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PROOF THEORY OF QUANTIFIED MODAL LOGICS

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Abstract

Indexed modal logics (IMLs) constitute the object of study of this thesis. IMLs generalize quantified modal logics (QMLs) in two respects: language and semantics. First of all, standard modal operators $\Box$ and $\Diamond$ are replaced by modal operators indexed by sets of variables $|x_1 \dots x_n|$ and $\langle x_1 \dots x_m \rangle$, or, more generally, by sets of pairs composed by a term and a variable: $|{}^{t_1}_{x_1} \dots {}^{t_n}_{x_n}|$ and $\langle{}^{t_1}_{x_1} \dots {}^{t_m}_{x_m}\rangle$. This allows us to distinguish between 'c is necessarily a P' and 'it is necessary that c is P', which are expressed by $|{}^{c}_{x}|P(x)$ and $\Box P(c)$, respectively. In this approach we can better control the interaction of first-order machinery (substitutions, quantifiers, and identity) with modalities. The second novelty is that Kripke semantics is replaced by the more general transition semantics, in which the relation of trans-world identity, used to evaluate modal open formulas, is replaced by an arbitrary relation between objects inhabiting possible worlds. This allows us to have a more fine-grained correspondence theory than that of Kripke semantics: many important formulas that are valid on every Kripke-frame correspond to particular classes of transition-frames.

IMLs are a major step in the model-theoretic understanding of quantified modal logics. Their proof theory has been confined to axiomatic systems, see [Cor09], for which completeness results are very hard to find and in most cases are still open problems. Our approach is different because we replace axiomatic systems with sequent calculi in the style of [Neg05], known as labelled sequent calculi. These calculi allow us to internalize transition semantics into the rules of inferences of the calculus, and to make use of the method of axioms-as-rules, which has already been used in [Neg05] for propositional modal logics. In this way we are able to define sequent calculi for many interesting semantically defined classes of transition-frames.

We prove, in a purely syntactical way, the following general results for our calculi:

(1) the structural rule of weakening is height-preserving admissible;
(2) all rules are height-preserving invertible;
(3) the structural rule of contraction is height-preserving admissible;
(4) the cut rule is admissible.

Within our approach completeness results are provable in a direct way and do not encounter the problems typical of Henkin-style proofs: we are able to produce a single strategy, a root-first proof search procedure, so that for any non theorem of the calculus under consideration, it produces a countermodel based on a transition-frame for the calculus. In this way we have a modular completeness result that encompasses all indexed extension of any propositional modal logic in the Lemmon-Scott fragment: we avoid many incompleteness results that vex axiomatic systems.

Our works is, to our knowledge, the first attempt to merge two of the more active fields of research in modal logics: that of generalizations of Kripke semantics and that of proof-theoretic studies of modal logics.

Contents

List of Figures

List of Tables

INTRODUCTION

This work is a proof-theoretic study of quantified modal logics (QMLs) in the context of indexed modalities and transition semantics. We hope in this way to dispel some 'locus communis' on the intractability of quantified modal logics. The core of our work is the introduction and the systematic study of labelled sequent calculi for Indexed Modal Logics (IMLs).

An IML is defined as the set of indexed modal formulas that are valid on a class of transition frames. In [Cor09] axiomatic systems for IMLs have been introduced, but, save for the minimal system that axiomatizes the formulas valid on the class of all transition frames, there exists no proof that such systems are complete with respect to the intended class of transition frames. This is an instance of a general problem for QMLs because completeness results are extremely difficult to find, and in most cases the quantified extensions of a complete propositional modal logic are not complete w.r.t. the intended semantics [SS90, Gar91, Cre95, Cre00, Cor02, Gol11].

The incompleteness phenomenon is a widespread phenomenon at the propositional level, even worst at the predicative level, see [She06]. Even if we limit our attention to the quantified extensions of those propositional modal logics (PMLs) which are complete and canonical, we find that most of them are incomplete. Of course the analysis needs to be more precise because 'quantified extensions' can mean different things: with/without classical quantification axioms, with/without the Barcan formula, with/without the Ghilardi formula or with/without the necessity of identity, etc. In general, given a PML **S** that is complete with respect to a given class of frames, two kinds of incompleteness phenomenons may arise for its quantified extensions: we have *incompletable* logics such as the non recursively axiomatizable **Q.GL** [Mon84] and *completable* logics such as **Q.S4.M** where, in order obtain a complete axiomatization, we need to add some *de re* axiom that regulates the interaction between modalities and the first-order machinery [Cre00].

We are interested in completable logics: with our approach we try to transfer at the predicate level (of some sort) a wide range of completeness results established at the propositional level. Thus we want to solve the following open problem for IMLs:

> how should we define a modular family of proof systems that characterize quantified extensions of a wide class of complete PMLs?

This will be done by introducing labelled sequent calculi for IMLs which behave extremely well from a proof-theoretic point of view —all the structural rules of inference are admissible in our calculi and the logical ones are invertible— and characterize the valid formulas of quantified extensions of any PML in the Lemmon-Scott fragment —i.e. whose characteristic axioms have the shape $\Diamond^n\Box^m p \to \Box^k\Diamond^i p$.

Observe that this result shows that for QMLs labelled sequents are stronger than axiomatic systems: for any completable —but incomplete— axiomatic system of the appropriate kind we can define a labelled sequent calculus that proves all theorems of the completion of the axiomatic system, and not only of the incomplete system. To illustrate, in [Cre95] it has been shown that the formula

$$\Diamond(\forall x(A(x) \to \Box A(x)) \wedge \Box\neg\forall x A(x)) \wedge \Diamond\forall x(A(x) \vee \Box A(x)) \wedge \forall x(\Diamond A(x) \to \Box A(x)) \quad (0.1)$$

is consistent in the axiomatic system **Q.2.BF** (i.e. Cresswell's *KG1+BF*), but unsatisfiable in the class $\mathcal{C}^{2,BF}$ of all Kripke frames for that logic, therefore proving the incompleteness of **Q.2.BF**, which is completable by adding some presently unknown axiom. On the other hand our modular completeness theorem entails that the labelled sequent calculus for $\mathcal{C}^{2,BF}$ is complete w.r.t. it, and therefore, as we will show in the last chapter, the negation of 0.1 is derivable in our labelled sequent calculus.

Before going into the complexities of IMLs, we briefly review the main tenets of the labelled sequent calculi for PMLs introduced in [Neg05, NP11].[1] Let a PML be defined as the set of propositional modal formulas that are valid on a class of frames $\mathcal{F}^p = \langle \mathcal{W}, \mathcal{R} \rangle$ (where the class is defined by a set of conditions on $\mathcal{R}$). The key idea of labelled sequent calculi is that by internalizing the possible-world semantics in the proof theory, we can talk about possible-world frames in proof-theoretic terms, and therefore we can define a modular family of calculi with well-behaved structural properties. The internalization is obtained by extending the language with a set of world labels $w, v, \ldots$ in order to replace propositional modal formulas ϕ with labelled ones $w : \phi$, where $w : \phi$ means that world w forces ϕ (ϕ is true at w), and to add relational atoms $w\mathscr{R}v$, where $w\mathscr{R}v$ means that world v is accessible from w. In this way we can extend the cut- and contraction-free sequent calculus **G3cp** by adding logical rules for the modal operators. These rules internalize the semantical explanations of the forcing relation for modal formulas. To illustrate, let's consider the 'only if' arrow of the forcing condition for $\Box\phi$

$$\text{for all } v \in \mathcal{W},\ w\mathcal{R}v \text{ implies } \models_v \phi, \text{ ONLY IF } \models_w \Box\phi \tag{0.2}$$

This condition is captured by the following rule (where v cannot occur in the conclusion)

$$\frac{w\mathscr{R}v, \Gamma \Rightarrow \Delta, v : \phi}{\Gamma \Rightarrow \Delta, w : \Box\phi}\ R\Box \tag{0.3}$$

Next we have to internalize the semantic conditions on $\mathcal{R}$ that define the PML under consideration by means of relational atoms. If we were to do so

[1] One of the main goals of sequent calculi for PMLs is that of finding calculi that allow to find derivations and to analyze the structural properties of derivations in a PMLs. Traditional sequent systems for PMLs fail to satisfy the required properties in many important cases, witness the difficulties in defining a cut-free system for **S5**. To overcome this limitation, various extensions of traditional sequent systems have been introduced, e.g., display logic [Bel82, Wan98], hyper(tree)- sequent calculi [Avr91, Pog11], and labelled sequent calculi [Neg05, NP11]. See [Neg11, Pog11] for comparisons of these approaches. Here we will deal with calculi of the latter kind because they are best suited for our purposes.

by adding a condition on $\mathcal{R}$ directly as a nonlogical axiom we would lose the admissibility of the structural rules because “the *Hauptsatz* fails for systems with proper axioms” [Gir87, p. 125]. Labelled sequent calculi overcome this difficulty by using the method of axioms-as-rules, introduced in [NP98] for universal axioms and extended in [Ne03] for geometric ones,[2] that allows to have a system that is equivalent to one with proper axioms without loosing the structural properties of the basic calculus. In this way we can define a well-behaved sequent calculus for any PML in the Lemmon-Scott fragment. To illustrate, the propositional logic **T**, which is the logic of reflexive frames, is characterized by the sequent calculus that with the left nonlogical rule

$$\frac{w\mathscr{R}w, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\, T \tag{0.4}$$

which, read bottom-up, says that for any arbitrary label w, $w\mathscr{R}w$ holds.

The key idea of this work is that labelled sequent calculi work equally well for IMLs, and therefore allow us to give a proof-theoretic characterization of a wide class of semantically defined logics. We conclude this introduction by recalling the fundamentals of the language and the semantics of IMLs, and by sketching how we will internalize the semantics in our labelled sequent calculi.

The main difference between the indexed language and the standard language of QMLs is the replacement of the propositional modal operators $\Box$ and $\Diamond$ with operators that are indexed by sets of pairs made out of terms and variables : $|^{t_1}_{x_1} \dots ^{t_n}_{x_n}|$ and $\langle^{t_1}_{x_1} \dots ^{t_m}_{x_m}\rangle$. In this way we can define an indexed modal formula as

$$|^{t_1}_{x_1} \dots ^{t_n}_{x_n}|A \tag{0.5}$$

or

$$\langle^{t_1}_{x_1} \dots ^{t_m}_{x_m}\rangle A \tag{0.6}$$

where, in both cases, the variables free in A must be among the variables

[2] A geometric formula is a formula $\forall x(A \to B)$ where neither A nor B contains $\to$ or $\forall$.

occurring in the lower row of the indexed operator. This proviso is needed because in transition semantics the satisfaction clause of a modal formula is parametric on the free variables of the formula in the scope of the indexed modal operator.

A transition frame is a propositional modal frame augmented with a set of domains indexed by possible worlds D_w[3] and by a set of binary relations, which are denoted as $\mathcal{T}_{(w,v)}$, between the objects of the domains of pairs of worlds w, v such that the second is accessible from the first.[4] A model over a transition frame is obtained by adding a local interpretation I_w of the signature over D_w for each and every possible world w. Assignments are local mappings σ_w from the variables to D_w. Note that terms and predicates have at every world w an extension that is defined over D_w irrespectively of their extension at the other worlds. By $\sigma_w(t)$ we denote the extension of the term t in w. In this way we can define the semantic clause for indexed modal formulas as follows:

$$\sigma_w \models_w |^{t_1}_{x_1} \dots ^{t_n}_{x_n}|A \quad iff \quad \text{for all } v \text{ s.t. } w\mathcal{R}v, \text{ and all } \sigma_v \text{ s.t } .\sigma_w(t_{i\leq\, n})\mathcal{T}_{(w,v)}\sigma_v(x_{i\leq\, n}),\ \sigma_v \models_v A \tag{0.7}$$

which informally means that the formula $|^{t_1}_{x_1} \dots ^{t_n}_{x_n}|A$ is true at w of the objects $a_1, \dots, a_n$ (the w-extension of the t_is) whenever the formula A is true in any accessible world v of every tuple of objects $b_1, \dots, b_n$ (taken from D_v) such that each b_i is a counterpart of a_i. Observe that in this way an indexed modal operator quantifies not on all accessible worlds, but only on those accessible worlds where there are counterparts of the objects we are talking about.

Clause 0.7 is equivalent to that of Kripke semantics for QMLs with respect to closed formulas in the scope of a modal operator indexed by the empty set of pairs of terms. But it is more general w.r.t. open formulas in the scope of an indexed modal operator, and therefore grants us a better control

[3] Although we will work with transition frames with double domains, in this introduction we deal with single domains for the sake of simplicity.

[4] The transition relation can be seen as a modal version of the counterpart relation introduced by David Lewis in [Lew68] to give a fist-order theory of modality.

over the interaction of the first-order machinery (quantifiers, substitutions and identity) with modalities. Many formulas that are valid on every Kripke frame correspond to particular conditions on the transition relations, and therefore transition semantics has a richer correspondence theory. By an IML we mean, as in [Cor09], the set of indexed formulas that are valid on a class of transition frames that is defined by some conditions on the accessiblity relation and/or on the transition relation.

In order to internalize transition semantics into the proof theory of **G3**-style sequent calculi we have to apply labels not only to formulas, but also to terms, which become expressions like t^w (because both formulas and terms have a locally defined extension at every world), and we have to add not only relational formulas, but also transitional formulas like $t^w \mathscr{T} s^v$ which informally means that the extension of s in v is a counterpart of the extension of t in w. We can now define the rules for the indexed modal operators as semantical explanations of the clauses of satisfaction, for example by the right-to-left implication of clause 0.7, we obtain the right rule (where v cannot occur in the conclusion)[5]

$$\frac{w\mathscr{R}v, t_1^w \mathscr{T} x_1^v, \ldots, t_n^w \mathscr{T} x_n^v, \Gamma \Rightarrow \Delta, v : A}{\Gamma \Rightarrow \Delta, w : |^{t_1^w}_{x_1} \ldots ^{t_n^w}_{x_n}| A} R|^{\vec{t}}_{\vec{x}}| \tag{0.8}$$

By having internalized transition semantics, we immediately have a sequent calculus for the minimal IML which has good structural properties because the addition of modal rules, being analogous to the rules for the quantifiers in **G3c**, doesn't affect the structural properties of the underlying calculus. Furthermore by applying the method of axioms-as-rules we can define a sequent calculus for any class of transition frames that is defined by means of universal and geometric properties of the accessibility relation and/or of the transition relations. For example let's consider all transition frames where the transition relation is surjective:

> for all $w, v \in \mathcal{W}$ and for all $b \in D_v$, if $w\mathcal{R}v$, then there is an $a \in D_w$ such that $a\mathcal{T}_{(w,v)}b$.

[5] As a consequence also the x_i^vs will respect the variable condition, and thus we don't have to add this as a further variable condition.

This is a condition that corresponds to the (indexed) Barcan Formula (BF): $\forall x|y_1, \dots y_n\, x|A \rightarrow |y_1 \dots y_n|\forall x A$. The labelled calculus for such class of transition frames is obtained by adding the rule (where x^w cannot occur in the conclusion)

$$\frac{x^w \mathscr{T} t^v, w\mathscr{R}v, \Gamma \Rightarrow \Delta}{w\mathscr{R}v, \Gamma \Rightarrow \Delta}\, BF \tag{0.9}$$

which, informally, means that $w\mathcal{R}v$ implies that each object of D_v (i.e. the extension of t in v) is a counterpart of some object of D_w, and therefore that the transition relation is surjective as desired.

By using these rather simple techniques we will be able to introduce, for each class of transition frames that is defined by universal and geometric conditions, sequent calculi where weakening and contraction are height-preserving admissible and where cut is admissible, and that, as we will show, give a proof-theoretic characterization of all the semantically defined IMLs we have considered. Furthermore we will not need to give a separate proof of completeness for each IMLs, as it often happens for completeness proofs of axiomatic systems for QMLs, but we will give a single modular proof that covers all the different cases.

To sum up, our proof-theoretic study will allow us to dissipate some of the main problems of QMLs, which are concisely stated in the following passage:

> [I]n many ways modal predicate logic is in the state today that modal propositional logic was itself in the late sixties and early seventies. We have a few results about particular systems, we lack results about other particular systems, and we have practically no general results at all about completeness in modal predicate logics.
> [Cre00, p. 178]

Synopsis. In Chapter 1 we introduce the indexed modal language, and explain its novelties w.r.t. the ordinary language of QMLs. We will give particular attention to the notions of bound variables and of substitution of terms for variables. In Chapter 2 we present transition semantics based on double domains transition frames and explain why transition semantics had to be coupled with the indexed modal language. Given that our focus will be the introduction of proof systems for semantically defined classes of transition frames, we will give particular attention to the semantic conditions that define many important IMLs. We stress that our main objective in the first two chapters is not that of justifying the introduction of IMLs, for which the reader is referred to [Cor09], but that of introducing all the notions that will be used in our proof-theoretic study of IMLs. For the same reason, we will not be interested in showing the possible concrete applications of IMLs, but only in showing that IMLs constitute a good theoretical framework for studying quantified modal logics.

