomposable kG-module with trivial Sylow restriction. In all but one of the proofs of Sections 5 and 6, this information is sufficient to obtain the asserted result. There is a unique case for which we need to compute $\rho^3(Q)$, relying on information gathered in [11].

Remark 3.3. It should be pointed out that conditions (B)(1) and (2) on the generators of H in the hypothesis of the theorem are not inherited by subgroups. That is, if J is subgroup of H also containing the normalizer of a Sylow p-subgroup of G, and H satisfies condition (B)(1) or (2), then we cannot conclude that J satisfies condition (B)(1) or (2) respectively.

4. Groups of Lie Type A

In this section we recall some known facts on the structure of the Sylow p-subgroups and their normalizers for finite groups of Lie type A in nondefining characteristic. More information can be found in [1, 4, 21, 26].

For convenience we set some notation that is used throughout the rest of paper.

Notation 4.1. *Let k be a field of prime characteristic p and q a prime power such that $\gcd(p, q) = 1$. Let e denote the least integer such that p divides $q^e - 1$ and write $q^e - 1 = p^t d$, where $\gcd(p, d) = 1$ and $t \geq 1$. Given a positive integer n, let r, f be integers such that $n = re + f$ and $0 \leq f < e$.*

Thus, e is the multiplicative order of q modulo p, and p^t is the highest power of p dividing $q^e - 1$. In particular, e is the smallest integer such that p divides the order of $\mathrm{GL}(e, q)$.

We start with the following useful elementary observations.

Proposition 4.2. *Suppose that G is a group such that $\mathrm{SL}(n, q) \subseteq G \subseteq \mathrm{GL}(n, q)$ and let S be a Sylow p-subgroup of G. Let $\mathrm{Det}(G) \subseteq \mathbb{F}_q^\times$ be the image of the determinant map.*

(a) *G is the subgroup of $\mathrm{GL}(n, q)$ consisting of all invertible matrices whose determinants are in $\mathrm{Det}(G)$.*
(b) *S is abelian if and only if and only if $n < pe$.*
(c) *The p-rank of G is r except in the case that p divides both n and $q - 1$. In that case, the p-rank is either r or $r - 1$, depending on whether the order of $\mathrm{Det}(G)$ is divisible by p.*

Proof. (a) is immediate. For (b) and (c), see [21] or [26]. □

In general, a Sylow p-subgroup S of G is a subgroup of a direct product of iterated wreath products. For $G = \mathrm{GL}(n, q)$, a Sylow p-subgroup S of G is the Sylow p-subgroup of a semi-direct product $(C_{p^t} \rtimes C_e)^r \rtimes \mathfrak{S}_r$, where $\mathfrak{S}_r$ is the symmetric group on r-letters ([1, Theorem VII.4.1]). For any $\mathrm{SL}(n, q) \subseteq G \subseteq \mathrm{GL}(n, q)$, a Sylow p-subgroup of G is the intersection of G with a Sylow p-subgroup of $\mathrm{GL}(n, q)$. Recall that a Sylow p-subgroup R of $\mathfrak{S}_r$ is a direct product of iterated wreath products as follows. Write $r = \sum_{0 \leq i \leq M} a_i p^i$ with $0 \leq a_i < p$ for each i. Then

$$R \cong \prod_{0 \leq i \leq M} \big(\underbrace{C_p \wr C_p \wr \cdots \wr C_p}_{i \text{ terms}} \big)^{a_i} = \prod_{0 \leq i \leq M} \left(C_p^{\wr i} \right)^{a_i}$$

where $C_p^{\wr i}$ is a Sylow p-subgroup of $\mathfrak{S}_{p^i}$.

Theorem 4.3. *Suppose that $p > 2$. With the above notation, the following hold.*

(a) $S \cong \prod_{0 \leq i \leq M} \left(C_{p^t} \wr (C_p^{\wr i}) \right)^{a_i}$
(b) *Each of the r factors C_{p^t} of S can be embedded as Sylow p-subgroup of a diagonal block $\mathrm{GL}(e, q)$ of $\mathrm{GL}(n, q)$, and the other generators of S can be embedded as permutation matrices of these blocks according to the p-adic expansion of r. In other words, S can be chosen in a Levi subgroup of $\mathrm{GL}(n, q)$*

with diagonal blocks of size

$$(\underbrace{e,\dots,e}_{a_0\ terms},\underbrace{ep,\dots,ep}_{a_1\ terms},\dots,\underbrace{ep^M,\dots,ep^M}_{a_M\ terms},\underbrace{1,\dots,1}_{f\ terms})$$

(c) *The normalizer $N_{\mathrm{GL}(n,q)}(S)$ of S is contained in the normalizer of the Levi subgroup containing S above.*

(d) *S contains a unique elementary abelian subgroup E of rank r, hence characteristic in S, and each elementary abelian subgroup of S is conjugate to a subgroup of E.*

Proof. See [4, Section 4], [1, Section VII], [21, Theorem 4.10.2 and Remark 4.10.4] and [26, Section 2]. □

The case $p = 2$ is handled separately, as the 2-local structure of $\mathrm{GL}(n, q)$ and subgroups is very different from the case $p > 2$.

The lemma below is well-known. We sketch a proof of the lemma because it is used several times.

Lemma 4.4. *Suppose that $n = rs$ for positive integers r and s, with $r > 1$. In $\widehat{G} = \mathrm{GL}(n, q)$ let $\widehat{L} \cong \mathrm{GL}(s, q)^r$ be the Levi subgroup of all elements that can be written as block diagonal $s \times s$ matrices. Let $L = \widehat{L} \cap G$ where $G = \mathrm{SL}(n, q)$, and let $N = N_G(L)$.*

(a) *If q is odd and $r = 2$, then the quotient $N/[N, N]$ is a Klein four group.*

(b) *If q is odd and $r > 2$, or if q is even, then the commutator subgroup of N has index 2 in N.*

Proof. The subgroup N is an extension

$$1 \longrightarrow L \longrightarrow N \longrightarrow \mathfrak{S}_r \longrightarrow 1$$

where the symmetric group $\mathfrak{S}_r$ acts on L by permuting the diagonal blocks. We know that $[L, L] \cong \mathrm{SL}(s, q)^r$ and can identify $L/[L, L]$ with the subgroup of $(\mathbb{F}_q^\times)^r$ given as $L/[L, L] \cong \{(a_1, \dots, a_r) \in (\mathbb{F}_q^\times)^r \mid a_1 \cdots a_r = 1\}$. Thus, $N/[L, L]$ is an extension

$$1 \longrightarrow L/[L, L] \longrightarrow N/[L, L] \longrightarrow \mathfrak{S}_r \longrightarrow 1,$$

where the symmetric group acts by permuting the places.

If $r = 2$, then $N/[L, L]$ is a dihedral group of order $2(q - 1)$ whose commutator subgroup is cyclic of index 4 if q is odd, and of index 2 if q is even. Therefore, the quotient group $N/[N, N]$ is a Klein four group if q is odd, respectively cyclic of order 2 if q is even.

Now assume that $r > 2$. It is easy to see that $L/[L, L]$ is generated by the element $\alpha = (a, a^{-1}, 1, \dots, 1)$ and its conjugates under the action of the symmetric group, where a is a generator for $\mathbb{F}_q^\times$. One of these conjugates is $\beta = (1, a, a^{-1}, 1, \dots, 1)$. For $\sigma = (1, 2) \in \mathfrak{S}_r$ we calculate $[\alpha\beta, \sigma] = \alpha\beta\sigma(\alpha\beta)^{-1}\sigma^{-1} = \alpha$. Hence, α and all of its conjugates under the action of the symmetric group are in the commutator subgroup of $N/[L, L]$ and hence $L \subseteq [N, N]$. On the other hand, the quotient group N/L is isomorphic to $\mathfrak{S}_r$ and as $N/[N, N]$ is the largest abelian quotient of N/L, and $N/[N, N]$ must have order 2. □

In the specific context of the section, Theorem 2.4 leads to the following observation.

Proposition 4.5. *Assume that Notation 4.1 holds. Let $n = n_1 + n_2 + \cdots + n_m$, where $n_1, \ldots, n_m$ are positive integers and let*

$$\widehat{L} \quad = \quad \prod_{i=1}^{m} \mathrm{GL}(n_i, q) \quad \subseteq \quad \mathrm{GL}(n, q)$$

be the Levi subgroup of diagonal blocks of sizes $n_1, \ldots, n_m$. Let $L = \mathrm{SL}(n, q) \cap \widehat{L}$. Assume further that

(a) *if p divides $q - 1$ then at least two of $n_1, \ldots, n_m$ are greater than one,*
(b) *if e is the smallest positive integer such that p divides $q^e - 1$ and $e > 1$, then at least two of $n_1, \ldots, n_m$ are greater than or equal to e.*

Then any indecomposable kL-module with trivial Sylow restriction has dimension one.

