DISCONNECTED RATIONAL HOMOTOPY THEORY

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ABSTRACT. We construct two algebraic versions of homotopy theory of rational disconnected topological spaces, one based on differential graded commutative associative algebras and the other one on complete differential graded Lie algebras. As an application of the developed technology we obtain results on the structure of Maurer-Cartan spaces of complete differential graded Lie algebras.

CONTENTS

Introduction

The purpose of this paper is to construct an algebraic theory of rational disconnected topological spaces (or simplicial sets), alluded to in [23, p. 67]. The corresponding theory for *connected* spaces was constructed in the seminal papers [26, 27]. The paper [26] related rational connected spaces to differential graded Lie algebras (dglas) whereas [27] took the perspective of commutative differential graded algebras (cdgas). In the present paper we pursue both points of view and construct both the dgla and cdga algebraic models for disconnected spaces. It is interesting that the differences between the two algebraic categories (which were somewhat hidden in the Quillen and Sullivan approaches) become more pronounced in our more general context; in particular we are naturally led to consider dglas endowed with a linearly compact topology, while our cdgas are still discrete.

The definitive reference establishing a correspondence between rational connected spaces and cdgas is [5], and our treatment relies heavily on the results of that paper. Recall that op. cit. constructed a closed model category structure on *non-negatively graded* cdgas and a Quillen adjunction between this category and the category of simplicial sets. This adjunction restricts to an equivalence between the homotopy categories of connected cdgas of finite $\mathbb{Q}$-type and connected rational nilpotent simplicial sets of finite $\mathbb{Q}$-type.

Our main innovation in establishing a cdga version of disconnected rational homotopy theory is that we remove the restriction that our cdgas be non-negatively graded and use the closed model structure on the category of $\mathbb{Z}$-graded cdgas [14]. This seemingly innocent modification has quite dramatic consequences. For example, any commutative algebra concentrated in degree zero is cofibrant in the Bousfield-Gugenheim category, but *not* in this extended category, unless it is a retract of a polynomial algebra. The closed model category of all cdgas appears more natural than the Bousfield-Gugenheim category; for instance it allows one to define Harrison-André-Quillen cohomology, cf. [3].

There is still a Quillen adjoint pair of functors between the categories of $\mathbb{Z}$-graded cdgas and simplicial sets giving rise to an adjunction on the level of homotopy categories. This adjunction restricts to an equivalence between the homotopy category of simplicial sets having finite number of connected components, each being rational, nilpotent and of finite $\mathbb{Q}$-type, and a certain subcategory of the homotopy category of $\mathbb{Z}$-graded cdgas. We give an explicit characterization of this subcategory.

To construct the second version of the disconnected rational homotopy theory (based on dglas) we need to relate the homotopy theory of commutative and Lie algebras. This relationship, which is sometimes referred to as *Koszul duality* was established in the work of Quillen [26]; it was formulated as the duality between differential graded coalgebras and dglas under certain (fairly severe) restrictions on the grading of the objects under consideration. These restrictions were subsequently removed in the seminal paper of Hinich [15]. In our context we need a result that is Koszul dual to Hinich's; it can be regarded as a Quillen equivalence between the categories of cdgas and differential graded Lie coalgebras (which can be dualized and viewed as complete dglas). We prove this result by a suitable adaptation of Hinich's methods. As a consequence we obtain an algebraic model of disconnected rational homotopy theory based on complete dglas. The condition of completeness is essential; it

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cannot be removed even when restricted to connected spaces (or simplicial sets). In the latter case our theory is close to, but still different from, the one constructed in the papers by Neisendorfer and Baues-Lemaire [2, 24]. In particular, our complete dglas admit minimal models even in the nonsimply-connected case, whereas simply-connectedness is an essential requirement for the existence of minimal models constructed by Neisendorfer and Baues-Lemaire.

One important application of the developed theory that we give in this paper, concerns the structure of Maurer-Cartan spaces. Recall (cf. [11]) that associated to any dgla $\mathfrak{g}$ is a simplicial set $\mathrm{MC}_\bullet(\mathfrak{g})$; in the case when $\mathfrak{g}$ is non-negatively graded and nilpotent, this is a simplicial set corresponding to $\mathfrak{g}$ under the Quillen-Sullivan correspondence. When $\mathfrak{g}$ is *not* non-negatively graded, the simplicial set $\mathrm{MC}_\bullet(\mathfrak{g})$ is much more mysterious; this is a fundamental object of study for deformation theory [17, 21, 23] and also comes up in modern approaches to quantum field theory [8]. It turns out that for differential graded Lie algebras satisfying an appropriate completeness condition there exists an operation of *disjoint product* corresponding to the operation of disjoint union of simplicial sets. The disjoint product of two dglas is never non-negatively graded and we prove that the Maurer-Cartan simplicial set of the disjoint product of complete dglas is weakly equivalent to the disjoint union of the corresponding simplicial sets. Furthermore, for an arbitrary complete dgla $\mathfrak{g}$ the simplicial set $\mathrm{MC}_\bullet(\mathfrak{g})$ naturally decomposes up to homotopy as a disjoint union of Maurer-Cartan spaces of certain *connected* dglas naturally associated with $\mathfrak{g}$. Along the way we establish a general result of independent interest, expressing the Chevalley-Eilenberg cohomology of free products of dglas through the Chevalley-Eilenberg cohomology of the individual pieces. It appears that this result is new even for ordinary Lie algebras.

Similar results on the Maurer-Cartan spaces of dglas were contained in the recent paper [6] by U. Buijs and A. Murillo, who used completely different methods, see, in particular, Theorem 5.5, Proposition 6.2 and Theorem 6.4 in op. cit. Their statements do not include any completeness assumptions. We have been unable to verify the claims made in that paper.

M. Golasiński published a series of papers, see e.g. [12], studying the equivariant rational homotopy type of spaces X with an action of a finite group G with possibly disconnected fixed-point sets X^H. When G is trivial, his theory describes spaces whose connected components are rational nilpotent via families of non-negatively graded complete nilpotent homologically connected cdgas indexed by the components of X. We do not think that Golasiński's theory implies any of our results.

Notation and conventions. All algebraic objects will be considered over a fixed field $\Bbbk$ of characteristic zero, at some places we require specifically $\Bbbk$ to be the field $\mathbb{Q}$ of rational numbers. The abbreviation 'dg' stands for 'differential graded'. Further, we will write 'cdga' and 'dgla' for 'commutative differential graded unital associative algebra' and 'differential graded Lie algebra' respectively. We allow also the terminal algebra $\mathbf{0}$ in which $1 = 0$. For a (co)cycle c, the symbol $[c]$ denotes its cohomology class. The degree of a homogeneous element a is denoted $|a|$.

All our cdgas will have cohomological grading with upper indices while dglas will have homological grading with lower indices. There will, however, be one important exception

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from this rule. At some places, we will need the tensor product of a homologically graded Lie algebra with the cohomologically graded cdga Ω of the Sullivan-de Rham forms. In this context, we consider Ω as homologically graded by $\Omega_* := \Omega^{-*}$. The tensor product will then be a homologically graded dgla as expected.

The *suspension* ΣV of a homologically graded vector space V is defined by the convention $\Sigma V_i = V_{i-1}$ resp. $\Sigma V^i = V^{i-1}$ for V cohomologically graded. The functor of taking the linear dual takes homologically graded vector spaces into cohomologically graded ones so that $(V^*)^i = (V_i)^*$; further we will write ΣV^* for $\Sigma(V^*)$; with this convention there is an isomorphism $(\Sigma V)^* \cong \Sigma V^*$. Regarding spectral sequences, we will use the terminology of [4]. We will often refer to a duality between the category of discrete vector spaces (colimits of finite-dimensional ones) and linearly compact spaces (limits of finite-dimensional ones), see e.g. [20].

Let $\mathscr{A}$ be the category of unital cdgas and $\mathscr{A}_{\geq 0}$ its full subcategory consisting of non-negatively graded cdgas. By [14] and [5] respectively, both $\mathscr{A}$ and $\mathscr{A}_{\geq 0}$ are closed model categories whose weak equivalences are morphisms inducing isomorphisms of cohomology, and fibrations are surjective morphisms.

Note that a cofibration between algebras in $\mathscr{A}_{\geq 0}$ need not be a cofibration in $\mathscr{A}$, therefore an algebra cofibrant in $\mathscr{A}_{\geq 0}$ need not be cofibrant in $\mathscr{A}$. For instance, every cdga concentrated in degree 0 is cofibrant in $\mathscr{A}_{\geq 0}$. Proposition 3.1 presents a wide class of cdgas that are cofibrant in $\mathscr{A}_{\geq 0}$ but not in $\mathscr{A}$.

As usual, a cdga $A = (A, d)$ is *connected* (resp. *homologically connected*)[1] if $A^0 = \Bbbk$ (resp. $H^0(A, d) = \Bbbk$) and $A^n = 0$ (resp. $H^n(A) = 0$) for $n < 0$. Each connected algebra $A \in \mathscr{A}_{\geq 0}$ admits a unique minimal model [5, §7]. Such an algebra A is of *finite $\mathbb{Q}$-type* if A is defined over $\mathbb{Q}$ and its minimal model has finitely many generators in each degree. Equivalently, the cohomology of $H(I/I^2)$, where $I \subset A$ is the ideal of positive-degree elements of A, is finite-dimensional in each degree, see [5, §9.2].

Let us recall some definitions of [5, §9]. The *lower central series* $G = \Gamma_1 G \supset \Gamma_2 G \supset \cdots$ of a group G is defined inductively by letting $\Gamma_{q+1} G$ be the subgroup of G generated by the commutators $\{xyx^{-1}y^{-1}; \ x \in G, y \in \Gamma_q G\}$. A group G is *nilpotent* if $\Gamma_q G = *$ for some $q \geq 1$. The *lower central series* $N = \Gamma_1 N \supset \Gamma_2 N \supset \cdots$ of a G-module N is defined inductively by letting $\Gamma_{q+1} N$ be the sub-module of N generated by $\{gn - n; \ g \in G, n \in \Gamma_q N\}$. A G-module N is *nilpotent* if $\Gamma_q N = 0$ for some $q \geq 1$.

A connected (pointed) Kan simplicial set S is *nilpotent* if $\pi_1(S)$ is a nilpotent group and $\pi_n(S)$ is a nilpotent $\pi_1(S)$-module for each $n \geq 2$.[2] This definition clearly does not depend on the choice of the base point.

A nilpotent simplicial set is *rational* if the groups $\pi_n(S)$ or, equivalently, the groups $H_n(S; \mathbb{Z})$, are uniquely divisible for each $n \geq 1$. Such a simplicial set is of *finite $\mathbb{Q}$-type* if the $\mathbb{Q}$-vector spaces $H_1(S; \mathbb{Q})$ and $\mathbb{Q} \otimes \pi_n(S)$ for $n \geq 2$ or, equivalently, the $\mathbb{Q}$-vector spaces $H_n(S; \mathbb{Q})$ for $n \geq 1$, are finite-dimensional.

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[1]It would be more logical to say *co*homologically connected, but we keep our terminology compatible with [5].

[2]So "nilpotent" always implies "Kan."

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Let $\mathscr{S}$ denote the category of simplicial sets and fN$\mathbb{Q}$-$\mathscr{S}^c$ its subcategory of connected nilpotent rational simplicial sets of finite $\mathbb{Q}$-type. We denote f$\mathbb{Q}$-$\mathscr{A}_{\geq 0}^c \subset \mathscr{A}_{\geq 0}$ the subcategory of homologically connected cdgas of finite $\mathbb{Q}$-type. It is well-known [5, Theorem 9.4] that there exists a pair of adjoint functors

$$(0.1) \qquad\qquad F : \mathscr{A}_{\geq 0} \rightleftarrows \mathscr{S} : A$$

that, for $\Bbbk = \mathbb{Q}$, induces an equivalence of the homotopy categories f$\mathbb{Q}$-ho$\mathscr{A}_{\geq 0}^c$ and fN$\mathbb{Q}$-ho$\mathscr{S}^c$. A similar adjunction and induced equivalence hold also for augmented and pointed versions of the above categories, see again Theorem 9.4 in op. cit.

We denote by $\mathscr{L}$ the category of dglas and by $\hat{\mathscr{L}}$ the category of *complete* dglas, i.e. inverse limits of finite-dimensional nilpotent dglas; the morphisms in $\hat{\mathscr{L}}$ are, naturally, continuous dgla maps. We will show in Section 9 that $\hat{\mathscr{L}}$ is a closed model category whose weak equivalences are maps $f : \mathfrak{g}' \to \mathfrak{g}''$ such that $\mathcal{C}(f) : \mathcal{C}(\mathfrak{g}'') \to \mathcal{C}(\mathfrak{g}')$ is a weak equivalence; here $\mathcal{C}(-)$ is the Chevalley-Eilenberg functor recalled in Definition 7.3. Fibrations in $\hat{\mathscr{L}}$ are surjective morphisms.

The free product of two dglas $\mathfrak{g}$ and $\mathfrak{h}$ will be denoted by $\mathfrak{g} * \mathfrak{h}$. If $\mathfrak{g}$ and $\mathfrak{h}$ are complete dglas then $\mathfrak{g} * \mathfrak{h}$ will stand for the *completed* free product of $\mathfrak{g}$ and $\mathfrak{h}$; it is thus a categorical coproduct of $\mathfrak{g}$ and $\mathfrak{h}$. Given a dgla $\mathfrak{g}$, a *Maurer-Cartan* element in $\mathfrak{g}$ is an element $\xi \in \mathfrak{g}_{-1}$ satisfying the Maurer-Cartan equation: $d\xi + \frac{1}{2}[\xi, \xi] = 0$. We will abbreviate the expression 'Maurer-Cartan' as 'MC'. The set of all MC elements in $\mathfrak{g}$ will be denoted by MC$(\mathfrak{g})$. This definition can be extended to give a simplicial set MC$_\bullet(\mathfrak{g})$ whose vertices are just the MC elements in $\mathfrak{g}$; a precise definition is recalled in the main text. Furthermore, given a dgla $\mathfrak{g}$ and an MC element $\xi \in $ MC$(\mathfrak{g})$ we can define a twisted differential d^ξ in $\mathfrak{g}$ by the formula $d^\xi(?) = d(?) + [?, \xi]$; we will write $\mathfrak{g}^\xi$ for the dgla $\mathfrak{g}$ supplied with the twisted differential.

Given a dgla $\mathfrak{g}$, we denote by $\mathfrak{g}\langle x \rangle$ the dgla obtained from $\mathfrak{g}$ by freely adjoining the variable x with $|x| = -1$ and $d(x) = -\frac{1}{2}[x, x]$. Clearly, $x \in $ MC$(\mathfrak{g}\langle x \rangle)$ and we will write $\mathfrak{g} \sqcup 0$ for the twisted dgla $(\mathfrak{g}\langle x \rangle)^x$. One should view the construction $\mathfrak{g} \sqcup 0$ as the Lie analogue of adjoining an isolated base point to a topological space. Furthermore, for two dglas $\mathfrak{g}$ and $\mathfrak{h}$ we set $\mathfrak{g} \sqcup \mathfrak{h} := (\mathfrak{g} \sqcup 0) * \mathfrak{h}$. The dgla $\mathfrak{g} \sqcup \mathfrak{h}$ will be called the *disjoint product* of $\mathfrak{g}$ and $\mathfrak{h}$; in the case when $\mathfrak{g}$ and $\mathfrak{h}$ are complete we will write $\mathfrak{g} \sqcup \mathfrak{h}$ for the corresponding completion. The operation of disjoint product equips the category of complete dglas with a non-unital monoidal structure. A non-complete version of the disjoint product was considered in [6].

For convenience of the reader, we include a glossary of notation at the end of Section 1.

1. Main results

We call a cdga $A = (A, d) \in \mathscr{A}$ homologically *disconnected* if $H^n(A) = 0$ for $n < 0$ and if $H^0(A)$ is isomorphic to the direct product $\prod_{i \in J} \Bbbk$ of copies of the ground field indexed by some *finite* set J. Let us denote by $\mathscr{A}^{dc}$ (resp. $\mathscr{A}_{\geq 0}^{dc}$) the full subcategory of $\mathscr{A}$ (resp. $\mathscr{A}_{\geq 0}$) consisting of homologically disconnected cdgas. We also denote by ho$\mathscr{A}^{dc}$ (resp. ho$\mathscr{A}_{\geq 0}^{dc}$) the full subcategory of ho$\mathscr{A}$ (resp. ho$\mathscr{A}_{\geq 0}$) whose objects are homologically disconnected cdgas.[3]

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[3]Model categories and their localizations are briefly recalled at the beginning of Section 2.

Theorem A. *The inclusion $\mathscr{A}_{\geq 0} \subset \mathscr{A}$ induces an equivalence of the homotopy categories* $\mathrm{ho}\mathscr{A}_{\geq 0}^{dc}$ *and* $\mathrm{ho}\mathscr{A}^{dc}$.

1.1. **Remark.** A related question is whether $\mathrm{ho}\mathscr{A}_{\geq 0}$ is a full subcategory of $\mathrm{ho}\mathscr{A}$. It is not difficult to show, for example, that for cdgas A and B with A concentrated in degree zero, the sets $\mathrm{ho}\mathscr{A}_{\geq 0}(A, B)$ and $\mathrm{ho}\mathscr{A}(A, B)$ are in natural bijective correspondence. In general however, we see no compelling reason for the homotopy classes of maps in both categories to be the same.

For a category $\mathcal{C}$ denote by $\mathrm{coProd}(\mathcal{C})$ the category whose objects are formal finite coproducts $A_1 \sqcup \cdots \sqcup A_s$, $s \geq 1$, of objects of $\mathcal{C}$ and the Hom-sets are

$$\mathrm{coProd}(\mathcal{C})\big(A_1 \sqcup \cdots \sqcup A_s, B_1 \sqcup \cdots \sqcup B_t\big) := \prod_{1 \leq i \leq s} \bigcup_{1 \leq j \leq t} \mathcal{C}(A_i, B_j),$$

with the obvious composition law.

1.2. **Remark.** The nature of our category $\mathrm{coProd}(\mathcal{C})$ bears some similarity to the category $\mathrm{inj}\text{-}\mathcal{C}$ of direct systems in $\mathcal{C}$, see e.g. [10, page 8]. This resemblance is however merely superficial, as in the definitions of $\mathrm{inj}\text{-}\mathcal{C}$ as well as in the dual definition of the category $\mathrm{proj}\text{-}\mathcal{C}$ of inverse systems in $\mathcal{C}$, one assumes the indexing small category to be left filtering, while we use discrete finite categories.

1.3. **Example.** Denote by $\mathscr{S}^c$ the category of connected simplicial sets and by $\mathscr{S}^{dc}$ the category of simplicial sets with finitely many components. Then clearly $\mathscr{S}^{dc} \cong \mathrm{coProd}(\mathscr{S}^c)$ and the same is obviously true also for the homotopy categories, i.e. $\mathrm{ho}\mathscr{S}^{dc} \cong \mathrm{coProd}(\mathrm{ho}\mathscr{S}^c)$.

Our next main theorem that states a similar result also for homotopy categories of cdgas requires a contravariant version of $\mathrm{coProd}(\mathcal{C})$. Namely, denote by $\mathrm{Prod}(\mathcal{C})$ the category[4] whose objects are formal finite products $A_1 \times \cdots \times A_s$, $s \geq 1$, of objects of $\mathcal{C}$, and morphisms are

$$\mathrm{Prod}(\mathcal{C})\big(A_1 \times \cdots \times A_s, B_1 \times \cdots \times B_t\big) := \prod_{1 \leq j \leq t} \bigcup_{1 \leq i \leq s} \mathcal{C}(A_i, B_j).$$

In the following theorem, $\mathscr{A}^c$ (resp. $\mathscr{A}_{\geq 0}^c$) denotes the full subcategory of $\mathscr{A}$ (resp. $\mathscr{A}_{\geq 0}$) consisting of homologically connected cdgas.

Theorem B. *Each homologically disconnected non-negatively graded cdga $A \in \mathscr{A}_{\geq 0}^{dc}$ is isomorphic to a finite product $\prod_{i \in J} A_i$ of homologically connected non-negatively graded cdgas $A_i \in \mathscr{A}_{\geq 0}^c$. This isomorphism extends to a natural equivalence of categories*

$$\mathrm{ho}\mathscr{A}_{\geq 0}^{dc} \sim \mathrm{Prod}(\mathrm{ho}\mathscr{A}_{\geq 0}^c).$$

Each homologically disconnected cdga $A \in \mathscr{A}^{dc}$ is weakly equivalent to a finite product $\prod_{i \in J} A_i$ of homologically connected cdgas $A_i \in \mathscr{A}^c$. As above, one has an equivalence

$$\mathrm{ho}\mathscr{A}^{dc} \sim \mathrm{Prod}(\mathrm{ho}\mathscr{A}^c).$$

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[4]The similarity of this category to the category of inverse systems is addressed in Remark 1.2.

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Our proofs of the above theorems use a proposition describing homotopy classes of maps whose domain is a localization of a cdga. We believe that this statement is of independent interest.

For a cocycle u of a cdga A we denote by $A[u^{-1}]$ the localization of A at u, i.e. at the multiplicative subset generated by u, see [1, Section 3]. Since u is a cocycle, $A[u^{-1}]$ bears the induced differential. Notice that each $\chi \in [A, D]_{\mathscr{A}} := \mathrm{ho}\mathscr{A}(A, D)$, for $A, D \in \mathscr{A}$, induces a map $\chi_* : H(A) \to H(D)$.

1.4. **Theorem.** *Let $A, D \in \mathscr{A}$ be cdga and $u \in A$ a cocycle. Denote*
$$[A, D]_{\mathscr{A}}^u := \big\{ \chi \in [A, D]_{\mathscr{A}} \mid \chi_*([u]) \in H(D) \text{ is invertible} \big\}.$$
There is a natural isomorphism $\big[A[u^{-1}], D\big]_{\mathscr{A}} \cong [A, D]_{\mathscr{A}}^u$.

Observe that each invertible odd-degree element x of a graded commutative algebra, i.e. one for which there exists y such that $xy = 1$, equals zero since $x^2 = 0$ by graded commutativity. So the only graded commutative algebra admitting invertible elements in odd degrees is the terminal one. Theorem 1.4 is thus meaningful only when the degree $|u|$ is even.

The next two applications of our theory describe the homotopy category of spaces with finitely many rational nilpotent components of finite $\mathbb{Q}$-type. We need the following definition related to the algebraic side.

1.5. **Definition.** Let A be a cdga. We say that a dg ideal I in A is an *augmentation ideal* if $A/I \cong \Bbbk$, the ground field. Further, A is said to have a *finite type* if for any augmentation ideal I of A the space $H(I/I^2)$ is finite dimensional in every degree.

Assume that the ground field $\Bbbk$ is the field $\mathbb{Q}$ of rational numbers and denote by $\mathrm{f}\mathbb{Q}\text{-}\mathscr{A}_{\geq 0}^{dc}$, resp. $\mathrm{f}\mathbb{Q}\text{-}\mathscr{A}^{dc}$, the subcategory of $\mathscr{A}_{\geq 0}^{dc}$, resp. $\mathscr{A}^{dc}$, consisting of algebras having a cofibrant replacement of finite type. In Section 4 we prove that this definition does not depend on the choice of a cofibrant replacement and relate it to the definition of finite $\mathbb{Q}$-type given in [5].

Theorem C. *The following three categories are equivalent.*

- *The homotopy category $\mathrm{fN}\mathbb{Q}\text{-}\mathrm{ho}\mathscr{S}^{dc}$ of simplicial sets with finitely many components that are nilpotent, rational and of finite $\mathbb{Q}$-type,*
- *the homotopy category $\mathrm{f}\mathbb{Q}\text{-}\mathrm{ho}\mathscr{A}_{\geq 0}^{dc}$ of homologically disconnected non-negatively graded cdgas of finite type over $\mathbb{Q}$, and*
- *the homotopy category $\mathrm{f}\mathbb{Q}\text{-}\mathrm{ho}\mathscr{A}^{dc}$ of homologically disconnected $\mathbb{Z}$-graded cdgas of finite type over $\mathbb{Q}$.*

Let us denote by $\mathrm{f}\mathbb{Q}\text{-}\mathrm{ho}\hat{\mathscr{L}}^{dc}$ the full subcategory of the homotopy category of $\hat{\mathscr{L}}$ consisting of disjoint products of complete non-negatively graded dglas with finite-dimensional homology in each dimension. We call objects of $\mathrm{f}\mathbb{Q}\text{-}\mathrm{ho}\hat{\mathscr{L}}^{dc}$ *disconnected* dglas. We have

Theorem D. *The following categories are equivalent:*

- *the homotopy category $\mathrm{fN}\mathbb{Q}\text{-}\mathrm{ho}\mathscr{S}_+^{dc}$ of pointed simplicial sets with finitely many components that are nilpotent, rational and of finite $\mathbb{Q}$-type, and*
- *the homotopy category $\mathrm{f}\mathbb{Q}\text{-}\mathrm{ho}\hat{\mathscr{L}}^{dc}$ of disconnected complete dglas of finite type.*

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Neisendorfer in [24, Proposition 7.3] proved that the subcategory $\text{fN}\mathbb{Q}\text{-ho}\mathscr{S}_+^c$ of $\text{fN}\mathbb{Q}\text{-ho}\mathscr{S}_+^{dc}$ consisting of *connected* spaces is equivalent to the homotopy category $\text{ho}(nDGLA)$ of non-negatively graded (discrete) dglas L whose homology $H(L)$ is of finite type and nilpotent. As a particular case of Theorem D we get another description of $\text{fN}\mathbb{Q}\text{-ho}\mathscr{S}_+^c$. Denote by $\text{f}\mathbb{Q}\text{-ho}\hat{\mathscr{L}}_{\geq 0}$ the full subcategory of $\text{f}\mathbb{Q}\text{-ho}\hat{\mathscr{L}}^{dc}$ of complete non-negatively graded dglas with finite dimensional homology in each degree.

1.6. Corollary. *The simplicial MC functor* $\text{MC}_\bullet(-)$ *induces an equivalence between the categories* $\text{f}\mathbb{Q}\text{-ho}\hat{\mathscr{L}}_{\geq 0}$ *and* $\text{fN}\mathbb{Q}\text{-ho}\mathscr{S}_+^c$.

A nice feature of the category $\hat{\mathscr{L}}$ is that each $\mathfrak{g} \in \hat{\mathscr{L}}$ has a *minimal model*, unique up to isomorphism, see Definition 9.17 and Theorem 9.18. To objects of $\text{f}\mathbb{Q}\text{-ho}\hat{\mathscr{L}}_{\geq 0}$ there correspond non-negatively graded minimal dglas $\mathscr{M}$ with homology of finite type. Corollary 1.6 implies a one-to-one correspondence between rational homotopy types of connected nilpotent spaces of finite $\mathbb{Q}$-type and isomorphism classes of minimal dglas $\mathscr{M}$ as above.

