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q-Hook length formulas for colored labeled forests

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Introduction

The *major index* has been deeply studied from the early 1900s. A classical result due to MacMahon [14] states that the major index is equidistributed with the length function on the symmetric group. In the last thirty years, this index has been generalized in two directions.

In 1989 Björner and Wachs [7] generalized the major index defining a new statistic on labeled forests (i.e., partially ordered sets whose Hasse diagram is a rooted forest) in a very natural way. They presented in particular two q-hook length formulas: one for the major index over permutations which correspond to linear extensions of a labeled forest, and the other for the new statistic over all labelings of a fixed forest.

In the early 2000s, Adin and Roichman [2] generalized the major index for colored permutation groups $G(r, n)$, which are wreath products of the form $\mathbb{Z}_r \wr \mathcal{S}_n$, where $\mathbb{Z}_r$ is the cyclic group of order r. They called this new statistic the *flag-major index* and showed that it is equidistributed with the length function for the classical Weyl group of type B (the case $r = 2$). In 2004 Biagioli and Caselli [6] defined an analogous statistic for the Weyl groups of type D and in 2007 Bagno and Biagioli [4] extended the definition of the flag-major index for complex reflection groups $G(r, p, n)$, which can be naturally identified as normal subgroups of index p of $G(r, n)$. Finally, in 2011 Caselli [8] introduced a new family of groups $G(r, p, q, n)$, the projective reflection groups, that can be described as quotients of $G(r, p, n)$ modulo the cyclic scalar subgroup C_q. Caselli introduced also the following notion of duality, which plays a crucial role in the theory of these groups: if $G = G(r, p, q, n)$, then we denote by $G^* = G(r, q, p, n)$ the dual group of G, obtained by simply exchanging the parameters p and q. Moreover, the definition of the flag-major index is generalized for these groups in [8].

Although its nature is combinatorial, the flag-major index also has important algebraic properties, in particular in the study of the action of reflection groups on polynomial rings ([2], [1], [6], [5]). We recall a very important property of projective reflection groups G ([8]), which generalizes and unifies in a very natural way several known results for wreath products and complex reflection groups: we can describe a monomial *descent basis* for the coinvariant algebra of a projective reflection group G by its dual group G^*. More precisely, we associate to any element $g \in G^*$ a monomial of degree equal to the flag-major index of g. We remark that this is just the first instance of the strict relation between the algebraic structure of G and the combinatorics of G^*, and it is the one we refer to in the present work.

In this thesis we give new definitions of labelings of a forest, which generalize the standard type in [7] and the signed type in [10]. In our context the labels are colored integers. We generalize the major index defined in [7] introducing the *flag-major index of a colored labeled forest.* This allows us to generalize in a natural way the two hook-length formulas recalled above. As particular cases of them, we recover some known results for the distribution of the flag-major index on projective reflection groups $G^* = G(r, n)/C_p$ [8] and on sets of cosets representatives for some special subgroups of G^* [9]. Finally, the study of colored labeled forests consisting of two linear trees (which has just apparently a simple combinatoric nature) allows us to show a notion of duality, in the sense introduced in [8], for two particular families of groups obtained from the direct product $G(r, n) \times G(r, m)$.

The thesis is structured as follows. In Chapter 1 we collect some notations and preliminaries for the necessary background. In Chapter 2 and 3 we introduce colored labelings and other particular generalizations of them. We define also the flag-major index for these labelings and we present an analogue of the q-hook length formula over all linear extensions of a colored labeled forests. In Chapter 4 we give a generalized version of the second q-hook length formula presented, computing also the cardinality of the set of all colored labelings of a fixed forest. Finally, in Chapter 5 we define two families of groups obtained from the product $G(r,n) \times G(r,m)$ and we show the strict relation between the combinatorics of one family and the invariant theory of the other.

The results appearing in this thesis has been done in collaboration with prof. Fabrizio Caselli.

Contents

Chapter 1

Notations and preliminaries

1.1 Some notations

Let $\mathbb{Z}$ be the set of integer numbers and $\mathbb{N}$ the set of non-negative integers. For $a, b \in \mathbb{Z}$, $a \leq b$, we let $[a, b] := \{a, a+1, \ldots, b\}$. For $n \in \mathbb{N}$, $n \neq 0$, we let also $[n] := [1, n]$. If q is an indeterminate, we let

$$[n]_q := \frac{1-q^n}{1-q} = 1 + q + q^2 + \cdots + q^{n-1}$$

be the *q-analogue* of n, and

$$[n]_q! := [1]_q [2]_q \cdots [n]_q.$$

We let

$$\mathscr{P}_n := \{f = (f_1, f_2, \ldots, f_n) \in \mathbb{N}^n : \, f_1 \geq f_2 \geq \cdots \geq f_n\}$$

be the set of *partitions* of *length* at most n, and $|f| := f_1 + f_2 + \cdots + f_n$ the *size* of f.

Let $\mathcal{S}_n$ be the symmetric group on n letters. A permutation $\sigma \in \mathcal{S}_n$ will be denoted by $\sigma = [\sigma_1, \sigma_2, \ldots, \sigma_n]$, where $\sigma_i = \sigma(i)$ for $i \in [n]$. We denote the *number of inversions* of σ by

$$\mathrm{inv}(\sigma) := |\{(i, j) : \, 1 \leq i < j \leq n \text{ and } \sigma_i > \sigma_j\}|,$$

the *descent set* of σ by

$$\mathrm{Des}(\sigma) := \{i \in [n-1] : \, \sigma_i > \sigma_{i+1}\},$$

and the *major index* of σ by

$$\mathrm{maj}(\sigma) := \sum_{i \in \mathrm{Des}(\sigma)} i.$$

If $r \in \mathbb{N}$, $r > 0$, we let $\mathbb{Z}_r := \mathbb{Z}/r\mathbb{Z}$. We simply denote by $\boldsymbol{a}$ the class of the integer a in $\mathbb{Z}_r$, since the integer r is always fixed in each context, and by $\mathrm{res}_r\,(a)$, or equivalently by $\mathrm{res}_r\,(\boldsymbol{a})$, the smallest non-negative representative of $\boldsymbol{a}$. We recall that an *r-colored integer* is a pair $(i, \boldsymbol{a})$, denoted also $i^{\boldsymbol{a}}$, where $i \in \mathbb{N} \setminus \{0\}$ and $\boldsymbol{a} \in \mathbb{Z}_r$. We let $|i^{\boldsymbol{a}}| := i$ and $c(i^{\boldsymbol{a}}) := \boldsymbol{a}$.
Finally, we denote by ζ_r the primitive r-th root of the unity $e^{2\pi i/r}$.

1.2 Complex reflection groups and $G(r, p, n)$

Let V be a complex vector space of finite dimension n and W a finite subgroup of $GL(V)$, the group of endomorphisms of V. An element $r \in GL(V)$ is called a *pseudo-reflection* if it has finite order and its fixed point space is of codimension 1. Then W is a (finite) *complex reflection group* if it is generated by pseudo-reflections.
Irreducible finite complex reflection groups have been completely classified in the fifties by Chevalley [11] and Shephard-Todd [17]. In this classification there are:

- an infinite family of groups $G(r, p, n)$, where r, p, n are positive integers with $p \mid r$;
- 34 other exceptional groups.

We will not deal with the 34 exceptional groups in this thesis. So we are going to describe the infinite family $G(r, p, n)$.

When $r = p = 1$, the group $G(1, 1, n)$ is the symmetric group $\mathcal{S}_n$, the group of the $n \times n$ permutation matrices.

When $p = 1$, the group $G(r, n) := G(r, 1, n)$ is the *wreath product* $\mathbb{Z}_r \wr \mathcal{S}_n$, also called *generalized symmetric group*, or *group of colored permutations*. $G(r, n)$ consists of all $n \times n$ matrices satisfying the following conditions:

- the entries are either 0 or r-th roots of unity;
- there is exactly one non-zero entry in every row and every column.

If p divides r, then $G(r, p, n)$ is the subgroup of $G(r, n)$ given by the matrices such that:

- the product of the non-zero entries is a r/p-th root of unity.

For our exposition it is more convenient to consider wreath products not as groups of complex matrices but as groups of colored permutations. So we recall the following alternative notation.

Notation 1.1. If $g \in G(r, n)$, we write $g = [\sigma_1^{\boldsymbol{c_1}}, \sigma_2^{\boldsymbol{c_2}}, \ldots, \sigma_n^{\boldsymbol{c_n}}]$ if the non-zero entry in the i-th row of g is $\zeta_r^{c_i}$ and appears in the σ_i-th column.

In this notation the element in the i-th position of g represents the r-colored integer $g(i^{\boldsymbol{0}}) = \sigma_i^{\boldsymbol{c_i}}$. We denote it also by g_i. So $G(r, n)$ is the group of permutations g of the set of r-colored integers $i^{\boldsymbol{a}}$, where $i \in [n]$ and $\boldsymbol{a} \in \mathbb{Z}_r$, such that if $g(i^{\boldsymbol{0}}) = j^{\boldsymbol{b}}$ then $g(i^{\boldsymbol{a}}) = j^{\boldsymbol{a+b}}$. In other words,

$$G(r, n) := \{[\sigma_1^{\boldsymbol{c_1}}, \sigma_2^{\boldsymbol{c_2}}, \ldots, \sigma_n^{\boldsymbol{c_n}}] : \sigma \in \mathcal{S}_n,\ \boldsymbol{c_i} \in \mathbb{Z}_r\}.$$

If $g \in G(r, n)$, we let $|g| := \sigma \in \mathcal{S}_n$ and we denote by

$$\mathrm{col}(g) := \sum_{i=1}^{n} c_i$$

the *color weight* of g, which is an integer defined only modulo r. We recall that

$$G(r, p, n) := \{g \in G(r, n) : \mathrm{col}(g) \equiv 0 \bmod p\}.$$

Note that $G(r, p, n)$ is a normal subgroup of $G(r, n)$ of index p, since it is the kernel of the map

$$G(r, n) \to \mathbb{Z}_p, \quad g \mapsto \mathrm{col}(g).$$

Example 1.2. $G(2, n)$ is the Coxeter group B_n of type B, also known as *group of signed permutations*, or *signed symmetric group*. We recall that a signed permutation on $[n]$ is a bijection β on the set $[-n, n] \setminus \{0\}$ such that $\beta(-i) = -\beta(i)$ for $i \in [-n, n] \setminus \{0\}$. We write $\beta = [\beta_1, \beta_2, \ldots, \beta_n] \in B_n$, where $\beta_i = \beta(i)$ for $i = 1, 2, \ldots, n$. If we identify (signed) non-zero integers with 2-colored integers in the following way:

$$m \mapsto \begin{cases} m^{\mathbf{0}} & \text{if } m > 0, \\ |m|^{\mathbf{1}} & \text{if } m < 0, \end{cases}$$

then $\beta = [|\beta_1|^{\boldsymbol{c_1}}, |\beta_2|^{\boldsymbol{c_2}}, \ldots, |\beta_n|^{\boldsymbol{c_n}}]$, where $\boldsymbol{c_i} \in \mathbb{Z}_2$. In the case $r = 2$, we will mainly use the signed notation.
For example, $\beta = [2,\ -4,\ 3,\ 5,\ 1] = [2^{\mathbf{0}},\ 4^{\mathbf{1}},\ 3^{\mathbf{0}},\ 5^{\mathbf{0}},\ 1^{\mathbf{0}}] \in G(2, 5)$.

Example 1.3. $G(2, 2, n)$ is the Coxeter group D_n of type D, also known as *group of even-signed permutations*, or *even-signed symmetric group*. D_n is the subgroup of B_n consisting of signed permutations with an even number of minus signs, or equivalently of 2-colored permutations in which the color **1** appears an even number of times:

$$D_n := \{g \in B_n : \operatorname{neg}(g) \equiv 0 \bmod 2\} = \{g \in B_n : \operatorname{col}(g) \equiv 0 \bmod 2\},$$

where $\operatorname{neg}(g) = |\{i \in [n] : g(i) < 0\}|$.
For example, $\gamma = [2,\ -4,\ 3,\ -5,\ 1] = [2^{\mathbf{0}},\ 4^{\mathbf{1}},\ 3^{\mathbf{0}},\ 5^{\mathbf{1}},\ 1^{\mathbf{0}}] \in G(2, 2, 5)$.

1.3 Projective reflection groups and $G(r, p, q, n)$

Let V be a complex vector space of finite dimension n and $S^q(V)$ the q-th symmetric power of V. Let C_q be the cyclic scalar subgroup of $GL(V)$ of order q generated by $\zeta_q I$. Finally, let G be a finite subgroup of $GL(S^q(V))$. Then, according to [8], we say that the pair (G, q) is a (finite) *projective reflection group* if there exists a finite complex reflection group $W \subset GL(V)$ such that $C_q \subseteq W$ and $G = W/C_q$.

In our work we will only consider those projective reflection groups arising as quotients (by scalar subgroups) of all non-exceptional irreducible complex reflection groups. More precisely,

Definition 1.4. Let r, p, q, n be positive integers such that $p \mid r$, $q \mid r$ and $pq \mid rn$. Then we let

$$G(r, p, q, n) := \frac{G(r, p, n)}{C_q},$$

where C_q is the cyclic group generated by $\zeta_q I$.

When $q = 1$, the group $G(r,p,1,n)$ is the complex reflection group $G(r,p,n)$.

Note the symmetry on the conditions for the parameters p and q in the definition of $G(r,p,q,n)$. This allows us to give the following:

Definition 1.5. Let $G = G(r,p,q,n)$. We denote by G^* the projective reflection group $G(r,q,p,n)$, where the roles of the parameters p and q are interchanged. We call G^* the *dual group* of G.

Following Notation 1.1, for an element $g \in G(r,p,q,n)$ we also write $g = [\sigma_1^{\boldsymbol{c_1}}, \sigma_2^{\boldsymbol{c_2}}, \ldots, \sigma_n^{\boldsymbol{c_n}}]$ to mean that g can be represented by $[\sigma_1^{\boldsymbol{c_1}}, \sigma_2^{\boldsymbol{c_2}}, \ldots, \sigma_n^{\boldsymbol{c_n}}]$ in $G(r,p,n)$. Recall that $\mathrm{col}(g)$ is defined modulo $\gcd(r,\, rn/q)$, which is a multiple of p.

Example 1.6. $G(2,1,2,n)$ is the group $B_n/\pm id$, where $id := id_{B_n}$ is the identity element of B_n. Note that $(B_n/\pm id)^* = D_n$. For example, $g = [2,\ -4,\ 3,\ 5,\ 1] \in G(2,1,2,5)$ can be represented by $g_1 = [2,\ -4,\ 3,\ 5,\ 1]$ or $g_2 = [-2,\ 4,\ -3,\ -5,\ -1]$ in $G(2,5)$.

1.4 Flag-major index on $G(r,p,q,n)$

Let $g = [\sigma_1^{\boldsymbol{c_1}}, \sigma_2^{\boldsymbol{c_2}}, \ldots, \sigma_n^{\boldsymbol{c_n}}] \in G(r,p,q,n)$. According to [8], we let

$$\mathrm{HDes}(g) := \{i \in [n-1] : \boldsymbol{c_i} = \boldsymbol{c_{i+1}} \text{ and } \sigma_i > \sigma_{i+1}\}$$

be the *homogeneous descent set* of g,

$$d_i(g) := |\{j \in [i, n-1] : j \in \mathrm{HDes}(g)\}|$$

for all $i \in [n]$, and

$$k_i(g) := \begin{cases} \mathrm{res}_{r/q}(\boldsymbol{c_n}) & \text{if } i = n, \\ k_{i+1}(g) + \mathrm{res}_r(\boldsymbol{c_i} - \boldsymbol{c_{i+1}}) & \text{if } i \in [n-1]. \end{cases}$$

Note that the sequence $d(g) := (d_1(g), d_2(g), \ldots, d_n(g))$ is a partition, and recall that $k(g) := (k_1(g), k_2(g), \ldots, k_n(g))$ is the smallest element in $\mathscr{P}_n$ (with respect to the entrywise order) such that

$$g = [\sigma_1^{\boldsymbol{k_1(g)}}, \sigma_2^{\boldsymbol{k_2(g)}}, \ldots, \sigma_n^{\boldsymbol{k_n(g)}}].$$

We also let

$$\lambda_i(g) := rd_i(g) + k_i(g)$$

for all $i \in [n]$, and similarly we note that $\lambda(g) := (\lambda_1(g), \lambda_2(g), \ldots, \lambda_n(g))$ is a partition such that

$$g = [\sigma_1^{\boldsymbol{\lambda_1(g)}}, \sigma_2^{\boldsymbol{\lambda_2(g)}}, \ldots, \sigma_n^{\boldsymbol{\lambda_n(g)}}].$$

Finally, we define the *flag-major index* of an element $g \in G(r,p,q,n)$ as

$$\mathrm{fmaj}(g) := |\lambda(g)|.$$

Note that these definitions do not depend on the choice of the representative of g in $G(r,p,n)$.

Example 1.7. Let $g = [2^{\mathbf{2}}, 7^{\mathbf{3}}, 6^{\mathbf{3}}, 4^{\mathbf{5}}, 8^{\mathbf{1}}, 1^{\mathbf{7}}, 5^{\mathbf{3}}, 3^{\mathbf{2}}] \in G(6,2,3,8)$. Then $\mathrm{HDes}(g) = \{2,5\}$, $d(g) = (2,2,1,1,1,0,0,0)$, $k(g) = (18,13,13,9,5,5,1,0)$, $\lambda(g) = (30,25,19,15,11,5,1,0)$ and $\mathrm{fmaj}(g) = 106$.

We recall that the flag-major index has the following distribution.

Theorem 1.8. ([8], consequence of Theorem 8.4) *Let t be an indeterminate. Then*

$$\sum_{g \in G(r,q,p,n)} t^{\mathrm{fmaj}(g)} = \mathrm{Deg}_q\big([d_1]_t [d_2]_t \cdots [d_n]_t\big),$$

where

$$\mathrm{Deg}_q\left(\sum_{k \geq 0} c_k\, t^k\right) := \sum_{k \geq 0} c_{kq}\, t^{kq},$$

$d_i = ri$ if $i < n$ and $d_n = rn/p$ are the fundamental degrees of $G(r,p,n)$ (see Section 1.5).

Corollary 1.9. *Let $q = 1$. Let $G = G(r,p,n)$ and $G^* = G(r,n)/C_p$. Then*

$$\sum_{g \in G^*} t^{\mathrm{fmaj}(g)} = [d_1]_t [d_2]_t \cdots [d_n]_t,$$

where d_i's are the fundamental degrees of G.

Corollary 1.10. ([2], Theorem 4.1) *Let $p = q = 1$. Then*

$$\sum_{g \in G(r,n)} t^{\mathrm{fmaj}(g)} = [d_1]_t [d_2]_t \cdots [d_n]_t,$$

where d_i's are the fundamental degrees of $G(r,n)$.

From now on, let $G = G(r,p,n)$ and $G^* = G(r,n)/C_p$. We recall that in [9] Caselli studied the distribution of the flag-major index on sets of cosets representatives for some special subgroups of G^*, defined as follows. For $k < n$, let

$$\mathscr{C}_k := \{[\sigma_1^{\mathbf{0}}, \sigma_2^{\mathbf{0}}, \ldots, \sigma_k^{\mathbf{0}}, g_{k+1}, \ldots, g_n] \in G^* : \sigma_1 < \sigma_2 < \cdots < \sigma_k\}. \qquad (1.1)$$

We note that the subgroup of G^* given by

$$\{g \in G^* : g = [g_1, g_2, \ldots, g_k, (k+1)^{\mathbf{0}}, \ldots, n^{\mathbf{0}}]\}$$

is isomorphic to $G(r,k)$ for all $k < n$. We may observe that $\mathscr{C}_k$ contains exactly p representatives for each (right) coset of $G(r,k)$ in G^*. Then we have the following distribution.

Theorem 1.11. ([9], Theorem 5.5) *Let $\mathscr{C}_k$ be defined as in (1.1). Then*

$$\sum_{g \in \mathscr{C}_k} t^{\mathrm{fmaj}(g^{-1})} = [p]_{t^{kr/p}} [(k+1)r]_t \, [(k+2)r]_t \cdots [(n-1)r]_t \, [nr/p]_t.$$

Corollary 1.12. ([9], Corollary 5.6) *If $p = 1$, then $\mathscr{C}_k$ is a complete system of coset representatives for the subgroup $G(r,k)$ and*

$$\sum_{g \in \mathscr{C}_k} t^{\mathrm{fmaj}(g^{-1})} = [(k+1)r]_t \, [(k+2)r]_t \cdots [nr]_t.$$

We recall now some results we will use in the present work.

Lemma 1.13. ([9], Lemma 5.1) *There exists a bijection*

$$G^* \times \mathscr{P}_n \times [0, p-1] \to \mathbb{N}^n, \qquad (g, \lambda, h) \mapsto f = (f_1, f_2, \ldots, f_n),$$

where $f_i = \lambda_{|g^{-1}(i)|}(g) + r\lambda_{|g^{-1}(i)|} + h\frac{r}{p}$ for all $i \in [n]$. In this case we say that f is g-compatible.

Lemma 1.14. ([9], Lemma 5.2) *If $g \in G^*$ we let S_g be the set of g-compatible vectors in $\mathbb{N}^n$. Then*

$$\sum_{f \in S_g} x_1^{f_1} x_2^{f_2} \cdots x_n^{f_n} =$$

$$= \frac{x_{|g_1|}^{\lambda_1(g)} x_{|g_2|}^{\lambda_2(g)} \cdots x_{|g_n|}^{\lambda_n(g)}}{(1 - x_{|g_1|}^r)(1 - x_{|g_1|}^r x_{|g_2|}^r) \cdots (1 - x_{|g_1|}^r \cdots x_{|g_{n-1}|}^r)(1 - x_{|g_1|}^{r/p} \cdots x_{|g_n|}^{r/p})}.$$

Lemma 1.15. ([9], Lemma 5.3) *If $g \in G^*$ then there exists $h \in [0, p-1]$ such that $\lambda_i(g) + \lambda_{|g_i|}(g^{-1}) \equiv h\frac{r}{p} \bmod r$, for $i \in [n]$.*

1.5 Invariants and descent basis

Let V be a complex vector space of finite dimension n and W a finite complex reflection groups. Then W is characterized by the structure of its invariant ring, in the following sense.
Let $S[V^*]$ be the symmetric algebra of polynomial functions on V. Any finite subgroup W of $GL(V)$ acts naturally on $S[V^*]$. Denote by $S[V^*]^W$ the invariant ring of W. Then Chevalley [11] and Shephard-Todd [17] proved that W is a complex reflection group if and only if $S[V^*]^W$ is generated by (1 and by) n algebraically independent homogeneous elements, called *basic invariants*. Although these polynomials are not uniquely determined, their degrees $d_1, \ldots, d_n$ are basic numerical invariants of W, and they are called *fundamental degrees* of W. Denote by $I(W)$ the ideal of $S[V^*]$ generated by the elements of strictly positive degree in $S[V^*]^W$. Then we recall that the *coinvariant algebra* of W is defined by

$$R(W) := \frac{S[V^*]}{I(W)}.$$

Since $I(W)$ is W-invariant, the group W acts naturally on $R(W)$. We recall that $R(W)$ is isomorphic to the left regular representation of W and in particular that its dimension as a $\mathbb{C}$-module is equal to $|W|$.

In [8] Caselli generalized this result to the case of projective reflection groups. Let $S_q[V^*]$ be the q-th Veronese subalgebra of $S[V^*]$, i.e., the algebra of polynomial functions on V generated by homogeneous polynomial functions of degree q. Let G be any finite subgroup of graded automorphisms of $S_q[V^*]$. Then (G, q) is a projective reflection group if and only if the invariant algebra $S_q[V^*]^G$ is generated by (1 and by) n algebraically independent homogeneous elements. See Theorem 2.1 in [8].

We denote by $I(G)$ the ideal of $S_q[V^*]$ generated by homogeneous elements of positive degree in $S_q[V^*]^G$. Then the *coinvariant algebra* of G is defined by

$$R(G) := \frac{S_q[V^*]}{I(G)}.$$

Let W be the complex reflection group such that $G = W/C_q$. We recall that

$$S_q[V^*]^G = S[V^*]^W. \tag{1.2}$$

See Proof of Theorem 2.1 in [8]. It follows that $R(G)$ is the subalgebra of $R(W)$ given by the elements of degree multiple of q. See Proof of Proposition 3.1 in [8].

Moreover, we recall that $R(G)$ is isomorphic to the group algebra $\mathbb{C}G$ and in particular that its dimension as a $\mathbb{C}$-module is equal to $|G|$. See Proposition 3.1 in [8].

