RANK REVERSAL PROPERTIES OF MULTICRITERIA DECISION MAKING MODELS

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ABSTRACT

Decision making problems in modern society are very important however complex. Therefore, they require strong solving techniques to handle. The AHP method attracts a lot attention for its advantages and has a very well structured methodology, while the PROMETHEE method of the European school is also widely accepted. Pairwise comparison is one of the fundamental methods for AHP, with its methodology track back to the definition of Perron-Frobenius theorem. Perron-Fronbenius explained the most fundamental structure of arbitrary pairwise comparison. The distributive model and the ideal model are widely accepted as a powerful tool in AHP multicriteria decision making problems based on pairwise comparison. In these models, it might happen that by introducing a new alternative, the original order of alternatives will change. Moreover, it is possible to introduce a new alternative, such that the order of the original alternatives will be given by 'almost any' criterion. In the latter part of this paper, we then give the detailed proofs based on these two models, and some examples which shows rank reversal caused by this new alternative could be harmful. As a continuous thinking of this case, in the second part, detailed proof are given about a method from Dr. S. Z. Nemeth, by defining the 'maximal box' in avoid causing rank reversal. These would then be followed with real life examples and results by using the software known as Experts' Choice, which consist the the rest part of this project.

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Chapter 1

Introduction

The aim of this project is to give some knowledge about decision making and introduce some existing mathematical methodology and techniques in solving multicriteria decision making problems. The structure of the project is ranked from general knowledge to more specified topics. Several real life examples included make the theoretical knowledge easy to be accepted as a more detailed, clear view of the cases.

Chapter two starts from 'what is decision making about' and 'how the decision is made', and continues with a brief introduction of the the history of decision making. In this part, some well known historical event about decision making happened in human history are given, showing how decision making evolved in human history. This topic then bring up to the modern decision making methodology evolved from mathematics and operational research. This includes two most popular methods with Analytical Hierarchy Process(AHP) the American school method along with PROMETHEE the European school method. This paper focus on the the models used in AHP rather than PROMETHEE method.

As the methodology of AHP is based on pairwise comparison, in chapter three, notations are explained about pairwise comparison.

It starts from stating the mathematical basis of AHP which is known as pairwise comparison. The definition of pairwise comparison is given in the first section. The mathematical structure and figure give a more straight forward image of this decentralized hierarchy structure. We then show its standard matrix form in the second section, which is behind the pairwise comparison. In section three, the famous theorem of matrix analysis the Perron-Frobenius theorem is cited and a detailed proof is given in a matter of matrix analysis as the cornerstone of pairwise comparison. In the last section, we also give the eigenvector method, which is not based on Perron-Frobenius theorem.

After finishing the theoretical background for AHP, in chapter four we state AHP in detail.

Distributive and ideal models are among the most powerful tools in AHP multicriteria decision making problems. In section one, we gives their definitions and mathematical expressions respectively. We then study on the stability of the order in section two, by introducing a new alternative, and it might happen that the original order of alternative will change. Moreover it is possible to introduce a new alternative, such that the order of the alternatives will be give by almost any criterion. Some real life examples given afterwards shows that in real life this could be harmful. Then it comes to the topic about the method we introduce to avoid rank reversal. It's a method based on the stability of AHP when introducing or eliminating an dominating alternative over the criterion, we designed a box box with artificially created numbers as the boundary of the box, where values taken from while avoiding potential rank reversal. Detailed proof are also given about the theoretical part of this case.

Meanwhile, there would always some cases in any group that individuals' judgments differ from other individuals in their group. Thus, we also discuss about AHP in group decision making in section seven. Theorem and proof then reflects the process how to aggregate the pairwise comparison matrices of the individual decision makers into group decision.

By using decision making software, we may greatly save the time for working complex decisions. So in chapter five we also include some examples and their running results in a specialized decision making software named Expert's Choice(EC).

At the end of the project, conclusions are given to give what are dealt in this project, how the result is expected, what further improvement could be done and several open questions listed related to be further investigated.

Chapter 2

Preliminaries

2.1 Brief History of Decision Making

The word 'decision' is nowadays very easily seen in modern society. It is widely used by individuals and groups in making and carrying out plan. Decision making is a natural phenomenon that is as old as the history of mankind. People in ancient days used to take advices from monarch, by predicting the outcome of the incident through personal experience, conditions and even mysterious religious ritual with some of them are rational while some not;and it's still one of the most popular ways judging by personal experience about circumstances one may face in certain case. And we admit the importance of personal experience play in human decision making.

The nature of decision making and its process is very complex, the way people make decisions has also gone through evolutionary process. In the times, societies consulted their elders for alternatives and experimental data about the probability of success for decision choices in similar situations. Villages, especially those underdeveloped rural areas in Africa and Asia, still holds their function from the elders. In the latter stage of a larger society, This role of advisory then shifts to religious astrologers and wizards, gives out the prediction of the day, usually in a religious ritual with goddess figures. One of the most famous historical event is that, Alexander the great(359-323BC) went for a great battle with the the Persian, he went for oracles and fortune teller for opinions which could give him some prediction about the war, rather than the war strategies. Great leaders like Alexander, they would usually go for different advice before important decisions made, but they are always the decision makers who made the final decisions. Astrologers, oracles or military counselors usually play the role as information providers.

A massive progress in division of labor and the emerge of modern science starts from the late 18th century, which also hugely impacted on decision making problems, with more factors and subjectives to consider, the decision problem is getting more complex and more difficult. Nowadays the term 'multiobjective' or 'multicriteria' decision making we use, traces its origin from Francis Y. Edgewoth and Vilfredo Pareto who coined the term multicriteria decision making in late 19th early 20th century.

The complexity in modern society determine we need more knowledge to understand how to make a 'better' decision subject to complex conditions, this is affected by numerous factors sometimes thousands which in most cases have conflicting objectives and no alternative is the best one on each criterion, so usually no optimal solution. Better quality implies a higher price. Let's take selecting university for example. You have some candidate universities in your mind, and trying to select the best option among them. These universities could be either the universities you are applying for, or you have already got offers from. You would might like to list these criterion which you would take into consideration: university ranking, major ranking, amount of tuition fees, teaching stuff and student ratio, location(city, campus), even first impression of the campus, and even someone you are familiar is/was in this university(say your girlfriend/parents),etc could all account. However with more alternatives and criterion taken into account, the system of decision making may become very large, and the decision making process may takes much longer than small decision problem.

All these then leads to the current systematically automated techniques supported by computers. The development of computer has a huge impact on the development of modern decision making, due to the significant growth in areas such as technology and telecommunication. With continuously development new programming algorithm developed by researchers, and 'supercomputer' which results could be made in minutes even seconds, decision making process is getting more efficient.

Nevertheless, it has always been the quest to find methods of supporting the structuring of complex decisions. Usually we divide a general decision making process into a hierarchy in following steps:

1. Planing.

Identity the decision problem to be considered. This is very important as it determines the overall structure of the system.

2. Requirements.

In the mathematical model, this is to find out the constraints and the feasible solutions limited in the feasible set.

3. Establish goals.

Goals are set up based on all previous requirements. This is usually known as the function we may use to find out solutions.

4. Identify alternatives.

Alternatives mean that different methods and algorithms applied to describe and approach the potential solution.

5. Define criteria.

Criteria is taken as the prametre or the objective measures of the goals. It is crucial in measuring how the alternatives would perform in achieving the goal.

6. Select decision making tool.

For our project, PROMETHEE and AHP are listed as more appropriate methods.

7. Evaluate alternatives against criteria.

When considering the criteria, the scale of measurement could be defined now to make the alternatives objective.

8. Validate solutions.

Alternatives defined in the decision making problem tools have to be validated to make it possible to carry out.

2.2 Preliminaries About Analytical Hierarchy Process (AHP) and PROMETHEE Method

AHP Method

The Analytic Hierarchy Process (known as 'AHP') is a structured method for dealing with complex decisions. This method was first proposed by Thomas L. Saaty in the 1980s and this field has been greatly developed since then, with wide application in economics, energy, management, environment, traffic, agriculture, industry, and the military. The root of AHP mainly comes from mathematics, which the decision problem is usually represented in a multilayer, tree structure. This tree structure is widely accepted in the modern world as it provides a detailed, clear representation of the problem. People in business sometimes draw 'spider' pictures to list ideas related to certain topic with independent subquestions and ideas lay at the same level. It is worthy to notice that the hierarchy structure quite matches the construction of human psychological nerve structure, especially that in human brain which consists a complex nerve system. Dantzig observes that the human mind has a sense to recognize the change on a small amount of objects when things are added or abstracted. This means human brain has the ability of distinguishing the degree of differences among objects according to their certain properties in common.[1]

Though under criticism, AHP is widely accepted as a strong tool in resolving real life problems in countries around the world. These topics covered ranges from measuring world influence of nations, estimating distances among cities, to energy allocation, and also applied to the area of conflict resolution by cost-benefit trade-off analysis in financial world.

In UK, AHP method is used to choose a better computer operating system for British Airways, between DB2 system and TPF system. This is measured in a range of scores like flexibility, database integrity, and programmer productivity. These scores are then given a relative importance which gives out the priorities of the systems on certain criterion, which the decision maker considered in adopting which system.

In Sudan, AHP is used in analyzing the allocation to products problems. Cotton is the main crop which is exported and allocated to the manufacturing sector. Thus the sections of agriculture, transportation and distribution and construction do not receive much agricultural products. By using AHP, they find out the priority by reallocating them to different sections, which helped the government making a better economic distribution decision.

In the United States, a research based on AHP was carried out to predict the future of the higher education. By the changes of each section like government and industry which would influence the higher education, predict are given based on the results got from AHP.

In Israel, analysis is also given on determine the scale of introducing new football teams to the football league consisting with sixteen teams. And they predicted the change of sequence in teams evaluation, after introducing new teams.

AHP was first known to Chinese in the year of 1982, and the study of AHP and application based on AHP methods were widely used since then which the logic of methodology quite match Chinese philosophy 'first to recognize the overall goal', and the Chinese social and political system structure with there's always someone on top makes a decision, which highly compatible with the traditional Chinese decision making framework.

PROMETHEE Method

PROMETHEE abbreviates for Preference Ranking Organization Method for Enrichment Evaluations, is introduced by Brans et al in 1982. This method is an outranking procedure to choose among the set of alternatives, find the most preferable one.

This method is based on the positive and negative preference flows for each alternative in the valued outranking relation to derive the ranking of alternatives. The positive flow is expressing how much an alternative is outranking the other ones, while the negative flow shows how much the other ones outrank it. The essence of PROMETHEE methods is to provide the possibility for alternatives pairs to be evaluated on an absolute scale with respect to different criteria and to determine degrees of preferences, by using the what we called generalized criteria.

Based on the preference flows, the PROMETHEE method can be split up into two methods based on a partial preorder and a complete preorder.First we give the definition of preorder.

Definition 2.1 *Consider some set P and a binary relation $\prec$ on P. Then $\prec$ is a preorder if it is reflexive and transitive, i.e. for all a, b and c in P, which we have:*

Reflexitivity: $a \prec a$ Transitivity: $a \prec b$ and $b \prec c \Leftrightarrow a \prec c$ A set that is equipped with a preorder is called a **preordered set.** *If the preorder is also antisymmetric as well as reflexive and transitive, that is if $a \prec b$ and $b \prec a$, it implies $a = b$, then it is called a* **partial order.**

A preference value of 1 is assigned if one alternative is preferred to the performance of another, with respect to a specific criterion, without considering the magnitude of the performance difference. A preference value of 0 is assigned if the alternative is equal or not better to the other alternative. Preference values are determined from the pairwise comparisons and are then analyzed to develop an overall rating value for each alternative.

Then the decision maker could choose among six preference functions could be used during the evaluation of alternatives, which are

1.Usual criterion function. 2.Quasi criterion function. 3.Criterion with linear preference and indifference function. 4.Level criterion function. 5.Criterion with linear preference function. 6.Gaussian criterion function.

These six functions are seen widely applied and fit for most of the practical cases which users would find the most appropriate function to describe their real life observations.

After the appropriate preference function has been chosen and carried out on the criteria, the next step for the decision maker is to calculate the overall rating between the alternatives. These overall rating values are on a scale of 0 to 1 which the latter implies that an alternative is strictly preferred to all other alternatives. A software package 'Decision Lab' is released to assist decision maker's analysis on real life cases.

Chapter 3

Analytical Hierarchy Process (AHP)

A hierarchy is the system which is based on the reconstruction of the identified elements. The new structure is grouped into several disjoint sets, which the elements of one level may only influence the elements of one other level. And the elements in each level are supposed to be independent from other elements at this level. Other situations also show the existence of combining both independent and dependent elements. For example, a central government usually have several ministries below, each providing a specialized public service, usually these include Ministry of Defense, Ministry of Foreign Affairs, Ministry of Finance, and Ministry of Health, with the top administrating boards varied from country to country. In UK it is generally known as prime minister, in the United States the word president is used, and chairman in China.

The AHP is a logical system which arrange the alternatives and criterion into several levels of a tree-structured hierarchy system. Clusters and subclusters are given by breaking the reality in a human way. It has several advantages which is obviously to see,

1. Hierarchy can be used to describe the changes in priority at higher level may affect priority of elements in lower levels.

2. The structure and function of the system is decomposed into different independent subquestions, this gives an great overview of the individuals and purposes and the relation among them at the same level. Elements at a lower level are given with detailed constraints, which ensured those elements at a higher level are satisfied.

3. Most natural systems could be described in AHP in a straightforward way, parental system for example.

4. Hierarchy system is stable in small changes which have limited effect and it is also flexible to fix that small changes into a well-established structure, and it would not disrupt the performance.

One of the difficulties of constructing this hierarchy is to recognize and understand at the highest level, with its interactions to other levels below, which sometimes may cause confusion in ranking the levels. The elements of each level below do not usually have direct interaction with the highest level. In practice, this interaction between the highest level and the smallest elements is solved by identifying their adjacent levels in most hierarchy structures.

Of course, there is no certain set procedure to determine the steps of setting up the system. In AHP, it is given with the goal listed at the highest level, criteria below goal, subcriteria below criteria, alternatives below subcriteria. The lowest level are usually the alternatives. This hierarchy structure is given in the Figure below.

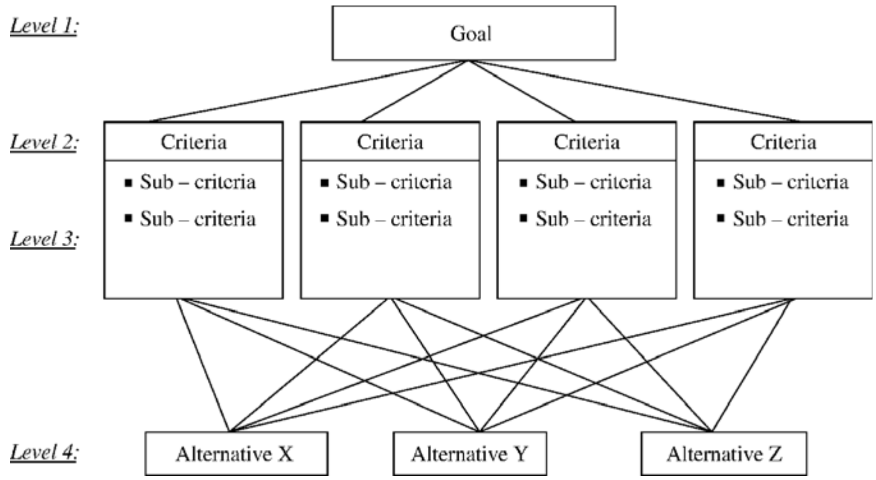


Table 3.1

General procedure in constructing AHP system

In AHP, a decision problem is first decomposed into a hierarchy of independent sub-problems.