The description of $\text{fN}\mathbb{Q}\text{-ho}\mathscr{S}_+^c$ given in Corollary 1.6 substantially differs from Neisendorfer's. Notice, for instance, that the category $\text{ho}(nDGLA)$ has more objects than $\text{f}\mathbb{Q}\text{-ho}\hat{\mathscr{L}}_{\geq 0}$. For example, the contractible free Lie algebra $\mathbb{L}(x, \partial x)$, $|x| = 1$, belongs to $\text{ho}(nDGLA)$ but not to $\text{f}\mathbb{Q}\text{-ho}\hat{\mathscr{L}}_{\geq 0}$; $\mathbb{L}(x, \partial x)$ is not complete.

Tracing the functors in [24], one can associate to a dgla $\mathfrak{g} \in \text{f}\mathbb{Q}\text{-}\hat{\mathscr{L}}_{\geq 0}$ the corresponding $L \in nDGLA$ as follows. The cdga $\mathcal{C}(\mathfrak{g})$ is connected and non-negatively graded, so it has the minimal model M_A. By assumption, M_A is a cdga of *finite type*, so we may take $L := \mathcal{L}(M_A)$, the *uncompleted* Quillen functor.

On the other hand, starting from $L \in nDGLA$, we take the cobar construction $\mathcal{C}^c(L)$ on the dgla A, i.e. the obvious coalgebra version of the uncompleted functor $\mathcal{C}(-)$, and its linear dual $\mathcal{C}^c(L)^*$.[5] Then $\mathfrak{g} := \hat{\mathcal{L}}(\mathcal{C}^c(L)^*)$ is the corresponding dgla in $\text{f}\mathbb{Q}\text{-}\hat{\mathscr{L}}_{\geq 0}$.

Another application concerns the general structure of the MC simplicial sets. The ground field $\Bbbk$ may again be an arbitrary field of characteristic zero.

1.7. Theorem. *Let* $\mathfrak{g}_i$, $i \in J$, *be a collection of complete dglas indexed by a finite set* J. *Then the simplicial set* $\text{MC}_\bullet\left(\bigsqcup_{i \in J} \mathfrak{g}_i\right)$ *is weakly equivalent to the disjoint union* $\bigcup_{i \in J} \text{MC}_\bullet(\mathfrak{g}_i)$.

Theorem 1.7 yields the following elementary corollary on the sets $\mathscr{MC}(-) := \pi_0 \text{MC}_\bullet(-)$ of connected components of the simplicial MC spaces. We do not know if it has a direct proof.

1.8. Corollary. *Let* $\mathfrak{g}_i, i \in J$, *be a collection of complete dglas indexed by a finite set* J. *Then there is a bijection* $\mathscr{MC}\left(\bigsqcup_{i \in J} \mathfrak{g}_i\right) \cong \bigcup_{i \in J} \mathscr{MC}(\mathfrak{g}_i)$ *of the MC moduli sets.*

The following theorem in a certain sense reverses Theorem 1.7. It uses the twisting $\mathfrak{g}^\xi$ of a complete dgla $\mathfrak{g}$ by an MC element $\xi \in \text{MC}(\mathfrak{g})$ and its connected cover $\overline{\mathfrak{g}^\xi}$ defined in (7.6).

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[5]Observe that $\mathcal{C}^c(L)^*$ exists while $\mathcal{C}(L)$ may not.

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1.9. **Theorem.** *For a complete dgla* $\mathfrak{g}$, *one has a weak equivalence*

$$(1.1) \qquad \mathrm{MC}_\bullet(\mathfrak{g}) \sim \bigcup_{[\xi] \in \mathscr{MC}(\mathfrak{g})} \mathrm{MC}_\bullet(\overline{\mathfrak{g}^\xi})$$

where the disjoint union in the right hand side runs over chosen representatives of the isomorphism classes in $\mathscr{MC}(\mathfrak{g})$. *If* $\mathscr{MC}(\mathfrak{g})$ *is finite, one furthermore has a weak equivalence*

$$\mathrm{MC}_\bullet(\mathfrak{g}) \sim \mathrm{MC}_\bullet \Big(\bigsqcup_{[\xi] \in \mathscr{MC}(\mathfrak{g})} \overline{\mathfrak{g}^\xi} \Big)$$

of simplicial sets.

Theorems A, B and C are proved in Section 5, Theorem D in Section 10. Theorem 1.4 is proved in Section 2 and Theorem 1.9 in Section 7. The proof of Theorem 1.7 occupies Section 8; it is surprisingly involved in that it relies, essentially, on all of the technology developed in the previous sections and the Appendix. The pointed (or augmented) versions of Theorems A, B and C are formulated in Section 6. We finish this part by a

Glossary of notation. We use the following notation for various categories:

$\mathscr{A}$,	the category of ($\mathbb{Z}$-graded) cdgas,
$\mathscr{A}_{\geq 0}$,	the category of non-negatively graded cdgas,
$\mathscr{A}^c$,	the category of homologically connected cdgas,
$\mathscr{A}_{\geq 0}^c$,	the category of non-negatively graded homologically connected cdgas,
$\mathscr{A}^{dc}$,	the category of homologically disconnected cdgas,
$\mathscr{A}_{\geq 0}^{dc}$,	the category of non-negatively graded homologically disconnected cdgas,
$\mathscr{L}$,	the category of dglas
$\hat{\mathscr{L}}$,	the category of complete dglas,
$\mathscr{S}$,	the category of simplicial sets,
$\mathscr{S}^c$,	the category of connected simplicial sets,
$\mathscr{S}^{dc}$,	the category of simplicial sets with finitely many components.

The prefix 'f$\mathbb{Q}$-' applied to a category of algebras means 'finite type over $\mathbb{Q}$' while the prefix 'fN$\mathbb{Q}$-' applied to a category of simplicial sets abbreviates 'nilpotent, rational components of finite $\mathbb{Q}$-type.' The subscript '$_+$' means 'pointed' for simplicial sets and 'augmented' for algebras.

Part 1. **The de Rham-Sullivan approach.**

In this part we describe our first version of disconnected rational homotopy theory based on cdgas.

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2. Homotopy properties of the localization and proof of Theorem 1.4

Recall [9, Sections 5,6] that the homotopy category ho$\mathscr{A}$ of the model category $\mathscr{A}$ of $\mathbb{Z}$-graded unital cdgas has the same objects as $\mathscr{A}$, and the morphism sets $[X, Y]_{\mathscr{A}}$ defined as

$$[X, Y]_{\mathscr{A}} := \pi(QX, QY)_{\mathscr{A}}, \ X, Y \in \mathscr{A},$$

where QX resp. QY is a cofibrant replacement of X resp. Y[6] and $\pi(-, -)_{\mathscr{A}}$ denotes the set of homotopy classes. By [9, Proposition 5.11], if A is cofibrant and Y fibrant, which in our situation means that Y is arbitrary, one has an isomorphism

(2.1) $[A, Y]_{\mathscr{A}} \cong \pi(A, Y)_{\mathscr{A}}.$

There is a functor $\gamma : \mathscr{A} \to \text{ho}\mathscr{A}$ which is the identity on objects and, for a morphism $f : X \to Y$ in $\mathscr{A}$, $\gamma(f)$ is the homotopy class of a lift $\tilde{f} : QX \to QY$ of f. The homotopy category ho$\mathscr{A}_{\geq 0}$ of $\mathscr{A}_{\geq 0}$ has an obvious similar description.

Let A be a cdga and $S \subset A$ a multiplicative subset of cocycles. Then the localization [1, Section 3][7] $S^{-1}A$ of A is a cdga and the canonical map $A \to S^{-1}A$ is a morphism of cdgas. The property crucial for us is the exactness [1, Proposition 3.3] of the functor $A \mapsto S^{-1}A$. If S is multiplicatively generated by a cocycle $u \in A$ we will write $A[u^{-1}]$ for $S^{-1}A$.

2.1. Remark. It is easy to see, using the exactness of the localization, that the homotopy type of $A[u^{-1}]$ depends only on the cohomology class of u in $H(A)$.

Let us prove Theorem 1.4 which we formulate in a slightly extended form as:

2.2. Theorem. *Let $A, D \in \mathscr{A}$, $u \in A$ a cocycle and $p : A \to A[u^{-1}]$ the localization map. Denote*

$$[A, D]^u_{\mathscr{A}} := \left\{ \chi \in [A, D]_{\mathscr{A}} \mid \chi_*([u]) \in H(D) \text{ is invertible} \right\}.$$

Then the map $p^\sharp : \left[A[u^{-1}], D \right]_{\mathscr{A}} \to [A, D]_{\mathscr{A}}$, $p^\sharp(\chi) := \chi \circ \gamma(p)$, induces an isomorphism

$$\left[A[u^{-1}], D \right]_{\mathscr{A}} \cong [A, D]^u_{\mathscr{A}}.$$

2.3. Example. A curious particular case is when u is cohomologous to zero 0 in A. Then $H\left(A[u^{-1}] \right)$ is the terminal algebra $\mathbf{0}$ in which $1 = 0$. By Remark 2.1, $A[u^{-1}]$ is isomorphic, in the homotopy category, to $\mathbf{0}$, so clearly

$$\left[A[u^{-1}], D \right]_{\mathscr{A}} = \begin{cases} \emptyset, & \text{if } 1 \neq 0 \text{ in } H(D), \text{ and} \\ \text{the one-point set,} & \text{if } H(D) = \mathbf{0}. \end{cases}$$

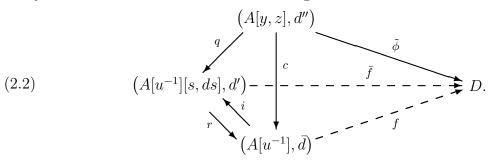It is immediate to see that the set $[A, D]^u_{\mathscr{A}}$ has the same description.

The rest of this section is devoted to the proof of Theorem 2.2. We say that $\phi \in [X, Y]_{\mathscr{A}}$ is *represented* by $f \in \mathscr{A}(X, Y)$ if $\phi = \gamma(f)$. We call a cocycle $u \in X \in \mathscr{A}$ *cohomologically invertible* if its cohomology class $[u] \in H(X)$ is invertible.

2.4. Proposition. *Assume that $A \in \mathscr{A}$ is cofibrant and $D \in \mathscr{A}$ an cdga whose each cohomologically invertible cocycle is invertible. Let $u \in A$ be a cocycle. Then each $\phi \in \left[A[u^{-1}], D \right]_{\mathscr{A}}$ is represented by some $f : A[u^{-1}] \to D$.*

---

[6]All objects in $\mathscr{A}$ are fibrant. In a general model category, the cofibrant replacement must be followed by the fibrant replacement.

[7]Notice that $S^{-1}A$ is in [1] called 'the ring of fractions with respect to the multiplicative subset S.'

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Proof. Let us look at the left half of the diagram

(2.2)

$$\left(A[y,z],d''\right)$$

with maps q, c, $\tilde{\phi}$, $\bar{f}$, f, i, r to $\left(A[u^{-1}][s,ds],d'\right)$, $\left(A[u^{-1}],\bar{d}\right)$ and D.

The symbols y,z,s,ds are new free generators with $|z| = |s| := -1$, $|y| := -|u|$. The cdga $\left(A[u^{-1}],\bar{d}\right)$ is the localization of $A = (A,d)$ at u with the induced differential. The differential d' of $A[u^{-1}][s,ds]$ is defined by

$$d'(\bar{a}) := \bar{d}(\bar{a}) \ \text{ for } \ \bar{a} \in A[u^{-1}], \ d'(s) := ds \ \text{ and } \ d'(ds) := 0,$$

and the differential d'' of $A[y,z]$ by

$$d''(a) := d(a) \ \text{ for } \ a \in A, \ d''(z) := uy - 1 \ \text{ and } \ d''(y) := 0.$$

Let $\bar{a}$ denote the image of $a \in A$ under the localization map $A \to A[u^{-1}]$. The map $c : A[y,z] \to A[u^{-1}]$ is then defined by

$$c(a) := \bar{a}, \ c(y) := u^{-1} \ \text{ and } \ c(z) = 0.$$

The map $q : A[y,z] \to A[u^{-1}][s,ds]$ is given by

$$q(a) := \bar{a}, \ q(y) := u^{-1}(ds+1) \ \text{ and } \ q(z) := s.$$

Finally, $i : A[u^{-1}] \to A[u^{-1}][s,ds]$ is the inclusion and $r : A[u^{-1}][s,ds] \to A[u^{-1}]$ the obvious retraction. It is routine to verify that all the maps above commute with the differentials, that $c = rq$ and that ri is the identity.

Since $\left(A[u^{-1}][s,ds],d'\right)$ is the tensor product of $\left(A[u^{-1}],\bar{d}\right)$ with the 'standard' acyclic cdga $\Bbbk[s,ds]$, one sees that both i and r are weak equivalences. A simple spectral sequence argument shows that also c is a weak equivalence. The cdga $\left(A[y,z],d''\right)$ was created from a cofibrant A by a cell attachment, it is therefore also cofibrant. As $A[u^{-1}]$ is generated by the image of A under the localization map and by u^{-1}, c is an epimorphism, i.e. a fibration in $\mathscr{A}$. The map c thus can be taken as a cofibrant replacement of $A[u^{-1}]$.

Let us inspect the localization of $\left(A[y,z],d''\right)$ at u. It is clear that $A[y,z][u^{-1}] \cong A[u^{-1}][y,z]$ with the differential $\bar{d}'$ given by

$$\bar{d}'(\bar{a}) := \bar{d}(\bar{a}) \ \text{ for } \ \bar{a} \in A[u^{-1}], \ \bar{d}'(z) := uy - 1 \ \text{ and } \ \bar{d}'(y) := 0.$$

It is simple to check that the formulas

$$\alpha(\bar{a}) := \bar{a} \ \text{ for } \ \bar{a} \in A[u^{-1}], \ \alpha(y) := u^{-1}(ds+1) \ \text{ and } \ \alpha(z) := s$$

define an isomorphism

$$\alpha : \left(A[u^{-1}][y,z],\bar{d}'\right) \xrightarrow{\cong} \left(A[u^{-1}][s,ds],d'\right)$$

such that $\alpha p = q$, where $p : A[y,z] \to A[u^{-1}][y,z]$ is the localization map. We can therefore take $\left(A[u^{-1}][y,z],\bar{d}'\right)$ as the localization of $\left(A[y,z],d''\right)$ at u, with q the localization map.

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Let $\tilde{\phi} : A[y, z] \to D$ as in (2.2) represents $\phi \in \left[A[u^{-1}], D \right]_{\mathscr{A}}$, i.e. $\phi = \gamma(\bar{\phi})$. By definition, $\phi_*([u]) \in H(D)$ is invertible, therefore the cocycle $\tilde{\phi}(u) \in D$ representing $\phi_*([u])$ is invertible, so $\tilde{\phi}$ factorizes via the localization map $q : A[y, z] \to A[u^{-1}][s, ds]$. We get, in (2.2), a unique map $\bar{f} : A[u^{-1}][s, ds] \to D$ such that $\tilde{\phi} = \bar{f}q$. Let finally $f := \bar{f}i : A[u^{-1}] \to D$.

We are going to prove that f represents ϕ. Applying the functor γ to the equation $fc = \bar{f}irq$ gives

$$(2.3) \qquad\qquad \gamma(f)\gamma(c) = \gamma(\bar{f})\gamma(i)\gamma(r)\gamma(q).$$

Since i and r are weak equivalences and $ri = \mathbb{1}$, $\gamma(i)$ and $\gamma(r)$ are mutually inverse isomorphisms in $\mathrm{ho}\mathscr{A}$. As c is our chosen cofibrant replacement of $A[u^{-1}]$, $\gamma(c)$ is the identity, thus (2.3) reduces to

$$\gamma(f) = \gamma(\bar{f})\gamma(q) = \gamma(\bar{f}q).$$

The proof is finished by recalling that $\bar{f}q = \tilde{\phi}$, hence $\phi = \gamma(\tilde{\phi}) = \gamma(f)$. □

2.5. **Lemma.** *For each cdga $X \in \mathscr{A}$ there exists $\widehat{X} \in \mathscr{A}$ and a weak equivalence $q : X \to \widehat{X}$ such that*

- (i) *each cohomologically invertible cocycle $u \in \widehat{X}$ is invertible, and*
- (ii) *each morphism $f : X \to Z$ whose target is an cdga $Z \in \mathscr{A}$ in which all cohomologically invertible cocycles are invertible, uniquely factorizes via $q : X \to \widehat{X}$.*

Proof. We start by observing that if, in an cdga $D = (D, d)$, all cocycles cohomologous to 1 are invertible, then all cohomologically invertible cocycles are invertible. Indeed, let $x \in D$ be cohomologically invertible, i.e. $xy = 1 + db$ for some $y, b \in D$. By assumption, $1 + db$ is invertible, so $x^{-1} := y(1 + db)^{-1}$ exists.

Denote by S the multiplicative set of all cohomologically invertible cocycles $u \in X$ and by $\widehat{X} := S^{-1}X$ the localization of $X = (X, d)$ at S with the induced differential $\bar{d}$. Let $q : X \to \widehat{X}$ be the localization map. To prove (i) it is, by the above observation, enough to show that each cocycle $x \in \widehat{X}$ cohomologous to 1 is invertible. Let $x = 1 + \bar{d}b$, $b \in \widehat{X}$. Clearly, $b = q(s)^{-1}q(a)$ for some $a \in X$ and $s \in S$, so $q(s)x = q(s) + q(da) = q(s + da)$. Since $[s + da] = [s]$, $q(s + da)$ is invertible in $\widehat{X}$ by the definition of S. We can therefore take $x^{-1} := q(s)q(s + da)^{-1}$.

Part (ii) follows from the standard universal property of the localization. □

2.6. **Lemma.** *Assume that all cohomologically invertible cocycles of $D \in \mathscr{A}$ are invertible. Then there exists a good path object in the sense of [9, §4.12]*

$$(2.4) \qquad\qquad D \xrightarrow{i} D^I \xrightarrow{(p_1, p_2)} D \times D$$

such that each cohomologically invertible cocycle of D^I is invertible.

Proof. Take any good path object $D \xrightarrow{\bar{i}} P \xrightarrow{(\bar{p}_1, \bar{p}_2)} D \times D$. Lemma 2.5 applied to P produces a cdga D^I such that all its cohomologically invertible cocycles are invertible, together with a weak equivalence $q : P \to D^I$. It is clear that, if all cohomologically invertible cocycles

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of D are invertible, $D \times D$ has the same property therefore, by (ii) of Lemma 2.5, $(\bar{p}_1, \bar{p}_2)$ factorizes as

$$(\bar{p}_1, \bar{p}_2) : P \xrightarrow{q} D^I \xrightarrow{(p_1, p_2)} D \times D.$$

We claim that (2.4) with $i := q\bar{i}$ is a good path object for D.

Firstly, i, being the composition of two weak equivalences, is a weak equivalence. Secondly, since P is a good path object, $(\bar{p}_1, \bar{p}_2)$ is a fibration in $\mathscr{A}$ i.e. an epimorphism, hence (p_1, p_2) must be an epimorphism, i.e. a fibration, as well. $\square$

2.7. **Proposition.** *Let $A, D \in \mathscr{A}$ and assume that A is cofibrant and each cohomologically invertible cocycle in D is invertible. Let $u \in A$ be a cocycle and $p : A \to A[u^{-1}]$ be the localization map. Assume that $f_i : A[u^{-1}] \to D$, $i = 1, 2$, are such that the compositions $f_1 p, f_2 p : A \to D$ are homotopic. Then f_1 and f_2 are right homotopic.*

Proof. Since A is cofibrant, by [9, Remark 4.23], we may assume that $f_1 p$ and $f_2 p$ are right homotopic via a good path object of Lemma 2.6. Let $h : A \to D^I$ be such a right homotopy, i.e.

(2.5) $$f_1 p = p_1 h \quad \text{and} \quad f_2 p = p_2 h.$$

It is clear that, for instance, $f_1 p(u)$ is an invertible element in D, with $f_1(u^{-1})$ as its inverse. By definition of a path object, $p_1 : D^I \to D$ is a weak equivalence, so the invertibility of $f_1 p(u) = p_1 h(u)$ implies the cohomological invertibility of $h(u) \in D$. By our choice of the cylinder D^I, $h(u) \in D$ is (strictly) invertible, thus the homotopy h factorizes as $h = \bar{h} p$ with some $\bar{h} : A[u^{-1}] \to D^I$.

It remains to prove that $\bar{h}$ is a right homotopy between f_1 and f_2, that is

(2.6) $$(f_1, f_2) = (p_1, p_2)\bar{h}.$$

To this end, we invoke the obvious fact that two morphisms, say $u_1, u_2 : A[u^{-1}] \to B$, agree if and only if their compositions $u_1 p, u_2 p : A \to B$ with the localization map $p : A \to A[u^{-1}]$ agree. To prove (2.6), it therefore suffices to show that $(f_1 p, f_2 p) = (p_1, p_2)\bar{h}p$, which follows from $h = \bar{h}p$ and (2.5). $\square$

Proof of Theorem 2.2. Let $c : \widetilde{A} \twoheadrightarrow A$ be a cofibrant replacement of A and $\tilde{u} \in \widetilde{A}$ a cocycle such that $c(\tilde{u}) = u$. Let $p : A \to A[u^{-1}]$ be, as in the theorem, the localization map for A at u and $\tilde{p} : \widetilde{A} \to \widetilde{A}[\tilde{u}^{-1}]$ the localization map for $\widetilde{A}$ at $\tilde{u}$. One has the induced morphism $\bar{c} : \widetilde{A}[\tilde{u}^{-1}] \to A[u^{-1}]$ that makes the diagram

$$
\begin{array}{ccc}
\widetilde{A} & \xrightarrow{\tilde{p}} & \widetilde{A}[\tilde{u}^{-1}] \\
c \downarrow & & \downarrow \bar{c} \\
A & \xrightarrow{p} & A[u^{-1}]
\end{array}
$$

commutative. Being a cofibrant replacement, the map c is as weak equivalence. By the exactness of the localization, $\bar{c}$ is a weak equivalence as well. Consider the induced diagram

(2.7)
$$
\begin{array}{ccc}
[\widetilde{A}, D]_{\mathscr{A}} & \xleftarrow{\tilde{p}^{\sharp}} & [\widetilde{A}[\tilde{u}^{-1}], D]_{\mathscr{A}} \\
c^{\sharp} \uparrow & & \uparrow \bar{c}^{\sharp} \\
[A, D]_{\mathscr{A}} & \xleftarrow{p^{\sharp}} & [A[u^{-1}], D]_{\mathscr{A}}
\end{array}
$$

in which $p^\sharp$ (resp. $\tilde{p}^\sharp$, resp. $c^\sharp$, resp. $\bar{c}^\sharp$) are the pre-compositions with $\gamma(p)$ (resp. $\gamma(\tilde{p})$, resp. $\gamma(c)$, resp. $\gamma(\bar{c})$).

Since c and $\bar{c}$ are weak equivalences, the induced maps $c^\sharp$ and $\bar{c}^\sharp$ are isomorphisms in ho$\mathscr{A}$ ($c^\sharp$ is in fact the identity). Clearly, $c^\sharp$ restricts to an isomorphism $[A, D]_{\mathscr{A}}^u \cong [\widetilde{A}, D]_{\mathscr{A}}^{\tilde{u}}$ and also the inclusions $Im(p^\sharp) \subset [A, D]_{\mathscr{A}}^u$ and $Im(\tilde{p}^\sharp) \subset [\widetilde{A}, D]_{\mathscr{A}}^u$ are obvious. Diagram (2.7) therefore restricts to

$$
\begin{array}{ccc}
[\widetilde{A}, D]_{\mathscr{A}}^{\tilde{u}} & \xleftarrow{\ \tilde{p}^\sharp\ } & \left[\widetilde{A}[\tilde{u}^{-1}], D\right]_{\mathscr{A}} \\
{\scriptstyle c^\sharp} \big\uparrow {\scriptstyle \cong} & & {\scriptstyle \cong} \big\uparrow {\scriptstyle \bar{c}^\sharp} \\
[A, D]_{\mathscr{A}}^u & \xleftarrow{\ p^\sharp\ } & \left[A[u^{-1}], D\right]_{\mathscr{A}}
\end{array}
$$

in which both vertical arrows are isomorphisms.

We conclude that the theorem will be proved for the localization $p : A \to A[u^{-1}]$ if we prove it for $\tilde{p} : \widetilde{A} \to \widetilde{A}[\tilde{u}^{-1}]$. We may thus assume from the beginning that A *is cofibrant*. An even simpler argument based on Lemma 2.5 shows that we may also assume, without loss of generality, that *each cohomologically invertible cocycle in D is invertible*.

To show that the image of $p^\sharp$ is $[A, D]_{\mathscr{A}}^u$ is now easy. Since A is cofibrant, each $\chi \in [A, D]_{\mathscr{A}}^u$ is represented by a map $w : A \to D$. As $\chi_*([u]) \in H(D)$ is, by assumption, invertible, $w(u)$ is invertible in D. Thus w factorizes via the localization map $p : A \to A[u^{-1}]$ as $w = fp$ for some $f : A[u^{-1}] \to D$. We then have $\chi = \gamma(w) = \gamma(fp) = p^\sharp(\gamma(f))$, so $\chi \in Im(p^\sharp)$.

Let us prove that $p^\sharp$ is injective. Assume that $\phi_i \in \left[A[\tilde{u}^{-1}], D\right]_{\mathscr{A}}$, $i = 1, 2$, are such that

$$(2.8) \qquad\qquad\qquad p^\sharp \phi_1 = p^\sharp \phi_2.$$

By Proposition 2.4, there exist $f_i : A[\tilde{u}^{-1}] \to D$ such that $\phi_i = \gamma(f_i)$. Equality (2.8) is then equivalent to $\gamma(f_1 p) = \gamma(f_2 p)$. Since A is cofibrant this, by (2.1), means that $f_1 p$ and $f_2 p$ are homotopic. By Proposition 2.7, f_1 and f_2 are right homotopic, which implies that $\gamma(f_1) = \gamma(f_2)$, i.e. $\phi_1 = \phi_2$. $\qquad\square$

3. Maps of direct products of dg commutative associative algebras

In this section we study maps, up to homotopy, whose source is a finite direct product of cdgas. Somewhat unexpectedly, it turns out that these maps can be completely understood in terms of (homotopy classes of) maps out of individual components of these direct products. Here is the first surprise.

3.1. Proposition. *Let A_i, $i \in J$, be cofibrant algebras in $\mathscr{A}_{\geq 0}$ indexed by a finite set J. Then the direct product $\prod_{i \in J} A_i$ is also cofibrant in $\mathscr{A}_{\geq 0}$.*

Proof. We prove the proposition for $J = \{1, 2\}$, the proof for an arbitrary finite indexing set will be similar. Let thus $A = A_1 \times A_2$. We need to prove that, for any epimorphism $p : E \twoheadrightarrow B$ in $\mathscr{A}_{\geq 0}$ which is also a weak equivalence, and for each $f : A \to B$, there exists a lift $\tilde{f} : A \to E$ making the diagram

$$(3.1)$$

$$
\begin{array}{ccc}
 & & E \\
 & {\scriptstyle \tilde{f}} \nearrow & \big\downarrow {\scriptstyle p} \\
A & \xrightarrow[f]{} & B
\end{array}
$$

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commutative.