In Chapter 3, which is the core of this work, we introduce labelled sequent calculi for indexed modal logics and study in great detail their structural properties. We will be able to introduce a sequent calculus for each semantic class of transition frames that has been introduced in Chapter 2. As we will show, in all our calculi the structural rules of weakening and contraction are height-preserving admissible, all rules are height-preserving invertible, and the structural rule of cut is admissible. We will be able to obtain all these proof-theoretic results in a straightforward way because all our proofs of (height-preserving) admissibility have the same general structure as the ones for the calculus **G3c** for classical first-order logic.

Finally, in Chapter 4, we prove, in a completely modular way, that each of our calculi is sound and complete with respect to the appropriate class of transition frames. After having shown that our calculi characterize all the IMLs we have considered, we prove that labelled sequent calculi are stronger than axiomatic systems in that if a formula A is a theorem of an axiomatic system for a quantified modal logic, then it is provable in the labelled sequent calculus for that logic, but the converse implication does not hold in general: we have completed many incomplete axiomatic systems.

CHAPTER 1

INDEXED MODAL LANGUAGE

This chapter introduces the indexed modal language, first proposed in [Cor09], which is based on changing the syntax of modal operators and of modal formulas by introducing indexed operators, i.e. operators that are parametrized by sets of terms, in order to gain a fine-grained control of free variables, and of substitutions, in modal contexts. With indexed operators we have formulas such as

$$|^{t_1}_{x} \dots {}^{t_n}_{x_n}| A(x_1, \dots, x_n) \ , \tag{0.1}$$

whose intended reading is 'It is necessary for $t_1, \dots, t_n$ that they are in relation $A(x_1, \dots, x_n)$'.

The chapter proceeds as follows: we begin by introducing the language $\mathcal{L}^{\boxdot}$ of indexed modal logics. Then, in Section 2, we introduce the notion of bound variables, where a variable may be bound by a quantifier or by an indexed operator. Finally, in Section 3, we define substitutions and prove some properties thereof.

1.1 Language

Definition 1.1 (Language)**.** The *language* $\mathcal{L}^{\boxdot}$ consists of

- A *signature*:
  - For every $n \in \mathbb{N}^+$, a (at most) denumerable set, REL, of n-ary relation symbols $R_1^n, R_2^n, \ldots$
  - A (at most) denumerable set, CON, of individual constants $c_1, c_2, \ldots$
  - For every $n \in \mathbb{N}^+$, a (at most) denumerable set, FUN, of n-ary function symbols $f_1^n, f_2^n, \ldots$
  - A binary relational constant $\doteq$ (equality).
- A (non-empty) denumerable set, VAR of (individual) *variables* $x_1, x_2, \ldots$
- The *logical symbols*, or *logical operators*, and their respective arity,
  - The propositional *connectives*: the 0-ary $\perp$ (falsum), and the binary $\wedge$ (and), $\vee$ (or), $\rightarrow$ (implies);
  - The unary *quantifiers*: $\forall$ (forall), $\exists$ (exists);
  - The unary *indexed operators*: $|\cdot|$ (it is necessary), $\langle\cdot\rangle$ (it is possible), respectively called 'box' and 'diamond'.
- The *auxiliary symbols* '(', ')', ',' and '$\star$'. ■

Definition 1.2 (Terms)**.** The set of *terms*, $TERM$, is defined inductively as:

1. If $x_i \in VAR$, then $x_i \in TERM$;
2. If $c_i \in CON$, then $c_i \in TERM$;
3. If $f_i^n \in FUN$ and $t_1, \ldots, t_n \in TERM$, then $f_i^n(t_1, \ldots, t_n) \in TERM$. ■

We will usually omit the superscript indicating the arity of a function, and assume that the number of arguments shown is correct.

Definition 1.3. Given a term t, the set $VAR(t)$ of the *variables occurring in t* is defined inductively by:

1. $VAR(x_i) = \{x_i\}$;

2. $VAR(c_i) = \emptyset$;

3. $VAR(f(t_1, \ldots, t_n)) = VAR(t_1) \cup \ldots \cup VAR(t_n)$.

Given a tuple $\vec{t}$ of terms, $VAR(\vec{t}) = \bigcup\{VAR(t_i) : t_i \in \vec{t}\}$. ■

By a *closed term* we mean any term t such that $VAR(t) = \emptyset$, and by an *open term* we mean a term that is not closed.

Definition 1.4 (Formulas)**.** We are now going to define, by a simultaneous induction, the sets (a) $FORM$ of *formulas* and the set (b) $FV(A)$ of the *free variables of a formula A*.

1. (a) If $R_i^n \in REL$ and $t_1, \ldots, t_n \in TERM$, then $R_i^n(t_1, \ldots, t_n) \in FORM$;
   (b) $FV(R_i^n(t_1, \ldots, t_n)) = VAR(t_1) \cup \ldots \cup VAR(t_n)$;

2. (a) $\bot \in FORM$;
   (b) $FV(\bot) = \emptyset$;

3. (a) If $A, B \in FORM$, then $(A \circ B) \in FORM$, where $\circ \in \{\wedge, \vee, \rightarrow\}$;
   (b) $FV((A \circ B)) = FV(A) \cup FV(B)$;

4. (a) If $A \in FORM$ and $x_i \in VAR$, then $(\mathcal{Q}x_iA) \in FORM$, where $\mathcal{Q} \in \{\forall, \exists\}$;
   (b) $FV((\mathcal{Q}x_iA)) = FV(A) - \{x_i\}$;

5. (a) If $A \in FORM$, $FV(A) \subseteq \{x_{i_1}, \ldots, x_{i_n}\}$, and $t_1, \ldots, t_n \in TERM$, then $(|^{t_1}_{x_{i_1}} \ldots ^{t_n}_{x_{i_n}}|A) \in FORM$, and $(\langle^{t_1}_{x_{i_1}} \ldots ^{t_n}_{x_{i_n}}\rangle A) \in FORM$;
   (b) $FV((|^{t_1}_{x_{i_1}} \ldots ^{t_n}_{x_{i_n}}|A)) = FV((\langle^{t_1}_{x_{i_1}} \ldots ^{t_n}_{x_{i_n}}\rangle A)) = VAR(t_1) \cup \ldots \cup VAR(t_n)$.

■

Definition 1.5. The set $FORM^=$ is defined as the set $FORM$ with the addition of:

8. (a) If $t, s \in TERM$, then $t \doteq s \in FORM^=$;
   (b) $FV(t \doteq s) = FV(t) \cup FV(s)$. ■

In practice parentheses can be left out in formulas formation following the usual conventions. As we did for functions, we will usually omit the superscript indicating the arity of a relation, and assume that the number of argument shown is correct. Whenever the distinction doesn't matter, by '*formulas*' we will mean both $FORM$ and $FORM^{=}$. The set $ATOM$ of *atomic formulas* is the set of all formulas that do not contain logical symbols. By a *closed formula*, or *sentence*, we mean any formula A such that $FV(A) = \emptyset$. If A, B are formulas, we wil use $\neg A$ as a shorthand for $A \to \bot$, and $A \leftrightarrow B$ for $(A \to B) \wedge (B \to A)$. If s, t are terms, we will use $s \not\doteq t$ for $\neg(s \doteq t)$.

We introduce the following notational conventions for indexed operators:

- The formula $|x_{i_1}, \dots x_{i_n}|A$ stands for $|^{x_{i_1}}_{x_{i_1}} \dots ^{x_{i_n}}_{x_{i_n}}|A$.
- When the index is the empty sequence, we write $|\star|A$ and $\langle\star\rangle A$ (this may happen only if A is a sentence).
- In a modal formula $|^{t_1}_{x_{i_1}} \dots ^{t_n}_{x_{i_1}}|A$, the tuple $t_1, \dots, t_n$ is called the *numerator* of that formula, and the tuple $x_{i_1}, \dots, x_{i_n}$ is called the *denominator* of that formula.

Note also that the free variables of an indexed formula are all and only the variables occurring inside its numerator, thus, in a sense that will become clearer later on, an indexed operator 'seals off' the formula occurring in its scope. As a matter of fact, the variables occurring in the denominator of an indexed operator and/or inside the formula in its scope, are neither free not bound in the indexed formula itself. This will allow us to introduce a fine-grained treatment of substitutions in modal contexts, and it will be of some importance for the indexed sequent calculi of Chapter XX.

In the following we will use the following *metavariables*, all possibly with subscripts taken from $\mathbb{N} - \{0\}$,

- x, y, z for variables, and $\vec{x}, \vec{y}, \vec{z}$ for tuples of variables;
- a, b, c for individual constants, and $\vec{a}, \vec{b}, \vec{c}$ for tuples of constants;
- f, g for function symbols;

- r, s, t for terms, and $\vec{r}, \vec{s}, \vec{t}$ for tuples of terms;
- A, B, C for formulas (possibly containing equality), and $A(\vec{x})$ for a formula whose free variables are among *vecx*;
- P, Q for atomic formulas (possibly containing equality), and $P(\vec{x})$ for an atomic formula whose free variables are among $\vec{x}$;
- p, q for 0-ary relation symbols, i.e. propositional atoms.

Whenever convenient, we will also denote logical operators by means of:

- '$\circ$' for binary propositional operators,
- '$\mathcal{Q}$' for the quantifiers,
- '$\boxdot$' for indexed operators.

Given a formula A, by a *subformula* of A we will mean any member of the construction tree of A, formally:

Definition 1.6 (Subformulas)**.** For any formula A the set $SF(A)$ of its *subformulas* is defined inductively by:

1. $SF(\bot) = \{\bot\}$;
2. $SF(R_i^n(t_1, \ldots, t_n)) = \{R_i^n(t_1, \ldots, t_n)\}$;
3. $SF(B \circ C) = SF(B) \cup SF(C) \cup \{B \circ C\}$;
4. $SF(\mathcal{Q}x_i B) = SF(B) \cup \{\mathcal{Q}x_i B\}$;
5. $SF(\boxdot B) = SF(B) \cup \{\boxdot B\}$. ■

In order to prove properties of terms and formulas, we will reason inductively on their height, which is defined as follows.

Definition 1.7 (Term-height)**.** The *term-height of a term* t, $h(t)$ is defined inductively as:

1. $h(x_i) = h(c_i) = 0$;

2. $h(f(t_1,\ldots,t_n)) = max\{h(t_1),\ldots,h(t_n)\}+1$. ■

Definition 1.8 (Formula-height)**.** The *formula-height of a formula* A, $h(A)$ is defined inductively as:

1. $h(\bot) = h(R_i^n(t_1,\ldots,t_n)) = h(s \doteq t) = 0$;
2. $h(\mathcal{Q}x_iB) = h(\boxdot B) = h(B)+1$;
3. $h(B \circ C) = max\{h(B), h(C)\}+1$. ■

1.2 Bound Variables

Both quantifiers and indexed operators are devices for anaphora: they create anaphoric linkages between variables that, in this process, are deprived of their referential role. To illustrate, the formula

$$|^{c}_{x}|\exists y P(x,y) \tag{2.2}$$

may be read as

c is necessarily P-ing something.[1]

Given that anaphoric linkages are expressed in first-order logic as bound variables, in indexed modal logics we have two different kinds of bound variables: variables —e.g. y in 2.2— bound by a quantifier and variables —e.g. x in 2.2— bound by an indexed modal operator; we will call them *q-bound* and *m-bound* variables, respectively. Although the notion of an m-bound variable is new, it can easily be understood in analogy to the well-known notion of a q-bound variable: given a formula

$$|^{t_1}_{x_1}\ldots{}^{t_n}_{x_n}|A\ , \tag{2.3}$$

the variables $x_1,\ldots,x_n$ occurring in the denominator of the indexed operator bound all occurrences of $x_1,\ldots x_n$ in the subformula A. The main difference

[1] Here we assume that $P(x,y)$ is a transitive verb, in general the suggested reading of a formula $|^{t_1}_{x_1}\ldots{}^{t_n}_{x_n}|R^n$ is 'it is necessary for $t_1,\ldots,t_n$ that they are in relation R^n'.

is that whereas q-bound variables have their anaphoric antecedent in a quantified expression $\mathcal{Q}x$, m-bound variables have their anaphoric antecedent in a term occurring in the numerator of an indexed operator. For example, in 2.3 each occurrences of $x_{i(\leq n)}$ in A has its anaphoric antecedent in the term $t_{i(\leq n)}$ occurring in the numerator of the modal operator.

In first-oder logic the purely anaphoric role of bound variables is captured by the fact that congruent formulas, i.e. formulas that are identical modulo a renaming of q-bound variables, may be identified: they are proof-theoretically inter-derivable and semantically co-satisfiable. As we will see, the same property holds for indexed formulas that are identical modulo a renaming of some of its m-bound variables, in this case we will talk of *i-congruent formulas*. Observe that our definition of formulas entails that no variable occurs free in A whenever A is in the scope of an indexed operator.[2] This is one motivation for our claim that an indexed operator seals off the formula in its scope.

These considerations on congruent and i-congruent formulas suggest that the only role of variables occurring in a formula A is that of expressing anaphoric linkages, unless they occur free in that formula. Thus, as long as we don't lose track of these linkages, we may dispense with these variables. One way of dispensing with them is by replacing formulas with skeletons, see [GSS09, p. 83-84], where the skeleton of a formula A is the expression that we obtain by replacing every occurrence of a variable in a formula A that is not free by a new expression that is graphically connected to its anaphoric antecedent. To show the difference between the two kinds of anaphoric linkages, we replace q-bound variables by $\bullet$ and connect 'from above' each $\bullet$ with its anaphoric antecedent, and we replace m-bound variables by $*$ and connect 'from below' each of its occurrence inside the scope of the operator with the corresponding occurrence in the denominator.[3] Formally

[2] To wit, the free occurrences of variables in $|\frac{\vec{t}}{\vec{x}}|A$ are all and only the occurrences of variables in the numerator $\vec{t}$, since all variables occurring in A are bound either by $|\frac{\vec{t}}{\vec{x}}|$ or by some other binding operator in its scope.

[3] Which, is anaphorically linked to the term (or $\bullet$ or $\star$) in the corresponding position of the numerator.

Definition 1.9 (Skeleton)**.** For any formula A, its *skeleton* $\ulcorner A \urcorner$ is defined inductively as follows:

1. $\ulcorner A \urcorner \quad = \quad A, \qquad$ for A atomic or $\bot$;

2. $\ulcorner (A \circ B) \urcorner \quad = \quad \ulcorner A \urcorner \circ \ulcorner B \urcorner$;

3. $\ulcorner \mathcal{Q} x A \urcorner$ is obtained by from $\mathcal{Q} x \ulcorner A \urcorner$ by replacing every occurrence of x with $\bullet$ and connecting it with the first occurrence of $\mathcal{Q}$;

4. $\ulcorner |^{t_1}_{x_1} \dots ^{t_n}_{x_n}| A \urcorner$ is obtained from $|^{t_1}_{x_1} \dots ^{t_n}_{x_n}| \ulcorner A \urcorner$ by replacing every occurrence of each x_i in the denominator and in $\ulcorner A \urcorner$ with $*$ and, then, connecting each occurrence of $\star$ in $\ulcorner A \urcorner$ with its occurrence inside the denominator.

5. $\ulcorner \langle^{t_1}_{x_1} \dots ^{t_n}_{x_n} \rangle A \urcorner$ is defined similarly to $\ulcorner |^{t_1}_{x_1} \dots ^{t_n}_{x_n}| A \urcorner$. ■

This definition is adapted from [GSS09, p. 84], where skeletons are called schemes. The same graphic representation of the anaphoric structure of a formula is used in [Kle67, p. 82]. The work [Bou68] uses systematically skeletons, there called *assemblies*, instead of formulas. But, even if it may be argued that "[skeletons] better correspond to human intuition about [quantified] logic[s]" [GSS09, p. 84], the use of formulas is simpler in that skeletons are more cumbersome to write. We will use skeletons only incidentally for explaining and justifying our notion of substitution of terms for variables: they represent, in a sense, the deep logical structure of indexed modal formulas, and hence we need to define substitution in such a way that it preserves the logical structure of the skeleton of a formula, and not the superficial aspect of the formula.

Definition 1.10 (i-congruence)**.** Two formulas are *i-congruent* whenever they have the same scheme: A is i-congruent to B iff $\ulcorner A \urcorner = \ulcorner B \urcorner$. ■

Here we present some illustrative examples of skeletons:

- The skeleton of $\exists x(P(x) \wedge \forall y Q(x, y, z))$ is:

$\exists \bullet (P(\bullet) \wedge \forall \bullet Q(\bullet, \bullet, z))$

- The skeleton of $|^{t\ s}_{x\ y}|(P(x) \wedge Q(x, y, c))$ is:

$|^{t\ s}_{*\ *}|(P(*) \wedge Q(*, *, c))$

- The skeleton of $\exists x(P(x) \wedge |^{x\ s}_{x\ y}|(P(y, x) \wedge \forall y Q(y, x)))$ is:

$\exists \bullet (P(\bullet) \wedge |^{\bullet\ t}_{*\ *}|(P(*, *) \wedge \forall \bullet Q(\bullet, *)))$

1.3 Substitution of Terms

Definition 1.11 (Substitution in terms)**.** Let $\vec{s} = s_1, \dots, s_n$ be an n-tuple of terms and $\vec{x} = x_1, \dots, x_n$ a n-tuple of pairwise distinct variables. The *simultaneous substitution of* $\vec{s}$ *for* $\vec{x}$ *in a term* t, $t[s_1, \dots, s_n/x_1, \dots, x_n]$ ($t[\vec{s}/\vec{x}]$ for short), is defined as:

- $y[\vec{s}/\vec{x}] = \begin{cases} y & \text{if } y \notin \vec{x}; \\ s_i & \text{if } y = x_i, \text{ with } i \leq n; \end{cases}$
- $c[\vec{s}/\vec{x}] = c;$
- $f(t_1, \dots, t_n)[\vec{s}/\vec{x}] = f(t_1[\vec{s}/\vec{x}], \dots, t_n[\vec{s}/\vec{x}]).$

A *simple substitution*, $t[s/x]$, is a simultaneous substitution such that $\vec{s} = s$ and $\vec{x} = x$. ■

Notational convention. Given a proviso C, by $t \stackrel{C}{=} s$ we mean that the equality between t and s depends on C.