Then the decision makers evaluate the elements by comparing them one another in pairs. By doing this, they can use their judgments which is based on data and personal experience, giving a relative importance to the elements. Both qualitative and quantitative criteria can be compared using informed judgments to derive weights and priorities. This is rather a mutual way of human judgment and scientific data. And AHP ensured the evaluations converted into numerical values.

In the final step, different weights are given to each alternative on its corresponding column. This makes the analysis of the decision problem straightforward.

This method would work through out the problem.

3.1 Definition of Pairwise Comparison

After setting up a hierarchy structure, the interaction between the levels is more or less clear, it is still unclear about what relationship of the elements in the same level could be. With the definition of the elements in the same level, they may usually have several factors to consider in common, or similar factor is shared among different objectives. This method is described in such a way. We first give the elements on arbitrary level with an criterion C at its adjacent higher level. We then compare the elements at this level in the pairwise manner, giving an relative strength of influence on this criterion C. For example, in the level of taste, apple, banana, and orange are compared for the question of which is better. The answer of preference may differ ranges from people to people, or what we called decision makers, but the way of choosing the 'best' is much the similar in the sense of comparison or priority. For example, Allen prefers apple three times to banana, prefers banana twice to orange, and prefer apple five time to orange.

How do we then determine the preference of one thing to the other in practice? At the this point, we use the most elementary method to convert the sense of priority into numerical expression. We introduce pairwise comparison to describe the preference or priority. 'Pairwise comparison generally refers to any process of comparing entities in pairs to judge which of each pair is preferred, or has a greater amount of some quantitative property.'[2] A general pairwise comparison decision problem is usually given in a matrix form with elements $a_i \geqslant 0, i = 1, 2, \cdots, n$.

$Weight$	$Criterion$	$Alternative$			
(w)	(C)	A_1	A_2	$\cdots$	A_n
w_1	C_1	a_{11}	a_{12}	$\cdots$	a_{1n}
w_2	C_2	a_{21}	a_{22}	$\cdots$	a_{2n}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
w_n	C_n	a_{m1}	a_{m2}	$\cdots$	a_{mn}

Table 3.1.1

Definition 3.1 *There are three expressions for alternatives x and y on A's preference:*

1.x is preferred over y by A: $x > y$

2.y is preferred over x by A: $y > x$

3.A share equal preference between x and y: $x = y$

In decision problems, we always need to give each criteria with weight respectively at first. By doing this, decision maker judge the priority of the criteria. There are usually two ways to give these weights.

$1 - 9$ **scale**

Verbal Scale	Numerical Values
Equally important/likely/preferred	1
Moderately more important/likely/preferred	3
Strongly more important/likely/preferred	5
Very Strongly important/likely/preferred	7
Extremely important/likely/preferred	9
Compromised intermediate values	$2, 4, 6, 8$

Table 3.1.2

By using this scale, a general pairwise comparison problem which compare the criterion to other is by assigning criteria with different weights ranked from 1 from 9, with $1, 3, 5, 7, 9$, represents judgment as Equally important', 'Moderately more important', 'Strongly more important', 'Very Strongly more important', and 'Extremely more important' or for short just listed as 'Equal', 'Moderate', 'Strong', 'Very Strong', and 'Extreme'. with $2, 4, 6, 8$, are compromises between them. This linear scaling method defined by Satty is not necessary 1 to 9.

This scale owns the advantage of reflecting human judgment on relative importance could be expressed in the way of simple numbers, which is straight forward to match visually and easy to handle numerically.

$1 - \alpha^n$ **scale**

Verbal Scale	Numerical Values
Equally important/likely/preferred	1
Very weakly more important/likely/preferred	α
Weakly more important/likely/preferred	α^2
$\vdots$	$\vdots$
Essentially important/likely/preferred	α^n

Table 3.1.3

With the linear scale, the decision maker cannot be consistent because the scale is not complete.[3] Take Allen's preference on fruits for example. He prefers apple twice to banana, prefers apple three times to orange, which is prefer banana to apple $1\frac{1}{2}$ times to orange. However his decision is constraint to choosing either 1 or 2 by the $'1 - 9'$ scale. So methods with flexible choice which could solve the problem of incompleteness. The method is defined with the scale of a consistent linear multiplicative scale $1, \alpha, \alpha^2, \alpha^3, \cdots, \alpha^n$, which α represents the smallest ratio of weights and α^n the largest. With these ratios numerically significant, similar verbal definition could be represented, α^2 equivalent to weak importance, α^4 essential importance, etc, up to α^n absolute importance. In the matter of consistency, $\alpha^2 \times \alpha^2 = \alpha^4$, which has a meaning of two weakly more important gives an essentially more important; Similarly, two essentially more importance would make an absolute importance α_8.

The advantage of this scale is it could be used in different models, with different α defined in different environment. However the chosen of 'meaningful' α and α^n might cause confusion, which the smallest α must be detectable and how well α^n could be estimated. In practice, it is usually left to decision maker to choose the minimum ratio α and maximum ratio α^n for a particular application. Using pairwise comparisons, similar ideas to above geometric or multiplicative scale are used in conflict resolution by cost-benefit trade-off analysis.

These methods of enumerating scale is very important in pairwise comparisons of the relative importance of one criterion over another. Indeed, it might cause confusion for decision maker that why they should not take numerical estimates for any numbers he can choose. This is much to the fact it should obey the consistency of pairwise comparison which we give a clear mathematical statement in the next section.

In this project, we use Satty's numerical scale 1 to 9, for qualitative data such as preference, ranking and subjective opinions.

3.2 Mathematical Background for Pairwise Comparison

Pairwise comparisons are essential in using AHP. The above scale defined with numbers represents the judgments of importance in making the comparisons. For a particular matrix of order n, the matrix is reciprocal and diagonal, the number of elements being compared, is n and the number of comparison is $\frac{n(n-1)}{2}$. On examining the pairwise comparison matrix A, a pair of elements (i, j) from a certain level of the hierarchy are compared in a common property or criterion which obtains less/more preference and how much, with the criterion usually locates at a parent element in the adjacent higher level. In matrix A, element i locates on the left side with j on the top, they are then compared in the matter of preference or importance under the given criterion, which gives the value a_{ij} in position (i, j). In the matter of matrix form of A, i and j are row and column index respectively. How to fill up the matrix is by using the following rules:

1. Actual judgment value on the right hand side of the diagonal elements.
2. Reciprocal value on the left side of the diagonal elements.

Take matrix A with the order of 3 for example. We put actual judgment value on the first row, second column, first row, third column, and second row, last column of the matrix. Then based on the decision maker's preference values, an arbitrary 3 by 3 matrix writes,

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

By following second rule above of the reciprocal pairwise comparison matrix, the reciprocal value is automatically entered for the transpose.

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ \frac{1}{a_{12}} & a_{22} & a_{23} \\ \frac{1}{a_{13}} & \frac{1}{a_{23}} & a_{33} \end{pmatrix}$$

The diagonal elements of arbitrary positive pairwise comparison matrix $a_{ij}, i = j$ are always 1; Since when $i = j$, the preference between the 'same' elements on the same criterion is equal. Then we have the completed pairwise comparison matrix form,

$$A = \begin{pmatrix} 1 & a_{12} & a_{13} \\ \frac{1}{a_{12}} & 1 & a_{23} \\ \frac{1}{a_{13}} & \frac{1}{a_{23}} & 1 \end{pmatrix}$$

Before we give the difference between theoretical pairwise comparison matrix and practical pairwise comparison matrix, we first need to define the notion of of reciprocal matrix and consistent matrix.

Definition 3.2 *Reciprocal Matrix,*

$$a_{ij} = \frac{1}{a_{ji}}, a_{ij} > 0, i, j, = 1, 2, \cdots, n;$$

Consistent Matrix,

$$a_{ik} = a_{ij}a_{jk}, a_{ij}, a_{ik}, a_{jk} > 0, i, j, k = 1, 2, \cdots, n.$$

A cycle of three elements could also be obtained from the definition in such a form

$$a_{ij}a_{jk}a_{ki} = 1, i, j, k = 1, \cdots, n.$$

	A_1	A_2	$\cdots$	A_n
A_1	$\frac{w_1}{w_1}$	$\frac{w_1}{w_2}$	$\cdots$	$\frac{w_1}{w_n}$
A_2	$\frac{w_2}{w_1}$	$\frac{w_2}{w_2}$	$\cdots$	$\frac{w_2}{w_n}$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
A_n	$\frac{w_n}{w_1}$	$\frac{w_n}{w_2}$	$\cdots$	$\frac{w_n}{w_n}$

Table 3.2.1

In general case the pairwise comparison matrices are of the above form.

The feature of pairwise comparison matrix is that its elements can be written as fractions of numbers w_i. We may also find that the following relation exists in pairwise comparison matrices,

$$Aw = nw, w \in R^n.$$

where A is the arbitrary positive pairwise comparison matrix. a_{ij} is the entry value of weights pair w_i and w_j, $a_{ij} = \frac{w_i}{w_j}, a_{ij}, w_i, w_j > 0$, and we would use it as the fraction of the weights in the latter part of this article, if not previous noted. with elements w_i from the vector w form the elements of A. R^n is the n-dimensional Euclidean space, n is the number of rows and also order of matrix A.

Theoretical Pairwise Comparison Matrix.

Definition 3.3 *A theoretical pairwise comparison matrix is perfectly reciprocal and consistent.*

$$a_{ij} = \frac{1}{a_{ji}}, a_{ij} > 0, a_{ik} = a_{ij}a_{jk}, i, j, k = 1, 2, \cdots, n.$$

It could also easily seen that if the matrix is positive and consistent, then

$$a_{ii} = 1, i = 1, 2, \cdots, n$$

Practical Pairwise Comparison Matrix.

In practice, when decision maker deals with intangibles, either the available judgment could be not sufficient or human judgment could be influenced by many factors which may not be necessarily important for the decision itself. Thus it is more likely the to be inconsistent in judgment. Consistency or a near consistency enhance the validity of the alternative priorities which could then be evaluated in a consistent matrix frame.

$$a_{ij} = \frac{1}{a_{ji}}, a_{ij} > 0, a_{ik} \approx a_{ij}a_{jk}, i, j, k = 1, 2, \cdots, n.$$

We continue with the same previous example. Allen prefers apple(A) three times to banana(B), prefers banana twice to orange(C), and prefer apple five times to orange. Their relative preference in a mathematical form is expressed as A=3B, B=2C, A=5C. Then Allen's preference on alternatives writes in a pairwise comparison form is given,

Fruit	Apple(A)	Banana(B)	Orange(C)
Apple(A)	1	$\frac{1}{3}$	$\frac{1}{5}$
Banana(B)	3	1	$\frac{1}{2}$
Orange(C)	5	2	1

From the above example, we may easily find out in the matter of preference, Allen prefers apple three times to banana, prefers banana twice to orange, so Allen 'should' prefers apple six times to orange, because A=3B, B=2C, which A=6C. However from his own judgment, he would make a five times preference on apple to orange. The difference between the two may cause a problem in consistency about his preference. Thus it is important to distinct a theoretical pairwise comparison matrix from a practical pairwise comparison matrix, and it's also important to find this difference and how it affects the result. The analysis of the 'acceptable errors' between them falls in the consistency of pairwise comparison which is discussed in detail in the later section.

3.3 Perron-Frobenius Theorem

In an arbitrary pairwise comparison matrix, we may find that eigenvector provides the priority ordering, and the eigenvalue is a measure of the consistency. And we already know that it exists

$$Aw = \lambda w$$

for any positive pairwise comparison matrix, such that the corresponding non-linearly independent eigenvector $(w_1, w_2, \cdots, w_n)$.

Theorem 3.4 *Every matrix of positive elements has positive eigenvalue of multiplication 1 which is larger than the absolute value of every other eigenvalue, the elements of the corresponding eigenvector are positive numbers and are determined up to the multiplication with a scalar.*

[4]

The theorem could be expressed in another way: An arbitrary positive matrix A has a real positive eigenvalue λ_n such that

$$r \geqslant |\lambda_i|,$$

for any eigenvalue λ_i of A. Furthermore, there is a positive eigenvector corresponding to λ_n. (The eigenvalue λ_n is called the maximal eigenvalue of A, and a positive eigenvector corresponding to λ_n is called a maximal eigenvector of A.)

Proof. Let A be an irreducible positive $n * n$ square matrix. Then it exists a vector w^0 such that

$$f_A(w^0) \geqslant f_A(w),$$

for all w in E^n. Let

$$\lambda_n = f_A(w^0),$$

that is,

$$\lambda_n = \{\max f_A(w) | x \in E^n\}.$$

We first show that λ_n is positive. Let $w = (1, 1, \cdots, 1)/n$. Then

$$\lambda_n \geqslant f_A(u)$$

$$= \min_i \frac{(Au)_i}{u_i}$$

$$= \min_i \sum_{j=1}^{n} a_{ij}$$

$$> 0,$$

since A has a nonzero row number. Next we show that λ_n is an eigenvalue of A. We certainly have

$$Aw^0 - \lambda w^0 \geqslant 0.(1)$$

Suppose that

$$Aw^0 - \lambda w^0 \neq 0.$$

Then,

$$(I_n + A^{n+1})(Aw^0 - \lambda w^0) > 0,$$

that is,

$$Ay^0 - \lambda_n y^0 > 0, (2)$$

where $y^0 = (I_n + A)^{n-1}x^0$. Since (2) is a strict inequality there exists a sufficient small positive number ε such that

$$Ay^0 - (\lambda_n + \varepsilon)y^0 \geqslant 0.$$

But it also exists that

$$\lambda + \varepsilon \leqslant f_A(y^0),$$

therefore,

$$\lambda \leqslant f_A(y^0),$$

which is a contradiction to the maximality of λ_n. Here (1) is an equality, λ_n is an eigenvalue, and w^0 is a nonnegative eigenvector corresponding to λ_n. Note that $w > 0$. Next, let $Aw = \lambda_i w$ where $w = (w_1, w_2, \cdots, w_n) > 0$. Then

$$\lambda_i Z_i = \sum_{j=1}^{n} a_{ij} z_{ij}, i, j = 1, 2, \cdots, n,$$

therefore we have

$$|\lambda_i||w_j| \leqslant \sum\nolimits_{j=1}^{n} a_{ij}|w_j|, i, j = 1, 2, \cdots, n.(3)$$

which is also

$$|\lambda_i||w| \leqslant A|w|.$$

By the definition of λ_n,

$$|\lambda_i| \leqslant f_A(|w|) \leqslant \lambda_n.$$

3.4 Eigenvector Method

There exists several pairwise comparison methods not based on Perron's theorem, but also widely used in AHP, eigenvector method is the most prominent one among them. In 1980 Dr. Satty demonstrated mathematically, the solution of eigenvector method was the best approach.

In practice, it is demonstrated that the practical pairwise comparison matrices are not consistent in many cases during the solution of decision problems, therefore a method to derive the vector of priorities from the matrix A, and extend the pairwise comparison to make it fit for the desired matrix class is desired.

As defined earlier, of any positive pairwise comparison matrix A, its elements a_{ij} satisfy $a_{ij}a_{jk} = a_{ik}$ with $i, j, k > 0$ in a perfect consistent case. With $a_{ij} = \frac{w_i}{w_j}, i, j = 1, 2, \cdots, n$ already known, we may have the following equation

$$\sum_{j=1}^{n} a_{ij}w_j = nw_i \ i = 1, 2, \cdots, n$$

this is equivalent to

$$Aw = nw$$

with full equation writes

$$A \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} = n \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix}.$$

However in practice, the entry a_{ij} are not based on exact measurement,however subjective judgments, it means there are small variations of a_{ij} from consistent case, which could be written as $a_{ij}a_{ik} \approx a_{ik}$.