Proof. Express $\{1, \ldots, m\} = A \cup B$ as a union of disjoint subsets such that in case (a) each of A and B contains some index i such that $n_i > 1$, or in case (b), each of A and B contains an index i such that $n_i \geq e$. Then let $H = \prod_{i \in A} \mathrm{GL}(n_i, q)$, $J = \prod_{i \in B} \mathrm{GL}(n_i, q)$. Then $\widehat{L} \cong H \times J$, and Theorem 2.4 proves the assertion for $L = \mathrm{SL}(n, q) \cap \widehat{L}$. Indeed, $\widehat{L}/L$ is abelian and the conditions (a) and (b) ensure that the orders of $H \cap L$ and $J \cap L$ are both divisible by p. □

We end the section by recalling the following result (cf. [11, Theorem 3.4]).

Theorem 4.6. *Let G be a group such that $\mathrm{SL}(n, q) \subseteq G \subseteq \mathrm{GL}(n, q)$. Suppose that a Sylow p-subgroup of G has p-rank at least 2. Then $TF(G) \cong \mathbb{Z}$.*

Note that the theorem excludes the groups $\mathrm{SL}(2, q)$ for $p = 2$, in which case a Sylow 2-subgroup is generalized quaternion.

Corollary 4.7. *Let G be a group such that $\mathrm{SL}(n, q) \subseteq G \subseteq \mathrm{GL}(n, q)$. Suppose that $Z \subseteq Z(G)$ and that G, Z satisfy the conditions of Theorem 1.1. Then $TF(G/Z) \cong \mathbb{Z}$.*

Proof. Let T be the subgroup of all elements of order p in the torus of diagonal $e \times e$ block matrices in $\mathrm{GL}(n, q)$. The point of the proof of Theorem 4.6 is that every elementary abelian p-subgroup of G is conjugate to a subgroup of $G \cap T$. From this it follows that if G has maximal elementary abelian subgroups of rank 2, then they are all conjugate to a subgroup of $G \cap T$ and the conclusion follows from Theorem 2.1. This is also true for G/Z if $T \cap Z$ is trivial.

Consequently, the only remaining cases occur when $T \cap Z$ is not trivial. This requires that p divide $q - 1$ or equivalently that $e = 1$. Now $T \cap Z$ is a cyclic central subgroup of G, so that it is still the case that every elementary abelian p-subgroup is conjugate to one generated by elements that are the classes modulo Z of diagonal matrices. If $n \geq 4$ then every maximal elementary abelian p-subgroup has rank at least 3 and again we are done. The same happens if $n = 3$ and either $p > 3$ or if

$n = p = 3$ and Z does not contain the Sylow p-subgroup of $Z(G)$. This proves the corollary. □

5. Endotrivial Modules for $\mathrm{SL}(n, q)$

The aim of this section is to prove Theorem 1.1 in the case that $G = \mathrm{SL}(n, q)$ and that Z is trivial. Throughout this section we assume Notation 4.1. Thus, $q^e - 1 = p^t d$ where d, e and t are positive integers such that p does not divide d, and e is the multiplicative order of q in the base field $\mathbb{F}_p \subseteq k$. The assumption that the Sylow p-subgroup of G is not cyclic is equivalent to the condition that $n \geq 2e$.

The proof is split into several cases. The first case is when e divides n but the quotient n/e is not a power of p.

Proposition 5.1. *Suppose that $G = \mathrm{SL}(re, q)$ for $r \geq 2$ not a power of p. Assume also that if $p = 2$, then $r \geq 4$. Then the trivial module k is the only indecomposable kG-module with trivial Sylow restriction. In particular, $TT(G) = \{0\}$.*

Proof. First notice that if $r < p$, then a Sylow p-subgroup is abelian and the proposition is proved in [11]. Hence, we may assume further that $n = re > pe$.

The proof is divided into three cases:

(i) $r = 2p^s$ for some $s \geq 1$ and $p > 2$,
(ii) $r = ap^s$ for $2 < a < p$, and
(iii) $r = ap^s + b$ for $1 \leq a < p$ and $1 \leq b < p^s$.

Note that in cases (i) and (ii) we may assume that $e > 1$ and that $p > 2$, as otherwise, p divides both n and $q - 1$. In that case, $\mathrm{SL}(n, q)$ is a perfect group with a nontrivial normal p-subgroup, and the proposition is a consequence of Proposition 2.2.

In the first two cases, let $m = p^s e$, so that $n = am$. Let

$$\widehat{L} \quad = \quad \widehat{L}(m, \ldots, m) \quad \cong \quad \mathrm{GL}(m, q)^a \quad \subseteq \quad \mathrm{GL}(n, q)$$

be the Levi subgroup consisting of a diagonal $m \times m$ blocks. Let $L = \widehat{L} \cap G$ and $N = N_G(L)$. The group N is an extension (perhaps not split) of the form

$$0 \longrightarrow L \longrightarrow N \longrightarrow \mathfrak{S}_a \longrightarrow 0,$$

where $\mathfrak{S}_a$ is the symmetric group on a letters. In addition, N contains the normalizer of a Sylow p-subgroup of G (cf. Theorem 4.3).

Case (i). Suppose that $a = 2$, and $n = am = 2p^s e$. The commutator subgroup $[N, N]$ must contain the perfect group $\mathrm{SL}(m, q) \times \mathrm{SL}(m, q)$. By Lemma 4.4, if q is odd, then the quotient group $N/[N, N]$ is a Klein four group, and we see that we can choose generators represented by the elements

$$\sigma = \begin{bmatrix} & I_m \\ -I_m & \end{bmatrix} \qquad \text{and} \qquad \tau = \begin{bmatrix} c & & & \\ & I_{m-1} & & \\ & & c^{-1} & \\ & & & I_{m-1} \end{bmatrix}$$

where c is a generator for the Sylow 2-subgroup of $\mathbb{F}_q^\times$. If q is even, then $N/[N, N]$ has order 2 and is generated by σ (remembering that $-1 = 1$). To invoke Theorem

3.1 it is enough to show that σ and τ, are in the commutator subgroup of the normalizer of some nontrivial p-subgroup of N.

There is an embedding $\varphi : \mathbb{F}_{q^e} \to \mathrm{Mat}_e(\mathbb{F}_q)$ where $\mathrm{Mat}_e(\mathbb{F}_q)$ is the algebra of $e \times e$ matrices over $\mathbb{F}_q$. This is given as the action of the algebra $\mathbb{F}_{q^e}$ on itself, but regarded as a vector space over $\mathbb{F}_q$. From this we get a homomorphism $\hat{\varphi} : \mathrm{GL}(2p^s, q^e) \to \mathrm{GL}(2p^s e, q)$. That is, the map $\hat{\varphi}$ replaces an element given by a matrix $(a_{i,j})$ by the block matrix $(\varphi(a_{i,j}))$. Again, $\hat{\varphi}$ can be obtained by taking the natural module for $\mathrm{GL}(2p^s, q^e)$ and writing it as a module over $\mathbb{F}_q$ of dimension $2p^s e$.

The group $\mathrm{SL}(2p^s, q^e)$ has a central element Y of order p since $s > 0$. Observe that $\hat{\varphi}(Y)$ is also in N. Let $H_1 = C_G(\hat{\varphi}(Y))$, which contains the image $\hat{\varphi}(\mathrm{SL}(2p^s, q^e))$. In particular, we have that $\varphi(-1) = -I_e$, and so for

$$X = \begin{bmatrix} & I_{p^s} \\ -I_{p^s} & \end{bmatrix} \qquad \text{then} \qquad \hat{\varphi}(X) = \begin{bmatrix} & I_m \\ -I_m & \end{bmatrix} = \sigma.$$

Note that X is in $\mathrm{SL}(2p^s, q^e)$, and hence σ is in the commutator subgroup of H_1. Moreover, because H_1 has a central element of order p, any indecomposable kH_1-module with trivial Sylow restriction has dimension one. Thus H_1 and $g_1 = \sigma$ satisfy condition (B)(2) of Theorem 3.1 with $H = N$. Clearly, any element of $[N, N]$, satisfies condition (B)(1) of Theorem 3.1, which implies that the proposition holds in case (i) if q is even, because $N = \langle [N, N], g_1 \rangle$.

To finish the proof for q odd, we prove the similar result for τ. Let $\widehat{H}_2 = \widehat{L}(2m - e, e) \subseteq \mathrm{GL}(n, q)$ be the Levi subgroup consisting of diagonal block matrices of sizes $2m - e$ and e, and let $H_2 = \widehat{H}_2 \cap G$. By Proposition 4.5, any indecomposable kH_2-module with trivial Sylow restriction has dimension one. Clearly, $H_2 \cap N$ has order divisible by p. The commutator subgroup $[H_2, H_2] \cong \mathrm{SL}(2m - e, q) \times \mathrm{SL}(e, q)$ contains the element τ. So condition (B)(2) of Theorem 3.1 is satisfied for $g_2 = \tau$ and H_2, and the proposition holds in case (i).