If we set $X := x_1, \ldots, x_n$ as a basis for V, then $S[V^*]$ and $S_q[V^*]$ can be identified respectively with the polynomial algebra $\mathbb{C}[X]$ and its subalgebra $S_q[X]$ generated by the monomials of degree q. Let now $W = G(r, p, n)$ and $G = G(r, p, q, n)$.

Observe that $G(r, n)$ acts on $\mathbb{C}[X]$ as follows:

$$[\sigma_1^{\mathbf{c_1}}, \sigma_2^{\mathbf{c_2}}, \ldots, \sigma_n^{\mathbf{c_n}}] \cdot P(X) = P\left(\zeta_r^{c_{\sigma_1}} x_{\sigma_1}, \zeta_r^{c_{\sigma_2}} x_{\sigma_2}, \ldots, \zeta_r^{c_{\sigma_n}} x_{\sigma_n}\right).$$

A set of basic invariants under this action is given by

$$e_i(x_1^r, \ldots, x_n^r), \quad i \in [n], \tag{1.3}$$

where the e_i's are the elementary symmetric functions. It follows that the fundamental degrees of $G(r, n)$ are

$$r,\ 2r,\ \ldots,\ nr.$$

Moreover, $\dim R\big(G(r, n)\big) = |G(r, n)| = n! r^n$.

Now, consider the restriction to W of the action of $G(r, n)$ on $\mathbb{C}[X]$. Let $d := r/p$. Then a set of basic invariants is given by

$$\begin{cases} e_i(x_1^r, \ldots, x_n^r) & \text{if } i \in [n-1] \\ x_1^d \cdots x_n^d & \text{if } i = n, \end{cases} \tag{1.4}$$

and the fundamental degrees of W are

$$r,\, 2r,\, \ldots,\, (n-1)r,\, nd.$$

Moreover, $\dim R(W) = |W| = n! r^{n-1} d$.

Finally, consider the action of G on $S_q[X]$. From (1.2) we recall that a set of basic invariants is given by (1.4). Moreover,

$$\dim R(G) = |G| = \frac{n! r^n}{pq} = |G^*|.$$

The following result shows that invariant theory of G is quite naturally described by its dual group G^*.

Theorem 1.16. ([8], Theorem 5.3) *Let $G = G(r, p, q, n)$. Then the set $\{a_g : g \in G^*\}$, where*

$$a_g(X) := \prod_{i=1}^{n} x_{|g_i|}^{\lambda_i(g)}$$

is a monomial of degree $\mathrm{fmaj}(g)$*, represents a basis for $R(G)$.*

Finally, we recall the following result.

Lemma 1.17. ([4], Equation (12)) *Let $W = G(r, p, n)$. Let M be a monomial in $S_d := \mathbb{C}[X]/(x_1^d \cdots x_n^d)$. Then M admits the following expression in $R(W)$:*

$$M = \sum_{g \in \Omega_n} \eta_g a_g,$$

where $\Omega_n := \{g \in G(r, n) : c(g_n) < \boldsymbol{d}\}$ and $\eta_g \in \mathbb{Z}$.

Corollary 1.18. ([5], Lemma 3.3) *Let $r = p = 2$ and $d = 1$. Let M be a monomial in $S_1 := \mathbb{C}[X]/(x_1 \cdots x_n)$. Then M admits the following expression in $R(D_n)$:*

$$M = \sum_{g \in \Delta_n} \eta_g a_g,$$

where $\Delta_n := \{g \in B_n : g(n) > 0\}$ and $\eta_g \in \mathbb{Z}$.

1.6 Labeled forests and q-hook length formulas

According to [7] we consider a finite poset F in which every element is covered by at most one element, or equivalently such that its Hasse diagram is a rooted forest with roots on top. For this reason we call also F a *forest* and we let $V(F)$ be its vertex set, $E(F)$ its edge set and $\prec$ the order relation in F. We can also denote an *edge* in $E(F)$ by an ordered pair (x, y) of elements of F such that x is covered by y. Let

$$h_x := |\{a \in F : a \preceq x\}|$$

be the *hook length* of the element x, for each $x \in F$, and

$$h_{(x,y)} := h_x$$

the hook length of the edge (x, y), for each $(x, y) \in E(F)$. Let

$$\mathscr{W}(F) := \{w : V(F) \to [n] \text{ s.t. } w \text{ is a bijection}\}$$

be the *set of labelings* of F. For $w \in \mathscr{W}(F)$ we denote the *number of inversions* of w by

$$\mathrm{inv}(w) := |\{(x, y) : x \prec y \text{ and } w(x) > w(y)\}|,$$

the *descent set* of w by

$$\mathrm{Des}(w) := \{(x, y) \in E(F) : w(x) > w(y)\},$$

the *major index* of w by

$$\mathrm{maj}(w) = \sum_{e \in \mathrm{Des}(w)} h_e,$$

and the *set of linear extensions* of w by

$$\mathscr{L}(w) = \{\sigma \in \mathcal{S}_n : \text{if } x \prec y \text{ then } \sigma^{-1}(w(x)) < \sigma^{-1}(w(y))\}.$$

Example 1.19. Let w be the labeling in Figure 1.1. Let $w_j^{-1} := w^{-1}(j)$ be the vertex with label j in w. Then $\mathscr{L}(w)$ is the following subset of $\mathcal{S}_5$:

$$\Big\{[2,3,5,4,1], [3,2,5,4,1], [3,5,2,4,1], [3,5,4,2,1], [3,5,4,1,2],$$
$$[2,5,3,4,1], [5,2,3,4,1], [5,3,2,4,1], [5,3,4,2,1], [5,3,4,1,2]\Big\}.$$

Moreover, $\mathrm{inv}(w) = |\{(w_4^{-1}, w_1^{-1}), (w_3^{-1}, w_1^{-1}), (w_5^{-1}, w_1^{-1}), (w_5^{-1}, w_4^{-1})\}| = 4$, $\mathrm{Des}(w) = \{(w_4^{-1}, w_1^{-1}), (w_5^{-1}, w_4^{-1})\}$ and $\mathrm{maj}(w) = 3 + 1 = 4$.

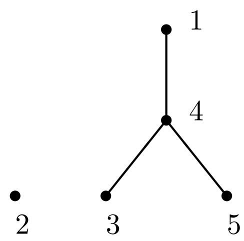

Figure 1.1: Example of labeling.

We are interested in the following important results:

Theorem 1.20. ([7], Theorem 1.2) *Let F be a finite forest with n elements and w a labeling of F. Then*

$$\sum_{\sigma \in \mathscr{L}(w)} q^{\mathrm{maj}(\sigma)} = q^{\mathrm{maj}(w)} \frac{[n]_q!}{\prod_{x \in F} [h_x]_q}.$$

Theorem 1.21. ([7], Theorem 1.3) *Let F be a finite forest with n elements and $\mathscr{W}(F)$ the set of all labelings of F. Then*

$$\sum_{w \in \mathscr{W}(F)} q^{\mathrm{maj}(w)} = \frac{n!}{\prod_{x \in F} h_x} \prod_{x \in F} [h_x]_q.$$

Chapter 2

Counting linear extensions of forest labelings: the r case

2.1 r-Colored labelings

Let F be a finite forest with n vertices (see Section 1.6).

Definition 2.1. We define the set of *r-colored labelings* of F as

$$\mathscr{W}_r(F) := \big\{ w : V(F) \to [n] \times \mathbb{Z}_r \text{ s.t. the projection on } [n] \text{ is a bijection} \big\},$$

so every element $x \in F$ is labeled by $w(x) = (\sigma_x, \boldsymbol{c_x})$ which represents the r-colored integer $\sigma_x^{\boldsymbol{c_x}}$.

We denote the label $w(x)$ also by w_x. We can identify a colored integer $i^{\mathbf{0}}$ with the integer i for each $i \in [n]$, and vice versa. Then for $w \in \mathscr{W}_r(F)$ we define the *set of linear extensions* of w as

$$\begin{aligned} \mathscr{L}(w) := \big\{ g \in G(r,n) :\ & c(g^{-1}(w_x)) = \mathbf{0} \text{ if } x \in F, \text{ and} \\ & \text{if } x \prec y \text{ then } g^{-1}(w_x) < g^{-1}(w_y) \big\}. \end{aligned}$$

If $x \in F$ and x is not a root, we let $p(x)$ be the element that covers x in the forest. For each $x \in F$ we let

$$z_x(w) := \begin{cases} \operatorname{res}_r(\boldsymbol{c_x}) & \text{if } x \text{ is a root of } F, \\ \operatorname{res}_r\big(\boldsymbol{c_x} - \boldsymbol{c_{p(x)}}\big) & \text{otherwise} \end{cases}$$

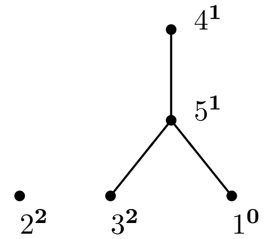

Figure 2.1: Example of 3-colored labeling.

and we define the *homogeneous descent set* of w as

$$\mathrm{HDes}(w) := \{(x,y) \in E(F) : \boldsymbol{c_x} = \boldsymbol{c_y} \text{ and } \sigma_x > \sigma_y\}.$$

Finally we define the *flag-major index* of w as

$$\mathrm{fmaj}(w) := \sum_{e\in E(F)} r\chi_e(w)h_e + \sum_{v\in V(F)} z_v(w)h_v,$$

where

$$\chi_e(w) := \begin{cases} 1 & \text{if } e \in \mathrm{HDes}(w), \\ 0 & \text{otherwise.} \end{cases}$$

Example 2.2. Let w be the 3-colored labeling in Figure 2.1. Then $\mathscr{L}(w)$ is the following subset of $G(3,5)$:

$$\Big\{[2^{\mathbf{2}},3^{\mathbf{2}},1^{\mathbf{0}},5^{\mathbf{1}},4^{\mathbf{1}}],[3^{\mathbf{2}},2^{\mathbf{2}},1^{\mathbf{0}},5^{\mathbf{1}},4^{\mathbf{1}}],[3^{\mathbf{2}},1^{\mathbf{0}},2^{\mathbf{2}},5^{\mathbf{1}},4^{\mathbf{1}}],[3^{\mathbf{2}},1^{\mathbf{0}},5^{\mathbf{1}},2^{\mathbf{2}},4^{\mathbf{1}}],$$
$$[3^{\mathbf{2}},1^{\mathbf{0}},5^{\mathbf{1}},4^{\mathbf{1}},2^{\mathbf{2}}],[2^{\mathbf{2}},1^{\mathbf{0}},3^{\mathbf{2}},5^{\mathbf{1}},4^{\mathbf{1}}],[1^{\mathbf{0}},2^{\mathbf{2}},3^{\mathbf{2}},5^{\mathbf{1}},4^{\mathbf{1}}],[1^{\mathbf{0}},3^{\mathbf{2}},2^{\mathbf{2}},5^{\mathbf{1}},4^{\mathbf{1}}],$$
$$[1^{\mathbf{0}},3^{\mathbf{2}},5^{\mathbf{1}},2^{\mathbf{2}},4^{\mathbf{1}}],[1^{\mathbf{0}},3^{\mathbf{2}},5^{\mathbf{1}},4^{\mathbf{1}},2^{\mathbf{2}}]\Big\},$$

$\mathrm{HDes}(w) = \big\{\big(w^{-1}(5^{\mathbf{1}}), w^{-1}(4^{\mathbf{1}})\big)\big\}$ and $\mathrm{fmaj}(w) = 3\cdot 3 + (1\cdot 4 + 1\cdot 1 + 2\cdot 1 + 2\cdot 1) = 18$.

Remark 2.3. If $r = 1$ then a 1-colored labeling w of F is a labeling $w \in \mathscr{W}(F)$, since $\boldsymbol{c_x} = \mathbf{0}$ for each $x \in F$ and $i^{\mathbf{0}}$ is the integer i. Then we have $\mathrm{HDes}(w) = \mathrm{Des}(w)$ and $\mathrm{fmaj}(w) = \mathrm{maj}(w)$. Moreover, if F is a *linear tree* (i.e., a totally ordered set $\{x_1, x_2, \dots, x_n\}$ in which $x_i \prec x_{i+1}$ for $i \in [n-1]$) we note that an r-colored labeling w of F can be thought as the unique linear extension $g \in G(r,n)$ of w. If we let $z_i(g) = z_{x_i}(w)$, $k_i(g) = \sum_{j\geq i} z_j(g)$, $d_i(g) = |\{j \geq i : (x_j, x_{j+1}) \in \mathrm{HDes}(w)\}|$ for all $i \in [n-1]$ and $d_n(g) = 0$, then we have $\mathrm{fmaj}(w) = \sum_{i\in[n]}(rd_i(g) + k_i(g)) = \mathrm{fmaj}(g)$.

Now we can give a generalized version of Theorem 1.20, which we can recover from the following result when $r = 1$:

Theorem 2.4. *Let F be a finite forest with n elements and w an r-colored labeling of F. Then*

$$\sum_{g \in \mathscr{L}(w)} q^{\mathrm{fmaj}(g)} = q^{\mathrm{fmaj}(w)} \frac{[d_1]_q [d_2]_q \cdots [d_n]_q}{\prod_{x \in F} [h_x r]_q},$$

where $d_i = ri$, $i = 1, \ldots, n$ are the fundamental degrees of $G(r, n)$.

We will give a proof of this result in a more general case (see Proof of Theorem 2.9).

Example 2.5. Let w be the 3-colored labeling in Figure 2.1. Then

$$\sum_{g \in \mathscr{L}(w)} q^{\mathrm{fmaj}(g)} = q^{18} + 2q^{21} + 2q^{24} + 2q^{27} + 2q^{30} + q^{33}$$

and

$$\begin{aligned} q^{\mathrm{fmaj}(w)} \frac{[r]_q [2r]_q \cdots [nr]_q}{\prod_{x \in F} [h_x r]_q} &= q^{18} \frac{[3]_q [6]_q [9]_q [12]_q [15]_q}{[3]_q [3]_q [3]_q [9]_q [12]_q} = \\ &= q^{18}(1 + q^3)(1 + q^3 + q^6 + q^9 + q^{12}). \end{aligned}$$

2.2 r-Starred labelings

Definition 2.6. We define the set of *r-starred labelings* of F as

$$\begin{aligned} \mathscr{S}_r(F) := \Big\{ & w : V(F) \to [n] \times \big(\mathbb{Z}_r \cup \{*\}\big) \text{ s.t.} \\ & \text{the projection on } [n] \text{ is a bijection, and the projection } \gamma \text{ on } \mathbb{Z}_r \cup \{*\} \\ & \qquad \text{is s.t., if } (x, y) \in E(F) \text{ and } \gamma(y) = *, \text{ then } \gamma(x) = * \Big\}. \end{aligned}$$

If w is an r-starred labeling, then every element $x \in F$ is labeled by $\sigma_x^{\boldsymbol{c_x}}$, where $\boldsymbol{c_x} \in \mathbb{Z}_r \cup \{*\}$ and the symbol $*$ represents any class in $\mathbb{Z}_r$ (i.e., the label $\sigma_x^{\boldsymbol{*}}$ represents r different colored integers $\sigma_x^{\mathbf{0}}, \sigma_x^{\mathbf{1}}, \ldots, \sigma_x^{\boldsymbol{r-1}}$). We require also that, if a vertex x has a starred label, then every vertex in the subtree rooted at x has a starred label. See Figure 2.2 for an example.

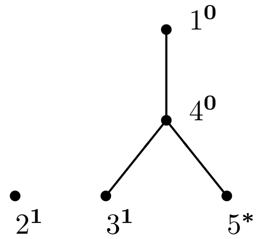

Figure 2.2: Example of 2-starred labeling.

Remark 2.7. An r-starred labeling without $*$ is an r-colored labeling.

We let $F_* := \{x \in F : \boldsymbol{c_x} = *\}$ and $F_r := \{x \in F : \boldsymbol{c_x} \in \mathbb{Z}_r\} = F - F_*$. For $w \in \mathscr{S}_r(F)$ we define the *set of linear extensions* of w as

$$\mathscr{L}(w) := \big\{g \in G(r,n) : c(g^{-1}(w_x)) = \mathbf{0} \text{ if } x \in F_r, \text{ and} \\ \text{if } x \prec y \text{ then } |g^{-1}(\sigma_x)| < |g^{-1}(\sigma_y)|\big\}$$

and for each $x \in F$ we let

$$z_x(w) := \begin{cases} 0 & \text{if } x \in F_*, \\ \operatorname{res}_r(\boldsymbol{c_x}) & \text{if } x \in F_r \text{ and } x \text{ is a root of } F, \\ \operatorname{res}_r\big(\boldsymbol{c_x} - \boldsymbol{c_{p(x)}}\big) & \text{otherwise.} \end{cases}$$

Now we let

$$\mathrm{HDes}(w) := \{(x,y) \in E(F) : x \in F_r,\ \boldsymbol{c_x} = \boldsymbol{c_y} \text{ and } \sigma_x > \sigma_y\}$$

be the *homogeneous descent set* of w and we define the *starred descent set* of w as

$$\mathrm{SDes}(w) := \{(x,y) \in E(F) : x \in F_* \text{ and } \sigma_x > \sigma_y\}$$

and finally the *flag-major index* of w as

$$\mathrm{fmaj}(w) := \sum_{e \in E(F)} (r\chi_e^r(w) + \chi_e^*(w))h_e + \sum_{v \in V(F)} z_v(w)h_v,$$

where

$$\chi_e^r(w) := \begin{cases} 1 & \text{if } e \in \mathrm{HDes}(w), \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad \chi_e^*(w) := \begin{cases} 1 & \text{if } e \in \mathrm{SDes}(w), \\ 0 & \text{otherwise.} \end{cases}$$

Example 2.8. Let w be the 2-starred labeling in Figure 2.2. Then $\mathscr{L}(w)$ is the following subset of $G(2,5)$:

$$\Big\{[2^{\mathbf{1}},3^{\mathbf{1}},5^{\boldsymbol{c}},4^{\mathbf{0}},1^{\mathbf{0}}],[3^{\mathbf{1}},2^{\mathbf{1}},5^{\boldsymbol{c}},4^{\mathbf{0}},1^{\mathbf{0}}],[3^{\mathbf{1}},5^{\boldsymbol{c}},2^{\mathbf{1}},4^{\mathbf{0}},1^{\mathbf{0}}],[3^{\mathbf{1}},5^{\boldsymbol{c}},4^{\mathbf{0}},2^{\mathbf{1}},1^{\mathbf{0}}],$$
$$[3^{\mathbf{1}},5^{\boldsymbol{c}},4^{\mathbf{0}},1^{\mathbf{0}},2^{\mathbf{1}}],[2^{\mathbf{1}},5^{\boldsymbol{c}},3^{\mathbf{1}},4^{\mathbf{0}},1^{\mathbf{0}}],[5^{\boldsymbol{c}},2^{\mathbf{1}},3^{\mathbf{1}},4^{\mathbf{0}},1^{\mathbf{0}}],[5^{\boldsymbol{c}},3^{\mathbf{1}},2^{\mathbf{1}},4^{\mathbf{0}},1^{\mathbf{0}}],$$
$$[5^{\boldsymbol{c}},3^{\mathbf{1}},4^{\mathbf{0}},2^{\mathbf{1}},1^{\mathbf{0}}],[5^{\boldsymbol{c}},3^{\mathbf{1}},4^{\mathbf{0}},1^{\mathbf{0}},2^{\mathbf{1}}]\Big\},$$

where $\boldsymbol{c} \in \{\mathbf{0},\mathbf{1}\}$. Moreover, $\mathrm{HDes}(w) = \big\{\big(w^{-1}(4^{\mathbf{0}}), w^{-1}(1^{\mathbf{0}})\big)\big\}$, $\mathrm{SDes}(w) = \big\{\big(w^{-1}(5^{*}), w^{-1}(4^{\mathbf{0}})\big)\big\}$ and $\mathrm{fmaj}(w) = (2\cdot 3+1)+(1\cdot 1+1\cdot 1) = 9$.

Now we can prove Theorem 2.4 by providing the proof for the following analogous version of that theorem holding for r-starred labelings:

Theorem 2.9. *Let F be a finite forest with n elements and w an r-starred labeling of F. Then*

$$\sum_{g\in\mathscr{L}(w)} q^{\mathrm{fmaj}(g)} = q^{\mathrm{fmaj}(w)} \frac{[d_1]_q[d_2]_q\cdots[d_n]_q}{\prod\limits_{x\in F_r}[h_x r]_q \prod\limits_{x\in F_*}[h_x]_q},$$

where $d_i = ri$, $i = 1,\dots,n$ are the fundamental degrees of $G(r,n)$.

Example 2.10. Let w be the 2-starred labeling in Figure 2.2. Then

$$\sum_{g\in\mathscr{L}(w)} q^{\mathrm{fmaj}(g)} = q^9 + q^{10} + 2\sum_{k=11}^{18} q^k + q^{19} + q^{20}$$

and

$$q^{\mathrm{fmaj}(w)} \frac{[r]_q[2r]_q\cdots[nr]_q}{\prod\limits_{x\in F_r}[h_x r]_q \prod\limits_{x\in F_*}[h_x]_q} = q^9\frac{[2]_q[4]_q[6]_q[8]_q[10]_q}{[2]_q[2]_q[6]_q[8]_q[1]_q} = q^9(1+q^2)\sum_{k=0}^{9} q^k.$$

Remark 2.11. Consider the poset $\{x_1, x_2, \dots, x_n\}$ with no order relation between any two different elements. The Hasse diagram V_n of this poset is a forest consisting of n disjoint vertices. Consider now the r-starred labeling w of V_n such that $w(x_i) = i^*$ for all $i \in [n]$. Then $\mathrm{fmaj}(w) = 0$ and $\mathscr{L}(w) = G(r,n)$. Therefore in this case Theorem 2.9 reduces to Corollary 1.10.

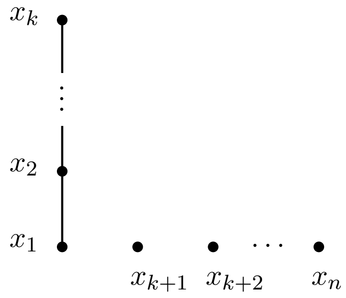

Figure 2.3: $T_{n,k}$ poset.

Remark 2.12. Let $k < n$. Consider the poset $\{x_1, x_2, \ldots, x_n\}$ with the ordering given by $x_i \prec x_j$ if and only if $i < j < k < n$. The Hasse diagram $T_{n,k}$ of this poset is a forest consisting of a linear tree of length k and $n-k$ disjoint vertices (see Figure 2.3). Consider now the r-starred labeling w of $T_{n,k}$ such that $w(x_i) = i^{\mathbf{0}}$ for $i \in [k]$ and $w(x_i) = i^*$ for $i = k+1, k+2, \ldots, n$. Then $h_{x_i} = i$ for $i \in [k]$ and $h_{x_i} = 1$ otherwise, $\mathrm{fmaj}(w) = 0$ and $\mathscr{L}(w) = \{g \in G(r,n) : c(g^{-1}(i)) = \mathbf{0}$ if $i \in [k]$ and $g^{-1}(1^{\mathbf{0}}) < g^{-1}(2^{\mathbf{0}}) < \cdots < g^{-1}(k^{\mathbf{0}})\}$. We finally note that if $g \in \mathscr{L}(w)$ then $g^{-1} \in \mathscr{C}_k$, where $\mathscr{C}_k$ is the same set defined in (1.1) when $p = 1$. Then in this case Theorem 2.9 reduces to Corollary 1.12.

Proof of the fmaj hook length formula of Theorem 2.9

Let w be a fixed r-starred labeling of F and

$$\begin{aligned}\mathscr{A} = \big\{ f \in \mathbb{N}^n : \; & f_{\sigma_x} \in \boldsymbol{c_x} \text{ if } x \in F_r, \text{ and } f_{\sigma_x} \geq f_{\sigma_y} \text{ for each } (x,y) \in E(F), \\ & \text{where } f_{\sigma_x} = f_{\sigma_y} \text{ implies } \boldsymbol{c_x} = \boldsymbol{c_y} \text{ or } x \in F_*, \text{ and } \sigma_x < \sigma_y \big\}.\end{aligned}$$

We show that the set $\mathscr{A}$ consists of all g-compatible vectors in $\mathbb{N}^n$ as g varies in the set $\mathscr{L}(w)$ of linear extensions of the r-starred labeling w:

Proposition 2.13. *Let $f \in \mathbb{N}^n$. Then $f \in \mathscr{A}$ if and only if f is g-compatible for some $g \in \mathscr{L}(w)$.*

Proof. We recall that f is g-compatible if and only if $f_i = \lambda_{|g^{-1}(i)|}(g) + r\lambda_{|g^{-1}(i)|}$ for all $i \in [n]$, where $\lambda \in \mathscr{P}_n$ (Lemma 1.13 when $p = 1$). We can divide the proof in two steps:

i) If $x \in F_r$, then $c(g^{-1}(w_x)) = \mathbf{0}$ if and only if $f_{\sigma_x} \in \boldsymbol{c_x}$.