By giving such a pairwise comparison square matrix A, we want to find a polynomial whose roots are precisely the eigenvalues of A. For a diagonal matrix A, the characteristic polynomial is easy to define: it exists such a corresponding eigenvector $w_1, w_2, \cdots, w_n$, the diagonalized matrix A writes

$$A = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}$$

We also find with diagonal entries $a_{ii} = 1$, it holds for

$$\lambda_1 + \lambda_2 + \cdots + \lambda_n = n.$$

This works because the diagonal entries are also the eigenvalues of this matrix.[5]

$$Av = \lambda v,$$

$$A \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = \lambda \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$$

In matrix theory, this means in a perfect consistent case, it exists a largest eigenvalue λ which is equal to n, with the corresponding eigenvector $(v_1, v_2, \cdots, v_n)$. The rest of the eigenvalues are all 0.

In practice, the small variations of the entries incurred keeps the largest eigenvalue $\lambda_{\max}$ close to n, the remaining eigenvalues close to 0. Now the problem becomes to find such a dominating eigenvector which satisfies

$$Aw = \lambda_{\max} w,$$

how good the dominating eigenvector would estimate w and how the small variance would affect the eigenvector.

Theorem 3.5 *Let $\lambda_{\max}$ be the dominating eigenvector of A, and let w be its corresponding right eigenvector with $\sum_{n=1}^{n} w_i = 1.\mu \equiv \frac{\lambda_{\max}-n}{n-1}$ is a measure of the average departure from consistency. [6]*

Proof. Let $a_{ij} = \left(\frac{w_i}{w_j}\right) \varepsilon_{ij}$, substituting in the expression for $\lambda_{\max}$, we may find the following,

$$\lambda_{\max} = \sum_{j=1}^{n} a_{ij} \frac{w_j}{w_i} = \sum_{j=1}^{n} \varepsilon_{ij},$$

$$n\lambda_{\max} = \sum_{i,j=1}^{n} \varepsilon_{ij} = n + \sum_{1 \leqslant i < j \leqslant n} \left(\varepsilon + \frac{1}{\varepsilon}\right),$$

$$\frac{\lambda_{\max} - n}{n-1} = -1 + \frac{1}{n(n-1)} \sum_{1 \leqslant i < j \leqslant n} \left(\varepsilon + \frac{1}{\varepsilon}\right)$$

As $\varepsilon_{ij} \to 1, \mu \to 0$, we may get a closer to the consistency. If we write $\varepsilon_{ij} = 1 + \delta_{ij}$, with $\delta_{ij} > -1$, above equation writes

$$\frac{\lambda_{\max} - n}{n-1} = \frac{1}{n(n-1)} \sum_{1 \leqslant i < j \leqslant n} \left(\delta_{ij}^2 - \frac{\delta_{ij}^3}{1+\delta_{ij}} \right)$$

$$\frac{\lambda_{\max} - n}{n-1} = \frac{1}{n(n-1)} \sum_{1 \leqslant i < j \leqslant n} \frac{\delta_{ij}^2}{1+\delta_{ij}}$$

Since $a_{ij} > 0$ with $\delta_{ij} > -1$, the above equation gives the result

$$\lambda_{\max} \geqslant n.$$

We may also find the 2μ is the variance of error incurred in estimating a_{ij}.

We also have the Viete's formulas,

$$a_{11} + a_{22} + \cdots + a_{nn} = \lambda_1 + \lambda_2 + \cdots + \lambda_n,$$

which is the say the diagonal entries equates the coefficients of λ^{n-1} in the general and diagonalized form.

In our case, $tr(A) = n$, then we have $a_{11} + a_{22} + \cdots + a_{nn} = \lambda_1 + \lambda_2 + \cdots + \lambda_n = n$. Without loss of generality that the dominating eigenvalue is λ_n, also called the maximum eigenvalue $\lambda_{\max}$, with $\lambda_n = n - (\lambda_1 + \lambda_2 + \cdots + \lambda_{n-1})$. It is used to estimate the consistency as it reflects the proportionality of preferences. The closer $\lambda_{\max}$ is to n, the more consistent is the result. The deviation from consistency could be represented by dividing $n-1$, which is the Consistency Index(C.I.), which is further discussed in the next section.

3.5 Consistency of Pairwise Comparison

Inconsistency Index

Based on Perron's theorem, the maximum eigenvalue $\lambda_{\max}$ of a positive reciprocal matrix should equal to the consistency of this matrix. However we notice from the example of Allen's preference on fruits that, A=3B, B=2C, A=5C. The third equation automatically should be A=6C deduced from the first two equation by transitivity, with the verbal meaning that Allen 'should' be six times preferring apple to orange instead of five times. This seems to fail in the consistency of pairwise comparison matrices. In practice, it happens that pairwise comparison matrices are not consistent and thus we introduce the **Consistency Index(C.I.)** to measure their inconsistency and in what extent the inconsistency could still be considered as acceptable.

n	1	2	3	4	5	6	7	8	9	10
Random Index	0.00	0.00	0.58	0.90	1.12	1.24	1.32	1.41	1.45	1.49

Table 3.5.1

Definition 3.6 *We call this deviation from consistency represented by* $\frac{\lambda_{\max}-n}{n-1}$ *the* **consistency Index(C.I.)***; The average of randomly generated pairwise comparison matrix named* **Random Index(R.I.)***, with the fraction CI divide by RI the* **Consistency Ratio(C.R.)**.

From the definition, we have the formula which could be used in calculating:

$$CI = \frac{\lambda_{\max} - n}{n-1} (1)$$

$$CR = \frac{CI}{RI} (2)$$

As we stated a little earlier, the result may obtain more consistency if $\lambda_{\max}$ is closer to n. A further comparison is made by using the difference $\lambda_{\max} - n$ divided by $n-1$, with R.I., we may then get the estimation of inconsistency ratio of this arbitrary pairwise comparison decision problem. IR would be generally considered acceptable or good in decision software Expert's Choice, if it's less than 0.10.

The inconsistency index could be further expressed,

$$\frac{\lambda_{\max} - n}{n-1} = \frac{n - \sum_{i=1}^{n-1} \lambda_i - n}{n-1} = \frac{\sum_{i=1}^{n-1} \lambda_i}{n-1},$$

$\sum_{i=1}^{n} \lambda_i = n$, while $\lambda_1 \leqslant 2 \leqslant \cdots \leqslant \lambda_n, i = 1, 2, \cdots, n$, are the eigenvalues of A.

3.6 Different Models (Distributive, Ideal, Ratings)

Distributive AHP Model

Distributive AHP model is one of the most popular ways to rank the alternatives based on pairwise comparison. It uses relative measurement method, which is to normalize alternative value under each criterion so that their sum is one. Distributive Model is used when there is dependence among the alternatives.

The expression of distributive AHP model is given as,

$$x_j^D = \sum_{i=1}^{m} \frac{w_i}{W} \frac{a_{ij}}{\sum_{k=1}^{n} a_{ik}} = \sum_{i=1}^{m} \left(\frac{w_i}{W} \frac{1}{\sum_{k=1}^{n} a_{ik}} \right) a_{ij}, j = 1, 2, \cdots, n.$$

while $w = \sum_{i=1}^{m} w_i$.

Ideal AHP Model

Ideal AHP model is much of the similar to the distributive AHP model, except it's expressed by dividing the value of each alternative by the value of the dominating alternative under each criterion. It is mostly used when the difference between alternatives is not so clear.

Similarly, we have the expression of ideal AHP model,

$$x_j^I = \sum_{i=1}^{m} \frac{w_i}{W} \frac{a_{ij}}{\max_k a_{ik}} = \sum_{i=1}^{m} \left(\frac{w_i}{W} \frac{1}{\max_k a_{ik}} \right) a_{ij}, j = 1, 2, \cdots, n.$$

Rating AHP Model

Unlike distributive and ideal AHP model, rating AHP model use an absolute measurement method, is also widely accepted in solving decision making problems. We have,

$$x_j^R = \sum_{i=1}^{m} \frac{w_i}{w} \frac{1}{a_i^*} A_{ij}, j = 1, 2, \cdots, n.$$

3.7 Rank Reversal

In real life decision making problems, the conditions or constraints may change from time to time considering the change of elements of certain decision which we may think about introduce a new alternative to the existing decision. However if the newly introduced alternative is 'dominating', it may cause rank reversal, which means the decision maker may ignore the overall value of the alternative, obviously it would cause a serious problem. In this literature, we would also discuss possible methods to avoid rank reversal by introducing alternatives with same order of magnitude, and stability index of rank reversal in the AHP ideal models.

Theorem 3.7 *If we use the distributive AHP model,it is possible to introduce a new alternative so that the original alternatives be ordered by the criterion C_1 only.[7]*

Proof. The arbitrary multicriteria decision problem of distributive AHP model writes in a matrix form,

$Weight$	$Criterion$	$Alternative$			
(w)	(C)	A_1	A_2	$\cdots$	A_n
w_1	C_1	a_{11}	a_{12}	$\cdots$	a_{1n}
w_2	C_2	a_{21}	a_{22}	$\cdots$	a_{2n}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
w_n	C_n	a_{m1}	a_{m2}	$\cdots$	a_{mn}

Table 3.7.1

using relative measurement method, we may have

$$\alpha_{ij} = \frac{w_i}{w} \frac{a_{ij}}{\sum_{k=1}^{n} a_{ik}}$$

for all $i = 1, 2, \cdots, m; j = 1, 2, \cdots, n$, while $w = \sum_{i=1}^{m} w_i$, and

$$C_i^{pq} = \alpha_{ip} - \alpha_{iq} = \frac{w_i}{w} \frac{a_{ip}}{\sum_{k=1}^{n} a_{ik}} - \frac{w_i}{w} \frac{a_{iq}}{\sum_{k=1}^{n} a_{ik}},$$

for all $p, q \in \{1, 2, \cdots, n\}$.

Then we have the value difference of original order,

$$x_p - x_q = \sum_{i=1}^{m} C_i^{pq} = \sum_{i=1}^{m} \frac{w_i}{w} \left(\frac{a_{ip}}{\sum_{k=1}^{n} a_{ik}} - \frac{a_{iq}}{\sum_{k=1}^{n} a_{ik}} \right),$$

note that x_p, x_q are the aggregated values on the p^th and q^th alternative respectively; $x_p^{'}$ and $x_q^{'}$ are their respective aggregated value after introducing the new alternative A_{n+1}.

Suppose that we introduce a new alternative A_{n+1}. Let $a_{i,n+1} > 0$ be the value of the new alternative on the i^{th} criterion, the new arbitrary multicriteria decision problem with $n + 1$ alternatives and m criteria writes,

$Weight$	$Criterion$	$Alternative$				
(w)	(C)	A_1	A_2	$\cdots$	A_n	A_{n+1}
w_1	C_1	a_{11}	a_{12}	$\cdots$	a_{1n}	$a_{1,n+1}$
w_2	C_2	a_{21}	a_{22}	$\cdots$	a_{2n}	$a_{2,n+1}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
w_n	C_n	a_{m1}	a_{m2}	$\cdots$	a_{mn}	$a_{m,n+1}$

Table 3.7.2

(1) suppose $a_{ip} > a_{iq}$, which means that in the i^{th} criterion A_p is better than A_q, with $p, q \in \{1, 2, \cdots, n\}$, we have

$$x_p^{'} - x_q^{'} = \frac{w_l}{w} \frac{l}{\sum a_{lk} + a_{l,n+1}} (a_{lp} - a_{lq}) > 0,$$

which means by taking new alternative $n+1$ in to account, the aggregated values of A_p is better than A_q. And for fixed $a_{l,n+1}$ we can also tell, we could decide the distinct values A_p and A_q by only considering the distinct value of p^{th} and q^{th} on the l^{th} criterion C_l, and the weight of l^{th} criterion w_l, as well as the aggregated values of a_{lj} on the l^{th} criterion, where $j \in \{1, 2, \cdots, n\}$.

(2) Easily we can see we can have the same result when $a_{lp} < a_{lq}$.

3.8 Group Decision-Making

It is usually the case that in modern society that for a decision problem, it consists several or even a group decision makers, company shareholders for example. With possibly different opinions, it increases the complexity of the decision process, which means computability needs to be concerned. In AHP pairwise comparison, this means we need to reduce the solutions of group decision problems to solution of individual decision problems. This is done by aggregating the pairwise comparison matrices of the individual decision makers. Aczel and Saaty showed us the finding in 1983 by using quasi arithmetic mean.

Theorem 3.8 *Consider an open interval $I \subset R_{++}$, where $R_{++} = x \in R : x > 0$. Now R^l_{++} is the positive orthant in R^l and $I^l = I \times I \times \cdots \times I$. Let $\phi : I \to R$ be a continuous and strictly increasing function and let the function $f : I \to R^l_{++}$ be of the form of a quasiarithmetical mean, i.e.*

$$f(y_1, \cdots, y_l) = \phi^{-1}\left(\frac{1}{n}\sum_{c=1}^{l} \phi(y_l)\right), \ (1)$$

If the following axioms are satisfied:

1)Reciprocal condition $f\left(\frac{1}{y_1}, \cdots, \frac{1}{y_l}\right) = \frac{1}{f(y_1, \cdots, y_l)}$, *where*

$$(y_1, \cdots, y_l), f\left(\frac{1}{y_1}, \cdots, \frac{1}{y_l}\right) \in I^l. \ (2)$$

2)Homogeneity condition

$$f(sy_1, \cdots, sy_l) = sf(y_1, \cdots, y_l) \ (4)$$

whenever $s > 0$, and $(y_1, \cdots, y_l), (sy_1, \cdots, sy_l) \in I^l$. Then, f should be of the form:

$$f(y_1, \cdots, y_l) = \prod_{c=1}^{l} y_c^{\frac{1}{l}} = \sqrt[l]{y_1, \cdots, y_l}, l \geqslant 2, (y_1, \cdots, y_l) \in I^l. \ (4)$$

This result is essential to the project and can be used as $l = 5 (> 2)$. It will ensure that a group decision can be made and the individual results found remain consistent within the AHP.

[8]

Proof. Let's Prove the expression (1) first. Define the influences of the individual judgments in a mapping,

$$f(y_1, y_2, \cdots, y_l) = g(y_1) \circ g(y_2) \circ \cdots \circ g(y_l)), \ (5)$$

where $\circ$ is a associative,continuous operation and in the predefined open interval $I \subset R_{++}$ the variables $y_1, y_2, \cdots, y_l$ take values. It is known for all continuous cancellative and associative operators on a real interval R_{++} could be expressed in the following form

$$y_i \circ y_j = \phi^{-1}[\phi(y_i) + \phi(y_j)] \ (y_i, y_j \in I, i \neq j). \ (6)$$

where $\phi : I \rightarrow R$ is an arbitrary continuous, strictly monotonic function. From (5) and (6), we now get

$$f(y_1, y_2, \cdots, y_l) = \phi^{-1}\left(\sum_{k=1}^{n} \phi[g(y_i)]\right) \ (7)$$

It is also easily to see that if all judgments have the same value y, the synthesized judgment should be y as well:

$$f(y, y, \cdots, y) = y, \ (8)$$

with $y \in I$ Combined with (7), we have

$$g(y) = \phi^{-1}\left[\frac{1}{n}\phi(y)\right], \ (9)$$

The expression (7) could also be further written in the quasi arithmetic mean form,

$$f(y_1, y_2, \cdots, y_l) = \phi^{-1}\left(\sum_{i=1}^{l} \phi(y_l)\right),$$

where $(y_1, y_2, \cdots, y_l) \in I^l \subset R^l$. We may also get a mapping $I \to nR$ from (9), that

$$R = \phi[g(I)] = \frac{1}{n}\phi(I).$$

Here R share the same openness properties as I, with both y_i and its reciprocal element $\frac{1}{y_i}$ in I, which satisfies (2) the reciprocal condition. Among the quasi arithmetic means, only the geometric mean

$$f(y_1, y_2, \cdots, y_l) = \prod_{i=1}^{l} y_i^{\frac{1}{l}}, l \geqslant 2, (y_1, y_2, \cdots, y_l) \in I^l, \quad (10)$$

satisfies the homogeneity condition (4) and reciprocal condition. Thus (10) is the standard form we would use for a group decision-making problem.