Proposition 1.12. *If $y \neq x$ and $y \notin VAR(s)$, then, for any term t*

$$(t[s/x])[r/y] = t[s, r/x, y]$$

Proof. The proof is by induction on the term-height $h(t)$ of t.
If $t = x$, then $(x[s/x])[r/y] = s[r/y] \overset{y \notin VAR(s)}{=} s \overset{y \neq x}{=} x[s, r/x, y]$.
If $t = y$, then $(y[s/x])[r/y] \overset{y \neq x}{=} y[r/y] = r \overset{y \neq x}{=} y[s, r/x, y]$.
If t is variable distinct from x and from y, or it is an individual constant, the proposition holds trivially.
If $t = f(t_1, \ldots, t)$, then $(f(t_1, \ldots, t_n)[s/x])[r/y] = f(t_1[s/x], \ldots, t_n[s/x])[r/y]$
$= f((t_1[s/x])[r/y], \ldots, (t_n[s/x])[r/y]) \overset{IH}{=} f(t_1[s, r/x, y], \ldots, t_n[s, r/x, y]) =$
$f(t_1, \ldots, t_n)[s, r/x, y]$. ■

For future reference we list some immediate facts about substitutions in terms.

Proposition 1.13. *For any term t:*

1. $t[x/x] = t$.
2. *If $x \notin VAR(t)$, then $t[s/x] = t$.*
3. *If $y \notin VAR(t)$, then $(t[y/x])[r/y] = t[r/x]$.*
4. *If $x \neq y$ and $x \notin VAR(r)$, then $(t[s/x])[r/y] = (t[r/y])[s[r/y]/x]$.*
5. *If $y \neq x$, $x \notin VAR(r)$, and $y \notin VAR(s)$, then $(t[s/x])[r/y] = (t[r/y])[s/x]$.*

Proof. The proofs are by induction on the term-height of t.

1. Straightforward.
2. Straightforward.
3. Let $t \neq y$ (since $y \notin VAR(t)$).
    - If $t = x$, then $(x[y/x])[r/y] = y[r/x]$.
    - If $t = z$ for some $z \neq x$, then $(z[y/x])[r/y] = z[r/y]$.
    - If $t = f(t_1, \ldots, t_n)$, then $(f(t_1, \ldots, t_n)[y/x])[r/y] = f(t_1[y/x], \ldots, t_n[y/x])[r/y]$
    $= f((t_1[y/x])[r/y], \ldots, (t_n[y/x])[r/y]) \overset{IH}{=} f(t_1[r/x], \ldots, t_n[r/x]) =$
    $f(t_1, \ldots, t_n)[r/x]$.
4. - If $t = x$, then $(x[s/x])[r/y] = s[r/y] = x[s[r/y]/x] \overset{y \neq x}{=} (x[r/y])[s[r/y]/x]$.

- If $t = y$, then $(y[s/x])[r/y] \stackrel{y \neq x}{=} y[r/y] = r \stackrel{x \notin VAR(r)}{=} r[s[r/y]/x] = (y[r/y])[s[r/y]/x]$.
- $t = z$, with $z \neq x$ and $z \neq y$. Trivial.
- If $t = f(t_1, \ldots, t_n)$, then $(f(t_1, \ldots, t_n)[s/x])[r/y] = f(t_1[s/x], \ldots, t_n[s/x])[r/y] = f((t_1[s/x])[r/y], \ldots, (t_n[s/x])[r/y]) \stackrel{IH}{=} f((t_1[r/y])[s[r/y]/x], \ldots, (t_n[r/y])[s[r/y]/x]) = f(t_1[r/y], \ldots, t_n[r/y])[s[r/y]/x] = (f(t_1, \ldots, t_n)[r/y])[s[r/y]/x]$.

5.
- If $t = x$, then $(x[s/x])[r/y] = s[r/y] \stackrel{y \notin VAR(s)}{=} s = x[s/x] \stackrel{y \neq x}{=} (x[r/y])[s/x]$.
- If $t = y$, then $(y[s/x])[r/y] \stackrel{y \neq x}{=} y[r/y] = r = y[r/y] \stackrel{x \notin VAR(r)}{=} (y[r/y])[s/x]$.
- $t = z$, with $z \neq x$ and $z \neq y$. Trivial.
- If $t = f(t_1, \ldots, t_n)$, then $(f(t_1, \ldots, t_n)[s/x])[r/y] = f(t_1[s/x], \ldots, t_n[s/x])[r/y] = f((t_1[s/x])[r/y], \ldots, (t_n[s/x])[r/y]) \stackrel{IH}{=} f((t_1[r/y])[s/x], \ldots, (t_n[r/y])[s/x]) = f(t_1[r/y], \ldots, t_n[r/y])[s/x] = (f(t_1, \ldots, t_n)[r/y])[s/x]$. ■

Definition 1.14 (Substitution)**.** Let $\vec{s} = s_1, \ldots, s_n$ be an n-tuple of terms and $\vec{x} = x_1, \ldots, x_n$ a n-tuple of pairwise distinct variables. The *simultaneous substitution of* $\vec{s}$ *for* $\vec{x}$ *in a formula* A, $A[\vec{s}/\vec{x}]$, is defined as:

- $(\perp)[\vec{s}/\vec{x}] = \perp$;
- $R_i(t_1, \ldots, t_n)[\vec{s}/\vec{x}] = R_i(t_1[\vec{s}/\vec{x}], \ldots, t_n[\vec{s}/\vec{x}])$;
- $(t_i \doteq t_j)[\vec{s}/\vec{x}] \quad = \quad t_i[\vec{s}/\vec{x}] \doteq t_j[\vec{s}/\vec{x}])$;
- $(B \circ C)[\vec{s}/\vec{x}] = B[\vec{s}/\vec{x}] \circ C[\vec{s}/\vec{x}]$;

- $(\mathcal{Q}yB)[\vec{s}/\vec{x}] = \begin{cases} \mathcal{Q}y(B[s_1 \dots y \dots s_n)/x_1 \dots y \dots x_n]) & \text{if } x_i = y \text{ and for all } j \neq i, \text{ if } x_j \neq y \\ & \text{and } x_j \in FV(B), \text{ then } y \notin VAR(s_j); \\ \mathcal{Q}z((B[z/y])[s_1 \dots z \dots s_n)/x_1 \dots z \dots x_n]) & \text{if } x_i = y \text{ and there is a } x_j \text{ s.t. } x_j \neq y, \\ & x_j \in FV(B), \text{ and } y \in VAR(s_j), \text{ and} \\ & z \text{ is a variable new to } \mathcal{Q}yB \text{ and to} \\ & s_1 \dots s_i \dots s_n; \\ \mathcal{Q}y(B[\vec{s}/\vec{x}]) & \text{if for all } x_i, x_i \neq y, \text{ and for all } x_j \in FV(B), \\ & y \notin VAR(s_j); \\ \mathcal{Q}z((B[z/y])[\vec{s}/\vec{x}]) & \text{if for all } x_i, x_i \neq y, \text{ and if there is a } x_j \\ & \text{s.t. } x_j \neq y, x_j \in FV(B), \text{ and} \\ & y \in VAR(s_j), \text{ and } z \text{ is a variable new} \\ & \text{to } \mathcal{Q}yB \text{ and to } s_1 \dots s_i \dots s_n; \end{cases}$
- $(|^{t_1}_{x_1} \dots ^{t_n}_{x_n}|B)[\vec{s}/\vec{x}] = |^{t_1[\vec{s}/\vec{x}]}_{x_1} \dots ^{t_n[\vec{s}/\vec{x}]}_{x_n}|B;$
- $(\langle^{t_1}_{x_1} \dots ^{t_n}_{x_n}\rangle B)[\vec{s}/\vec{x}] = \langle^{t_1[\vec{s}/\vec{x}]}_{x_1} \dots ^{t_n[\vec{s}/\vec{x}]}_{x_n}\rangle B.$

A *simple substitution*, $A[s/x]$ is a simultaneous substitution such that $\vec{s} = s$ and $\vec{x} = x$. ■

In the above definition when renaming is needed, the notion of substitution is defined modulo the choice of a new variable, but in general this is unproblematic, and, if needed, we can make it unambiguous by assuming that we always rename with the first variable in our enumeration of VAR that is new, as done, e.g. in [KD06].

Even if we have defined substitution without restrictions on the terms to be substituted, we will make extended use of the usual notion of 'a term being free for a variable in a formula, where t is said to be *free for* x *in* A, if no free occurrences of x in A are within the scope of an expression $\mathcal{Q}y$ where y is any variable occurring in t. More precisely:

Definition 1.15. $t_1, \dots, t_n$ *are free respectively for* $x_1, \dots x_n$ *in* A whenever:

1. A is atomic, $\perp$, or $\boxdot B$;
2. A is $B \circ C$ and each $t_{i(\leq n)}$ is free for $x_{i(\leq n)}$ in B and in C;
3. A is $\mathcal{Q}yB$, and each $t_{i(\leq n)}$ is free for $x_{i\leq n)}$ in B, and (either $x_i \notin FV(B)$ or $y \notin VAR(t_i)$.) ■

To better understand our definition of substitution of a term for a variable in a quantified formula, we present explicitly the case of a simple substitution in a quantified formula, $(\mathcal{Q}yB)[s/x]$, where our definition is equivalent to the following one:

$$(\mathcal{Q}yB)[s/x] = \begin{cases} \mathcal{Q}yB & \text{if } x = y; \\ \mathcal{Q}z((B[z/y])[s/x]) & \text{if } x \neq y, \text{ and } s \text{ isn't free for } x \text{ in } \mathcal{Q}yB, \\ & \text{where } z \text{ is a variable new to } \mathcal{Q}yB; \\ \mathcal{Q}y(B[s/x]) & \text{if } x \neq y, \text{ and } s \text{ is free for } x \text{ in } \mathcal{Q}yB. \end{cases} \tag{3.4}$$

The modal cases are defined in such a way that indexed formulas behave like atomic formulas w.r.t. substitution: the substitution is performed inside the numerator of the operator, and not inside the formula which is in the scope of the operator; formula whose free variables are all m-bound by the indexed operator. Thus, e.g., if we apply the substitution $[t/x]$ to the formula

$$|^{y\ x}_{x\ z}|P(x,z)\ , \tag{3.5}$$

we obtain the formula

$$|^{y\ t}_{x\ z}|P(x,z)\ , \tag{3.6}$$

and not the formula

$$|^{y\ x}_{x\ z}|P(t,z)\ , \tag{3.7}$$

that would be obtained by permuting the substitution with the operator, as done for the standard quantified modal language $\mathcal{L}^{\Box}$, see Appendix A. From a syntactical point of view this approach is justified by our understanding of formulas as skeletons: the skeleton corresponding to the formula in 3.5 is

$$|^{y\ x}_{*\ *}(P(*,*)$$

and by applying to it the substitution $[t/x]$ we shall obtain the skeleton of the formula in 3.6, which is

$|^{y\ t}_{*\ *}(P(*,*)$

and not that of the formula in 3.7, which is

$|^{y\ x}_{*\ *}(P(t,*)$

For future reference we list some facts about substitutions in formulas that will be useful later on.

Proposition 1.16. *If $y \neq x$, $y \notin VAR(s)$ and s is free for x in A, then*

$$(A[s/x])[r/y] = A[s, r/x, y]$$

Proof. The proof is by induction on the formula-height $h(A)$ of A.

If $h(A) = 0$, then if A is $\perp$ there is nothing to prove, if A is atomic the proposition follows from Proposition 1.12 thanks to the provisos of the lemma.

If $h(A) = n + 1$, we argue by cases.

- If $A = B \circ C$, then $(((B \circ C)[s/x])[r/y] = (B[s/x] \circ C[s/x])[r/y] = (B[s/x])[r/y] \circ (C[s/x])[r/y] \overset{IH}{=} B[s, r/x, y] \circ C[s, r/x, y] = (B \circ C)[s, r/x, y]$.
- If $A = \mathcal{Q}zB$, we have three cases.
  - $z = x$. $((\mathcal{Q}xB)[s/x])[r/y] = (\mathcal{Q}xB)[r/y] = (\mathcal{Q}xB)[s, r/x, y]$.
  - $z = y$. $((\mathcal{Q}yB)[s/x])[r/y] \overset{s\, is\, free\, for\, x\, in\, A}{=} (\mathcal{Q}y(B[s/x]))[r/y] = \mathcal{Q}y(B[s/x]) \overset{s\, is\, free\, for\, x\, in\, A}{=} (\mathcal{Q}yB)[s/x] = (\mathcal{Q}yB)[s, r/x, y]$.
  - $z \neq x$ and $z \neq y$. $((\mathcal{Q}zB)[s/x])[r/y] \overset{s\, is\, free\, for\, x\, in\, A}{=} (\mathcal{Q}z(B[s/x]))[r/y] =$

$$= \begin{cases} \text{if } r \text{ is free for } y \text{ in } \mathcal{Q}z(B[s/x]), & \mathcal{Q}z((B[s/x])[r/y]) \overset{IH}{=} \mathcal{Q}z(B[s, r/x, y] = (\mathcal{Q}zB)[s, r/x, y] \\ & . \\ \text{else,} & \mathcal{Q}z_1(((B[z_1/z])[s/x])[r/y])) \overset{IH}{=} \mathcal{Q}z_i((B[z_i/z])[s, r/x, z]) = \\ & (\mathcal{Q}zB)[s, r/x, y]. \end{cases}$$

- If $A = |^{t_1}_{x_1} \dots ^{t_n}_{x_n}|B$, then $((|^{t_1}_{x_1} \dots ^{t_n}_{x_n}|B)[s/x])[r/y] = (|^{t_1[s/x]}_{x_1} \dots ^{t_n[s/x]}_{x_n}|B)[r/y] = |^{(t_1[s/x])[r/y]}_{x_1} \dots ^{(t_n[s/x])[r/y]}_{x_n}|B \stackrel{1.12}{=} |^{t_1[s,r/x,y]}_{x_1} \dots ^{t_n[s,r/x,y]}_{x_n}|B = (|^{t_1}_{x_1} \dots ^{t_n}_{x_n}|B)[s,r/x,y]$.

- If $A = \langle^{t_1}_{x_1} \dots ^{t_n}_{x_n}\rangle B$, we proceed as for $|^{t_1}_{x_1} \dots ^{t_n}_{x_n}|B$. ■

Proposition 1.17. *For any formula A:*

1. *$A[x/x] = A$.*
2. *If $x \notin VAR(A)$, then $A[s/x] = A$.*
3. *If $y \notin VAR(A)$ and s is free for x in A, then $(A[y/x])[s/y] = A[s/x]$.*
4. *If $x \neq y$, $x \notin VAR(r)$, and s and r are free for x and y in A, then $(A[s/x])[r/y] = (A[r/y])[s[r/y]/x]$.*
5. *If $y \neq x$, $x \notin VAR(r)$, $y \notin VAR(s)$, and s and r are free respectively for x and y in A, then $(A[s/x])[r/y] = (A[r/y])[s/x]$.*
6. *If $y \notin VAR(A[s/x])$, then $(A[y/x])[s/y] = A[s/x]$.*

Proof. The proofs are by induction on the formula-height of A, $h(A)$. Note that for the indexed case we won't need to use IH, but we will prove the properties directly as we did for formulas with formula-height 0. This happens because indexed formulas behave as atomic ones w.r.t. substitution of terms for variables.

1. Straightforward.
2. Straightforward.
3. If $h(A) = 0$, then if A is $\perp$ the proof is trivial, and if it is an atom, say $P(t_1, \dots, t_n)$, $(A[y/x])[s/y] = P((t_1[y/x])[s/y], \dots, (t_n[y/x])[s/y])$ and — given that $y \notin VAR(A)$ — the property holds by Proposition 1.13.3.

   If $h(A) = n + 1$, we argue by cases.