Since $f_{\sigma_x} \in \boldsymbol{c_x}$ if and only if $\lambda_{|g^{-1}(\sigma_x)|}(g) \in \boldsymbol{c_x}$, then $\lambda_{\sigma_x}(g^{-1}) \in -\boldsymbol{c_x}$ from Lemma 1.15 (for $p = 1$), and this is equivalent to $c(g^{-1}(\sigma_x)) = -\boldsymbol{c_x}$. So the result follows.

ii) If $(x, y) \in E(F)$, then $|g^{-1}(\sigma_x)| < |g^{-1}(\sigma_y)|$ if and only if $f_{\sigma_x} \geq f_{\sigma_y}$, where $f_{\sigma_x} = f_{\sigma_y}$ implies $\boldsymbol{c_x} = \boldsymbol{c_y}$ or $x \in F_*$, and $\sigma_x < \sigma_y$.

$\Leftarrow$) If $f_{\sigma_x} > f_{\sigma_y}$ then $|g^{-1}(\sigma_x)| < |g^{-1}(\sigma_y)|$ since $\lambda(g)$ and λ are both partitions. If $f_{\sigma_x} = f_{\sigma_y}$ then $\lambda_{|g^{-1}(\sigma_x)|}(g) = \lambda_{|g^{-1}(\sigma_y)|}(g)$. Since $\sigma_x < \sigma_y$, then the definition of the statistics $\lambda_i(g)$ implies that $|g^{-1}(\sigma_x)| < |g^{-1}(\sigma_y)|$.
$\Rightarrow$) If $|g^{-1}(\sigma_x)| < |g^{-1}(\sigma_y)|$ then $f_{\sigma_x} \geq f_{\sigma_y}$ since $\lambda(g)$ and λ are both partitions. Moreover, we note that $f_{\sigma_x} \neq f_{\sigma_y}$ either if $\boldsymbol{c_x} = \boldsymbol{c_y}$ or $x \in F_*$, and $\sigma_x > \sigma_y$, or if $\boldsymbol{c_x} \neq \boldsymbol{c_y}$, for $x \in F_r$. So the result follows by contradiction. $\square$

For $x \in F$ we let $\mathscr{F}_x = \{a \in F : a \succeq x\}$ be the filter at x, which is a chain, and $\mathscr{E}_x = \{(y, z) \in E(F) : y \in \mathscr{F}_x\}$ the set of edges of $\mathscr{F}_x$. We let also

$$\chi_y^r(w) = \begin{cases} 1 & \text{if } y \in F_r, \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad \chi_y^*(w) = \begin{cases} 1 & \text{if } y \in F_*, \\ 0 & \text{otherwise} \end{cases}$$

and finally

$$\mathscr{B} = \Bigg\{ f \in \mathbb{N}^n : f_{\sigma_x} = \sum_{y \in \mathscr{F}_x} (z_y + r m_y \chi_y^r + m_y \chi_y^*) + \sum_{e \in \mathscr{E}_x} (r\chi_e^r + \chi_e^*), \qquad \text{for each } x \in F,\ m_y \in \mathbb{N} \Bigg\}$$

where we omitted the dependence from w. We show that $\mathscr{A}$ and $\mathscr{B}$ are the same set, so in particular $\mathscr{B}$ consists of all g-compatible vectors as $g \in \mathscr{L}(w)$:

Proposition 2.14. $\mathscr{A} = \mathscr{B}$.

Proof. $\supseteq$) Let $f \in \mathscr{B}$ and $x \in F$. By definition, $f_{\sigma_x} = f_{\sigma_y} + \big(z_x + r m_x \chi_x^r + m_x \chi_x^* + r\chi_{(x,y)}^r + \chi_{(x,y)}^*\big)$, where $y = p(x)$. We note that $f_{\sigma_x} \in \boldsymbol{c_x}$ if $x \in F_r$ (by an inductive argument) and $f_{\sigma_x} \geq f_{\sigma_y}$, where $f_{\sigma_x} = f_{\sigma_y}$ implies $m_x = 0$, $\boldsymbol{c_x} = \boldsymbol{c_y}$ or $x \in F_*$, and $\sigma_x < \sigma_y$. Then $f \in \mathscr{A}$.

$\supseteq$) Let u be a root. If $u \in F_r$ then $f_{\sigma_u} \in \boldsymbol{c_u}$, so there exists $m_u \in \mathbb{N}$ such that $f_{\sigma_u} = \mathrm{res}_r(\boldsymbol{c_u}) + rm_u$. Otherwise if $u \in F_*$ there exists $m_u \in \mathbb{N}$ such that $f_{\sigma_u} = m_u$. Then $f_{\sigma_u} = z_u + rm_u\chi_u^r + m_u\chi_u^*$. Let x be an element covered by u. If $x \in F_r$ then $u \in F_r$ and there exists $m_x \in \mathbb{N}$ such that $f_{\sigma_x} = f_{\sigma_u} + \mathrm{res}_r(\boldsymbol{c_x} - \boldsymbol{c_u}) + r\chi_{(x,u)}^r + rm_x$. We note that $f_{\sigma_x} \in \boldsymbol{c_x}$. Otherwise if $x \in F_*$ there exists $m_x \in \mathbb{N}$ such that $f_{\sigma_x} = f_{\sigma_u} + \chi_{(x,u)}^* + m_x$. Then $f_{\sigma_x} = f_{\sigma_u} + z_x + r\chi_{(x,u)}^r + rm_x\chi_x^r + \chi_{(x,u)}^* + m_x\chi_x^*$. We finally obtain the result extending this argument to every $x \in F$. $\square$

Now we are ready to prove the main result of this section:

Proof of Theorem 2.9. We consider the formal power series $\sum_{f\in\mathscr{A}} q^{|f|}$ and we compute it in two different ways. In the first computation we use Lemma 1.14 (for $p = 1$) and Proposition 2.13 and we have

$$\sum_{f\in\mathscr{A}} q^{|f|} = \sum_{g\in\mathscr{L}(w)} \frac{q^{\lambda_1(g)}q^{\lambda_2(g)}\cdots q^{\lambda_n(g)}}{(1-q^r)(1-q^{2r})\cdots(1-q^{nr})} = $$
$$= \frac{\sum_{g\in\mathscr{L}(w)} q^{\mathrm{fmaj}(g)}}{(1-q^r)(1-q^{2r})\cdots(1-q^{nr})}.$$

In the second computation we use directly the definition of $\mathscr{B}$ and Proposition 2.14: using the same notations, we have

$$|f| = \sum_{x\in F} f_{\sigma_x} = \sum_{v\in V(F)} z_v h_v + r\sum_{x\in F_r} m_x h_x + \sum_{x\in F_*} m_x h_x +$$
$$+ \sum_{e\in E(F)} (r\chi_e^r + \chi_e^*)h_e = \mathrm{fmaj}(w) + r\sum_{x\in F_r} m_x h_x + \sum_{x\in F_*} m_x h_x,$$

where $m_x \in \mathbb{N}$, and then

$$\sum_{f\in\mathscr{A}} q^{|f|} = \sum_{f\in\mathscr{B}} q^{|f|} = \sum_{m_x\in\mathbb{N}} q^{\mathrm{fmaj}(w) + r\sum\limits_{F_r} m_x h_x + \sum\limits_{F_*} m_x h_x} =$$
$$= q^{\mathrm{fmaj}(w)} \frac{1}{\prod\limits_{x\in F_r}(1-q^{rh_x}) \prod\limits_{x\in F_*}(1-q^{h_x})}.$$

Therefore

$$\sum_{g\in\mathscr{L}(w)} q^{\mathrm{fmaj}(g)} = q^{\mathrm{fmaj}(w)} \frac{(1-q^r)(1-q^{2r})\cdots(1-q^{nr})}{\prod\limits_{x\in F_r}(1-q^{rh_x}) \prod\limits_{x\in F_*}(1-q^{h_x})}.$$

$\square$

We can reformulate Theorem 2.9 in this way:

Theorem 2.15. *Let F be a finite forest with n elements and w an r-starred labeling of F. Then*

$$\sum_{g\in\mathscr{L}(w)} q^{\mathrm{fmaj}(g)} = q^{\mathrm{fmaj}(w)} \frac{[r]_q[2r]_q\cdots[nr]_q}{\prod\limits_{x\in F}[h_x i_x]_q}, \qquad \textit{where } i_x = \begin{cases} 1 & \textit{if } x\in F_*, \\ r & \textit{otherwise.}\end{cases}$$

2.3 r-Partial labelings

Starting from Theorem 2.15, we can further generalize the result introducing a new notion of labeling. First, let m be a positive integer and d a positive divisor of m. Let also π_d^m be the projection

$$\pi_d^m : \mathbb{Z}_m \to \mathbb{Z}_d, \qquad \boldsymbol{z} \mapsto \boldsymbol{c},$$

where c is the remainder of the division of z by d. We note that

$$(\pi_d^m)^{-1}(\boldsymbol{c}) = \left\{\boldsymbol{c} + \boldsymbol{kd} :\ k\in\mathbb{Z},\, 0\le k < \frac{m}{d}\right\},$$

so $|(\pi_d^m)^{-1}(\boldsymbol{c})| = m/d$ for each $\boldsymbol{c}\in\mathbb{Z}_d$.

Definition 2.16. We define the set of *r-partial labelings* of F as

$$\begin{aligned}\mathscr{P}_r(F) := \Big\{ & w : V(F) \to [n]\times\left(\textstyle\bigcup_{i|r}\mathbb{Z}_i\right) \text{ s.t.} \\ & \text{the projection on } [n] \text{ is a bijection, and the projection } \gamma \text{ on } \left(\textstyle\bigcup_{i|r}\mathbb{Z}_i\right) \\ & \quad \text{is s.t., if } (x,y)\in E(F) \text{ and } \gamma(y)\in\mathbb{Z}_{i_y}, \text{ then } \gamma(x)\in\left(\textstyle\bigcup_{i_x|i_y}\mathbb{Z}_{i_x}\right)\Big\}.\end{aligned}$$

If w is an r-partial labeling, then every element $x\in F$ is labeled by $\sigma_x^{i_x,\boldsymbol{j_x}}$, where i_x is a positive divisor of r and $\boldsymbol{j_x}$ is a class in $\mathbb{Z}_{i_x}$. So the label w_x represents r/i_x different r-colored integers:

$$\sigma_x^{(\pi_{i_x}^r)^{-1}(\boldsymbol{j_x})} = \left\{\sigma_x^{\boldsymbol{j_x}}, \sigma_x^{\boldsymbol{j_x}+\boldsymbol{i_x}}, \ldots, \sigma_x^{\boldsymbol{j_x}+\boldsymbol{r}-\boldsymbol{i_x}}\right\}.$$

We require also that, if the vertex y is covered by x, then i_y is a divisor of i_x.

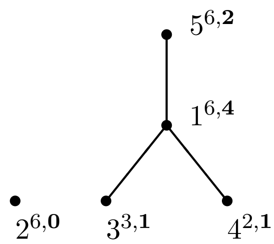

Figure 2.4: Example of 6-partial labeling.

Remark 2.17. If $i_x = r$ the color of w_x can be identified with the class $\boldsymbol{j_x}$ in $\mathbb{Z}_r$, so $\sigma_x^{r,\boldsymbol{j_x}} = \sigma_x^{\boldsymbol{j_x}}$. If $i_x = 1$ the color of w_x is any class in $\mathbb{Z}_r$, so $\sigma_x^{1,\mathbf{0}} = \sigma_x^{*}$. Moreover, if $i_x = r$ for each $x \in F$ then an r-partial labeling is an r-colored labeling and if r is a prime number then an r-partial labeling is an r-starred labeling.

For $w \in \mathscr{P}_r(F)$ we define the *set of linear extensions* of w as

$$\begin{aligned}\mathscr{L}(w) := \{g \in G(r,n) :\ & \pi^r_{i_x}\left(c(g^{-1}(\sigma_x))\right) = -\boldsymbol{j_x} \text{ if } x \in F, \text{ and} \\ & \text{if } x \prec y \text{ then } |g^{-1}(\sigma_x)| < |g^{-1}(\sigma_y)|\}\end{aligned}$$

and for each $x \in F$ we let

$$z_x(w) := \begin{cases} \operatorname{res}_{i_x}(\boldsymbol{j_x}) & \text{if } x \text{ is a root of } F, \\ \operatorname{res}_{i_x}\left(\boldsymbol{j_x} - \boldsymbol{j_{p(x)}}\right) & \text{otherwise.} \end{cases}$$

Finally we let

$$\operatorname{HDes}(w) := \{(x,y) \in E(F) :\ \boldsymbol{j_x} = \pi^{i_y}_{i_x}(\boldsymbol{j_y}) \text{ and } \sigma_x > \sigma_y\}$$

be the *homogeneous descent set* of w and we define the *flag-major index* of w as

$$\operatorname{fmaj}(w) := \sum_{e \in E(F)} i_e \chi_e(w) h_e + \sum_{v \in V(F)} z_v(w) h_v,$$

where

$$i_{(x,y)} := i_x \text{ for each } (x,y) \in E(F) \qquad \text{and} \qquad \chi_e(w) := \begin{cases} 1 & \text{if } e \in \operatorname{HDes}(w), \\ 0 & \text{otherwise.} \end{cases}$$

Example 2.18. Let w be the 6-partial labeling in Figure 2.4. Then $\mathscr{L}(w)$ is the following subset of $G(6,5)$:

$$\Big\{[2^{\mathbf{0}},3^{\boldsymbol{a}},4^{\boldsymbol{b}},1^{\mathbf{4}},5^{\mathbf{2}}],[3^{\boldsymbol{a}},2^{\mathbf{0}},4^{\boldsymbol{b}},1^{\mathbf{4}},5^{\mathbf{2}}],[3^{\boldsymbol{a}},4^{\boldsymbol{b}},2^{\mathbf{0}},1^{\mathbf{4}},5^{\mathbf{2}}],[3^{\boldsymbol{a}},4^{\boldsymbol{b}},1^{\mathbf{4}},2^{\mathbf{0}},5^{\mathbf{2}}],$$
$$[3^{\boldsymbol{a}},4^{\boldsymbol{b}},1^{\mathbf{4}},5^{\mathbf{2}},2^{\mathbf{0}}],[2^{\mathbf{0}},4^{\boldsymbol{b}},3^{\boldsymbol{a}},1^{\mathbf{4}},5^{\mathbf{2}}],[4^{\boldsymbol{b}},2^{\mathbf{0}},3^{\boldsymbol{a}},1^{\mathbf{4}},5^{\mathbf{2}}],[4^{\boldsymbol{b}},3^{\boldsymbol{a}},2^{\mathbf{0}},1^{\mathbf{4}},5^{\mathbf{2}}],$$
$$[4^{\boldsymbol{b}},3^{\boldsymbol{a}},1^{\mathbf{4}},2^{\mathbf{0}},5^{\mathbf{2}}],[4^{\boldsymbol{b}},3^{\boldsymbol{a}},1^{\mathbf{4}},5^{\mathbf{2}},2^{\mathbf{0}}]\Big\},$$

where $\boldsymbol{a}\in\{\mathbf{1},\mathbf{4}\}$ and $\boldsymbol{b}\in\{\mathbf{1},\mathbf{3},\mathbf{5}\}$.
Moreover, $\mathrm{HDes}(w)=\big\{\big(w^{-1}(3^{3,\mathbf{1}}),w^{-1}(1^{6,\mathbf{4}})\big)\big\}$ and $\mathrm{fmaj}(w)=(3\cdot 1)+(2\cdot 4+2\cdot 3+1\cdot 1)=18$.

We can generalize again Theorem 2.4 in this way:

Theorem 2.19. *Let F be a finite forest with n elements and w an r-partial labeling of F. Then*

$$\sum_{g\in\mathscr{L}(w)} q^{\mathrm{fmaj}(g)} = q^{\mathrm{fmaj}(w)}\frac{[d_1]_q[d_2]_q\cdots[d_n]_q}{\prod\limits_{x\in F}[h_x i_x]_q},$$

where $d_i=ri$, $i=1,\dots,n$ are the fundamental degrees of $G(r,n)$.

Example 2.20. Let w be the 6-partial labeling in Figure 2.4. Then

$$\sum_{g\in\mathscr{L}(w)} q^{\mathrm{fmaj}(g)} = q^{18}+\sum_{k=20}^{23}q^k+2q^{24}+q^{25}+2\sum_{k=26}^{47}q^k+q^{48}+2q^{49}+\sum_{k=50}^{53}q^k+q^{55}$$

and

$$q^{\mathrm{fmaj}(w)}\frac{[r]_q[2r]_q\cdots[nr]_q}{\prod\limits_{x\in F}[h_x i_x]_q} = q^{18}\frac{[6]_q[12]_q[18]_q[24]_q[30]_q}{[6]_q[3]_q[2]_q[18]_q[24]_q} = q^{18}\sum_{k=0}^{5}q^{2k}\cdot\sum_{k=0}^{9}q^{3k}.$$

Proof of the fmaj hook length formula of Theorem 2.19

Let now w be a fixed r-partial labeling of F and

$$\mathscr{A}=\big\{f\in\mathbb{N}^n:\ f_{\sigma_x}\in\boldsymbol{j_x}\text{ if }x\in F,\text{ and }\ f_{\sigma_x}\ge f_{\sigma_y}\text{ for each }(x,y)\in E(F),$$
$$\text{where }f_{\sigma_x}=f_{\sigma_y}\text{ implies }\boldsymbol{j_x}=\pi_{i_x}^{i_y}(\boldsymbol{j_y})\text{ and }\sigma_x<\sigma_y\big\}.$$

As in the case of an r-starred labeling, we show that the set $\mathscr{A}$ consists of all g-compatible vectors in $\mathbb{N}^n$ as g varies in the set $\mathscr{L}(w)$ of linear extensions of the r-partial labeling w:

Proposition 2.21. *Let $f \in \mathbb{N}^n$. Then $f \in \mathscr{A}$ if and only if f is g-compatible for some $g \in \mathscr{L}(w)$.*

Proof. Again we divide the proof in two steps:

i) If $x \in F$, then $\pi^r_{i_x}\big(c(g^{-1}(\sigma_x))\big) = -\boldsymbol{j_x}$ if and only if $f_{\sigma_x} \in \boldsymbol{j_x}$.

Since $f_{\sigma_x} \in \boldsymbol{j_x}$ if and only if $\lambda_{|g^{-1}(\sigma_x)|}(g) \in \boldsymbol{j_x}$, then $\lambda_{\sigma_x}(g^{-1}) \in -\boldsymbol{j_x}$ from Lemma 1.15 (for $p = 1$), and this is equivalent to $\pi^r_{i_x}\big(c(g^{-1}(\sigma_x))\big) = -\boldsymbol{j_x}$.

ii) If $(x, y) \in E(F)$, then $|g^{-1}(\sigma_x)| < |g^{-1}(\sigma_y)|$ if and only if $f_{\sigma_x} \geq f_{\sigma_y}$, where $f_{\sigma_x} = f_{\sigma_y}$ implies $\sigma_x < \sigma_y$ and $\boldsymbol{j_x} = \pi^{i_y}_{i_x}(\boldsymbol{j_y})$.

$\Leftarrow$) If $f_{\sigma_x} > f_{\sigma_y}$ then $|g^{-1}(\sigma_x)| < |g^{-1}(\sigma_y)|$ since $\lambda(g)$ and λ are both partitions. If $f_{\sigma_x} = f_{\sigma_y}$ then $\lambda_{|g^{-1}(\sigma_x)|}(g) = \lambda_{|g^{-1}(\sigma_y)|}(g)$. Since $\sigma_x < \sigma_y$, then the definition of the statistics $\lambda_i(g)$ implies that $|g^{-1}(\sigma_x)| < |g^{-1}(\sigma_y)|$.
$\Rightarrow$) If $|g^{-1}(\sigma_x)| < |g^{-1}(\sigma_y)|$ then $f_{\sigma_x} \geq f_{\sigma_y}$ since $\lambda(g)$ and λ are both partitions. Moreover, we note that $f_{\sigma_x} \neq f_{\sigma_y}$ if either $\boldsymbol{j_x} = \pi^{i_y}_{i_x}(\boldsymbol{j_y})$ and $\sigma_x > \sigma_y$, or $\boldsymbol{j_x} \neq \pi^{i_y}_{i_x}(\boldsymbol{j_y})$. So the result follows by contradiction. $\square$

We let now

$$\mathscr{B} = \left\{ f \in \mathbb{N}^n : \, f_{\sigma_x} = \sum_{y \in \mathscr{F}_x} (z_y + i_y m_y) + \sum_{e \in \mathscr{E}_x} i_e \chi_e, \, m_y \in \mathbb{N}, \, x \in F \right\}$$

where we omitted the dependence from w. Similarly we show that $\mathscr{A}$ and $\mathscr{B}$ are the same set, so in particular $\mathscr{B}$ consists of all g-compatible vectors as $g \in \mathscr{L}(w)$:

Proposition 2.22. $\mathscr{A} = \mathscr{B}$.

Proof. $\supseteq$) Let $f \in \mathscr{B}$ and $x \in F$. By definition, $f_{\sigma_x} = f_{\sigma_y} + \big(z_x + i_x m_x + i_x \chi_{(x,y)}\big)$, where $y = p(x)$. Then $f \in \mathscr{A}$.
$\subseteq$) Let u be a root. Then $f_{\sigma_u} \in \boldsymbol{j_u}$, so there exists $m_u \in \mathbb{N}$ such that $f_{\sigma_u} = \mathrm{res}_{i_u}(\boldsymbol{j_u}) + i_u m_u = z_u + i_u m_u$. Let x be an element covered by u. Then there exists $m_x \in \mathbb{N}$ such that $f_{\sigma_x} = f_{\sigma_u} + \mathrm{res}_{i_x}(\boldsymbol{j_x} - \boldsymbol{j_u}) + i_x \chi_{(x,u)} + i_x m_x = f_{\sigma_u} + z_x + i_x \chi_{(x,u)} + i_x m_x$. We note that $f_{\sigma_x} \in \boldsymbol{j_x}$. We obtain the result extending this argument to every $x \in F$. $\square$

Now we are ready to prove this more general version of our main result:

Proof of Theorem 2.19. We compute the formal power series $\sum_{f\in\mathscr{A}} q^{|f|}$ in two different ways, as above. In the first computation we use Lemma 1.14 (for $p=1$) and Proposition 2.21 and we have

$$\sum_{f\in\mathscr{A}} q^{|f|} = \sum_{g\in\mathscr{L}(w)} \frac{q^{\lambda_1(g)}q^{\lambda_2(g)}\cdots q^{\lambda_n(g)}}{(1-q^r)(1-q^{2r})\cdots(1-q^{nr})} = $$
$$= \frac{\sum_{g\in\mathscr{L}(w)} q^{\mathrm{fmaj}(g)}}{(1-q^r)(1-q^{2r})\cdots(1-q^{nr})}.$$

In the second computation we use directly the definition of $\mathscr{A}$ and Proposition 2.22: using the same notations, we have

$$|f| = \sum_{x\in F} f_{\sigma_x} = \sum_{v\in V(F)} (z_v+i_v m_v)h_v + \sum_{e\in E(F)} i_e\chi_e h_e = \mathrm{fmaj}(w) + \sum_{x\in F} i_x m_x h_x,$$

where $m_x \in \mathbb{N}$, and then

$$\sum_{f\in\mathscr{A}} q^{|f|} = \sum_{f\in\mathscr{B}} q^{|f|} = \sum_{m_x\in\mathbb{N}} q^{\mathrm{fmaj}(w)+\sum\limits_{x\in F} i_x m_x h_x} = q^{\mathrm{fmaj}(w)} \frac{1}{\prod\limits_{x\in F}(1-q^{i_x h_x})}.$$

Therefore

$$\sum_{g\in\mathscr{L}(w)} q^{\mathrm{fmaj}(g)} = q^{\mathrm{fmaj}(w)} \frac{(1-q^r)(1-q^{2r})\cdots(1-q^{nr})}{\prod\limits_{x\in F}(1-q^{i_x h_x})}.$$

$\square$

Chapter 3

Counting linear extensions of forest labelings: the (r, p) case

3.1 (r, p)-Colored labelings

Let F be a finite forest with n vertices (see Section 1.6) and G^* the projective reflection group $G(r, n)/C_p$ (see Section 1.3).
Consider the action of C_p on the set $\mathscr{W}_r(F)$ of labelings defined by

$$([1^{\boldsymbol{kr/p}}, 2^{\boldsymbol{kr/p}}, \ldots, n^{\boldsymbol{kr/p}}], \sigma_x^{\boldsymbol{c_x}}) \longmapsto \sigma_x^{\boldsymbol{c_x}+\boldsymbol{kr/p}}$$

for each $x \in F$ and $k = 0, 1, \ldots, p-1$. Note that this is simply the action of the cyclic subgroup of $\mathbb{Z}_r$ of order p generated by $\boldsymbol{r/p}$ on the set $(\mathbb{Z}_r)^n$ of colors. Let $x_1, x_2, \ldots, x_n$ be a linear extension of F and denote by $\boldsymbol{c_i}$ the color of $w(x_i)$ in $w \in \mathscr{W}_r(F)$, for $i \in [n]$. Then every orbit of $(\mathbb{Z}_r)^n$ is an arithmetic progression α on $(\mathbb{Z}_r)^n$, in which the common difference is the n-tuple $(\boldsymbol{r/p}, \boldsymbol{r/p}, \ldots, \boldsymbol{r/p})$:

$$\alpha(\boldsymbol{c_1}, \boldsymbol{c_2}, \ldots, \boldsymbol{c_n}) := \big\{(\boldsymbol{c_1} + \boldsymbol{kr/p},\, \boldsymbol{c_2} + \boldsymbol{kr/p}, \ldots,\, \boldsymbol{c_n} + \boldsymbol{kr/p})\big\}_{k=0}^{p-1}$$

where $(\boldsymbol{c_1}, \boldsymbol{c_2}, \ldots, \boldsymbol{c_n}) \in (\mathbb{Z}_r)^n$.