3.9 Sensitivity and Stability

Sensitivity Analysis

Sensitivity analysis is well known for its importance in the validation and calibration of numerical models. It is a tool to check the robustness of the final outcome against changes to the previous status, which could help reduce uncertainty.

Sensitivity analysis of a multicriteria decision problem is the analysis on the results, when small change is made to alternatives, either their weights or values with respect to criteria. In our case, we use sensitivity analysis to help us find out how much is changed in AHP distributive model after eliminating the dominating the dominating alternative to achieve the third goal. We use a relative measurement based on comparison of two status before and after introducing or eliminating an alternative, identifying criteria which is sensitive and dependable to the change of alternative in the model. This could be written in the following expression,

$$SA = \frac{\frac{x_i'}{x_j'}}{\frac{x_i}{x_j}} = \frac{x_i'}{x_j'}\frac{x_j}{x_i}$$

, where x_i is the value assigned to alternative A_i, x_j the value of alternative A_j of the original status, and $x_i{}'$ the value of alternative A_i, $x_j{}'$ the value of alternative A_j in the latter status respectively. The result indicates how much change on alternatives, which numerically is seen by its closeness to 1.

Stability of models after eliminating the 'dominating' alternative.[9]

When considering a multicriteria decision making problem, we usually have three goals: to eliminate some worst alternatives, to choose some best alternatives, or to rank the alternatives. We already know the values of weights and alternative with respect to criteria is usually approximate, it then becomes important for us to get the same or very close solutions. This problem is called stability. In our case, we analyze stability of the rank of alternatives to achieve the third goal.

We can see from last section, rank reversal caused by the value of newly introduced alternative dominating on certain criterion, which will change the original order could be harmful. Then a method is designed to introduce an artificially created area which the value of this new alternative on certain criterion could take value within to keep the original order of alternatives. We call it the stability of the order of alternatives which the evaluation vector of the new alternative can move without causing rank reversal, with the so-called 'maximal box' the artificially created area for the new alternative to take values within.

Again, we first start from giving an arbitrary multicriteria decision problem with n alternatives, we have,

$Weight$	$Criterion$	$Alternative$					
(w)	(C)	A_1	$\cdots$	A_j	A_{j+1}	$\cdots$	A_n
w_1	C_1	a_{11}	$\cdots$	a_{1j}	$a_{1,j+1}$	$\cdots$	a_{1n}
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
w_n	C_n	a_{1m}	$\cdots$	a_{mj}	$a_{m,j+1}$	$\cdots$	a_{mn}

Table 3.7.3

which $A_1, A_2, \cdots, A_n$ are n alternatives, $C_1, C_2, \cdots, C_m$ are m criterion, $w_1, w_2, \cdots, w_m$ are the weights of criterion $C_1, C_2, \cdots, C_m$ respectively. we have

$$x_j = \sum_{i=1}^{m} \widehat{w}_i \widehat{a}_{ij}, j = 1, 2, \cdots, n,$$

where

$$\widehat{w}_i = \frac{w_i}{\sum_{l=1}^{m} w_l}$$

and

$$\widehat{a}_{ij} = \frac{a_{ij}}{\sum_{k=1}^{n} a_{ik}}, \left(\widehat{a}_{ij} = \frac{a_{ij}}{\max\{a_{ik}, k = 1, 2, \cdots, n\}}\right).$$

when introducing a new alternative Z. Let $Z_i > 0$ be the value of Z on the i^{th} criterion. The new aggregated values will be

$$x_j^{'} = \sum_{i=1}^{m} \widehat{w}_i \widehat{a}_{ij}^{'}, j = 1, 2, \cdots, n,$$

then

$$\widehat{a}_{ij}^{'} = \frac{a_{ij}}{Z_i + \sum_{i=1}^{n} a_{ik}}, \left(\widehat{a}_{ij} = \frac{a_{ij}}{\max\{Z_i, \max(a_{ik}, k = 1, 2, \cdots, n)}\right)$$

$Weight$	$Criterion$	$Alternative$						
(w)	(C)	A_1	$\cdots$	A_j	A_{j+1}	$\cdots$	A_n	$Z = A_{n+1}$
w_1	C_1	a_{11}	$\cdots$	a_{1j}	$a_{1,j+1}$	$\cdots$	a_{1n}	$Z_1 = a_{1,n+1}$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
w_i	C_i	a_{i1}	$\cdots$	a_{ij}	$a_{i,j+1}$	$\cdots$	a_{in}	$Z_i = a_{i,n+1}$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
w_n	C_n	a_{1m}	$\cdots$	a_{mj}	$a_{m,j+1}$	$\cdots$	a_{mn}	$Z_m = a_{m,n+1}$

Table 3.7.4

As we already have

$$\widehat{a}_{ij}^{'} = \frac{1}{Z_i + \sum_{i=1}^{n} a_{ik}} a_{ij},$$

and

$$Z_i\,(Z_i > 0) \mapsto \frac{1}{Z_i + \sum_{i=1}^{n} a_{ik}}$$

is a monotone decreasing function, which we may have

$$\lim_{Z_i \to \infty} \frac{1}{Z_i + \sum_{i=1}^{n} a_{ik}} = 0.$$

So the new aggregated value is

$$x_j^{'} = \sum_{i=1}^{m} \widehat{w}_i a_{ij} \frac{1}{Z_i + \sum_{i=1}^{n} a_{ik}},$$

We set a permutation τ of the set $(1, 2, \cdots, n)$ by the decreasing function

$$\frac{1}{Z_i + \sum_{i=1}^{n} a_{ik}},$$

we may have $x_{\tau(1)} \geqslant x_{\tau(2)} \geqslant \cdots x_{\tau(n)}$, we can see the use of this permutation in the later part of this paper.

Denote by $D(E)$ the artificially created 'minimal' ('maximal') alternative of components

$$d_i = \min\{a_{ij}, i = 1, 2, \cdots, m; j = 1, 2, \cdots, n\},$$

$$e_i = \max\{a_{ij}, i = 1, 2, \cdots, m; j = 1, 2, \cdots, n\}.$$

Theorem 3.9 *The following relations hold:*

$$\alpha_j = \max(\lambda \in I : \sum_{i \in N_i} w_i(a_{i\tau(j)} - a_{i,\tau(j+1)})\phi_i(e_i\lambda) + \sum_{i \in N_j} w_i(a_{i\tau(j)} - a_{i,\tau(j+1)})\phi_i\left(\frac{d_i}{\lambda}\right) \geqslant 0,$$

and

$$\alpha = min\left\{a_{j:}j = 1, 2, \cdots, n\right\},$$

for $j = 1, 2, \cdots, n$.

We define the function $\varphi_j : I \to R$ *by*

$$\varphi_j(\lambda) = \sum_{i \in P_j} w_i(a_{i\tau(j)} - a_{i,\tau(j+1)})\phi_i(e_i\lambda) + \sum_{i \in N_j} w_i(a_{i\tau(j)} - a_{i,\tau(j+1)})\phi_i\left(\frac{d_i}{\lambda}\right)$$

It is easy to see that ψ_j *is decreasing and by using* $lim_{t \to +\infty}\phi_i(t) = 0$, *we have*

$$\lim_{\lambda \to +\infty} \psi_j(\lambda) \leqslant 0$$

for all $j = 1, 2, \cdots, n$.

If $\psi(\sqrt{\max\{\frac{d_i}{e_i} : i = 1, 2, \cdots, m})}) < 0$, *then by convention we put* $\alpha_j = -\infty$ *(since the supremum of an empty set is* $-\infty$.*) In this case we have* $\alpha = -\infty$.

We first set the ratio λ as the order of magnitudes of the alternative Z_i and alternatives $A_{ij}(j = 1, 2, \cdots, n)$,

$$\lambda \in \left[\sqrt{max\{\frac{d_i}{e_i} : i = 1, 2, \cdots, m\}, +\infty} \right],$$

Definition 3.10 *To any $\lambda \geqslant 0, \frac{d_i}{\lambda} < \lambda e_i, Z_i \in (\frac{d_i}{\lambda_i}, \lambda_i e_i), i = 1, 2, \cdots, m,$ we take the maximal box $(\frac{d_i}{\lambda_i}, \lambda_i e_i)$ in which the evaluation vector of the new alternative can move without creating rank reversal, with λ the* **Stability Index,** *with respect to the order $x_{\tau(j)} \geqslant x_{\tau(j+1)}, j \in \{1, 2, \cdots, n-1\}$.*

then by using the last convention get from the theorem above, we have:

Theorem 3.11 *If $\psi_j \left(\sqrt{\max\{\frac{d_i}{e_i} : i = 1, 2, \cdots, m\}} \right) \geqslant 0$, then α_j is the smallest root*[2] *of the equation,*

$$\sum_{i \in P_j} w_i(a_{i\tau(j)} - a_{i,\tau(j+1)})\phi_i(e_i\lambda) + \sum_{i \in N_j} w_i(a_{i\tau(j)} - a_{i,\tau(j+1)})\phi_i \left(\frac{d_i}{\lambda} \right) = 0.$$

If $\psi_j \left(\sqrt{\max\{\frac{d_i}{e_i} : i = 1, 2, \cdots, m\}} \right) < 0$, then $\alpha_j = -\infty$.

we ma have such inequality

$$\lambda_{\max} \geqslant \sqrt{\frac{d_i}{e_i}}.\lambda^2 > \frac{d_i}{e_i}.$$

The idea is to find what we called the **Global stability** α, which holds $\alpha = \min\{\lambda_j : j = 1, 2, \cdots, n\}$,

Proof. similar to case 1,combine the situation (1) and (2),

$$P_j = \{i \in \{1, 2, \cdots, m\} : a_{i\tau(j)} \geqslant a_{i,\tau(j+1)}\},$$

$$N_j = \{i \in \{1, 2, \cdots, m\} : a_{i\tau(j)} \leqslant a_{i,\tau(j+1)}\},$$

and we have,

$$\alpha_j = \max\{\lambda \in I : \min\{x'_{\tau(j)} - x'_{\tau(j+1)} : Z_i \in I^i_\lambda, i = 1, 2, \cdots, m\} \geqslant 0\},$$

where $I = \left(\sqrt{\max\{\frac{d_i}{e_i} : i = 1, 2, \cdots, m\}, +\infty}\right)$ and $I^i_\lambda = [\frac{d_i}{\lambda}, e_i\lambda]$,

$$\alpha_j = \lambda \in I : \sum_{i \in P_j} w_i(a_{i\tau(j)} - a_{i,\tau(j+1)}) \frac{1}{e_i\lambda + \sum_{i=1}^n a_{ik}} +$$

$$\sum_{i \in N_j} w_i(a_{i\tau(j)} - a_{i,\tau(j+1)}) \frac{1}{\frac{d_i}{\lambda} + \sum_{i=1}^n a_{ik}} \geqslant 0,$$

where $Z_i \in \left(\frac{d_i}{\lambda}, e_i\lambda\right)$.

Since we are trying to find the smallest root of α_j, we have

$$\sum_{i \in P_j} w_i(a_{i\tau(j)} - a_{i,\tau(j+1)}) \frac{1}{e_i\lambda + \sum_{i=1}^n a_{ik}} + \sum_{i \in N_j} w_i(a_{i\tau(j)} - a_{i,\tau(j+1)}) \frac{1}{\frac{d_i}{\lambda} + \sum_{i=1}^n a_{ik}} = 0$$

which is,

$$\sum_{i \in P_j} \left(\frac{w_i}{w} \frac{1}{e_i\lambda + \sum_{i=1}^n a_{ik}}\right) (a_{i\tau(j)} - a_{i,\tau(j+1)}) + \sum_{i \in N_j} \left(\frac{w_i}{w} \frac{1}{\frac{d_i}{\lambda} + \sum_{i=1}^n a_{ik}}\right) (a_{i\tau(j)} - a_{i,\tau(j+1)}) = 0$$

$$\frac{w_i}{w}(a_{i\tau(j)} - a_{i\tau(j+1)}) \left(\frac{1}{e_i\lambda + \sum_{i=1}^n a_{ik}} + \frac{1}{\frac{d_i}{\lambda} + \sum_{i=1}^n a_{ik}}\right) = 0.$$

With $a_{i\tau(j)}, a_{i,\tau(j+1)}, w_i, w$, are the given values and weights, we may have the following equations,

$$\frac{1}{e_i\lambda + \sum_{i=1}^{n} a_{ik}} + \frac{1}{\frac{d_i}{\lambda} + \sum_{i=1}^{n} a_{ik}} = 0$$

$$e_i\lambda + \sum_{i=1}^{n} a_{ik} = -(\frac{d_i}{\lambda} + \sum_{i=1}^{n} a_{ik})\,(1)$$

$$\frac{1}{e_i\lambda + \sum_{i=1}^{n} a_{ik}} = 0\,(2)$$

$$\frac{1}{\frac{d_i}{\lambda} + \sum_{i=1}^{n} a_{ik}} = 0\,(3)$$

we have,

$$x'_{\tau(j)} - x'_{\tau(j+1)} = \sum f_{i\tau(j)}\phi_i(Z_i) = \sum_{i=1}^{m}(b_{i\tau(j)} - b_{i,\tau(j+1)})\phi_i(Z_i)$$

$$x'_{\tau(j)} - x'_{\tau(j+1)} = \sum_{i=1}^{m}(\widehat{w}_i a_{i\tau(j)} - \widehat{w}_i a_{i,\tau(j+1)})\phi_i(Z_i) = \sum_{n+1}^{m} \frac{w_i}{\sum_{i=1}^{m} w_l}(a_{i\tau(j)} - a_{i,\tau(j+1)})\phi_i(Z_i)$$

We already know that the weight $w_l^{l=1} > 0$ is given, so that the aggregated weights $\sum_{l=1}^{m} w_l$ could be easily calculated. Then the above equation could be expressed as,

$$\frac{1}{\sum_{l=1}^{m} w_l}\sum_{i=1}^{m} w_i(a_{i\tau(j)} - a_{i\tau(j+1)})\phi_i(Z_i) \geqslant 0,$$

$$\frac{1}{\sum_{l=1}^{m} w_l}\left(\sum_{i\in P_j} w_i(a_{i\tau(j)} - a_{i,\tau(j+1)})\phi_i(Z_i)\right) - \frac{1}{\sum_{l=1}^{m} w_l}\left(\sum_{i\in N_j} w_i(a_{i,\tau(j+1)} - a_{i\tau(j)})\phi_i(Z_i)\right) \geqslant 0.$$

As $\phi_i : Z_i \mapsto \frac{1}{Z_i + \sum_{k=1}^{n} a_{ik}}$ is a monotone decreasing function when $Z_i \in [\frac{d_i}{\lambda}, e_i\lambda] > 0$, when d_i, e_i, λ are positive value as defined,

$$\lim_{Z_i \to +\infty} \phi_i = \lim_{Z_i \to +\infty} \frac{1}{Z_i + \sum_{k=1}^{n} a_{ik}} = 0$$

$$\psi_j(\lambda) = \sum_{i \in P_j} w_i(a_{i\tau(j)} - a_{i,\tau(j+1)})\phi_i(e_i\lambda) - \sum_{i \in N_j} (a_{i,\tau(j+1)} - a_i\tau(j))\phi_i\left(\frac{d_i}{\lambda}\right) = 0,$$

which is the smallest root of the equation,

with $\lim_{e_i\lambda \to +\infty} \phi_i(e_i\lambda) = 0, \lim_{\frac{d_i}{\lambda} \to +\infty} \left(\frac{d_i}{\lambda}\right) = 0,$

Then,

$$\frac{1}{\sum_{l=1}^{m} w_l}\left(\sum_{i \in P_j} w_i(a_{i\tau(j)} - a_{i,\tau(j+1)})\phi_i(e_i\lambda)\right) - \frac{1}{\sum_{l=1}^{m} w_l}\left(\sum_{i \in N_j} w_i(a_{i,\tau(j+1)} - a_{i\tau(j)})\phi_i\left(\frac{d_i}{\lambda}\right)\right) = 0.$$

Now we only need to solve this equation of λ with $w_l, w_i, a_i, \tau(j), \tau(j+1), e_i, d_i$ all given, with global stability α_j being the only variable to be calculated.