Let $e_1 := (1,0) \in A^0 = A_1^0 \times A_2^0$ and $e_2 := (0,1) \in A^0 = A_1^0 \times A_2^0$. Clearly

(3.2) $\qquad e_1 + e_2 = 1$ (the unit of A), $e_1^2 = e_1$, $e_2^2 = e_2$ and $e_1 e_2 = 0$.

Let $u_i := f(e_i)$, $i = 1, 2$. These elements satisfy an obvious analogue of (3.2), moreover $de_1 = de_2 = 0$. There are three possibilities:

Case 1.: $u_1 = 1$, $u_2 = 0$. Then f restricted to $0 \times A_2 \subset A$ is trivial. Indeed, for $(0, x_2) \in 0 \times A_2$ one has $(0, x_2) = (0, 1)(0, x_2)$, therefore

$$f(0, x_2) = f(0, 1)f(0, x_2) = u_2 f(0, x_2) = 0.$$

In other words, f factorizes via the projection $\pi_1 : A_1 \times A_2 \to A_1$ as

$$f : A_1 \times A_2 \xrightarrow{\pi_1} A_1 \xrightarrow{f_1} B,$$

where $f_1(x_1) := f(x_1, 0)$ for $x_1 \in A_1$. Since A_1 is cofibrant by assumption, one has a lift $\widetilde{f_1} : A_1 \to E$ in the diagram

$\widetilde{f} := \widetilde{f_1}\pi$ then clearly solves the lifting problem (3.1).

Case 2.: $u_1 = 0$, $u_2 = 1$. This 'mirror image' of Case 1 can be treated analogously.

Case 3.: $u_1, u_2 \neq 0$. Since p is a weak equivalence and since there are no 0-dimensional boundaries, p induces an isomorphism of 0-cocycles $Z^0(E) \cong Z^0(B)$.[8] In particular, one has cocycles $\widetilde{u}_1, \widetilde{u}_2 \in E^0$ such that $p(\widetilde{u}_i) = u_i$, $i = 1, 2$, satisfying conditions analogous to (3.2). Put

$$B_i := u_i B \quad \text{and} \quad E_i := \widetilde{u}_i E, \ i = 1, 2.$$

It is clear that then

$$B \cong B_1 \times B_2, \ E \cong E_1 \times E_2$$

and that, under the above isomorphisms, also the maps f and p split,

$$f = (f_1, f_2) : A_1 \times A_2 \to B_1 \times B_2, \ p = (p_1, p_2) : E_1 \times E_2 \to B_1 \times B_2,$$

with $f_i := u_i f \iota_i$, where $\iota_i : A_i \hookrightarrow A_1 \times A_2$ are inclusions given by[9]

$$\iota_1(x_1) := (x_1, 0), \ \iota_2(x_2) := (0, x_2), \ x_i \in A_i, i = 1, 2.$$

The maps p_1 and p_2 are defined in the obvious similar way. Since both p_1 and p_2 must clearly be weak equivalences and epimorphisms, one has, for $i = 1, 2$, the lifts $\widetilde{f}_i$ in the diagrams

The map $\widetilde{f} := (\widetilde{f}_1, \widetilde{f}_2) : A_1 \times A_2 \to E_1 \times E_2$ solves the lifting problem (3.1). $\qquad \square$

---

[8]At this place we need B and E to be non-negatively graded.

[9]These inclusions are homomorphisms of non-unital cdgas, but the composition f_i preserves units.

Let us formulate the following simple principle whose proof is straightforward.

3.2. **Principle.** *Assume A_i, $i \in J$, are arbitrary (unital) cdgas indexed by a finite set J and D a cdga such that 1 is the only nontrivial idempotent in $Z^0(D)$. Then the projections*

$$(3.3) \qquad \pi_i : \prod_{s \in J} A_s \to A_i, \ i \in J,$$

from the cartesian product induce monomorphisms of the homomorphisms sets

$$\pi_i^* : \mathscr{A}(A_i, D) \hookrightarrow \mathscr{A}\big(\prod_{s \in J} A_s, D\big), \ i \in J,$$

which in turn induce a decomposition

$$(3.4) \qquad \mathscr{A}\big(\prod_{s \in J} A_s, D\big) \cong \bigcup_{s \in J} \mathscr{A}(A_s, D) \quad \text{(the disjoint union)}.$$

Together with Proposition 3.1, Principle 3.2 gives:

3.3. **Theorem.** *Let $A_i \in \mathscr{A}_{\geq 0}$ be cdgas indexed by a finite set J and $D \in \mathscr{A}_{\geq 0}$ be such that $1 \in H^0(D)$ is the only nontrivial idempotent. Then the projections (3.3) induce a decomposition of the set of homotopy classes*

$$(3.5) \qquad \big[\prod_{s \in J} A_s, D\big]_{\mathscr{A}_{\geq 0}} \cong \bigcup_{s \in J} [A_s, D]_{\mathscr{A}_{\geq 0}} \quad \text{(the disjoint union)}.$$

The same statement holds also with $\mathscr{A}$ in place of $\mathscr{A}_{\geq 0}$.

Proof. Let us prove the first part. Since D is non-negatively graded, $H^0(D) = Z^0(D)$, so D fulfills the assumptions of Principle 3.2. Moreover, any homotopy $h : \prod_{s \in J} A_s \to D[t, dt]$ factors through a unique homotopy $h_i : A_i \to D[t, dt]$ since the cdga $D[t, dt]$ also satisfies the assumptions of Principle 3.2. Consider the diagram

$$\begin{array}{ccc}
\bigcup_{s \in J} \mathscr{A}_{\geq 0}(A_s, D) & \xrightarrow{\cong} & \mathscr{A}_{\geq 0}\big(\prod_{s \in J} A_s, D\big) \\
\downarrow & & \downarrow \\
\bigcup_{s \in J} [A_s, D]_{\mathscr{A}_{\geq 0}} & \longrightarrow & \big[\prod_{s \in J} A_s, D\big]_{\mathscr{A}_{\geq 0}}
\end{array}$$

where the vertical arrows are natural quotient maps, associating to a morphism its homotopy class. It follows that the lower horizontal arrow, making the diagram commutative, exists, is unique and and bijective; this finishes the proof for the category $\mathscr{A}_{\geq 0}$.

The second part with $\mathscr{A}$ in place of $\mathscr{A}_{\geq 0}$ must be proved differently. The reason is that, firstly, no statement analogous to Proposition 3.1 holds in $\mathscr{A}$ and, secondly, even if $1 \in H^0(D)$ is the only nontrivial idempotent, there may be many nontrivial idempotents in $Z^0(D)$, so Principle 3.2 does not apply.

To simplify the exposition, we assume again that $J = \{1, 2\}$, the proof for a general finite J is similar. Put

$$u_1 := 1 \times 0 \in A_1 \times A_2 \ \text{ and } \ u_1 := 0 \times 1 \in A_1 \times A_2,$$

and define

$$[A_1 \times A_2, D]_{\mathscr{A}}^i \subset [A_1 \times A_2, D]_{\mathscr{A}}, \ i = 1, 2,$$

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as the subset of $\chi \in [A_1 \times A_2, D]$ such that $\chi_*([u_i]) = 1$. It is clear that

$$[A_1 \times A_2, D]_{\mathscr{A}} = [A_1 \times A_2, D]^1_{\mathscr{A}} \cup [A_1 \times A_2, D]^2_{\mathscr{A}} \quad \text{(the disjoint union)}.$$

So all we need to prove is that the projections induce an isomorphism

(3.6) $$[A_i, D]_{\mathscr{A}} \cong [A_1 \times A_2, D]^i_{\mathscr{A}}, \ i = 1, 2.$$

Notice that

$$(A_1 \times A_2)[u_i^{-1}] \cong A_i, \ i = 1, 2.$$

The isomorphism (3.6) is thus a consequence of Theorem 1.4 taken with $A = A_1 \times A_2$ and $u = u_i$. $\qquad\square$

For $A \in \mathscr{A}_{\geq 0}$ consider its cofibrant replacement $c' : Q'A \twoheadrightarrow A$ in $\mathscr{A}_{\geq 0}$ and take a cofibrant replacement $c : QA \twoheadrightarrow Q'A$ of $Q'A$ in $\mathscr{A}$. The composition $c'c : QA \twoheadrightarrow A$ is clearly a cofibrant replacement of A in $\mathscr{A}$. Notice also that, for $D \in \mathscr{A}_{\geq 0}$, the 'standard' path object

(3.7) $$D[t, dt] := D \otimes \Bbbk[t, dt], \ |t| := 0, \ d(t) := dt,$$

with the projections $p_1, p_2 : D[t, dt] \to D$ given by the evaluation at 0 resp 1, is a good path object in the sense of [9, §4.12] for D in both categories $\mathscr{A}_{\geq 0}$ and $\mathscr{A}$. Therefore, if $f_1, f_2 : Q'A \to D$ are right homotopic in $\mathscr{A}_{\geq 0}$, then $f_1 c, f_2 c$ are right homotopic in $\mathscr{A}$. By (2.1),

$$[A, D]_{\mathscr{A}_{\geq 0}} \cong \pi(Q'A, D)_{\mathscr{A}_{\geq 0}} \quad \text{and} \quad [A, D]_{\mathscr{A}} \cong \pi(QA, D)_{\mathscr{A}},$$

so the pre-composition with c defines a natural map

(3.8) $$K_{A,D} : [A, D]_{\mathscr{A}_{\geq 0}} \to [A, D]_{\mathscr{A}}.$$

3.4. **Lemma.** *Let $A \in \mathscr{A}^c_{\geq 0}$ and $D \in \mathscr{A}_{\geq 0}$. Then $K_{A,D}$ is an isomorphism. In particular, $\mathrm{ho}\mathscr{A}^c_{\geq 0}$ is a full subcategory of $\mathrm{ho}\mathscr{A}$.*

Proof. The cdga A admits a minimal model M by [5, Proposition 7.7]. As M is, by definition, weakly equivalent to A,

(3.9) $$[A, D]_{\mathscr{A}_{\geq 0}} \cong [M, D]_{\mathscr{A}_{\geq 0}} \quad \text{and} \quad [A, D]_{\mathscr{A}} \cong [M, D]_{\mathscr{A}}.$$

The cdga M is clearly cofibrant in both categories $\mathscr{A}_{\geq 0}$ and $\mathscr{A}$, therefore

$$[M, D]_{\mathscr{A}} = \pi(M, D)_{\mathscr{A}_{\geq 0}} \quad \text{and} \quad [M, D]_{\mathscr{A}_{\geq 0}} = \pi(M, D)_{\mathscr{A}}$$

by (2.1). Since (3.7) is a good path object for D in both categories $\mathscr{A}_{\geq 0}$ and $\mathscr{A}$,

$$\pi(M, D)_{\mathscr{A}_{\geq 0}} \cong \pi(M, D)_{\mathscr{A}}.$$

The lemma is an obvious combination of the above isomorphisms. $\qquad\square$

3.5. **Proposition.** *The categories $\mathrm{ho}\mathscr{A}^c_{\geq 0}$ and $\mathrm{ho}\mathscr{A}^c$ are equivalent.*

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Proof. If $\mathcal{C}'$ is a full subcategory of $\mathcal{C}$ with the property that each object of $\mathcal{C}'$ is isomorphic to some object of $\mathcal{C}$, then both categories are equivalent. In light of Lemma 3.4, it is enough to prove that each $A \in \mathscr{A}^c$ is weakly equivalent to an cdga in $\mathscr{A}_{\geq 0}$.

Proposition 7.7 of [5] states the existence of a minimal model generated by elements of degrees ≥ 1 of each cdga $A \in \mathscr{A}^c_{\geq 0}$. One can easily verify that the proof of this proposition leads to a minimal model of an *arbitrary* homologically connected cdga A, i.e. of an arbitrary $A \in \mathscr{A}^c$. The minimal model M of $A \in \mathscr{A}^c$ is then a connected cdga belonging to $\mathscr{A}_{\geq 0}$, weakly equivalent to A. This finishes the proof. $\square$

Theorem 3.3 has the following important consequence.

3.6. Theorem. *Let A_i, $i \in J$, be cdgas in $\mathscr{A}^c_{\geq 0}$ indexed by a finite set and $D \in \mathscr{A}_{\geq 0}$ be such that $1 \in H^0(D)$ is the only nontrivial idempotent in $H^0(D)$. Then the map in (3.8) with $A = \prod_{s \in J} A_s$ is an isomorphism*

$$\big[\prod_{s \in J} A_s, D \big]_{\mathscr{A}_{\geq 0}} \cong \big[\prod_{s \in J} A_s, D \big]_{\mathscr{A}}.$$

Proof. By Lemma 3.4, $[A_i, D]_{\mathscr{A}_{\geq 0}} \cong [A_i, D]_{\mathscr{A}}$, $i \in J$. The rest follows from Theorem 3.3. $\square$

4. DG commutative associative algebras of finite type

In this section we investigate properties of cdgas having finite type in the sense of Definition 1.5. The main results are Propositions 4.2 and 4.4. For a cdga B and a map $f : A \to B$ we will denote by $\mathrm{Der}_f(A, B)$ the dg space of derivations of A with values in B, where B is viewed as a dg A-module via f; if the map f is clear from the context we will write simply $\mathrm{Der}(A, B)$ for $\mathrm{Der}_f(A, B)$.

Note that having an augmentation ideal I in A is equivalent to specifying a map $\epsilon : A \to \Bbbk$ (an augmentation). Furthermore, the dual dg space $(I/I^2)^*$ is naturally identified with the dg space $\mathrm{Der}_\epsilon(A, \Bbbk)$. Thus, A is of finite type if and only if for any augmentation $\epsilon : A \to \Bbbk$ the dg space $\mathrm{Der}_\epsilon(A, \Bbbk)$ has finite-dimensional cohomology in each positive degree.

We now discuss the homotopy invariance of the notion of finite type. To this end, note that for a cofibrant cdga A and a dg A-module M the dg space $\mathrm{Der}(A, M)$ is quasi-isomorphic to $C_{AQ}(A, M)$, the Andrè-Quillen cohomology complex of A with coefficients in M, cf. [3, Theorem 2.4].

4.1. Lemma. *Let A be a cofibrant cdga, B is a cdga and $f, g : A \to B$ are two Sullivan homotopic maps. Then the dg vector spaces $\mathrm{Der}_f(A, B)$ and $\mathrm{Der}_g(A, B)$ are quasi-isomorphic.*

Proof. Let $h : A \to B[t, dt]$ be a Sullivan homotopy from f to g. The two evaluation maps $|_{0,1} : B[t, dt] \to B$ determine maps of dg vector spaces $\mathrm{Der}_h(A, B[t, dt]) \to \mathrm{Der}_f(A, B)$ and $\mathrm{Der}_h(A, B[t, dt]) \to \mathrm{Der}_g(A, B)$. Comparing the corresponding spectral sequences ([3, Corollary 2.5]) we conclude that both maps are quasi-isomorphisms, giving the desired conclusion. $\square$

4.2. Proposition. *Let A and A' be two quasi-isomorphic cofibrant cdgas. Then A is of finite type if and only if A' is of finite type.*

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Proof. Let A be of finite type and I' be an augmentation ideal of A'. The given quasi-isomorphism $A \to A'$ determines a dg map $A \to A' \to A'/I' \cong \Bbbk$. Comparing the corresponding spectral sequences in [3, Corollary 2.5] we obtain that $\mathrm{Der}(A', \Bbbk)$ is quasi-isomorphic to $\mathrm{Der}(A, \Bbbk)$ and thus, $\mathrm{Der}(A', \Bbbk)$ has finite-dimensional cohomology, as desired. Since each quasi-isomorphism of cofibrant algebras is homotopy invertible, the rôles of A and A' can be exchanged. This finishes the proof. $\square$

In the following statement $\hat{\mathcal{L}}A$ is the completed Harrison complex of an augmented cdga A recalled in Definition 7.3.

4.3. Lemma. *Suppose that A is a cofibrant cdga and I is an augmentation ideal in A. Then $(I/I^2)^*$ is quasi-isomorphic to $\Sigma\hat{\mathcal{L}}A$.*

Proof. Without loss of generality one can assume that the cdga A is free as a graded associative commutative algebra. It is easy to show that it is then isomorphic, as a non-differential algebra, to the graded polynomial ring generated by I. Lemma 4.3 therefore appears as a version of [24, Proposition 4.2(a)], so we omit its proof. $\square$

Observe that a non-negatively graded homologically connected cdga A is of finite type if and only if it is of finite type in the sense of [5, §9.2]. We can now formulate a criterion for a homologically disconnected cdga to be of finite type.

4.4. Proposition. *Let A be a homologically disconnected cdga, i.e. A is quasi-isomorphic, by Theorem B, to $A_1 \times \cdots \times A_n$ where A_i are connected cdgas, $i = 1, 2, \ldots, n$. Then A is of finite type if and only if each A_i is of finite type.*

Proof. We assume, without loss of generality, that A is cofibrant and each A_i is also cofibrant. Suppose that every A_i is of finite type and let I be an augmentation ideal in A. By Theorem B, the map $A \to A/I \cong \Bbbk$ must factor in the homotopy category through a map $A_i \to \Bbbk$ for some $i = 1, \ldots, n$. Since by Lemma 4.1 the homology of the space $\mathrm{Der}(A, \Bbbk)$ does not depend on the homotopy class of the map $A \to \Bbbk$ we might as well assume that the map $A \to \Bbbk$ factors through A_i on the nose. Denote by I_i the kernel of the corresponding map $A_i \to \Bbbk$.
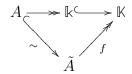
Consider the dga $B := \hat{T}\Sigma I^*$, the reduced cobar-construction of A with the augmentation given by $A \to A/I \cong \Bbbk$. Note that B computes $\mathrm{Ext}_A(\Bbbk, \Bbbk)$, the differential Ext of the cdga A. Similarly, the reduced cobar-construction $B_i := \hat{T}\Sigma I_i^*$ calculates $\mathrm{Ext}_{A_i}(\Bbbk, \Bbbk)$.

One clearly has $\mathrm{Ext}_A(\Bbbk, \Bbbk) \cong \mathrm{Ext}_{A\otimes_A A_i}(\Bbbk \otimes_A A_i, \Bbbk)$ while, by standard homological algebra, $\mathrm{Ext}_{A\otimes_A A_i}(\Bbbk \otimes_A A_i, \Bbbk)$ is isomorphic to $\mathrm{Ext}_{A_i}(\Bbbk, \Bbbk)$. Looking at the primitive elements in the spaces $\mathrm{Ext}_A(\Bbbk, \Bbbk)$ resp. $\mathrm{Ext}_{A_i}(\Bbbk, \Bbbk)$ that are described as the homology of B resp. B_i we conclude that the complete dglas $\hat{\mathcal{L}}(A)$ and $\hat{\mathcal{L}}(A_i)$ are quasi-isomorphic. Therefore, by Lemma 4.3, I/I^2 is quasi-isomorphic to I_i/I_i^2. This proves that if each A_i is of finite type, then so is A.

Conversely, suppose that A is of finite type and let I_i be an augmentation ideal in A_i; then $\Bbbk$ becomes an A_i-module via the augmentation $A_i \to A_i/I_i \cong \Bbbk$. Moreover, $\Bbbk$ is also an A module via the composition $A \xrightarrow{\pi_i} A_i \to A_i/I_i \cong \Bbbk$ in which π_i realizes the projection $p_i : A_1 \times \cdots \times A_n \to A_i$ in the homotopy category $\mathrm{ho}\mathscr{A}$. The same argument as above shows that the dg spaces I_i/I_i^2 and I/I^2 are quasi-isomorphic and thus A_i is of finite type. $\square$

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4.5. **Remark.** The quasi-isomorphism $\hat{\mathcal{L}}(A) \sim \hat{\mathcal{L}}(A_i)$ was crucial for the proof of Proposition 4.4. It can be established differently. Assume, for simplicity, that $n = i = 2$. Since, by a standard spectral sequence argument, $\hat{\mathcal{L}}(-)$ preserves quasi-isomorphisms, we only need to prove that $\hat{\mathcal{L}}(A_1 \times A_2)$ is quasi-isomorphic to $\hat{\mathcal{L}}(A_2)$, where $A_1 \times A_2$ is augmented via the composition $A_1 \times A_2 \to A_2 \to A_2/I_2 \cong \Bbbk$. By Proposition 9.19 proved in the second part, $\hat{\mathcal{L}}(A_1 \times A_2) \cong \hat{\mathcal{L}}(A_1) \sqcup \hat{\mathcal{L}}(A_2)$. Moreover, by Remark 7.15, $\hat{\mathcal{L}}(A_1) \sqcup \hat{\mathcal{L}}(A_2)$ is quasi-isomorphic to $\hat{\mathcal{L}}(A_2)$. This gives the requisite statement.

4.6. **Remark.** It is *not true* that for a cofibrant cdga A of finite type and any maximal ideal I of A the quotient I/I^2 has finite dimensional cohomology. This is not even true for a homologically connected A. Indeed, let A be cofibrant cdga supplied with an augmentation $A \to \Bbbk$. Let $\mathbb{K} \supset \Bbbk$ be an infinite dimensional field extension of $\Bbbk$. The factorization axiom in the closed model category $\mathscr{A}$ expresses the composite map $A \to \mathbb{K}$ as

$$A \rightarrowtail \Bbbk \hookrightarrow \mathbb{K}$$
$$\sim \searrow \quad \nearrow f$$
$$\tilde{A}$$

with a surjective map $f : \tilde{A} \to \mathbb{K}$, where $\tilde{A}$ is quasi-isomorphic to A and still cofibrant. The kernel of f is a maximal ideal $\tilde{I}$ in $\tilde{A}$ and $\tilde{A}/\tilde{I} \cong \mathbb{K}$. Clearly

$$\mathrm{Der}(A, \mathbb{K}) \cong \mathrm{Der}(A, \Bbbk) \otimes_{\Bbbk} \mathbb{K}.$$

On the other hand, $\mathrm{Der}(A, \mathbb{K})$ is quasi-isomorphic to $\mathrm{Der}(\tilde{A}, \mathbb{K}) \cong (\tilde{I}/\tilde{I}^2)^*$. We see that $H(\tilde{I}/\tilde{I}^2)$ may be infinite-dimensional even when $\tilde{A}$ is of finite type.

5. Proofs of Theorems A, B and C

Proof of Theorem A. It is an obvious combination of Proposition 3.5 and Theorem B which we prove below. $\square$

Proof of Theorem B. We start by proving the more difficult second part of the theorem. Assume that $A \in \mathscr{A}^{dc}$ and let $\iota_i : \Bbbk \to \prod_{i \in J} \Bbbk \cong H^0(A)$ be the canonical inclusion into the ith factor. Denote by $e_i \in H^0(A)$, $i \in J$, the idempotent $\iota_i(1)$ and by $H(A)[e_i^{-1}]$ the localization of $H(A)$ at the multiplicative subset generated by e_i. By elementary algebra,

$$(5.1) \qquad\qquad H(A) \cong \prod_{i \in J} H(A)[e_i^{-1}].$$

Choose a cochain $u_i \in A^0$ representing e_i. The localization is exact, therefore $H\big(A[u_i^{-1}]\big) \cong H(A)[e_i^{-1}]$ and the natural cdga map

$$(5.2) \qquad\qquad A \to \prod_{i \in J} A[u_i^{-1}]$$

induces the isomorphism (5.1) of cohomology.

As $H^0(A[u_i^{-1}]) \cong H^0(A)[e_i^{-1}] \cong \Bbbk$, each $A[u_i^{-1}]$ is homologically connected; its cohomology in negative degrees clearly vanishes. Therefore (5.2) shows that each $A \in \mathscr{A}^{dc}$ is weakly equivalent to a finite product of cdgas from $\mathscr{A}^c$. Denote, for the purposes of this proof,

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the full subcategory of $\mathscr{A}^{dc}$ whose objects are these products by $\bar{\mathscr{A}}^{dc}$. It follows from the above that the corresponding homotopy categories $\mathrm{ho}\mathscr{A}^{dc}$ and $\mathrm{ho}\bar{\mathscr{A}}^{dc}$ are equivalent. Theorem 3.3 then implies that $\mathrm{ho}\bar{\mathscr{A}}^{dc}$ is equivalent to $\mathrm{Prod}(\mathrm{ho}\mathscr{A}^{c})$. This finishes the proof of the second part.

The first part can be proved in exactly the same way, but the situation admits a simplification. Assume the same notation as above. The cdga A is now non-negatively graded, so there are no 0-boundaries in A^0, thus u_i is unique and it is an idempotent. The multiplication with u_i defines the projection $\pi_i : A \twoheadrightarrow u_iA$ which represents the localization map $A \to A[u_i^{-1}]$. Since $\{u_i\}_{i\in J}$ are orthogonal idempotents whose sum is 1, the system $\{\pi_i\}_{i\in J}$ defines, instead of just a weak equivalence (5.2), a strict isomorphism

$$A \cong \prod_{i\in J} u_iA,$$

in which $u_iA \in \mathscr{A}_{\geq 0}^c$. So $\mathscr{A}_{\geq 0}^{dc}$ is equivalent to its full subcategory $\bar{\mathscr{A}}_{\geq 0}^{dc}$ whose objects are finite products of cdgas from $\mathscr{A}_{\geq 0}^c$, and the same is true also for the corresponding homotopy categories, that is, $\mathrm{ho}\mathscr{A}_{\geq 0}^{dc} \sim \mathrm{ho}\bar{\mathscr{A}}_{\geq 0}^{dc}$. The proof is finished with the aid of Theorem 3.3. □

Proof of Theorem C. The equivalence of Example 1.3 clearly restrict to the equivalence

$$\mathrm{fN\mathbb{Q}\text{-}ho}\mathscr{S}^{dc} \sim \mathrm{coProd}(\mathrm{fN\mathbb{Q}\text{-}ho}\mathscr{S}^c).$$

By Proposition 4.4, the equivalences of Theorem B restrict to the equivalences

(5.3) $\mathrm{f\mathbb{Q}\text{-}ho}\mathscr{A}^{dc} \sim \mathrm{Prod}(\mathrm{f\mathbb{Q}\text{-}ho}\mathscr{A}^c)$ and $\mathrm{f\mathbb{Q}\text{-}ho}\mathscr{A}_{\geq 0}^{dc} \sim \mathrm{Prod}(\mathrm{f\mathbb{Q}\text{-}ho}\mathscr{A}_{\geq 0}^c).$

The equivalence $\mathrm{fN\mathbb{Q}\text{-}ho}\mathscr{S}^{dc} \sim \mathrm{f\mathbb{Q}\text{-}ho}\mathscr{A}_{\geq 0}^{dc}$ then follows from the classical equivalence between $\mathrm{f\mathbb{Q}\text{-}ho}\mathscr{A}_{\geq 0}^c$ and $\mathrm{fN\mathbb{Q}\text{-}ho}\mathscr{S}^c$, cf. [5, Theorem 9.4]. It is easy to show that it is in fact induced by adjunction (0.1). The equivalence $\mathrm{f\mathbb{Q}\text{-}ho}\mathscr{A}_{\geq 0}^{dc} \sim \mathrm{f\mathbb{Q}\text{-}ho}\mathscr{A}^{dc}$ is a combination of (5.3) with Proposition 3.5. □

6. Augmented dg commutative associative algebras and pointed spaces.

In this section we briefly outline the relationship between the homotopy theory of augmented cdgas and pointed spaces. The results formulated here will be used in Section 10. They are more or less obvious analogues of the non-augmented theory developed in the previous sections.