- Let $A = B \circ C$. Then $= ((B \circ C)[y/x])[s/y]) = ((B[y/x] \circ C[y/x])[s/y] = ((B[y/x])[s/y]) \circ ((C[y/x])[s/y]) \stackrel{IH}{=} (B[s/x]) \circ (C[s/x]) = (B \circ C)[s/x]$.
- Let $A = \mathcal{Q}zB$. Since $y \notin VAR(A)$ we know that $z \neq y$ and that y is free for x in $\mathcal{Q}zB$.
  - If $z = x$, then $(A[y/x])[s/y] = ((\mathcal{Q}xB)[y/x])[s/y] = (\mathcal{Q}xB)[s/y] \stackrel{y \notin VAR(A)}{=} \mathcal{Q}xB = (\mathcal{Q}xB)[s/x] = A[s/x]$.
  - If $z \neq x$, $(A[y/x])[s/y] = ((\mathcal{Q}zB)[y/x])[s/y]) \stackrel{y \text{ is free for } x \text{ in } \mathcal{Q}zB}{=} (\mathcal{Q}z(B[y/x])[s/y] \stackrel{s \text{ is free for } x \text{ in } A}{=} \mathcal{Q}z((B[y/x])[s/y]) \stackrel{IH}{=} \mathcal{Q}z(B[s/x]) \stackrel{s \text{ is free for } x \text{ in } A}{=} (\mathcal{Q}zB)[s/x] = A[s/x]$.
- If $A = |^{t_1}_{x_1} \dots ^{t_n}_{x_n}|B$, then $(A[y/x])[s/y] = |^{(t_1[y/x])[s/y]}_{x_1} \dots ^{(t_n[y/x])[s/y]}_{x_n}|B \stackrel{y \notin VAR(A), 1.13.3}{=} |^{t_1[s/x]}_{x_1} \dots ^{t_n[s/x]}_{x_n}|B$.
- If $A = \langle^{t_1}_{x_1} \dots ^{t_n}_{x_n}\rangle B$, the proof is analogous.

4. Let $h(A) = 0$. If $A = \perp$ the proof is trivial, if it is $P(t_1, \dots, t_n)$, then $(A[s/x])[r/y] = P((t_1[s/x])[r/y], \dots (t_n[s/x])[r/y])$, and —given that $x \neq y$ and $x \notin VAR(r)$ — the property holds by Proposition 1.13.4.

   Let $h(A) = n + 1$.

   - If $A = B \circ C$. Then $((B \circ C))[s/x])[r/y] = (B[s/x] \circ C[s/x])[r/y] = (B[s/x])[r/y] \circ (C[s/x])[r/y] \stackrel{IH}{=} (B[r/y])[s[r/y]/x] \circ (C[r/y])[s[r/y]/x] = (B \circ C)[r/y])[s[r/y]/x]]$. Three cases to be considered.
   - If $A = \mathcal{Q}zB$, three cases are possible:
     - $z = x$. Then $((\mathcal{Q}xB)[s/x])[r/y] = (\mathcal{Q}xB)[r/y] \stackrel{x \neq y, \text{ and } r \text{ is free for } y \text{ in } A}{=} \mathcal{Q}x(B[r/y]) = (\mathcal{Q}x(B[r/y]))[s[r/y]/x] = (A[r/y])[s[r/y]/x]$.
     - $z = y$. Then $((\mathcal{Q}yB)[s/x])[r/y] \stackrel{x \neq y, \text{ and } s \text{ is free for } x \text{ in } A}{=} (\mathcal{Q}y(B[s/x]))[r/y] = \mathcal{Q}y(B[s/x]) \stackrel{x \neq y, \text{ and } s \text{ is free for } x \text{ in } A}{=} (\mathcal{Q}yB)[s/x] \stackrel{s \text{ is free for } x \text{ in } A}{=} (\mathcal{Q}yB)[s[r/y]/x] = ((\mathcal{Q}yB)[r/y])[s[r/y]/x]$.
     - $z \neq x$ and $z \neq y$. Then $((\mathcal{Q}zB)[s/x])[r/y] \stackrel{s \text{ is free for } x \text{ in } A}{=} (\mathcal{Q}z(B[s/x]))[r/y] \stackrel{r \text{ is free for } y \text{ in } A}{=} \mathcal{Q}z((B[s/x])[r/y] \stackrel{IH}{=} \mathcal{Q}z((B[r/y])[s[r/y]/x]) \stackrel{s\ (r) \text{ is free for } x\ (y) \text{ in } A}{=} (\mathcal{Q}z(B[r/y]))[s[r/y]/x] \stackrel{r \text{ is free for } y \text{ in } A}{=} ((\mathcal{Q}zB)[r/y])[s[r/y]/x] = (A[r/y])[s[r/y]/x]$.

- If $A = |^{t_1}_{x_1} \dots ^{t_n}_{x_n}|B$, then $(A[s/x])[r/y] = |^{(t_1[s/x])[r/y]}_{x_1} \dots ^{(t_n[s/x])[r/y]}_{x_n}|B \overset{1.13.4}{=} |^{(t_1[r/y])[s[r/y]/x]}_{x_1} \dots ^{(t_n[r/y])[s[r/y]/x]}_{x_n}|B$.
- If $A = \langle^{t_1}_{x_1} \dots ^{t_n}_{x_n}\rangle B$, the proof is analogous.

5. Let $h(A) = 0$. If $A = \perp$ the proof is trivial. If A is $P(t_1, \dots, t_n)$, then $(A[s/x])[r/y] = P((t_1[s/x])[r/y], \dots (t_n[s/x])[r/y])$, and —given that $x \neq y$, $x \notin VAR(r)$ and $y \notin VAR(s)$— the property holds by Proposition 1.13.5.

   If $h(A) = n + 1$, we argue by cases.

   - If $A = B \circ C$, then$((B \circ C))[s/x])[r/y] = (B[s/x] \circ C[s/x])[r/y] = (B[s/x])[r/y] \circ (C[s/x])[r/y]$
     $\overset{IH}{=} (B[r/y])[s/x] \circ (C[r/y])[s/x] = (B \circ C)[r/y])[s/x]$.
   - If $A = \mathcal{Q}zB$, three are possible:
     - $z = x$. Then $((\mathcal{Q}xB)[s/x])[r/y] = (\mathcal{Q}xB)[r/y] \overset{r \text{ is free for } y \text{ in } A}{=} \mathcal{Q}x(B[r/y]) = (\mathcal{Q}x(B[r/y])[s/x] = ((\mathcal{Q}xB)[r/y])[s/x]$.
     - $z = y$. Then $((\mathcal{Q}yB)[s/x])[r/y] \overset{s \text{ is free for } x \text{ in } A}{=} (\mathcal{Q}y(B[s/x]))[r/y]$ $= \mathcal{Q}y(B[s/x]) \overset{s \text{ is free for } x \text{ in } A}{=} (\mathcal{Q}yB)[s/x] = ((\mathcal{Q}yB)[r/y])[s/x]$.
     - $z \neq x$ and $z \neq y$. Then $((\mathcal{Q}zB)[s/x])[r/y] \overset{s \text{ is free for } x \text{ in } A}{=}$ $(\mathcal{Q}z(B[s/x]))[r/y] \overset{r \text{ is free for } y \text{ in } A, \text{ and } y \notin VAR(s)}{=} \mathcal{Q}z((B[s/x])[r/y]) \overset{IH}{=}$ $\mathcal{Q}z((B[r/y])[s/x]) \overset{s \text{ is free for } x \text{ in } A, \text{ and } x \notin VAR(r)}{=} (\mathcal{Q}z(B[r/y]))[s/x]$ $\overset{r \text{ is free for } y \text{ in } A}{=} ((\mathcal{Q}zB)[r/y])[s/x]$.
   - If $A = |^{t_1}_{x_1} \dots ^{t_n}_{x_n}|B$, then $(A[s/x])[r/y] = |^{(t_1[s/x])[r/y]}_{x_1} \dots ^{(t_n[s/x])[r/y]}_{x_n}|B$ $\overset{1.13.5}{=} |^{(t_1[r/y])[s/x]}_{x_1} \dots ^{(t_n[r/y])[s/x]}_{x_n}|B = (A[r/y])[s/x]$.
   - If $A = \langle^{t_1}_{x_1} \dots ^{t_n}_{x_n}\rangle B$, the proof is analogous.

6. The proof is that of property 3 with the addition of the following case: A is QzB, $z \neq x$, and s is not free for x in it. Now, $((\mathcal{Q}zB)[y/x])[s/y])$ $\overset{y \text{ is free for } x \text{ in } A}{=} (\mathcal{Q}z(B[y/x])[s/y] \overset{z_1 \text{new}}{=} \mathcal{Q}z_1(((B[y/x])[z_1/z])[s/y]) \overset{1.17.5}{=}$ $\mathcal{Q}z_1(((B[z_1/z])[y/x])[s/y]) \overset{IH}{=} \mathcal{Q}z_1((B[z_1/z])[s/x]) = (\mathcal{Q}zB)[s/x]$. ■

CHAPTER 2

TRANSITION SEMANTICS

In this chapter we introduce transition semantics, which is a counterpart-theoretic generalization of Kripke-semantics, and was first introduced in [Cor09]. Its ancestor is the so-called Lewis semantics presented in [Cor01] which, in its turn, is the set-theoretic version of the categorial semantics for quantified modal logic introduced in [GM88]. The main difference between transition semantics and Kripke-semantics is the way a modal formula with free variables is evaluated: whereas the latter makes use of trans-world identity, the former makes use of an arbitrary *transition* relation between objects inhabiting possible worlds. To illustrate, a formula (of the standard language) $\Box A(x)$ is true of an object o inhabiting w whenever it is true of that very object in every world accessible from w; a modal open $\mathcal{L}^{\boxdot}$-formula $|x|A(x)$, instead, is true of an object o inhabiting w whenever $A(x)$ is true of every counterpart of o (in all accessible worlds where there are counterparts of o). The reader is referred to [BG06, GSS09] for generalizations of Kripke semantics.

This chapter proceeds as follows: Section 1 introduces transition semantics. Then in Section 2 we explain some of its main features and show why wthe indexed modal language is needed. Section 3 is a technical one with results about the relations between substitutions and assignments. Section

4 presents some basic results of the Correspondence Theory for transition semantics. Section 5 introduces two constraints on the notion of interpretation of closed terms that allow to validate the two arrows of the equivalence between *de re* and *de dicto* formulas that is characteristic of Kripke-frames with rigid designators. Finally Section 6 introduces some conditions that we can impose on the domains of quantification.

2.1 Transition Frames

Let $\mathcal{F}^p = \langle \mathcal{W}, \mathcal{R} \rangle$ be a (relational) frame for propositional modal logics —i.e. $\mathcal{W}$ is a non-empty set of *worlds*, and $\mathcal{R} \subseteq \mathcal{W} \times \mathcal{W}$ is an *accessibility relation* over $\mathcal{W}$. By a *system of (double) domains* over $\mathcal{F}^p$ we mean a quadruple $\langle \mathcal{W}, \mathcal{R}, \mathcal{U}, \mathcal{D} \rangle$, where $\mathcal{U} = \biguplus_{w \in \mathcal{W}} \{U_w : U_w \neq \emptyset\}$ is the disjoint union of a family of non-empty sets, and $\mathcal{D} = \biguplus_{w \in \mathcal{W}} \{D_w : D_w \subseteq U_w\}$ is the disjoint union of a family of, possibly empty, subset thereof. U_w (resp. D_w) is called the *outer* (resp. *inner*) *domain* of the world w. If $a \in U_w$ (resp. $a \in D_w$) we say that a *inhabits* (resp. *exists in*) w. We will use w, v and u — possibly with numerical subscripts— as metavariables for possible worlds.

By a *transition relation* over a system of domains $\langle \mathcal{W}, \mathcal{R}, \mathcal{U}, \mathcal{D} \rangle$ we mean any binary relation over $\mathcal{U} \times \mathcal{U}$ such that:

$$\text{if } w\mathcal{R}v, \text{ then } \mathcal{T}_{(w,v)} \subseteq U_w \times U_v, \text{ else } \mathcal{T}_{(w,v)} = \emptyset\,. \tag{1.1}$$

If $a\mathcal{T}_{(w,v)}b$ then we say that b is a (w, v)-transition, or a *counterpart*, of a.

Observe that we had to define $\mathcal{U}$ and $\mathcal{D}$ by means of disjoint unions because one and the same object may be a member of the outer (inner) domain of two different worlds while being a counterpart of some object in one of them, but not in the other. One way to avoid this complication would be to assume that $w \neq u$ implies $U_w \cap U_v = \emptyset$ —i.e. that the domains are pairwise disjoints, but, for the sake of generality, we will not do it here.

Definition 2.1 (t-frame)**.** A *transition frame* (‘t-frame’ for short.) is a quintuple $\mathcal{F} = \langle \mathcal{W}, \mathcal{R}, \mathcal{U}, \mathcal{D}, \mathcal{T} \rangle$ where $\langle \mathcal{W}, \mathcal{R}, \mathcal{U}, \mathcal{D} \rangle$ is a system of domains, and

$\mathcal{T} = \biguplus_{w,v \in \mathcal{W}} \{\mathcal{T}_{(w,v)}\}$ is the disjoint union of a set made of a transition relation $\mathcal{T}_{(w,v)}$ for every pair of worlds $w, v \in \mathcal{W}$. ■

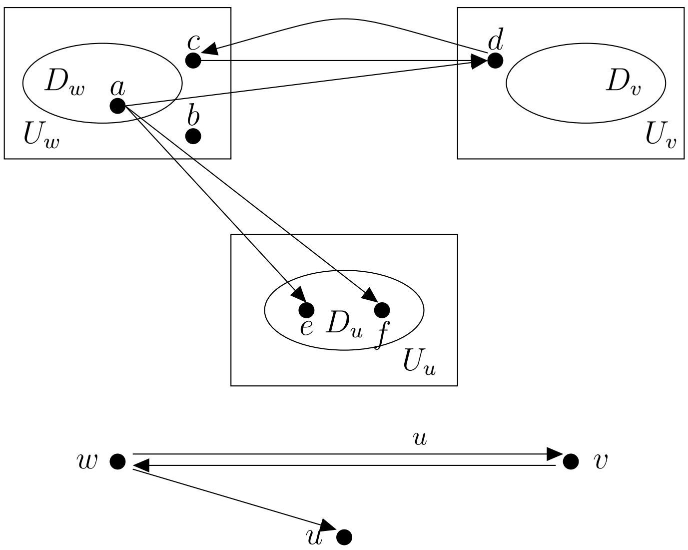

Figure 2.1: A t-frame $\mathcal{F}$.

Figure 2.1 shows an example of a t-frame, made of three world w, v, u and the respective domains. The transition relation is almost as arbitrary as the accessibility relation is: the only constraint on it is that if $not(w_1 \mathcal{R} w_2)$ then $\mathcal{T}_{(w_1,w_2)} = \emptyset$. This t-frame illustrates many of the possibilities that we have in t-frames:

- an object (d) is a transition of two different objects inhabiting the same world (a and c);
- two objects inhabiting one world (e and f) are transitions of one and the same object (a);
- an object (b) has no transition whatsoever in the accessible worlds.

We stress that all these possibilities make the transition relation far more general than the trans-world identity relation on which Kripke-semantics is based.

Definition 2.2 (t-model)**.** Let $\mathcal{F} = \langle \mathcal{W}, \mathcal{R}, \mathcal{U}, \mathcal{D}, \mathcal{T} \rangle$ be a t-frame. A *transition model* ('t-model' for short.) over $\mathcal{F}$ is a pair $\mathcal{M} = \langle \mathcal{F}, \mathcal{I} \rangle$ where $\mathcal{I} = \{I_w : w \in \mathcal{W}\}$ is a family of classical interpretation functions defined over U_w, i.e. for each $w \in \mathcal{W}$

- $I_w(c) \in U_w$;
- $I_w(f^n) : (U_w)^n \to U_w$;
- $I_w(P^n) \subseteq (U_w)^n$, we assume $(U_w)^0 = \{w\}$, thus for any propositional atom p, $\mathcal{I}_w(p)$ is either $\{w\}$ or $\emptyset$. ■

Definition 2.3 (Assignments)**.** Let $\mathcal{F} = \langle \mathcal{W}, \mathcal{R}, \mathcal{U}, \mathcal{D}, \mathcal{T} \rangle$ be a t-frame. An *assignment* over $\mathcal{F}$ is a function $\sigma : (VAR \times \mathcal{W}) \to \mathcal{U}$ such that, for all $x_i \in VAR$ and $w \in \mathcal{W}$, $\sigma(x_i, w) \in U_w$ —i.e. it is a function mapping each pair made of a variable and a world to an inhabitant of that world. By a w-assignment σ_w we mean the restriction of σ to the world w, i.e $\sigma_w : VAR \to U_w$. Let σ_w be a w-assignment, then for any $a \in U_w$, by $\sigma_w^{x \triangleright a}$ we mean the w-assignment behaving like σ save for x that is mapped to a.

■

We will use lower case Greek letters $\sigma, \pi \dots$ to denote assignments.

One striking feature of transition semantics is that interpretations and assignments are defined locally at each possible world. This is one of the main difference with respect to Kripke-semantics where we have to introduce functions that are either common to all worlds, or, a least, that satisfy some global constraint —e.g. we have to impose that $w\mathcal{R}v$ implies that any w-assignment is also a v-assignment, see [vB83, vB10a] and Section 2.2.3 for an explanation of the role of this constraint. The absence of this restriction allows us to consider a transition model as a set of (double-domain) Tarskian models endowed with arrows between the Tarskian models (given by $\mathcal{R}$) and arrows between the objects of the different Tarskian models (given by $\mathcal{T}$).

Definition 2.4 (Extensions of terms)**.** Let $\mathcal{M} = \langle \mathcal{F}, \mathcal{I} \rangle$ be a t-model, σ an assignment over $\mathcal{F}$, and $w \in \mathcal{W}$. The *w-extension of a term t in $\mathcal{M}$ under σ*, I_w^σ, is defined as:

- $I^\sigma_w(x) = \sigma_w(x)$;
- $I^\sigma_w(c) = I_w(c)$;
- $I^\sigma_w(f(t_1, \ldots, t_n)\,) = I_w(f)(I^\sigma_w(t_1), \ldots, I^\sigma_w(t_n)\,)$.