Lemma 3.12 *By Cauchy Theorem, we have* $\lim_{\lambda \to +\infty} \psi_j(\lambda) \leqslant 0$, *then to the equation*

$$\psi_j(\lambda) = \sum_{i \in P_j} w_i(a_i\tau(j) - a_{i,\tau(j+1)})\phi_i(e_i\lambda) - \sum_{i \in N_j} w_i(a_{i,\tau(j+1)} - a_{i\tau(j)})\phi_i\left(\frac{d_i}{\lambda}\right),$$

if $\psi_j \geqslant 0$, *then* α_j *is the smallest root of equation, which is the global stability;*

if $\psi_j < 0$, *then* $\alpha_j = -\infty$;

if $a_{i\tau(j)} \geqslant a_{i,\tau(j+1)}$ *for* $i = 1, 2, \cdots, m, A_{\tau(j)}$ *Pareto dominates* $A_{\tau(j+1)}$, *we then have* $\alpha_j = +\infty$, *which means the order of alternatives* $A_{\tau(j)}, A_{\tau(j+1)}$ *doesnt change when a new alternative is introduced, the existing order is already Pareto ordered and global stability* α_j *is not necessary to be introduced in this case.*

We could also find similar result could be obtained when a alternative is eliminated for the proofs above.

Chapter 4

Case Studies

Every year there are thousands of oversea students choose to study in UK, it becomes a decision making problem to choose a proper university among a wide range of UK universities and institutes, due to a lack of information. For international student who don't have the opportunity to live in UK before they come to study in UK to have an overall image about UK universities, for example, by attending the university open days to get an insight of the design of courses in the department and facilities on campus. Most of their knowledge about universities comes from two sources.

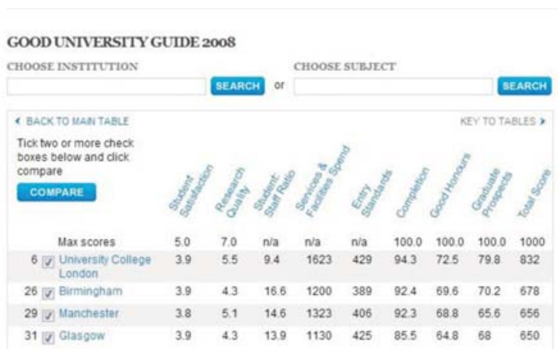
Firstly, annual 'best university ranking'. medias best known about university rankings include Times and the Guardian.

Secondly, information comes from Internet, UK universities discussion board. It includes opinions from people who is studying or has studied in UK institutes, some information could be provided based on their personal experiences.

Based on these information source, people who intend to attend these universities could get a relative rational measurement while considering their own conditions. The motivation of this project is by introducing AHP method, it would assist overseas students in selecting a proper UK university. As the alternatives includes many different universities and departments, in our case, we choose the mathematics department in four different the University of Birmingham, the University of Glasgow, University College London and the University of Manchester from four UK major cities as the candidate universities to narrow down the alternatives, which could also be seen as 'a tale of four cities'.

1. **Birmingham University**
2. **Glasgow University**
3. **Manchester University**
4. **UCL**

We may find their positions from 'Good University Guide 2008' and 'Good University Guide 2010' on Times Online website respectively.[10]



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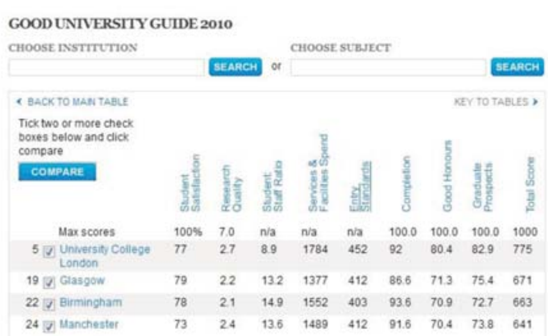
Tick two or more check boxes below and click compare

COMPARE

		Student Satisfaction	Research Quality	Student: Staff Ratio	Services & Facilities Spend	Entry Standards	Completion	Good Honours	Graduate Prospects	Total Score
	Max scores	5.0	7.0	n/a	n/a	n/a	100.0	100.0	100.0	1000
6	University College London	3.9	5.5	9.4	1623	429	94.3	72.5	79.8	832
26	Birmingham	3.9	4.3	16.6	1200	389	92.4	69.6	70.2	678
29	Manchester	3.8	5.1	14.6	1323	406	92.3	68.8	65.6	656
31	Glasgow	3.9	4.3	13.9	1130	425	85.5	64.8	68	650



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COMPARE

		Student Satisfaction	Research Quality	Student: Staff Ratio	Services & Facilities Spend	Entry Standards	Completion	Good Honours	Graduate Prospects	Total Score
	Max scores	100%	7.0	n/a	n/a	n/a	100.0	100.0	100.0	1000
5	University College London	77	2.7	8.9	1784	452	92	80.4	82.9	775
19	Glasgow	79	2.2	13.2	1377	412	86.6	71.3	75.4	671
22	Birmingham	78	2.1	14.9	1552	403	93.6	70.9	72.7	663
24	Manchester	73	2.4	13.6	1489	412	91.6	70.4	73.8	641

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MATHEMATICS

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COMPARE

		Pure		Statistics	Entry Standards	Graduate Prospects	Total Score
9	University College London	5B	5B	5B	462	82%	89.5
17	Glasgow	5D	5A	5A	418	68%	83.7
23	Manchester	5A	5A	4B	427	59%	81.9
31	Birmingham	5C	5B	4B	403	65%	80.6

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MATHEMATICS

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Tick two or more check boxes below and click compare

COMPARE

		Pure	Applied	Statistics	Entry Standards	National Student Survey	Graduate Prospects	Total Score
12	Glasgow	2.6	2.3	2.5	413	84%	79%	88
13	University College London	3	2.3	2.3	471	78%	82%	87.9
18	Birmingham	2.6	2.1		430	83%	76%	86.7
=41	Manchester	3	3.2	2.8	444	75%	59%	79.8

Similar ranking could be found on The Guardian website.[11]

Based on the questionnaire comes from 5 potential undergraduate students, a clustered averages were calculated using the Group Decision Making Theory given in the previous chapter. We take the result in four decimal places to increase the accuracy of calculation. Then the problem become constructing and analyzing in AHP using Expert Choice.

It's also important to know the questionnaire is designed in two parts.

For criterion 1 'University Ranking' and criterion 2 'Mathematics Ranking', the preference over each alternative are based on 'hard' data.

These data are calculated according to the positions of each university. For example, on Good University guide 2008, UCL ranks 6th while Glasgow University ranks 30th, then we woulde say UCL is 5 times better than Glasgow University. Here we would take both rankings from Time Online and the Guardian equally important, at the same time taking the rankings of the year 2008 and 2010 from each of them equally important as well, then we would calculate the average rankings by taking the geometric mean of these four. by taking the geometric mean, we would give a relative objective judgment and preference over the rankings.

For criterion 3 'Tuition Fee, Living Expenses and Funding Opportunities' and criterion 4 'Social life', are based on 'soft' judgment of respondent, as the judgment may vary from people to people.

Result

The questionnaires are distributed to 5 potential international students who want to study in mathematics to give their preference over these candidate universities.

Criterion 1: University ranking - 'Hard' data from Times Online and The Guardian. To make the rank more accurate and reflect the position of the universities in the ranking table better, we take the ranking of both the year 2008 and 2010 into consideration, while considering the ranking of each media and year equally. However modification would very according to different questionnairer.

	Bham	Glasgow	Manchester	UCL
Bham	1	1.000	1.000	0.250
Glasgow	1.000	1	1.000	0.250
Manchester	1.000	1.000	1	0.250
UCL	4.000	4.000	4.000	1

Criterion 2: Mathematics ranking - 'Hard' data from Times Online and The Guardian. To make the rank more accurate and reflect the position of the universities in the ranking table better, we take the ranking of both the year 2008 and 2010 into consideration, while considering the ranking of each media and year equally. However modification would very according to different questionnaire r.

	Bham	Glasgow	Manchester	UCL
Bham	1	1.000	2.000	0.660
Glasgow	1.000	1	2.000	0.660
Manchester	0.500	0.500	1	0.330
UCL	1.516	1.516	3.031	1

Criterion 3: Tuition Fee, Living Expenses and Funding Opportunities - To most of the universities, expenses varied according to three main levels: Non laboratory-based courses, Laboratory-based courses, Clinical courses. Here we discuss tuition fee level in mathematics, which usually falls in non laboratory-based courses tuition fee category; Living expenses usually include housing, mostly house renting, food, transport and other including shopping, traveling etc. Most of the expenses are dependable to the living standard of the city. Funding Opportunity mainly includes overseas scholarships and other funding sources for international students.

	Bham	Glasgow	Manchester	UCL
Bham	1	1.246	0.530	2.862
Glasgow	0.803	1	0.500	2.491
Manchester	1.888	2.000	1	4.644
UCL	0.349	0.401	0.215	1

Criterion 4: Social life - It covers many activities such as city environment, tourism, night life, etc. To international students, popularity of the university in its country, career related opportunities and crime in the city are also widely considered.

	Bham	Glasgow	Manchester	UCL
Bham	1	0.758	0.871	0.322
Glasgow	1.320	1	1.149	0.401
Manchester	1.149	0.871	1	0.370
UCL	3.104	2.491	2.702	1

Using the same method, we then need to compare the criterion themselves to each other, listing their relative importance against each other. For example, we say criterion 2 mathematics ranking is three times more important to criterion 3 tuition fee and living expenses.

	Criterion 1	Criterion 2	Criterion 3	Criterion 4
Criterion 1	1	0.574	1.320	2.702
Criterion 2	1.741	1	2.297	4.704
Criterion 3	0.758	0.435	1	2.169
Criterion 4	0.370	0.213	0.461	1

We then use the software Matlab to find the maximum eigenvalue λ_{max} which gives us the listing result.

$$\begin{pmatrix} 0.4586 \\ 0.7984 \\ 0.3526 \\ 0.1674 \end{pmatrix}.$$

By definition 3.5, Consistency Ratio is calculated using equation $\frac{\lambda_{max}-n}{n-1}$, with $n = 4$ in our case. Then we C.I is $\frac{4.0006-4}{4-1} = 0.0002$, then $C.R = \frac{C.I}{R.I} = \frac{0.0002}{0.90} = 0.0002$ while R.I is 0.90 for $n = 4$. The Consistency Ratio is 0.02% which is much less than acceptable

10% by EC.

Easily we calculated the normalized eigenvector w by summing up the aboding eigenvector elements and dividing them by the sum, which we get

$$\begin{pmatrix} 0.2581 \\ 0.4493 \\ 0.1984 \\ 0.0942 \end{pmatrix}.$$

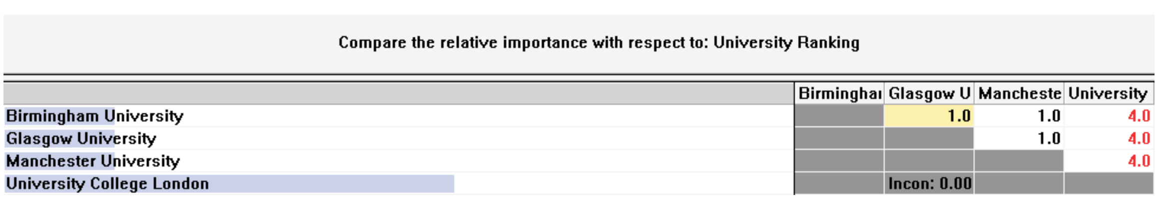
which is done by using Maple Calculator. Similar calculation could also be done using software Excel with code $= sum(A1 : A4)$ which gives the sum in $A5$ and normalized by using code $= Ai/A5, i = 1, 2, 3, 4$ respectively.

Expert Choice Results

By using the software Expert Choice, we could obtain the rank of alternatives with AHP distributive and ideal models. The goal is set as 'The use of AHP in university selection'.

We first insert the group decision results on the comparison of each criterion over another, the weighting of each criterion is automatically calculated which we could get

University Ranking: 0.2581 Maths Ranking: 0.4493 Tuition Fee, Living Expenses and Scholarship: 0.1984 Social Life: 0.0942


Compare the relative importance with respect to: University Ranking

	Birminghai	Glasgow U	Mancheste	University
Birmingham University		1.0	1.0	4.0
Glasgow University			1.0	4.0
Manchester University				4.0
University College London		Incon: 0.00		

Compare the relative importance with respect to: Maths Ranking

	Birminghai	Glasgow U	Mancheste	University
Birmingham University		1.0	2.0	1.516
Glasgow University			2.0	1.516
Manchester University				3.031
University College London		Incon: 0.00		

Compare the relative importance with respect to: Tution Fee, Living Expense & Scholarship

	Birminghai	Glasgow U	Mancheste	University
Birmingham University		1.084	1.888	2.862
Glasgow University			2.0	2.491
Manchester University				4.644
University College London		Incon: 0.00		

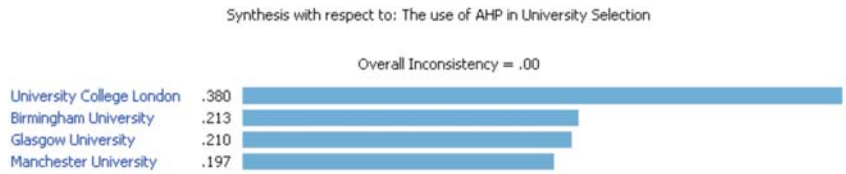
Compare the relative importance with respect to: Social Life

	Birminghai	Glasgow U	Mancheste	University
Birmingham University		1.32	1.149	3.104
Glasgow University			1.149	2.491
Manchester University				2.702
University College London		Incon: 0.00		

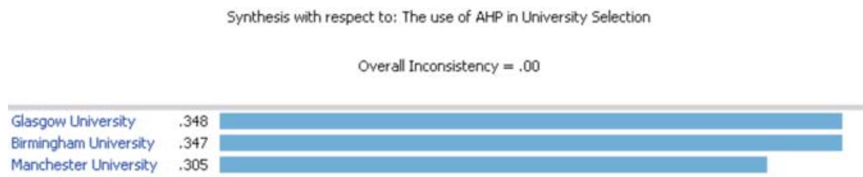
Similarly for each criterion, values are inserted in each criterion table. We could see from the table, values in black mean that the alternative on the left column is taken as 'better' or 'more important' to the corresponding row alternatives, while values in red mean the exacting the opposite. The Consistency Ratio over each criterion are all 0.00, which are strictly less than 10%, which mean the results obtained over each criterion are within the acceptable range of consistency.

The next step is to synthesize them into the overall results for both distributive model and ideal model.