Recall (e.g. [16, Proposition 1.1.8]) that for a given closed model category $\mathcal{C}$ and an object $O \in \mathcal{C}$, the overcategory of O and the undercategory of O are themselves closed model categories with fibrations, cofibrations and weak equivalences created in the category $\mathcal{C}$. Now consider the overcategory $\mathscr{A}_+$ of the initial object $\Bbbk$ in $\mathscr{A}$; its objects are *augmented* unital cdgas; i.e. cdgas A supplied with an augmentation $A \to \Bbbk$. The morphisms in $\mathscr{A}_+$ are cdga maps respecting the augmentation. The category $\mathscr{A}_+$ has a closed model structure inherited from $\mathscr{A}$.

Inside the category $\mathscr{A}_+$ is the category $\mathscr{A}_+^c$ consisting of homologically connected cdgas and the category $\mathscr{A}_+^{dc}$ consisting of augmented homologically disconnected cdgas. We also have the corresponding subcategories of non-negatively graded augmented cdgas, indicated by ≥ 0 in the subscript. Similarly we have the category $\mathscr{S}_+$ of pointed simplicial sets, the *undercategory* of the terminal simplicial set; it has a closed model structure inherited from $\mathscr{S}$. Let us formulate an augmented version of Theorem A:

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Theorem A_+. *The inclusion $\mathscr{A}_{\geq 0_+} \subset \mathscr{A}_+$ induces an equivalence of the homotopy categories $\mathrm{ho}\mathscr{A}_{\geq 0_+}^{dc}$ and $\mathrm{ho}\mathscr{A}_+^{dc}$.*

For a category $\mathcal{C}$ with the terminal object $*$ denote by $\mathrm{coProd}(\mathcal{C})_+$ the category whose objects are the formal finite coproducts $A_1 \sqcup \cdots \sqcup A_s$, $s \geq 1$, such that $A_1, \ldots, A_{s-1}$ are objects of $\mathcal{C}$, A_s is an object of the undercategory $*/\mathcal{C}$ of $\mathcal{C}$, and the Hom-sets are

$$\mathrm{coProd}(\mathcal{C})_+\big(A_1 \sqcup \cdots \sqcup A_s, B_1 \sqcup \cdots \sqcup B_t\big)$$
$$:= \ \mathrm{coProd}(\mathcal{C})(A_1 \sqcup \cdots \sqcup A_{s-1}, B_1 \sqcup \cdots \sqcup B_t) \times */\mathcal{C}(A_s, B_t),$$

with the obvious composition law.

Assume that $\mathcal{C}$ is a closed model category and consider $*/\mathcal{C}$ with the induced closed model category structure. Since the categories $\mathrm{ho}(*/\mathcal{C})$ and $*/\mathrm{ho}\mathcal{C}$ are not equivalent in general, unlike the un-pointed case, we cannot apply the above construction to the homotopy category $\mathrm{ho}(*/\mathcal{C})$ directly. We need to modify the above definition by taking $\mathrm{coProd}\big(\mathrm{ho}(*/\mathcal{C})\big)_+$ the category with the same objects as $\mathrm{coProd}(*/\mathcal{C})$, but with the morphism sets

$$\mathrm{coProd}\big(\mathrm{ho}(*/\mathcal{C})\big)_+\big(A_1 \sqcup \cdots \sqcup A_s, B_1 \sqcup \cdots \sqcup B_t\big)$$
$$:= \ \mathrm{coProd}(\mathrm{ho}\mathcal{C})(A_1 \sqcup \cdots \sqcup A_{s-1}, B_1 \sqcup \cdots \sqcup \overline{B}_t) \times \mathrm{ho}(*/\mathcal{C})(A_s, B_t),$$

where $\overline{B}_t$ denotes the object $B_t \in */\mathcal{C}$ considered as an object of $\mathcal{C}$ by forgetting the coaugmentation $* \to B_t$. The above definitions are designed to model the category of disconnected *pointed* simplicial sets:

6.1. **Example.** Denote by $\mathscr{S}_+^c$ the category of pointed connected simplicial sets and by $\mathscr{S}_+^{dc}$ the category of pointed simplicial sets with finitely many components. One then has $\mathscr{S}_+^{dc} \cong \mathrm{coProd}_+(\mathscr{S}^c)$ and the same is obviously true for the homotopy categories, i.e. $\mathrm{ho}\mathscr{S}_+^{dc} \cong \mathrm{coProd}_+(\mathrm{ho}\mathscr{S}_+^c)$.

Note that the weak equivalences in $\mathscr{S}_+^{dc}$ are pointed maps inducing bijections on the sets of connected components as well as a weak equivalence on each connected component.

Dually, let $\mathcal{C}$ be a category with an initial object $*$. Denote by $\mathrm{Prod}(\mathcal{C})_+$ the category whose object are formal finite products $A_1 \times \cdots \times A_s$, $s \geq 1$, where $A_1, \ldots, A_{s-1}$ are objects of $\mathcal{C}$ and A_s is an object of the overcategory $\mathcal{C}/*$ of $\mathcal{C}$. The morphism sets are

$$\mathrm{Prod}(\mathcal{C})_+\big(A_1 \times \cdots \times A_s, B_1 \times \cdots \times B_t\big)$$
$$:= \ \mathrm{Prod}(\mathcal{C})(A_1 \times \ldots \times A_s, B_1 \times \ldots B_{t-1}) \times \mathcal{C}/*(A_s, B_t).$$

The category $\mathrm{Prod}\big(\mathrm{ho}(\mathcal{C}/*)\big)_+$ is defined by an obvious dualization of the definition of $\mathrm{coProd}\big(\mathrm{ho}(*/\mathcal{C})\big)_+$ given above. The following theorems are proved analogously to the non-augmented case.

Theorem B_+. *Each homologically disconnected non-negatively graded augmented cdga $A \in \mathscr{A}_{\geq 0_+}^{dc}$ is isomorphic to a finite product $A_1 \times \cdots \times A_s$ of homologically connected cdgas $A_1, \ldots, A_{s-1} \in \mathscr{A}_{\geq 0}^c$ and a homologically connected augmented cdga $A_s \in \mathscr{A}_{\geq 0_+}^c$. This isomorphism extends to a natural equivalence of categories*

$$\mathrm{ho}\mathscr{A}_{\geq 0_+}^{dc} \sim \mathrm{Prod}(\mathrm{ho}\mathscr{A}_{\geq 0}^c)_+.$$

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Each homologically disconnected augmented cdga $A \in \mathscr{A}_+^{dc}$ is weakly equivalent to a finite product $A_1 \times \cdots \times A_s$ of homologically connected cdgas $A_1, \ldots, A_{s-1} \in \mathscr{A}^c$ and a homologically connected augmented cdga $A_s \in \mathscr{A}_+^c$. As above, one has an equivalence

$$\mathrm{ho}\mathscr{A}_+^{dc} \sim \mathrm{Prod}(\mathrm{ho}\mathscr{A}_+^c)_+.$$

Theorem C$_+$. *The following three categories are equivalent.*

- *The homotopy category fN$\mathbb{Q}$-ho$\mathscr{S}_+^{dc}$ of pointed simplicial sets with finitely many components that are rational and of finite $\mathbb{Q}$-type,*
- *the homotopy category f$\mathbb{Q}$-ho$\mathscr{A}_{\geq 0}^{dc}{}_+$ of homologically disconnected non-negatively graded augmented cdgas of finite type over $\mathbb{Q}$, and*
- *the homotopy category f$\mathbb{Q}$-ho$\mathscr{A}_+^{dc}$ of homologically disconnected $\mathbb{Z}$-graded augmented cdgas of finite type over $\mathbb{Q}$.*

Part 2. The Lie-Quillen approach

In this part we give an application of the developed theory to the structure of MC spaces and describe the second version of disconnected rational homotopy theory based on dglas.

7. The simplicial Maurer-Cartan space

We write $\mathscr{A}_+$ for the category of *augmented* cdgas; it is, thus, an overcategory of $\Bbbk \in \mathscr{A}$. As such, it inherits from $\mathscr{A}$ the structure of a closed model category. The weak equivalences and fibrations are still quasi-isomorphisms and surjective maps respectively. Note that the product of two augmented cdgas A and B is their *fiber product* $A \times_{\Bbbk} B$. We denote the augmentation ideal of $A \in \mathscr{A}_+$ by A_+; it is a possibly non-unital cdga. Conversely, given a non-unital cdga B one can form a unital algebra B_e obtained by adjoining the unit; $B_e \cong \Bbbk \oplus B$. Thus, the category $\mathscr{A}_+$ is equivalent to the category of non-unital cdgas.

7.1. **Definition.** A *complete* dgla is an inverse limit of finite-dimensional nilpotent dglas. The category of complete dglas and their continuous homomorphisms will be denoted by $\hat{\mathscr{L}}$.

7.2. **Remark.** The functor of linear duality establishes an anti-equivalence between the category $\hat{\mathscr{L}}$ and that of conilpotent Lie coalgebras, cf. [3] where complete Lie algebras were called *pronilpotent* Lie algebras; we feel that this terminology might not be ideal since, e.g. an abelian Lie algebra on a countably dimensional vector space is not pronilpotent under this convention.

Let us show, however, that complete dglas $\mathfrak{g} \in \hat{\mathscr{L}}$ are pronilpotent in the classical sense,

$$(7.1) \qquad\qquad\qquad \mathfrak{g} = \lim_k \mathfrak{g}/\mathfrak{g}_k,$$

where $\mathfrak{g} = \mathfrak{g}_1 \supset \mathfrak{g}_2 \supset \cdots$ is the lower central series. Assume $\mathfrak{g} = \lim_n \mathfrak{g}^n$ where $\mathfrak{g}^n$, $n \geq 0$, are finite-dimensional nilpotent. Since the filtered limit of finite-dimensional vector spaces is exact, we easily verify that $(\mathfrak{g}^n)_k \cong \lim_n (\mathfrak{g}^n)_k$ and that $\mathfrak{g}/\mathfrak{g}_k \cong \lim_n \mathfrak{g}^n/(\mathfrak{g}^n)_k$ for each $k \geq 1$. The nilpotence of $\mathfrak{g}^n$ implies $\lim_k \mathfrak{g}^n/(\mathfrak{g}^n)_k \cong \mathfrak{g}^n$, therefore

$$\lim_k \mathfrak{g}/\mathfrak{g}_k \cong \lim_k \lim_n \mathfrak{g}^n/(\mathfrak{g}^n)_k \cong \lim_n \lim_k \mathfrak{g}^n/(\mathfrak{g}^n)_k \cong \lim_n \mathfrak{g}^n = \mathfrak{g}$$

as desired.

7.3. Definition. Let $\hat{\mathcal{L}} : \mathscr{A}_+ \mapsto \hat{\mathscr{L}}$ be the functor associating to an augmented cdga A the complete dgla $\hat{\mathscr{L}}(A)$ whose underlying graded Lie algebra is $\hat{\mathbb{L}}\Sigma^{-1}A_+^*$, the completed free Lie algebra on $\Sigma^{-1}A_+^*$. The differential d in $\hat{\mathscr{L}}(A)$ is defined as $d = d_I + d_{II}$ where d_I is induced by the internal differential in A_+ and d_{II} is determined by its restriction onto $\Sigma^{-1}A^*$, which is, in turn, induced by the product map $A \otimes A \to A$.

Dually, let $\mathcal{C} : \hat{\mathscr{L}} \mapsto \mathscr{A}_+$ be the functor associating to a complete dgla $\mathfrak{g}$ the (discrete) augmented cdga $\mathcal{C}(A)$ whose underlying graded Lie algebra is $S\Sigma\mathfrak{g}^*$, the symmetric algebra on $\Sigma\mathfrak{g}^*$. The differential d in $\mathcal{C}(\mathfrak{g})$ is defined as $d = d_I + d_{II}$ where d_I is induced by the internal differential in $\mathfrak{g}$ and d_{II} is determined by its restriction onto $\Sigma\mathfrak{g}^*$, which is, in turn, induced by the bracket map $\mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$.

7.4. Remark. The dgla $\hat{\mathcal{L}}(A)$ is also known as the *Harrison complex* of an augmented cdga A whereas $\mathcal{C}(\mathfrak{g})$ is an analogue of the Chevalley-Eilenberg complex of $\mathfrak{g}$, except that usually $\mathfrak{g}$ is not assumed to be complete (and in that case $\mathcal{C}(\mathfrak{g})$ has to be completed).

The following result follows directly from the definition.

7.5. Proposition. The functors $\mathcal{C}$ and $\hat{\mathcal{L}}$ are adjoint, so there is a natural isomorphism for $A \in \mathscr{A}_+$ and $\mathfrak{g} \in \hat{\mathscr{L}}$:

$$\hat{\mathscr{L}}\big(\hat{\mathcal{L}}(A), \mathfrak{g}\big) \cong \mathscr{A}_+\big(\mathcal{C}(\mathfrak{g}), A\big).$$

7.6. Remark. In Section 9 we endow $\hat{\mathscr{L}}$ with the structure of a closed model category, in such a way that the functors $\hat{\mathcal{L}}$ and $\mathcal{C}$ will become inverse equivalences of the corresponding homotopy categories. We will also see that each cdga of the form $\mathcal{C}(\mathfrak{g})$ is cofibrant.

7.1. Simplicial mapping space for cdgas and dglas. To construct a simplicial Hom in the category of complete dglas we are forced, as an intermediate step, to deal with dglas endowed with a linear topology, but which are *not complete*. An example of such a dgla is a tensor product $A \otimes \mathfrak{g}$ where $\mathfrak{g}$ is a complete dgla and A a discrete cdga that is infinite dimensional. Our convention is that the tensor product of a discrete vector space W and a complete vector space $V = \lim_\alpha V_\alpha$ is the topological vector space $W \otimes V := \lim_\alpha (W \otimes V_\alpha)$. Whenever two dglas $\mathfrak{g}$ and $\mathfrak{h}$ are endowed with a linear topology we will write $\mathrm{Hom}_{\mathrm{dgla}}(\mathfrak{g}, \mathfrak{h})$ for the set of *continuous* Lie homomorphisms from $\mathfrak{g}$ into $\mathfrak{h}$.

Let $\mathfrak{g}$ and $\mathfrak{h}$ be two complete dglas. We are going to construct a simplicial set of maps $\hat{\mathscr{L}}(\mathfrak{g}, \mathfrak{h})_\bullet$. As expected, the simplicial enrichment will be obtained by tensoring the dgla $\mathfrak{h}$ with the Sullivan-de Rham algebra $\Omega(\Delta^\bullet)$ of polynomial forms on the standard topological cosimplicial simplex, cf. [5, §2]. According to our conventions, $\Omega(\Delta^\bullet)$ will be considered as a homologically graded cdga, so $\Omega(\Delta^\bullet) \otimes \mathfrak{h}$ will be a homologically graded dgla. In particular, $\Omega^p(\Delta^\bullet) \otimes \mathfrak{h}_q$ will be placed in homological degree $q - p$.

7.7. Definition. Let $\hat{\mathscr{L}}(\mathfrak{g}, \mathfrak{h})_\bullet := \mathrm{Hom}_{\mathrm{dgla}}\big(\mathfrak{g}, \Omega(\Delta^\bullet) \otimes \mathfrak{h}\big)$ with the faces and degeneracy maps coming from the corresponding geometric maps on the cosimplicial simplex Δ^*.

7.8. Remark. Note that $\Omega(\Delta^n)$, $n \geq 0$, is a discrete infinite-dimensional cdga whereas $\mathfrak{h}$ is a complete dgla; thus $\Omega(\Delta^n) \otimes \mathfrak{h}$ is a topological, but not a complete dgla.

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Recall that the category $\mathscr{A}_+$ also has a simplicial Hom: for two augmented cdgas A and B let $\mathscr{A}_+(A, B)_\bullet$ be the simplicial set for which
$$\mathscr{A}_+(A, B)_n = \mathscr{A}_+\big(A, (\Omega(\Delta^n) \otimes B_+)_e\big).$$

The following result is an enriched version of Proposition 7.5; its proof is a straightforward inspection.

7.9. **Proposition.** For any augmented cdga A and a complete dgla $\mathfrak{g}$ there is a natural isomorphism of simplicial sets:

(7.2)
$$\hat{\mathscr{L}}(\hat{\mathcal{L}}(A), \mathfrak{g})_\bullet \cong \mathscr{A}_+(\mathcal{C}(\mathfrak{g}), A)_\bullet .$$

7.10. **Remark.** We defined the simplicial mapping space between two augmented cdgas in this way in order for the adjunction (7.2) to hold. The simplicial mapping space in the category of (unital) cdgas is different; for two (unital) cdgas A and B we have

(7.3)
$$\mathscr{A}(A, B)_\bullet := \mathscr{A}\big(A, B \otimes \Omega(\Delta^\bullet)\big).$$

It is easy to see that if A is an augmented cdga and B is a (unital) cdga then there is an isomorphism between simplicial sets
$$\mathscr{A}_+(A, B_e)_\bullet \cong \mathscr{A}(A, B)_\bullet .$$

Denote by $\mathfrak{s}$ the dgla spanned by two vectors x and $[x, x]$ with $|x| = -1$ and the differential $d(x) = -\frac{1}{2}[x, x]$. The dgla $\mathfrak{s}$ is designed to model the topological space S^0, the zero-dimensional sphere (which is a disjoint union of two points). Note that $\mathfrak{s}$ is isomorphic to $\hat{\mathcal{L}}(\Bbbk \times \Bbbk)$, where $\Bbbk \times \Bbbk$ is the augmented cdga (with vanishing differential) obtained from the ground field $\Bbbk$ by adjoining a unit.

7.11. **Definition.** Let $\mathfrak{g}$ be a complete cdga. Its *MC simplicial set* $\mathrm{MC}_\bullet(\mathfrak{g})$ is the simplicial mapping space:
$$\mathrm{MC}_\bullet(\mathfrak{g}) := \hat{\mathscr{L}}(\mathfrak{s}, \mathfrak{g})_\bullet .$$
In other words, $\mathrm{MC}_\bullet(\mathfrak{g})$ is the simplicial space of MC elements in $\mathfrak{g} \otimes \Omega(\Delta^\bullet)$. The set $\pi_0 \mathrm{MC}_\bullet(\mathfrak{g})$ is called the *MC moduli set* of $\mathfrak{g}$ and will be denoted by $\mathscr{MC}(\mathfrak{g})$.

7.12. **Example.** Fix a finite set S and assign, to each $s \in S$, a degree -1 generator x_s. Consider the complete dgla $\mathfrak{g}_S := \hat{\mathbb{L}}(x_s; \; s \in S)$ freely generated by the set $\{x_s\}_{s \in S}$, with the differential given by $dx_s := -\frac{1}{2}[x_s, x_s]$. Let us describe the MC elements in $\mathfrak{g}_S \otimes \Omega(\Delta^n)$.

The vector space $\big(\mathfrak{g}_S \otimes \Omega(\Delta^n)\big)_{-1}$ consists of expressions
$$x = \sum_{s \in S} x_s \otimes \alpha_s, \; \alpha_s \in \Omega^0(\Delta^n).$$

The MC condition for x reads
$$\sum_s \left(-\frac{1}{2}[x_s, x_s] \otimes \alpha_s - x_s \otimes d\alpha_s + \frac{1}{2}[x_s, x_s] \otimes \alpha_s^2 \right) + \sum_{s' \neq s''} [x_{s'}, x_{s''}] \otimes \alpha_{s'}\alpha_{s''} = 0.$$

By the freeness of $\mathfrak{g}_S$, the above equation is satisfied if and only if

(7.4a)
$$d\alpha_s = 0 \text{ for each } s \in S,$$

(7.4b)
$$\alpha_s(1 - \alpha_s) = 0 \text{ for each } s \in S \text{ and}$$

(7.4c)
$$\alpha_{s'}\alpha_{s''} = 0 \text{ for each } s', s'' \in S, \; s' \neq s''.$$

Equation (7.4a) implies that α_s is a constant, by (7.4b) the only possible values of this constant are 0 or 1. Equation (7.4c) then implies that $\alpha_s = 1$ for at most one s.

We conclude that $\mathrm{MC}_\bullet(\mathfrak{g}_S)$ is the constant simplicial set representing the discrete space $S \cup \{*\}$. If S is arbitrary, we define $\mathfrak{g}_S := \lim_F \mathfrak{g}_F$, the obvious limit over finite subsets $F \subset S$. It is easy to check that the above calculation remains valid, so we see that each non-empty discrete space can be realized as the MC space of some complete dg-Lie algebra. An interpretation of this example is given in Remark 7.19 below.

7.13. Remark. In light of Proposition 7.9 we see that the MC simplicial set of a complete dgla $\mathfrak{g}$ is isomorphic to

$$\mathscr{A}_+\big(\mathcal{C}(\mathfrak{g}), \Bbbk \times \Bbbk\big)_\bullet \cong \mathscr{A}\big(\mathcal{C}(\mathfrak{g}), \Bbbk\big)_\bullet.$$

It is easy to see that if two complete $\mathfrak{g}$ and $\mathfrak{h}$ are related by a filtered quasi-isomorphism, then $\mathrm{MC}_\bullet(\mathfrak{g})$ and $\mathrm{MC}_\bullet(\mathfrak{h})$ are weakly equivalent simplicial sets. We will not need this result.

7.2. Disjoint products of dglas and their MC spaces. Recall that an MC element $\xi \in \mathfrak{h}_{-1}$ in a dgla $\mathfrak{h}$ allows one to twist the differential d in $\mathfrak{h}$ according to the formula $d^\xi(?) = d(?) + [?, \xi]$. The graded Lie algebra $\mathfrak{h}$ supplied with the twisted differential d^ξ will be denoted by $\mathfrak{h}^\xi$. The same construction applies when $\mathfrak{h}$ is a complete dgla, in that case $\mathfrak{h}^\xi$ will likewise be complete.

Given an arbitrary complete dgla $\mathfrak{g}$ consider the complete dgla $\mathfrak{g} * \mathfrak{s}$ where $*$ stands for the coproduct in the category $\hat{\mathscr{L}}$. Abusing the notation, we will write x for the image of $x \in \mathfrak{s}$ inside $\mathfrak{g} * \mathfrak{s}$ under the inclusion of dglas $\mathfrak{s} \hookrightarrow \mathfrak{g} * \mathfrak{s}$. It is clear that x is an MC element in $\mathfrak{g} * \mathfrak{s}$. The twisted dgla $(\mathfrak{g} * \mathfrak{s})^x$ is analogous to adjoining a base point to a topological space. Generalizing this construction, we define a *disjoint product* of two complete dglas; note that it also makes sense for ordinary (non-complete) dglas.

7.14. Definition. Let $\mathfrak{g}$ and $\mathfrak{h}$ be two complete dglas. Their disjoint product is the complete dgla $(\mathfrak{g} * \mathfrak{s})^x * \mathfrak{h}$; it will be denoted by $\mathfrak{g} \sqcup \mathfrak{h}$.

In particular, for a dgla $\mathfrak{g}$ the twisted free product $(\mathfrak{g} * \mathfrak{s})^x$ is now denoted as $\mathfrak{g} \sqcup 0$. Even if $\mathfrak{g}$ is non-negatively graded, $\mathfrak{g} \sqcup 0$ is not; this obviously applies also to more general disjoint products of dglas.

7.15. Remark. The operation of a disjoint product of two dglas was considered (without this name) in [6] and, in a more general, operadic context in [7]. It is not hard to prove that for any dgla $\mathfrak{g}$ (complete or not) the dgla $\mathfrak{g} \sqcup 0$ is acyclic, cf. [7, Theorem 5.7] or [6, Lemma 6.1]. It follows that for any two dglas $\mathfrak{g}$ and $\mathfrak{h}$ the dgla $\mathfrak{g} \sqcup \mathfrak{h}$ is quasi-isomorphic to $\mathfrak{h}$.

In particular, the operation of a disjoint product of two (complete) dglas is very far from being commutative. This phenomenon has the following topological explanation, or analogue. Given two *pointed* topological spaces X and Y define their 'disjoint product' as the disjoint union $X \sqcup Y$ having its base point in Y. Clearly, the operation of disjoint product of two topological spaces is likewise noncommutative, owing to the asymmetric placement of the base point.

The following statement 'measures' the non-commutativity of the disjoint product.

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7.16. **Proposition.** *The disjoint products* $\mathfrak{g} \sqcup \mathfrak{h}$ *and* $\mathfrak{h} \sqcup \mathfrak{g}$ *are related by a twist; namely the element* $-x$ *is an MC element in* $\mathfrak{g} \sqcup \mathfrak{h} \cong (\mathfrak{g} * \mathfrak{s})^x * \mathfrak{h}$ *and there is an isomorphism of dglas*

$$(\mathfrak{g} \sqcup \mathfrak{h})^{-x} \cong \mathfrak{h} \sqcup \mathfrak{g}.$$

Proof. The isomorphism in the proposition is the identity map on $\mathfrak{g}$ and $\mathfrak{h}$ and takes $x \in \mathfrak{g} \sqcup \mathfrak{h}$ to $-x \in \mathfrak{h} \sqcup \mathfrak{g}$. It is straightforward to check that it has the desired property. $\square$

However, the following result shows that the operation of the disjoint product of dglas is associative, just as its topological counterpart.

7.17. **Proposition.** For any three dglas $\mathfrak{g}, \mathfrak{h}, \mathfrak{a}$ there is a natural isomorphism of dglas:

$$(\mathfrak{g} \sqcup \mathfrak{h}) \sqcup \mathfrak{a} \cong \mathfrak{g} \sqcup (\mathfrak{h} \sqcup \mathfrak{a}).$$

Proof. We have an isomorphism of graded Lie algebras, disregarding the differential:

(7.5) $$\mathfrak{g} \sqcup (\mathfrak{h} \sqcup \mathfrak{a}) \cong (\mathfrak{g} * \mathfrak{s}) * ((\mathfrak{h} * \mathfrak{s}) * \mathfrak{a}).$$

Denote the generator of the first copy of $\mathfrak{s}$ by x and the generator of the second copy of $\mathfrak{s}$ by y in (7.5). Then we have by definition an isomorphism of dglas:

$$\mathfrak{g} \sqcup (\mathfrak{h} \sqcup \mathfrak{a}) \cong (\mathfrak{g} * \mathfrak{s})^x * ((\mathfrak{h} * \mathfrak{s})^y * \mathfrak{a}).$$

Similarly there is an isomorphism of dglas

$$(\mathfrak{g} \sqcup \mathfrak{h}) \sqcup \mathfrak{a} \cong (((\mathfrak{g} * \mathfrak{s})^{x'} * \mathfrak{h}) * \mathfrak{s})^{y'} * \mathfrak{a},$$

where x' and y' have an obvious meaning.