Case (ii). Now suppose that $2 < a < p$. In this case, the quotient group $N/[N, N]$ has order 2 and a generator is represented by the element

$$\sigma = \begin{bmatrix} & I_m & \\ -I_m & & \\ & & I_{(a-2)m} \end{bmatrix}$$

Again, it is enough to show that σ, is in the commutator subgroup of an appropriate subgroup of G to invoke Theorem 3.1. Let $\widehat{L} = \widehat{L}(2m, (a-2)m) \cong \mathrm{GL}(2m, q) \times \mathrm{GL}((a-2)m, q)$ be the Levi subgroup of diagonal block matrices of size $2m$ and $(a-2)m$, for $m = p^s e$. Let $H_1 = \widehat{L} \cap G$. Every indecomposable kH_1-module with trivial Sylow restriction has dimension one, by Proposition 4.5. Clearly, $H_1 \cap N$ has order divisible by p, and σ, is in $[H_1, H_1] \cong \mathrm{SL}(2m, q) \times \mathrm{SL}((a-2)m, q)$. Again Condition (B)(2) of Theorem 3.1 holds for σ, and H_1. So the proposition is proved also in case (ii).

Case (iii). Let $\widehat{L} = \widehat{L}(ap^se, be) \cong \mathrm{GL}(ap^se, q) \times \mathrm{GL}(be, q)$ be the Levi subgroup of blocks of size ap^se and be, and put $N = \widehat{L} \cap G$. Observe that, N contains the normalizer of a Sylow p-subgroup of G. Thus by Proposition 4.5, any indecomposable kN-module with trivial Sylow restriction has dimension one.

The commutator subgroup of N is the direct product $\mathrm{SL}(ap^se, q) \times \mathrm{SL}(be, q)$, implying that $N/[N, N] \cong \mathbb{F}_q^\times$. Hence, N is generated by $[N, N]$ and a diagonal matrix σ with diagonal entries $1, 1, \dots, 1, w, w^{-1}, 1, \dots, 1$ where w is a generator of $\mathbb{F}_q^\times$ and the nonidentity entries occur in rows ap^se and $ap^se + 1$.

Now let $\widehat{H}_1 = \widehat{L}(ap^se-1, be+1) \cong \mathrm{GL}(ap^se-1, q) \times \mathrm{GL}(be+1, q)$, the Levi subgroup of blocks of size $ap^se - 1$ and $be + 1$. Let $H_1 = \widehat{H}_1 \cap G$. It is straightforward to show that condition (B)(2) of Theorem 3.1 is satisfied for $g_1 = \sigma$, H_1 and $H = N$, and the proposition holds in case (iii). This completes the proof. □

The next step is the following.

Proposition 5.2. *Suppose that $G = \mathrm{SL}(p^se, q)$ and $s \geq 1$. Then any indecomposable kG-module with trivial Sylow restriction has dimension one. Thus, if $p^se > 2$, then $TT(G) = \{0\}$.*

Proof. First we should notice that if $e = 1$, that is, if p divides $q - 1$, then G has a central subgroup of order p, and we are done by Proposition 2.2. So assume that $e > 1$. This assumption requires that $p > 2$.

Let $\theta : \mathrm{GL}(e, q)^{p^s} \to \mathrm{GL}(ep^s, q)$ be the injective group homomorphism given by letting $\theta(A_1, \dots, A_{p^s})$ be the block diagonal matrix of $e \times e$ blocks $A_1, \dots, A_{p^s}$:

$$\theta(A_1, \dots, A_{p^s}) = \begin{bmatrix} A_1 & & & \\ & A_2 & & \\ & & \cdots & \\ & & & A_{p^s} \end{bmatrix}$$

Choose an element $u \in \mathrm{SL}(e, q)$ of order p. For $i = 1, \dots, p^s$, let $x_i = \theta(A_1, \dots, A_{p^s})$ where $A_i = u$ and $A_j = I_e$, the $e \times e$ identity matrix, for $j \neq i$. Let $Q = \langle x_1, \dots, x_{p^s} \rangle$.

Since p is odd, Q is the unique elementary abelian subgroup of rank p^s in some given Sylow p-subgroup S of G and each elementary abelian subgroup of S is conjugate to a subgroup of Q, by Theorem 4.3. Thus Q is characteristic in S, which implies that $N_G(S) \subseteq N_G(Q)$. Hence, we may apply Theorem 3.1 to $H = N_G(Q)$.

Write $S = C \wr R$ where $u \in C$ and $C \cong C_{p^t}$ is a Sylow p-subgroup of $\mathrm{GL}(e, q)$, and where $R \cong (C_p)^{\wr s}$ is a Sylow p-subgroup of $\mathfrak{S}_{p^s}$. Note that $\langle u \rangle \subseteq C$ with equality if and only if $t = 1$.

From [11, Section 6], we have $N_{\mathrm{GL}(e,q)}(\langle u \rangle) = N_{\mathrm{GL}(e,q)}(C) = \langle w, g \rangle \cong C_{q^e-1} \rtimes C_e$ where ${}^g w = w^q$. Hence, $H = N_G(Q)$ is an extension

$$1 \longrightarrow J \longrightarrow H \longrightarrow \mathfrak{S}_{p^s} \longrightarrow 1$$

where

$$J = N_{\mathrm{GL}(e,q)}(C)^{p^s} \cap G = \{\theta(A_1, \dots, A_{p^s}) \mid A_i \in N_{\mathrm{GL}(e,q)(C)}\ ,\ \prod_{1 \leq i \leq p^s} \mathrm{Det}(A_i) = 1\}$$

Thus J is generated by conjugates under $\mathfrak{S}_{p^s}$ of elements of the form

$$a = \theta(A_1, A_2, I_e, \ldots, I_e) \quad \text{where}$$

$A_1, A_2 \in N_{\mathrm{GL}(e,q)}(C)$ and $\mathrm{Det}(A_1)\,\mathrm{Det}(A_2) = 1$. So H is generated by J and elements of the form

$$X_i = \begin{bmatrix} I_{e(i-1)} & & \\ & \tau & \\ & & I_{e(p^s-i-1)} \end{bmatrix} \quad \text{with} \quad \tau = \begin{bmatrix} & I_e \\ -I_e & \end{bmatrix}$$

for $1 \leq i \leq p^s - 1$. Note that all X_i are conjugate.

For $i = 1, \ldots, p^s - 1$, let $R_i = \langle x_i, x_{i+1} \rangle \subset S$. Then $N_G(R_i)$ is an extension

$$1 \longrightarrow (N_{\mathrm{GL}(e,q)}(C) \wr \mathfrak{S}_2) \cap G \longrightarrow N_G(R_i) \longrightarrow \mathrm{SL}(e(p^s-2), q) \longrightarrow 1$$

If $p^s > 3$, then the elements a and X_1 lie in the commutator subgroup of $N_G(R_3)$. A similar condition holds for any conjugates of a and X_1 under the action of $\mathfrak{S}_{p^s}$. By applying Theorem 3.1 to $H = N_G(Q)$ and the generators given above, we are done.

We are left with the case $p^s = 3$ and $e = 2$. A computer calculation shows that for $G = \mathrm{SL}(6, 2)$ with $p = 3$, we have $\rho^2(Q) \neq N_G(Q)$. Hence, the method of Theorem 3.1 fails. One the other hand, [11, Proposition 7.9] shows that $a, X_1 \in \rho^2(R_1)$ and therefore all their conjugates are in $\rho^3(Q)$. Thus $N_G(Q) = \rho^3(Q)$, and Corollary 4.6 of [16] asserts that $TT(G) = \{0\}$ in this case. □

We are now ready for the proof of the main theorem of the section.

Theorem 5.3. *Assume Notation 4.1. Let $G = \mathrm{SL}(n, q)$ with $n \geq 2e$ if p is odd, or $n \geq 3$ if $p = 2$. Then the trivial kG-module is the unique indecomposable kG-module with trivial Sylow restriction.*

Proof. Let $n = re + f$ with $r \geq 2$ and $0 \leq f < e$. By Propositions 5.1 and 5.2, the theorem is true if $f = 0$. Hence, we assume that $f > 0$ and thus also $e > 1$. There is a natural embedding of $\mathrm{SL}(n-1, q) \hookrightarrow \mathrm{SL}(n, q)$. It is an easy exercise to show that the index of $\mathrm{SL}(n-1, q)$ in $\mathrm{SL}(n, q)$ is prime to p and, hence, $\mathrm{SL}(n-1, q)$ contains a Sylow p-subgroup of $\mathrm{SL}(n, q)$. By [11, Theorem 9.6], the restriction map $T(\mathrm{SL}(n, q)) \to T(\mathrm{SL}(n-1, q))$ is injective since $e > f \geq 1$. Therefore, by induction on f, the proof of the theorem is complete. □

6. Proof of Theorem 1.1

The proof of Theorem 1.1 is a consequence of Theorem 5.3 and a case by case inspection depending on the p-part of the central subgroup Z of G in Theorem 1.1. The next proposition is an essential step in the general proof.

Proposition 6.1. *Let $G = \mathrm{SL}(n, q)$ where $n = rp \geq 3$ for some $r \geq 1$ and assume that p divides $q - 1$. If $n = p = 3$, assume further that 9 divides $q - 1$. Let Z be a nontrivial central subgroup of G. Then the trivial $k(G/Z)$-module is the unique indecomposable $k(G/Z)$-module with trivial Sylow restriction.*

Proof. Recall that, in general, if A, B, C are groups such that $C \subseteq B \subseteq A$ and C is normal in A, then $N_{A/C}(B/C) = N_A(B)/C$. This fact is used to identify normalizers.