Distributive Model Result



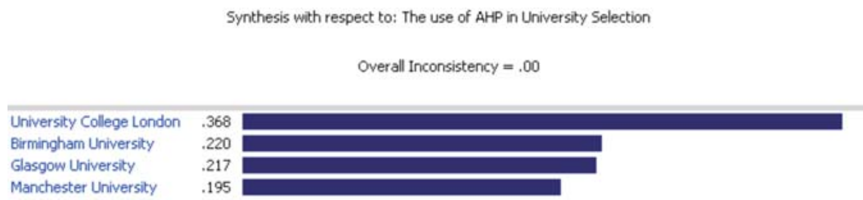
Performance after the dominating alternative is eliminated



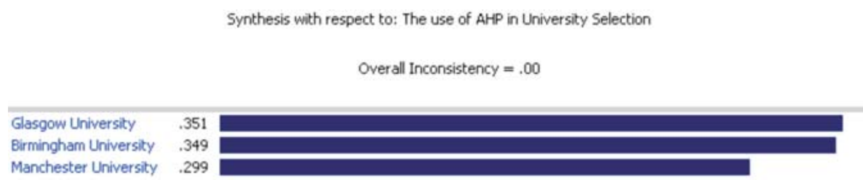
We can see the overall inconsistency is 0.00, which is again strictly less than the allowed 0.1. The result also show that School of Mathematics in University College London outperform other three maths institutes, as the best candidate for oversea student to select. We also notice that there's slight difference among the other three candidates, Birmingham University, Glasgow University and Manchester University, with Birmingham University the second best and Glasgow University the third.

It is very important for us to notice the rank of alternatives changed from Birmingham University is better than Glasgow University, to Glasgow University is better than Birmingham University after the alternative UCL is eliminating from selection choices. This is what we discussed before the rank reversal situation which is caused by the dependency of alternatives in ranking after an dominating alternative is eliminated.

Ideal Model Result



Performance after the dominating alternative is eliminated



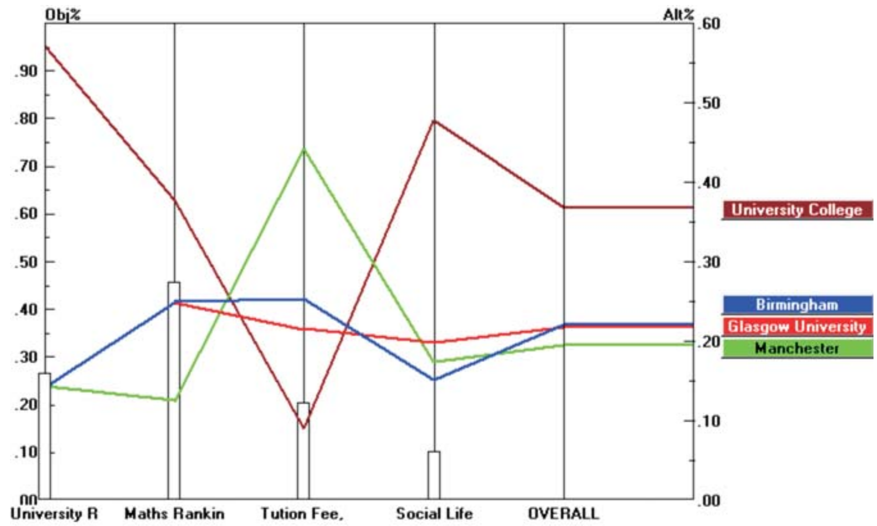
Again the overall inconsistency is 0.00, strictly less than 0.1. Again, the result shows that University College London is still ranked as the best candidate, with the same preference order UCL>Bham University>Glasgow University>Manchester University.

Similarly, rank reversal also happened after eliminating the dominating alternative UCL in ideal model.

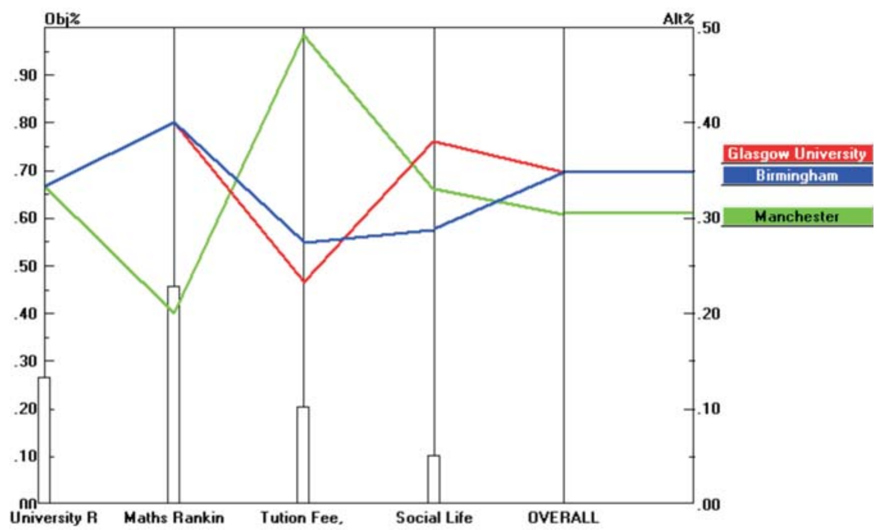
Sensitivity Analysis

We now have got the overall performance of alternatives, the next stage for us is to see whether and how much the ranking of alternatives change, when the dominating alternative is eliminated, which is performed by using sensitivity analysis. Here we are listing five different types of sensitivity analysis.

1. Performance Sensitivity



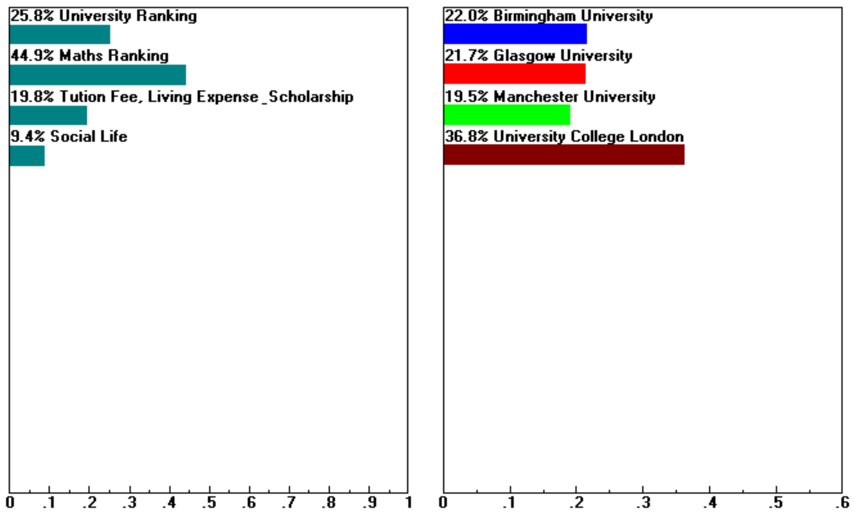
Performance after the dominating alternative is eliminated



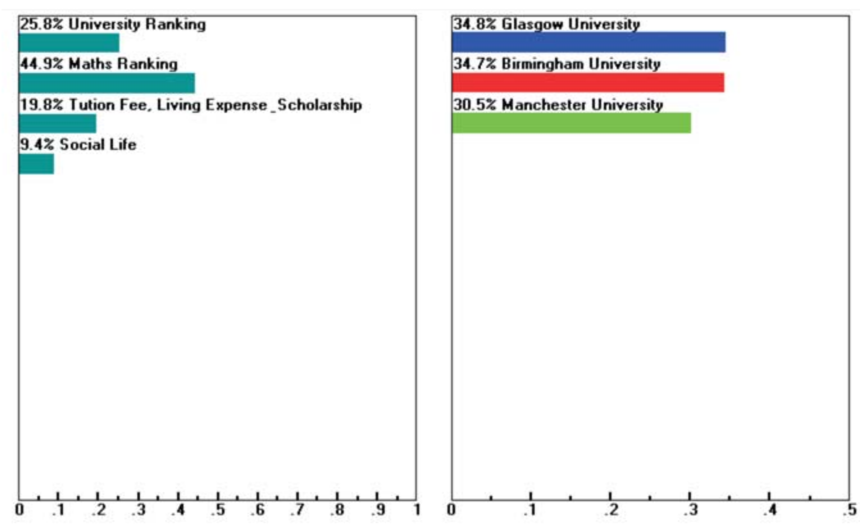
The performance sensitivity gives us the preference of each alternative over each criteria, and the overall preference at the same time. The values are determined by the priority of each criterion and are represented with rectangular bars which we could see from above graphs. The graph on the top shows us the status of preference for each alternative before the dominating alternative UCL is eliminated, and is given by a top-down sequence which

Birmingham University is slightly prior to Glasgow University; in the next graph, we can tell after the dominating alternative UCL is eliminated, we can see the sequence change to Glasgow University on top of Birmingham University, which means rank reversal occured.

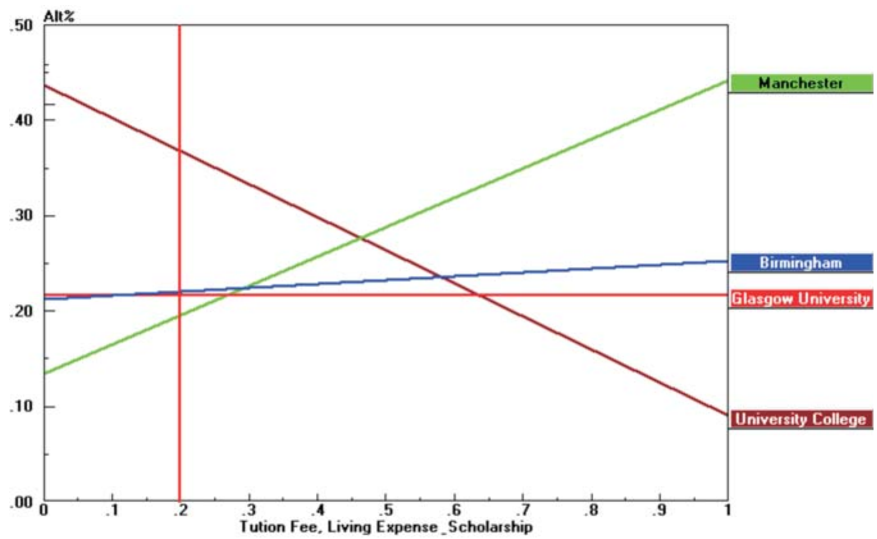
2. **Dynamic Sensitivity**



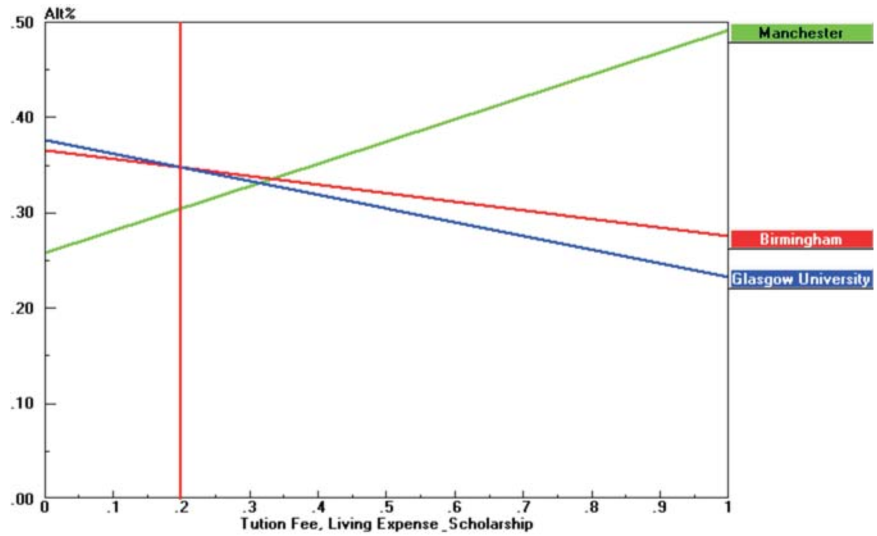
Performance after the dominating alternative is eliminated



Dynamic sensitivity analysis would show us in exact percentage of both the weight of each criterion and the overall status of preference. The priority of criterion and the alternatives are shown in the aboving two separated graphs. As we can see in the graphs, the percentage of each criterion doesn't change before and after rank reversal. However, in the first graph, Birmingham university shares a 22.0% overall preference, which has a larger percentage than Glasgow University; it changed to Glasgow University outranks Birmingham University with 0.1% in the second graph. This change of overall preference between Birmingham University and Glasgow University again demonstrates the occurrence of rank reversal. **3. Gradient Sensitivity**

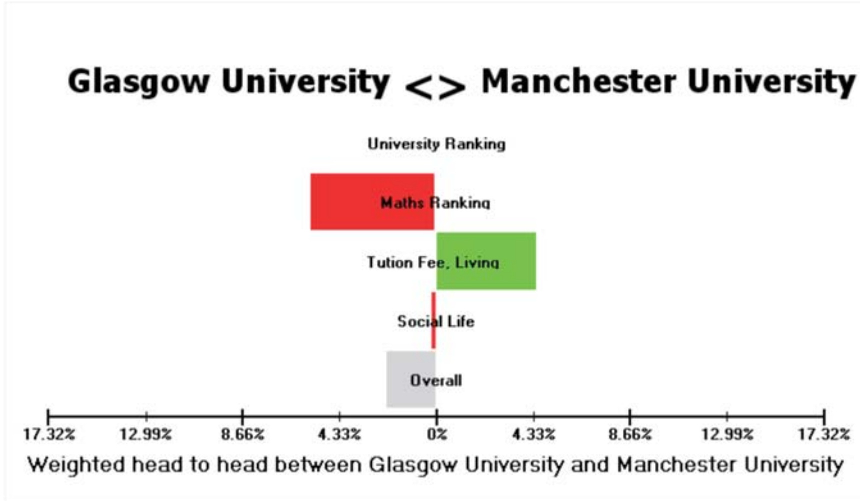


Performance after the dominating alternative is eliminated

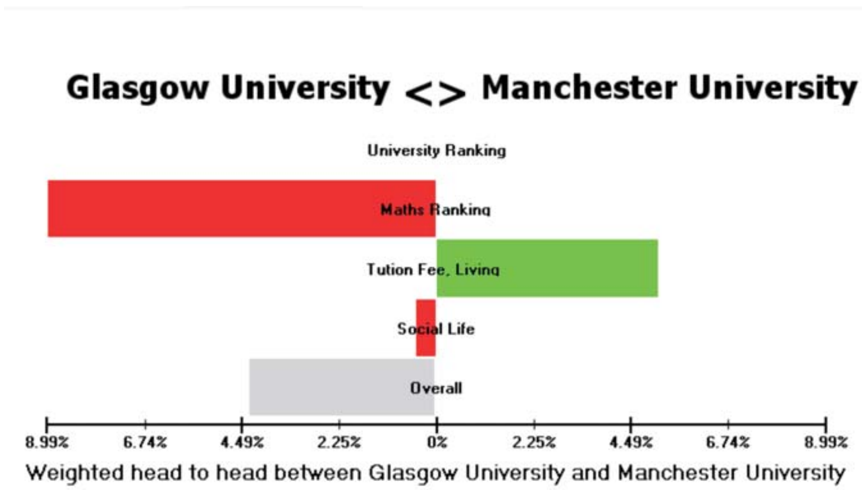


Gradient sensitivity means it's the performance of each alternative over a certain criterion. In our example, we analysis the performance over criterion 3 tuition fee, living expense and scholarship. From the first graph, we could see Manchester University have the best performance over this criterion with Birmingham University the second, Glasgow University the third. The red colour vertical line which accross with alternative lines shows how much the priority of the current criterion needs to be increased if we want to change the ranking of alternatives criterion. We can see from the aboving graph, the preference of alternatives over this criterion doesn't change before and after rank reversal, which Manchester University is the best alternative whilst Birmingham University and Glasgow University still the second and the third.