Now consider the map $f : \mathfrak{g} \sqcup (\mathfrak{h} \sqcup \mathfrak{a}) \cong (\mathfrak{g} \sqcup \mathfrak{h}) \sqcup \mathfrak{a}$ which is the identity on $\mathfrak{g}, \mathfrak{h}$ and $\mathfrak{a}$, $f(x) := x' + y'$ and $f(y) := y'$. A straightforward calculation shows that f is compatible with the differentials and is, therefore, an isomorphism as claimed. $\square$

7.18. **Remark.** The operation of disjoint product *does not* make the category of dglas into a monoidal category, since there is no unit object. However, one can check that the pentagon condition for the operation $\sqcup$ is satisfied and thus, in any expression involving multiple disjoint products the arrangement of brackets does not matter, up to a natural isomorphism.

7.19. **Remark.** As an exercise we recommend to verify that for finite S the algebra $\mathfrak{g}_S$ from Example 7.19 is isomorphic to the disjoint product of 0 with itself, iterated $\mathrm{card}(S)$-times, that is

$$\mathfrak{g}_\emptyset \cong 0, \ \ \mathfrak{g}_{\{*\}} \cong 0 \sqcup 0, \ \ \mathfrak{g}_{\{*,*\}} \cong (0 \sqcup 0) \sqcup 0 \cong 0 \sqcup (0 \sqcup 0), \dots$$

The fact that $\mathfrak{g}_S$ represents the disjoint union $S \cup \{*\}$ established in Example 7.19 is thus corroborated by Theorem 1.7.

Define a *connected cover* $\overline{\mathfrak{h}}$ of a dgla $\mathfrak{h} = (\mathfrak{h}, [-,-], \partial)$ as the sub-dgla of $\mathfrak{h}$ given by

(7.6) $$\overline{\mathfrak{h}}_n := \begin{cases} \mathfrak{h}_n & \text{for } n > 0, \\ \mathrm{Ker}(\partial : \mathfrak{h}_0 \to \mathfrak{h}_{-1}) & \text{for } n = 0, \text{and} \\ 0 & \text{for } n < 0. \end{cases}$$

If $\mathfrak{h}$ is complete then $\overline{\mathfrak{h}}$ is clearly complete as well. The map $H_n(\overline{\mathfrak{h}}) \to H_n(\mathfrak{h})$ induced by the inclusion $\iota : \overline{\mathfrak{h}} \hookrightarrow \mathfrak{h}$ is an isomorphism for $n \geq 0$.

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For an MC element $\xi \in \mathfrak{h}_{-1}$ denote by $\phi^\xi : \mathrm{MC}(\mathfrak{h}^\xi) \to \mathrm{MC}(\mathfrak{h})$ the isomorphism $\eta \mapsto \eta + \xi$, $\eta \in \mathrm{MC}(\mathfrak{h}^\xi)$, and by $\phi_\bullet^\xi : \mathrm{MC}_\bullet(\mathfrak{h}^\xi) \to \mathrm{MC}_\bullet(\mathfrak{h})$ the obvious induced map of simplicial sets. Our proof of Theorem 1.9 will be based on

7.20. Proposition. *Let $\mathfrak{g}$ be a complete dgla and $\xi \in \mathfrak{g}_{-1}$ an MC element. Then the composition*

$$\mathrm{MC}_\bullet(\overline{\mathfrak{g}^\xi}) \xrightarrow{\mathrm{MC}_\bullet(\iota)} \mathrm{MC}_\bullet(\mathfrak{g}^\xi) \xrightarrow{\phi_\bullet^\xi} \mathrm{MC}_\bullet(\mathfrak{g})$$

induces a weak equivalence between $\mathrm{MC}_\bullet(\overline{\mathfrak{g}^\xi})$ and the connected component of $\mathrm{MC}_\bullet(\mathfrak{g})$ containing $\xi \in \mathrm{MC}(\mathfrak{g})$.

Proof. Let $\mathfrak{h}$ be a complete dgla. By Remark 7.13 we have an isomorphism of simplicial spaces $\mathrm{MC}_\bullet(\mathfrak{h}) \cong \mathscr{A}\big(\mathcal{C}(\mathfrak{h}), \Bbbk\big)_\bullet$, so

$$\pi_n\big(\mathrm{MC}_\bullet(\mathfrak{h})\big) \cong \pi_n\big(\mathscr{A}(\mathcal{C}(\mathfrak{h}), \Bbbk)_\bullet\big), \ n \geq 1.$$

As shown in Section 9, the cdga $\mathcal{C}(\mathfrak{h})$ is cofibrant, so the homotopy groups of the simplicial space $\mathscr{A}\big(\mathcal{C}(\mathfrak{h}), \Bbbk\big)_\bullet$ can be, for $n \geq 1$, calculated using [5, Proposition 8.12] with $X := \mathcal{C}(\mathfrak{h})$. One gets

$$\pi_n\big(\mathscr{A}(\mathcal{C}(\mathfrak{h}), \Bbbk)_\bullet\big) \cong \mathrm{Hom}\big(H^n(\Sigma\mathfrak{h}^*), \Bbbk)\big) \cong H_{n-1}(\mathfrak{h}).$$

Since all isomorphisms above are functorial, they combine into a functorial isomorphism

$$(7.7) \qquad\qquad \pi_n\big(\mathrm{MC}_\bullet(\mathfrak{h})\big) \cong H_{n-1}(\mathfrak{h}), \ n \geq 1,$$

valid for an arbitrary complete dgla. This isomorphism is for $n \geq 2$ an isomorphism of abelian groups, but we will not need this result.

Let us return to the proof of our theorem. It is clear that the the simplicial isomorphism $\phi_\bullet^\xi : \mathrm{MC}_\bullet(\mathfrak{g}^\xi) \cong \mathrm{MC}_\bullet(\mathfrak{g})$ induces an isomorphism between the connected component containing the trivial MC element $0 \in \mathrm{MC}(\mathfrak{g}^\xi)$ and the connected component containing $\xi \in \mathfrak{g}_{-1}$. Furthermore, it is clear that the simplicial set $\mathrm{MC}_\bullet(\overline{\mathfrak{g}^\xi})$ is connected, with the trivial MC element $0 \in \overline{\mathfrak{g}^\xi}_{-1}$ its only 0-simplex. It is therefore enough to show that the inclusion $\iota : \overline{\mathfrak{g}^\xi} \hookrightarrow \mathfrak{g}^\xi$ induces and isomorphism

$$\pi_n\big(\mathrm{MC}_\bullet(\overline{\mathfrak{g}^\xi})\big) \cong \pi_n\big(\mathrm{MC}_\bullet(\mathfrak{g}^\xi)\big),$$

for arbitrary $n \geq 1$. By (7.7), this is the same as showing that ι induces an isomorphism

$$H_{n-1}(\overline{\mathfrak{g}^\xi}) \cong H_{n-1}(\mathfrak{g}^\xi),$$

for each $n \geq 1$. The last isomorphism easily follows from the definition (7.6) of the connected cover $\overline{\mathfrak{g}^\xi}$ of $\mathfrak{g}^\xi$. $\qquad\square$

Proof of Theorem 1.9. The isomorphism (1.1) follows from Proposition 7.20, the weak equivalence from (1.1) and Theorem 1.7. $\qquad\square$

[June 7, 2014]

8. Proof of Theorem 1.7

We start this section by the following auxiliary statement.

8.1. **Lemma.** *Let A, B and C be unital cdgas with A and B cofibrant and such that $H^0(C)$ has no idempotents different from 1. Denote by $\overline{A \times B}$ a cofibrant approximation of $A \times B$. Then there is a weak equivalence of simplicial sets*

$$\mathscr{A}(\overline{A \times B}, C)_\bullet \simeq \mathscr{A}(A, C)_\bullet \cup \mathscr{A}(B, C)_\bullet .$$

Proof. Let S be a finite, connected simplicial set and let $\Omega(S)$ be the cdga of Sullivan-de Rham forms on S (denoted by $A(S)$ in [5]). Then we have the following standard adjunction isomorphism, cf. [5, Lemma 5.2]:

$$(8.1) \qquad \mathscr{S}\big(S, \mathscr{A}(\overline{A \times B}, C)_\bullet\big) \cong \mathscr{A}\big(\overline{A \times B}, \Omega(S) \otimes C\big).$$

Observe that this adjunction automatically extends to simplicially enriched Homs, that is

$$(8.2) \qquad \mathscr{S}\big(S, \mathscr{A}(\overline{A \times B}, C)_\bullet\big)_\bullet \cong \mathscr{A}\big(\overline{A \times B}, \Omega(S) \otimes C\big)_\bullet.$$

Indeed, by (8.1) one has for each $n \geq 0$,

$$\mathscr{S}\big(S \times \Delta^n, \mathscr{A}(\overline{A \times B}, C)_\bullet\big) \cong \mathscr{A}\big(\overline{A \times B}, \Omega(S \times \Delta^n) \otimes C\big) \cong \mathscr{A}\big(\overline{A \times B}, \Omega(S) \otimes C \otimes \Omega(\Delta^n)\big),$$

consequently

$$\mathscr{S}\big(S, \mathscr{A}(\overline{A \times B}, C)_\bullet\big)_n \cong \mathscr{A}\big(\overline{A \times B}, \Omega(S) \otimes C\big)_n$$

for each $n \geq 0$. By naturality, the above individual isomorphisms assemble into an isomorphism (8.2) of simplicial sets. Since $\overline{A \times B}$ is cofibrant, the simplicial set $\mathscr{A}(\overline{A \times B}, C)_\bullet$ is Kan. Adjunction (8.2) induces an isomorphism between the sets of connected components of the corresponding simplicial sets, therefore

$$\big[S, \mathscr{A}(\overline{A \times B}, C)_\bullet\big] \cong \big[\overline{A \times B}, \Omega(S) \otimes C\big] \cong \big[A \times B, \Omega(S) \otimes C\big].$$

The cdga $\Omega(S) \otimes C$ is bigraded and the first grading is non-negative. It is easy to show that in this situation any idempotent in its cohomology must be contained in the $(0,0)$-graded part which is isomorphic to $H^0(S) \otimes H^0(C) \cong H^0(C)$. But $H^0(C)$ has no non-trivial idempotents other than 1 by our assumption. According to Theorem 2.3

$$\big[A \times B, \Omega(S) \otimes C\big] \cong \big[A, \Omega(S) \otimes C\big] \cup \big[B, \Omega(S) \otimes C\big] \cong \big[S, \mathscr{A}(A, C)_\bullet\big] \cup \big[S, \mathscr{A}(A, C)_\bullet\big].$$

Note that, because S is connected, we have a natural bijection

$$\mathscr{S}\big(S, \mathscr{A}(A, C)_\bullet \cup \mathscr{A}(B, C)_\bullet\big) \cong \mathscr{S}\big(S, \mathscr{A}(A, C)_\bullet\big) \cup \mathscr{S}\big(S, \mathscr{A}(A, C)_\bullet\big)$$

and, therefore,

$$\big[S, \mathscr{A}(\overline{A \times B}, C)_\bullet\big] \cong \big[A \times B, \Omega(S)\big].$$

Thus, we have a natural bijection in the homotopy category of simplicial sets, for any connected finite simplicial set S:

$$\big[S, \mathscr{A}(A, C)_\bullet \cup \mathscr{A}(B, C)_\bullet\big] \cong \big[S, \mathscr{A}(\overline{A \times B}, C)_\bullet\big].$$

The desired conclusion follows. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

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8.2. **Remark.** Note that the above lemma is an enriched analogue of Theorem 3.3. Since $A \times B$ is not cofibrant in general, the result will not be true without taking a cofibrant replacement. Also note that the lemma can be easily generalized to a finite direct product of cofibrant cdgas.

We need one preliminary result on the MC twisting of dglas. Let $\mathfrak{g}$ be a complete dgla and consider the cdga $\mathcal{C}(\mathfrak{g})$. It is clear that the set of MC elements in $\mathfrak{g}$ is in 1-1 correspondence with cdga maps (augmentations) $\mathcal{C}(\mathfrak{g}) \to \Bbbk$, in particular the canonical augmentation corresponds to the zero MC element.

Let us explain this correspondence explicitly. The evaluation at $\xi \in \mathfrak{g}_{-1}$ determines a degree $+1$ linear map $\alpha_\xi : \mathfrak{g}^* \to \Bbbk$ which in turn defines a degree 0 linear map (denoted by the same symbol) $\alpha_\xi : \Sigma\mathfrak{g}^* \to \Bbbk$ from the space of generators of $\mathcal{C}(\mathfrak{g})$ to the ground field. The latter map extends to a unique morphisms of graded algebras $\epsilon_\xi : \mathcal{C}(\mathfrak{g}) \to \Bbbk$ which commutes with the differentials, i.e. makes $\mathcal{C}(\mathfrak{g})$ an augmented cdga, if and only if ξ is Maurer-Cartan.

Likewise, define a degree 0 linear map $\beta_\xi : \Sigma\mathfrak{g}^* \to \Bbbk \oplus \Sigma\mathfrak{g}^* \subset \mathcal{C}(\mathfrak{g}^\xi)$ as $\beta_\xi := \alpha_\xi + \mathbb{1}_{\Sigma\mathfrak{g}^*}$. One can easily verify that β_ξ extends to a cdga's isomorphism $\phi_\xi : \mathcal{C}(\mathfrak{g}) \to \mathcal{C}(\mathfrak{g}^\xi)$ such that the diagram

(8.3)

$$
\begin{array}{ccc}
\mathcal{C}(\mathfrak{g}) & \xrightarrow{\phi_\xi} & \mathcal{C}(\mathfrak{g}^\xi) \\
& {}_{\epsilon_\xi}\searrow \quad \swarrow_{\epsilon_0} & \\
& \Bbbk &
\end{array}
$$

commutes.

The maps ϵ_ξ and ϕ_ξ above have a nice geometric meaning if we interpret elements $f \in \mathcal{C}(\mathfrak{g})$ or $f \in \mathcal{C}(\mathfrak{g}^\xi)$ as polynomial functions on $\Sigma\mathfrak{g}$. Then ϵ_ξ is the evaluation at $\vartheta := \Sigma\xi$ while ϕ_ξ is given by the shift by ϑ, that is

$$\epsilon_\xi(f) = f(\vartheta) \quad \text{and} \quad \phi_\xi(f)(u) = f(u + \vartheta), \ u \in \Sigma\mathfrak{g}.$$

Checking the commutativity of (8.3) in this language boils down to

$$\epsilon_0\big(\phi_\xi(f)\big) = \phi_\xi(f)(0) = f(\vartheta) = \epsilon_\xi(f).$$

Given $\xi \in \mathrm{MC}(\mathfrak{g})$ there is a bijective correspondence $\mathrm{MC}(\mathfrak{g}) \to \mathrm{MC}(\mathfrak{g}^\xi)$ defined by $\eta \mapsto \eta - \xi$. In particular, the element $\xi \in \mathrm{MC}(\mathfrak{g})$ corresponds to the zero element in $\mathrm{MC}(\mathfrak{g}^\xi)$.

8.3. **Lemma.** *Let $\mathfrak{g}$ be a complete dgla. Then the augmented cdga $\mathcal{C}(\mathfrak{g} \sqcup 0)$ is quasi-isomorphic to $\mathcal{C}(\mathfrak{g}) \times \Bbbk$, with the augmentation given by the projection to the* second *factor.*

Proof. Consider the cdga $\mathcal{C}(\mathfrak{g} * \mathfrak{s})$. According to Corollary A.7 the augmented cdga $\mathcal{C}(\mathfrak{g} * \mathfrak{s})$ is quasi-isomorphic to $\mathcal{C}(\mathfrak{g}) \times_\Bbbk (\Bbbk \times \Bbbk) \cong \mathcal{C}(\mathfrak{g}) \times \Bbbk$; moreover, the augmentation of $\mathcal{C}(\mathfrak{g}) \times \Bbbk$ factors through the projection onto the *first* factor. On the other hand, there is, as in (8.3), an isomorphism of cdgas (not respecting the augmentation) $\phi_x : \mathcal{C}(\mathfrak{g} * \mathfrak{s}) \to \mathcal{C}(\mathfrak{g} * \mathfrak{s})^x$ such that the induced augmentation $\mathcal{C}(\mathfrak{g} * \mathfrak{s}) \to \mathcal{C}(\mathfrak{g} * \mathfrak{s})^x \to \Bbbk$ corresponds to the MC element $x \in \mathrm{MC}(\mathfrak{g} * \mathfrak{s})$.

Note that the augmentation of $\mathcal{C}(\mathfrak{g} * \mathfrak{s})$ corresponding to the MC element $x \in \mathrm{MC}(\mathfrak{g} * \mathfrak{s})$ translates via the above quasi-isomorphism into the augmentation of $\mathcal{C}(\mathfrak{g}) \times \Bbbk$ given by the projection onto the second factor. This proves the desired statement. $\qquad\square$

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We now return to the proof of Theorem 1.7. It suffices to consider the case when J consists of two elements. Let $\mathfrak{g}$ and $\mathfrak{h}$ be two complete dglas. Then by Theorem A.7 and Lemma 8.3 the augmented cdga $\mathcal{C}(\mathfrak{g} \sqcup \mathfrak{h}) = \mathcal{C}\big((\mathfrak{g} \sqcup 0) * \mathfrak{h}\big)$ is quasi-isomorphic to

$$\mathcal{C}(\mathfrak{g} \sqcup 0) \times_{\Bbbk} \mathcal{C}(\mathfrak{h}) \simeq (\mathcal{C}(\mathfrak{g}) \times \Bbbk) \times_{\Bbbk} \mathcal{C}(\mathfrak{h}) \cong \mathcal{C}(\mathfrak{g}) \times \mathcal{C}(\mathfrak{h}).$$

We conclude that the simplicial set $\mathrm{MC}_\bullet(\mathfrak{g} \sqcup \mathfrak{h}) = \mathscr{A}_+(\mathcal{C}(\mathfrak{g} \sqcup \mathfrak{h}), \Bbbk \times \Bbbk)_\bullet$ is weakly equivalent to $\mathscr{A}_+(\overline{\mathcal{C}(\mathfrak{g}) \times \mathcal{C}(\mathfrak{h})}, \Bbbk \times \Bbbk)_\bullet$. The latter simplicial Hom is taken in the category of augmented cdgas; it is isomorphic to the simplicial mapping space $\mathscr{A}(\overline{\mathcal{C}(\mathfrak{g}) \times \mathcal{C}(\mathfrak{h})}, \Bbbk)_\bullet$ taken in the category of (unital) non-augmented cdgas. Now we have the following weak equivalence of simplicial sets, which follows from Lemma 8.1:

$$\mathscr{A}\big(\overline{\mathcal{C}(\mathfrak{g}) \times \mathcal{C}(\mathfrak{h})}, \Bbbk\big)_\bullet \simeq \mathscr{A}(\mathcal{C}(\mathfrak{g}), \Bbbk)_\bullet \cup \mathscr{A}(\mathcal{C}(\mathfrak{h}), \Bbbk)_\bullet \ .$$

This finishes the proof of Theorem 1.7.

We close this section by two versions of an example which is a dgla analogue of a disjoint union of a circle S^1 and an isolated point. This example is simple enough to be worked out by hand, although the calculations are still nontrivial. We will see that, in this particular case, an analogue of Theorem 1.7 actually holds without a completion. The general claim, in the non-completed context, was made in [6, Theorem 6.4], but we have not been able to parse the proof in op. cit.

8.4. **Example.** Let x (resp. a) be a generator of degree -1 (resp. 0). Denote by $\mathfrak{f}$ the free non-complete Lie algebra $\mathbb{L}(x, a)$ generated by x and a, with the differential given by $dx := -\frac{1}{2}[x, x]$, $da := 0$. In this example we describe MC elements t in $\mathfrak{f} \otimes \Omega(\Delta^n)$, for an arbitrary $n \geq 0$.

Let us start by observing that $\mathfrak{f}_{-1}$ has a basis $\{e_i\}_{i \geq 0}$, with

$$e_0 := x, \ e_1 := [a, x], \ e_2 := [a, [a, x]], \ e_3 := [a, [a, [a, x]]], \ldots$$

From degree reasons, each element of $\big(\mathfrak{f} \otimes \Omega(\Delta^n)\big)_{-1}$ has the form

$$(8.4) \qquad\qquad t = \sum_{i \geq 0} e_i \otimes \alpha_i + a \otimes \omega,$$

for some $\alpha_i \in \Omega^0(\Delta^n)$, $i \geq 0$, and $\omega \in \Omega^1(\Delta^n)$, with the assumption that only *finitely many* α_i's are $\neq 0$. The MC equation for t reads

$$\sum_{i \geq 0}(de_i \otimes \alpha_i - e_i \otimes d\alpha_i) + a \otimes d\omega + \tfrac{1}{2}\sum_{i,j \geq 0}[e_i, e_j] \otimes \alpha_i \alpha_j - \sum_{i \geq 0}[a, e_i] \otimes \omega \alpha_i = 0$$

(the term $\frac{1}{2}[a, a] \otimes \omega^2$ clearly vanishes). Singling out the parts in $\mathfrak{f}_0 \otimes \Omega^2(\Delta^n), \mathfrak{f}_{-1} \otimes \Omega^1(\Delta^n)$ and $\mathfrak{f}_{-2} \otimes \Omega^0(\Delta^n)$, respectively, one gets the equations

$$(8.5a) \qquad\qquad\qquad\qquad\qquad a \otimes d\omega \ = \ 0,$$

$$(8.5b) \qquad\qquad \sum_{i \geq 0}(e_i \otimes d\alpha_i + [a, e_i] \otimes \omega \alpha_i) \ = \ 0,$$

$$(8.5c) \qquad \sum_{i \geq 0} de_i \otimes \alpha_i + \tfrac{1}{2}\sum_{i,j \geq 0}[e_i, e_j] \otimes \alpha_i \alpha_j \ = \ 0.$$

Since, by definition, $e_{i+1} = [a, e_i]$, (8.5b) implies that $d\alpha_0 = 0$, i.e. $\alpha_0 \in \Bbbk$ *is a constant*, while

$$(8.6) \qquad\qquad\qquad d\alpha_{i+1} = -\omega \alpha_i, \quad \text{for } i \geq 0.$$

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Equation (8.5c) means that $\sum_{i\geq 0} e_i \otimes \alpha_i$ is an MC element in $\mathfrak{f} \otimes \Omega^0(\Delta^n)$, with $\Omega^0(\Delta^n)$ taken with the trivial differential. It expands to the system

(8.7a) $$de_0 \otimes \alpha_0 \;=\; -\tfrac{1}{2}[e_0, e_0] \otimes \alpha_0^2,$$

(8.7b) $$de_1 \otimes \alpha_1 \;=\; -[e_0, e_1] \otimes \alpha_0 \alpha_1,$$

(8.7c) $$de_2 \otimes \alpha_2 \;=\; -[e_0, e_2] \otimes \alpha_0 \alpha_2 - \tfrac{1}{2}[e_1, e_1] \otimes \alpha_1^2,$$

(8.7d) $$de_2 \otimes \alpha_3 \;=\; -[e_0, e_3] \otimes \alpha_0 \alpha_3 - [e_1, e_2] \otimes \alpha_1 \alpha_2,$$

$$\vdots$$

Since $e_0 = x$ and $dx = -\tfrac{1}{2}[x, x]$, (8.7a) is equivalent to $\alpha_0 = \alpha_0^2$, therefore $\alpha_0 \in \Omega^0(\Delta^n)$ is a constant that equals either 0 or 1.

Case $\alpha_0 = 0$. In this case (8.7b) reduces to $de_1 \otimes \alpha_1 = 0$. Since $de_1 \neq 0$, it implies $\alpha_1 = 0$. Then (8.7c) reduces to $de_2 \otimes \alpha_2 = 0$ so, by the same argument, $\alpha_2 = 0$ and (8.7d) reduces to $de_3 \otimes \alpha_3 = 0$. We prove inductively that $\alpha_n = 0$ for each $n \geq 0$. While (8.5b) is automatically satisfied, (8.5a) requires $d\omega = 0$. We conclude that the MC elements t in $\mathfrak{f} \otimes \Omega(\Delta^n)$ with $\alpha_0 = 0$ are precisely closed forms in $\Omega^1(\Delta^n)$.

Case $\alpha_0 = 1$. Observe first that $\xi := e_0 \otimes 1 = x \otimes 1$ is an MC element of this form. We are going to prove that it is the *only* MC-element of $\mathfrak{f} \otimes \Omega(\Delta^n)$ with $\alpha_0 = 1$.

The Lie algebra $\mathfrak{f}_0$ is one-dimensional, spanned by a. Consider, only for the purposes of this example, the completion $\hat{\mathfrak{f}}$ of $\mathfrak{f}$ with respect to the grading by the number of a's. Then $\exp(\mathfrak{f}_0)$ acts on the set of MC elements in $\hat{\mathfrak{f}}$, in particular,

$$\exp(a)e_0 = e_0 + e_1 + \frac{1}{2!}e_2 + \frac{1}{3!}e_3 + \frac{1}{4!}e_4 + \cdots$$

is an MC element in $\hat{\mathfrak{f}}$. This means that

$$de_0 \;=\; -\tfrac{1}{2}[e_0, e_0],$$
$$de_1 \;=\; -[e_0, e_1]$$
$$de_2 \;=\; -[e_0, e_2] - [e_1, e_1],$$
$$de_3 \;=\; -[e_0, e_3] - 3[e_1, e_2],$$
$$\vdots$$

Substituting this to (8.7a)–(8.7d), we get

(8.8a) $$[e_0, e_0] \otimes \alpha_0 \;=\; [e_0, e_0] \otimes \alpha_0^2,$$

(8.8b) $$[e_0, e_1] \otimes \alpha_1 \;=\; [e_0, e_1] \otimes \alpha_0 \alpha_1,$$

(8.8c) $$([e_0, e_2] + [e_1, e_1]) \otimes \alpha_2 \;=\; [e_0, e_2] \otimes \alpha_0 \alpha_2 + \tfrac{1}{2}[e_1, e_1] \otimes \alpha_1^2,$$

(8.8d) $$([e_0, e_3] + 3[e_1, e_2]) \otimes \alpha_3 \;=\; [e_0, e_3] \otimes \alpha_0 \alpha_3 + [e_1, e_2] \otimes \alpha_1 \alpha_2,$$

$$\vdots$$

If $\alpha_1 = 0$, (8.8c) reduces to $[e_1, e_1] \otimes \alpha_2 = 0$, which implies that $\alpha_2 = 0$. Then (8.8d) implies that $[e_1, e_2] \otimes \alpha_3 = 0$, so $\alpha_3 = 0$. Continuing this process we prove that actually $\alpha_n = 0$ for all $n \geq 1$. Equation (8.6) with $i = 0$ gives $\omega = 0$. So the only MC element in $\hat{\mathfrak{f}} \otimes \Omega(\Delta^n)$ with $\alpha_0 = 1$ and $\alpha_1 = 0$ is $\xi := x \otimes 1$.

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Let now t be an MC element in $\mathfrak{f} \otimes \Omega(\Delta^n)$ with $\alpha_0 = 1$ and $\alpha_1 \neq 0$. Then $\exp(-a \otimes \alpha_1)t$ is clearly an MC element in $\hat{\mathfrak{f}} \otimes \Omega(\Delta^n)$ with $\alpha_1 = 0$, so $\exp(-a \otimes \alpha_1)t = x \otimes 1$ or, equivalently,

$$(8.9) \qquad t = \exp(a \otimes \alpha_1)x = a \otimes d\alpha_1 + e_0 \otimes 1 + e_1 \otimes \alpha_1 + \frac{1}{2!}e_2 \otimes \alpha_1^2 + \frac{1}{3!}e_3 \otimes \alpha_1^3 + \cdots$$

In particular, t has *infinitely many* non-trivial elements, so *it is not* an MC element in the uncompleted dgla $\mathfrak{f} \otimes \Omega(\Delta^n)$. So α_1 must be 0 therefore the only MC element of $\mathfrak{f} \otimes \Omega(\Delta^n)$ with $\alpha_0 = 1$ is $x \otimes 1$. We arrive at the following conclusion.