Definition 3.1. We call (r, p)*-colored labelings* of F the orbits of $\mathscr{W}_r(F)$ under the action of C_p and we define the set of these labelings as

$$\mathscr{W}_{r,p}(F) := \mathscr{W}_r(F)/C_p.$$

See an example in Figure 3.1.

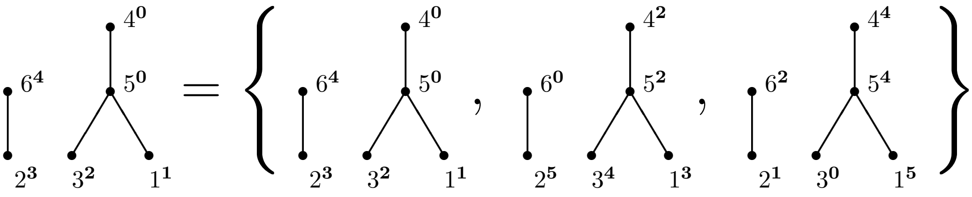

Figure 3.1: Example of $(6,3)$-colored labeling.

Remark 3.2. Note that if $(x,y) \in E(F)$ then the difference $c(\widetilde{w}_x) - c(\widetilde{w}_y)$ does not depend on the choice of $\widetilde{w}$ lift of w in $\mathscr{W}_r(F)$. Then we can define

$$\boldsymbol{c_{(x,y)}} := c(\widetilde{w}_x) - c(\widetilde{w}_y) \in \mathbb{Z}_r.$$

For $w \in \mathscr{W}_{r,p}(F)$ we define the *set of linear extensions* of w as

$$\begin{aligned}\mathscr{L}(w) := \big\{g \in G^* :\ &\text{for each } \widetilde{g} \text{ lift of } g \text{ in } G(r,n), \text{there exists } \widetilde{w} \text{ lift of } w \\ &\text{in } \mathscr{W}_r(F) \text{ s.t. } c(\widetilde{g}^{-1}(\sigma_x)) = -c(\widetilde{w}_x) \text{ if } x \in F, \text{ and} \\ &\qquad\qquad \text{if } x \prec y \text{ then } |g^{-1}(\sigma_x)| < |g^{-1}(\sigma_y)|\big\}.\end{aligned}$$

Note that a linear extension of a labeling is now an element of G^*.

Example 3.3. Let w be the $(6,3)$-colored labeling in Figure 3.1. For example the element $g = [1^{\mathbf{1}}, 3^{\mathbf{2}}, 5^{\mathbf{0}}, 4^{\mathbf{0}}, 2^{\mathbf{3}}, 6^{\mathbf{4}}] \in G(6,3,6)^*$ is a linear extension of w. A lift of g in $G(6,6)$ is an element

$$\widetilde{g} \in \left\{[1^{\boldsymbol{1+2k}}, 3^{\boldsymbol{2+2k}}, 5^{\boldsymbol{2k}}, 4^{\boldsymbol{2k}}, 2^{\boldsymbol{3+2k}}, 6^{\boldsymbol{4+2k}}],\ k = 0,1,2\right\}.$$

Then

$$\widetilde{g}^{-1} \in \left\{[1^{\boldsymbol{-1+2k}}, 5^{\boldsymbol{-3+2k}}, 2^{\boldsymbol{-2+2k}}, 4^{\boldsymbol{2k}}, 3^{\boldsymbol{2k}}, 6^{\boldsymbol{-4+2k}}],\ k = 0,1,2\right\}$$

is the inverse of $\widetilde{g} \in G(6,6)$.

We let

$$\mathrm{HDes}(w) := \{(x,y) \in E(F) :\ \boldsymbol{c_{(x,y)}} = \mathbf{0} \text{ and } \sigma_x > \sigma_y\}$$

be the *homogeneous descent set* of w and finally we define the *flag-major index* of w as the multiset

$$\mathrm{Fmaj}(w) := \\ = \left\{\!\!\left\{ \sum_{e\in E(F)} r\chi_e^{r,p}(w)h_e + \sum_{v\in V(F)} z_v(\widetilde{w})h_v, \text{ for each } \widetilde{w} \text{ lift of } w \text{ in } \mathscr{W}_r(F) \right\}\!\!\right\},$$

where

$$\chi_e^{r,p}(w) := \begin{cases} 1 & \text{if } e \in \mathrm{HDes}(w), \\ 0 & \text{otherwise.} \end{cases}$$

Remark 3.4. Note that the previous definition is equivalent to the following:

$$\mathrm{Fmaj}(w) = \left\{\!\!\left\{ \mathrm{fmaj}(\widetilde{w}), \text{ for each } \widetilde{w} \text{ lift of } w \text{ in } \mathscr{W}_r(F) \right\}\!\!\right\}.$$

Example 3.5. Let w be the $(6,3)$-colored labeling in Figure 3.1. We show that the flag-major index of w is a multiset:

$$\begin{aligned} \mathrm{Fmaj}(w) = \{\!\{ 6\cdot 3 + (2\cdot \mathrm{res}_6(4+2k) + 5 + 4\cdot \mathrm{res}_6(2k) + 2 + 1), \\ k = 0,1,2 \}\!\} = \{\!\{ 34, 34, 46 \}\!\}. \end{aligned}$$

We can generalize again Theorem 2.4 in the following:

Theorem 3.6. *Let F be a finite forest with n elements and w an (r,p)-colored labeling of F. Then*

$$\sum_{g\in\mathscr{L}(w)} q^{\mathrm{fmaj}(g)} = \sum_{s\in \mathrm{Fmaj}(w)} q^s \frac{[d_1]_q[d_2]_q\cdots[d_n]_q}{\prod_{x\in F}[h_x r]_q},$$

where $d_i = ri$ if $i < n$ and $d_n = rn/p$ are the fundamental degrees of G.

We will give a proof of this result in the most general case (see Proof of Theorem 3.20).

3.2 (r,p)-Starred labelings

Consider the action of C_p on the set $\mathscr{S}_r(F)$ defined by

$$([1^{\boldsymbol{kr/p}}, 2^{\boldsymbol{kr/p}}, \ldots, n^{\boldsymbol{kr/p}}], \sigma_x^{\boldsymbol{c_x}}) \longmapsto \begin{cases} \sigma_x^{*} & \text{if } x \in F_*, \\ \sigma_x^{\boldsymbol{c_x + kr/p}} & \text{otherwise} \end{cases}$$

for each $x \in F$ and $k = 0, 1, \ldots, p-1$.

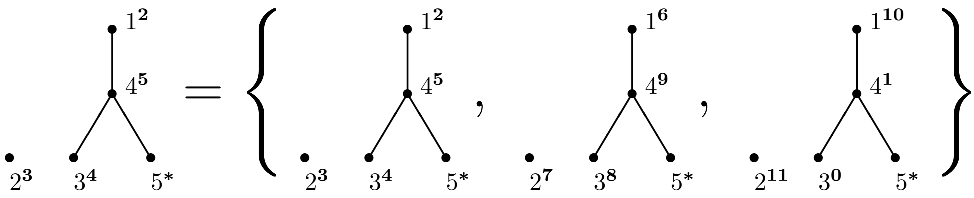

Figure 3.2: Example of $(12,3)$-starred labeling.

Definition 3.7. We call (r,p)-*starred labelings* of F the orbits of $\mathscr{S}_r(F)$ under the action of C_p and we define the set of these labelings as

$$\mathscr{S}_{r,p}(F) := \mathscr{S}_r(F)/C_p.$$

See an example in Figure 3.2.

Remark 3.8. If $F_* = F$ then the action of C_p on $\mathscr{S}_r(F)$ is trivial, i.e., if w is an r-starred labeling in which each label has color $*$, then its orbit contains only w.

We analyze (r,p)-starred labelings as a particular case of a more general type of labelings, described in the following section.

3.3 (r,p)-Partial labelings

Consider now the action of C_p on the set $\mathscr{P}_r(F)$ of labelings defined by

$$([1^{\boldsymbol{kr/p}}, 2^{\boldsymbol{kr/p}}, \ldots, n^{\boldsymbol{kr/p}}], \sigma_x^{i_x, \boldsymbol{j_x}}) \longmapsto \sigma_x^{i_x, \boldsymbol{j_x} + \boldsymbol{kr/p}}$$

for each $x \in F$ and $k = 0, 1, \ldots, p-1$. As in the colored case, we can read it as the action of the cyclic subgroup of $\mathbb{Z}_r$ of order p generated by $\boldsymbol{r/p}$ on the set Γ of colors, each defined as a residue class modulo a divisor of r. Let $x_1, x_2, \ldots, x_n$ be a linear extension of F and denote by $\boldsymbol{j_l} \in \mathbb{Z}_{i_l}$ the color of $w(x_l)$ in $w \in \mathscr{P}_r(F)$, for $l \in [n]$. Then every orbit of $\Gamma = \mathbb{Z}_{i_1} \times \mathbb{Z}_{i_2} \times \ldots \times \mathbb{Z}_{i_n}$, is an arithmetic progression α on Γ, in which the common difference is the n-tuple $(\boldsymbol{r/p}, \boldsymbol{r/p}, \ldots, \boldsymbol{r/p})$:

$$\alpha(\boldsymbol{j_1}, \boldsymbol{j_2}, \ldots, \boldsymbol{j_n}) = \left\{(\boldsymbol{j_1} + \boldsymbol{kr/p},\, \boldsymbol{j_2} + \boldsymbol{kr/p}, \ldots,\, \boldsymbol{j_n} + \boldsymbol{kr/p})\right\}_{k=0}^{p-1}$$

where $(\boldsymbol{j_1}, \boldsymbol{j_2}, \ldots, \boldsymbol{j_n}) \in \Gamma$.

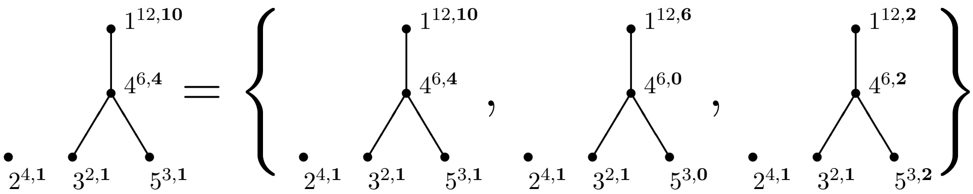

Figure 3.3: Example of $(24, 3)$-partial labeling.

Definition 3.9. We call (r, p)-*partial labelings* of F the orbits of $\mathscr{P}_r(F)$ under the action of C_p and we define the set of these labelings as

$$\mathscr{P}_{r,p}(F) := \mathscr{P}_r(F)/C_p.$$

See an example in Figure 3.3.

The following lemma is useful to determine the cardinality of these orbits:

Lemma 3.10. *Let F be a forest and $v_1, v_2, \ldots, v_l$ its roots. Let $w \in \mathscr{P}_r(F)$ and consider the action of C_p on $\mathscr{P}_r(F)$ defined as above. Then the orbit of w contains p/d distinct elements, where*

$$d := \gcd\left(\frac{r}{\operatorname{lcm}(i_1, i_2, \ldots, i_l)}, p\right), \tag{3.1}$$

and i_t denotes i_{v_t}, for $t \in [l]$.

Proof. We consider first the case in which F is a tree and then the case of a general forest.

• <u>F tree</u>
Let F be a tree and v its root. Consider $w \in \mathscr{P}_r(F)$. So, by definition, $i_x \mid i_v$ for every $x \in F$. Then the cardinality of the orbit of w depends only on the choice of i_v, as we can see in the following claim.

Claim 3.11. *If F is a tree and v is its root, then*

$$\gcd\left(\frac{r}{i_v}, p\right) = d \geq 1$$

if and only if $\boldsymbol{j_v + kr/p}$ are p/d distinct residue classes in $\mathbb{Z}_{i_v}$, for $k \in \{0, 1, \ldots, p-1\}$. Equivalently, the orbit of w contains p/d elements.

It is enough to show that the period of $\boldsymbol{r/p}$ in $\mathbb{Z}_{i_v}$ is p/d, i.e.,

$$\frac{i_v}{\gcd(r/p,\, i_v)} = \frac{p}{d}.$$

In fact,

$$\frac{i_v}{\gcd(r/p,\, i_v)} = \frac{i_v\, p}{\gcd(r,\, i_v\, p)} = \frac{p}{\gcd(r/i_v,\, p)} = \frac{p}{d}.$$

Note that, if $\gcd(r/i_v,\, p/d) = 1$, then we can replace p with p/d since the period of $\boldsymbol{rd/p}$ in $\mathbb{Z}_{i_v}$ is p/d. Otherwise, if $\gcd(r/i_v,\, p/d) = d' > 1$, then consider p/dd' and repeat the same argument.

• <u>F forest</u>

Now let F be a forest with components $T_1, T_2, \ldots, T_l$ and roots $v_1, v_2, \ldots, v_l$. Let $w \in \mathscr{P}_r(F)$. From Claim 3.11 we know that the orbit of w restricted to T_t has p/d_t elements, where $d_t = \gcd(r/i_t,\, p)$ and $i_t = i_{v_t}$, for $t \in [l]$. Then in this case the orbit of w contains as many elements as

$$\operatorname{lcm}\left(\frac{p}{d_1}, \frac{p}{d_2}, \ldots, \frac{p}{d_l}\right).$$

Then we can conclude with the following claim.

Claim 3.12. *Let F be a forest and $v_1, v_2, \ldots, v_l$ its roots. Then*

$$\gcd\left(\frac{r}{\operatorname{lcm}(i_1, i_2, \ldots, i_l)},\, p\right) = d \geq 1$$

if and only if

$$\operatorname{lcm}\left(\frac{p}{d_1}, \frac{p}{d_2}, \ldots, \frac{p}{d_l}\right) = p/d,$$

where $d_t = \gcd(r/i_t,\, p)$ and p/d_t is the period of $\boldsymbol{r/p}$ in $\mathbb{Z}_{i_t}$.

Let π be a prime that divides p. Let a and b be positive integers and c a non-negative integer, $c \leq a$, such that $\pi^a \parallel p$, $\pi^b \parallel r$ and $\pi^c \parallel d$, where the symbol $\parallel$ means "exactly divides".

$\Rightarrow$) By hypothesis

$$\pi^{c+1} \nmid \frac{r}{\operatorname{lcm}(i_1, i_2, \ldots, i_l)},$$

so there exists $t \in [l]$ such that $\pi^{b-c} \mid i_t$. Then $\pi^c \parallel d_t$ and $\pi^{a-c} \mid p/d_t$. So

$$\pi^{a-c} \mid \operatorname{lcm}\left(\frac{p}{d_1}, \frac{p}{d_2}, \ldots, \frac{p}{d_l}\right).$$

By repeating the same argument for each prime in the factorization of p, we have

$$\frac{p}{d} \mid \operatorname{lcm}\left(\frac{p}{d_1}, \frac{p}{d_2}, \dots, \frac{p}{d_l}\right).$$

The result follows, since $d \mid d_t$ and we have

$$\operatorname{lcm}\left(\frac{p}{d_1}, \frac{p}{d_2}, \dots, \frac{p}{d_l}\right) \mid \frac{p}{d}.$$

$\Leftarrow$) By hypothesis there exists $t \in [l]$ such that $\pi^{a-c} \mid p/d_t$. Then $\pi^c \mid d_t$ and $\pi^{b-c} \mid i_t$. So

$$\pi^c \mid \gcd\left(\frac{r}{\operatorname{lcm}(i_1, i_2, \dots, i_l)}, p\right)$$

and, by repeating this argument for each prime in the factorization of d, we have

$$d \mid \gcd\left(\frac{r}{\operatorname{lcm}(i_1, i_2, \dots, i_l)}, p\right).$$

Suppose that

$$d' = \gcd\left(\frac{r}{\operatorname{lcm}(i_1, i_2, \dots, i_l)}, p\right)$$

where $d \mid d'$. Then there exists a positive integer c' such that $c < c' \le a$ and $\pi^{c'} \parallel d'$. If $\pi^{c+1} \nmid d'$ we can replace π with any of the other primes in the factorization of p. Then there exists $t \in [l]$ such that $\pi^{b-c'} \parallel i_t$, so $c' = c$. We conclude that $d' = d$. $\square$

Remark 3.13. Let d be defined as in (3.1). Then the l-tuple $\left(\boldsymbol{\frac{r}{p}}, \boldsymbol{\frac{r}{p}}, \dots, \boldsymbol{\frac{r}{p}}\right)$ has period p/d in $\mathbb{Z}_{i_1} \times \mathbb{Z}_{i_2} \times \cdots \times \mathbb{Z}_{i_l}$. Moreover, by the definition of partial labeling, we have

$$\gcd\left(\frac{r}{\operatorname{lcm}(\{i_x : x \in F\})}, p\right) = d.$$

Example 3.14. Let w be the first labeling in Figure 3.4. Note that

$$d = \gcd\left(\frac{30}{\operatorname{lcm}(3,6)}, 6\right) = \gcd(5,6) = 1.$$

Consider the colors of the two roots: $(\mathbf{2}, \mathbf{1}) \in \mathbb{Z}_3 \times \mathbb{Z}_6$. These represent in the orbit of w the following colors:

$$\left\{(\mathbf{2} + \mathbf{5}\boldsymbol{k}, \mathbf{1} + \mathbf{5}\boldsymbol{k}) \in \mathbb{Z}_3 \times \mathbb{Z}_6,\ k = 0, 1, \dots, 5\right\} =$$
$$= \left\{(\mathbf{2},\mathbf{1}), (\mathbf{1},\mathbf{0}), (\mathbf{0},\mathbf{5}), (\mathbf{2},\mathbf{4}), (\mathbf{1},\mathbf{3}), (\mathbf{0},\mathbf{2})\right\},$$

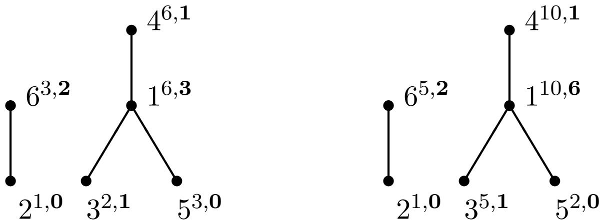

Figure 3.4: Examples of $(30, 6)$-partial labelings.

and then the cardinality of the orbit is 6. Let now w be the second labeling in Figure 3.4. We have

$$d = \gcd\left(\frac{30}{\operatorname{lcm}(5,10)}, 6\right) = \gcd(3,6) = 3.$$

Consider the colors of the two roots: $(\mathbf{2}, \mathbf{1}) \in \mathbb{Z}_5 \times \mathbb{Z}_{10}$. These represent in the orbit of w the following colors:

$$\left\{(\mathbf{2} + \mathbf{5}\boldsymbol{k}, \mathbf{1} + \mathbf{5}\boldsymbol{k}) \in \mathbb{Z}_5 \times \mathbb{Z}_{10},\ k = 0, 1, \ldots, 5\right\} = \left\{(\mathbf{2}, \mathbf{1}), (\mathbf{2}, \mathbf{6})\right\},$$

and then the cardinality of the orbit is 2.

Let $u \in \mathscr{P}_r(F)$ and denote by $j(u_x)$ the color $\boldsymbol{j_x} \in \mathbb{Z}_{i_x}$ in the label of x.

Remark 3.15. Let $w \in \mathscr{P}_{r,p}(F)$. If $(x, y) \in E(F)$, then we can consider the difference $j(\widetilde{w}_x) - j(\widetilde{w}_y)$ modulo i_x and we note that it does not depend on the choice of $\widetilde{w}$ lift of w in $\mathscr{P}_r(F)$. Then we can define

$$\boldsymbol{j_{(x,y)}} := \pi_{i_x}^{i_y}\left(j(\widetilde{w}_x) - j(\widetilde{w}_y)\right) \in \mathbb{Z}_{i_x}.$$

For $w \in \mathscr{P}_{r,p}(F)$ we define the *set of linear extensions* of w as

$$\begin{aligned}\mathscr{L}(w) := \big\{g \in G^* :\ &\text{for each } \widetilde{g} \text{ lift of } g \text{ in } G(r,n), \text{ there exists } \widetilde{w} \text{ lift of } w \\ &\text{in } \mathscr{P}_r(F) \text{ s.t. } \pi_{i_x}^r\left(c(\widetilde{g}^{-1}(\sigma_x))\right) = -j(\widetilde{w}_x) \text{ if } x \in F, \text{ and} \\ &\qquad\qquad \text{if } x \prec y \text{ then } |g^{-1}(\sigma_x)| < |g^{-1}(\sigma_y)|\big\},\end{aligned}$$

where d is defined as in (3.1).

Example 3.16. Let w be the second labeling in Figure 3.4. For example the element $g = [5^{2,\mathbf{0}}, 3^{5,\mathbf{1}}, 1^{10,\mathbf{6}}, 4^{10,\mathbf{1}}, 2^{1,\mathbf{0}}, 6^{5,\mathbf{2}}] \in G(30,6,6)^*$ is a linear extension of w. A lift of g in $G(30,6)$ is an element

$$\widetilde{g} \in \left\{[5^{2,\mathbf{5}\boldsymbol{k}}, 3^{5,\mathbf{1}+\mathbf{5}\boldsymbol{k}}, 1^{10,\mathbf{6}+\mathbf{5}\boldsymbol{k}}, 4^{10,\mathbf{1}+\mathbf{5}\boldsymbol{k}}, 2^{1,\mathbf{5}\boldsymbol{k}}, 6^{5,\mathbf{2}+\mathbf{5}\boldsymbol{k}}],\ k = 0, 1\right\}.$$

Then

$$\widetilde{g}^{-1} \in \left\{[3^{10,\boldsymbol{-6+5k}}, 5^{1,\boldsymbol{5k}}, 2^{5,\boldsymbol{-1+5k}}, 4^{10,\boldsymbol{-1+5k}}, 1^{2,\boldsymbol{5k}}, 6^{5,\boldsymbol{-2+5k}}],\ k=0,1\right\}$$

is the inverse of $\widetilde{g} \in G(30,6)$.

We let

$$\mathrm{HDes}(w) := \{(x,y) \in E(F) : \boldsymbol{j_{(x,y)}} = \mathbf{0} \text{ and } \sigma_x > \sigma_y\}$$

be the *homogeneous descent set* of w and finally we define the *flag-major index* of w as the multiset

$$\mathrm{Fmaj}(w) := \\ = \left\{\!\!\left\{ \sum_{e \in E(F)} i_e \chi_e(w) h_e + \sum_{v \in V(F)} z_v(\widetilde{w}) h_v, \text{ for each } \widetilde{w} \text{ lift of } w \text{ in } \mathscr{P}_r(F) \right\}\!\!\right\},$$

where

$$\chi_e(w) := \begin{cases} 1 & \text{if } e \in \mathrm{HDes}(w), \\ 0 & \text{otherwise.} \end{cases}$$

Remark 3.17. Note that the previous definition is equivalent to the following:

$$\mathrm{Fmaj}(w) = \left\{\!\!\left\{ \mathrm{fmaj}(\widetilde{w}), \text{ for each } \widetilde{w} \text{ lift of } w \text{ in } \mathscr{W}_r(F) \right\}\!\!\right\}.$$

Remark 3.18. Let d be defined as in (3.1). Then $|\mathrm{Fmaj}(w)| = p/d$.