When no ambiguity arises, we use σ_w as a shorthand for I^σ_w, and $\sigma_w(t_1, \ldots, t_n)$ for $(I^\sigma_w(t_1), \ldots, I^\sigma_w(t_n))$. ∎

Definition 2.5 (Satisfaction)**.** Let $\mathcal{M} = \langle \mathcal{F}, \mathcal{I} \rangle$ be a t-model, and σ_w be a w-assignment for some $w \in \mathcal{W}$. We define the *satisfaction* of a formula $A \in FORM$ at w under σ_w in $\mathcal{M}$, $\sigma_w \models^{\mathcal{M}}_w A$, as follows:

$\sigma_w \not\models^{\mathcal{M}}_w \bot$;

$\sigma_w \models^{\mathcal{M}}_w p \iff w \in I_w(p)$;

$\sigma_w \models^{\mathcal{M}}_w P^k(t_1, ..., t_k) \iff \langle \sigma_w(t_1), ..., \sigma_w(t_k) \rangle \in I_w(P^k)$ (for $k \leq 1$);

$\sigma_w \models^{\mathcal{M}}_w B \wedge C \iff \sigma_w \models^{\mathcal{M}}_w B$ and $\sigma_w \models^{\mathcal{M}}_w C$;

$\sigma_w \models^{\mathcal{M}}_w B \vee C \iff \sigma_w \models^{\mathcal{M}}_w B$ or $\sigma_w \models^{\mathcal{M}}_w C$;

$\sigma_w \models^{\mathcal{M}}_w B \to C \iff \sigma_w \not\models^{\mathcal{M}}_w B$ or $\sigma_w \models^{\mathcal{M}}_w C$;

$\sigma_w \models^{\mathcal{M}}_w \forall x B \iff$ for all $a \in D_w$, $\sigma^{x \triangleright a}_w \models^{\mathcal{M}}_w B$;

$\sigma_w \models^{\mathcal{M}}_w \exists x B \iff$ for some $a \in D_w$, $\sigma^{x \triangleright a}_w \models^{\mathcal{M}}_w B$;

$\sigma_w \models^{\mathcal{M}}_w |^{t_1 \cdots t_n}_{x_1 \cdots x_n}| B \iff$ for all $v \in \mathcal{W}$ s.t. $w\mathcal{R}v$, and all assignments τ s.t. $\sigma_w(t_i)\mathcal{T}_{(w,v)}\tau_v(x_i)$ for all $i \in \{1, \ldots, n\}$, $\tau_v \models^{\mathcal{M}}_v B$;

$\sigma_w \models^{\mathcal{M}}_w \langle^{t_1 \cdots t_n}_{x_1 \cdots x_n}\rangle B \iff$ there is a $v \in \mathcal{W}$ s.t. $w\mathcal{R}v$, and there is an assignment τ s.t. $\sigma_w(t_i)\mathcal{T}_{(w,v)}\tau_v(x_i)$ for all $i \in \{1, \ldots, n\}$ and $\tau_v \models^{\mathcal{M}}_v B$.

If $A \in FORM^{=}$, we define $\sigma_w \models^{\mathcal{M}^=}_w A$ as $\sigma_w \models^{\mathcal{M}}_w A$ with the addition of

$$\sigma_w \models^{\mathcal{M}^=}_w t \doteq s \quad \Longleftrightarrow \quad \sigma_w(t) = \sigma_w(s).$$

When no ambiguity arises, we write $\sigma \models_w A$, or $\sigma_w \models A$, as shorthand for either $\sigma_w \models^{\mathcal{M}}_w A$ or $\sigma_w \models^{\mathcal{M}^=}_w A$. ■

For the sake of completeness, we present explicitly also the clause for negation which, as we know, is defined as $\neg A = A \to \bot$.

$$\sigma_w \models^{\mathcal{M}}_w \neg A \quad \Longleftrightarrow \quad \sigma_w \not\models^{\mathcal{M}}_w A.$$

Definition 2.6. Let $\mathcal{C}^\star$ be some class of t-frames, $\mathcal{F}$ be a t-frame, and $\mathcal{M}$ be some t-model with or without equality (depending on whether $A \in FORM$ or $A \in FORM^{=}$). We say that:

- A is *true at* w in $\mathcal{M}$, $\models^{\mathcal{M}}_w A$, iff for all assignments σ, $\sigma \models^{\mathcal{M}}_w A$;
- A is *true in* $\mathcal{M}$, $\models^{\mathcal{M}} A$, iff for all $w \in \mathcal{W}$, $\models^{\mathcal{M}}_w A$;
- A is *valid on* $\mathcal{F}$, $\mathcal{F} \models A$, iff for all $\mathcal{M}$ based on $\mathcal{F}$, $\models^{\mathcal{M}} A$;
- A is *valid on* $\mathcal{C}^\star$, $\mathcal{C}^\star \models A$, iff for all $\mathcal{F} \in \mathcal{C}^\star$, $\mathcal{F} \models A$. ■

2.2 Indexed Operators and Transition Semantics

Now that we have introduced the basic notions of transition semantics, we can explain briefly some of its peculiarities and show why we had to introduce indexed operators in the syntax of quantified modal logics.

2.2.1 De Re/De Dicto Distinction

In transitions semantics terms are not rigid designator —i.e. we don't impose that a term t has the same extension in all possible worlds of a given model. Thus, in evaluating an indexed formula, we have to know where —i.e. in which world— to determine the extension of any term occurring in that formula. To illustrate, to evaluate at a world w the formulas

$$|^c_x|P(x) \tag{2.2}$$

and

$$|\star|P(c) \tag{2.3}$$

we have to apply different procedures. In the former case we have first to determine the w-extension of c and then to move to each accessible world v to see if all c's (w, v)-transitions satisfy the open formula $P(x)$. In the latter we have first to move to each accessible world v and then to see if the sentence $P(c)$ is true therein. Observe that in this second case the transition relation $\mathcal{T}_{(w,v)}$ has no role because in transition semantics the v-extension of a constant c doesn't depend necessarily on the (w, v)-transitions of the w-extension of c. The difference between the two procedures is, chiefly, the order in which the two steps of moving to accessible worlds and of determining the extension of terms are made. In modal semantics that are based on trans-world identity, if terms are rigid designators, this distinction is irrelevant since terms denote the same object in all (accessible) worlds, but it becomes crucial once we have allowed for non-rigid designators. Note that this distinction – which is a *de re* vs. *de dicto* distinction— seems extremely relevant from an intuitive point of view: the natural reading of the formula 2.2 is 'c is necessarily P-ing', and that of 2.3 is 'it is necessary that $P(c)$'. In the first case we are asking if an object has some property by necessity, whereas in the second we are asking if a sentence expresses a necessary truth. It should be immediately clear that these two questions can have different answers.

The need to express this de re/de dicto distinction constitutes the first motivation for introducing indexed operators because

> *formal syntax drawn from that of classical logic cannot distinguish modal meanings we can readily distinguish intuitively.*
>
> [FM98, p. 190]

The problem, in a nutshell, is that if c is a non-rigid designator, then in evaluating $\Box P(c)$ at w we don't know where to determine the extension of c. That formula can stand either for its *de re* interpretation or for its *de dicto* one, but the basic modal language lacks the expressive power needed to show which of the two readings is being considered on a particular occasion,

> consequently, when non-rigid designators have been treated at all, one of the readings has been disallowed, thus curtailling expressive power.
>
> [Fit91, p. 114]

It is immediate to realize that the introduction of indexed operators allows us to overcome such limitation of the basic modal language. For example the two formulas 2.2 and 2.3 express the two possible readings of $\Box P(c)$.

2.2.2 Box-Distribution

A key idea of the Definition 2.5 of satisfaction is that a modal indexed operator works as a quantifier restricted not only by the accessibility relation $\mathcal{R}$, but also by the transition relation $\mathcal{T}$. In fact, $\sigma_w \models_w |^{t_1}_{x_1} \dots {}^{t_n}_{x_n}|A$ iff the subformula A is satisfied by all n-tuples of transitions of $\sigma_w(t_1, \dots, t_n)$ in each and every accessible world where there are such n-tuples. As a consequence, we may have $\sigma_w \models_w |x\,y|P(x)$, but not $\sigma_w \models_w |x|P(x)$. To wit, the first formula holds iff in every accessible world v where there are (w, v)-transitions of both $\sigma_w(x)$ and $\sigma_w(y)$, all (w, v)-transitions of $\sigma_w(x)$ satisfy $P(x)$. The second formula holds iff in every accessible world v where there are (w, v)-transitions of $\sigma_w(x)$, all (w, v)-transitions of $\sigma_w(x)$ satisfy $P(x)$. In a model where there is some world u containing some transition o of $\sigma_w(x)$ but not of $\sigma_w(y)$ and such that P is false of o at u, the second formula is false, but the first one may nonetheless be true, as it happens in Figure 2.2.

As a consequence a distribution such as the following one

$$\frac{\sigma_w \models_w |x\,y|(A(x, y) \to B(x)) \qquad \sigma_w \models_w |x\,y|A(x, y)}{|x|B(x)}$$

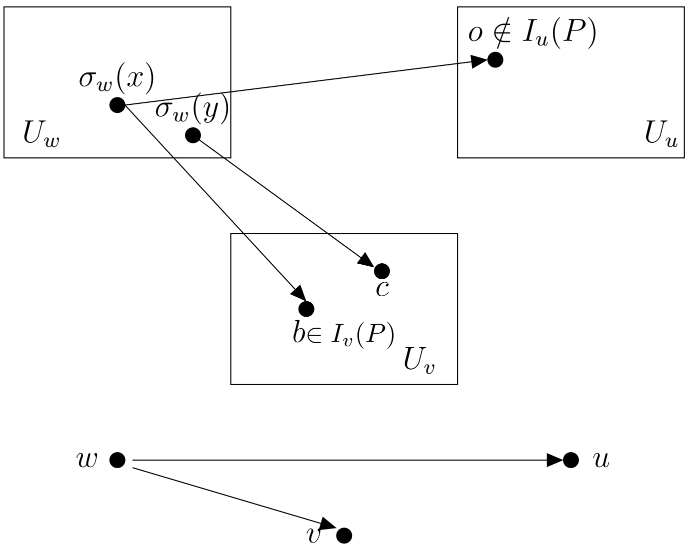

Figure 2.2: A t-model s.t. $\sigma_w \models_w |x\,y|P(x)$, but not $\sigma_w \models_w |x|P(x)$.

is not a valid rule of inference because in the conclusion we have lost trace of the restriction to worlds where there are transitions of the object $\sigma_w(y)$.

The satisfaction of a modal formula can depend on some free variable that doesn't occur in any of its atomic sub-formulas, let's call them *ghost-variables*. Thanks to indexed modalities we can keep trace of ghost-variables and therefore express $\boxdot$-distribution as

$$\frac{\sigma_w \models_w |x\,y|(A(x,y) \to B(x)) \qquad \sigma_w \models_w |x\,y|A(x,y)}{|x\,y|B(x)}$$

which, instead, is a valid inference rule.

Other languages in which it is possible to track ghost-variables are the typed languages, see [Cor01, Ghi01, BG06], but typed languages are a major departure from the ordinary language and are not easy to work with:

> The problem of finding a good linguistic presentation is important[..]. From this point of view we cannot say that the solution presented here [by means of typed languages] is satisfactory. Its main defect lies in the fact that only quantification on the variable having the greatest index is allowed. This does not affect

> the expressive power of the language, because it simply eliminates alphabetic variants, although sometimes it may make the deductions unnatural. Something more liberal would be preferable.
> [Ghi01, pp. 111–112]

The problem of keeping track of ghost-variables is more important than one could think at first sight, as we can see by examining van Benthem's approach to the semantics of modal logics.

2.2.3 van Benthem's Semantics

In [vB83, vB10b] van Benthem introduces a semantics for the standard modal language (with modalities without indices) which is very interesting from our point of view because it is a particular case of transition semantics. In brief, it is a semantics in which each world is endowed with a single domain and the transition relation is the identity relation, but, contrary to Kripke semantics, it is not everywhere defined.

van Benthem defines satisfaction as follows

$$\sigma \models_w \Box A(\vec{x}) \qquad iff \qquad \text{for all } v \text{ s.t. } w\mathcal{R}v \text{ and s.t. } \sigma(\vec{x}) \in D_v,\ \sigma \models_v A(\vec{x}) \tag{2.4}$$

This clause is a particular case of our clause in that we have to move to all accessible worlds where there are transitions of the objects $\sigma(\vec{x})$, the only difference is that here the only possible transition of an object is that very same object. We stress that the possibility of having a non-everywhere defined transition relation is extremely natural for many interpretations of the modal operators, such as the temporal one.

> To use an ordinary language example, 'She is always angry' does not mean that, at all points in time she is angry; but that she is angry at all points in time *during her existence*.
> [vB83, pp. 136-137]

van Benthem tells us [vB10b, Prop. 8] that in his approach

- CBF : $\Box\forall x A \to \forall x \Box A$ corresponds to Tautology
- GF : $\exists x \Box A \to \Box\exists x A$ corresponds to incresing domains: $wRv \to D_w \subseteq D_v$

Two possible axiomatic derivations of CBF and GF are presented in Figure 2.3. Looking at both proofs it is clear that whenever CBF is derivable, GF is derivable too and viceversa. So how is it that one corresponds to tautology and the other to increasing domains? In our analysis it is the box-distribution that needs to be examined.

$$
\dfrac{\dfrac{\dfrac{}{\forall x Px \to Px}\,Ax}{\Box(\forall x Px \to Px)}\,Nec \qquad \dfrac{}{\Box(\forall x Px \to Px) \to (\Box\forall x Px \to \Box Px)}\,Ax}{\dfrac{\Box\forall x Px \to \Box Px}{\Box\forall x Px \to \forall x \Box Px}\,UG}\,MP
$$

$$
\dfrac{\dfrac{\dfrac{}{Px \to \exists x Px}\,Ax}{\Box(Px \to \exists x Px)}\,Nec \qquad \dfrac{}{\Box(Px \to \exists x Px) \to (\Box Px \to \Box\exists x Px)}\,Ax}{\dfrac{\Box Px \to \Box\exists x Px}{\exists x \Box Px \to \Box\exists x Px}\,UG}\,MP
$$

Figure 2.3: Axiomatic derivations of CBF and GF

In the case of CBF we have an 'innocuous' distribution because we distribute over an implication whose 'if' clause has less free variables then its 'then' clause. Thus it says that

> if in all accessible world where there are transitions of $\sigma_w(x)$, $\forall x Px$ implies Px, then if in all accessible worlds $\forall x Px$ is true, in all accessible worlds where there are transitions of $\sigma_w(x)$ they satisfy Px.

Whereas in the case of GF we have the opposite, and thus a 'dangerous' distribution since it says

> if in all accessible worlds where there are transitions of $\sigma_w(x)$, Px implies $\exists xPx$, then if in all accessible worlds where there are transition of $\sigma_w(x)$ that satisfy Px, in all accessible worlds $\exists xPx$ is true.

Since the transition relation is not totally defined the first distribution is valid —and therefore CBF is valid on every frame — and the second is not valid —and therefore GF is valid only on particular classes of frames. Without being able to discriminate between innocuous and dangerous instances of $\Box$-distribution we cannot axiomatize the set of formulas valid on van Benthem's frames by means of a quantified extensions of normal PMLs.

If we rephrase both proofs in a language with indexed modalities, where $\boxdot$-distribution holds only for the innocuous instances, we see immediately where the problem lies, see Figure 2.4.

$$
\dfrac{\dfrac{}{|\star|\forall xPx \to |x|\forall xPx}\,LNGT \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\forall xPx \to Px}\,Ax}{|x|(\forall xPx \to Px)}\,Nec \quad \dfrac{}{|x|(\forall xPx \to Px) \to (|x|\forall xPx \to |x|Px)}\,Ax}{|x|\forall xPx \to |x|Px}\,MP}{|x|\forall xPx \to \forall x|x|Px}\,UG}{|\star|\forall xPx \to \forall x|x|Px}
$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{Px \to \exists xPx}\,Ax}{|x|(Px \to \exists xPx)}\,Nec \quad \dfrac{}{|x|(Px \to \exists xPx) \to (|x|Px \to |x|\exists xPx)}\,Ax}{|x|Px \to |x|\exists xPx}\,MP}{\exists x|x|Px \to |x|\exists xPx}\,UG \qquad \dfrac{}{|x|\exists xPx \to |\star|\exists xPx}\,SHRT}{\exists x|x|Px \to |\star|\exists xPx}
$$

Figure 2.4: Axiomatic derivations of CBF and GF in IMLs

In a language with indexed modalities, the proof of CBF requires a principle, $LNGT$, that is valid on all transition frames, and thus is an axiom of the basic axiomatic system **Q.Kim**,[1] whereas, the proof of GF requires a

[1] All innocuous instances of distribution where the 'if' clause has less free variables than the 'then' one are derivable by the axiom of $\boxdot$-distribution and an instance of $LNGT$.

principle, $SHRT$, that is valid only on frames where the transition relation is everywhere defined. Of course if we limit ourselves to frames where the transition relation is the identity relation, as in van Benthem's semantics, GF seems to correspond to incresing domains since an everywhere defined identity relation is the subset relation.