Claim. *The only MC elements of $\mathfrak{f} \otimes \Omega(\Delta^n)$ are either x or closed 1-forms in $\Omega^1(\Delta^n)$. In other words, one has an isomorphism of simplicial sets*

$$\mathrm{MC}_\bullet(\mathfrak{f}) \cong \mathrm{MC}_\bullet(\mathfrak{a}) \cup \mathrm{MC}_\bullet(0),$$

where 0 is the trivial Lie algebra and $\mathfrak{a}$ the one-dimensional abelian Lie algebra generated by a. So $\mathfrak{f}$ serves as an algebraic model for the disjoint union $S^1 \cup \{*\}$.

8.5. **Example.** Let us analyze the completed version of the previous example, i.e. the MC elements in $\hat{\mathfrak{f}} \widehat{\otimes} \Omega(\Delta^n)$, where $\hat{\mathfrak{f}}$ is the completion of the Lie algebra $\mathfrak{f} = \mathbb{L}(x, a)$. Of course, we know the answer from Theorem 1.7 but the explicit calculation that could be performed in this simple case is still instructive. Most of the work done in the previous example applies in the complete situation with only small modifications. For instance, elements of $(\hat{\mathfrak{f}} \widehat{\otimes} \Omega(\Delta^n))_{-1}$ are of the form (8.4), but the sum may now have *infinitely many* nontrivial terms.

Equations (8.5a)–(8.7d) still take place, as does the separation into Case $\alpha_0 = 0$ and Case $\alpha_0 = 1$. Precisely as in Example 8.4 we show that the only MC element with $\alpha_0 = 0$ is the trivial one. In contrast with the uncomplete case we, however, do not require the sum (8.9) to be *finite*, so α_1 there may be an arbitrary function from $\Omega^0(\Delta^\bullet)$.

We conclude that an MC element t in $\hat{\mathfrak{f}} \otimes \Omega(\Delta^n)$ is either $1 \otimes \omega$ for a closed form $\omega \in \Omega^1(\Delta^\bullet)$ if $\alpha_0 = 0$, or

$$t = \exp(a \otimes \alpha)x, \quad \text{for some } \alpha \in \Omega^0(\Delta^n),$$

if $\alpha_0 = 1$. Since the isotropy subgroup of x under the action of $\exp\left(a \otimes \Omega^0(\Delta^n)\right)$ is trivial, we conclude

Claim. *One has an isomorphism of simplicial sets*

$$\mathrm{MC}_\bullet(\hat{\mathfrak{f}}) \cong \mathrm{MC}_\bullet(\mathfrak{a}) \cup \Omega^0(\Delta^\bullet).$$

By [5, Proposition 1.1.], $\Omega^0(\Delta^\bullet)$ is contractible, therefore $\mathrm{MC}_\bullet(\hat{\mathfrak{f}})$ has the simplicial homotopy type of $\mathrm{MC}_\bullet(\mathfrak{a}) \cup \mathrm{MC}_\bullet(0)$. Thus $\hat{\mathfrak{f}}$ also serves as an algebraic model for the disjoint union $S^1 \cup \{*\}$. Note that the simplicial sets $\mathrm{MC}_\bullet(\hat{\mathfrak{f}})$ and $\mathrm{MC}_\bullet(\mathfrak{f})$ are weakly equivalent, but *not* isomorphic.

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9. Dual Hinich correspondence

The purpose of this section is to establish an analogue to the main result of Hinich's paper [15], giving a closed model category structure on cocomplete cocommutative dg coalgebras. A formal dualization of Hinich's result states that the category of complete cdgas has a closed model structure, cf. [18] where this formulation was explicitly spelled out. The result proved in this section should be viewed as the Koszul dual to Hinich's. We prove this Koszul dual version by suitably adapting Hinich's methods. The use of the associative version of this result contained in [19, 25] allows us to shorten the proof in several places.

Recall that in Definition 7.3 we introduced an adjoint pair of functors $\mathcal{C}$ and $\hat{\mathcal{L}}$ between the categories $\mathscr{A}_+$ of augmented cdgas and $\hat{\mathscr{L}}$ of complete dglas. We will write $i_\mathfrak{g} : \hat{\mathcal{L}}\mathcal{C}(\mathfrak{g}) \to \mathfrak{g}$ and $i_A : \mathcal{C}\hat{\mathcal{L}}(A) \to A$ for the counits of this adjunction. Our goal is to establish a closed model category structure on $\hat{\mathscr{L}}$ in such a way that the functors $\mathcal{C}$ and $\hat{\mathcal{L}}$ would induce an equivalence on the level of homotopy categories.

The adjoint functors $\hat{\mathcal{L}}$ and $\mathcal{C}$ can be 'embedded' into an adjunction between bigger categories of *associative* algebras. We will denote the category of augmented dg associative (discrete) algebras (dgas) by $\mathscr{A}ss_+$; algebras in $\mathscr{A}ss_+$ will be *cohomologically graded*. The augmentation ideal in an augmented dga A will be denoted by A_+, as in the commutative case. Let us remind the details of the corresponding adjunction following [19]. First, we need a relevant definition.

9.1. **Definition.** A *complete augmented* dga is, by definition, an inverse limit of finite-dimensional nilpotent augmented dgas.[10] The category of complete dgas and their continuous homomorphisms will be denoted by $\widehat{\mathscr{A}ss}_+$. Algebras in $\widehat{\mathscr{A}ss}_+$ are assumed to be *homologically graded*.

Algebras of Definition 9.1 are complete in the sense of [26, Appendix A.1]. For instance, repeating the arguments used in Remark 7.1 to prove that complete dglas are pronilpotent, one may easily show that $\lim_k A/A_+^k \cong A$, which is Condition (c) on page 265 of that paper.

9.2. **Remark.** The functor of linear duality establishes an anti-equivalence between the category $\widehat{\mathscr{A}ss}_+$ and that of conilpotent dg coalgebras, cf. [13] where complete dgas were called *formal* dgas; again, we opted to change this terminology since formality often has a different meaning in homological algebra.

9.3. **Definition.** Let $\hat{\mathcal{B}} : \mathscr{A}ss_+ \mapsto \widehat{\mathscr{A}ss}_+$ be the functor associating to a dga A the complete dga $\hat{\mathcal{B}}(A)$ whose underlying graded algebra is $\hat{T}\Sigma^{-1}A_+^*$, the completed tensor algebra on $\Sigma^{-1}A_+^*$. The differential d in $\hat{T}\Sigma^{-1}A_+^*$ is defined as $d = d_I + d_{II}$ where d_I is induced by the internal differential in A and d_{II} is determined by its restriction onto $\Sigma^{-1}A_+^*$, which is, in turn, induced by the product map $A_+ \otimes A_+ \to A_+$.

Likewise, let $\mathcal{B} : \widehat{\mathscr{A}ss}_+ \mapsto \mathscr{A}ss_+$ be the functor associating to a complete dga C the (discrete) dga $\mathcal{B}(C)$ whose underlying graded algebra is $T\Sigma C_+^*$, the tensor algebra on ΣC_+^*. The differential d in $\mathcal{B}(C)$ is defined as $d = d_I + d_{II}$ where d_I is induced by the internal differential in C and d_{II} is determined by its restriction onto ΣC_+^*, which is, in turn, induced by the product $C_+ \otimes C_+ \to C_+$.

---

[10]We call an augmented algebra nilpotent if its augmentation ideal is nilpotent in the usual sense.

9.4. **Remark.** We will refer to either functor $\mathcal{B}$ or $\hat{\mathcal{B}}$ as the *cobar-construction*. Note that for a dga A its cobar-construction $\hat{\mathcal{B}}(A)$ is a complete Hopf algebra, whose space $\Sigma^{-1}A_+^*$ of algebra generators consists of primitive elements. If A is commutative, then the differential of $\hat{\mathcal{B}}(A)$ takes $\Sigma^{-1}A_+^*$ to primitives and thus makes $\hat{\mathcal{B}}(A)$ a dg Hopf algebra.

9.5. **Proposition.**

(1) *Let A be a cdga. Then there is a natural isomorphism of complete dglas:*

$$\mathcal{P}\hat{\mathcal{B}}(A) \cong \hat{\mathcal{L}}(A).$$

(2) *Let $\mathfrak{g}$ be a complete dgla and $\hat{U}\mathfrak{g}$ be its completed universal enveloping algebra. Then there is a quasi-isomorphism of dgas*

$$\mathcal{B}(\hat{U}\mathfrak{g}) \simeq \mathcal{C}(\mathfrak{g}).$$

Proof. The first statement is a consequence of the completed version of the well-known fact that the primitive elements in the tensor algebra on a graded vector space form the free Lie algebra on the same vector space which easily follows from the Appendix to [26].

To prove the second statement, note that the inclusion of $\mathfrak{g}$ into $\hat{U}\mathfrak{g}$ as the space of primitive elements induces a map of dgas $\mathcal{B}(\hat{U}\mathfrak{g}) \to \mathcal{C}(\mathfrak{g})$. To see that the latter map is a quasi-isomorphism it suffices to assume that $\mathfrak{g}$ is finite-dimensional nilpotent; the general result will be obtained by passing to the limit. Denote by $\tilde{\mathfrak{g}}$ the graded Lie algebra with the same underlying space as $\mathfrak{g}$ and the vanishing differential. Then we have spectral sequences with E_1 terms $H\big(\mathcal{B}(\hat{U}\tilde{\mathfrak{g}})\big)$ and $H\big(\mathcal{C}(\tilde{\mathfrak{g}})\big)$, which converge to $\mathcal{B}(\hat{U}\mathfrak{g})$ and $\mathcal{C}(\mathfrak{g})$ respectively. The map $\mathcal{B}(\hat{U}\mathfrak{g}) \to \mathcal{C}(\mathfrak{g})$ gives a map between these spectral sequences. It is therefore enough to prove that the map $\mathcal{B}(\hat{U}\tilde{\mathfrak{g}}) \to \mathcal{C}(\tilde{\mathfrak{g}})$ is a quasi-isomorphism.

As in the proof of Lemma A.5 we use the filtration of $\tilde{\mathfrak{g}}^*$ induced by the shifted lower central series of $\tilde{\mathfrak{g}}$. This filtration induces, in the usual way, increasing exhaustive and complete filtrations of $\mathcal{B}(\hat{U}\tilde{\mathfrak{g}})$ and $\mathcal{C}(\tilde{\mathfrak{g}})$ compatible with the map $\mathcal{B}(\hat{U}\tilde{\mathfrak{g}}) \to \mathcal{C}(\tilde{\mathfrak{g}})$. This brings us to the case when $\tilde{\mathfrak{g}}$ is abelian, the desired result then follows from the calculation of $\mathrm{Tor}_{\mathcal{C}(\tilde{\mathfrak{g}})}(\Bbbk, \Bbbk)$ via the Koszul complex. $\qquad\square$

The category $\mathscr{A}ss_+$ has a closed model category structure, by a general result of Hinich, cf. [14]. Namely, weak equivalences in $\mathscr{A}ss_+$ are quasi-isomorphisms of augmented dgas and fibrations are surjective homomorphisms. The category $\widehat{\mathscr{A}ss}_+$ of complete dgas also admits the structure of a closed model category, as follows.

9.6. **Definition.** A morphism $f : A \to B$ in $\widehat{\mathscr{A}ss}_+$ is called

(1) a *weak equivalence* if $\mathcal{B}(f) : \mathcal{B}(B) \to \mathcal{B}(A)$ is a quasi-isomorphism in $\mathscr{A}ss_+$;
(2) a *fibration* if f is surjective; if, in addition, f is a weak equivalence then f is called an *acyclic fibration*;

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(3) a *cofibration* if f has the left lifting property with respect to all acyclic fibrations. That means that in any commutative square

$$
\begin{array}{ccc}
A & \longrightarrow & C \\
f \downarrow & \nearrow & \downarrow g \\
B & \longrightarrow & D
\end{array}
$$

where g is an acyclic fibration there exists a dotted arrow making the whole diagram commutative.

9.7. Theorem. *The category $\widehat{\mathscr{A}ss}_+$ is a closed model category with fibrations, cofibrations and weak equivalences defined as above. Moreover, it is Quillen equivalent to the closed model category $\mathscr{A}ss_+$ via the adjunctions $\mathcal{B}$ and $\hat{\mathcal{B}}$.*

Proof. This is just a reformulation of Théorème 1.3.1.2 of [19], taking into account the anti-equivalence between complete dgas and conilpotent dg coalgebras. $\square$

9.8. Remark. The above result is an associative analogue of Hinich's theorem [15] on the existence of a closed model category on dg conilpotent coalgebras or, equivalently, on complete cdgas. All objects in $\widehat{\mathscr{A}ss}_+$ are fibrant and cofibrant objects correspond precisely to A_∞ algebras, cf., for example, [13] for a treatment of A_∞ algebras relevant to the present context.

We will now construct a closed model category structure on $\hat{\mathscr{L}}$ using Theorem 9.7 as a shortcut; a more direct approach, essentially repeating the original Hinich's argument in the dual context is also possible.

9.9. Definition. A morphism $f : \mathfrak{g} \to \mathfrak{h}$ in $\hat{\mathscr{L}}$ is called

(1) a *weak equivalence* if $\mathcal{C}(f) : \mathcal{C}(\mathfrak{h}) \to \mathcal{C}(\mathfrak{g})$ is a quasi-isomorphism in $\mathscr{A}_+$;
(2) a *fibration* if f is surjective; if, in addition, f is a weak equivalence then f is called an *acyclic fibration*;
(3) a *cofibration* if f has the left lifting property with respect to all acyclic fibrations.

We will prove that the above structures make $\hat{\mathscr{L}}$ a closed model category. Our proof will be based on the following results.

9.10. Proposition.

(1) *Let A be an augmented cdga; then $i_A : \mathcal{C}\hat{\mathcal{L}}(A) \to A$ is a quasi-isomorphism.*
(2) *Let $\mathfrak{g}$ be a complete dgla; then $i_\mathfrak{g} : \hat{\mathcal{L}}\mathcal{C}(\mathfrak{g}) \to \mathfrak{g}$ is a weak equivalence, i.e. $\mathcal{C}(i_\mathfrak{g}) : \mathcal{C}(\mathfrak{g}) \to \mathcal{C}(\hat{\mathcal{L}}\mathcal{C}(\mathfrak{g}))$ is a quasi-isomorphism.*

Proof. There is a quasi-isomorphism $\mathcal{C}\hat{\mathcal{L}}(A) \simeq \mathcal{B}(\hat{U}\hat{\mathcal{L}}(A))$ in $\mathscr{A}ss_+$ by Proposition 9.5(2). Furthermore, the dga $\mathcal{B}(\hat{U}\hat{\mathcal{L}}(A)) = \mathcal{B}\hat{\mathcal{B}}(A)$ is quasi-isomorphic to A by Theorem 9.7. This proves (1).

Let us prove that the induced map $\mathcal{C}(i_\mathfrak{g}) : \mathcal{C}(\hat{\mathcal{L}}\mathcal{C}(\mathfrak{g})) \to \mathcal{C}(\mathfrak{g})$ is a quasi-isomorphism of augmented cdgas. By Proposition 9.5(2) we have a quasi-isomorphism of dgas

$$\mathcal{C}(\hat{\mathcal{L}}\mathcal{C}(\mathfrak{g})) \simeq \mathcal{B}\hat{U}(\hat{\mathcal{L}}\mathcal{C}(\mathfrak{g})) = \mathcal{B}\hat{\mathcal{B}}\mathcal{C}(\mathfrak{g})$$

and, by Theorem 9.7, $\mathcal{B}\hat{\mathcal{B}}\mathcal{C}(\mathfrak{g})$ is quasi-isomorphic to $\mathcal{C}(\mathfrak{g})$ as required. $\square$

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The following statement is an analogue of the 'Key Lemma' of [15, p. 223].

9.11. Lemma. *Let A be a cdga, $\mathfrak{g}$ be a complete dgla and $f : A \to \mathcal{C}(\mathfrak{g})$ be a surjective map. Consider the pushout diagram*

$$
\begin{array}{ccc}
\hat{\mathcal{L}}\mathcal{C}(\mathfrak{g}) & \xrightarrow{\hat{\mathcal{L}}(f)} & \hat{\mathcal{L}}(A) \\
\downarrow{\scriptstyle i_{\mathfrak{g}}} & & \downarrow{\scriptstyle j} \\
\mathfrak{g} & \longrightarrow & \mathfrak{a} \; .
\end{array}
$$

Then $\mathcal{C}(j) : \mathcal{C}(\mathfrak{a}) \to \mathcal{C}\hat{\mathcal{L}}(A)$ is a quasi-isomorphism of cdgas.

In the proof of Lemma 9.11 we will use the following technical statement.

9.12. Lemma. *Let $\phi : (A', d') \to (A'', d'')$ be a chain map. Assume that $A' = \bigcup_{p,q} F_{p,q}A'$ is a finite double filtration which is descending in the sense that*

$$F_{p+1,q}A' \cup F_{p,q+1}A' \subset F_{p,q}A' \;\; \text{for each} \;\; p, q.$$

Assume also that d' is the sum $d'_1 + d'_2$ of two differentials such that

$$d'_1(F_{p,q}A') \subset F_{p,q}A' \;\; \text{and} \;\; d'_2(F_{p,q}A') \subset F_{p-1,q+1}A' \;\; \text{for each} \;\; p, q.$$

Suppose that A'' has a double filtration with the similar properties and that ϕ is compatible with these filtrations. If the induced map

$$\phi_{p,q} : \frac{F_{p,q}A'}{F_{p+1,q}A' \cup F_{p,q+1}A'} \longrightarrow \frac{F_{p,q}A''}{F_{p+1,q}A'' \cup F_{p,q+1}A''}$$

of the quotients, with the differentials induced by the 'untwisted' parts d'_1 resp. d''_1, is a quasi-isomorphism for each p and q, then ϕ is a quasi-isomorphism, too.

Proof. Consider the spectral sequences in the q-direction. The E_1-sheets of these spectral sequences are clearly the same as if the twisted parts d'_2 resp. d''_2 of the full differentials vanish. We may therefore assume $d'_2 = d''_2 = 0$ from the beginning, in which case is the claim obvious. $\square$

We will also need to know that the functor $\mathcal{C}(-)$ preserves filtered quasi-isomorphisms. The proof of the following statement is a harmless modification of [15, Proposition 4.4.4].

9.13. Lemma. *Assume that $\mathfrak{u}$ and $\mathfrak{v}$ are filtered complete dglas, with complete filtrations*

$$(9.1) \qquad \mathfrak{u} = F_1\mathfrak{u} \supset F_2\mathfrak{u} \supset F_3\mathfrak{u} \supset \cdots \;\; resp. \;\; \mathfrak{v} = F_1\mathfrak{v} \supset F_2\mathfrak{v} \supset F_3\mathfrak{v} \supset \cdots .$$

Let $\phi : \mathfrak{u} \to \mathfrak{v}$ be a morphism, compatible with the filtrations, such that the induced map $\phi_n : \mathfrak{u}/F_n\mathfrak{u} \to \mathfrak{v}/F_n\mathfrak{v}$ is a quasi-isomorphism for each $n \geq 1$. Then $\mathcal{C}(\phi) : \mathcal{C}(\mathfrak{v}) \to \mathcal{C}(\mathfrak{u})$ is a quasi-isomorphism, too.

9.14. Remark. The map $\phi_n : \mathfrak{u}/F_n\mathfrak{u} \to \mathfrak{v}/F_n\mathfrak{v}$ is a chain map in the category of linearly compact spaces. By saying it is a quasi-isomorphism we mean that it is a quasi-isomorphism in the underlying category of dg vector spaces. It is easy to show that ϕ_n is a quasi-isomorphism if and only if its dual ϕ_n^* is a quasi-isomorphism in the usual sense, compare Remark 10.3.

Proof of Lemma 9.11. Our proof will be similar to that of [15, pp. 224-225] or [19, pp. 42-43]. The kernel B of the map $f : A \to \mathcal{C}(\mathfrak{g})$ is a non-unital cdga. Since $\mathcal{C}(\mathfrak{g})$ is free as a non-differential algebra, choosing an algebra splitting of f, we obtain an isomorphism

$$A \cong B \oplus \mathcal{C}(\mathfrak{g})$$

of graded algebras, in which B is closed under the differential. Therefore $\hat{\mathcal{L}}(A)$ is isomorphic, as a complete graded Lie algebra, to $\hat{\mathcal{L}}(B_e) * \hat{\mathcal{L}}\mathcal{C}(\mathfrak{g})$, the (completed) free product of $\hat{\mathcal{L}}(B_e)$ and $\hat{\mathcal{L}}\mathcal{C}(\mathfrak{g})$. The differential in $\hat{\mathcal{L}}(A)$ consists of three parts: the differential in $\hat{\mathcal{L}}(B_e)$, the differential in $\hat{\mathcal{L}}\mathcal{C}(\mathfrak{g})$ and the 'twisted' part mapping the generators of $\hat{\mathcal{L}}(B_e)$ into $\hat{\mathcal{L}}\mathcal{C}(\mathfrak{g})$.

We clearly have the following isomorphisms of complete graded Lie algebras (again, disregarding the differentials):

$$\mathfrak{a} \cong \hat{\mathcal{L}}(A) *_{\hat{\mathcal{L}}\mathcal{C}(\mathfrak{g})} \mathfrak{g} \cong \left(\hat{\mathcal{L}}(B_e) * \hat{\mathcal{L}}\mathcal{C}(\mathfrak{g})\right) *_{\hat{\mathcal{L}}\mathcal{C}(\mathfrak{g})} \mathfrak{g} \cong \hat{\mathcal{L}}(B_e) * \mathfrak{g}.$$

The differential in $\hat{\mathcal{L}}(B_e) * \mathfrak{g}$ is, under this isomorphism, also the sum of three parts: the differential in $\hat{\mathcal{L}}(B_e)$, the differential in $\mathfrak{g}$ and the twisted part, which maps the generators of $\hat{\mathcal{L}}(B_e)$ into $\mathfrak{g}$. It is clear that the map $j : \hat{\mathcal{L}}(A) \to \mathfrak{a}$, under the above identifications, equals

$$(9.2) \qquad \mathbb{1} * i_{\mathfrak{g}} : \hat{\mathcal{L}}(B_e) * \hat{\mathcal{L}}\mathcal{C}(\mathfrak{g}) \to \hat{\mathcal{L}}(B_e) * \mathfrak{g},$$

but the differentials differ by the twistings from those of the free products.

Before we proceed, we need some notation. If $\mathfrak{u}$ and $\mathfrak{v}$ are filtered complete dglas as in (9.1), then their completed free product $\mathfrak{u} * \mathfrak{v}$ possesses an induced double filtration with $F_{p,q}(\mathfrak{u} * \mathfrak{v})$ the closure of the subspace generated by all iterated brackets of $u_i \in F_{p_i}\mathfrak{u}$ and $v_j \in F_{q_j}\mathfrak{v}$, $1 \le i \le a$, $1 \le j \le b$, such that

$$p_1 + \cdots + p_a = p \quad \text{and} \quad q_1 + \cdots + p_b = q.$$

One also has the associated total filtration with $F_n(\mathfrak{u} * \mathfrak{v}) := \bigcup_{p+q=n} F_{p,q}(\mathfrak{u} * \mathfrak{v})$.

For graded dglas one defines the bigrading on their completed free product in the obvious analogous manner. We then have the following formula for the associated bigraded dgla:

$$(9.3) \qquad \mathrm{Gr}_{p,q}(\mathfrak{u} * \mathfrak{v}) := \frac{F_{p,q}(\mathfrak{u} * \mathfrak{v})}{F_{p+1,q}(\mathfrak{u} * \mathfrak{v}) \cup F_{p,q+1}(\mathfrak{u} * \mathfrak{v})} \cong \left(\mathrm{Gr}(\mathfrak{u}) * \mathrm{Gr}(\mathfrak{v})\right)_{p,q}.$$

Let us return to our proof. The lower central series $\mathfrak{g}_1 \supset \mathfrak{g}_2 \supset \mathfrak{g}_3 \supset \cdots$ of $\mathfrak{g}$ induces, in the standard manner, a filtration of $\hat{\mathcal{L}}\mathcal{C}(\mathfrak{g})$ while the complete free Lie algebra $\hat{\mathcal{L}}(B_e)$ is filtered by its lower central series. As explained above, one has the induced double filtrations of the completed free products $\hat{\mathcal{L}}(B_e) * \hat{\mathcal{L}}\mathcal{C}(\mathfrak{g})$ and $\hat{\mathcal{L}}(B_e) * \mathfrak{g}$. The associated total filtrations are stable under the differentials and the map (9.2) factorizes, for each n, into a map

$$(9.4) \qquad \phi = \phi_n : \frac{\hat{\mathcal{L}}(B_e) * \hat{\mathcal{L}}\mathcal{C}(\mathfrak{g})}{F_n(\hat{\mathcal{L}}(B_e) * \hat{\mathcal{L}}\mathcal{C}(\mathfrak{g}))} \longrightarrow \frac{\hat{\mathcal{L}}(B_e) * \mathfrak{g}}{F_n(\hat{\mathcal{L}}(B_e) * \mathfrak{g})}$$

of dglas. By Lemma 9.13, it suffices to prove that ϕ is a quasi-isomorphism for each n.

Fix n and consider the double filtrations of the quotients in (9.4) induced by the double filtrations $F_{p,q}\left(\hat{\mathcal{L}}(B_e) * \hat{\mathcal{L}}\mathcal{C}(\mathfrak{g})\right)$ resp. $F_{p,q}\left(\hat{\mathcal{L}}(B_e) * \mathfrak{g}\right)$. It is straightforward though technically involved to prove that these filtrations and the map ϕ satisfy the assumptions of Lemma 9.12. Taking the quotients was needed to make these double filtrations finite.

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By Lemma 9.12, it is enough to prove that the bigraded components ϕ_{pq} of the map ϕ are quasi-isomorphisms. Using (9.3), one easily identifies ϕ_{pq} with the map

$$(9.5) \qquad (\mathbb{1} * i_{\mathrm{Gr}(\mathfrak{g})})_{p,q} : \big(\mathrm{Gr}(\hat{\mathcal{L}}(B_e)) * \hat{\mathcal{L}}\mathcal{C}(\mathrm{Gr}(\mathfrak{g}))\big)_{p,q} \longrightarrow \big(\mathrm{Gr}(\hat{\mathcal{L}}(B_e)) * \mathrm{Gr}(\mathfrak{g})\big)_{p,q}$$

if $p + q < n$ while $\phi_{pq} = 0$ otherwise.