Example 3.19. Let w be the first labeling in Figure 3.4. Then the flag-major index of w is the multiset:

$$\mathrm{Fmaj}(w) = \{\!\{ (2\cdot 1 + 3\cdot 1) + (2\cdot \mathrm{res}_3(2+5k) + 4\cdot \mathrm{res}_6(1+5k) + 3\cdot 2), \\ k = 0,1,\ldots,5 \}\!\} = \{\!\{19, 13, 31, 31, 25, 19\}\!\}.$$

Let w be the second labeling in Figure 3.4. Then the flag-major index of w is the multiset:

$$\mathrm{Fmaj}(w) = \{\!\{ (5\cdot 1 + 2\cdot 1) + (2\cdot \mathrm{res}_5(2+5k) + 4\cdot \mathrm{res}_{10}(1+5k) + 3\cdot 5), \\ k = 0,1 \}\!\} = \{\!\{30, 50\}\!\}.$$

Now we can generalize Theorem 2.4 in this way:

Theorem 3.20. *Let F be a finite forest with n elements and w an (r,p)-partial labeling of F. Then*

$$\sum_{g\in\mathscr{L}(w)} q^{\mathrm{fmaj}(g)} = \sum_{s\in\mathrm{Fmaj}(w)} q^s \frac{[d_1]_q[d_2]_q\cdots[d_n]_q}{\prod\limits_{x\in F}[h_x i_x]_q},$$

where $d_i = ri$ if $i<n$ and $d_n = rn/p$ are the fundamental degrees of G.

Remark 3.21. Consider the poset $\{x_1,x_2,\ldots,x_n\}$ with no order relation between any two different elements. The Hasse diagram V_n of this poset is a forest consisting of n disjoint vertices. Consider now the (r,p)-partial labeling w of V_n such that $w(x_i)=i^{1,\mathbf{0}}$ for all $i\in[n]$. This is equivalent to consider the (r,p)-starred labeling w of V_n such that $w(x_i)=i^{*}$ for all $i\in[n]$. Then $\mathrm{Fmaj}(w)=\{0\}$ and $\mathscr{L}(w)=G^*$. Therefore in this case Theorem 3.20 reduces to Corollary 1.9.

Remark 3.22. Let $k<n$. Consider the poset $\{x_1,x_2,\ldots,x_n\}$ with the ordering given by $x_i\prec x_j$ if and only if $i<j<k<n$. We called its Hasse diagram $T_{n,k}$ (see again Figure 2.3). Consider now the (r,p)-partial labeling w of $T_{n,k}$ such that $w(x_i)=i^{r,\mathbf{0}}$ for $i\in[k]$ and $w(x_i)=i^{1,\mathbf{0}}$ for $i=k+1,k+2,\ldots,n$. Then $h_{x_i}=i$ for $i\in[k]$ and $h_{x_i}=1$ otherwise, $\mathrm{Fmaj}(w)=\{0,kr/p,2kr/p,\ldots,(p-1)kr/p\}$ and $\mathscr{L}(w)=\{g\in G^* : \exists\, k\in\{0,1,\ldots,p-1\}$ s.t. $c(\widetilde{g}^{-1}(i))=\boldsymbol{k}\frac{\boldsymbol{r}}{\boldsymbol{p}}$ for each $\widetilde{g}$ lift of g in $G(r,n)$, $i\in[k]$ and $|g^{-1}(1)|<|g^{-1}(2)|<\cdots<|g^{-1}(k)|\}$. We finally note that if $g\in\mathscr{L}(w)$ then $g^{-1}\in\mathscr{C}_k$, where $\mathscr{C}_k$ is the same set defined in (1.1). Then in this case Theorem 3.20 reduces to Theorem 1.11.

Proof of the fmaj hook length formula of Theorem 3.20

Let now w be a fixed (r,p)-partial labeling of F. Let

$$\begin{aligned}\mathscr{A}=\big\{f\in\mathbb{N}^n : \exists\,\widetilde{w} \text{ lift of } w \text{ in } \mathscr{P}_r(F) \text{ s.t. } f_{\sigma_x}\in j(\widetilde{w}_x) \text{ if } x\in F, \text{ and}\\ f_{\sigma_x}\geq f_{\sigma_y} \text{ for each } (x,y)\in E(F),\ \text{where } f_{\sigma_x}=f_{\sigma_y}\\ \text{implies } \boldsymbol{j_{(x,y)}}=\mathbf{0} \text{ and } \sigma_x<\sigma_y\big\}\end{aligned}$$

and we show that $\mathscr{A}$ consists of all g-compatible vectors in $\mathbb{N}^n$ as g varies in the set $\mathscr{L}(w)$ of linear extensions of the (r,p)-partial labeling w.

Proposition 3.23. *Let $f \in \mathbb{N}^n$. Then $f \in \mathscr{A}$ if and only if f is g-compatible for some $g \in \mathscr{L}(w)$.*

Proof. We recall that f is g-compatible if and only if $f_i = \lambda_{|g^{-1}(i)|}(g) + r\lambda_{|g^{-1}(i)|} + h\frac{r}{p}$ for all $i \in [n]$, where $\lambda \in \mathscr{P}_n$ and $h \in \{0,1,\ldots,p-1\}$ (Lemma 1.13). We divide the proof in two steps:

i) For each $x \in F$, there exists $\widetilde{w}$ lift of w in $\mathscr{P}_r(F)$ such that

$$\pi^r_{i_x}\left(c(\widetilde{g}^{-1}(\sigma_x))\right) = -j(\widetilde{w}_x)$$

for each $\widetilde{g}$ lift of g in $G(r,n)$ if and only if there exists $\widehat{w}$ lift of w in $\mathscr{P}_r(F)$ such that $f_{\sigma_x} \in j(\widehat{w}_x)$.

Since $f_{\sigma_x} \in j(\widehat{w}_x)$ if and only if $\lambda_{|g^{-1}(\sigma_x)|}(g) + hr/p \in j(\widehat{w}_x)$, then for Lemma 1.15 there exists $k \in \{0,1,\ldots,p-1\}$ such that $\lambda_{\sigma_x}(g^{-1}) + kr/p \in -j(\widehat{w}_x)$ and this is equivalent to say that for each $\widetilde{g}$ lift of g in $G(r,n)$ there exists $\widetilde{w}$ lift of w in $\mathscr{P}_r(F)$ such that $\pi^r_{i_x}\left(c(\widetilde{g}^{-1}(\sigma_x))\right) = -j(\widetilde{w}_x)$.

ii) If $(x,y) \in E(F)$, then $|g^{-1}(\sigma_x)| < |g^{-1}(\sigma_y)|$ if and only if $f_{\sigma_x} \geq f_{\sigma_y}$, where $f_{\sigma_x} = f_{\sigma_y}$ implies $\sigma_x < \sigma_y$ and $\boldsymbol{j_{(x,y)}} = \mathbf{0}$.

$\Leftarrow$) If $f_{\sigma_x} > f_{\sigma_y}$ then $|g^{-1}(\sigma_x)| < |g^{-1}(\sigma_y)|$ since $\lambda(g)$ and λ are both partitions and the result follows. If $f_{\sigma_x} = f_{\sigma_y}$ then $\lambda_{|g^{-1}(\sigma_x)|}(g) = \lambda_{|g^{-1}(\sigma_y)|}(g)$. Then the definition of the statistics $\lambda_i(g)$ implies that $|g^{-1}(\sigma_x)| < |g^{-1}(\sigma_y)|$, since $\sigma_x < \sigma_y$.
$\Rightarrow$) If $|g^{-1}(\sigma_x)| < |g^{-1}(\sigma_y)|$ then $f_{\sigma_x} \geq f_{\sigma_y}$ since $\lambda(g)$ and λ are both partitions. Moreover, we note that $f_{\sigma_x} \neq f_{\sigma_y}$ if $\boldsymbol{j_{(x,y)}} = \mathbf{0}$ and $\sigma_x > \sigma_y$, or if $\boldsymbol{j_{(x,y)}} \neq \mathbf{0}$. So the result follows by contradiction. $\square$

We let now

$$\mathscr{B} = \Bigg\{ f \in \mathbb{N}^n : \exists\, \widetilde{w} \text{ lift of } w \text{ s.t.}$$
$$f_{\sigma_x} = \sum_{y \in \mathscr{F}_x} (z_y(\widetilde{w}) + i_y m_y) + \sum_{e \in \mathscr{E}_x} i_e \chi_e, \text{ for each } x \in F,\ m_y \in \mathbb{N} \Bigg\}$$

where we omitted the dependence from w. Again we show that $\mathscr{A}$ and $\mathscr{B}$ are the same set, so in particular $\mathscr{B}$ consists of all g-compatible vectors as $g \in \mathscr{L}(w)$:

Proposition 3.24. $\mathscr{A} = \mathscr{B}$.

Proof. $\supseteq$) Let $f \in \mathscr{B}$ and $x \in F$. By definition, $f_{\sigma_x} = f_{\sigma_y} + \big(z_x(\widetilde{w}) + i_x m_x + i_x \chi_{(x,y)}\big)$, where $y = p(x)$. Then $f \in \mathscr{A}$.
$\subseteq$) Let u be a root. Then there exists $\widetilde{w}$ lift of w such that $f_{\sigma_u} \in j(\widetilde{w}_u)$, so there exists $m_u \in \mathbb{N}$ such that $f_{\sigma_u} = \operatorname{res}_{i_u}(j(\widetilde{w}_u)) + i_u m_u = z_u(\widetilde{w}) + i_u m_u$. Let x be an element covered by u. Then there exists $m_x \in \mathbb{N}$ such that $f_{\sigma_x} = f_{\sigma_u} + \operatorname{res}_{i_x}\big(\boldsymbol{j_{(x,y)}}\big) + i_x \chi_{(x,u)} + i_x m_x = f_{\sigma_u} + z_x(\widetilde{w}) + i_x \chi_{(x,u)} + i_x m_x$. We note that $f_{\sigma_x} \in j(\widetilde{w}_x)$. We obtain the result extending this argument to every $x \in F$. □

Now we are ready to prove the most general version of our main result:

Proof of Theorem 3.20. We compute the formal power series $\sum_{f \in \mathscr{A}} q^{|f|}$ in two different ways. In the first computation we use Lemma 1.14 and Proposition 3.23 and we have

$$\sum_{f \in \mathscr{A}} q^{|f|} = \sum_{g \in \mathscr{L}(w)} \frac{q^{\lambda_1(g)} q^{\lambda_2(g)} \cdots q^{\lambda_n(g)}}{(1-q^r)(1-q^{2r}) \cdots (1-q^{(n-1)r})(1-q^{nr/p})}$$
$$= \frac{\sum_{g \in \mathscr{L}(w)} q^{\operatorname{fmaj}(g)}}{(1-q^r)(1-q^{2r}) \cdots (1-q^{(n-1)r})(1-q^{nr/p})}.$$

In the second computation we use directly the definition of $\mathscr{A}$ and Proposition 3.24: using the same notations, we have

$$|f| = \sum_{x \in F} f_{\sigma_x} = \sum_{v \in V(F)} \left(z_v(\widetilde{w}) + i_v m_v\right) h_v + \sum_{e \in E(F)} i_e \chi_e h_e = s + \sum_{x \in F} i_x m_x h_x,$$

where $m_x \in \mathbb{N}$, $s \in \operatorname{Fmaj}(w)$, and then

$$\sum_{f \in \mathscr{A}} q^{|f|} = \sum_{f \in \mathscr{B}} q^{|f|} = \sum_{m_x \in \mathbb{N}} \left(\sum_{s \in \operatorname{Fmaj}(w)} q^s \right) q^{\sum\limits_{x \in F} i_x m_x h_x}$$
$$= \sum_{s \in \operatorname{Fmaj}(w)} q^s \frac{1}{\prod\limits_{x \in F} (1 - q^{i_x h_x})}.$$

Therefore

$$\sum_{g \in \mathscr{L}(w)} q^{\operatorname{fmaj}(g)} = \sum_{s \in \operatorname{Fmaj}(w)} q^s \frac{(1-q^r)(1-q^{2r}) \cdots (1-q^{(n-1)r})(1-q^{nr/p})}{\prod\limits_{x \in F} (1 - q^{i_x h_x})}.$$

□

Chapter 4

Counting forest labelings

4.1 q-Counting colored labelings

Let F be a finite forest with n vertices (see Section 1.6). In this chapter we generalize the result in Theorem 1.21 by q-counting the set of all labelings of a fixed forest F using the fmaj statistic, for each type of labeling defined in Chapters 2 and 3. We recall from [7] that, for any fixed $\sigma \in \mathcal{S}_n$, there are

$$\frac{n!}{\prod_{x \in F} h_x}$$

labelings w of F such that σ is a linear extension of w, since there is a bijection between the set $\{w \in \mathscr{W}(F) : \sigma \in \mathscr{L}(w)\}$ and the set $\mathscr{L}(F)$ of linear extensions of F. The same argument also applies to any element $g \in G(r, n)$, respectively $g \in G^*$, where $G = G(r, p, n)$. So we have the following result.

Remark 4.1. Let $g \in G(r, n)$ and $u \in \mathscr{W}_r(F)$. Then there exists a bijection

$$\{w \in \mathscr{W}_r(F) : g \in \mathscr{L}(w)\} \to \mathscr{L}(u).$$

Similarly, let now $g \in G^*$ and $u \in \mathscr{W}_{r,p}(F)$. Then there exists a bijection

$$\{w \in \mathscr{W}_{r,p}(F) : g \in \mathscr{L}(w)\} \to \mathscr{L}(u).$$

Moreover, if $w \in \mathscr{W}_r(F)$ or $w \in \mathscr{W}_{r,p}(F)$, then we have

$$|\mathscr{L}(w)| = \frac{n!}{\prod_{x \in F} h_x}.$$

To see this, let $x_1, x_2, \ldots, x_n$ be a linear extension of F. Let $g = [g_1, g_2, \ldots, g_n]$, where $g_i = \sigma_i^{\boldsymbol{c_i}}$ for $i = 1, 2, \ldots, n$. Then the labeling w, defined by $w(x_i) = g_i$ for $i = 1, 2, \ldots, n$, satisfies $g \in \mathscr{L}(w)$. With this labeling we associate the linear extension h such that $h_i = w(x_i)$ for $i = 1, 2, \ldots, n$. Then $h = g$. Vice versa, consider the element h such that $h_i = u(x_i)$ for $i = 1, 2, \ldots, n$. Clearly $h \in \mathscr{L}(u)$. With this linear extension we associate the labeling w such that $w(x_i) = h_i$. Then $w = u$. Thus, the map is a bijection.
Note that we did not need to specify $g \in G(r, n)$ and $u \in \mathscr{W}_r(F)$, or $g \in G^*$ and $u \in \mathscr{W}_{r,p}(F)$, since the proof is the same. Note also that $|\mathscr{L}(w)| = |\mathscr{L}(F)|$, so this cardinality does not depend on the choice of the colored labeling w.

Theorem 4.2. *Let F be a finite forest with n elements and $\mathscr{W}_r(F)$ the set of all r-colored labelings of F. Then*

$$\sum_{w \in \mathscr{W}_r(F)} q^{\mathrm{fmaj}(w)} = \frac{n!}{\prod_{x \in F} h_x} \prod_{x \in F} [h_x r]_q.$$

Remark 4.3. For $r = 2$, the result was given in [10] (Theorem 2.3).

Proof. We consider the double sum

$$\sum_{w \in \mathscr{W}_r(F)} \sum_{g \in \mathscr{L}(w)} q^{\mathrm{fmaj}(g)}$$

and we evaluate it in two different ways. In the first computation we use Theorem 2.4 and we have

$$\begin{aligned}
\sum_{w \in \mathscr{W}_r(F)} \sum_{g \in \mathscr{L}(w)} q^{\mathrm{fmaj}(g)} &= \sum_{w \in \mathscr{W}_r(F)} q^{\mathrm{fmaj}(w)} \frac{[r]_q [2r]_q \cdots [nr]_q}{\prod_{x \in F} [h_x r]_q} \\
&= \frac{[r]_q [2r]_q \cdots [nr]_q}{\prod_{x \in F} [h_x r]_q} \sum_{w \in \mathscr{W}_r(F)} q^{\mathrm{fmaj}(w)}.
\end{aligned}$$

In the second computation we exchange the order of summations and use Remark 4.1 and Corollary 1.10. Let χ denotes the indicator function which has value 1 when the argument is true and 0 otherwise. Then we have

$$\begin{aligned}\sum_{w\in\mathscr{W}_r(F)}\ \sum_{g\in\mathscr{L}(w)} q^{\mathrm{fmaj}(g)} &= \sum_{w\in\mathscr{W}_r(F)}\ \sum_{g\in G(r,n)} q^{\mathrm{fmaj}(g)}\,\chi(g\in\mathscr{L}(w)) = \\ &= \sum_{g\in G(r,n)}\ \sum_{w\in\mathscr{W}_r(F)} q^{\mathrm{fmaj}(g)}\,\chi(g\in\mathscr{L}(w)) = \\ &= \sum_{g\in G(r,n)} q^{\mathrm{fmaj}(g)} \sum_{w\in\mathscr{W}_r(F)} \chi(g\in\mathscr{L}(w)) = \\ &= |\mathscr{L}(F)| \sum_{g\in G(r,n)} q^{\mathrm{fmaj}(g)} = \\ &= \frac{n!}{\prod_{x\in F} h_x}[r]_q[2r]_q\cdots[nr]_q.\end{aligned}$$

Therefore by equating

$$\frac{[r]_q[2r]_q\cdots[nr]_q}{\prod_{x\in F}[h_x r]_q}\sum_{w\in\mathscr{W}_r(F)} q^{\mathrm{fmaj}(w)} = \frac{n!}{\prod_{x\in F} h_x}[r]_q[2r]_q\cdots[nr]_q$$

and we have the result. □

Theorem 4.4. *Let F be a finite forest with n elements and $\mathscr{W}_{r,p}(F)$ the set of all (r,p)-colored labelings of F. Then*

$$\sum_{w\in\mathscr{W}_{r,p}(F)}\ \sum_{s\in\mathrm{Fmaj}(w)} q^s = \frac{n!}{\prod_{x\in F} h_x}\prod_{x\in F}[h_x r]_q.$$

Proof. Again we consider the double sum

$$\sum_{w\in\mathscr{W}_{r,p}(F)}\ \sum_{g\in\mathscr{L}(w)} q^{\mathrm{fmaj}(g)}$$

and we evaluate it in two different ways. In the first computation by Theorem 3.6 we have

$$\begin{aligned}&\sum_{w\in\mathscr{W}_{r,p}(F)}\ \sum_{g\in\mathscr{L}(w)} q^{\mathrm{fmaj}(g)} = \\ &\qquad = \frac{[r]_q[2r]_q\cdots[(n-1)r]_q[nr/p]_q}{\prod_{x\in F}[h_x r]_q}\sum_{w\in\mathscr{W}_{r,p}(F)}\ \sum_{s\in\mathrm{Fmaj}(w)} q^s.\end{aligned}$$

In the second computation by exchanging the order of summations and using Remark 4.1 and Corollary 1.9 we have

$$\begin{aligned}\sum_{w\in\mathscr{W}_{r,p}(F)}\sum_{g\in\mathscr{L}(w)} q^{\mathrm{fmaj}(g)} &= \sum_{w\in\mathscr{W}_{r,p}(F)}\sum_{g\in G^*} q^{\mathrm{fmaj}(g)}\,\chi(g\in\mathscr{L}(w)) = \\ &= \sum_{g\in G^*} q^{\mathrm{fmaj}(g)} \sum_{w\in\mathscr{W}_{r,p}(F)} \chi(g\in\mathscr{L}(w)) = \\ &= \frac{n!}{\prod\limits_{x\in F} h_x}[r]_q[2r]_q\cdots[(n-1)r]_q[nr/p]_q.\end{aligned}$$

Therefore by equating we have the result. □

Let now T be a linear tree and w an arbitrary (r,p)-colored labeling of T. We let

$$\mathrm{fmaj}(w) := \min\ \mathrm{Fmaj}(w)$$

be the smallest value of the multiset $\mathrm{Fmaj}(w)$. If $g\in G^*$ is the unique linear extension of w, then $\mathrm{fmaj}(w)=\mathrm{fmaj}(g)$. If $p=1$, see Remark 2.3. We have the following result.

Corollary 4.5. *Let T be a linear tree with n elements and $\mathscr{W}_{r,p}(T)$ the set of all (r,p)-colored labelings of T. Then*

$$\sum_{w\in\mathscr{W}_{r,p}(T)} q^{\mathrm{fmaj}(w)} = \prod_{k=1}^{n-1}[kr]_q[rn/p]_q.$$

Proof. Note that

$$\sum_{w\in\mathscr{W}_{r,p}(T)} q^{\mathrm{fmaj}(w)} = \sum_{g\in G^*} q^{\mathrm{fmaj}(g)}.$$

Then the result follows from Corollary 1.9. □

4.2 q-Counting partial labelings

We can generalize the previous results to partial labelings of a fixed forest F in the following way. Let $x_1,x_2,\ldots,x_n$ be a linear extension of F. We fix the vector $I:=(i_1,i_2,\ldots,i_n)\in\mathbb{N}^n$, where i_k is a positive divisor of r for $k=1,2,\ldots,n$, and i_j is a divisor of i_k if x_j is covered by x_k in the forest

F. We let $\Gamma_I := \mathbb{Z}_{i_1} \times \mathbb{Z}_{i_2} \times \ldots \times \mathbb{Z}_{i_n}$ and denote by

$$\mathscr{P}_{r,I}(F) := \{w \in \mathscr{P}_r(F) : (\gamma(x_1), \gamma(x_2), \ldots, \gamma(x_n)) \in \Gamma_I, \\ \text{for each } x_1, x_2 \ldots, x_n \in \mathscr{L}(F)\},$$

where γ is given in Definition 2.16, the set of all r-partial labelings w of F in which each color is defined as a residue class modulo a fixed divisor of r. Let now $w \in \mathscr{P}_{r,I}(F)$ and consider the following equivalence relation on $\mathscr{L}(w)$: if $g, h \in \mathscr{L}(w)$ then

$$g \sim h \quad \text{if and only if} \quad |g| = |h| \in \mathcal{S}_n.$$

We denote by $\mathscr{L}(w)/\!\sim$ the set of all equivalence classes.

Proposition 4.6. *Let $g \in G(r,n)$ and $u \in \mathscr{P}_{r,I}(F)$. Then there exists a bijection*

$$\{w \in \mathscr{P}_{r,I}(F) : g \in \mathscr{L}(w)\} \to \mathscr{L}(u)/\!\sim .$$

Moreover, if $w \in \mathscr{P}_{r,I}(F)$, then we have

$$|\mathscr{L}(w)| = \frac{n!\, r^n}{\prod\limits_{x \in F} h_x i_x}.$$

Proof. We use the same idea shown in Remark 4.1. So let $x_1, x_2, \ldots, x_n$ be a linear extension of F and $g = [g_1, g_2, \ldots, g_n] \in G(r,n)$, where $g_k = \sigma_k^{\boldsymbol{c_k}}$ for $k = 1, 2, \ldots, n$. Then the partial labeling w, defined by

$$w(x_k) = \sigma_k^{\pi^r_{i_k}(\boldsymbol{c_k})} = \sigma_k^{i_k, \boldsymbol{j_k}}$$

for $k = 1, 2, \ldots, n$, satisfies $g \in \mathscr{L}(w)$. With this labeling we associate the equivalent class $\overline{h}$ of linear extensions such that, for each $h \in \overline{h}$, we have $|h| = \sigma$ and $c(h_k) \in (\pi^r_{i_k})^{-1}(\boldsymbol{j_k})$ for $k = 1, 2, \ldots, n$. Then $\overline{h} = \overline{g}$. Vice versa, let u be the labeling defined by $u(x_k) = \tau_k^{i_k, \boldsymbol{j_k}}$ for $k = 1, 2, \ldots, n$ and consider the class $\overline{h}$ of linear extensions such that, for each $h \in \overline{h}$, we have $|h| = \tau$ and $c(h_k) = \boldsymbol{c_k} \in (\pi^r_{i_k})^{-1}(\boldsymbol{j_k})$ for $k = 1, 2, \ldots, n$. Clearly $h \in \mathscr{L}(u)$. With this class we associate the labeling w such that $w(x_k) = \tau_k^{\pi^r_{i_k}(\boldsymbol{c_k})} = \tau_k^{i_k, \boldsymbol{j_k}}$. Then $w = u$. Thus, the map is a bijection.