2.3 Substitution and Satisfaction

In this section we prove a series of useful lemmas that clarify the connection between substitutions and assignments. We will end up in showing that i-congruent formulas (see Definition 1.10) are not semantically distinguishable one from another. This constitutes a semantical justification for our claim that skeletons represent the deep logical structures of formulas.

Lemma 2.7 (Substitution and extension of terms)**.** *Let s, t be terms, $\mathcal{M} = \langle \mathcal{F}, \mathcal{I} \rangle$ a t-model, σ a w-assignment. It holds that*

$$1. \qquad \sigma_w(t[s/x]) \quad = \quad \sigma_w^{x \rhd \sigma_w(s)}(t)\ .$$

If $z \notin VAR(t)$

$$2. \qquad \sigma^{z \rhd a}(t[z/x]) \quad = \quad \sigma^{x \rhd a}(t)\ .$$

Proof. The proofs are by induction on the term-height $h(t)$ of t.

1.
   - If $h(t) = 0$ and t is either a constant or a variable different from x the lemma holds trivially.
   - If $h(t) = 0$ and $t = x$ then $\sigma_w(x[s/x]) = \sigma_w(s) = \sigma_w^{x \rhd \sigma_w(s)}(x)$.
   - If $h(t) = n + 1$, then $t = f(t_1, \ldots, t_n)$ and $\sigma_w(f(t_1, \ldots, t_n)[s/x]) = I_w(f)(\sigma_w(t_1[s/x]), \ldots, \sigma_w(t_n[s/x])) \overset{IH}{=} I_w(f)(\sigma_w^{x \rhd \sigma_w(s)}(t_1), \ldots, \sigma_w^{x \rhd \sigma_w(s)}(t_n)) = \sigma_w^{x \rhd \sigma_w(s)}(f(t_1, \ldots, t_n))$.

2.
   - If $h(t) = 0$ and t is either a constant or a variable different from x the lemma holds trivially, note that $t = z$ is forbidden.
   - If $h(t) = 0$ and $t = x$, then $\sigma_w^{z \rhd a}(x[z/x]) \overset{2.7.1}{=} \sigma_w^{z \rhd a,\, x \rhd \sigma_w^{z \rhd a}(z)}(x) = \sigma_w^{z \rhd a,\, x \rhd a}(x) = \sigma_w^{x \rhd a}(x)$ since $z \neq x$.

- $h(t) = n+1$ and $t = f(t_1, \ldots, t_n)$, then $\sigma^{z \rhd a}(f(t_1, \ldots, t_n)[z/x]) = I_w(f)(\sigma_w^{z \rhd a}(t_1[z/x]), \ldots, \sigma_w^{x \rhd a}(t_n[z/x])) \stackrel{IH}{=} I_w(f)(\sigma_w^{x \rhd a}(t_1), \ldots, \sigma_w^{x \rhd a}(t_n)) = \sigma_w^{x \rhd a}(f(t_1, \ldots, t_n))$. ■

Lemma 2.8. *If $z \notin VAR(A)$, then, for all worlds w of each model $\mathcal{M} = \langle \mathcal{F}, \mathcal{I} \rangle$, all assignment σ over $\mathcal{F}$, and all object $a \in U_w$,*

$$\sigma_w^{x \rhd a} \models_w^{\mathcal{M}} A \iff \sigma_w^{z \rhd a} \models_w^{\mathcal{M}} A[z/x]$$

Proof. By induction on the formula-height $h(A)$ of A.

If $h(A) = 0$, then either A is $\perp$ and the lemma holds trivially, or A is an atom, say $P(t_1, \ldots, t_n)$, and we have $\sigma_w^{x \rhd a} \models_w P(t_1, \ldots, t_n) \iff \sigma_w^{x \rhd a}(t_1, \ldots, t_n) \in I_w(P) \stackrel{2.7.2}{\iff} \sigma_w^{z \rhd a}(t_1[z/x], \ldots, t_n[z/x]) \in I_w(P) \iff \sigma_w^{z \rhd a} \models_w P(t_1[z/x], \ldots, t_n[z/x]) \iff \sigma_w^{z \rhd a} \models_w P(t_1, \ldots, t_n)[z/x]$.

If $h(a) = n+1$, we argue by cases and we leave out the propositional ones.

- $A = \forall y B$ (observe that $y = z$ is excluded, and that z is free for x in A).
  - $y = x$. Then $\sigma_w^{x \rhd a} \models \forall x B \iff \sigma_w \models (\forall x B)[z/x] \iff \sigma_w^{z \rhd a} \models (\forall x B)[z/x]$.
  - $y \neq x$. Then $\sigma_w^{x \rhd a} \models_w \forall y B \iff \forall b \in D_w\, (\sigma_w^{x \rhd a,\, y \rhd b} \models_w B) \stackrel{x \neq y}{\iff} \forall b \in D_w\, (\sigma_w^{y \rhd b,\, x \rhd a} \models_w B) \stackrel{IH}{\iff} \forall b \in D_w\, (\sigma_w^{y \rhd b,\, z \rhd a} \models_w B[z/x]) \stackrel{y \neq z}{\iff} \forall b \in D_w (\sigma_w^{z \rhd a,\, y \rhd b} \models_w B[z/x]) \iff \sigma_w^{z \rhd a} \models_w \forall y (B[z/x]) \stackrel{y \neq z}{\iff} \sigma_w^{z \rhd a} \models_w (\forall y B)[z/x]$.
- $A = \exists y B$. We proceed as for $\forall y B$.
- $A = |_{x_1}^{t_1} \ldots {}_{x_n}^{t_n}| B$. Then $\sigma_w^{x \rhd a} \models_w |_{x_1}^{t_1} \ldots {}_{x_n}^{t_n}| B \iff \forall v \in \mathcal{W}, \forall \tau\, (w \mathcal{R} v\, \&\, \bigwedge_{i=1}^n \sigma_w^{x \rhd a}(t_i) \mathcal{T}_{(w,v)} \tau_v(x_i) \Rightarrow \tau_v \models_v B) \stackrel{2.7.2}{\iff} \forall v \in \mathcal{W}, \forall \tau\, (w \mathcal{R} v\, \&\, \bigwedge_{i=1}^n \sigma_w^{z \rhd a}(t_i[z/x]) \mathcal{T}_{(w,v)} \tau_v(x_i) \Rightarrow \tau_v \models_v B) \iff \sigma_w^{z \rhd a} \models_w |_{x_1}^{t_1[z/x]} \ldots {}_{x_n}^{t_n[z/x]}| B \iff \sigma^{z \rhd a} \models_w (|_{x_1}^{t_1} \ldots {}_{x_n}^{t_n}| B)[z/x]$.
- $A = \langle_{x_1}^{t_1} \ldots {}_{x_n}^{t_n} \rangle B$. As for the previous case. ■

Corollary 2.9. *If y is new to $\forall xA$, then for all $w, \mathcal{M}, \sigma$,*

$$\sigma_w \models_w^{\mathcal{M}} \forall xA \quad\iff\quad \sigma_w \models_w^{\mathcal{M}} \forall y(A[y/x])$$

Proof. $\sigma \models_w^{\mathcal{M}} \forall xA \iff \forall a \in D_w\,(\sigma_w^{x \triangleright a} \models_w^{\mathcal{M}} A) \overset{2.8}{\iff} \forall a \in D_w\,(\sigma_w^{y \triangleright a} \models_w^{\mathcal{M}} A[y/x]) \iff \sigma_w \models_w^{\mathcal{M}} \forall y(A[y/x])$. ■

Lemma 2.10 (Substitution and satisfaction)**.** *For all $w, \mathcal{M}, \sigma$*

$$\sigma_w \models_w^{\mathcal{M}} A[s/x] \quad\iff\quad \sigma_w^{x \triangleright \sigma_w(s)} \models_w^{\mathcal{M}} A$$

Proof. The proof is by induction on $h(A)$.

$h(A) = 0$. If $A = \bot$, the lemma holds trivially, and, if A is atomic, say $P(t_1, \ldots, t_n)$, then $\sigma_w \models_w P(t_1, \ldots, t_n)[s/x] \iff \sigma_w \models_w P(t_1[s/x], \ldots, t_n[s/x]) \iff \sigma_w(t_1[s/x], \ldots, t_n[s/x]) \in I_w(P) \overset{2.7.1}{\iff} \sigma_w^{x \triangleright \sigma(s)}(t_1, \ldots, t_n) \in I_w(P) \iff \sigma_w^{x \triangleright \sigma(s)} \models_w P(t_1, \ldots, t_n)$.

If $h(A) = n + 1$, we argue by cases.

- $A = B \circ C$. Then $\sigma_w \models_w (B \circ C)[s/x] \iff \sigma_w \models_w B[s/x] \circ C[s/x] \iff \sigma_w \models_w B[s/x]$ and/or/implies $\sigma_w \models_w C[s/x] \overset{IH}{\iff} \sigma_w^{x \triangleright \sigma(s)} \models_w B$ and/or/implies $\sigma_w^{x \triangleright \sigma(s)} \models_w C \iff \sigma_w^{x \triangleright \sigma(s)} \models_w B \circ C$.
- $A = \forall yB$. We distinguish three cases.
  - $y = x$. Then $\sigma_w \models (\forall xB)[s/x] \iff \sigma_w \models \forall xB \iff \sigma_w^{x \triangleright \sigma_w(s)} \models \forall xB$.
  - $y \neq x$ and s is free for x in A. Then $\sigma_w \models (\forall yB)[s/x] \iff \sigma_w \models \forall y(B[s/x]) \iff \forall a \in D_w\,(\sigma_w^{y \triangleright a} \models_w B[s/x]) \overset{IH}{\iff} \forall a \in D_w\,(\sigma^{y \triangleright a,\, x \triangleright \sigma_w^{y \triangleright a}(s)} \models B) \iff \forall a \in D_w\,(\sigma_w^{x \triangleright \sigma_w(s),\, y \triangleright a} \models B) \iff \sigma_w^{x \triangleright \sigma_w(s)} \models \forall yB$.
  - $y \neq x$ and s isn't free for x in A. Then, for some z such that (†) z is new to A and to s, $\sigma_w \models (\forall yB)[s/x] \iff \sigma_w \models_w \forall z((B[z/y])[s/x]) \iff \forall a \in D_w\,(\sigma_w^{z \triangleright a} \models (B[z/y])[s/x]) \overset{IH}{\iff} \forall a \in D_w\,(\sigma_w^{z \triangleright a,\, x \triangleright \sigma_w^{z \triangleright a}(s)} \models B[z/y]) \overset{\dagger}{\iff} \forall a \in D_w\,(\sigma_w^{x \triangleright \sigma_w(s),\, z \triangleright a} \models B[z/y]) \overset{2.8}{\iff} \forall a \in D_w\,(\sigma_w^{x \triangleright \sigma_w(s),\, y \triangleright a} \models B) \iff \sigma_w^{x \triangleright \sigma_w(s)} \models \forall yB$.

- $A = |^{t_1}_{y_1} \dots ^{t_n}_{y_n}|B$. Then $\sigma_w \models (|^{\vec{t}}_{\vec{y}}|B)[s/x] \iff \sigma_w \models |^{\vec{t}[s/x]}_{\vec{y}}|B \iff \forall v \in \mathcal{W}, \forall \tau\, (w\mathcal{R}v \,\&\, \bigwedge_{i=1}^{n} \sigma_w(t_i[s/x])\mathcal{T}_{(w,v)}\tau_v(y_i) \Rightarrow \tau_v \models B) \overset{2.7.1}{\iff} \forall v \in \mathcal{W}, \forall \tau\, (w\mathcal{R}v \,\&\, \bigwedge_{i=1}^{n} \sigma_w^{x \triangleright \sigma_w(s)}(t_i)\mathcal{T}_{(w,v)}\tau_v(y_i) \Rightarrow \tau_v \models B) \iff \sigma_w^{x \triangleright \sigma_w(s)} \models |^{\vec{t}}_{\vec{y}}|B$.

- Analogously for $A = \langle^{t_1}_{y_1} \dots ^{t_n}_{y_n}\rangle B$. ■

Corollary 2.11. *If each of $\vec{x}$ and $\vec{y}$ is an n-tuple of pairwise disjoint variables, then for all $w, \mathcal{M}, \sigma$*

$$\sigma_w \models^{\mathcal{M}}_w |^{\vec{t}}_{\vec{x}}|A \quad \iff \quad \sigma_w \models^{\mathcal{M}}_w |^{\vec{t}}_{\vec{y}}|(A[\vec{y}/\vec{x}]) \ .$$

Proof. $\sigma_w \models^{\mathcal{M}}_w |^{\vec{t}}_{\vec{x}}|A$

$\iff$

$\forall v \in \mathcal{W}, \forall \tau\, (w\mathcal{R}v \,\&\, \bigwedge_{i=1}^{n} \sigma_w(t_i)\mathcal{T}_{(w,v)}\tau_v(x_i) \Rightarrow \tau_v \models^{\mathcal{M}}_v A)$

$\overset{2.7.1}{\iff}$

$\forall v \in \mathcal{W}, \forall \tau\, (w\mathcal{R}v \,\&\, \bigwedge_{i=1}^{n} \sigma_w(t_i)\mathcal{T}_{(w,v)}\tau_v^{y_i \triangleright \tau_v(x_i)}(y_i) \Rightarrow \tau_v \models^{\mathcal{M}}_v A)$

$\overset{2.10}{\iff}$

$\forall v \in \mathcal{W}, \forall \tau\, (w\mathcal{R}v \,\&\, \bigwedge_{i=1}^{n} \sigma_w(t_i)\mathcal{T}_{(w,v)}\tau_v^{y_i \triangleright \tau_v(x_i)}(y_i) \Rightarrow \tau_v^{y_i \triangleright \tau_v(x_i)} \models^{\mathcal{M}}_v A[\vec{y}/\vec{x}])$

$\iff$

$\sigma_w \models^{\mathcal{M}}_w |^{\vec{t}}_{\vec{y}}|(A[\vec{y}/\vec{x}])$. ■

Theorem 2.12 (Satisfaction and i-congruent formulas)**.** *If A and B are i-congruent formulas, they are co-satisfiable —i.e. for any formula A and B, for any world w of a model $\mathcal{M} = \langle \mathcal{F}, \mathcal{I} \rangle$, and for any assignment σ over $\mathcal{F}$*

$$\textit{if } \ulcorner A \urcorner = \ulcorner B \urcorner, \quad \textit{then} \quad \sigma_w \models^{\mathcal{M}}_w A \iff \sigma_w \models^{\mathcal{M}}_w B \ .$$

Proof. By induction on the formula-height $h(A)$ of A (and of B).

$h(A) = 0$. Then A is either an atom or $\bot$, and the theorem holds since $\ulcorner A \urcorner = \ulcorner B \urcorner$ entails that $A = B$.

$h(A) = n + 1$

- $A = C \circ D$, for some propositional connective $\circ$. Then $\ulcorner A \urcorner = \ulcorner B \urcorner$ entails that $B = E \circ F$, with $\ulcorner C \urcorner = \ulcorner E \urcorner$ and $\ulcorner D \urcorner = \ulcorner F \urcorner$. By IH,

$\sigma \models^{\mathcal{M}}_w C \iff \sigma \models^{\mathcal{M}}_w E$ and $\sigma \models^{\mathcal{M}}_w D \iff \sigma \models^{\mathcal{M}}_w F$, and we can conclude that $\sigma \models^{\mathcal{M}}_w A \iff \sigma \models^{\mathcal{M}}_w B$.

- $A = \mathcal{Q}xC$. Then $\ulcorner A \urcorner = \ulcorner B \urcorner$ entails that B is either $\mathcal{Q}xD$ or, for some y new to A, $\mathcal{Q}y(D[y/x])$, and $\ulcorner C \urcorner = \ulcorner D \urcorner$.

  In the first case we have $\sigma_w \models \mathcal{Q}xC \iff \forall a \in D_w\, (\sigma_w^{x \rhd a} \models C) \overset{IH}{\iff} \forall a \in D_w\, (\sigma_w^{x \rhd a} \models D) \iff \sigma_w \models \forall x D$.

  In the second case, since $\sigma_w \models \mathcal{Q}xC \overset{2.9}{\iff} \sigma_w \models \forall y(C[y/x])$, we can proceed as in the first one.

- $A = |\frac{\vec{t}}{\vec{x}}|C$. Then $\ulcorner A \urcorner = \ulcorner B \urcorner$ entails that B is either $|\frac{\vec{t}}{\vec{x}}|D$, or it is $|\frac{\vec{t}}{\vec{y}}|(D[\vec{y}/\vec{x}])$, with $\ulcorner C \urcorner = \ulcorner D \urcorner$.

  In the first case, $\sigma_w \models |\frac{\vec{t}}{\vec{x}}|C$

  $\iff$

  $\forall w \in \mathcal{W}, \forall \tau\, (w\mathcal{R}v\, \&\, \bigwedge_{i=1}^{n} \sigma_w(t_i)\mathcal{T}_{(w,v)}\tau_v(x_i) \Rightarrow \tau_v \models C)$

  $\overset{IH}{\iff}$

  $\forall w \in \mathcal{W}, \forall \tau\, (w\mathcal{R}v\, \&\, \bigwedge_{i=1}^{n} \sigma_w(t_i)\mathcal{T}_{(w,v)}\tau_v(x_i) \Rightarrow \tau_v \models D)$

  $\iff$

  $\sigma_w \models |\frac{\vec{t}}{\vec{x}}|D$.