Since the gradings of both $\mathrm{Gr}(\hat{\mathcal{L}}(B_e))$, $\hat{\mathcal{L}}\mathcal{C}(\mathrm{Gr}(\mathfrak{g}))$ and $\mathrm{Gr}(\mathfrak{g})$ are *positive*, the (p, q)-components of the free products in (9.5) are spanned by iterated brackets of length $\leq p + q$. We may thus in (9.5) disregard the topologies and consider *uncompleted* free products.

In the proof of Theorem A.4 we established that the uncompleted free product is an exact functor. To finish the proof, it is therefore enough to establish that the canonical map $\hat{\mathcal{L}}\mathcal{C}\big(\mathrm{Gr}(\mathfrak{g})\big)_p \to \mathrm{Gr}(\mathfrak{g})_p$ is a quasi-isomorphism for each p. It is immediate to see that $\hat{\mathcal{L}}\mathcal{C}(\mathrm{Gr}(\mathfrak{g}))_p$ coincides with the degree p-part $\mathscr{L}\mathscr{C}(\mathrm{Gr}(\mathfrak{g}))_p$ of the composition of the adjoint functors $\mathscr{C}$ and $\mathscr{L}$ (see [15, 2.2] for their definitions) applied to $\mathrm{Gr}(\mathfrak{g})$ considered as a discrete dgla. The proof is finished by of [15, Proposition 3.3.2] by which the canonical map $\mathscr{L}\mathscr{C}(\mathrm{Gr}(\mathfrak{g})) \to \mathrm{Gr}(\mathfrak{g})$ is a quasi-isomorphism. $\qquad\qquad\square$

Let us prove another auxiliary statement.

9.15. **Lemma.** *The functor* $\hat{\mathcal{L}} : \mathscr{A}_+ \to \hat{\mathscr{L}}$ *preserves weak equivalences. Moreover, it maps fibrations to cofibrations and cofibrations to fibrations.*

Proof. Let $f : A \to B$ be a weak equivalence, i.e. a quasi-isomorphism, in $\mathscr{A}_+$. By definition, $\hat{\mathcal{L}}(f) : \hat{\mathcal{L}}(B) \to \hat{\mathcal{L}}(A)$ is a weak equivalence if the induced map $\mathcal{C}\hat{\mathcal{L}}(f) : \mathcal{C}\hat{\mathcal{L}}(A) \to \mathcal{C}\hat{\mathcal{L}}(B)$ is a quasi-isomorphism. This follows from the diagram

$$
\begin{array}{ccc}
\mathcal{C}\hat{\mathcal{L}}(A) & \xrightarrow{\mathcal{C}\hat{\mathcal{L}}(f)} & \mathcal{C}\hat{\mathcal{L}}(B) \\
{\scriptstyle i_A}\downarrow & & \downarrow{\scriptstyle i_B} \\
A & \xrightarrow{\quad f \quad} & B
\end{array}
$$

in which the vertical maps are quasi-isomorphisms by Proposition 9.10(1).

Assume that f is a fibration in $\mathscr{A}_+$, i.e. an epimorphism. To prove that $\hat{\mathcal{L}}(f)$ is a cofibration, we must find a dotted arrow in each diagram

$$
\begin{array}{ccc}
\hat{\mathcal{L}}(B) & \longrightarrow & \mathfrak{g} \\
{\scriptstyle \hat{\mathcal{L}}(f)}\downarrow & \nearrow & \downarrow{\scriptstyle u} \\
\hat{\mathcal{L}}(A) & \longrightarrow & \mathfrak{h}
\end{array}
$$

in which $u : \mathfrak{g} \to \mathfrak{h}$ is an acyclic fibration in $\hat{\mathscr{L}}$. Using the adjunction between $\hat{\mathcal{L}}$ and $\mathcal{C}$, we see that we may equivalently seek for a dotted arrow in the diagram

$$
\begin{array}{ccc}
\mathcal{C}(\mathfrak{h}) & \longrightarrow & A \\
{\scriptstyle \mathcal{C}(u)}\downarrow & \nearrow & \downarrow{\scriptstyle f} \\
\mathcal{C}(\mathfrak{g}) & \longrightarrow & B
\end{array}
$$

in which f is a fibration in $\mathscr{A}_+$ by assumption. Since $\mathcal{C}(u)$ is a quasi-isomorphism by the definition of weak equivalences in $\hat{\mathscr{L}}$, all we need to show is that $\mathcal{C}(u)$ is a cofibration in $\mathscr{A}_+$.

Denote by $\mathfrak{g} = \mathfrak{g}_1 \subset \mathfrak{g}_2 \subset \mathfrak{g}_2 \subset \cdots$ the lower central series of $\mathfrak{g}$. In Remark 7.2 we verified that $\mathfrak{g} = \lim \mathfrak{g}/\mathfrak{g}_n$. Since $K := \mathrm{Ker}(f) \subset \mathfrak{g}$ is a closed subspace, by standard properties of filtered limits we verify that the limit of the tower

$$(9.6) \qquad \mathfrak{h} \cong \mathfrak{g}/(\mathfrak{g}_1 \cap K) \overset{\pi_1}{\leftarrow} \mathfrak{g}/(\mathfrak{g}_2 \cap K) \overset{\pi_2}{\leftarrow} \mathfrak{g}/(\mathfrak{g}_3 \cap K) \overset{\pi_3}{\leftarrow} \cdots$$

equals $\mathfrak{g}$, therefore the colimit of the diagram

$$\mathcal{C}(\mathfrak{h}) \hookrightarrow \mathcal{C}(\mathfrak{g}/\mathfrak{g}_2 \cap K) \hookrightarrow \mathcal{C}(\mathfrak{g}/\mathfrak{g}_3 \cap K) \hookrightarrow \mathcal{C}(\mathfrak{g}/\mathfrak{g}_4 \cap K) \hookrightarrow \cdots$$

equals $\mathcal{C}(\mathfrak{g})$. It would therefore suffice to prove that the maps $\mathcal{C}(\pi_n)$, where π_n are as in (9.6), are cofibrations in $\mathscr{A}_+$ for each $n \geq 1$.

To this end, observe that the kernel of π_n is an abelian dgla $(\mathfrak{g}_n \cap K)/(\mathfrak{g}_{n+1} \cap K)$. With this knowledge, it is easy to see as in the proof of the lemma in [15, §5.2.2] that $\mathcal{C}(\pi_n)$ is a standard cofibration obtained by adding free generators to $\mathcal{C}(\mathfrak{g}/\mathfrak{g}_n \cap K)$.

To prove the last part of the lemma, note the standard fact that cofibrations in $\mathscr{A}_+$ are monomorphisms, while $\hat{\mathcal{L}}(-)$ clearly converts monomorphisms to epimorphisms, i.e. fibrations in $\hat{\mathscr{L}}$. $\qquad \square$

In the proof of Lemma 9.15 we established that the functor $\mathcal{C}(-)$ converts fibrations to cofibrations. As each $\mathfrak{g} \in \hat{\mathcal{L}}$ is fibrant, this in particular implies that the dglas $\mathcal{C}(\mathfrak{g})$ are cofibrant in $\mathscr{A}_+$.

9.16. **Theorem.** *The category $\hat{\mathscr{L}}$ is a closed model category with fibrations, cofibrations and weak equivalences as in Definition 9.9. Moreover, it is Quillen equivalent to the closed model category $\mathscr{A}_+$ via the adjunctions $\hat{\mathcal{L}}$ and $\mathcal{C}$.*

Proof. The arguments are precisely dual to those of Hinich [15, pp. 223–224]. The category $\hat{\mathscr{L}}$ admits arbitrary limits and colimits. Indeed, the limits are created in the category of linearly compact dg vector spaces. The coequalizer of two maps $f, g : \mathfrak{g} \to \mathfrak{h}$ is the quotient of $\mathfrak{h}$ by the closed ideal in $\mathfrak{h}$ generated by the elements $f(a) - g(a)$ where $a \in \mathfrak{g}$. Finally, the coproduct of a family of complete dglas is constructed by taking their free product, and then completing. This proves axiom CM1; the axioms CM2 and CM3 are obvious, and one half of CM4 holds by definition.

Let us prove the factorization axiom CM5. Given a map $f : \mathfrak{g} \to \mathfrak{h}$ in $\hat{\mathscr{L}}$, consider the corresponding map of cdgas $\mathcal{C}(f) : \mathcal{C}(\mathfrak{h}) \to \mathcal{C}(\mathfrak{g})$. Suppose we factorize $\mathcal{C}(f)$ as

$$(9.7) \qquad \mathcal{C}(\mathfrak{h}) \overset{i}{\to} A \overset{p}{\to} \mathcal{C}(\mathfrak{g})$$

where p is a surjective map, i.e. a fibration, and i a cofibration in $\mathscr{A}_+$. Then one has an induced factorization of f:

$$(9.8) \qquad \mathfrak{g} \overset{\tilde{i}}{\longrightarrow} \hat{\mathcal{L}}(A) *_{\hat{\mathcal{L}}\mathcal{C}(\mathfrak{g})} \mathfrak{g} \overset{\tilde{p}}{\longrightarrow} \mathfrak{h}$$

where $\tilde{p}$ is easily seen to be an epimorphism, i.e. a fibration in $\hat{\mathscr{L}}$. By construction, $\tilde{i}$ is obtained by a cobase-change from $\hat{\mathcal{L}}(p)$ which is a cofibration by Lemma 9.15. Thus $\tilde{i}$ is a cofibration as well.

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Since $\mathscr{A}_+$ is a closed model category, factorization (9.7) exists and can be chosen such that p is a quasi-isomorphism. Then the corresponding $\tilde{i}$ in (9.8) is acyclic; this follows (just as in [15]) from Lemma 9.11 and Proposition 9.10(1). This proves the first factorization of CM5.

The proof of the second factorization property of CM5 is similar, we only choose this time i in (9.7) to be a quasi-isomorphism. The map $\tilde{p}$ is then acyclic by Lemma 9.11.

It remains to prove the second half of CM4, i.e. that any acyclic cofibration in $\hat{\mathscr{L}}$ has the LLP (left lifting property) with respect to all fibrations. Let $f : \mathfrak{g} \to \mathfrak{h}$ be such a cofibration. Factorize it as in (9.8): $f = \tilde{p} \circ \tilde{i}$ where $\tilde{p}$ is an acyclic fibration and $\tilde{i}$ is an acyclic cofibration obtained by a cobase-change from a map $\hat{\mathcal{L}}(p)$, where p is an acyclic fibration in $\mathscr{A}_+$.

By Lemma 9.15, $\hat{\mathcal{L}}(p)$ is an acyclic cofibration in $\hat{\mathscr{L}}$, so it has the LLP with respect to all fibrations in $\hat{\mathscr{L}}$, thus $\tilde{i}$ has the same property. Since $\tilde{p}$ has the LLP with respect to f it follows that f is a retract of $\tilde{i}$ and so, it has the LLP with respect to all fibrations in $\hat{\mathscr{L}}$ as required. This finishes our proof that $\hat{\mathscr{L}}$ is a closed model category.

The statement that the adjoint functors $\hat{\mathcal{L}}$ and $\mathcal{C}$ form a Quillen pair, i.e. satisfy assumptions of [9, Theorem 9.7], follows at once from Proposition 9.10 and Lemma 9.15. □

An interesting feature of the closed model structure of Definition 9.9 is the existence and uniqueness of minimal models.

9.17. Definition. A dgla $\mathscr{M} \in \hat{\mathscr{L}}$ of the form $\mathscr{M} = \big(\hat{\mathbb{L}}(M), \partial\big)$ is *minimal* if ∂ induces the trivial differential on M. A *minimal model* of a dgla $\mathfrak{g} \in \hat{\mathscr{L}}$ is a minimal dgla $\mathscr{M}$ together with a weak equivalence $\mathscr{M} \to \mathfrak{g}$.

9.18. Theorem. *Each $\mathfrak{g} \in \hat{\mathscr{L}}$ has a minimal model unique up to an isomorphism.*

Proof. For $\mathfrak{g} \in \hat{\mathscr{L}}$ denote by H the cohomology of $\mathcal{C}(\mathfrak{g})$ and choose a homotopy equivalence $f : \mathcal{C}(\mathfrak{g}) \to H$ in the category of dg vector spaces; here H is considered with the trivial differential. By [22, Move M1, p. 133], there exists a C_∞-algebra C with the underlying dg-vector space H, and a C_∞-morphism $F : \mathcal{C}(\mathfrak{g}) \to C$ extending f. Let $\mathscr{M} = \big(\hat{\mathbb{L}}(\Sigma^{-1}H^*), \partial\big)$ be the complete dgla corresponding to the C_∞-algebra C. It is minimal and the C_∞-morphism F translates into a weak equivalence $\varphi : \mathscr{M} \to \hat{\mathcal{L}}\mathcal{C}(\mathfrak{g})$. The canonical map $i_{\mathfrak{g}} : \hat{\mathcal{L}}\mathcal{C}(\mathfrak{g}) \to \mathfrak{g}$ is, by Proposition 9.10(2), a weak equivalence, too. The composition

$$\rho : \mathscr{M} \xrightarrow{\varphi} \hat{\mathcal{L}}\mathcal{C}(\mathfrak{g}) \xrightarrow{i_{\mathfrak{g}}} \mathfrak{g}$$

is then the desired minimal model of $\mathfrak{g}$.

Let us prove uniqueness. Suppose that $\rho' : \mathscr{M}' \to \mathfrak{g}$ is another minimal model. Since $\mathcal{C}(\mathfrak{g})$ is fibrant as every object of $\mathscr{A}_+$, $\hat{\mathcal{L}}\mathcal{C}(\mathfrak{g})$ is cofibrant by Proposition 9.15. Although ρ' need not be a fibration, it is still a map between cofibrant objects. By e.g. [28, Lemma 3.5], there exists the dotted arrow α in the diagram

$$
\begin{array}{ccc}
\mathscr{M} & \xrightarrow{\varphi} \hat{\mathcal{L}}\mathcal{C}(\mathfrak{g}) & \dashrightarrow^{\alpha} \mathscr{M}' \\
 & \llap{$i_{\mathfrak{g}}$}\searrow & \downarrow{\rho'} \\
 & & \mathfrak{g}
\end{array}
$$

making the right triangle homotopy commutative. The composition $\alpha\varphi$ is a weak equivalence of minimal dglas, so it induces an isomorphism of the generators; it is therefore itself an isomorphism. $\qquad\square$

9.1. **Non-unital monoidal structures on $\hat{\mathscr{L}}$ and $\mathscr{A}_+$.** In Subsection 7.2 we constructed a non-unital monoidal structure on the category $\hat{\mathscr{L}}$ of complete dglas. An analogous, but simpler, structure on the category $\mathscr{A}_+$ is given by the product $A \times B$ of augmented cdgas $A, B \in \mathscr{A}_+$ equipped with the augmentation through the projection onto the *second* factor. The operation $A, B \mapsto A \times B$ is coherently associative, but not commutative and not unital. The two structures correspond to each other under the adjoint functors $\hat{\mathcal{L}}$ and $\mathcal{C}$:

9.19. **Proposition.**

(1) *For two augmented cdgas A and B there is a natural isomorphism in $\hat{\mathscr{L}}$:*

$$\hat{\mathcal{L}}(A \times B) \cong \hat{\mathcal{L}}(A) \sqcup \hat{\mathcal{L}}(B).$$

(2) *For two complete dglas $\mathfrak{g}$ and $\mathfrak{h}$ there is a natural quasi-isomorphism in $\mathscr{A}_+$:*

$$\mathcal{C}(\mathfrak{g} \sqcup \mathfrak{h}) \simeq \mathcal{C}(\mathfrak{g}) \times \mathcal{C}(\mathfrak{h}).$$

Proof. Starting from part (1), take B to be the trivial augmented cdga: $B = \Bbbk$. Disregarding the differentials, we have the following isomorphisms of complete graded algebras:

$$\hat{\mathcal{L}}(A \times \Bbbk) \cong \hat{\mathbb{L}}(\Sigma^{-1}A^*) \cong \hat{\mathbb{L}}(\Sigma^{-1}A_+^*) * \hat{\mathbb{L}}(\Sigma^{-1}\Bbbk) \cong \hat{\mathcal{L}}(A) * \hat{\mathcal{L}}(\Bbbk) \cong \hat{\mathcal{L}}(A) * \mathfrak{s}.$$

Direct inspection shows that, under the above isomorphism, the differential in $\hat{\mathcal{L}}(A \times \Bbbk)$ corresponds to the twisted differential d^x in $\hat{\mathcal{L}}(A) * \mathfrak{s} \cong \hat{\mathcal{L}}(A) \sqcup 0$, so the dglas $\hat{\mathcal{L}}(A \times \Bbbk)$ and $\hat{\mathcal{L}}(A) \sqcup 0$ are isomorphic. For a general cdga B we have:

$$\hat{\mathcal{L}}(A \times B) \cong \hat{\mathcal{L}}\big((A \times \Bbbk) \times_{\Bbbk} B\big) \cong \hat{\mathcal{L}}(A \times \Bbbk) * \hat{\mathcal{L}}(B) \cong \big(\hat{\mathcal{L}}(A) \sqcup 0\big) * \hat{\mathcal{L}}(B) \cong \hat{\mathcal{L}}(A) \sqcup \hat{\mathcal{L}}(B).$$

Here we used the fact that $\hat{\mathcal{L}}$ transforms categorical products in $\mathscr{A}_+$ into categorical coproducts in $\hat{\mathscr{L}}$, which follows formally from its adjointness property.

Part (2) was established in the proof of Theorem 1.7 on page 31. $\qquad\square$

10. Disconnected spaces and dg Lie algebras

In this section we prove Theorem D. The ground field will always be the field $\mathbb{Q}$ of rational numbers.

Let us define a functor $\mathcal{Q} : \mathscr{S}_+ \mapsto \hat{\mathscr{L}}$ as the composition $\mathcal{Q}(S) := \hat{\mathcal{L}}\Omega(S)$, where $\Omega(S)$ is the polynomial Sullivan-de Rham algebra of a pointed simplicial set $S \in \mathscr{S}_+$ [5, §2], with the augmentation $A(S) \to \mathbb{Q}$ induced by the inclusion of the base point of S. The simplicial MC space functor $\mathrm{MC}_\bullet : \hat{\mathscr{L}} \mapsto \mathscr{S}_+$ was recalled in Definition 7.11, the base point of $\mathrm{MC}_\bullet(\mathfrak{g})$ is the 0-simplex corresponding to the trivial MC-element $0 \in \mathfrak{g}$. We then have the following result.

10.1. **Proposition.** *The functors $\mathrm{MC}_\bullet : \hat{\mathscr{L}} \mapsto \mathscr{S}_+$ and $\mathcal{Q} : \mathscr{S}_+ \mapsto \hat{\mathscr{L}}$ form an adjoint pair, i.e. there is a natural isomorphism*

$$\mathscr{S}_+\big(S, \mathrm{MC}_\bullet(\mathfrak{g})\big) \cong \hat{\mathscr{L}}\big(\mathcal{Q}(S), \mathfrak{g}\big),$$

for each pointed simplicial set S and a complete dgla $\mathfrak{g}$.

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Proof. Note that, by definition, there is an isomorphism of simplicial sets $\mathrm{MC}(\mathfrak{g}) \cong F\big(\mathcal{C}(\mathfrak{g})\big)$, where F is the Bousfield-Kan functor associating to a cdga B the simplicial set $\mathscr{A}(B, \mathbb{Q})_\bullet$ recalled in (7.3).

Note also that if B is an augmented cdga then $F(B)$ is naturally a pointed simplicial set. The proposition now follows from the fact that the functors MC and $\mathcal{Q}$ are the compositions of the upper resp. lower functors in the diagram

$$(10.1) \qquad \hat{\mathscr{L}} \underset{\hat{\mathcal{L}}}{\overset{\mathcal{C}}{\rightleftarrows}} \mathscr{A}_+ \underset{\Omega}{\overset{F}{\rightleftarrows}} \mathscr{S}_+$$

in which the functors $\hat{\mathcal{L}}$ and $\mathcal{C}$ are adjoint by Proposition 7.5, while the functors F and Ω form an adjoint pair by [5, §8.9] (the functor Ω of Sullivan-De Rham forms was denoted by A in [5]). $\qquad \square$

10.2. **Definition.** The subcategory in the homotopy category of $\hat{\mathscr{L}}$ formed by the disjoint products of finitely many non-negatively graded complete dglas whose homology are finite-dimensional in each degree will be denoted by $\mathrm{f}\mathbb{Q}\text{-ho}\hat{\mathscr{L}}^{dc}$.

10.3. **Remark.** Note that $\mathfrak{g} = (\mathfrak{g}, \partial) \in \hat{\mathscr{L}}$ is a chain complex in the category of linearly compact spaces. Since ∂ is continuous, $\mathrm{Ker}(\partial)$, and also $\mathrm{Im}(\partial)$ as a continuous image of a linearly compact space, are closed subspaces of $\mathfrak{g}$. So $H(\mathfrak{g}) = \mathrm{Ker}(\partial)/\mathrm{Im}(\partial)$ is a linearly compact graded space. Clearly $H(\mathfrak{g})^* \cong H(\mathfrak{g}^*)$, where $H(\mathfrak{g}^*)$ is the ordinary cohomology of the discrete dual $(\mathfrak{g}^*, \partial^*)$.

10.4. **Remark.** The functor $\mathfrak{g}, \mathfrak{h} \mapsto \mathfrak{g} \sqcup \mathfrak{h}$ lifts to the homotopy category of $\hat{\mathscr{L}}$. Indeed, there is a quasi-isomorphism $\mathcal{C}(\mathfrak{g}) \times \mathcal{C}(\mathfrak{h}) \simeq \mathcal{C}(\mathfrak{g} \sqcup \mathfrak{h})$, by Proposition 9.19(2). If $\mathfrak{g}$ and $\mathfrak{g}'$ are weakly equivalent complete dglas then, by definition, the cdgas $\mathcal{C}(\mathfrak{g})$ and $\mathcal{C}(\mathfrak{g}')$ are quasi-isomorphic, thus the cdgas $\mathcal{C}(\mathfrak{g}) \times \mathcal{C}(\mathfrak{h})$ and $\mathcal{C}(\mathfrak{g}') \times \mathcal{C}(\mathfrak{h})$ are likewise quasi-isomorphic. Therefore the complete dglas $\mathfrak{g} \sqcup \mathfrak{h}$ and $\mathfrak{g}' \sqcup \mathfrak{h}$ are weakly equivalent.

It follows from Theorem 9.16 resp. [5, Lemma 8.5] that the composite functors $\mathrm{MC}_\bullet$ and $\mathcal{Q}$ induce an adjoint pair of functors between the homotopy categories of $\hat{\mathscr{L}}$ and $\mathscr{S}_+$. The following result implies Theorem D.

10.5. **Theorem.** *The functors* $\mathrm{MC}_\bullet$ *and* $\mathcal{Q}$ *determine mutually inverse equivalences between the categories* $\mathrm{f}\mathbb{Q}\text{-ho}\hat{\mathscr{L}}^{dc}$ *and* $\mathrm{fN}\mathbb{Q}\text{-ho}\mathscr{S}_+^{dc}$.

Proof. We use the pointed (resp. augmented) versions of the results of Part 1 formulated in Section 6. As the first step, observe that the category $\mathrm{f}\mathbb{Q}\text{-ho}\hat{\mathscr{L}}^{dc}$ is equivalent to the auxiliary 'extended' category $\mathrm{extf}\mathbb{Q}\text{-ho}\hat{\mathscr{L}}^{dc}$ whose objects are dglas weakly equivalent to objects of $\mathrm{f}\mathbb{Q}\text{-ho}\hat{\mathscr{L}}^{dc}$. We claim that the left adjunction of (10.1) restricts to the adjunction

$$(10.2) \qquad \mathrm{extf}\mathbb{Q}\text{-ho}\hat{\mathscr{L}}^{dc} \underset{\hat{\mathcal{L}}}{\overset{\mathcal{C}}{\rightleftarrows}} \mathrm{f}\mathbb{Q}\text{-ho}\mathscr{A}_+^{dc},$$

where $\mathrm{f}\mathbb{Q}\text{-ho}\mathscr{A}_+^{dc}$ is the full subcategory of $\mathrm{ho}\mathscr{A}_+$ consisting of augmented homologically disconnected algebras of finite type. To this end, we must check that

$$(10.3) \qquad \mathcal{C}(\mathfrak{g}) \in \mathrm{f}\mathbb{Q}\text{-ho}\mathscr{A}_+^{dc} \quad \text{and} \quad \hat{\mathcal{L}}(A) \in \mathrm{extf}\mathbb{Q}\text{-ho}\hat{\mathscr{L}}^{dc}$$

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whenever $\mathfrak{g} \in \mathrm{f}\mathbb{Q}\text{-ho}\hat{\mathscr{L}}^{dc}$ and $A \in \mathrm{f}\mathbb{Q}\text{-ho}\mathscr{A}_+^{dc}$.

Let us look at the first condition. Assume that $\mathfrak{g} = \mathfrak{g}_1 \sqcup \cdots \sqcup \mathfrak{g}_k$, where $\mathfrak{g}_i \in \mathrm{f}\mathbb{Q}\text{-ho}\hat{\mathscr{L}}^{dc}$ are non-negatively graded. Since, by Proposition 9.19(2), $\mathcal{C}(\mathfrak{g}_1 \sqcup \cdots \sqcup \mathfrak{g}_k)$ is quasi-isomorphic to $\mathcal{C}(\mathfrak{g}_1) \times \cdots \times \mathcal{C}(\mathfrak{g}_k)$, it is enough to show that $\mathcal{C}(\mathfrak{g}) \in \mathrm{f}\mathbb{Q}\text{-ho}\mathscr{A}_+^{dc}$ whenever $\mathfrak{g} \in \mathrm{f}\mathbb{Q}\text{-ho}\hat{\mathscr{L}}^{dc}$ is non-negatively graded. By Remark 7.6, $\mathcal{C}(\mathfrak{g})$ is cofibrant non-negatively graded connected cdga whose only augmentation ideal I is the ideal generated by $\Sigma\mathfrak{g}^*$, so $I/I^2 \cong \Sigma\mathfrak{g}^*$. Therefore $\mathcal{C}(\mathfrak{g})$ is connected of finite type since $H(\mathfrak{g}^*)$ is such by assumption. This proves the first condition of (10.3).

To prove the second condition, observe that, by definition, the algebra A is weakly equivalent (i.e. related by a ziz-zag of quasi-isomorphisms) to a finite product of non-negatively graded connected algebras of finite type, $A \simeq A_1 \times \cdots \times A_k$, with the augmentation given by the projection to the last factor. Since, by Proposition 9.19(1),

$$\hat{\mathcal{L}}(A_1 \times \cdots \times A_k) \cong \hat{\mathcal{L}}(A_1) \sqcup \cdots \sqcup \hat{\mathcal{L}}(A_k),$$

it is enough to verify that $\hat{\mathcal{L}}(A) \in \mathrm{extf}\mathbb{Q}\text{-ho}\hat{\mathscr{L}}^{dc}$ whenever A is connected, cofibrant non-negatively graded cdga of finite type. Since in this case $\hat{\mathcal{L}}(A)$ is non-negatively graded, it suffices to prove that the homology of $\hat{\mathcal{L}}(A)$ is finite-dimensional in each degree. This, however, immediately follows from Lemma 4.3.