Let now $\overline{g} \in \mathscr{L}(w)/\!\sim$ and note that the cardinality of $\overline{g}$ is

$$\mathscr{C} := \prod_{x \in F} \frac{r}{i_x},$$

since $\boldsymbol{j_k} \in \mathbb{Z}_{i_k}$ represents r/i_k distinct classes in $\mathbb{Z}_r$, for each $k = 1, 2, \ldots, n$. Then we have

$$|\mathscr{L}(w)| = \mathscr{C}\, |\mathscr{L}(w)/\!\sim| = \mathscr{C}\, |\mathscr{L}(F)|,$$

so this cardinality does not depend on the choice of the partial labeling w. $\square$

Theorem 4.7. *Let F be a finite forest with n elements and $\mathscr{P}_{r,I}(F)$ the set of all r-partial labelings of F where vector I is fixed. Then*

$$\sum_{w \in \mathscr{P}_{r,I}(F)} q^{\mathrm{fmaj}(w)} = \frac{n!\, r^n}{\prod_{x \in F} h_x i_x} \prod_{x \in F} [h_x i_x]_q.$$

Proof. We consider the double sum

$$\sum_{w \in \mathscr{P}_{r,I}(F)} \sum_{g \in \mathscr{L}(w)} q^{\mathrm{fmaj}(g)}$$

and we evaluate it in two different ways. In the first computation by Theorem 2.19 we have

$$\sum_{w \in \mathscr{P}_{r,I}(F)} \sum_{g \in \mathscr{L}(w)} q^{\mathrm{fmaj}(g)} = \frac{[r]_q [2r]_q \cdots [nr]_q}{\prod_{x \in F} [h_x i_x]_q} \sum_{w \in \mathscr{P}_{r,I}(F)} q^{\mathrm{fmaj}(w)}.$$

In the second computation by exchanging the order of summations and using Proposition 4.6 and Corollary 1.10 we have

$$\begin{aligned}
\sum_{w \in \mathscr{P}_{r,I}(F)} \sum_{g \in \mathscr{L}(w)} q^{\mathrm{fmaj}(g)} &= \sum_{w \in \mathscr{P}_{r,I}(F)} \sum_{g \in G(r,n)} q^{\mathrm{fmaj}(g)}\, \chi(g \in \mathscr{L}(w)) = \\
&= \sum_{g \in G(r,n)} q^{\mathrm{fmaj}(g)} \sum_{w \in \mathscr{P}_{r,I}(F)} \chi(g \in \mathscr{L}(w)) = \\
&= \mathscr{C}\, |\mathscr{L}(F)| \sum_{g \in G(r,n)} q^{\mathrm{fmaj}(g)} = \\
&= \frac{n!\, r^n}{\prod_{x \in F} h_x i_x} [r]_q [2r]_q \cdots [nr]_q.
\end{aligned}$$

Therefore by equating we have the result. $\square$

We denote now by

$$\mathscr{P}_{r,p,I}(F) := \mathscr{P}_{r,I}(F)/C_p$$

the set of all (r,p)-partial labelings w of F in which each color is defined as a residue class modulo a fixed divisor of r. Let now $w \in \mathscr{P}_{r,p,I}(F)$ and consider again the equivalence relation on $\mathscr{L}(w)$ such that, if $g, h \in \mathscr{L}(w)$, then

$$g \sim h \quad \text{if and only if} \quad |g| = |h| \in \mathcal{S}_n.$$

We denote by $\mathscr{L}(w)/\sim$ the set of all equivalence classes.

Proposition 4.8. *Let $g \in G^*$ and $u \in \mathscr{P}_{r,p,I}(F)$. Then there exists a bijection*

$$\{w \in \mathscr{P}_{r,p,I}(F) : g \in \mathscr{L}(w)\} \to \mathscr{L}(u)/\sim .$$

Moreover, if $w \in \mathscr{P}_{r,p,I}(F)$, then we have

$$|\mathscr{L}(w)| = \frac{n!\, r^n}{d \prod\limits_{x \in F} h_x i_x},$$

where

$$d = \gcd\left(\frac{r}{\operatorname{lcm}(i_1, i_2, \ldots, i_n)}, p\right). \tag{4.1}$$

Proof. For the first part the proof is the same as in Proposition 4.6, where now $g \in G^*$ and $u \in \mathscr{P}_{r,p,I}(F)$.
Moreover, the cardinality $|\mathscr{L}(w)|$ does not depend on the choice of the partial labeling w, since we have

$$|\mathscr{L}(w)| = \overline{\mathscr{C}}\, |\mathscr{L}(w)/\sim| = \overline{\mathscr{C}}\, |\mathscr{L}(F)|,$$

where $\overline{\mathscr{C}}$ is the cardinality of class $\overline{g} \in \mathscr{L}(w)/\sim$ as a subset of $\mathscr{L}(w)$. We just need to prove that

$$\overline{\mathscr{C}} = \frac{r^n}{d \prod\limits_{x \in F} i_x}.$$

To see this, we can compute the number $\mathscr{C}$ of (distinct) lifts of $\overline{g}$ in $G(r,n)$ and then divide this number for p, to obtain the number of (distinct) representatives of these lifts in G^*. This is equivalent to prove that

$$\mathscr{C} = \frac{r^n p}{d \prod\limits_{x \in F} i_x}.$$

Consider the n-tuple of colors $(\boldsymbol{j_1}, \boldsymbol{j_2}, \ldots, \boldsymbol{j_n}) \in \Gamma_I$. From Remark 3.13 we know that the period of the n-tuple $(\boldsymbol{r/p}, \boldsymbol{r/p}, \ldots, \boldsymbol{r/p})$ in Γ_I is p/d. Therefore, the set

$$J := \left\{ \left(\boldsymbol{j_1} + \boldsymbol{k}\frac{\boldsymbol{r}}{\boldsymbol{p}},\, \boldsymbol{j_2} + \boldsymbol{k}\frac{\boldsymbol{r}}{\boldsymbol{p}},\, \ldots,\, \boldsymbol{j_n} + \boldsymbol{k}\frac{\boldsymbol{r}}{\boldsymbol{p}} \right) \in \Gamma_I \,:\, k = 0, 1, \ldots, p-1 \right\}$$

contains p/d distinct elements. Now we note that each $\boldsymbol{j_l} \in \mathbb{Z}_{i_l}$ represents r/i_l distinct classes in $\mathbb{Z}_r$, for $l \in [n]$. So each element of J corresponds to

$$\frac{r^n}{i_1 i_2 \cdots i_n}$$

distinct elements in $(\mathbb{Z}_r)^n$. Then the result follows. □

Theorem 4.9. *Let F be a finite forest with n elements and $\mathscr{P}_{r,p,I}(F)$ the set of all (r,p)-partial labelings of F where vector I is fixed. Then*

$$\sum_{w \in \mathscr{P}_{r,p,I}(F)} \sum_{s \in \mathrm{Fmaj}(w)} q^s = \frac{n!\, r^n}{d \prod_{x \in F} h_x i_x} \prod_{x \in F} [h_x i_x]_q,$$

where d is defined as in (4.1).

Proof. Again we consider the double sum

$$\sum_{w \in \mathscr{P}_{r,p,I}(F)} \sum_{g \in \mathscr{L}(w)} q^{\mathrm{fmaj}(g)}$$

and we evaluate it in two different ways. In the first computation by Theorem 3.20 we have

$$\sum_{w \in \mathscr{P}_{r,p,I}(F)} \sum_{g \in \mathscr{L}(w)} q^{\mathrm{fmaj}(g)} = \\ = \frac{[r]_q [2r]_q \cdots [(n-1)r]_q [nr/p]_q}{\prod_{x \in F} [h_x i_x]_q} \sum_{w \in \mathscr{P}_{r,p,I}(F)} \sum_{s \in \mathrm{Fmaj}(w)} q^s.$$

In the second computation by exchanging the order of summations and using

Proposition 4.8 and Corollary 1.9 we have

$$\begin{aligned}\sum_{w\in\mathscr{P}_{r,p,I}(F)}\sum_{g\in\mathscr{L}(w)} q^{\mathrm{fmaj}(g)} &= \sum_{w\in\mathscr{P}_{r,p,I}(F)}\sum_{g\in G^*} q^{\mathrm{fmaj}(g)}\,\chi(g\in\mathscr{L}(w)) = \\ &= \sum_{g\in G^*} q^{\mathrm{fmaj}(g)} \sum_{w\in\mathscr{P}_{r,p,I}(F)} \chi(g\in\mathscr{L}(w)) = \\ &= \overline{\mathscr{C}}\,|\mathscr{L}(F)| \sum_{g\in G^*} q^{\mathrm{fmaj}(g)} = \\ &= \frac{n!\,r^n}{d\prod\limits_{x\in F} h_x i_x}[r]_q[2r]_q\cdots[(n-1)r]_q[nr]_q.\end{aligned}$$

Therefore by equating we have the result. $\square$

4.3 A particular case: the disjoint union of two linear trees

Consider the case in which the poset F is the disjoint union of two totally ordered sets, i.e., F consists of two linear trees T_1 and T_2. Let n be the size of T_1 and m the size of T_2, so $n+m$ is the size of F. For $i=1,2$, let v_i be the root of T_i. If w is an arbitrary r-colored labeling of F, let w_i be the restriction of w to the linear tree T_i. Note that, if u_i is an r-colored labeling of T_i such that $c(u_i(x)) = c(w_i(x))$ for each $x\in T_i$ and $\mathrm{HDes}(u_i) = \mathrm{HDes}(w_i)$, then $\mathrm{fmaj}(u_i) = \mathrm{fmaj}(w_i)$. Then

$$\mathrm{fmaj}(w) = \mathrm{fmaj}(u_1) + \mathrm{fmaj}(u_2),$$

from the definition of fmaj. Finally, by noting that $|\mathscr{W}_r(T_1)| = n!\,r^n$, $|\mathscr{W}_r(T_2)| = m!\,r^m$ and $|\mathscr{W}_r(F)| = (n+m)!\,r^{n+m}$, from Theorem 4.2 we have

$$\begin{aligned}\sum_{w\in\mathscr{W}_r(F)} q^{\mathrm{fmaj}(w)} &= \frac{(n+m)!}{n!\,m!}\cdot\prod_{i=1}^{2}\sum_{u_i\in\mathscr{W}_r(T_i)} q^{\mathrm{fmaj}(u_i)} \\ &= \binom{n+m}{n}\cdot\prod_{k=1}^{n}[kr]_q\cdot\prod_{l=1}^{m}[lr]_q.\end{aligned}$$

Then, for this q-counting, consider two independent labelings for T_1 and T_2 is equivalent to consider a total labeling for F, up to a constant. For this reason, in this section we study the product $\mathscr{W}_r(T_1)\times\mathscr{W}_r(T_2)$ and two particular sets obtained from it.

Let C_p be the cyclic subgroup of $G(r,n) \times G(r,m)$ of order p generated by

$$([1^{\boldsymbol{r/p}}, 2^{\boldsymbol{r/p}}, \ldots, n^{\boldsymbol{r/p}}],\ [1^{\boldsymbol{r/p}}, 2^{\boldsymbol{r/p}}, \ldots, m^{\boldsymbol{r/p}}]).$$

For any $w \in \mathscr{W}_r(T_1) \times \mathscr{W}_r(T_2)$ we denote by

$$\mathrm{col}(w) := \sum_{x \in F} c(w_x)$$

the *color weight* of w. We consider two particular sets obtained from $\mathscr{W}_r(T_1) \times \mathscr{W}_r(T_2)$: its subset

$$\begin{aligned} \mathscr{G} &:= \Gamma_p(\mathscr{W}_r(T_1) \times \mathscr{W}_r(T_2)) \\ &:= \{w \in \mathscr{W}_r(T_1) \times \mathscr{W}_r(T_2) : \mathrm{col}(w) \equiv 0 \bmod p\}, \end{aligned}$$

and its quotient

$$\mathscr{H} := \frac{\mathscr{W}_r(T_1) \times \mathscr{W}_r(T_2)}{C_p},$$

where the action of C_p on the set $\mathscr{W}_r(T_1) \times \mathscr{W}_r(T_2)$ is defined by adding the same multiple of r/p to all the colors of the labels of F.
Consider the subset $\mathscr{G}$ and q-count all its elements according to the fmaj index. Then we have the following result:

Proposition 4.10. *Let T_1 and T_2 be linear trees of size n and m, respectively. Let $\mathscr{G} = \Gamma_p(\mathscr{W}_r(T_1) \times \mathscr{W}_r(T_2))$. Then*

$$\sum_{w \in \mathscr{G}} q^{\mathrm{fmaj}(w)} = \mathrm{Deg}_p\left(\prod_{k=1}^{n} [kr]_q \cdot \prod_{l=1}^{m} [lr]_q\right),$$

where

$$\mathrm{Deg}_p\left(\sum_{k \geq 0} c_k\, q^k\right) := \sum_{k \geq 0} c_{kp}\, q^{kp}.$$

Proof. From the definition of fmaj and Theorem 4.2, we have

$$\sum_{w \in \mathscr{W}_r(T_1) \times \mathscr{W}_r(T_2)} q^{\mathrm{fmaj}(w)} = \prod_{i=1}^{2} \sum_{u_i \in \mathscr{W}_r(T_i)} q^{\mathrm{fmaj}(u_i)} = \prod_{k=1}^{n} [kr]_q \cdot \prod_{l=1}^{m} [lr]_q.$$

Moreover, by definition

$$\begin{aligned} \mathrm{fmaj}(w) &\equiv \sum_{x \in F} z_x(w) h_x \bmod r \\ &\equiv \mathrm{col}(w) \bmod r. \end{aligned}$$

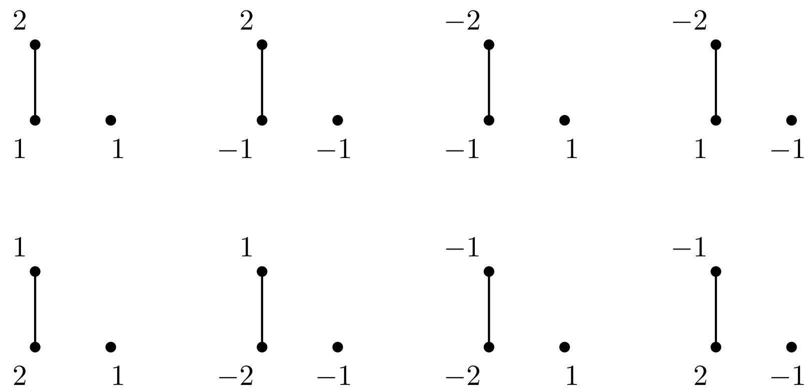

Figure 4.1: Example of $\mathscr{G}$, with $r = p = 2$, $n = 2$, $m = 1$.

Then we take exactly the monomials of

$$\prod_{k=1}^{n}[kr]_q \cdot \prod_{l=1}^{m}[lr]_q$$

of degree multiple of p. $\square$

Example 4.11. Let $\mathscr{G}$ be the set of labelings in Figure 4.1. Then

$$\begin{aligned}\sum_{w\in\mathscr{G}} q^{\mathrm{fmaj}(w)} &= 1 + q^2 + q^2 + q^4 + q^2 + q^2 + q^4 + q^4 \\ &= 1 + 4q^2 + 3q^4,\end{aligned}$$

and

$$\begin{aligned}\mathrm{Deg}_2\left([2]_q[4]_q \cdot [2]_q\right) &= \mathrm{Deg}_2\left((1+q)^2 \cdot (1+q+q^2+q^3)\right) \\ &= \mathrm{Deg}_2\left(1+3q+4q^2+4q^3+3q^4+q^5)\right) \\ &= 1+4q^2+3q^4.\end{aligned}$$

Consider now the set $\mathscr{H}$ and let $w \in \mathscr{H}$. For $i = 1, 2$, let $\boldsymbol{c_i}$ be the color of the root v_i of T_i in w (to mean that w can be represented by its lift in $\mathscr{W}_r(T_1) \times \mathscr{W}_r(T_2)$ such that $\boldsymbol{c_i}$ is the color of the root v_i), and let $\mu_i \in [0, p-1]$ such that

$$\boldsymbol{c_i} \in [\mu_i \boldsymbol{d}, (\mu_i + 1)\boldsymbol{d} - \boldsymbol{1}]. \tag{4.2}$$

We define the *H-flag-major index* of w as the following subset of $\mathrm{Fmaj}(w)$:

$$\mathrm{Hfmaj}(w) := \begin{cases} \Big\{ \min \mathrm{Fmaj}(w) \Big\} & \text{if } \mu_1 = \mu_2, \\ \Big\{\!\!\Big\{ \mathrm{hfmaj}_0(w),\, \mathrm{hfmaj}_1(w) \Big\}\!\!\Big\} & \text{if } \mu_1 \neq \mu_2, \end{cases}$$

where

$$\begin{aligned} \mathrm{hfmaj}_0(w) := \sum_{e \in E(F)} & \big(r\chi_e^{r,p}(w) + \mathrm{res}_r\left(\boldsymbol{c_e}\right) \big) h_e \\ & + n \cdot \mathrm{res}_{r/p}\left(c_1\right) + m \cdot \mathrm{res}_r\left(\mathrm{res}_{r/p}\left(c_1\right) + c_2 - c_1\right), \end{aligned}$$

$$\begin{aligned} \mathrm{hfmaj}_1(w) := \sum_{e \in E(F)} & \big(r\chi_e^{r,p}(w) + \mathrm{res}_r\left(\boldsymbol{c_e}\right) \big) h_e \\ & + m \cdot \mathrm{res}_{r/p}\left(c_2\right) + n \cdot \mathrm{res}_r\left(\mathrm{res}_{r/p}\left(c_2\right) + c_1 - c_2\right). \end{aligned}$$

Remark 4.12. If $\mu_1 = \mu_2$, note that

$$\begin{aligned} \min \mathrm{Fmaj}(w) = \sum_{e \in E(F)} & \big(r\chi_e^{r,p}(w) + \mathrm{res}_r\left(\boldsymbol{c_e}\right) \big) h_e \\ & + n \cdot \mathrm{res}_{r/p}\left(c_1\right) + m \cdot \mathrm{res}_{r/p}\left(c_2\right). \end{aligned}$$

If $\mu_1 \neq \mu_2$, then

$$\mathrm{res}_{r/p}\left(c_i\right) = c_i - kr/p = \mathrm{res}_r\left(c_i - kr/p\right)$$

for some $k \in [0, p-1]$, and

$$\begin{aligned} \mathrm{res}_r\left(\mathrm{res}_{r/p}\left(c_i\right) + c_j - c_i\right) &= \mathrm{res}_r\left(\mathrm{res}_{r/p}\left(c_i\right) + c_j - \mathrm{res}_{r/p}\left(c_i\right) - kr/p\right) \\ &= \mathrm{res}_r\left(c_j - kr/p\right), \end{aligned}$$

where $(i,j) = (1,2)$ or $(i,j) = (2,1)$. So $\mathrm{Hfmaj}(w) \subseteq \mathrm{Fmaj}(w)$.

Now we q-count all the elements of $\mathscr{H}$ according to the Hfmaj index. Then we have the following result:

Proposition 4.13. *Let T_1 and T_2 be linear trees of size n and m, respectively. Let $\mathscr{H} = \mathscr{W}_r(T_1) \times \mathscr{W}_r(T_2)/C_p$. Then*

$$\begin{aligned} \sum_{w \in \mathscr{H}} \sum_{s \in \mathrm{Hfmaj}(w)} q^s = \\ = \prod_{k=1}^{n-1} [kr]_q [rn/p]_q \cdot \prod_{l=1}^{m-1} [lr]_q [rm/p]_q \cdot \Big([p]_{q^{rm/p}} + [p]_{q^{rn/p}} - 1 \Big). \end{aligned}$$

Proof. Let $\mathscr{U} := \{u \in \mathscr{W}_r(T_1) \times \mathscr{W}_r(T_2) : c_1 < d \text{ or } c_2 < d\}$. Let now $\mathscr{H}_0 := \{w \in \mathscr{H} : \mu_1 = \mu_2\}$ and $\mathscr{H}_1 := \{w \in \mathscr{H} : \mu_1 \neq \mu_2\}$, where μ_i is given in (4.2), and note that $\mathscr{H} = \mathscr{H}_0 \cup \mathscr{H}_1$. Then there exists a bijection of multisets

$$\phi : \left\{\!\!\left\{ s \in \mathrm{Hfmaj}(w) : w \in \mathscr{H} \right\}\!\!\right\} \to \left\{\!\!\left\{ \mathrm{fmaj}(u) : u \in \mathscr{U} \right\}\!\!\right\},$$

where

$$\phi(s) \in \left\{\!\!\left\{ \mathrm{fmaj}(u) : u \in \mathscr{W}_r(T_1) \times \mathscr{W}_r(T_2) \text{ s.t. } c_1 < d \text{ and } c_2 < d \right\}\!\!\right\}$$

if $w \in \mathscr{H}_0$, and

$$\begin{aligned}\phi(s) \in &\left\{\!\!\left\{ \mathrm{fmaj}(u) : u \in \mathscr{W}_r(T_1) \times \mathscr{W}_r(T_2) \text{ s.t. } c_1 < d \text{ and } c_2 \geq d \right\}\!\!\right\} \\ &\cup \left\{\!\!\left\{ \mathrm{fmaj}(u) : u \in \mathscr{W}_r(T_1) \times \mathscr{W}_r(T_2) \text{ s.t. } c_1 \geq d \text{ and } c_2 < d \right\}\!\!\right\}\end{aligned}$$

if $w \in \mathscr{H}_1$. If we identify a class of labelings with its minimal representative (with a slight abuse of notation), then by using the above bijection ϕ, from Theorem 4.2 and Corollary 4.5, we have

$$\begin{aligned}
&\sum_{w \in \mathscr{H}} \sum_{s \in \mathrm{Hfmaj}(w)} q^s = \\
&= \sum_{w_1 \in \mathscr{W}_{r,p}(T_1)} q^{\mathrm{fmaj}(w_1)} \sum_{w_2 \in \mathscr{W}_r(T_2)} q^{\mathrm{fmaj}(w_2)} \\
&\quad + \sum_{w_1 \in \mathscr{W}_r(T_1)} q^{\mathrm{fmaj}(w_1)} \sum_{w_2 \in \mathscr{W}_{r,p}(T_2)} q^{\mathrm{fmaj}(w_2)} \\
&\quad - \sum_{w_1 \in \mathscr{W}_{r,p}(T_1)} q^{\mathrm{fmaj}(w_1)} \sum_{w_2 \in \mathscr{W}_{r,p}(T_2)} q^{\mathrm{fmaj}(w_2)} \\
&= \prod_{k=1}^{n-1} [kr]_q [rn/p]_q \cdot \prod_{l=1}^{m} [lr]_q + \prod_{k=1}^{n} [kr]_q \cdot \prod_{l=1}^{m-1} [lr]_q [rm/p]_q \\
&\quad - \prod_{k=1}^{n-1} [kr]_q [rn/p]_q \cdot \prod_{l=1}^{m-1} [lr]_q [rm/p]_q \\
&= \prod_{k=1}^{n-1} [kr]_q [rn/p]_q \cdot \prod_{l=1}^{m-1} [lr]_q [rm/p]_q \cdot \left(\frac{[rm]_q}{[rm/p]_q} + \frac{[rn]_q}{[rn/p]_q} - 1 \right) \\
&= \prod_{k=1}^{n-1} [kr]_q [rn/p]_q \cdot \prod_{l=1}^{m-1} [lr]_q [rm/p]_q \cdot \left([p]_{q^{rm/p}} + [p]_{q^{rn/p}} - 1 \right).
\end{aligned}$$

□

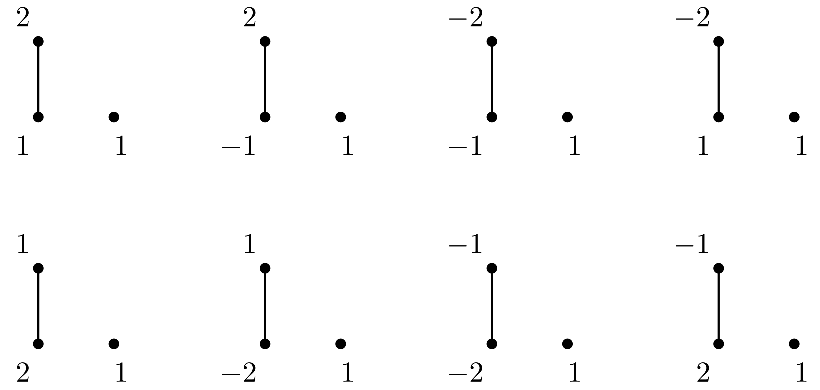

Figure 4.2: Example of $\mathscr{H}$, with $r = p = 2$, $n = 2$, $m = 1$.