4. Head to Head Sensitivity

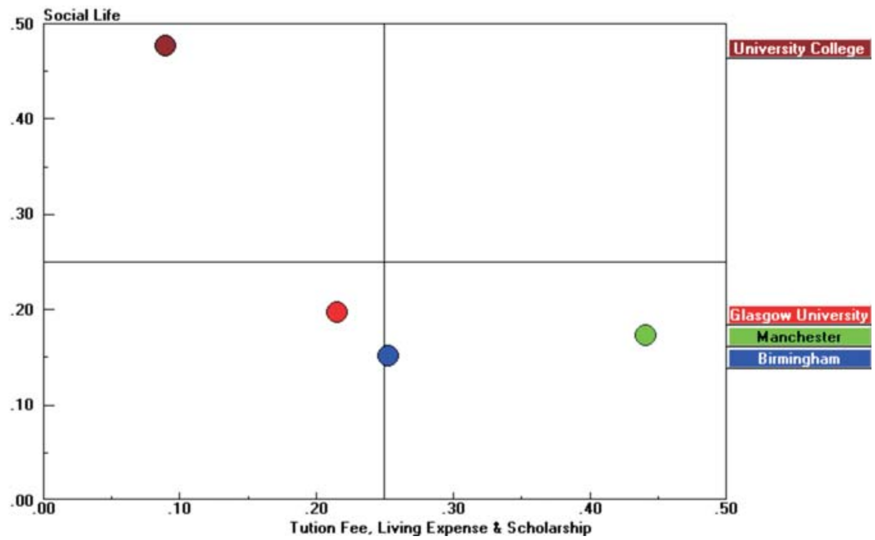


Performance after the dominating alternative is eliminated

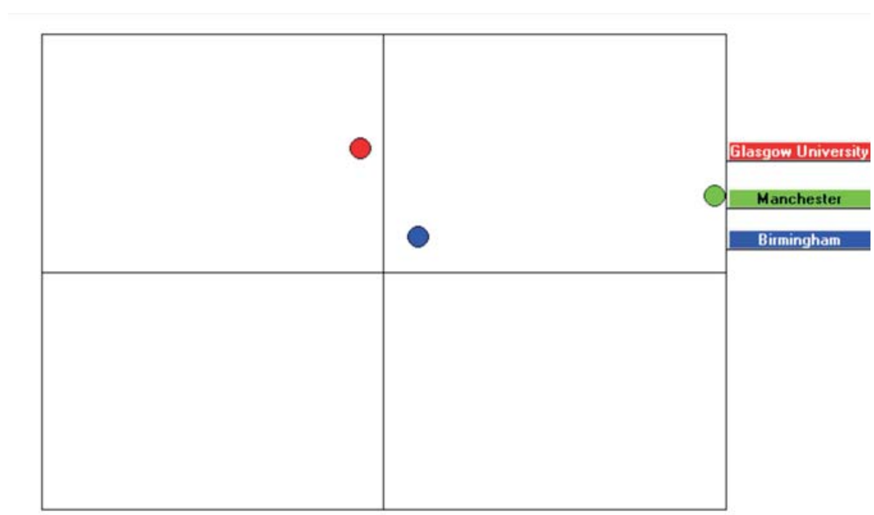


Head to head sensitivity is the analysis of two alternatives over each criterion. It uses horizontal rectangular bars to give an outperform typed view over each criterion and the overall criterion at the same time. We can see from the aboving head to head graphs that Glasgow University is much better than Manchester University over criterion mathematics ranking and slightly better than Manchester University over social life; Manchester University is dominating on criterion tuition fees, living expense and scholarship. The overall preference is towards Glasgow University.

5. Two-Dimentional Sensitivity



Performance after the dominating alternative is eliminated



This analysis is different, which analyze on the performance analysis over two chosen criterion, with the performance of each alternative marked as a point on the graph. In our case, we choose criterion tuition fee, living expense and scholarship and social life as the

X and Y axises. From the aboving graphs we can easily find out, UCL is the best over criterion social life whilst Manchester University dominates in tuition fee, living expense and scholarship.

However using pareto optimality analysis, we may find out Birmingham University and Glasgow University are overall optimal solutions. Further distinction of optimality between Birmingham University and Glasgow University could be found using other types of sensitivity analysis.

After listing the aboving five types of sensitivity analysis, our next step is to test and calculate the degree of sensitivity after eliminating the dominating alternative UCL, by comparing the performance of any other two alternatives, using a comparison of values over this two alternatives before and after rank reversal.

$$\frac{x_2{}'}{x_1{}'}\frac{x_1}{x_2} = \frac{0.233}{0.275}\frac{0.213}{0.210} = 0.859,$$

We also have

$$\frac{x_3{}'}{x_1{}'}\frac{x_1}{x_3} = \frac{0.492}{0.275}\frac{0.213}{0.197} = 1.934,$$

Usually we define the degree of change by see how close the result is to 1. If the number obtained is suffiently large enough than 1, for example 20, then we would say the ranking of alternatives is greatly changed.

In our case for the aboving results, we would give the conclusion that there is very slight change of between A_1 and A_2, and the change between A_1 and A_3 is also not extinct.

Chapter 5

Conclusion

In my project, we gave the history of decision making, and some well known historical event about decision making happened in human history. Two most popular methods with Analytical Hiechachy Process(AHP) the American school method along with PROMETHEE the European school method are then given as nowadays the most popular methods widely used. This paper focus on the the models used in AHP rather than PROMETHEE method.

It continues with a more detailed explanation of AHP and pairwise comparison, which AHP method is based on. With the elements of pairwise comparison are positive, it maintains the characters of both reciprocal and consistent, we gave the standard pairwise matrix form and what is behind the pairwise comparison. Perron-Frobenius theorem is cited to prove pairwise comparison in the frame of matrix analysis. In the last section, four different eigenvector method is given as complementarity method to pairwise comparison method which based on Perron-Frobenius theorem.

Later, distributive and ideal models are given as the most powerful tools in AHP pairwise comparison problems. We gives their definitions and mathematical expressions respectively. We then proved the stability of the order in section two, by introducing a new alternative, and it might happen that the original order of alternative will change.

Moreover it is possible to introduce a new alternative, such that the order of the alternatives will be give by almost any criterion. Some real life examples are also given to show in real life this could be harmful. I gave the detailed proof of 'maximal box method' based on my supervisor Dr. Nemeth's previous work.

These theorems and proofs are then reflected in the results of selected example, which was run from EC.

The results also show that: The differences between the first two optimal results are not quite significant;

No significant differences neither in telling the differences between distributive and ideal model, which alternative is better or optimal; however, we may notice ideal shows slightly preference between the two results.

When introducing a new alternative in the distributive or ideal model,order of the original alternatives could be determined by a insignificant criterion. To avoid manipulations, decision makers are supposed to neglect it rather than take it into consideration. If this criterion is significant, we need to use another model instead.

For the models mentioned in the paper, we may also find out that both models are not strong enough in distinguishing the difference between the two alternatives for priority, further research could be done. So, it is another important issue we should always consider to choose a proper model.

Chapter 6

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Chapter 7

Appendix A

Appendix A - Design of The Questionnaire.

Verbal Scale	Numerical Values
Equally important/likely/preferred	1
Moderately more important/likely/preferred	3
Strongly more important/likely/preferred	5
Very Strongly important/likely/preferred	7
Extremely important/likely/preferred	9
Compromised intermediate values	$2, 4, 6, 8$

Criterion 1: University ranking - 'Hard' data from Times Online and The Gardian. To make the rank more accurate and reflect the position of the universities in the ranking table better, we take the ranking of most recent two years into consideration, while considering the ranking of each media and year equally. However modification would very according to different questionnaire r.

	Bham	Glasgow	Manchester	UCL
Bham	1			
Glasgow		1		
Manchester			1	
UCL				1

Criterion 2: Mathematics ranking - 'Hard' data from Times Online and The Gardian. To make the rank more accurate and reflect the position of the universities in the ranking table better, we take the ranking of both the year 2008 and 2010 into consideration, while considering the ranking of each media and year equally. However modification would very according to different questionnaire r.

	Bham	Glasgow	Manchester	UCL
Bham	1			
Glasgow		1		
Manchester			1	
UCL				1

Criterion 3: Tuition Fee, Living Expenses and Funding Opportunities - To most of the universities, expenses varied according to three main levels: Non laboratory-based courses, Laboratory-based courses, Clinical courses. Here we discuss tuition fee level in mathematics, which usually falls in non laboratory-based courses tuition fee category; Living expenses usually include housing, mostly house renting, food, transport and other including shopping, traveling etc. Most of the expenses are dependable to the living standard of the city. Funding Opportunity mainly includes overseas scholarships and other funding sources for international students.

	Bham	Glasgow	Manchester	UCL
Bham	1			
Glasgow		1		
Manchester			1	
UCL				1

Criterion 4: Social life - It covers many activities such as city environment, tourism, night life, etc. To international students, popularity of the university in its country, career related opportunities and crime in the city are also widely considered.

	Bham	Glasgow	Manchester	UCL
Bham	1			
Glasgow		1		
Manchester			1	
UCL				1

Using the same method, we then need to compare the criterion themselves to each other, listing their relative importance against each other. For example, we say criterion 2 mathematics ranking is three times more important to criterion 3 tuition fee and living expenses.

	Criterion 1	Criterion 2	Criterion 3	Criterion 4
Criterion 1	1			
Criterion 2		1		
Criterion 3			1	
Criterion 4				1

Chapter 8

Appendix B

Appendix B - Questionnaire

Student 1

Criterion 1: University ranking - 'Hard' data from Times Online and The Gardian. To make the rank more accurate and reflect the position of the universities in the ranking table better, we take the ranking of both the year 2008 and 2010 into consideration, while considering the ranking of each media and year equally. However modification would very according to different questionnaire r.

	Bham	Glasgow	Manchester	UCL
Bham	1	1	1	$\frac{1}{4}$
Glasgow	1	1	1	$\frac{1}{4}$
Manchester	1	1	1	$\frac{1}{4}$
UCL	4	4	4	1

Criterion 2: Mathematics ranking - 'Hard' data from Times Online and The Gardian. To make the rank more accurate and reflect the position of the universities in the ranking table better, we take the ranking of both the year 2008 and 2010 into consideration, while considering the ranking of each media and year equally. However modification would very

according to different questionnaire r.

	Bham	Glasgow	Manchester	UCL
Bham	1	1	2	$\frac{1}{2}$
Glasgow	1	1	2	$\frac{1}{2}$
Manchester	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{4}$
UCL	2	2	4	1

Criterion 3: Tuition Fee, Living Expenses and Funding Opportunities - To most of the universities, expenses varied according to three main levels: Non laboratory-based courses, Laboratory-based courses, Clinical courses. Here we discuss tuition fee level in mathematics, which usually falls in non laboratory-based courses tuition fee category; Living expenses usually include housing, mostly house renting, food, transport and other including shopping, traveling etc. Most of the expenses are dependable to the living standard of the city. Funding Opportunity mainly includes overseas scholarships and other funding sources for international students.

	Bham	Glasgow	Manchester	UCL
Bham	1	1	$frac12$	2
Glasgow	1	1	$\frac{1}{2}$	2
Manchester	2	2	1	4
UCL	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{4}$	1

Criterion 4: Social life - It covers many activities such as city environment, tourism, night life, etc. To international students, popularity of the university in its country, career related opportunities and crime in the city are also widely considered.

	Bham	Glasgow	Manchester	UCL
Bham	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{4}$
Glasgow	2	1	1	$\frac{1}{2}$
Manchester	2	1	1	$\frac{1}{2}$
UCL	4	2	2	1

Using the same method, we then need to compare the criterion themselves to each other, listing their relative importance against each other. For example, we say criterion 2 mathematics ranking is three times more important to criterion 3 tuition fee and living expenses.

	Criterion 1	Criterion 2	Criterion 3	Criterion 4
Criterion 1	1	1	2	4
Criterion 2	1	1	2	4
Criterion 3	$\frac{1}{2}$	$\frac{1}{2}$	1	2
Criterion 4	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	1

Student 2

Criterion 1: University ranking - 'Hard' data from Times Online and The Gardian. To make the rank more accurate and reflect the position of the universities in the ranking table better, we take the ranking of both the year 2008 and 2010 into consideration, while considering the ranking of each media and year equally. However modification would very according to different questionnaire r.

	Bham	Glasgow	Manchester	UCL
Bham	1	1	1	$\frac{1}{4}$
Glasgow	1	1	1	$\frac{1}{4}$
Manchester	1	1	1	$\frac{1}{4}$
UCL	4	4	4	1

Criterion 2: Mathematics ranking - 'Hard' data from Times Online and The Gardian. To make the rank more accurate and reflect the position of the universities in the ranking table better, we take the ranking of both the year 2008 and 2010 into consideration, while considering the ranking of each media and year equally. However modification would very according to different questionnaire r.

	Bham	Glasgow	Manchester	UCL
Bham	1	1	2	1
Glasgow	1	1	2	1
Manchester	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$
UCL	1	1	2	1

Criterion 3: Tuition Fee, Living Expenses and Funding Opportunities - To most of the universities, expenses varied according to three main levels: Non laboratory-based courses, Laboratory-based courses, Clinical courses. Here we discuss tuition fee level in mathematics, which usually falls in non laboratory-based courses tuition fee category; Living expenses usually include housing, mostly house renting, food, transport and other including shopping, traveling etc. Most of the expenses are dependable to the living standard of the city. Funding Opportunity mainly includes overseas scholarships and other funding sources for international students.

	Bham	Glasgow	Manchester	UCL
Bham	1	2	1	4
Glasgow	$\frac{1}{2}$	1	$\frac{1}{2}$	2
Manchester	1	2	1	3
UCL	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{3}$	1

Criterion 4: Social life - It covers many activities such as city environment, tourism, night life, etc. To international students, popularity of the university in its country, career related opportunities and crime in the city are also widely considered.

	Bham	Glasgow	Manchester	UCL
Bham	1	$\frac{1}{2}$	1	$\frac{1}{3}$
Glasgow	2	1	2	$\frac{1}{2}$
Manchester	1	$\frac{1}{2}$	1	$\frac{1}{3}$
UCL	3	2	3	1

Using the same method, we then need to compare the criterion themselves to each other, listing their relative importance against each other. For example, we say criterion 2 mathematics ranking is three times more important to criterion 3 tuition fee and living expenses.

	Criterion 1	Criterion 2	Criterion 3	Criterion 4
Criterion 1	1	$\frac{1}{2}$	1	2
Criterion 2	2	1	2	4
Criterion 3	1	$\frac{1}{2}$	1	2
Criterion 4	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{2}$	1

Student 3

Criterion 1: University ranking - 'Hard' data from Times Online and The Gardian. To make the rank more accurate and reflect the position of the universities in the ranking table better, we take the ranking of both the year 2008 and 2010 into consideration, while considering the ranking of each media and year equally. However modification would very according to different questionnaire r.

	Bham	Glasgow	Manchester	UCL
Bham	1	1	1	$\frac{1}{4}$
Glasgow	1	1	1	$\frac{1}{4}$
Manchester	1	1	1	$\frac{1}{4}$
UCL	4	4	4	1

Criterion 2: Mathematics ranking - 'Hard' data from Times Online and The Gardian. To make the rank more accurate and reflect the position of the universities in the ranking table better, we take the ranking of both the year 2008 and 2010 into consideration, while considering the ranking of each media and year equally. However modification would very according to different questionnaire r.

	Bham	Glasgow	Manchester	UCL
Bham	1	1	2	$\frac{1}{2}$
Glasgow	1	1	2	$\frac{1}{2}$
Manchester	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{4}$
UCL	2	2	4	1

Criterion 3: Tuition Fee, Living Expenses and Funding Opportunities - To most of the universities, expenses varied according to three main levels: Non laboratory-based courses, Laboratory-based courses, Clinical courses. Here we discuss tuition fee level in mathematics, which usually falls in non laboratory-based courses tuition fee category;

Living expenses usually include housing, mostly house renting, food, transport and other including shopping, traveling etc. Most of the expenses are dependable to the living standard of the city. Funding Opportunity mainly includes overseas scholarships and other funding sources for international students.