First, we consider the case that Z is a nontrivial p-group. Let T be the torus of diagonal matrices in G, and let Q be a Sylow p-subgroup of T. We choose S to be a Sylow p-subgroup of G that contains Q. We note that Q is characteristic in S, it being the unique abelian subgroup isomorphic to $(C_{p^t})^{n-1}$, where t is the highest power of p that divides $q-1$. Therefore, $N_G(S) \subseteq N_G(Q)$ and we have an extension

$$1 \longrightarrow T \longrightarrow N_G(Q) \longrightarrow \mathfrak{S}_n \longrightarrow 1.$$

There is an inclusion

$$N_{G/Z}(S/Z) = N_G(S)/Z \subseteq N_G(Q)/Z = N_{G/Z}(Q/Z)$$

where $N_{G/Z}(Q/Z)$ is an extension

$$1 \longrightarrow T/Z \longrightarrow N_{G/Z}(Q/Z) \longrightarrow \mathfrak{S}_n \longrightarrow 1.$$

Let $N = N_G(Q)$. Then $[N, N]$ has index 2 in N, and so $(N/Z)/[N/Z, N/Z]$ has order two and is generated by the class of the element

$$X = \begin{bmatrix} U & \\ & I_{n-2} \end{bmatrix}, \quad \text{where} \quad U = \begin{bmatrix} & 1 \\ -1 & \end{bmatrix}.$$

By Theorem 3.1, the proof of the theorem is complete in this case if we show that the class of X is in the commutator subgroup of some nontrivial p-subgroup R/Z of Q/Z. For this let R be the subgroup generated by

$$Y = \begin{bmatrix} V & \\ & I_{n-3} \end{bmatrix}, \quad \text{where} \quad V = \begin{bmatrix} \zeta & & \\ & \zeta & \\ & & \zeta^{-2} \end{bmatrix},$$

where ζ is a generator for the Sylow p-subgroup of $\mathbb{F}_q^\times$. Note that the matrix V is not a scalar matrix. Then the normalizer of R contains the Levi subgroup

$$L = (\mathrm{GL}(2, q) \times \mathrm{GL}(1, q) \times \mathrm{GL}(n-3, q)) \cap G.$$

It follows that the class of X in G/Z is in $[L/Z, L/Z]$ which is contained in the commutator subgroup $[N_{G/Z}(R/Z), N_{G/Z}(R/Z)]$. By Theorem 3.1, the trivial $k(G/Z)$-module is the unique indecomposable module with trivial Sylow restriction.

Next assume that Z is an arbitrary nontrivial central subgroup of G. Let $\widehat{Z}$ denote the Sylow p-subgroup of Z. If M is an indecomposable $k(G/Z)$-module with trivial Sylow restriction, then M inflates to an indecomposable trivial Sylow restriction $k(G/\widehat{Z})$-module $\mathrm{Inf}_{G/Z}^{G/\widehat{Z}} M$ on which $Z/\widehat{Z}$ acts trivially. But we have just shown that $\mathrm{Inf}_{G/Z}^{G/\widehat{Z}} M$ must have dimension one. Thus M is trivial. $\square$

The following is also required.

Proposition 6.2. *Suppose that $n \geq 2$, p divides $q-1$, and if $p = n = 3$ assume that 9 divides $q-1$. Let S be a Sylow p-subgroup of $G = \mathrm{SL}(n,q)$ that contains the torus T of diagonal matrices of order p. Then $N_G(T)$ is generated by a collection of elements, each of which is in the commutator subgroup of the normalizer of a subgroup of T that is not central in G.*

Proof. There is no loss of generality in assuming that S contains the Sylow p-subgroup of the torus of diagonal matrices of determinant one, and so contains T. Let $Y \in S$ be as in the proof of Proposition 6.1. Then the commutator subgroup of the normalizer $N_G(\langle Y\rangle)$ of the subgroup generated by Y contains any element of the form

$$X = \begin{bmatrix} U & \\ & I_{n-2} \end{bmatrix}, \quad \text{for} \quad U \in \mathrm{SL}(2,q).$$

Any conjugate of X, under a permutation matrix P, is contained in the commutator subgroup of the normalizer of $PYP^{-1} \in S$. It is not difficult to show that $N_G(T)$ is generated by elements of this form. □

We can now prove the main theorem.

Proof of Theorem 1.1. Assume the notation of Theorem 1.1. Note that if $e = 1$, $n = 2$, $p > 2$, and Z does not contain a Sylow p-subgroup of $Z(G)$, then a Sylow p-subgroup of G is abelian of p-rank 2, and G/Z has a nontrivial normal p-subgroup. Thus Theorem 1.1 holds by Corollary 4.7 and Proposition 2.2. Likewise when $n = p = 3$ divides $q-1$ and Z does not contain a Sylow 3-subgroup of $Z(G)$, then G/Z has a nontrivial normal 3-subgroup and the conclusion of the theorem follows. If 3 does not divide $q-1$, then $n < 2e$ and the theorem does not apply. In the rest of the proof, we assume that $n \geq 3$ and that if $n = 3$ then $p > 3$.

Let $\widehat{G} = Z \cdot \mathrm{SL}(n,q)$ and $\widehat{Z} = Z \cap \mathrm{SL}(n,q)$. Then $\widehat{G}/Z \cong \mathrm{SL}(n,q)/\widehat{Z}$. We first prove the theorem for $G = \widehat{G}$. There are two cases to consider.

Assume first that p does not divide the order of $\widehat{Z}$. Then any indecomposable $k(\widehat{G}/Z)$-module with trivial Sylow restriction inflates to a $k\,\mathrm{SL}(n,q)$-module with trivial Sylow restriction. By Theorem 5.3, this must be the trivial module.

Suppose that p divides the order of $\widehat{Z}$. Because any element of $\widehat{Z}$ is a scalar matrix, p must divide $q-1$ and n. By hypothesis, if $p = 2$, then $n > 3$; while if $p = 3$, then $n > 3$. In all cases $n \geq p$. Hence, by Proposition 6.1, the trivial module is the unique indecomposable $\widehat{G}/Z$-module with trivial Sylow restriction.

Next suppose that the index of $\widehat{G}$ in G is a power of p, so that $(G/Z)/(\widehat{G}/Z)$ is a p-group. In this case, p divides $q-1$. For convenience, let $K = G/Z$ and $J = \widehat{G}/Z$ so that K/J is a p-group. Let S be a Sylow p-subgroup of K and $S' = J \cap S$, a Sylow p-subgroup of J. Recall that $J \cong \mathrm{SL}(n,q)/\widehat{Z}$. We may assume that S' contains the image T (modulo Z) of the torus of diagonal matrices of order p in $\mathrm{SL}(n,q)$, and that T is normal in S and S'. Thus, Proposition 6.2 says that $N_J(T)$ is generated by a collection of elements, each of which is in the commutator subgroup of the normalizer of some nontrivial subgroup of T, which is not central in G, and therefore cannot be contained in Z. By Theorem 3.1, with $H = N_K(T) = SN_J(T)$, we conclude that the trivial module is the unique indecomposable kK-module with trivial Sylow restriction.

Finally, suppose that there is a subgroup H such that $\widehat{G} \subseteq H \subseteq G$, and such that $H/\widehat{G}$ is a Sylow p-subgroup of $G/\widehat{G}$. Note that by hypothesis, $H/\widehat{G}$ is non-trivial. Theorem 5.3 and Proposition 6.1 show that the trivial module is the unique indecomposable $k(H/Z)$-module with trivial Sylow restriction. Note that the index of H in G is prime to p. Suppose that M is an $k(G/Z)$-module with trivial Sylow restriction. Then

$$M_{\downarrow H/Z} \quad \cong \quad k \oplus (\text{proj}),$$

implying that M is a direct summand of $(k_{H/Z})^{\uparrow G/Z}$. However, the restriction of $(k_{H/Z})^{\uparrow G/Z}$ to H/Z is a direct sum of copies of k, since H/Z is normal in G/Z and has index coprime to p. Both conditions can only occur if M has dimension one.

We have shown that if S is a Sylow p-subgroup of G/Z, then the kernel of the restriction map $T(G/Z) \to T(S)$ is $X(G/Z)$ the group of one-dimensional $k(G/Z)$-modules. The proof of Theorem 1.1 is completed using Corollary 4.7. □

7. Type A_1 in characteristic $p \geq 3$

In the case that $n = 2$ and p is odd, the Sylow p-subgroup of a subquotient of $\mathrm{GL}(2, q)$ can be cyclic, and so the structure of (G/Z) changes accordingly. In this section, we briefly discuss some cases that were not included in the results of [11] and are also excluded from Theorem 1.1 by condition (b) of the hypothesis. The techniques are well known, so only a sketch of the proof is given. As before, write $\mathrm{Det}(H)$ for the image under the determinant map of a subgroup H of G .