As the functors $\hat{\mathcal{L}}$ and $\mathcal{C}$ form a Quillen pair by Theorem 9.16, their restrictions in (10.2) are mutually inverse equivalences of categories. The rest of the proof immediately follows from Theorem C_+ of Section 6 by which the category $\mathrm{f}\mathbb{Q}\text{-ho}\mathscr{A}_+^{dc}$ is equivalent to the category $\mathrm{fN}\mathbb{Q}\text{-ho}\mathscr{S}_+^{dc}$. $\qquad\qquad\square$

10.6. Remark. The reader may wonder why Theorem 10.5 relates *augmented* algebras to dg-Lie algebras with no additional structure. The answer is that the dg Lie-analogue of an augmentation is a choice of an MC-element. From this perspective, each dgla is canonically augmented by the trivial MC-element 0. Our definition of the disjoint product $\mathfrak{g} \sqcup \mathfrak{h}$ is such that its augmentation is induced by the augmentation of the last factor.

10.7. Remark. Note that Theorem 10.5 allows one to say something new even for connected spaces (or simplicial sets). Indeed, the category of *unpointed* connected spaces is a subcategory of *disconnected* pointed spaces. Namely, this subcategory consists of spaces, consisting of two connected components, one of which is the basepoint. We see, therefore, that the homotopy category of rational connected unpointed spaces is equivalent to a certain subcategory of $\mathrm{f}\mathbb{Q}\text{-ho}\hat{\mathscr{L}}^{dc}$, whose objects are of the form $\mathfrak{g} \sqcup 0$ for a non-negatively graded complete dgla $\mathfrak{g}$. This subcategory is not full; in fact it is easy to see that there is a natural bijection

$$\big[\mathfrak{g} \sqcup 0, \mathfrak{h} \sqcup 0\big] \cong \big[\mathrm{MC}_\bullet(\mathfrak{g}), \mathrm{MC}_\bullet(\mathfrak{h})\big] \cup \{*\}$$

where $*$ denotes an isolated basepoint corresponding to the zero map $\mathfrak{g} \sqcup 0 \to \mathfrak{h} \sqcup 0$.

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Appendix A. Cohomology of free products of dg Lie algebras

The purpose of this appendix is to express the Chevalley-Eilenberg cohomology of free products of dglas in terms of the Chevalley-Eilenberg cohomology of the factors. The main results are Theorem A.4 for the non-complete case and Theorem A.7 for the compete one. These results seem completely standard but we have not found them in the literature. There is an analogous result for group cohomology, but it relies on the construction of the classifying space of a group; such a construction has no analogue for an arbitrary (i.e. not necessarily nilpotent) dgla, so we needed to develop a certain algebraic machinery instead. First, introduce some notation.

Recall that $\mathscr{L}$ is the category of (discrete) dglas. We denote by $\hat{\mathscr{A}}$ the category of complete cdgas; its objects are inverse limits of finite dimensional nilpotent *non-unital* cdgas.

A.1. **Definition.** Let $\hat{\mathcal{C}} : \mathscr{L} \mapsto \hat{\mathscr{A}}$ be the functor associating to a discrete dgla $\mathfrak{g} \in \mathscr{L}$ the complete cdga $\hat{\mathcal{C}}(\mathfrak{g})$ whose underlying non-unital cdga is $\hat{S}_+\Sigma\mathfrak{g}^*$, the completed non-unital symmetric algebra on $\Sigma\mathfrak{g}^*$. The differential d in $\hat{\mathcal{C}}(\mathfrak{g})$ is defined as $d = d_I + d_{II}$, where d_I is induced by the internal differential in $\mathfrak{g}$ and d_{II} is determined by its restriction onto $\Sigma\mathfrak{g}^*$, which is, in turn, induced by the bracket map $\mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$.

A.2. **Remark.** The construction $\hat{\mathcal{C}}(\mathfrak{g})$ is a double complex with the horizontal differential d_{II} and the vertical differential d_I. As such, it has two spectral sequences associated with it; one converging to the direct product totalization, the other to the direct sum totalization. Since $\hat{\mathcal{C}}(\mathfrak{g})$ is constructed using direct products, only the first spectral sequence is relevant. We denote this spectral sequence by $E'(\mathfrak{g})$; we have $E'_2(\mathfrak{g}) = H\big(\hat{\mathcal{C}}(H(\mathfrak{g}))\big)$. Using this spectral sequence, we easily prove that the functor $\hat{\mathcal{C}} : \mathscr{L} \mapsto \hat{\mathscr{A}}$ preserves quasi-isomorphisms.

The proof of Theorem A.4 below will use the following lemma.

A.3. **Lemma.** *Let $f : \mathfrak{u} \to \mathfrak{v}$ be a morphism of dglas such that the induced morphism $\mathcal{U}(f) : \mathcal{U}(\mathfrak{u}) \to \mathcal{U}(\mathfrak{v})$ of their universal enveloping algebras is a quasi-isomorphism. Then f is a quasi-isomorphism too.*

Proof. By [26, Theorem 4.5], the functor $\mathcal{U}$ from dglas to cocommutative connected dg Hopf-algebras admits a quasi-inverse, denoted by $\mathcal{P}$, which associates to a dg Hopf algebra its dgla of primitive elements. Consider the following commutative diagram of canonical maps:

(A.1)
$$
\begin{array}{ccc}
H\mathcal{P}\mathcal{U}(\mathfrak{u}) & \xrightarrow{H\mathcal{P}\mathcal{U}(f)} & H\mathcal{P}\mathcal{U}(\mathfrak{v}) \\
\downarrow & & \downarrow \\
\mathcal{P}H\mathcal{U}(\mathfrak{u}) & \xrightarrow{\mathcal{P}H\mathcal{U}(f)} & \mathcal{P}H\mathcal{U}(\mathfrak{v}).
\end{array}
$$

Since $\mathcal{P}\mathcal{U}(\mathfrak{u}) \cong \mathfrak{u}$ one has $H\mathcal{P}\mathcal{U}(\mathfrak{u}) \cong H(\mathfrak{u})$ and, since $\mathcal{U}H \cong H\mathcal{U}$ by [26, Theorem 2.1], one has $\mathcal{P}H\mathcal{U}(\mathfrak{u}) \cong \mathcal{P}\mathcal{U}H(\mathfrak{u}) \cong H(\mathfrak{u})$; similarly for $\mathfrak{v}$ in place of $\mathfrak{u}$. We conclude that both the vertical arrows in (A.1) are isomorphisms, while the bottom map $\mathcal{P}H\mathcal{U}(f)$ is an isomorphism by assumption. Thus $H(f) = H\mathcal{P}\mathcal{U}(f)$ must be an isomorphism too. $\qquad\square$

A.4. **Theorem.** *Let $\mathfrak{g}$ and $\mathfrak{h}$ be two discrete dglas. Then there is a quasi-isomorphism $\hat{\mathcal{C}}(\mathfrak{g} * \mathfrak{h}) \cong \hat{\mathcal{C}}(\mathfrak{g}) \times \hat{\mathcal{C}}(\mathfrak{h})$ where $\mathfrak{g} * \mathfrak{h}$ is the free (non-completed) product of $\mathfrak{g}$ and $\mathfrak{h}$.*

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Proof. As the first step we prove that the functor $\mathfrak{g} \mapsto \mathfrak{g} * \mathfrak{h}$ preserves quasi-isomorphisms. Our proof of this fact will use the isomorphism $\mathcal{U}(\mathfrak{g} * \mathfrak{h}) \cong \mathcal{U}(\mathfrak{g}) * \mathcal{U}(\mathfrak{h})$ for universal enveloping algebras, where the symbol $*$ in the right hand side stands for the free product of dgas.

Let $\mathfrak{g}'$ be a dgla quasi-isomorphic to $\mathfrak{g}$. Since $\mathcal{U}$ preserves quasi-isomorphisms by [26, Theorem 2.1], it suffices to prove that the functor of taking the free product on the category of dgas preserves quasi-isomorphisms. Indeed, if it is so, then we have a chain of isomorphisms and quasi-isomorphisms:

$$\mathcal{U}(\mathfrak{g} * \mathfrak{h}) \cong \mathcal{U}(\mathfrak{g}) * \mathcal{U}(\mathfrak{h}) \simeq \mathcal{U}(\mathfrak{g}') * \mathcal{U}(\mathfrak{h}) \cong \mathcal{U}(\mathfrak{g}' * \mathfrak{h}).$$

Lemma A.3 then implies that $\mathfrak{g} * \mathfrak{h}$ is quasi-isomorphic to $\mathfrak{g}' * \mathfrak{h}$.

To show the exactness of the free product functor for associative algebras, observe that the product $A * B$ of dgas A and B decomposes as a direct sum of tensor products of A and B:

$$A * B \cong A \oplus B \oplus (A \otimes B) \oplus (B \otimes A) \oplus (B \otimes A \otimes B) \oplus (A \otimes B \otimes A) \oplus \cdots$$

so the statement follows from the exactness of the tensor product of dgas over a field of characteristic zero.

Recall that the functor $\mathfrak{g} \mapsto \hat{\mathcal{C}}(\mathfrak{g})$ preserves quasi-isomorphisms by Remark A.2. Since $\mathscr{L}$ is a closed model category, we conclude that it suffices to prove the statement of our theorem in the case when both $\mathfrak{g}$ and $\mathfrak{h}$ are standard cofibrant dglas, i.e. when they are obtained from the trivial dgla by a sequence of cell attachments. In that case $\mathfrak{g} * \mathfrak{h}$ is likewise a standard cofibrant dgla.

Further, we claim that if $\mathfrak{a}$ is a cofibrant dgla then $\hat{\mathcal{C}}(\mathfrak{a})$ is quasi-isomorphic to the space $\mathrm{Der}(\mathfrak{a}, \Bbbk)$ of derivations of $\mathfrak{a}$ with coefficients in the trivial one-dimensional $\mathfrak{a}$-module $\Bbbk$ – note that $\mathrm{Der}(\mathfrak{a}, \Bbbk)$ is isomorphic to $(\mathfrak{a}/[\mathfrak{a}, \mathfrak{a}])^*$, the dual space of indecomposables of $\mathfrak{a}$. This is a standard fact which can be proved e.g. by $\Bbbk$-linear dualization of [15, Proposition 9.1.1]. Finally, it is clear that $\mathrm{Der}(\mathfrak{g} * \mathfrak{h}, \Bbbk) \cong \mathrm{Der}(\mathfrak{g}, \Bbbk) \times \mathrm{Der}(\mathfrak{h}, \Bbbk)$. The desired result is proved. $\square$

Recall that we are really interested in *complete* dglas whereas the last result concerns non-complete ones. Somewhat surprisingly, it also holds for complete dglas, as a *consequence* of Theorem A.4. In the formulation of the complete case we have to take the fiber product of the corresponding CE complexes $\mathcal{C}(\mathfrak{g})$ and $\mathcal{C}(\mathfrak{h})$ since the latter are *augmented* cdgas. Let us prove some preliminary statements. From now on we will write $\mathcal{C}_+(\mathfrak{g})$ for the augmentation ideal of the cdga $\mathcal{C}(\mathfrak{g})$.

A.5. **Lemma.** *Let $\mathfrak{g}$ and $\mathfrak{h}$ be finite dimensional nilpotent graded Lie algebras, viewed as complete dglas. Then there is a quasi-isomorphism $\mathcal{C}_+(\mathfrak{g} * \mathfrak{h}) \simeq \mathcal{C}_+(\mathfrak{g}) \times \mathcal{C}_+(\mathfrak{h})$.*

Proof. The statement of the lemma is similar to the quasi-isomorphism of Theorem A.4, however the essential difference is that $\mathfrak{g}$ and $\mathfrak{h}$ are regarded as *complete* dglas (with vanishing differentials) and so, their free product $\mathfrak{g} * \mathfrak{h}$ is likewise completed. However, it turns out not to influence the result. The natural maps $\mathfrak{g} \to \mathfrak{g} * \mathfrak{h}$ and $\mathfrak{h} \to \mathfrak{g} * \mathfrak{h}$ induce a map

(A.2) $\mathcal{C}_+(\mathfrak{g} * \mathfrak{h}) \to \mathcal{C}_+(\mathfrak{g}) \times \mathcal{C}_+(\mathfrak{h});$

we want to show that this map is a quasi-isomorphism.

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We start the proof by reducing the statement to the case when $\mathfrak{g}$ and $\mathfrak{h}$ are abelian. First observe that $\mathfrak{g}$, being finite dimensional nilpotent, has finite filtration, whose associated graded is abelian (graded) Lie algebra which will be denoted by $\mathfrak{g}^\natural$.

Indeed, consider the lower central series $\mathfrak{g} = \mathfrak{g}_1 \supset \mathfrak{g}_2 \supset \cdots \supset \mathfrak{g}_{s+1} = 0$ of $\mathfrak{g}$. Although the associated graded of this descending filtration need not be abelian, the shifted filtration

(A.3) $$\mathfrak{g} = F_0\mathfrak{g} \supset F_1\mathfrak{g} \supset \cdots \supset F_s\mathfrak{g} = 0$$

with $F_n\mathfrak{g} := \mathfrak{g}_{n+1}$, $0 \leq n \leq s$, does have this property. Let $G^n\mathfrak{g}^* := (F_{n+1}\mathfrak{g})^\perp = \mathfrak{g}_{n+2}^\perp$ be the annihilator of $F_{n+1}\mathfrak{g}$ in $\mathfrak{g}^*$. Then

(A.4) $$0 \subset G^0\mathfrak{g}^* \subset G^1\mathfrak{g}^* \subset \cdots \subset G^s\mathfrak{g}^* = \mathfrak{g}^*$$

is a finite ascending filtration of $\mathfrak{g}^*$. The subspaces

$$G^n\mathcal{C}_+(\mathfrak{g}) := \bigoplus_{k \geq 1} \bigoplus_{n = n_1 + \cdots + n_k} \Sigma G^{n_1}\mathfrak{g}^* \cdots \Sigma G^{n_k}\mathfrak{g}^* \subset \mathcal{C}_+(\mathfrak{g})$$

form an ascending filtration

$$0 \subset G^0\mathcal{C}_+(\mathfrak{g}) \subset G^1\mathcal{C}_+(\mathfrak{g}) \subset G^2\mathcal{C}_+(\mathfrak{g}) \subset \cdots \subset \mathcal{C}_+(\mathfrak{g})$$

which is exhaustive, Hausdorff and complete. The E_1-term of the induced spectral sequence clearly equals $\mathcal{C}_+(\mathfrak{g}^\natural)$ and this spectral sequence strongly converges to $\mathcal{C}_+(\mathfrak{g})$.

Let us turn our attention to the completed free product $\mathfrak{g} * \mathfrak{h}$. One has a natural epimorphism $\pi : \mathbb{L}(\mathfrak{g}, \mathfrak{h}) \twoheadrightarrow \mathfrak{g} \tilde{*} \mathfrak{h}$, where $\mathbb{L}(\mathfrak{g}, \mathfrak{h})$ is the free graded Lie algebra generated by $\mathfrak{g} \oplus \mathfrak{h}$, and $\mathfrak{g} \tilde{*} \mathfrak{h}$ denotes, only in this proof, the *uncompleted* free product. Let $\mathbb{L}^{\geq m}(\mathfrak{g}, h)$ be the ideal spanned by products of at least m elements, and $(\mathfrak{g} \tilde{*} \mathfrak{h})^{\geq m} := \pi\big(\mathbb{L}^{\geq m}(\mathfrak{g}, h)\big)$. Since π is an epimorphism, $(\mathfrak{g} \tilde{*} \mathfrak{h})^{\geq m}$ is an ideal, and

$$\mathfrak{g} * \mathfrak{h} = \lim_m (\mathfrak{g} \tilde{*} \mathfrak{h})/(\mathfrak{g} \tilde{*} \mathfrak{h})^{\geq m}.$$

Filtration (A.3) induces, in the standard manner, a filtration $F_n\mathbb{L}(\mathfrak{g}, \mathfrak{h})$, $n \geq 0$, of $\mathbb{L}(\mathfrak{g}, \mathfrak{h})$. Denote finally by $F_n(\mathfrak{g} \tilde{*} \mathfrak{h}) := \pi\big(F_n\mathbb{L}(\mathfrak{g}, \mathfrak{h})\big)$ the induced filtration of the uncompleted free product and by $F_n(\mathfrak{g} * \mathfrak{h})$ its closure in $\mathfrak{g} * \mathfrak{h}$. It follows from the continuity of the bracket in $\mathfrak{g} * \mathfrak{h}$ that

$$\mathfrak{g} * \mathfrak{h} = F_0(\mathfrak{g} * \mathfrak{h}) \supset F_1(\mathfrak{g} * \mathfrak{h}) \supset F_2(\mathfrak{g} * \mathfrak{h}) \supset \cdots$$

is a descending filtration by ideals. The formula $G^n(\mathfrak{g} * \mathfrak{h})^* := \big(F_{n+1}(\mathfrak{g} * \mathfrak{h})\big)^\perp$ defines an ascending filtration

(A.5) $$0 \subset G^0(\mathfrak{g} * \mathfrak{h})^* \subset G^1(\mathfrak{g} * \mathfrak{h})^* \subset G^2(\mathfrak{g} * \mathfrak{h})^* \subset \cdots (\mathfrak{g} * \mathfrak{h})^*.$$

Recall that $(\mathfrak{g} * \mathfrak{h})^*$, by definition, consists of *continuous* linear functionals. By continuity, every such functional α factors through the canonical epimorphism

$$(\mathfrak{g} * \mathfrak{h}) \twoheadrightarrow (\mathfrak{g} \tilde{*} \mathfrak{h})/(\mathfrak{g} \tilde{*} \mathfrak{h})^{\geq m}$$

for $m \gg 0$. By the finiteness of (A.3), the induced filtration of $(\mathfrak{g} \tilde{*} \mathfrak{h})/(\mathfrak{g} \tilde{*} \mathfrak{h})^{\geq m}$ is finite, so α annihilates $F_n(\mathfrak{g} * \mathfrak{h})$ for $n \gg 0$. This implies that the filtration (A.5) is exhaustive.

Now we proceed as in the case of $\mathcal{C}_+(\mathfrak{g})$. The filtration (A.5) induces an exhaustive, Hausdorff and complete filtration of $\mathcal{C}_+(\mathfrak{g} * \mathfrak{h})$. The induced spectral sequence strongly converges to $\mathcal{C}_+(\mathfrak{g} * \mathfrak{h})$ and its E_1-term is $\mathcal{C}_+(\mathfrak{g}^\natural * \mathfrak{h})$.

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Since the canonical map (A.2) is compatible with the filtrations, by the comparison theorem for spectral sequences, it is enough to prove that (A.2) is a quasi-isomorphism with $\mathfrak{g}$ replaced by $\mathfrak{g}^\natural$. Repeating the same steps with $\mathfrak{h}$ in place of $\mathfrak{g}$, we prove that the desired statement would follow if we can prove that the map $\mathcal{C}_+(\mathfrak{g}^\natural * \mathfrak{h}^\natural) \to \mathcal{C}_+(\mathfrak{g}^\natural) \times \mathcal{C}_+(\mathfrak{h}^\natural)$ is a quasi-isomorphism. In other words, we reduced the statement of the lemma to the case when both $\mathfrak{g}$ and $\mathfrak{h}$ are *abelian*.

The dg space $\mathcal{C}(\mathfrak{g} * \mathfrak{h}) = S\Sigma(\mathfrak{g} * \mathfrak{h})^*$ consists of symmetric tensors in $\Sigma(\mathfrak{g} * \mathfrak{h})^*$; we can introduce a grading on $\mathcal{C}(\mathfrak{g} * \mathfrak{h})$ as follows. Let $x \in \mathfrak{g} * \mathfrak{h}$. Then x has weight $n+1$ if x is a sum of Lie monomials of bracket length n; this grading lifts to $\mathcal{C}(\mathfrak{g} * \mathfrak{h})$.

It is clear that the differential in $\mathcal{C}(\mathfrak{g} * \mathfrak{h})$ preserves the weight grading and so the dg space $\mathcal{C}(\mathfrak{g} * \mathfrak{h})$ decomposes as an infinite direct sum of subcomplexes consisting of elements of fixed weight. It follows that the map (A.2) is a quasi-isomorphism if and only if it is a quasi-isomorphism for each weight component. This, in turn, holds if and only if and only if a similar statement holds when the infinite direct sum over all positive weights is replaced by the corresponding infinite direct product. Observe that the obtained completed dg space has the form $\hat{\mathcal{C}}(\mathfrak{g} * \mathfrak{h})^* = \hat{S}\Sigma(\mathfrak{g} * \mathfrak{h})^*$ where in the last formula $\mathfrak{g} * \mathfrak{h}$ stands for the *uncompleted* free product. It is, therefore, nothing but the standard complex computing the Chevalley-Eilenberg cohomology of the uncompleted Lie algebra $\mathfrak{g} * \mathfrak{h}$. Thus, the statement is reduced to computing the usual Chevalley-Eilenberg cohomology of $\mathfrak{g} * \mathfrak{h}$ and, therefore, follows from Theorem A.4. □

A.6. **Lemma.** *Let $\mathfrak{a}$ be a complete dgla and let $\tilde{\mathfrak{a}}$ be the complete dgla with vanishing differential and the same graded Lie bracket as $\mathfrak{a}$. Then there exists a spectral sequence $E''(\mathfrak{a})$ converging strongly to $\mathcal{C}_+(\mathfrak{a})$ such that $E_1''(\mathfrak{a}) = H\big(\mathcal{C}_+(\tilde{\mathfrak{a}})\big)$.*

Proof. As graded vector spaces, $\mathcal{C}_+(\mathfrak{a}) = \bigoplus_{p \geq 1} S^p(\Sigma^{-1}\mathfrak{a}^*)$, where $S^p(-)$ denotes the subspace of the symmetric algebra consisting of elements of homogeneity p. It is obvious that

$$(\text{A.6}) \qquad F_n\mathcal{C}_+(\mathfrak{a}) := \bigoplus_{q \geq n,\ p \geq 1} \big[S^p(\Sigma^{-1}\mathfrak{a}^*)\big]^{p+q}$$

forms a decreasing exhaustive filtration of the cochain complex $\mathcal{C}_+(\mathfrak{a})$. Its degree m component equals

$$F_n\mathcal{C}_+(\mathfrak{a})^m := \bigoplus_{q \geq n,\ m > q} \big[S^{m-q}(\Sigma^{-1}\mathfrak{a}^*)\big]^m.$$

Clearly $F_n\mathcal{C}_+(\mathfrak{a})^m = 0$ if $n > m$, the filtration (A.6) is thus Hausdorff and complete. The induced spectral sequence therefore converges strongly and obviously has the properties stated in the lemma. □

A.7. **Theorem.** *Let $\mathfrak{g}$ and $\mathfrak{h}$ be two complete dglas. Then there is a quasi-isomorphism $\mathcal{C}_+(\mathfrak{g} * \mathfrak{h}) \simeq \mathcal{C}_+(\mathfrak{g}) \times \mathcal{C}_+(\mathfrak{h})$.*

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Proof. Assume first that $\mathfrak{g}$ and $\mathfrak{h}$ are finite-dimensional nilpotent. Arguing as in the proof of Lemma A.5 we construct a map $\mathcal{C}_+(\mathfrak{g} * \mathfrak{h}) \longrightarrow \mathcal{C}_+(\mathfrak{g}) \times \mathcal{C}_+(\mathfrak{h})$; the desired statement is equivalent to proving that this is a quasi-isomorphism. Consider a map of spectral sequences

$$E_1''(\mathfrak{g} * \mathfrak{h}) = H(\mathcal{C}_+(\widetilde{\mathfrak{g} * \mathfrak{h}})) \to E_1''(\tilde{\mathfrak{g}}) \times E_1''(\tilde{\mathfrak{h}}) = H\big(\mathcal{C}_+(\tilde{\mathfrak{g}})\big) \times H\big(\mathcal{C}_+(\tilde{\mathfrak{h}})\big)$$

of Lemma A.6. The map in the middle is an isomorphism by Lemma A.5, so the desired result for finite dimensional nilpotent $\mathfrak{g}$ and $\mathfrak{h}$ follows.

Assume now that $\mathfrak{g} = \lim_\alpha \mathfrak{g}_\alpha$ and $\mathfrak{h} = \lim_\beta \mathfrak{h}_\beta$ are limits of finite-dimensional nilpotent algebras. Then the continuous duals equal $\mathfrak{g}^* = \operatorname{colim}_\alpha \mathfrak{g}_\alpha^*$, $\mathfrak{h}^* = \operatorname{colim}_\beta \mathfrak{h}_\beta^*$ which readily implies that

$$\mathcal{C}_+(\mathfrak{g}) = \operatorname*{colim}_\alpha \mathcal{C}_+(\mathfrak{g}_\alpha) \quad \text{and} \quad \mathcal{C}_+(\mathfrak{h}) = \operatorname*{colim}_\beta \mathcal{C}_+(\mathfrak{h}_\beta).$$

By the standard properties of the filtered limits, one has

$$(A.7) \qquad \mathfrak{g} * \mathfrak{h} \cong \lim_{\alpha,\beta,m} (\mathfrak{g}_\alpha \,\tilde{*}\, \mathfrak{h}_\beta)/(\mathfrak{g}_\alpha \,\tilde{*}\, \mathfrak{h}_\beta)^{\geq m},$$

where $\tilde{*}$ denotes, as in the proof of Lemma A.5, the uncompleted free product. The isomorphism (A.7) represents $\mathfrak{g} * \mathfrak{h}$ as a limit of finite dimensional spaces, so

$$\begin{aligned} (\mathfrak{g} * \mathfrak{h})^* &\cong \operatorname*{colim}_{\alpha,\beta,m} \big[(\mathfrak{g}_\alpha \,\tilde{*}\, \mathfrak{h}_\beta)/(\mathfrak{g}_\alpha \,\tilde{*}\, \mathfrak{h}_\beta)^{\geq m}\big]^* \cong \operatorname*{colim}_{\alpha,\beta} \operatorname*{colim}_m \big[(\mathfrak{g}_\alpha \,\tilde{*}\, \mathfrak{h}_\beta)/(\mathfrak{g}_\alpha \,\tilde{*}\, \mathfrak{h}_\beta)^{\geq m}\big]^* \\ &\cong \operatorname*{colim}_{\alpha,\beta} (\mathfrak{g}_\alpha * \mathfrak{h}_\beta)^* \end{aligned}$$

which easily implies that

$$\mathcal{C}_+(\mathfrak{g} * \mathfrak{h}) = \operatorname*{colim}_{\alpha,\beta} \mathcal{C}_+(\mathfrak{g}_\alpha * \mathfrak{h}_\beta).$$

Since the canonical map (A.2) is compatible with these colimits and the colimits are exact functors, the general case follows from the finite-dimensional nilpotent one. $\qquad\square$

A.8. **Remark.** Note that the quasi-isomorphism of Theorem A.7 can also be formulated as $\mathcal{C}(\mathfrak{g} * \mathfrak{h}) \simeq \mathcal{C}(\mathfrak{g}) \times_\Bbbk \mathcal{C}(\mathfrak{h})$.

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