Example 4.14. Let $\mathscr{H}$ be the set of labelings in Figure 4.2. Then

$$\begin{aligned}\sum_{w\in\mathscr{H}} \sum_{s\in\mathrm{Hfmaj}(w)} q^s =& 1 + q + (q + q^2) + (q^2 + q^3) + q^2 + q \\ &+ (q^3 + q^4) + (q^2 + q^3) = 1 + 3q + 4q^2 + 3q^3 + q^4,\end{aligned}$$

and

$$\begin{aligned}[2]_q[4/2]_q \cdot [2/2]_q \cdot \big([2]_q + [2]_{q^2} - 1\big) &= (1 + q)^2 \cdot (1 + q + 1 + q^2 - 1) \\ &= 1 + 3q + 4q^2 + 3q^3 + q^4.\end{aligned}$$

Remark 4.15. If T is a linear tree of size n, note that the following maps are bijections:

$$\mathscr{W}_r(T) \to G(r, n), \qquad w \mapsto g,$$

and

$$\mathscr{W}_{r,p}(T) \to G^*, \qquad w \mapsto g,$$

where g is the unique linear extension of the labeling w.
In the following chapter we extend this result to $\mathscr{G}$ and $\mathscr{H}$.

Chapter 5

Invariants and products

5.1 The product $B_n \times B_m$

Let $n, m \in \mathbb{N}$, $n, m > 0$. Let $\mathbb{C}[X, Y] := \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_m]$ and denote by $S_k[X, Y]$ the algebra of polynomials in $\mathbb{C}[X, Y]$ generated by (1 and by) the monomials of degree k. Let $B_n \times B_m$ the direct product of two Coxeter groups of type B. We consider the following two groups obtained from $B_n \times B_m$: its subgroup

$$D(B_n \times B_m) := \{(g, h) \in B_n \times B_m : \operatorname{neg}(g) + \operatorname{neg}(h) \equiv 0 \bmod 2\},$$

and its quotient

$$\frac{B_n \times B_m}{\pm id},$$

where $id := (id_{B_n}, id_{B_m})$ is the identity element of $B_n \times B_m$.

Remark 5.1. $(B_n \times B_m)/ \pm id$ is a projective reflection group: it is the quotient of a reflection group modulo the cyclic subgroup $\pm id$ of order 2. We know that it acts on the algebra $S_2[X, Y]$ and its invariants coincide with the invariants of $B_n \times B_m$, which are

$$\mathbb{C}[e_1(x_1^2, \ldots, x_n^2), \ldots, e_n(x_1^2, \ldots, x_n^2)] \otimes \mathbb{C}[e_1(y_1^2, \ldots, y_m^2), \ldots, e_m(y_1^2, \ldots, y_m^2)],$$

where the e_j's are the elementary symmetric functions. Then the invariant ring of $(B_n \times B_m)/ \pm id$ is generated as a $\mathbb{C}$-algebra by $n + m$ algebraically independent homogeneous polynomials (together with 1). See Section 1.5.

Denote by $I((B_n \times B_m)/\pm id)$ the ideal of $S_2[X,Y]$ generated by the invariants of (strictly) positive degree and let

$$R\left(\frac{B_n \times B_m}{\pm id}\right) = \frac{S_2[X,Y]}{I((B_n \times B_m)/\pm id)}$$

be the coinvariant algebra of $(B_n \times B_m)/\pm id$. We define the *flag-major index* of an element $\gamma \in D(B_n \times B_m)$ as

$$\mathrm{fmaj}(\gamma) := \mathrm{fmaj}(g) + \mathrm{fmaj}(h),$$

where g and h are the projections of γ on B_n and B_m, respectively. We now associate to any γ a monomial $a_\gamma \in \mathbb{C}[X,Y]$ of degree $\mathrm{fmaj}(\gamma)$ such that

$$a_\gamma(X,Y) := a_g(X)a_h(Y) = \prod_{i=1}^{n} x_{|g(i)|}^{\lambda_i(g)} \prod_{j=1}^{m} y_{|h(j)|}^{\lambda_j(h)}.$$

Proposition 5.2. *The set $\{a_\gamma : \gamma \in D(B_n \times B_m)\}$ represents a basis for the coinvariant algebra $R((B_n \times B_m)/\pm id)$.*

Proof. Recall that $R((B_n \times B_m)/\pm id)$ is the subalgebra of $R(B_n \times B_m) = \mathbb{C}[X,Y]/I(B_n \times B_m)$ given by the elements of even degree. Then $R((B_n \times B_m)/\pm id)$ has a basis given by

$$\{a_g a_h : (g,h) \in B_n \times B_m \text{ and } \deg(a_g a_h) \equiv 0 \bmod 2\}.$$

We note that

$$\deg(a_g a_h) = \mathrm{fmaj}(g) + \mathrm{fmaj}(h) = \sum_i \lambda_i(g) + \sum_j \lambda_j(h)$$

and then

$$\deg(a_g a_h) \equiv \mathrm{neg}(g) + \mathrm{neg}(h) \bmod 2,$$

since $\sum_i \lambda_i(g) \equiv \sum_i k_i(g) \equiv \mathrm{neg}(g) \bmod 2$. Then the basis is exactly the set $\{a_\gamma : \gamma \in D(B_n \times B_m)\}$. □

Moreover,

$$\dim R\left(\frac{B_n \times B_m}{\pm id}\right) = |D(B_n \times B_m)| = \left|\frac{B_n \times B_m}{\pm id}\right|.$$

Example 5.3. Let $n = 2$ and $m = 1$. The elements of $D(B_2 \times B_1)$ are

$$\begin{array}{llll} ([1,2],\,[1]) & ([-1,-2],\,[1]) & ([-1,2],\,[-1]) & ([-2,1],\,[-1]) \\ ([2,1],\,[1]) & ([1,-2],\,[-1]) & ([2,-1],\,[-1]) & ([-2,-1],\,[1]). \end{array}$$

The corresponding monomials

$$\begin{array}{llll} 1 & x_1x_2 & x_1y_1 & x_2y_1 \\ x_2^2 & x_1^2x_2y_1 & x_1x_2^2y_1 & x_1x_2^3 \end{array}$$

form a basis for $S_2[x_1, x_2, y_1]/(x_1^2 + x_2^2,\, x_1^2x_2^2,\, y_1^2)$, that is the coinvariant algebra of $(B_2 \times B_1)/\pm id$.

Consider now $D(B_n \times B_m)$ and note that

$$\mathrm{Inv}(B_n \times B_m) \subset \mathrm{Inv}\big(D(B_n \times B_m)\big) \subset \mathrm{Inv}(D_n \times D_m),$$

since $D_n \times D_m \subset D(B_n \times B_m) \subset B_n \times B_m$.

Claim 5.4. *The invariant ring of $D(B_n \times B_m)$ is generated as a $\mathbb{C}$-algebra by (1 and by) $n + m + 1$ homogeneous polynomials, which are*

- *the n elementary symmetric functions $e_i(x_1^2, \ldots, x_n^2)$ for $i \in [n]$,*
- *the m elementary symmetric functions $e_j(y_1^2, \ldots, y_m^2)$ for $j \in [m]$,*
- *the monomial e_Xe_Y, where $e_X := x_1 \cdots x_n$ and $e_Y := y_1 \cdots y_m$.*

Equivalently, $\mathrm{Inv}(D(B_n \times B_m))$ *is generated by the basic invariants of $B_n \times B_m$ and e_Xe_Y.*

To prove Claim 5.4 we need the following result.

Lemma 5.5. *Let G be a finite group and V a complex vector field of finite dimension n. Consider a representation of G on V and suppose that such representation is monomial, i.e., there exists a basis $B = \{b_1, b_2, \ldots, b_n\}$ of V such that $g(b_i) = c_{ji}b_j$, where $c_{ji} \in \mathbb{C}$, for every $g \in G$. Let $v = a_1b_1 + \ldots + a_nb_n$ be an invariant element of V and suppose that there exists a subgroup H of G and $l \in [n]$ such that*

$$\sum_{h \in H} h(b_l) = 0.$$

Then $a_l = 0$.

Proof. Consider any G-orbit of the basis B and consider the projection of $v = \sum a_i b_i$ on the elements of this G-orbit. This element is still invariant. Then we can suppose that the action of G on B is transitive. Let S be a set of representatives of (left) cosets of H in G, i.e., $G = S \cdot H = \biguplus_{s \in S} sH$, where $\biguplus$ denotes the disjoint union. Then

$$\sum_{g \in G} g(b_l) = \sum_{s \in S} \sum_{h \in H} sh(b_l) = 0.$$

This holds for every element $b_j \in B$: since the representation is monomial and G is transitive, there exists an element $\widetilde{g} \in G$ such that $b_j = c\,\widetilde{g}(b_l)$ for a suitable $c \in \mathbb{C}$. So

$$\sum_{g \in G} g(b_j) = \sum_{g \in G} g(c\,\widetilde{g}(b_l)) = \sum_{g \in G} c\, g\widetilde{g}(b_l) = c \sum_{g' \in G} g'(b_l) = 0.$$

Then, since v is invariant,

$$v = \frac{1}{|G|} \sum_{g \in G} g(v) = \frac{1}{|G|} \sum_{g \in G} \sum_i a_i g(e_i) = 0.$$

Finally $a_i = 0$ for each $i \in [n]$. □

Proof of Claim 5.4. Let P be a $D(B_n \times B_m)$-invariant polynomial. Then P is $D_n \times D_m$-invariant. Suppose that P is homogeneous: if not, then its homogeneous components are still invariant (from the uniqueness of the decomposition in homogeneous components). If $e_X e_Y$ divides P, then we proceed by induction. If $e_X e_Y$ does not divide P, then there exists a monomial $M = x_1^{d_1} \cdots x_n^{d_n} y_1^{f_1} \cdots y_m^{f_m}$ in P such that at least one of the d_i's or f_j's is 0. We can clearly assume $d_1 = 0$. Again suppose that at least one of the d_i's or f_j's in M is odd. Suppose $d_2 \equiv 1 \bmod 2$. Consider now the element $\gamma := (\gamma_1, id_{B_m}) \in D(B_n \times B_m)$ such that $\gamma_1(i) = -i$ if $i = 1, 2$ and $\gamma_1(j) = j$ if $j \in [3, n]$. Then $\gamma(M) = -M$, i.e., $\gamma(M) + M = 0$. Since $< \gamma >$ has order 2, from Lemma 5.5 the coefficient of M in P is 0.

Otherwise, suppose $f_1 \equiv 1 \bmod 2$. Then consider the element $\gamma := (\gamma_1, \gamma_2) \in D(B_n \times B_m)$ such that $\gamma_1(1) = -1$, $\gamma_2(1) = -1$ and $\gamma_1(j) = j$ if $j \in [2, n]$, $\gamma_2(i) = i$ if $i \in [2, m]$. Repeat the same argument.

We can conclude that all the d_i's and f_j's in each monomial M of P are even. Then P is $B_n \times B_m$-invariant, since it is $D_n \times D_m$-invariant. □

Note again that, since

$$\mathrm{Inv}(B_n \times B_m) \subset \mathrm{Inv}(D(B_n \times B_m)) \subset \mathrm{Inv}(D_n \times D_m),$$

then $R(D_n \times D_m)$ is a quotient of $R(D(B_n \times B_m))$ which in turn is a quotient of $R(B_n \times B_m)$. In particular

$$R\big(D(B_n \times B_m)\big) = \frac{R(B_n \times B_m)}{(e_X e_Y)},$$

where $(e_X e_Y)$ is the ideal generated by $e_X e_Y$ in $R(B_n \times B_m)$.

Proposition 5.6. *Consider the set $\{a_g a_h : (g,h) \in B_n \times B_m\}$. Then:*

- *the subset of elements $a_g a_h$ such that $g(n) < 0$ and $h(m) < 0$ is a basis for the ideal $(e_X e_Y)$ in $R(B_n \times B_m)$,*
- *all the other elements $a_g a_h$ form a basis for $R(D(B_n \times B_m))$.*

To prove Proposition 5.6 we need the following result.

Lemma 5.7. *Let $M \in \mathbb{C}[X]$ be a monomial such that $x_1 \cdots x_n$ divides M. Then M admits the following expansion in $R(B_n)$:*

$$M = \sum_{g \in -\Delta_n} \eta_g a_g,$$

where $\eta_g \in \mathbb{Z}$ and $\Delta_n = \{g \in B_n : g(n) > 0\}$.

Proof. If $x_1^2 \cdots x_n^2 \mid M$, then $M = 0$ in $R(B_n)$. So we can suppose

$$M = x_1 \cdots x_n \cdot N$$

and $x_1 \cdots x_n \nmid N$. From Corollary 1.18, N admits the following expansion in $R(D_n)$:

$$N = \sum_{g \in \Delta_n} \eta_g a_g,$$

where $\eta_g \in \mathbb{Z}$. Since $R(D_n) = R(B_n)/(x_1 \cdots x_n)$, then in $R(B_n)$ we have

$$N = \sum_{g \in \Delta_n} \eta_g a_g + P \cdot x_1 \cdots x_n,$$

where $P \in R(B_n)$. Then in $R(B_n)$

$$\begin{aligned} M &= \sum_{g \in \Delta_n} \eta_g a_g \cdot x_1 \cdots x_n + P \cdot x_1^2 \cdots x_n^2 \\ &= \sum_{g \in \Delta_n} \eta_g a_g \cdot x_1 \cdots x_n = \sum_{g \in \Delta_n} \eta_g \prod_{i=1}^{n} x_{|g(i)|}^{\lambda_i(g)+1}. \end{aligned}$$

Note that if $g \in \Delta_n$, then $k_n(g) = 0$ and $k_i(g) = k_{i+1}(g) + \varepsilon_i(g)$ if $i \in [n-1]$, where

$$\varepsilon_i(g) := \begin{cases} 1 & \text{if } g(i) \cdot g(i+1) < 0, \\ 0 & \text{otherwise.} \end{cases}$$

Consider now the element $h := -g \in B_n$. Note that $h(n) < 0$ and $\varepsilon_i(h) = \varepsilon_i(g)$ for $i \in [n-1]$. By definition, we have $\mathrm{HDes}(h) = \mathrm{HDes}(g)$ and so $d_i(h) = d_i(g)$ for each $i \in [n]$. Moreover, $k_n(h) = 1$ and $k_i(h) = k_{i+1}(h) + \varepsilon_i(g)$ if $i \in [n-1]$, so $k_i(h) = k_i(g) + 1$ for $i \in [n]$. Then

$$\lambda_i(h) = 2d_i(h) + k_i(h) = 2d_i(g) + k_i(g) + 1 = \lambda_i(g) + 1.$$

Finally, we have

$$M = \sum_{g \in \Delta_n} \eta_g \prod_{i=1}^{n} x_{|g(i)|}^{\lambda_i(g)+1} = \sum_{h \in -\Delta_n} \eta_h \prod_{i=1}^{n} x_{|h(i)|}^{\lambda_i(h)} = \sum_{h \in -\Delta_n} \eta_h a_h,$$

where $\eta_h \in \mathbb{Z}$. $\square$

Proof of Proposition 5.6. If $g(n) < 0$ and $h(m) < 0$, then $\lambda_n(g) \neq 0$ and $\lambda_m(h) \neq 0$. Recall that $\lambda(g)$ and $\lambda(h)$ are partitions, so $\lambda_i(g) \neq 0$ and $\lambda_j(h) \neq 0$ for each $i \in [n]$, $j \in [m]$. Then $e_X e_Y$ divides $a_g a_h$. Moreover, from Lemma 5.7 we note that in $R(B_n \times B_m)$ a monomial in which all the variables appear is a linear combination of elements $a_g a_h$ such that $g(n) < 0$ and $h(m) < 0$.

Note that the elements $a_g a_h$ such that $g(n)$ and $h(m)$ are not both negative are equivalently the monomials $a'_g a'_h, e_X a'_g a'_h, e_Y a'_g a'_h$ such that $(g, h) \in D_n^* \times D_m^*$. These $a'_g a'_h$ are independent in $R(D_n \times D_m)$ since they form a basis for it. Then they are independent in $R(D(B_n \times B_m))$, since $R(D_n \times D_m)$ is a quotient of it. $\square$

Consider the group $(B_n \times B_m)/\pm id$. Let $\delta \in (B_n \times B_m)/\pm id$ and let g and h be the projections of a representative of δ on B_n and B_m, respectively. We define the *H-flag-major index* of an element $\delta \in (B_n \times B_m)/\pm id$ as the following multiset:

$$\mathrm{Hfmaj}(\delta) := \begin{cases} \left\{\mathrm{hfmaj}_0(\delta)\right\} & \text{if } g(n)h(m) > 0, \\ \left\{\!\!\left\{\mathrm{hfmaj}_0(\delta),\, \mathrm{hfmaj}_1(\delta)\right\}\!\!\right\} & \text{if } g(n)h(m) < 0, \end{cases}$$

where

$$\mathrm{hfmaj}_0(\delta) := \sum_{i=1}^{n+m} \lambda_i^{(0)}(\delta), \qquad \mathrm{hfmaj}_1(\delta) := \sum_{i=1}^{n+m} \lambda_i^{(1)}(\delta),$$

$$\lambda_i^{(0)}(\delta) := 2 \cdot d_i(\delta) + k_i^{(0)}(\delta), \qquad \lambda_i^{(1)}(\delta) := 2 \cdot d_i(\delta) + k_i^{(1)}(\delta),$$

$$d_i(\delta) := \big(d_1(g), d_2(g), \dots, d_n(g), d_1(h), d_2(h), \dots, d_m(h)\big),$$

$$k_i^{(0)}(\delta) := \begin{cases} 0 & \text{if } i = n+m, \\ k_{i+1}^{(0)}(\delta) + \varepsilon_{i-n}(h) & \text{if } i \in [n+1, n+m-1], \\ \varepsilon(\delta) & \text{if } i \in n, \\ k_{i+1}^{(0)}(\delta) + \varepsilon_i(g) & \text{if } i \in [n-1], \end{cases}$$

$$k_i^{(1)}(\delta) := \begin{cases} 1 & \text{if } i = n+m, \\ k_{i+1}^{(1)}(\delta) + \varepsilon_{i-n}(h) & \text{if } i \in [n+1, n+m-1], \\ 0 & \text{if } i \in n, \\ k_{i+1}^{(1)}(\delta) + \varepsilon_i(g) & \text{if } i \in [n-1], \end{cases}$$

$$\varepsilon(\delta) := \begin{cases} 1 & \text{if } g(n) \cdot h(m) < 0, \\ 0 & \text{otherwise}, \end{cases}$$

$$\varepsilon_i(g) = \begin{cases} 1 & \text{if } g(i) \cdot g(i+1) < 0, \\ 0 & \text{otherwise}, \end{cases}$$

and $d_j(g) = |\{i \in [j, n-1] : i \in \mathrm{HDes}(g)\}|$ defined as in Section 1.3.

Let $U := \{(\alpha, \beta) \in B_n \times B_m : \alpha(n) > 0 \text{ or } \beta(m) > 0\}$ and $\mathrm{fmaj}(\alpha, \beta) :=$

$\mathrm{fmaj}(\alpha) + \mathrm{fmaj}(\beta)$ for $(\alpha, \beta) \in B_n \times B_m$. Note that there exists a bijection of multisets

$$\phi : \Big\{\Big\{ s \in \mathrm{Hfmaj}(\delta) : \delta \in (B_n \times B_m)/\pm id \Big\}\Big\} \to \Big\{\Big\{ \mathrm{fmaj}(\alpha, \beta) : (\alpha, \beta) \in U \Big\}\Big\},$$

where

$$\phi(s) \in \Big\{\Big\{ \mathrm{fmaj}(\alpha, \beta) : (\alpha, \beta) \in B_n \times B_m \text{ s.t. } \alpha(n) > 0 \text{ and } \beta(m) > 0 \Big\}\Big\}$$

if $g(n)h(m) > 0$, and

$$\begin{aligned} \phi(s) \in \Big\{\Big\{ &\mathrm{fmaj}(\alpha, \beta) : (\alpha, \beta) \in B_n \times B_m \text{ s.t. } \alpha(n) > 0 \text{ and } \beta(m) < 0 \Big\}\Big\} \\ &\cup \Big\{\Big\{ \mathrm{fmaj}(\alpha, \beta) : (\alpha, \beta) \in B_n \times B_m \text{ s.t. } \alpha(n) < 0 \text{ and } \beta(m) > 0 \Big\}\Big\} \end{aligned}$$

if $g(n)h(m) < 0$.

Remark 5.8. Using the bijection ϕ, we can conclude that a basis for $R(D(B_n \times B_m))$ is the set

$$\begin{aligned} &\Big\{ a_\delta^{(0)} : \delta \in (B_n \times B_m)/\pm id \text{ s.t. } \delta(n)\delta(n+m) > 0 \Big\} \\ &\qquad \cup \Big\{ a_\delta^{(0)}, a_\delta^{(1)} : \delta \in (B_n \times B_m)/\pm id \text{ s.t. } \delta(n)\delta(n+m) < 0 \Big\}, \end{aligned}$$

where

$$a_\delta^{(0)}(X,Y) := \prod_{i=1}^{n} x_{|\delta(i)|}^{\lambda_i^{(0)}(\delta)} \prod_{j=n+1}^{n+m} y_{|\delta(j)|}^{\lambda_j^{(0)}(h)},$$

$$a_\delta^{(1)}(X,Y) := \prod_{i=1}^{n} x_{|\delta(i)|}^{\lambda_i^{(1)}(\delta)} \prod_{j=n+1}^{n+m} y_{|\delta(j)|}^{\lambda_j^{(1)}(h)}.$$

Moreover,

$$\dim R\big(D(B_n \times B_m)\big) = \frac{3}{2} \cdot \left| \frac{B_n \times B_m}{\pm id} \right| = \frac{3}{2} \cdot |D(B_n \times B_m)|.$$

Example 5.9. Let $n = 2$ and $m = 1$. The elements δ of $(B_2 \times B_1)/\pm id$ are

$$\begin{array}{llll} ([1,2],\,[1]) & ([-1,2],\,[1]) & ([-2,1],\,[1]) & ([2,1],\,[1]) \\ ([-1,-2],\,[1]) & ([1,-2],\,[1]) & ([2,-1],\,[1]) & ([-2,-1],\,[1]). \end{array}$$

The corresponding monomials are

$$\begin{array}{llll} 1 & x_1 & x_2 & x_2^2 \\ x_1x_2 & x_1^2x_2 & x_1x_2^2 & x_1x_2^3 \\ y_1 & x_1y_1 & x_2y_1 & x_2^2y_1, \end{array}$$

and they form a basis for $\mathbb{C}[x_1, x_2, y_1]/(x_1^2 + x_2^2,\ x_1^2x_2^2,\ y_1^2,\ x_1x_2y_1)$, that is the coinvariant algebra of $D(B_2 \times B_1)$.

Proposition 5.2 and Remark 5.8 show a duality between the groups $(B_n \times B_m)/\pm id$ and $D(B_n \times B_m)$. Let us generalize this behavior.

5.2 The product $G(r,n) \times G(r,m)$

Let $n, m, r \in \mathbb{N}$, $n, m, r > 0$, and denote by ζ_r the primitive r-th root of the unity. Consider the direct product $G(r,n) \times G(r,m)$ of two groups of r-colored permutations. Let p be a positive divisor of r. We consider the following two groups obtained from $G(r,n) \times G(r,m)$: its subgroup

$$\begin{aligned} G &:= \Gamma_p(G(r,n) \times G(r,m)) \\ &:= \{(g,h) \in G(r,n) \times G(r,m) :\ \mathrm{col}(g) + \mathrm{col}(h) \equiv 0 \bmod p\}, \end{aligned}$$

and its quotient

$$H := \frac{G(r,n) \times G(r,m)}{C_p},$$

where C_p is the cyclic subgroup of $G(r,n) \times G(r,m)$ of order p generated by

$$([1^{\boldsymbol{r/p}}, 2^{\boldsymbol{r/p}}, \ldots, n^{\boldsymbol{r/p}}],\ [1^{\boldsymbol{r/p}}, 2^{\boldsymbol{r/p}}, \ldots, m^{\boldsymbol{r/p}}]).$$

H is a projective reflection group, since it is the quotient of a reflection group modulo a cyclic scalar subgroup of order p. So it acts on the algebra $S_p[X,Y]$ and its invariants coincide with the invariants of $G(r,n) \times G(r,m)$, which are

$$\mathbb{C}[e_1(x_1^r, \ldots, x_n^r), \ldots, e_n(x_1^r, \ldots, x_n^r)] \otimes \mathbb{C}[e_1(y_1^r, \ldots, y_m^r), \ldots, e_m(y_1^r, \ldots, y_m^r)],$$

where the e_j's are the elementary symmetric functions. Then the invariant ring of H is generated as a $\mathbb{C}$-algebra by $n + m$ algebraically independent homogeneous polynomials (together with 1). See Section 1.5.