Theorem 7.1. *Assume that $p > 2$ and that p divides $q-1$. Suppose that G is a group such that $\mathrm{SL}(2, q) \subseteq G \subseteq \mathrm{GL}(2, q)$. Let $Z \subseteq Z(G)$ be a central subgroup of G. Then $|Z|$ divides $2 \cdot |\mathrm{Det}(G)|$.*

(a) *If p divides $|Z(G) : Z|$, that is, if Z does not contain the Sylow p-subgroup of $Z(G)$, then $T(G/Z) = X(G/Z) \oplus \mathbb{Z}$.*
(b) *Otherwise, $T(G)$ is an extension*

$$0 \longrightarrow X(N_{G/Z}(S)) \longrightarrow T(G) \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

where S is a Sylow p-subgroup of G/Z and the right-hand map in the sequence is the restriction onto $T(S) \cong \mathbb{Z}/2\mathbb{Z}$.

Proof. The subgroup Z consists of scalar matrices. Let I_2 denote the identity matrix in G. If $aI_2 \in Z$, then $a^2 \in \mathrm{Det}(G)$. It follows that $|Z|$ divides $2|\mathrm{Det}(G)|$ as asserted.

A Sylow p-subgroup S of G/Z is cyclic if and only if Z contains the Sylow p-subgroup of $Z(G)$. In this case S is isomorphic to a Sylow p-subgroup of $\mathrm{SL}(2, q)$ and S is a TI subgroup of G/Z, implying that the stable categories of G/Z and $N_{G/Z}(S)$ are equivalent. Part (b) of the theorem follows from [23, Theorems 3.2 and 3.6].

Otherwise, i.e. if Z does not contains the Sylow p-subgroup of $Z(G)$, then S is not cyclic and Theorem 1.1 applies, proving part (a) of the theorem. □

8. Type A_2 in Characteristic 3

In this section we consider the endotrivial modules for the groups excluded by condition (c) of the hypothesis of Theorem 1.1. Throughout the section we assume the following. Let $n = p = 3$, and let q be a prime power such that 3 divides $q - 1$ (i.e., $e = 1$). Let $\mathrm{SL}(3, q) \subseteq G \subseteq \mathrm{GL}(3, q)$ and Z a central subgroup of G containing the Sylow 3-subgroup of the center $Z(G)$ of G.

Note that if $n = p = 3$ does not divide $q-1$ and Z contains the Sylow 3-subgroup of the center $Z(G)$ of G, then a Sylow 3-subgroup of G/Z is cyclic and therefore $T(G/Z)$ is known by Theorem 11.1. (A similar situation occurs in type A_1, with $n = 2 < 3 = p$.)

Lemma 8.1. *The group G/Z decomposes as a direct product $G/Z \cong H \times V$ where $Z \cdot \mathrm{SL}(3, q)/Z \subset H$ has index a power of 3 in H and 3 does not divide the order of V. In particular, $T(G/Z) \cong T(H) \oplus X(G/Z)$, where $X(G/Z) \cong X(V)$ is the group of one-dimensional kV-modules.*

Proof. Since $\mathrm{Det}(G) \subseteq \mathbb{F}_q^\times$ is an abelian group, we can write $\mathrm{Det}(G) = U' \times V'$ and $\mathrm{Det}(Z) = U'' \times V''$ where U', U'' are 3-groups and V' and V'' are $3'$-groups. Let $V = V'/V''$. Since $U'' \subseteq U'$, we have $V \cong \mathrm{Det}(G)/(U' \cdot \mathrm{Det}(Z))$. Consider the group homomorphism $\psi : V \to G/Z$ defined by $\psi(a) = aI_3Z \in G/Z$ for each class $a \in V \subseteq \mathbb{F}_q^\times/U' \mathrm{Det}(G)$. Consider also the homomorphism $\vartheta : G/Z \to V$, given as the composition of the induced determinant map on the quotient group, i.e., $\mathrm{Det}(xZ) = \mathrm{Det}(x)\mathrm{Det}(Z) \in \mathrm{Det}(G)/\mathrm{Det}(Z)$ for all $x \in G$, with the quotient onto $\mathrm{Det}(G)/U' \mathrm{Det}(Z) \cong V$. We have $\vartheta\psi(a) = a^3$, which is an automorphism of V because V is a $3'$-group. Since $\psi(V)$ is in the center of G/Z, we conclude that V is a direct factor of G/Z. So the first part of the claim holds with H the kernel of ϑ.

For the last part of the statement, we observe that the kernel of the restriction map $T(G/Z) \to T(H)$ is generated by the isomorphism classes of indecomposable modules in the induction $k_H^{\uparrow G/Z}$ of the trivial kH-module to G/Z. Since the index $|G/Z : H| = |V|$ is not divisible by 3 and the factor group V is abelian, the induced module $k_H^{\uparrow G/Z}$ is a direct sum of one-dimensional modules on which H acts trivially. Therefore, the kernel of the restriction map $T(G/Z) \to T(H)$ is isomorphic to $X(V) \cong X(G/H)$ as required. □

The next result provides a description of H and V under our assumptions.

Proposition 8.2. *For G and Z as above, one of the two situations occurs.*

(a) *If 3 does not divide $(q-1)/|\mathrm{Det}(G)|$, i.e., if Z contains the Sylow 3-subgroup of $Z(\mathrm{GL}(3, q))$, then $G/Z \cong \mathrm{PGL}(3, q) \times V$ where $V \cong \mathrm{Det}(G)/\mathrm{Det}(Z)$.*

(b) *Otherwise, $G/Z \cong \mathrm{PSL}(3, q) \times V$ where V is the 3-complement in $\mathrm{Det}(G)/\mathrm{Det}(Z)$.*

In both cases, $T(G/Z) \cong T(H) \oplus X(G/Z)$, where $X(G/Z) \cong X(V)$, the group of one-dimensional kV-modules and H is either $\mathrm{PGL}(3, q)$ or $\mathrm{PSL}(3, q)$ as appropriate.

Proof. Suppose that 3 does not divide $(q - 1)/|\mathrm{Det}(G)|$. By Lemma 8.1 and its proof, we may assume that $\mathrm{Det}(G) = G/\mathrm{SL}(3, q)$ is a 3-group. In the case that $\mathrm{Det}(G)$ is a Sylow 3-subgroup of $\mathbb{F}_q^\times$, we must have $G/Z \cong \mathrm{PGL}(3, q)$, which proves (a).

Otherwise, $\mathrm{Det}(G)$ is not a Sylow 3-subgroup of $\mathbb{F}_q^\times$, and so there exists an element $\gamma \in \mathbb{F}_q^\times$, $\gamma \notin \mathrm{Det}(G)$ such that γ^3 is in $\mathrm{Det}(G)$. Then the scalar matrix $X = \gamma I_3$ is an element of G with the property that $\mathrm{Det}(X)$ generates $\mathrm{Det}(G)$, because $\mathbb{F}_q^\times$ is a cyclic group. Since Z contains the Sylow 3-subgroup of $\mathrm{Det}(G)$, it follows that $X \in Z$, and $Z \cdot \mathrm{SL}(3,q) = G$. Hence,

$$G/Z \cong Z \cdot \mathrm{SL}(3,q)/Z \cong \mathrm{SL}(3,q)/(Z \cap \mathrm{SL}(3,q)) \cong \mathrm{PSL}(3,q),$$

which proves (b).

The last statement, about the group of endotrivial modules, follows because a complete set of nonisomorphic simple kV-modules all have dimension one and define different blocks of $k(G/Z)$. Thus, any indecomposable endotrivial $k(G/Z)$-module is the (outer) tensor product of a one-dimensional kV-module and an indecomposable endotrivial kH-module. Finally it should be noted that H has a nontrivial normal 3-subgroup, since, by construction, H is an extension of $Z\cdot\mathrm{SL}(3,q)/Z$ by a nontrivial 3-group. Thus, $X(H)$ is trivial. □

By Theorem 2.1, the torsion free rank of $T(G)$ is related to the number of conjugacy classes of maximal elementary abelian 3-subgroups of rank 2. The following calculation is important to determine $TF(G)$.

Proposition 8.3. *Let G and Z be as above. The group G/Z has 3-rank 2. In addition,*

(a) *The group $\mathrm{PGL}(3,q)$ has three conjugacy classes of maximal elementary abelian 3-subgroups.*
(b) *If $q \equiv 1 \pmod 9$ then $\mathrm{PSL}(3,q)$ has four conjugacy classes of maximal elementary abelian 3-subgroups.*
(c) *If $q \equiv 4,7 \pmod 9$ then a Sylow 3-subgroup of $\mathrm{PSL}(3,q)$ is elementary abelian of order 9.*

Proof. Write $q-1 = 3^t d$ where 3 does not divide d, and suppose that $|Z| \geq 3$. A Sylow 3-subgroup S of $G = \mathrm{GL}(3,q)$ is generated by elements

$$X_1 = \begin{bmatrix} \zeta & & \\ & 1 & \\ & & 1 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 1 & & \\ & \zeta & \\ & & 1 \end{bmatrix}, \quad X_3 = \begin{bmatrix} 1 & & \\ & 1 & \\ & & \zeta \end{bmatrix}, \quad Y = \begin{bmatrix} & 1 & \\ & & 1 \\ 1 & & \end{bmatrix},$$

where ζ is a primitive 3^t root of unity in $\mathbb{F}_q$. Let x_1, x_2, x_3 and y denote the images of X_1, X_2, X_3 and Y (respectively) in $G = \mathrm{PGL}(3,q)$. Note that $S \cap Z(\mathrm{GL}(3,q)) = Z(S) = \langle X_1X_2X_3 \rangle$.