  In the second case $\sigma_w \models |\frac{\vec{t}}{\vec{x}}|C$

  $\overset{2.11}{\iff}$

  $\sigma_w \models |\frac{\vec{t}}{\vec{y}}|(C[\vec{y}/\vec{x}])$

  $\iff$

  $\forall w \in \mathcal{W}, \forall \tau\, (w\mathcal{R}v\, \&\, \bigwedge_{i=1}^{n} \sigma_w(t_i)\mathcal{T}_{(w,v)}\tau_v(y_i) \Rightarrow \tau_v \models C[\vec{y}/\vec{x}])$

  $\overset{IH}{\iff}$

  $\forall w \in \mathcal{W}, \forall \tau\, (w\mathcal{R}v\, \&\, \bigwedge_{i=1}^{n} \sigma_w(t_i)\mathcal{T}_{(w,v)}\tau_v(y_i) \Rightarrow \tau_v \models D[\vec{y}/\vec{x}])$

  $\iff$

  $s_w \models |\frac{\vec{t}}{\vec{y}}|(D[\vec{y}/\vec{x}])$.

- Analogously for $A = \langle\frac{\vec{t}}{\vec{x}}\rangle B$. ■

2.4 Correspondence Theory

Correspondence between formulas of IML and properties of transition frames encompasses correspondence results between formulas of $L^{\boxdot}$ and properties of R and/or of $\mathcal{T}$. We have two kinds of correspondence results in IMLs: 'propositional' ones that are the indexed extension of correspondence results in propositional modal logics, and 'transitional' ones where some formula that governs the interaction of quantifiers and identity with indexed modal operators.

2.4.1 Propositional Correspondence Results

The indexed modal extension of any correspondence result in propositional modal logics holds whenever both the accessibility relation and the transition relation have the relevant property. To illustrate, validity of $|x|P(x) \to P(x)$ on a transition frame tells us not only that the relation $\mathcal{R}$ is reflexive but also that $\mathcal{T}$ is reflexive. That $\mathcal{R}$ is reflexive obtains trivially since $|\star|p \to p$ is a particular case of $|x|P(x) \to P(x)$ and we know from standard correspondence theory that validity of $\Box p \to p$ corresponds to reflexivity of $\mathcal{R}$. We state, without proof, the propositional correspondence results for some of the most important formulas in the Lemmon-Scott fragment.

Theorem 2.13 (T^t)**.** *The following two conditions on $\mathcal{F}$ are equivalent*

1. $\mathcal{F} \models |\vec{x}|A \to A$
2. *$\mathcal{F}$ is t-reflexive, i.e. such that*
   - *$\mathcal{R}$ is reflexive: for all $w \in \mathcal{W}$, $w\mathcal{R}w$;*
   - *$\mathcal{T}$ is reflexive: for all $a \in \mathcal{U}_w$, $a\mathcal{T}_{(w,w)}a$.*

Theorem 2.14 (4^t)**.** *The following two conditions on $\mathcal{F}$ are equivalent*

1. $\mathcal{F} \models \langle\vec{x}\rangle\langle\vec{x}\rangle A \to \langle\vec{x}\rangle A$
2. *$\mathcal{F}$ is t-transitive, i.e. such that*

- *$\mathcal{R}$ is transitive: for all $w, v, u \in \mathcal{W}$, $w\mathcal{R}v$ and $v\mathcal{R}u$ imply $w\mathcal{R}u$;*
- *$\mathcal{T}$ is transitive: for all $a \in \mathcal{U}_w, b \in U_v, c \in U_u$, $a\mathcal{T}_{(w,v)}b$ and $b\mathcal{T}_{(v,u)}c$ imply $a\mathcal{T}_{(w,u)}c$.*

Theorem 2.15 (5^t)**.** *The following two conditions on $\mathcal{F}$ are equivalent*

1. *$\mathcal{F} \models \langle\vec{x}\rangle A \to |\vec{x}|\,\langle\vec{x}\rangle A$*
2. *$\mathcal{F}$ is t-euclidean, i.e. such that*
   - *$\mathcal{R}$ is euclidean: for all $w, v, u \in \mathcal{W}$, $w\mathcal{R}v$ and $w\mathcal{R}u$ imply $v\mathcal{R}u$;*
   - *$\mathcal{T}$ is euclidean for all $a \in \mathcal{U}_w, b \in U_v, c \in U_u$, $a\mathcal{T}_{(w,v)}b$ and $a\mathcal{T}_{(w,u)}c$ imply $b\mathcal{T}_{(v,u)}c$.*

Theorem 2.16 (B^t)**.** *The following two conditions on $\mathcal{F}$ are equivalent*

1. *$\mathcal{F} \models A \to |\vec{x}|\,\langle\vec{x}\rangle A$*
2. *$\mathcal{F}$ is t-symmetric, i.e. such that*
   - *$\mathcal{R}$ is symmetric: for all $w, v \in \mathcal{W}$, $w\mathcal{R}v$ implies $v\mathcal{R}w$;*
   - *$\mathcal{T}$ is symmetric: for all $a \in \mathcal{U}_w, b \in U_v$, $a\mathcal{T}_{(w,v)}b$ implies $b\mathcal{T}_{(v,w)}a$.*

Theorem 2.17 (D^t)**.** *The following two conditions on $\mathcal{F}$ are equivalent*

1. *$\mathcal{F} \models |\vec{x}|A \to \langle\vec{x}\rangle A$*
2. *$\mathcal{F}$ is t-serial, i.e. such that*
   - *$\mathcal{R}$ is serial: for all $w \in \mathcal{W}$ there is $v \in \mathcal{W}$ such that $w\mathcal{R}v$;*
   - *$\mathcal{T}$ is serial: for all $a_1, \ldots, a_n \in \mathcal{U}_w$ there are $b_1, \ldots, b_n \in U_v$ such that $a_1\mathcal{T}_{(w,v)}b_1$ and... and $a_n\mathcal{T}_{(w,v)}b_n$.*

Theorem 2.18 (2^t)**.** *The following two conditions on $\mathcal{F}$ are equivalent*

1. *$\mathcal{F} \models \langle\vec{x}\rangle|\vec{x}|A \to |\vec{x}|\langle\vec{x}\rangle A$*
2. *$\mathcal{F}$ is t-directed, i.e. such that*

- *$\mathcal{R}$ is directed: for all $w, v, u \in \mathcal{W}$ if $w\mathcal{R}v$ and $w\mathcal{R}u$, then there is $w' \in \mathcal{W}$ such that $v\mathcal{R}w'$ and $u\mathcal{R}w'$;*
- *$\mathcal{T}$ is directed: for all $a_1, \ldots, a_n \in \mathcal{U}_w$, all $b_1, \ldots, b_n \in U_v$, and all $c_1, \ldots, c_n \in U_u$, if, for all i s.t. $1 \leq i \leq n$, $a_i\mathcal{T}_{(w,v)}b_i$ and $a_i\mathcal{T}_{(w,u)}c_i$, then there are $d_1, \ldots, d_n \in U_{w'}$ such that $b_i\mathcal{T}_{(v,w')}d_i$ and $c_i\mathcal{T}_{(u,w')}d_i$.*

2.4.2 Transitional Correspondence Results

We state here the main results in transitional correspondence theory, the proofs are analogous to those in [Cor09].

Theorem 2.19 (Converse Barcan Formula CBF)**.** *The following two conditions on $\mathcal{F}$ are equivalent*

1. $\mathcal{F} \models |\vec{x}|\forall y A \rightarrow \forall y|\vec{x}\, y|A$
2. *$\mathcal{F}$ is $\mathcal{D}$-preservative, i.e. such that*
   - *for all $a \in D_w$ and all $b \in U_v$, if $a\mathcal{T}_{(w,v)}b$, then $b \in D_v$*

Theorem 2.20 (Necessity of Identity NI)**.** *The following two conditions on $\mathcal{F}$ are equivalent*

1. $\mathcal{F} \models x \doteq y \rightarrow |x\, y|x \doteq y$
2. *$\mathcal{F}$ is $\mathcal{U}$-functional, i.e. such that*
   - *for all $a \in U_w$ and all $b, c \in U_v$, if $a\mathcal{T}_{(w,v)}b$ and $a\mathcal{T}_{(w,v)}c$, then $b = c$*

Theorem 2.21 (Necessity of Distinctness ND)**.** *The following two conditions on $\mathcal{F}$ are equivalent*

1. $\mathcal{F} \models x \not\doteq y \rightarrow |x\, y|x \not\doteq y$
2. *$\mathcal{F}$ is $\mathcal{U}$-not-convergent, i.e. such that*
   - *for all $a, b \in U_w$ and all $c \in U_v$, if $a\mathcal{T}_{(w,v)}c$ and $b\mathcal{T}_{(w,v)}c$, then $a = b$*

Theorem 2.22 (Barcan Formula BF)**.** *The following two conditions on $\mathcal{F}$ are equivalent*

1. $\mathcal{F} \models \forall y|\vec{x}\,y|A \to |\vec{x}|\forall y A$

2. $\mathcal{F}$ *is* $\mathcal{D}$*-surjective, i.e. such that*

   - *for all* $w, v \in \mathcal{W}$ *and all* $b \in U_v$*, if* $w\mathcal{R}v$ *and* $b \in D_v$*, then there is an* $a \in U_w$ *such that* $a \in D_w$ *and* $a\mathcal{T}_{(w,v)}b$

Theorem 2.23 (Ghilardi Formula GF)**.** *The following two conditions on* $\mathcal{F}$ *are equivalent*

1. $\mathcal{F} \models \exists y|\vec{x}\,y|A \to |\vec{x}|\exists y A$

2. $\mathcal{F}$ *is* $\mathcal{D}$*-totally-defined, i.e. such that*

   - *for all* $w, v \in \mathcal{W}$ *and all* $a \in U_w$*, if* $w\mathcal{R}v$ *and* $a \in D_w$*, then there is* $b \in U_v$ *such that* $b \in D_v$ *and* $a\mathcal{T}_{(w,v)}b$

Theorem 2.24 (Shortening $SHRT$)**.** *The following two conditions on* $\mathcal{F}$ *are equivalent*

1. $\mathcal{F} \models |\vec{x}\,y|A \to |\vec{x}|A$

2. $\mathcal{F}$ *is* $\mathcal{U}$*-totally-defined, i.e. such that*

   - *for all* $w, v \in \mathcal{W}$ *and all* $a \in U_w$*, if* $w\mathcal{R}v$*, then there is an* $b \in U_v$ *such that* $a\mathcal{T}_{(w,v)}b$

Observe that if we have the classical axiomatization of the quantifiers, then, as shown in Figure 2.4, CBF is a theorem of the minimal logic and GF is derivable from $SHRT$. From the semantic side, if we restrict our attention to single domains transition frames, CBF correspond to a trivial condition and GF and $SHRT$ correspond to the same condition on $\mathcal{T}$.

In Tables 2.1 and 2.2 we have reported all propositional and transitional correspondence results presented in this section. We have separated the results involving existential quantifiers from those involving only universal quantifiers for a reason that has to do with labelled sequent calculi.

Table 2.1: Propositional correspondence results

Formula	Class of t-frames	Conditions
$\lvert\vec{x}\rvert A \to A$	t-reflexive ($\mathcal{C}^T$)	$(T^{\mathcal{R}})\ \forall w(w\mathcal{R}w)$ $(T^{\mathcal{T}})\ \forall a_w(a\mathcal{T}_{(w,w)}a)$
$\lvert\vec{x}\rvert A \to \lvert\vec{x}\rvert\lvert\vec{x}\rvert A$	t-transitive ($\mathcal{C}^4$)	$(4^{\mathcal{R}})\ \forall w,v,z(w\mathcal{R}v \,\&\, v\mathcal{R}z \to w\mathcal{R}z)$ $(4^{\mathcal{T}})\ \forall a_w,b_v,c_z(a\mathcal{T}_{(w,v)}b \,\&\, b\mathcal{T}_{(v,u)}c \to a\mathcal{T}_{(w,u)}c)$
$\langle\vec{x}\rangle A \to \lvert\vec{x}\rvert\langle\vec{x}\rangle A$	t-euclidean ($\mathcal{C}^5$)	$(5^{\mathcal{R}})\ \forall w,v,z(w\mathcal{R}v \,\&\, w\mathcal{R}z \to v\mathcal{R}z)$ $(5^{\mathcal{T}})\ \forall a_w,b_v,c_z(a\mathcal{T}_{(w,v)}b \,\&\, a\mathcal{T}_{(w,u)}c \to b\mathcal{T}_{(v,u)}c)$
$A \to \lvert\vec{x}\rvert\langle\vec{x}\rangle A$	t-symmetric ($\mathcal{C}^B$)	$(B^{\mathcal{R}})\ \forall w,v(w\mathcal{R}v \to v\mathcal{R}w)$ $(B^{\mathcal{T}})\ \forall a_w,b_v(a\mathcal{T}b \to b\mathcal{T}a)$
$\lvert\vec{x}\rvert A \to \langle\vec{x}\rangle A$	t-serial ($\mathcal{C}^D$)	$(D^{\mathcal{R}})\ \forall w\exists v(w\mathcal{R}v)$ $(D^{\mathcal{T}})\ \forall w\forall\vec{a_w}\exists v\exists\vec{b_v}(w\mathcal{R}v \,\&\, \vec{a}\mathcal{T}_{(w,v)}\vec{b})$
$\langle\vec{x}\rangle\lvert\vec{x}\rvert A \to \lvert\vec{x}\rvert\langle\vec{x}\rangle A$	t-directed ($\mathcal{C}^2$)	$(2^{\mathcal{R}})\ \forall w,v,u(w\mathcal{R}v \,\&\, w\mathcal{R}u \to \exists w'(v\mathcal{R}w' \,\&\, u\mathcal{R}w'))$ $(2^{\mathcal{T}})\ \forall\vec{a_w},\vec{b_v},\vec{c_u}(\vec{a}\mathcal{T}_{(w,v)}\vec{b} \,\&\, \vec{a}\mathcal{T}_{(w,v)}\vec{c} \to$ $\exists w', \exists\vec{d_{w'}}(\vec{b}\mathcal{T}_{(v,w')}\vec{d} \,\&\, \vec{c}\mathcal{T}^{(u,w')}\vec{d}))$

Table 2.2: Transitional correspondence results

Formula	Class of t-frames	Condition
$\lvert\vec{x}\rvert\forall yA \to \forall y\lvert\vec{x}\,y\rvert A$	$\mathcal{D}$-preservative ($\mathcal{C}^{CBF}$)	$\forall a_w,b_v(a \in D_w \& a\mathcal{T}_{(w,v)}b \Rightarrow b \in D_w)$
$x \doteq y \to \lvert x\,y\rvert x \doteq y$	$\mathcal{U}$-functional ($\mathcal{C}^{NI}$)	$\forall a_w,b_v,c_v(a\mathcal{T}_{(w,v)}b \,\&\, a\mathcal{T}_{(w,v)}c \to b = c)$
$x \not\doteq y \to \lvert x\,y\rvert x \not\doteq y$	$\mathcal{U}$-not-convergent ($\mathcal{C}^{ND}$)	$\forall a_w,b_w,c_v(a\mathcal{T}_{(w,v)}c \,\&\, b\mathcal{T}_{(w,v)}c \to a = b)$
$\forall y\lvert\vec{x}\,y\rvert A \to \lvert\vec{x}\rvert\forall yA$	$\mathcal{D}$-surjective ($\mathcal{C}^{BF}$)	$\forall w,v\forall b_v(w\mathcal{R}v \& b \in D_v \to \exists a_w(a\mathcal{T}_{(w,v)}b \& a \in D_w))$
$\exists y\lvert\vec{x}\,y\rvert A \to \lvert\vec{x}\rvert\exists yA$	$\mathcal{D}$-totally-defined ($\mathcal{C}^{GF}$)	$\forall w,v\forall a_w(w\mathcal{R}v \& a \in D_w \to \exists b_v(a\mathcal{T}_{w,v)}b \& b \in D_v))$
$\lvert\vec{x}\,y\rvert A \to \lvert\vec{x}\rvert A$	$\mathcal{U}$-totally-defined ($\mathcal{C}^{SH}$)	$\forall w,v,\forall a_w(w\mathcal{R}v \to \exists b_v a\mathcal{T}_{(w,v)}b)$

2.5 Rigidity of Terms

In transition semantics the denotation of a closed terms is defined locally at every world independently of the accessibility and of the transition relations. As a consequence the (world relative) semantic value of a *de re* sentence such as $|^c_x|A(x)$ and that of a *de dicto* one such as $|\star|A(c)$ are not interrelated in any way. But we can define two conditions on $\mathcal{I}$ each of which validates one arrow of the *de re*/*de dicto* equivalence for closed terms.

We define a condition, called t-rigidity that validates the *de-re*-to-*de-dicto* implication $|^t_x \dots |A \to | \dots |(A[t/x])$. This is obtained by imposing that the w-extension of any closed terms t has the v-extension of that same term t as one of its transitiosn in every accessible world v. This condition imposes that the transition relation is $\mathcal{U}$-totally defined for closed terms.