	Bham	Glasgow	Manchester	UCL
Bham	1	1	$\frac{1}{3}$	2
Glasgow	1	1	$\frac{1}{2}$	4
Manchester	3	2	1	5
UCL	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{5}$	1

Criterion 4: Social life - It covers many activities such as city environment, tourism, night life, etc. To international students, popularity of the university in its country, career related opportunities and crime in the city are also widely considered.

	Bham	Glasgow	Manchester	UCL
Bham	1	1	2	$\frac{1}{3}$
Glasgow	1	1	2	$\frac{1}{3}$
Manchester	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{6}$
UCL	3	3	6	1

Using the same method, we then need to compare the criterion themselves to each other, listing their relative importance against each other. For example, we say criterion 2 mathematics ranking is three times more important to criterion 3 tuition fee and living expenses.

	Criterion 1	Criterion 2	Criterion 3	Criterion 4
Criterion 1	1	$\frac{1}{2}$	$\frac{1}{2}$	3
Criterion 2	2	1	1	6
Criterion 3	2	1	1	6
Criterion 4	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{6}$	1

Student 4

Criterion 1: University ranking - 'Hard' data from Times Online and The Gardian. To make the rank more accurate and reflect the position of the universities in the ranking table better, we take the ranking of both the year 2008 and 2010 into consideration, while considering the ranking of each media and year equally. However modification would very according to different questionnaire r.

	Bham	Glasgow	Manchester	UCL
Bham	1	1	1	$\frac{1}{4}$
Glasgow	1	1	1	$\frac{1}{4}$
Manchester	1	1	1	$\frac{1}{4}$
UCL	4	4	4	1

Criterion 2: Mathematics ranking - 'Hard' data from Times Online and The Gardian. To make the rank more accurate and reflect the position of the universities in the ranking table better, we take the ranking of both the year 2008 and 2010 into consideration, while considering the ranking of each media and year equally. However modification would very according to different questionnaire r.

	Bham	Glasgow	Manchester	UCL
Bham	1	1	2	$\frac{1}{2}$
Glasgow	1	1	2	$\frac{1}{2}$
Manchester	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{4}$
UCL	2	2	4	1

Criterion 3: Tuition Fee, Living Expenses and Funding Opportunities - To most of the universities, expenses varied according to three main levels: Non laboratory-based courses, Laboratory-based courses, Clinical courses. Here we discuss tuition fee level in mathematics, which usually falls in non laboratory-based courses tuition fee category; Living expenses usually include housing, mostly house renting, food, transport and other including shopping, traveling etc. Most of the expenses are dependable to the living standard of the city. Funding Opportunity mainly includes overseas scholarships and other funding sources for international students.

	Bham	Glasgow	Manchester	UCL
Bham	1	$\frac{1}{2}$	$\frac{1}{2}$	3
Glasgow	2	1	1	6
Manchester	2	1	1	6
UCL	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{6}$	1

Criterion 4: Social life - It covers many activities such as city environment, tourism, night life, etc. To international students, popularity of the university in its country, career related opportunities and crime in the city are also widely considered.

	Bham	Glasgow	Manchester	UCL
Bham	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{4}$
Glasgow	2	1	1	$\frac{1}{2}$
Manchester	2	1	1	$\frac{1}{2}$
UCL	4	2	2	1

Using the same method, we then need to compare the criterion themselves to each other, listing their relative importance against each other. For example, we say criterion 2 mathematics ranking is three times more important to criterion 3 tuition fee and living expenses.

	Criterion 1	Criterion 2	Criterion 3	Criterion 4
Criterion 1	1	$\frac{1}{2}$	2	2
Criterion 2	2	1	4	4
Criterion 3	$\frac{1}{2}$	$\frac{1}{4}$	1	1
Criterion 4	$\frac{1}{2}$	$\frac{1}{4}$	1	1

Student 5

Criterion 1: University ranking - 'Hard' data from Times Online and The Gardian. To make the rank more accurate and reflect the position of the universities in the ranking table better, we take the ranking of both the year 2008 and 2010 into consideration, while considering the ranking of each media and year equally. However modification would very according to different questionnaire r.

	Bham	Glasgow	Manchester	UCL
Bham	1	1	1	$\frac{1}{4}$
Glasgow	1	1	1	$\frac{1}{4}$
Manchester	1	1	1	$\frac{1}{4}$
UCL	4	4	4	1

Criterion 2: Mathematics ranking - 'Hard' data from Times Online and The Gardian. To make the rank more accurate and reflect the position of the universities in the ranking table better, we take the ranking of both the year 2008 and 2010 into consideration, while considering the ranking of each media and year equally. However modification would very according to different questionnaire r.

	Bham	Glasgow	Manchester	UCL
Bham	1	1	2	1
Glasgow	1	1	2	1
Manchester	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$
UCL	1	1	2	1

Criterion 3: Tuition Fee, Living Expenses and Funding Opportunities - To most of the universities, expenses varied according to three main levels: Non laboratory-based courses, Laboratory-based courses, Clinical courses. Here we discuss tuition fee level in mathematics, which usually falls in non laboratory-based courses tuition fee category; Living expenses usually include housing, mostly house renting, food, transport and other including shopping, traveling etc. Most of the expenses are dependable to the living standard of the city. Funding Opportunity mainly includes overseas scholarships and other funding sources for international students.

	Bham	Glasgow	Manchester	UCL
Bham	1	3	$\frac{1}{2}$	4
Glasgow	$\frac{1}{3}$	1	$\frac{1}{4}$	1
Manchester	2	6	1	6
UCL	$\frac{1}{4}$	1	$\frac{1}{6}$	1

Criterion 4: Social life - It covers many activities such as city environment, tourism, night life, etc. To international students, popularity of the university in its country, career related opportunities and crime in the city are also widely considered.

	Bham	Glasgow	Manchester	UCL
Bham	1	2	1	$\frac{1}{2}$
Glasgow	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{4}$
Manchester	1	2	1	$\frac{1}{2}$
UCL	2	4	2	1

Using the same method, we then need to compare the criterion themselves to each other, listing their relative importance against each other. For example, we say criterion 2 mathematics ranking is three times more important to criterion 3 tuition fee and living expenses.

	Criterion 1	Criterion 2	Criterion 3	Criterion 4
Criterion 1	1	$\frac{1}{2}$	2	3
Criterion 2	2	1	4	6
Criterion 3	$\frac{1}{2}$	$\frac{1}{4}$	1	2
Criterion 4	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{2}$	1

Chapter 9

Appendix C

Matlab Calculation

$>> A = [10.5741.3202.702; 1.74112.2974.704; 0.7580.43512.169; 0.3700.2130.4611]$

$A =$

1.0000	0.5740	1.3200	2.7020
1.7410	1.0000	2.2970	4.7040
0.7580	0.4350	1.0000	2.1690
0.3700	0.2130	0.4610	1.0000

$>> [V, D] = eig(A)$

$V =$

-0.4586	-0.5184	$0.4050 + 0.0117i$	$0.4050 - 0.0117i$
-0.7984	0.8550	0.6940	0.6940
-0.3526	0.0172	$-0.3142 + 0.4360i$	$-0.3142 - 0.4360i$
-0.1674	0.0018	$-0.1441 - 0.2111i$	$-0.1441 + 0.2111i$

$D =$

$$\begin{matrix} 4.0006 & 0 & 0 & 0 \\ 0 & 0.0003 & 0 & 0 \\ 0 & 0 & -0.0005 + 0.0412i & 0 \\ 0 & 0 & 0 & -0.0005 - 0.0412i \end{matrix}$$

from which we find the maximum eigenvalue is 4.0006 and the corresponding eigenvector is

$$\begin{pmatrix} 0.4586 \\ 0.7984 \\ 0.3526 \\ 0.1674 \end{pmatrix}.$$

Chapter 10

Appendix D

Group Decision Calculation

Based on the questionnaires we sent out to potential international students, a result is calculated by using the Group Decision Making method we mentioned in earlier chapter based on each criterion.

Criterion 1: University ranking. - 'Hard' data from Times Online and The Gardian. To make the rank more accurate or reflect the position of the universities in the ranking table better, we take the ranking of both the year 2008 and 2009 into consideration, while considering the ranking of each year equally. However modification would very according to different questionnaire r.

	Bham	Glasgow	Manchester	UCL
Bham	1	$\sqrt[5]{1*1*1*1*1}$	$\sqrt[5]{1*1*1*1*1}$	$\sqrt[5]{\frac{1}{4}*\frac{1}{4}*\frac{1}{4}*\frac{1}{4}*\frac{1}{4}}$
Glasgow	$\sqrt[5]{1*1*1*1*1}$	1	$\sqrt[5]{1*1*1*1*1}$	$\sqrt[5]{\frac{1}{4}*\frac{1}{4}*\frac{1}{4}*\frac{1}{4}*\frac{1}{4}}$
Manchester	$\sqrt[5]{1*1*1*1*1}$	$\sqrt[5]{1*1*1*1*1}$	1	$\sqrt[5]{\frac{1}{4}*\frac{1}{4}*\frac{1}{4}*\frac{1}{4}*\frac{1}{4}}$
UCL	$\sqrt[5]{4*4*4*4*4}$	$\sqrt[5]{4*4*4*4*4}$	$\sqrt[5]{4*4*4*4*4}$	1

Criterion 2: Mathematics ranking - 'Hard' data from Times Online and The Gardian. To make the rank more accurate and reflect the position of the universities in the ranking

table better, we take the ranking of both the year 2008 and 2010 into consideration, while considering the ranking of each media and year equally. However modification would very according to different questionnaire r.

	Bham	Glasgow	Manchester	UCL
Bham	1	$\sqrt[5]{1*1*1*1*1}$	$\sqrt[5]{2*2*2*2*2}$	$\sqrt[5]{\frac{1}{2}*1*\frac{1}{2}*\frac{1}{2}*1}$
Glasgow	$\sqrt[5]{1*1*1*1*1}$	1	$\sqrt[5]{2*2*2*2*2}$	$\sqrt[5]{\frac{1}{2}*1*\frac{1}{2}*\frac{1}{2}*1}$
Manchester	$\sqrt[5]{\frac{1}{2}*\frac{1}{2}*\frac{1}{2}*\frac{1}{2}*\frac{1}{2}}$	$\sqrt[5]{\frac{1}{2}*\frac{1}{2}*\frac{1}{2}*\frac{1}{2}*\frac{1}{2}}$	1	$\sqrt[5]{\frac{1}{4}*\frac{1}{2}*\frac{1}{4}*\frac{1}{4}*\frac{1}{2}}$
UCL	$\sqrt[5]{2*1*2*2*1}$	$\sqrt[5]{2*1*2*2*1}$	$\sqrt[5]{4*2*4*4*2}$	1

Criterion 3: Tuition Fee, Living Expenses and Funding Opportunities - To most of the universities, expenses varied according to three main levels: Non laboratory-based courses, Laboratory-based courses, Clinical courses. Here we discuss tuition fee level in mathematics, which usually falls in non laboratory-based courses tuition fee category; Living expenses usually include housing, mostly house renting, food, transport and other including shopping, traveling etc. Most of the expenses are dependable to the living standard of the city. Funding Opportunity mainly includes overseas scholarships and other funding sources for international students.

	Bham	Glasgow	Manchester	UCL
Bham	1	$\sqrt[5]{1*2*1*\frac{1}{2}*3}$	$\sqrt[5]{\frac{1}{2}*1*\frac{1}{3}*\frac{1}{2}*\frac{1}{2}}$	$\sqrt[5]{2*4*2*3*4}$
Glasgow	$\sqrt[5]{1*\frac{1}{2}*2*1*\frac{1}{3}}$	1	$\sqrt[5]{\frac{1}{2}*\frac{1}{2}*\frac{1}{2}*1*\frac{1}{4}}$	$\sqrt[5]{2*2*4*6*1}$
Manchester	$\sqrt[5]{2*1*3*2*2}$	$\sqrt[5]{2*2*2*1*6}$	1	$\sqrt[5]{4*3*5*6*6}$
UCL	$\sqrt[5]{\frac{1}{2}*\frac{1}{4}*\frac{1}{2}*\frac{1}{3}*\frac{1}{4}}$	$\sqrt[5]{\frac{1}{2}*\frac{1}{2}*\frac{1}{4}*\frac{1}{6}*1}$	$\sqrt[5]{\frac{1}{4}*\frac{1}{3}*\frac{1}{5}*\frac{1}{6}*\frac{1}{6}}$	1

Criterion 4: Social life - It covers many activities such as city environment, tourism,

night life, etc. To international students, popularity of the university in its country, career related opportunities and crime in the city are also widely considered.

	Bham	Glasgow	Manchester	UCL
Bham	1	$\sqrt[5]{\frac{1}{2} * \frac{1}{2} * 1 * \frac{1}{2} * 2}$	$\sqrt[5]{\frac{1}{2} * 1 * 2 * \frac{1}{2} * 1}$	$\sqrt[5]{\frac{1}{4} * \frac{1}{3} * \frac{1}{3} * \frac{1}{4} * \frac{1}{2}}$
Glasgow	$\sqrt[5]{2 * 2 * 1 * 2 * \frac{1}{2}}$	1	$\sqrt[5]{1 * 2 * 2 * 1 * \frac{1}{2}}$	$\sqrt[5]{\frac{1}{2} * \frac{1}{2} * \frac{1}{3} * \frac{1}{2} * \frac{1}{4}}$
Manchester	$\sqrt[5]{2 * 1 * \frac{1}{2} * 2 * 1}$	$\sqrt[5]{1 * \frac{1}{2} * \frac{1}{2} * 1 * 2}$	1	$\sqrt[5]{\frac{1}{2} * \frac{1}{3} * \frac{1}{6} * \frac{1}{2} * \frac{1}{2}}$
UCL	$\sqrt[5]{4 * 3 * 3 * 4 * 2}$	$\sqrt[5]{2 * 2 * 3 * 2 * 4}$	$\sqrt[5]{2 * 3 * 6 * 2 * 2}$	1

Using the same method, we then need to compare the criterion themselves to each other, listing their relative importance against each other. For example, we say criterion 2 mathematics ranking is three times more important to criterion 3 tuition fee and living expenses.

	Criterion 1	Criterion 2	Criterion 3	Criterion 4
Criterion 1	1	$\sqrt[5]{1 * \frac{1}{2} * \frac{1}{2} * \frac{1}{2} * \frac{1}{2}}$	$\sqrt[5]{2 * 1 * \frac{1}{2} * 2 * 2}$	$\sqrt[5]{4 * 2 * 3 * 2 * 3}$
Criterion 2	$\sqrt[5]{1 * 2 * 2 * 2 * 2}$	1	$\sqrt[5]{2 * 2 * 1 * 4 * 4}$	$\sqrt[5]{4 * 4 * 6 * 4 * 6}$
Criterion 3	$\sqrt[5]{\frac{1}{2} * 1 * 2 * \frac{1}{2} * \frac{1}{2}}$	$\sqrt[5]{\frac{1}{2} * \frac{1}{2} * 1 * \frac{1}{4} * \frac{1}{4}}$	1	$\sqrt[5]{2 * 2 * 6 * 1 * 2}$
Criterion 4	$\sqrt[5]{\frac{1}{4} * \frac{1}{2} * \frac{1}{3} * \frac{1}{2} * \frac{1}{3}}$	$\sqrt[5]{\frac{1}{4} * \frac{1}{4} * \frac{1}{6} * \frac{1}{4} * \frac{1}{6}}$	$\sqrt[5]{\frac{1}{2} * \frac{1}{2} * \frac{1}{6} * 1 * \frac{1}{2}}$	1