Thus a Sylow 3-subgroup of G has a presentation

$$S/Z(S) = \langle x_1, x_2, y \mid x_i^{3^t} = y^3 = 1\ ,\ {}^y x_1 = x_1^{-1}x_2^{-1}\ ,\ {}^y x_2 = x_1 \rangle \cong (C_{3^t} \times C_{3^t}) \rtimes C_3$$

The only central subgroup of order 3 in $S/Z(S)$ is generated by the element $x_1^r x_2^{-r}$ for $r = 3^{t-1}$.

The 3-group $S/Z(S)$ has rank 2 and each noncyclic elementary abelian subgroup has the form $\langle z, x \rangle$ for some noncentral element $x \in S/Z(S)$ of order 3. Note that $(x_i^j y)^3 = 1$ for $i = 1,2$ and any $0 \leq j < 3^t$. Moreover the unique subgroup $C_3 \times C_3$ in the normal subgroup $C_{3^t} \times C_{3^t}$ of S generated by x_1 and x_2 is characteristic in S. The other maximal elementary abelian subgroups have the form $\langle z, x\rangle$ with $x \notin \langle x_1, x_2\rangle$. All such elements x have order 3. A routine calculation shows that there are three S-conjugacy classes of these, namely

$$\langle z, y\rangle, \qquad \langle z, x_1y\rangle \quad \text{and} \quad \langle z, x_1y^2\rangle$$

This is determined, for example, by looking at the monomial matrices obtained by conjugating X_1Y, and it gives us a total of four $S/Z(S)$-conjugacy classes of elementary abelian subgroups of $S/Z(S)$ of order 9.

In $\mathrm{GL}(3,q)$, with the above elements X_1, X_2, X_3, Y we get that

$$X_1Y = \begin{bmatrix} & \zeta & \\ & & 1 \\ 1 & & \end{bmatrix} = {}^T(X_1Y^2) \quad \text{where} \quad T = \begin{bmatrix} 1 & & \\ & & 1 \\ & 1 & \end{bmatrix}.$$

Thus $\langle Z, X_1Y\rangle$ is G-conjugate to $\langle Z, X_1Y^2\rangle$. The same holds in $G = \mathrm{PGL}(3,q)$, that is, there are exactly three G-conjugacy classes of $C_3 \times C_3$. This finishes the proof of (a)

For (b), we refer the reader to the results in [20], where S is a 3-group studied by N. Blackburn and denoted $B(3, 2t; 0, 0, 0)$. Then the 3-fusion system defined by $\mathrm{PSL}(3,q)$ on S stabilizes the three conjugacy classes of 3-centric radical elementary abelian subgroups of order 9 of S, and there is a single conjugacy class of elementary abelian subgroups of order 9 that are not 3-centric radical in S. As a consequence, no two of the four conjugacy classes of elementary abelian subgroups of S of order 9 fuse in G/Z (cf. [20, Theorem 5.10 and Tables 2 and 4]).

Finally, (c) is immediate from the observations that 9 is the highest power of 3 which divides $|\mathrm{PSL}(3,q)|$ for $q-1 \equiv 3 \pmod 9$ and that $\mathrm{PSL}(3,q)$ has no element of order 9. □

The following is the main result of the section.

Theorem 8.4. *Suppose that* $n = p = 3$*, and that* q *a prime power such that* 3 *divides* $q-1$ *(i.e.,* $e = 1$*). Let* $\mathrm{SL}(3,q) \subseteq G \subseteq \mathrm{GL}(3,q)$ *and* Z *a central subgroup of* G *containing the Sylow* 3*-subgroup of the center* $Z(G)$ *of* G*. Then the following hold.*

(a) *If* 3 *does not divide* $(q-1)/|\mathrm{Det}(G)|$ *then* $T(G/Z) \cong \mathbb{Z}^3 \oplus X(G/Z)$.

(b) *If* $q \equiv 1 \pmod 9$ *and if* 3 *divides* $(q-1)/|\mathrm{Det}(G)|$ *then* $T(G/Z) \cong \mathbb{Z}^4 \oplus X(G/Z)$.

(c) *If* $q \equiv 4, 7 \pmod 9$ *and if* 3 *divides* $(q-1)/|\mathrm{Det}(G)|$ *then* $T(G/Z) \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus X(G/Z)$.

In every case, $X(G/Z) \cong X(V)$ *is the character group of* V*, the normal* 3*-complement in the cyclic group* $\mathrm{Det}(G)/\mathrm{Det}(Z)$*.*

Proof. The factors $X(G/Z)$ are determined in Proposition 8.2. The ranks of the torsion free parts of $T(G/Z)$ are established in Proposition 8.3. The only question is the kernel $K(G/Z)$ of the restriction $T(G/Z) \to T(S)$ where S is a Sylow 3-subgroup of G/Z, which is isomorphic to either $\mathrm{PGL}(3,q)$ or $\mathrm{PSL}(3,q)$. In cases (a) and (b), we compute $K(G/Z)$ using Theorem 3.1 with H being the normalizer of the image of the torus in $\mathrm{GL}(3,q)$ or $\mathrm{SL}(3,q)$ respectively. Note here that the normalizer of the image of the torus is equal to the image of the normalizer of the torus. The calculation is very similar to that in the proofs of Propositions 5.1 and 5.2, and we leave it to the reader to fill in the details. The result is that $K(G/Z) = TT(G/Z) = \{0\}$ for $G/Z = \mathrm{PGL}(3, q)$ in case (a) and for $G/Z = \mathrm{PSL}(3, q)$ in case (b).

The only thing left is the calculation of $K(G/Z) = TT(G/Z)$ in case (c), where $G/Z = \mathrm{PSL}(3, q)$ and $q \equiv 4, 7 \pmod 9$. In this situation, a Sylow 3-subgroup S of G/Z is elementary abelian of order 9 and the methods of [16] apply. More precisely, [16, Theorem 8.4] shows that $K(G/Z) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. □

9. Type A_2 in Characteristic 2

Throughout this section let $p = 2$ and G be a group such that $\mathrm{SL}(3, q) \subseteq G \subseteq \mathrm{GL}(3, q)$. Let Z be a central subgroup of G. Our objective is to determine $T(G/Z)$ under these assumptions, namely addressing the first part of the cases of Theorem 1.1 excluded by condition (d) of the hypothesis.

We begin with a decomposition of G/Z (similar to that for $p = 3$ and Lemma 8.1).

Lemma 9.1. *Let G and Z be as given above. Then $G/Z \cong H/Z_3 \times W_2 \times W$ where*

(a) *W is the direct product of the Sylow ℓ-subgroups of $\mathrm{Det}(G)/\mathrm{Det}(Z)$ for ℓ not equal to 2 or 3,*

(b) *W_2 is the Sylow 2-subgroup of $\mathrm{Det}(G)/\mathrm{Det}(Z)$,*

(c) *Z_3 is the Sylow 3-subgroup of Z, and*

(d) *H is an extension*

$$1 \longrightarrow \mathrm{SL}(3, q) \longrightarrow H \longrightarrow V_3 \longrightarrow 1$$

where V_3 is the Sylow 3-subgroup of $\mathrm{Det}(G)/\mathrm{Det}(Z)$.

Moreover, $T(G/Z) \cong T(H/Z_3 \times W_2) \oplus X(W)$.

Proof. The proof follows the same line of reasoning as in Lemma 8.1. That is, the composition

$$Z(G) \to G \to \mathrm{Det}(G)$$

induces an isomorphism from the Sylow ℓ-subgroup of $Z(G)$ to the Sylow ℓ-subgroup of $\mathrm{Det}(G)$ for all primes $\ell \neq 3$. The same holds for the induced composition

$$Z(G)/Z \to G/Z \to \mathrm{Det}(G)/\mathrm{Det}(Z).$$

One can finish the proof by letting H be the inverse image of the Sylow 3-subgroup of $\mathrm{Det}(G)/\mathrm{Det}(Z)$ under the map induced by the determinant. □

Now we can state the main theorem of the section.

Theorem 9.2. *Let G be a group such that $\mathrm{SL}(3, q) \subseteq G \subseteq \mathrm{GL}(3, q)$ and let $Z \subseteq Z(G)$. Assume that the field k has characteristic 2.*

(a) *Suppose that 2 divides the order of $\mathrm{Det}(G)/\mathrm{Det}(Z)$. Then $T(G/Z) \cong \mathbb{Z} \oplus X(G/Z)$.*

(b) *Suppose that 2 does not divide the order of $\mathrm{Det}(G)/\mathrm{Det}(Z)$. Then*

   (i) *if 4 divides $q - 1$, then $T(G/Z) \cong \mathbb{Z} \oplus X(G/Z)$,*

   (ii) *if 4 divides $q + 1$, then $T(G/Z) \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus X(G/Z)$.*

Proof. Let H, Z_3, W_2 and W be as in Lemma 9.1. Suppose that 2 divides the order of $\mathrm{Det}(G)/\mathrm{Det}(Z)$. Observe that G/Z has 2-rank 3, since H has 2-rank 2 (an easy fact that is apparent later in this proof). Moreover, the center of a Sylow 2-subgroup of G has 2-rank 2. So every maximal elementary abelian subgroup has 2-rank 3. This means that $T(G/Z)$ has torsion-free rank one. In addition, G/Z has a nontrivial normal 2-subgroup, and hence every indecomposable $k(G/Z)$-module with trivial Sylow restriction has dimension one. This proves (a).