Denote by $I(H)$ the ideal of $S_p[X,Y]$ generated by the invariants of (strictly) positive degree and let

$$R(H) = \frac{S_p[X,Y]}{I(H)}$$

be the coinvariant algebra of H. We define the *flag-major index* of an element $\gamma \in G$ as

$$\mathrm{fmaj}(\gamma) := \mathrm{fmaj}(g) + \mathrm{fmaj}(h),$$

where g and h are the projections of γ on $G(r,n)$ and $G(r,m)$, respectively. We now associate to any element $\gamma \in G$ a monomial $a_\gamma \in \mathbb{C}[X,Y]$ of degree $\mathrm{fmaj}(\gamma)$ such that

$$a_\gamma(X,Y) := a_g(X)a_h(Y) = \prod_{i=1}^{n} x_{|g_i|}^{\lambda_i(g)} \prod_{j=1}^{m} y_{|h_j|}^{\lambda_j(h)}.$$

Proposition 5.10. *The set $\{a_\gamma : \gamma \in G\}$ represents a basis for the coinvariant algebra $R(H)$.*

Proof. Recall that $R(H)$ is the subalgebra of

$$R\big(G(r,n) \times G(r,m)\big) = \frac{\mathbb{C}[X,Y]}{I\big(G(r,n) \times G(r,m)\big)}$$

given by the elements of degree multiple of p. Then $R(H)$ has a basis given by

$$\{a_g a_h : (g,h) \in G(r,n) \times G(r,m) \text{ and } \deg(a_g a_h) \equiv 0 \bmod p\}.$$

We note that

$$\deg(a_g a_h) = \mathrm{fmaj}(g) + \mathrm{fmaj}(h) = \sum_i \lambda_i(g) + \sum_j \lambda_j(h)$$

and then

$$\deg(a_g a_h) \equiv \mathrm{col}(g) + \mathrm{col}(h) \bmod r,$$

since $\sum_i \lambda_i(g) \equiv \sum_i k_i(g) \equiv \mathrm{col}(g) \bmod r$. Then the basis is exactly the set $\{a_\gamma : \gamma \in G\}$. □

Moreover,

$$\dim R(H) = |G| = |H|.$$

Consider now G and note that

$$\mathrm{Inv}\big(G(r,n) \times G(r,m)\big) \subset \mathrm{Inv}(G) \subset \mathrm{Inv}\big(G(r,p,n) \times G(r,p,m)\big),$$

since $G(r,p,n) \times G(r,p,m) \subset G \subset G(r,n) \times G(r,m)$.

Claim 5.11. *Let $d = r/p$. The invariant ring of G is generated as a $\mathbb{C}$-algebra by (1 and by) $n+m+1$ homogeneous polynomials, which are*

- *the n elementary symmetric functions $e_i(x_1^r, \ldots, x_n^r)$ for $i \in [n]$,*
- *the m elementary symmetric functions $e_j(y_1^r, \ldots, y_m^r)$ for $j \in [m]$,*
- *the monomial $e_X^d e_Y^d = x_1^d \cdots x_n^d y_1^d \cdots y_m^d$.*

Equivalently, $\mathrm{Inv}(G)$ *is generated by the basic invariants of $G(r,n) \times G(r,m)$ and $e_X^d e_Y^d$.*

Proof. Let P be a G-invariant polynomial. Then P is $G(r,p,n) \times G(r,p,m)$-invariant. Suppose that P is homogeneous: if not, then its homogeneous components are still invariant. If $e_X^d e_Y^d$ divides P, then we proceed by induction, since $e_X^d e_Y^d$ is clearly G-invariant. If $e_X^d e_Y^d$ does not divide P, then there exists a monomial $M = x_1^{d_1} \cdots x_n^{d_n} y_1^{f_1} \cdots y_m^{f_m}$ in P such that at least one of the d_i's or f_j's is less than d: we can assume $0 \leq d_1 < d$. Again suppose that there exists $i \in [2,n]$, respectively $j \in [m]$, such that $r \nmid d_i$, respectively $r \nmid f_j$: suppose $r \nmid d_2$. Consider now the element $\gamma := \big(\gamma_1, id_{G(r,m)}\big) \in G$ such that $\gamma_1(1) = 1^{\mathbf{1}}$, $\gamma_1(2) = 2^{\boldsymbol{p-1}}$ and $\gamma_1(j) = j$ if $j \in [3,n]$. Let $s := d_1 + (p-1)d_2$. Then $\gamma^k(M) = \zeta_r^{ks} M$ for $k \in \mathbb{N}$. Let $\rho := \gcd(r,s)$ and $\alpha := r/\rho$, $\beta := s/\rho$. Then the subgroup $< \gamma >$ has order r and we have

$$\sum_{h \in <\gamma>} h(M) = \sum_{k=0}^{r-1} \gamma^k(M) = \frac{r}{\alpha} \cdot \sum_{k=0}^{\alpha-1} \gamma^k(M) = 0,$$

since

$$1 + \zeta_r^s + \zeta_r^{2s} + \ldots + \zeta_r^{(\alpha-1)s} = 1 + \zeta_\alpha^\beta + \zeta_\alpha^{2\beta} + \ldots + \zeta_\alpha^{(\alpha-1)\beta} = 0.$$

Then, from Lemma 5.5, the coefficient of M in P is 0.
Otherwise, suppose $r \nmid f_1$. Then consider the element $\gamma := (\gamma_1, \gamma_2) \in G$ such that $\gamma_1(1) = 1^{\mathbf{1}}$, $\gamma_2(1) = 1^{\boldsymbol{p-1}}$ and $\gamma_1(j) = j$ if $j \in [2,n]$, $\gamma_2(i) = i$ if $i \in [2,m]$. Let $s := d_1 + (p-1)f_1$ and repeat the same argument.
Again, suppose now $0 < d_1 < d$ and $r \mid d_i$, $r \mid f_j$ for each $i \in [2,n]$, $j \in [m]$ in M. Then $\gamma^k(M) = \zeta_r^{kd_1} M$ for $k \in \mathbb{N}[0, r-1]$. As in the previous case, M does not appear in P and we can conclude that all the d_i's and f_j's in each monomial M of P are multiple of r (or 0). Then P is $G(r,n) \times G(r,m)$-invariant, since it is $G(r,p,n) \times G(r,p,m)$-invariant. □

Note again that, since

$$\mathrm{Inv}\big(G(r,n) \times G(r,m)\big) \subset \mathrm{Inv}(G) \subset \mathrm{Inv}\big(G(r,p,n) \times G(r,p,m)\big),$$

then $R(G(r,p,n) \times G(r,p,m))$ is a quotient of $R(G)$ which in turn is a quotient of $R(G(r,n) \times G(r,m))$. In particular

$$R(G) = \frac{R\big(G(r,n) \times G(r,m)\big)}{(e_X^d e_Y^d)},$$

where $(e_X^d e_Y^d)$ is the ideal generated by $e_X^d e_Y^d$ in $R(G(r,n) \times G(r,m))$.

Proposition 5.12. *Let $d = r/p$. Consider the set $\{a_g a_h :\ (g,h) \in G(r,n) \times G(r,m)\}$. Then:*

- *the subset of elements $a_g a_h$ such that $c(g_n) \geq \boldsymbol{d}$ and $c(h_m) \geq \boldsymbol{d}$ is a basis for the ideal $(e_X^d e_Y^d)$ in $R(G(r,n) \times G(r,m))$,*
- *all the other elements $a_g a_h$ form a basis for $R(G)$.*

To prove Proposition 5.12 we need the following result.

Lemma 5.13. *Let $M \in \mathbb{C}[X]$ be a monomial such that $e_X^d = x_1^d \cdots x_n^d$ divides M. Then M admits the following expansion in $R(G(r,n))$:*

$$M = \sum_{g \in \overline{\Omega_n}} \eta_g a_g,$$

where $\eta_g \in \mathbb{Z}$ and $\overline{\Omega_n} := \{g \in G(r,n) :\ c(g_n) \geq \boldsymbol{d}\}$.

Proof. If $x_1^r \cdots x_n^r \mid M$, then $M = 0$ in $R(G(r,n))$. So we can suppose

$$M = (x_1^d \cdots x_n^d)^s \cdot N,$$

where $s \in [p-1]$ and $x_1^d \cdots x_n^d \nmid N$, i.e., if $N = x_1^{d_1} \cdots x_n^{d_n}$, at least one of the d_i's is less than d. From Lemma 1.17, N admits the following expansion in $R(G(r,p,n))$:

$$N = \sum_{g \in \Omega_n} \eta_g a_g,$$

where $\eta_g \in \mathbb{Z}$. Since

$$R\big(G(r,p,n)\big) = \frac{R\big(G(r,n)\big)}{(x_1^d \cdots x_n^d)},$$

then in $R(G(r,n))$ we have

$$N = \sum_{g \in \Omega_n} \eta_g a_g + P \cdot x_1^d \cdots x_n^d,$$

where $P \in R(G(r,n))$. Then in $R(G(r,n))$

$$M = \sum_{g \in \Omega_n} \eta_g a_g \cdot (x_1^d \cdots x_n^d)^s + P \cdot (x_1^d \cdots x_n^d)^{s+1}.$$

Write $P = c_1 m_1 + \cdots + c_k m_k$, where m_i is a monomial and $c_i \in \mathbb{C}$. Then we have two possibilities:

i) $x_1^d \cdots x_n^d \nmid m_i$,

ii) $x_1^d \cdots x_n^d \mid m_i$, so $m_i = (x_1^d \cdots x_n^d)^t \cdot n_i$, where $t \in [p-s-2]$ and $x_1^d \cdots x_n^d \nmid n_i$.

In the other cases $m_i \cdot (x_1^d \cdots x_n^d)^{s+1} = 0$ in $R(G(r,n))$. Now we can write the previous expansion for m_i in case *i*), and for n_i in case *ii*). We can apply this argument recursively until we obtain the following expansion for M in $R(G(r,n))$:

$$M = \sum_{g \in \Omega_n} \eta_{s,g}\, a_g \cdot e_X^{sd} + \sum_{g \in \Omega_n} \eta_{s+1,g}\, a_g \cdot e_X^{(s+1)d} + \cdots + \sum_{g \in \Omega_n} \eta_{p-1,g}\, a_g \cdot e_X^{(p-1)d},$$

with suitable coefficients $\eta_{i,g} \in \mathbb{Z}$ for $i \in [s, p-1]$. Then

$$M = \sum_{j=s}^{p-1} \sum_{g \in \Omega_n} \eta_{j,g}\, a_g \cdot e_X^{jd} = \sum_{j=s}^{p-1} \sum_{g \in \Omega_n} \eta_{j,g} \prod_{i=1}^{n} x_{|g(i)|}^{\lambda_i(g) + jd}.$$

Note that if $g \in \Omega_n$, then $k_n(g) = c(g_n) < \boldsymbol{d}$. Consider now the element $h \in G(r,n)$ such that $|h| = |g|$ and $c(h_i) = c(g_i) + \mu\boldsymbol{d}$, where $\mu \in [p-1]$, for each $i \in [n]$. Then

$$c(h_n) \in [\mu\boldsymbol{d}, (\mu+1)\boldsymbol{d} - \boldsymbol{1}].$$

By definition we have $\mathrm{HDes}(h) = \mathrm{HDes}(g)$ and so $d_i(h) = d_i(g)$ for each $i \in [n]$. Moreover, $k_n(h) = c(g_n) + \mu d$ and

$$k_i(h) = k_{i+1}(h) + \mathrm{res}_r\,(c(h_i) - c(h_{i+1})) = k_{i+1}(h) + \mathrm{res}_r\,(c(g_i) - c(g_{i+1}))$$

if $i \in [n-1]$, so $k_i(h) = k_i(g) + \mu d$ for $i \in [n]$. Then

$$\lambda_i(h) = rd_i(h) + k_i(h) = rd_i(g) + k_i(g) + \mu d = \lambda_i(g) + \mu d.$$

Finally, let $\Omega_n(a,b) := \{g \in G(r,n) : c(g_n) \in [a,b]\}$. In this notation $\Omega_n = \Omega_n(0, d-1)$ and $\overline{\Omega_n} = \Omega_n(d, r-1)$. We have

$$\begin{aligned} M &= \sum_{j=s}^{p-1} \sum_{g \in \Omega_n} \eta_{j,g} \prod_{i=1}^{n} x_{|g(i)|}^{\lambda_i(g)+jd} \\ &= \sum_{j=s}^{p-1} \sum_{h \in \Omega_n(jd,(j+1)d-1)} \eta_{j,h} \prod_{i=1}^{n} x_{|h(i)|}^{\lambda_i(h)} = \sum_{h \in \overline{\Omega_n}} \eta_h a_h, \end{aligned}$$

where $\eta_h \in \mathbb{Z}$. $\square$

Proof of Proposition 5.12. If $c(g_n) \geq \boldsymbol{d}$ and $c(h_m) \geq \boldsymbol{d}$, then $\lambda_n(g) \geq d$ and $\lambda_m(h) \geq d$. Recall that $\lambda(g)$ and $\lambda(h)$ are partitions, so $\lambda_i(g) \geq d$ and $\lambda_j(h) \geq d$ for each $i \in [n]$, $j \in [m]$. Then $e_X^d e_Y^d$ divides $a_g a_h$. Moreover, from Lemma 5.13 we note that in $R(G(r,n) \times G(r,m))$ a monomial in which all the variables appear with exponent at least d is a linear combination of elements $a_g a_h$ such that $c(g_n) \geq \boldsymbol{d}$ and $c(h_m) \geq \boldsymbol{d}$.
Note that the elements $a_g a_h$ such that $(g,h) \in G(r,n) \times G(r,m)$ and $c(g_n)$ and $c(h_m)$ are not both $\geq \boldsymbol{d}$ are equivalently the monomials

$$\begin{aligned} &a'_g a'_h,\ e_X^d \cdot a'_g a'_h,\ e_X^{2d} \cdot a'_g a'_h,\ \ldots,\ e_X^{(p-1)d} \cdot a'_g a'_h, \\ &\qquad\qquad e_Y^d \cdot a'_g a'_h,\ e_Y^{2d} \cdot a'_g a'_h,\ \ldots,\ e_Y^{(p-1)d} \cdot a'_g a'_h \end{aligned}$$

such that $(g,h) \in G(r,p,n)^* \times G(r,p,m)^*$. These $a'_g a'_h$ are independent in $R(G(r,p,n) \times G(r,p,m))$ since they form a basis for it. Then they are independent in $R(G)$, since $R(G(r,p,n) \times G(r,p,m))$ is a quotient of it. $\square$

Consider the group H. Let $\delta \in H$ and let g and h be the projections of a representative of δ on $G(r,n)$ and $G(r,m)$, respectively. Again let $\boldsymbol{c_i} := c(g_i)$ and $\boldsymbol{z_j} := c(h_j)$ for $i \in [n]$, $j \in [m]$. Let $\mu, \nu \in [0, p-1]$ such that

$$\boldsymbol{c_n} \in [\mu\boldsymbol{d}, (\mu+1)\boldsymbol{d}-\boldsymbol{1}] \quad \text{and} \quad \boldsymbol{z_m} \in [\nu\boldsymbol{d}, (\nu+1)\boldsymbol{d}-\boldsymbol{1}]. \tag{5.1}$$

We define the *H-flag-major index* of an element $\delta \in H$ as the following multiset:

$$\mathrm{Hfmaj}(\delta) := \begin{cases} \Big\{\mathrm{hfmaj}_0(\delta)\Big\} & \text{if } \mu = \nu, \\ \Big\{\!\!\Big\{\mathrm{hfmaj}_0(\delta),\, \mathrm{hfmaj}_1(\delta)\Big\}\!\!\Big\} & \text{if } \mu \neq \nu, \end{cases}$$

where

$$\mathrm{hfmaj}_0(\delta) := \sum_{i=1}^{n+m} \lambda_i^{(0)}(\delta), \qquad \mathrm{hfmaj}_1(\delta) := \sum_{i=1}^{n+m} \lambda_i^{(1)}(\delta),$$

$$\lambda_i^{(0)}(\delta) := 2\cdot d_i(\delta) + k_i^{(0)}(\delta), \qquad \lambda_i^{(1)}(\delta) := 2\cdot d_i(\delta) + k_i^{(1)}(\delta),$$

$$d_i(\delta) := \big(d_1(g), d_2(g), \ldots, d_n(g), d_1(h), d_2(h), \ldots, d_m(h)\big),$$

$$k_i^{(0)}(\delta) := \begin{cases} \mathrm{res}_{r/p}\left(z_m\right) & \text{if } i = n+m, \\ k_{i+1}^{(0)}(\delta) + \mathrm{res}_r\left(z_i - z_{i+1}\right) & \text{if } i \in [n+1, n+m-1], \\ \mathrm{res}_r\left(k_{n+m}^{(0)}(\delta) + c_n - z_m\right) & \text{if } i \in n, \\ k_{i+1}^{(0)}(\delta) + \mathrm{res}_r\left(c_i - c_{i+1}\right) & \text{if } i \in [n-1], \end{cases}$$

$$k_i^{(1)}(\delta) := \begin{cases} \mathrm{res}_{r/p}\left(c_n\right) & \text{if } i = n, \\ k_{i+1}^{(1)}(\delta) + \mathrm{res}_r\left(c_i - c_{i+1}\right) & \text{if } i \in [1, n-1], \\ \mathrm{res}_r\left(k_n^{(1)}(\delta) + z_m - c_n\right) & \text{if } i \in n+m, \\ k_{i+1}^{(1)}(\delta) + \mathrm{res}_r\left(z_i - z_{i+1}\right) & \text{if } i \in [n+1, n+m-1], \end{cases}$$

and $d_j(g) = |\{i \in [j, n-1] : \, i \in \mathrm{HDes}(g)\}|$ defined as in Section 1.3.

Let $U := \{(\alpha,\beta) \in G(r,n)\times G(r,m) : \, c(\alpha_n) < \boldsymbol{d} \text{ or } c(\beta_m) < \boldsymbol{d}\}$ and recall that $\Omega_n = \{\alpha \in G(r,n) : \, c(\alpha_n) < \boldsymbol{d}\}$ and $\overline{\Omega_n} = \{\alpha \in G(r,n) : \, c(\alpha_n) \geq \boldsymbol{d}\}$. Let $\mathrm{fmaj}(\alpha,\beta) := \mathrm{fmaj}(\alpha) + \mathrm{fmaj}(\beta)$ for $(\alpha,\beta) \in G(r,n)\times G(r,m)$. Note that there exists a bijection of multisets

$$\phi : \Big\{\!\!\Big\{s \in \mathrm{Hfmaj}(\delta) : \, \delta \in H\Big\}\!\!\Big\} \to \Big\{\!\!\Big\{\mathrm{fmaj}(\alpha,\beta) : \, (\alpha,\beta) \in U\Big\}\!\!\Big\},$$

where
$$\phi(s) \in \left\{\!\!\left\{ \mathrm{fmaj}(\alpha,\beta) : (\alpha,\beta) \in \Omega_n \times \Omega_m \right\}\!\!\right\}$$
if $\mu = \nu$, and
$$\phi(s) \in \left\{\!\!\left\{ \mathrm{fmaj}(\alpha,\beta) : (\alpha,\beta) \in \Omega_n \times \overline{\Omega_m} \right\}\!\!\right\} \cup \left\{\!\!\left\{ \mathrm{fmaj}(\alpha,\beta) : (\alpha,\beta) \in \overline{\Omega_n} \times \Omega_m \right\}\!\!\right\}$$
if $\mu \neq \nu$, where μ and ν are given by (5.1).

Remark 5.14. If $\mu \neq \nu$, then
$$k^{(0)}_{n+m}(\delta) = \mathrm{res}_{r/p}(z_m) = z_m - kr/p = \mathrm{res}_r(z_m - kr/p)$$
for some $k \in [0, p-1]$, and
$$\begin{aligned} k^{(0)}_n(\delta) &= \mathrm{res}_r\left(\mathrm{res}_{r/p}(z_m) + c_n - z_m\right) \\ &= \mathrm{res}_r\left(\mathrm{res}_{r/p}(z_m) + c_n - \mathrm{res}_{r/p}(z_m) - kr/p\right) \\ &= \mathrm{res}_r(c_n - kr/p). \end{aligned}$$
In the same way,
$$k^{(1)}_n(\delta) = \mathrm{res}_r(c_n - kr/p)$$
for some $k \in [0, p-1]$, and
$$k^{(1)}_{n+m}(\delta) = \mathrm{res}_r(z_m - kr/p).$$
If $\mu = \nu$, then
$$k^{(0)}_{n+m}(\delta) = \mathrm{res}_{r/p}(z_m)$$
and
$$k^{(0)}_n(\delta) = \mathrm{res}_{r/p}(c_n).$$

Remark 5.15. For any $\delta \in H$, let $\mu_i \in [0, p-1]$ such that
$$c(\delta_i) \in [\mu_i \boldsymbol{d}, (\mu_i + 1)\boldsymbol{d} - \boldsymbol{1}].$$
Using the bijection ϕ, we can conclude that a basis for $R(G)$ is the set
$$\left\{ a^{(0)}_\delta : \delta \in H \text{ s.t. } \mu_n = \mu_{n+m} \right\} \cup \left\{ a^{(0)}_\delta, a^{(1)}_\delta : \delta \in H \text{ s.t. } \mu_n \neq \mu_{n+m} \right\},$$

where

$$a_\delta^{(0)}(X,Y) := \prod_{i=1}^{n} x_{|\delta(i)|}^{\lambda_i^{(0)}(\delta)} \prod_{j=n+1}^{n+m} y_{|\delta(j)|}^{\lambda_j^{(0)}(h)},$$

$$a_\delta^{(1)}(X,Y) := \prod_{i=1}^{n} x_{|\delta(i)|}^{\lambda_i^{(1)}(\delta)} \prod_{j=n+1}^{n+m} y_{|\delta(j)|}^{\lambda_j^{(1)}(h)}.$$

Moreover,

$$\dim R(G) = \frac{2p-1}{p} \cdot |H| = \frac{2p-1}{p} \cdot |G|.$$

Finally, let us consider a finitely generated graded commutative algebra A over $\mathbb{C}$, which is generated by elements of positive degree. So

$$A = \bigoplus_{k \geq 0} A_k,$$

where $A_0 = \mathbb{C}$. Recall that the Hilbert function of A is the map

$$\mathrm{HF}_A : \mathbb{N} \to \mathbb{N}, \qquad k \mapsto \dim_{\mathbb{C}} A_k,$$

and the Hilbert series of A is the formal series

$$\mathrm{HS}_A(q) := \sum_{k \geq 0} \mathrm{HF}_A(k)\, q^k.$$

According to the notes in Remark 4.15 and the notation in Section 4.3, we have the following two results:

Proposition 5.16.

$$\sum_{w \in \mathscr{G}} q^{\mathrm{fmaj}(w)} = \mathrm{HS}_{R(H)}(q).$$

Proof. Note that there exists a bijection:

$$\{\mathrm{fmaj}(\gamma) : \gamma \in G\} \to \{\mathrm{fmaj}(w) : w \in \mathscr{G}\}.$$

Then

$$\sum_{\gamma \in G} q^{\mathrm{fmaj}(\gamma)} = \sum_{w \in \mathscr{G}} q^{\mathrm{fmaj}(w)},$$

so from Propositions 4.10 and 5.10 we have the result. □

Example 5.17. See Examples 4.11 and 5.3.

Proposition 5.18.

$$\sum_{w \in \mathscr{H}} \sum_{s \in \mathrm{Hfmaj}(w)} q^s = \mathrm{HS}_{R(G)}(q).$$

Proof. Recall Remarks 4.12 and 5.14. Note that there exists a bijection:

$$\{s \in \mathrm{Hfmaj}(\delta) : \delta \in H\} \to \{s \in \mathrm{Hfmaj}(w) : w \in \mathscr{H}\}.$$

Then

$$\sum_{\delta \in H} \sum_{s \in \mathrm{Hfmaj}(\delta)} q^s = \sum_{w \in \mathscr{H}} \sum_{s \in \mathrm{Hfmaj}(w)} q^s,$$

so from Proposition 4.13 and Remark 5.15 we have the result. □

Example 5.19. See Examples 4.14 and 5.9.

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