For the rest of the proof assume that 2 does not divide the order of $\mathrm{Det}(G)/\mathrm{Det}(Z)$. That is, the group W_2 of Lemma 9.1 is trivial. Then, a Sylow 2-subgroup of G/Z is the image of a Sylow 2-subgroup of $\mathrm{SL}(3,q)$ under the map $\mathrm{SL}(3,q) \to (Z\cdot\mathrm{SL}(3,q))/Z \hookrightarrow G/Z$. Thus, G/Z has 2-rank two and any two maximal elementary abelian 2-subgroups are conjugate in $\mathrm{SL}(2,q)$ and also in G/Z. It follows that $TF(G/Z) \cong \mathbb{Z}$ and generated by the class of $\Omega(k)$.

Let S denote the Sylow 2-subgroup of G/Z. We distinguish two cases. First, suppose that $q \equiv 1 \pmod 4$. Then $q-1 = 2^t d$ where d is odd and $|S| = 2^{2t+1}$. We can assume that S is generated by the classes (modulo Z) of the elements

$$X_1 = \begin{bmatrix} \zeta & & \\ & \zeta^{-1} & \\ & & 1 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 1 & & \\ & \zeta & \\ & & \zeta^{-1} \end{bmatrix}, \text{ and } \quad Y = \begin{bmatrix} & 1 & \\ -1 & & \\ & & 1 \end{bmatrix}$$

where $\zeta \in \mathbb{F}_q$ is a root of unity of order 2^t. The group S is a wreath product and has a unique G/Z-conjugacy class of Klein four subgroups. Therefore $TF(G/Z) \cong \mathbb{Z}$.

For the other case, we suppose that $q \equiv 3 \pmod 4$ and that $q+1 = 2^t d$ where d is odd and $|S| = 2^{t+2}$. We can choose the Sylow 2-subgroup S of G/Z that is the collection of classes (modulo Z) of all block matrices of the form

$$\begin{bmatrix} X & \\ & r \end{bmatrix}$$

where $r = \mathrm{Det}(X)^{-1}$ and where X runs through the elements of some fixed Sylow 2-subgroup of $\mathrm{GL}(2,q)$. Thus, S is isomorphic to a Sylow 2-subgroup of $\mathrm{GL}(2,q)$, since the two groups have the same order. It is well known that S is semi-dihedral (see also [3, 17]). Hence, $T(S) \cong \mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$. By [13], the restriction map $T(G/Z) \to T(S)$ is a split surjection.

The only issue left to prove is that every indecomposable $k(G/Z)$-module with trivial Sylow restriction has dimension one. Observe by Lemma 9.1, we may assume that $G = H$ is an extension of $\mathrm{SL}(3,q)$ by a cyclic group whose order is a power of 3. With this assumption $G/Z = H/Z_3$. The asserted result is obtained in two steps.

Let $J = (Z_3\cdot\mathrm{SL}(3,q))/Z_3 \cong \mathrm{SL}(3,q)/(Z_3\cap\mathrm{SL}(3,q))$. The first step is to show that $T(J) = T(S)$, or equivalently, that every indecomposable endotrivial kJ-module with trivial Sylow restriction has dimension one.

Notice that $Z_3 \cap \mathrm{SL}(3,q)$ is in the center of $\mathrm{SL}(3,q)$ and has order either 1 or 3. The normalizer of the Sylow 2-subgroup S of J has the form $N_J(S) = S \times Z(J)$ with $Z(J)$ of order 1 or 3. Suppose that $|Z(J)| = 1$, that is, either $Z_3 \neq \{1\}$ or 3 does not divide $q-1$. Then $T(J) = T(S)$ by Proposition 2.3 and the first step is complete in this case.

Now assume that $Z(J)$ has order 3. Then 3 divides $q-1$, $Z_3 = \{1\}$ and $J = \mathrm{SL}(3,q)$. Let u be a primitive cube root of one. We fix the following elements of J

$$X = \begin{bmatrix} u & & \\ & u^2 & \\ & & 1 \end{bmatrix}, \quad Y = \begin{bmatrix} 1 & & \\ & u^2 & \\ & & u \end{bmatrix}, \quad A = \begin{bmatrix} -1 & & \\ & -1 & \\ & & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & & \\ & -1 & \\ & & -1 \end{bmatrix}.$$

Note that XY generates the Sylow 3-subgroup of $Z(J)$. Moreover, X is in the commutator subgroup (which is isomorphic to $\mathrm{SL}(2,q)$) of $N_J(\langle A\rangle)$ and, similarly, Y is in the commutator subgroup of $N_J(\langle B\rangle)$. Furthermore, $S \subseteq \rho^2(\langle A\rangle)$ by an argument similar to the one used in the proof of Proposition 6.1. So in the notation of Section 3, $X \in \rho^1(\langle A\rangle)$ and $Y \in \rho^1(\langle B\rangle)$. It follows that $Y \in \rho^2(\langle A\rangle)$ and $XY \in \rho^3(S)$. Thus, $\rho^3(S) = N_J(S)$, and from [16] we have that the only indecomposable endotrivial kJ-module with trivial Sylow restriction is the trivial module. This completes the first step.

For the second step and to finish the proof of the theorem, suppose that M is a $k(G/Z)$-module with trivial Sylow restriction. Then $M_{\downarrow J} \cong k \oplus (\mathrm{proj})$. This implies that M is a direct summand of $(k_J)^{\uparrow G/Z}$ which is a direct sum of one-dimensional modules, since J is a normal subgroup of G/Z of odd index. Therefore, M has dimension one. □

10. Type A_1 in Characteristic 2

Throughout this section let $p = 2$ and let G be a group such that $\mathrm{SL}(2,q) \subseteq G \subseteq \mathrm{GL}(2,q)$. Let Z be a central subgroup of G. Our objective is to determine $T(G/Z)$ under these assumptions, namely addressing the second part of the cases of Theorem 1.1 excluded by condition (d) of the hypothesis.

This case is more tedious than the previous one because the group G/Z can have dihedral (including Klein four), semi-dihedral or generalized quaternion Sylow 2-subgroups. The differences in the 2-local structure of G/Z lead to as many distinct outcomes for the structure of $T(G/Z)$, which we now detail.

As in the previous two sections, we start with a useful decomposition of G/Z. The proof of the lemma below is similar to that of Lemma 9.1 and therefore left to the reader.

Lemma 10.1. *For G and Z as above, $G/Z \cong H/Z_2 \times W$ where*

(a) *W is the odd part of $\mathrm{Det}(G)/\mathrm{Det}(Z)$,*
(b) *Z_2 is the Sylow 2-subgroup of Z, and*
(c) *H is an extension*

$$1 \longrightarrow \mathrm{SL}(2,q) \longrightarrow H \longrightarrow V_2 \longrightarrow 1$$

where V_2 is the Sylow 2-subgroup of $\mathrm{Det}(G)/\mathrm{Det}(Z)$.

In addition $T(G/Z) \cong T(H/Z_2) \oplus X(W)$, with $X(H/Z_2) = \{1\}$ and $X(G/Z) \cong X(W)$.

We can now state and prove the main theorem of this section.

Theorem 10.2. *Let G be a group such that $\mathrm{SL}(2,q) \subseteq G \subseteq \mathrm{GL}(2,q)$ and $Z \subseteq Z(G)$. Assume that the field k has characteristic 2. Write $|G : \mathrm{SL}(2,q)| = 2^s m_1$ and $|Z| = 2^r m_2$ where m_1 and m_2 are odd integers. The group of endotrivial modules $T(G/Z)$ is described as follows.*

- (A) *Suppose that $q \equiv 3 \pmod 4$. Write $q+1 = 2^t d$ where d is odd. Note that $r, s \in \{0,1\}$.*
  - (1) *Assume that $s = 0$.*
    - (a) *If $r = 0$, then $T(G/Z) \cong \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus X(G/Z)$.*
    - (b) *If $r = 1$, then*
      - (i) *if $q \equiv 3 \pmod 8$, then $T(G/Z) \cong \mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \oplus X(G/Z)$,*
      - (ii) *if $q \equiv 7 \pmod 8$, then $T(G/Z) \cong \mathbb{Z}^2 \oplus X(G/Z)$.*
  - (2) *Assume that $s = 1$.*
    - (a) *If $r = 0$ then $T(G/Z) \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus X(G/Z)$.*
    - (b) *If $r = 1$ then $T(G/Z) \cong \mathbb{Z}^2 \oplus X(G/Z)$.*
- (B) *Suppose that $q \equiv 1 \pmod 4$. Write $q-1 = 2^t d$ where d is odd. Note that $0 \le r \le s+1$, $r \le t$ and $s \le t$.*
  - (1) *Assume that $r = 0$.*
    - (a) *If $s = 0$, then $T(G/Z) \cong \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus X(G/Z)$.*
    - (b) *If $s > 0$, then $T(G/Z) \cong \mathbb{Z} \oplus X(G/Z)$.*
  - (2) *Assume that $r > 0$.*
    - (a) *If $0 < r < s+1 \le t$, then $T(G/Z) \cong \mathbb{Z} \oplus X(G/Z)$.*
    - (b) *If $r = s+1 \le t$, then*
      - (i) *if $q \equiv 1 \pmod 8$, then $T(G/Z) \cong \mathbb{Z}^2 \oplus X(G/Z)$,*
      - (ii) *if $q \equiv 5 \pmod 8$, then $T(G/Z) \cong \mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \oplus X(G/Z)$.*
    - (c) *If $r = s = t$, then $T(G/Z) \cong \mathbb{Z}^2 \oplus X(G/Z)$.*

Proof. For the purposes of the proof let H and Z_2 be as in Lemma 10.1. Hence, it suffices to find $T(H/Z_2)$ in each of the above cases.

Let K denote the kernel of the restriction map $T(H/Z_2) \to T(S)$, where S is a Sylow 2-subgroup of H/Z_2. We should first note that if $r = 0$ or if $r < s+1 \le t$ or if $r < s = t$, then $K = \{1\}$, and the restriction map is injective. The reason is that in each of these cases H/Z_2 has a nontrivial central 2-subgroup and $X(H/Z_2) = \{1\}$, since $\mathrm{SL}(2,q)$ is a perfect group. Thus $K = \{1\}$ by Proposition 2.2 and Lemma 10.1. It follows that the only cases in which K might not be trivial are (A)(1)(b), (B)(1)(b), and (B)(2)(c).

Suppose first that $q+1 = 2^t d$ for $t > 1$ and d odd. A Sylow 2-subgroup of $\mathrm{GL}(2,q)$ is a semi-dihedral group of order 2^{t+2} and it is self-normalizing, by [3, 17]. This is the Sylow 2-subgroup in the case (A)(2)(a). So the restriction map $T(H/Z_2) \to T(S) \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ is an isomorphism by [13]. In case (A)(1)(a), a Sylow 2-subgroup of H/Z_2 is generalized quaternion, as for $\mathrm{SL}(2,q)$, and so the restriction map $T(H/Z_2) \to T(S) \cong \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ is an isomorphism by [13].

If $r = 1$, the group Z_2 is the center of the Sylow 2-subgroup of H. Thus, the Sylow 2-subgroup S of H/Z_2 is a dihedral group, possibly a Klein four group. In cases (A)(1)(b)(ii), and (A)(2)(b), S is dihedral of order at least 8. In these cases the group H/Z_2 has two conjugacy classes of (maximal) elementary abelian 2-subgroups.

Hence, the torsion-free rank of $T(H/Z_2)$ is two, by Theorem 2.1. Note also that S is self-normalizing. Thus by Proposition 2.3, $T(H/Z_2) \cong \mathbb{Z}^{\times 2}$ as asserted.

In the case (A)(1)(b)(i), $H/Z_2 \cong \mathrm{PSL}(2,q)$, and S is a Klein four group with normalizer $N_{H_2/Z}(S) \cong S \rtimes C_3$ of order 12. The Green correspondents of the non-trivial $kN_{H/Z_2}(S)$-modules of dimension one are $k(H/Z_2)$-modules with trivial Sylow restriction of dimension greater than one. The detailed computation of $T(H/Z_2)$ is carried out in [16].

Suppose now that $q-1 = 2^t d$ for $t > 1$ and d odd. A Sylow 2-subgroup of $\mathrm{GL}(2,q)$ is a wreath product, which we can choose to be generated by

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} \zeta & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 \\ 0 & \zeta \end{bmatrix}$$

where ζ is a 2^t-root of unity in $\mathbb{F}_q^\times$. A Sylow 2-subgroup of $\mathrm{SL}(2,q)$ is a generalized quaternion group [3]. Hence, if $r = s = 0$, then we have the same situation as in case (A)(1)(a). If $r = 0 < s$, then the subgroup consisting of diagonal matrices with entries 1 and -1, has rank 2 and every involution is H/Z_2-conjugate to an element of this subgroup. Consequently, H/Z_2 has a unique conjugacy class of maximal elementary abelian 2-subgroups, all of which have order 4. Hence, the torsion-free rank of $T(G)$ is one and the proof of (B)(1) is complete.

If $0 < r < s+1 \leq t$, then the group S has an elementary abelian subgroup of rank 3, generated by the classes (modulo Z_2) of the elements

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \zeta & 0 \\ 0 & \zeta \end{bmatrix}$$

where ζ is a 2^{s+1}-root of 1 in $\mathbb{F}_q^\times$. In addition, the last two elements above are central in S and hence S has no maximal elementary abelian subgroups of rank 2. Because H/Z_2 has a nontrivial normal 2-subgroup, any indecomposable $k(H/Z_2)$-module with trivial Sylow restriction has dimension 1 and the claim holds in case (B)(2)(a).

If $r = s+1 \leq t$ then the composition

$$\mathrm{SL}(2,q) \to (Z_2 \cdot \mathrm{SL}(2,q))/Z_2 \hookrightarrow H/Z_2$$

is surjective and has kernel $Z(\mathrm{SL}(2,q))$. Thus $H/Z_2 \cong \mathrm{PSL}(2,q)$. Its Sylow 2-subgroup is a dihedral group or, in the case that $q \equiv 5 \pmod 8$, a Klein four group. Hence, we have the same situation as in (A)(1)(b) with the same result. In particular, the results of [16] apply in case (B)(2)(b)(ii).

Finally, in case (B)(2)(c), a Sylow 2-subgroup S of H/Z_2 is isomorphic to the quotient of a Sylow 2-subgroup of $\mathrm{GL}(2,q)$ by its center. So S is a dihedral group of order at least 8 (cf. [17]). Hence, the conclusion is the same as in case (A)(1)(b)(ii). □

11. Appendix: Classification of Endotrivial Modules in the Cyclic Sylow Subgroup Setting

The following result summarizes one of the main results of [11] and provides a classification of the group of endotrivial modules for finite groups of Lie type A in the case when a Sylow p-subgroup of G is cyclic.

Theorem 11.1. *Suppose that* $\mathrm{SL}(n,q) \subseteq G \subseteq \mathrm{GL}(n,q)$ *and that* $Z \subseteq Z(G)$. *Assume that the Sylow* p*-subgroup* S *of* G *is cyclic and let* $N = N_G(S)$. *Then* $T(G/Z) \cong T(\widehat{N})$ *where* $\widehat{N} = N_{G/Z}(\widehat{S})$ *and* $\widehat{S}$ *is a Sylow* p*-subgroup of* G/Z. *Moreover,* $T(\widehat{N})$ *is the middle term of a not necessarily split extension*

$$1 \longrightarrow X(\widehat{N}) \longrightarrow T(\widehat{N}) \longrightarrow T(\widehat{S}) \longrightarrow 0 \qquad (1)$$

where $X(\widehat{N}) \cong N/(Z[N,N])$ *is the group of isomorphism classes of* $k\widehat{N}$*-modules of dimension one. Let* $D = \mathrm{Det}(G) \cong G/\mathrm{SL}(n,q)$ *and let* $d = |D|$. *In the case that* $Z = \{1\}$ *we have the following.*

(a) *If* $p = 2$ *then* $n = 1$, *and* $T(G) \cong D/\mathrm{Det}(S)$.
(b) *Suppose that* $p > 2$ *divides* $q-1$. *If* p *divides* d, *then* $n = 1$ *and* $T(G) \cong \mathbb{Z}/a\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, *where* $d = ap^t$ *for* a *relatively prime to* p.
(c) *If* $p > 2$ *divides* $q-1$ *and* p *does not divide* d, *then there are two possibilities:*
  (i) *assuming that 2 does not divide* $(q-1)/d$, *then* $T(G) \cong \mathbb{Z}/d\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$.
  (ii) *assuming that 2 divides* $(q-1)/d$, *then* $T(G) \cong \mathbb{Z}/d\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.
(d) *Suppose that* p *does not divide* $q-1$. *Let* e *be the least integer such that* p *divides* $q^e - 1$. *Then* $n = e + f$ *for some* f *with* $0 \leq f < e$. *Let* $m = (q-1)/d$ *and* $\ell = \gcd\big(m(q-1), q^e - 1\big)/m$. *Then we have two possibilities:*
  (i) *if* $f = 0$ *then* $T(G) \cong \mathbb{Z}/\ell\mathbb{Z} \oplus \mathbb{Z}/2e\mathbb{Z}$,
  (ii) *while if* $f > 0$, *then* $T(G) \cong \mathbb{Z}/2e\mathbb{Z} \oplus \mathbb{Z}/(q-1)\mathbb{Z} \oplus \mathbb{Z}/d\mathbb{Z}$ *(except that* $T(G) \cong \mathbb{Z}/2e\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ *if both* $f = 2$ *and* $q = 2$*).*

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