We define another condition, called stability, that validates the *de-dicto*-to-*de-re* implication $| \dots |(A[t/x]) \to |^t_x \dots |A$. This is obtained by imposing that for any worlds w and v, if the w-extension of a closed term has some (w, v)-transition then this object is the v-extension of that closed terms. This condition imposes that the transition relation is $\mathcal{U}$-functional for closed terms.

Definition 2.25. A *t-rigid model* $\mathcal{M}^r$ is any t-model where $\mathcal{I}$ is such that if the w-extension of some term t is a, then, in any world v that is accessible from w, the v-extension of t is a (w, v)-transition of a. More precisely a t-model is rigid whenever, for all $w, v \in \mathcal{W}$,

- $w\mathcal{R}v$ implies $I_w(c)\mathcal{T}_{(w,v)}I_v(c)$, and
- for all $a_1, \dots, a_n \in U_w$ and all $b_1, \dots, b_n \in U_v$, $\bigwedge_{i=1}^n a_i\mathcal{T}_{(w,v)}b_i$ implies $I_w(f)(a_1, \dots, a_n)\mathcal{T}_{(w,v)}I_v(f)(b_1, \dots, b_n)$. ■

Definition 2.26. A *stable model* $\mathcal{M}^s$ is any t-model where $\mathcal{I}$ is such that if a is the w-extension of a closed term t, and if b is a (w, v)-transition of a, then b is the v-extension of t. More precisely a t-model is stable whenever, for all $w, v \in \mathcal{W}$,

- $I_w(c)\mathcal{T}_{(w,v)}o$ implies $o = I_v(c)$, and

- if $t_1, \ldots, t_n \in U_w$ are closed terms and $I_w(f)(I_w(t_1, \ldots, t_n))\mathcal{T}_{(w,v)}o$, then $o = I_v(f)(I_v(t_1, \ldots, t_n))$. ■

In the following we will use $\mathcal{F} \models^r$, $\mathcal{F} \models^s$ and $\mathcal{F} \models^{rs}$ for the notions of truth in all t-rigid, stable, both rigid and stable models based on $\mathcal{F}$, respectively.

Observe that for applications it would be better to have not an all-or-nothing distinction between models that are t-rigid and/or stable and models that are not so, but a more fine-grained distinction between terms that are t-rigid, stable, both t-rigid and stable, and neither t-rigid nor stable. Here we work with the coarser distinction for the sake of simplicity, but it is possible to adopt the finer one by sorting the terms.

Lemma 2.27. *Let $\mathcal{M}^r$ be a t-rigid model. If $\vec{y}$ is (a tuple made from the elements of) the set of all variables occurring in the tuple $\vec{t}$, then*

$$\sigma_w \models_w^{\mathcal{M}^r} |\tfrac{\vec{t}}{\vec{x}}|A \to |\vec{y}|(A[\vec{t}/\vec{x}]) \tag{5.5}$$

Proof. We prove that for all w and σ, $\sigma_w \models_w^{\mathcal{M}^r} \langle y_1 \ldots y_m\rangle(A[t_1, \ldots, t_n/x_1, \ldots, x_n])$ (with $\{y_1, \ldots, y_m\} \subseteq \{t_1, \ldots, t_n\}$) implies that $\sigma_w \models_w^{\mathcal{M}^r} \langle\tfrac{t_1}{x_1} \ldots \tfrac{t_n}{x_n}\rangle A$.

Let's assume there is a v and a τ_v such that

$$\bigwedge_{i=1}^{m} \sigma_w(y_i)\mathcal{T}(w,v)\tau_n(y_i) \;\&\; \tau_v \models_v^{\mathcal{M}^r} (A[t_1, \ldots, t_n/x_1, \ldots, x_n])$$

Since $y_1, \ldots, y_m$ are all the variables occurring in $t_1, \ldots, t_m$, from the definition of t-rigidity we have that

$$\bigwedge_{j=1}^{n} \sigma_w(t_j)\mathcal{T}_{(w,v)}\tau_v(t_j)$$

which, by Lemma 2.7.1, entails that

$$\bigwedge_{j=1}^{n} \sigma_w(t_j)\mathcal{T}_{(w,v)}\tau_v^{x_j \triangleright \tau_v(t_j)}(x_j)$$

By Lemma 2.10, $\tau_v \models_v^{\mathcal{M}^r} (A[t_1, \ldots, t_n/x_1, \ldots, x_n]$ implies

$$\tau_v^{x_1 \triangleright \tau_v(t_1), \ldots, x_n \tau_v(t_n)} \models_v^{\mathcal{M}^r} A$$

and we conclude that $\sigma_w \models_w^{\mathcal{M}^r} \langle {}^{t_1}_{x_1} \ldots {}^{t_n}_{x_n} \rangle A$. ■

Lemma 2.28. *Let $\mathcal{M}^s$ be a stable model. If $\vec{t}$ is a tuple of closed terms, then*

$$\sigma_w \models_w^{\mathcal{M}^r} |\vec{y}|(A[\vec{t}/\vec{x}]) \to |\vec{y}\ {}^{\vec{t}}_{\vec{x}}|A \tag{5.6}$$

Proof. We show that $\sigma_w \models_w^{\mathcal{M}^s} \langle\ y_1 \ldots y_m\ {}^{t_1}_{x_1} \ldots {}^{t_n}_{x_n} \rangle A$, with $t_1, \ldots, t_n$ closed terms, implies that $\sigma_w \models_w^{\mathcal{M}^s} \langle\ y_1 \ldots y_m \rangle (A[t_1, \ldots, t_n/x_1, \ldots, x_n])$.

Let's assume there is a v and a τ_v such that

$$\bigwedge_{i=1}^{m} \sigma_w(y_i) \mathcal{T}_{(w,v)} \tau_n(y_i)\ \&\ \bigwedge_{j=1}^{n} \sigma_w(t_j) \mathcal{T}_{(w,v)} \tau_v(x_j)\ \&\ \tau_v \models_v^{\mathcal{M}^r} A$$

For each j such that $1 \leq j \leq n$, since t_j is a closed term and $\mathcal{M}^s$ is stable, we have that $\tau_v(x_j) = \tau_v(t_j)$, and therefore

$$\tau_v^{x_1 \triangleright \tau_v(t_1), \ldots, x_n \triangleright \tau_v(t_n)} \models_v^{\mathcal{M}^r} A$$

By Lemma 2.10,

$$\tau_v \models_v^{\mathcal{M}^r} A[t_1, \ldots, t_n/x_1, \ldots, x_n]$$

and we conclude that $\sigma_w \models_w^{\mathcal{M}^s} \langle\ y_1 \ldots y_m \rangle (A[t_1, \ldots, t_n/x_1, \ldots, x_n])$. ■

Some considerations are in order. First of all we stress that variables are, in a sense, always t-rigid since they validate 5.5 (where $\vec{t}$ is a tuple of pairwise distinct variables) on every t-frame —it is a case of i-congruence, but they are not stable since they don't validate 5.6.

A second point worth noticing is the relation between the notion of t-rigidity (and stability) and the correspondence results of the previous section. t-rigidity imposes that all objects that are the w-extension of a closed term have at least one (w, v)-transition in each and every accessible world v —namely the v-extension of that closed term, therefore it is a restricted form of total-definedness. Stability imposes that all objects that are the w-extension of a closed term have at most one (w, v)- transition in each and every accessible world v —namely the v-extension of that closed term, therefore it is a restricted form of functionality. Whenever we are working on t-rigid and/or stable models based on a t-frame where $\mathcal{T}$ is both $\mathcal{U}$-functional and $\mathcal{U}$-totally defined, the indexes of an indexed operator are dispensable because an indexed modal formula like $|^{t_1}_{x_1} \dots ^{t_n}_{x_n}|A$ is equivalent to $|y_1 \dots y_m|(A[t_1, \dots, t_n/x_1, \dots, x_n]$ where $y_1, \dots, y_m$ are all the variables among $x_1, \dots, x_n$ that occurs free in A —i.e. substitutions commute with indexed operators and each operator can be indexed by all and only the variables free in its scope.

Note, finally, that from the relation between these notions and correspondence results, it follows that every t-rigid model based on a functional t-frame is also stable, and that every stable model based on a totally defined t-structure is also t-rigid. This fact allows us to answer a question posed in [FM98] where, after noticing that in the language with the λ-machinery the following formulas are semantically equivalent

$$\langle \lambda x.\Box A(x)\rangle(t) \leftrightarrow \Box(\langle \lambda x.A(x)\rangle(t)) \tag{5.7}$$

$$\langle \lambda x.\Box A(x)\rangle(t) \leftarrow \Box(\langle \lambda x.A(x)\rangle(t)) \tag{5.8}$$

$$\langle \lambda x.\Box A(x)\rangle(t) \rightarrow \Box(\langle \lambda x.A(x)\rangle(t)) \;, \tag{5.9}$$

we read

> [5.7] essentially says that the lack of *de re* and *de dicto* distinction is characteristic of rigidity. What is somehow unexpected is the further equivalence between this item and items [5.8] and [5.9]. These latter say that either half of the equivalence in item [5.7] suffices. The deeper significance of this technical result is not understood (at least not yet, at least not by us).
> [FM98, p. 213]

This technical result holds in Kripke-semantics because it assumes a transition relation that is both totally defined and functional: it is the subset relation determined by trans-world identity. In this context the semantic condition that validates one of the two implications entails the other. In this we see that the only way to have a modal logic where only one half of 5.7 holds is by adopting a semantics that allows for a non-functional and non-totally defined relation of counterpart —i.e. we need transition semantics coupled with the indexed modal language.

2.6 Domains and Existence

The distinction between single and double domains modal semantics is often labelled as a distinction between *possibilist* and *actualist* quantification, see [FM98]. The transition semantics we have presented here is a generalization of that of [Cor09] in that we have distinguished an inner domain (of quantification) and an outer domain (of interpretation of the descriptive symbols). A similar semantics, called 'Lewis-semantics', coupled with a typed language, was adopted in [Cor03]. A further generalization of double domain transition semantics is introduced in [CO13] for indexed epistemic logics.

In modal contexts the distinction between double domains structures and single domains ones is usually referred as a distinction between *actualist* and *possibilist* quantification, see [HC96, FM98]. By having adopted double domains, we make valid the principles of free logic, and not that of classical logic: we allow for a predicate to be satisfied at a world by objects that inhabit that world without existing therein.[2] This means, e.g., that the step from '$P(c)$ is true at w' to '$\exists x P(x)$ is true at w' is not valid because terms have no existential import, and the formula $\forall x A \to \exists x A$ is not valid since we allow for worlds with an empty inner domain of quantification.[3] See [Ben01] for an introduction to free logics, and [Gar91] for the advantages of free logics in modal contexts. Our main motivations for using actualist instead of possiblist quantification are technical: we want to prove some

[2] To be more precise, we have the principles of positive free logic because predication is independent from existence.

[3] See [NP11, Chap. 12.1] for a proof-theoretic analysis of empty inner domains in QMLs.

incompleteness theorems in Kripke-semantics bases on double domains, and we can easily define single domains t-frames as a limit case of double domains ones. As shown in [HC96], it is possible also to do the opposite and start with possibilist quantification and define the actualist quantifiers by means of an existence predicate.

Semantically there are two possible assumptions for changing the behavior of quantification that are worth mentioning. First of all we can consider t-frames where the inner domains are all non-empty. On these t-frames, let's call them *existential* t-frames, we have that $\forall x A \to \exists x A$ is valid. A second, and stronger, restriction is to consider the t-frames where each inner domain coincides with the corresponding outer one —i.e. $\forall w \in \mathcal{W}$, $D_w = U_w$. These are, obviously, the class of all single domain t-frames, let's call them *classical* t-frames, introduced in [Cor09] that validates the classical quantification theory. Note that the restriction to classical t-frames has the following consequences for the transitional correspondence results. First of all the notion of $\mathcal{D}$-preservativeness becomes trivial, and thus CBF is valid on all classical t-frames. This is an expected results since, as we noted before, CBF is derivable from an innocuous $\boxdot$-distribution over a theorem of classical quantification theory. Furthermore we lose the distinction between the notions of $\mathcal{D}$-totally-defined and $\mathcal{U}$-totally-defined transition relation, therefore GF and $SHRT$ correspond to the same class of classical t-structures. Also this is an expected result since in [Cor09] it is shown that these two formulas are inter-derivable. From our point of view, see Section 2.2.3, in classical t-frames both corresponds to a dangerous instance of $\boxdot$-distribution that loses track of a ghost-variable; in double domain t-frames they differ because GF, but not $SHRT$, depends also on an existence claim.

Observe that normally the adoption of quantified modal logics based on double domains and free logic is based on the need to falsify some principles governing the interaction of quantifiers and modalities such as CBF and GF (and BF for symmetric Kripke-frames). Given that GF is not valid in the class of all classical t-frames, in transition semantics the adoption of actualist quantification is meant mainly to falsify CBF.[4] For example, we used it in

[4] Note that BF doesn't become valid on symmetric t-frames where CBF holds because

[CO13] for epistemic indexed logic to falsify the doxastic version of CBF because

> I may believe that 'all the basket players are taller than myself' simply because I am not aware of all of them, for example I am not aware of Muggsy Bogues, so why should I be compelled by the logic to conclude that 'I believe of each basket player that he is taller than myself'?
> [CO13, p. 1170]

It is, nevertheless, possible to generalize transition semantics in such a way that CBF is invalid despite the fact that the underlying quantificational theory is the classical one. CBF is valid in classical t-frames because we have stated that a (w, v)-transition is a relation between pairs of objects and not between n-tuples thereof, therefore the satisfaction clause for a formula $|x_1 \ldots x_n|A$ says that it is satisfied (at w under σ) whenever $A(x_1, \ldots, x_n)$ is satisfied in every accessible world v by any v-assignment that maps each x_i to a (w, v)-transition of $\sigma_w(x_i)$ —i.e. a (w, v)-transition of an n-tuple of objects is any n-tuple made of a (w, v)-transitions for each and every individual member of that n-tuple of objects. But we can define a more general semantics in which the transition relation is defined between n-tuples of objects, and not between single objects, in such a way that if $m \neq n$, then the set of (w, v)-transitions of n-tuples is independent of that of m-tuples —i.e. a pair of object $\langle o_1, o_2 \rangle$ can be a (w, v)-transition of a pair $\langle a_1, a_2 \rangle$ without o_1 (o_2) being a (w, v)-transition of a_1 (a_2), and vice-versa. By an easy calculation, it is possible to show that on such semantics CBF is not valid: in axiomatic system for indexed modal logics CBF is inter-derivable with the formula $LNGT := |\vec{x}|A \to |\vec{x}\, y|A$, see [Cor09] for the derivations, which is obviously not valid in this generalization of transition semantics given that the transitions of an $n+1$-tuple have no relation whatsoever with that of an n-tuple of its elements. Metaframe semantics [GSS09] has a counterpart relation that is defined directly between n-tuples. But it doesn't allow to

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the semantic condition corresponding to BF is the dual of that for GF, and not of that for CBF.

falsify CBF because it assumes that if two n-tuples are in the counterpart relation, then every initial segment thereof are in the counterpart relation.

CHAPTER 3

LABELLED SEQUENT CALCULI

Axiomatic systems have been until very recent times the more general and modular kind of proof system for modal logics, but now there are techniques to develop modular proof systems for modal logics based on natural deduction, tableaux and sequent caluli. Labelled proof systems [Gab96] allow to internalize possible-world semantics within the syntax of the rules of inferences, and, thus, to develop modular calculi for most kinds of modal logics. Labelled natural deduction has been used in [Rus96, Vig00] for propositional modal logics and for QMLs with rigid designators defined over the $\mathcal{L}^{\Box}$ language. Labelled tableaux have been used in [Kup12] for the logic of modal metaframes with constant domain and rigid designators. Here we will use labelled sequent calculi, which have been introduced in [Neg05] for propositional modal logics and have been applied in [NP11] for QMLs with rigid designators defined over the language $\mathcal{L}^{\Box}$. These calculi, which are obtained by adding to a basic calculus rules for the accessibility relation $\mathcal{R}$, are particularly interesting because they have well-behaved structural properties, since weakening and contraction are height-preserving admissible and cut is admissible for all logics definable by means of Lemmon-Scott formulas – i.e. **K** (or $\mathbf{Q}^{(\circ)}.\mathbf{K}$) extended with axioms of shape $\Diamond^n\Box^m A \to \Box^k\Diamond^i A$. The strategy, roughly, is that of transforming the first-order conditions on $\mathcal{R}$ that correspond to a set of Lemmon-Scott formulas into a set of left nonlogical rules. Within this approach the admissibility results follow from general results for the extensions of **G3**-style sequent calculi with universal [NP98] and geometric [Ne03] nonlogical axioms.

Given that the classes of t-frames we are interested in are defined not only by conditions on $\mathcal{R}$ but also on $\mathcal{T}$, and that we have not imposed that terms are rigid designators, we have to generalize labelled sequent calculi in order to introduce rules for the transition relation and express where (i.e. in which world) a term has to be evaluated. Since all the correspondence results and the rules that we have to introduce to model 'rigid' designators and assumptions over the domains of existence are either universal or geometric, we will be able to show labelled sequent calculi for IMLs have the same well-behaved structural properties of those for propositional modal logics.

The chapter proceed as follows: in Sect. 1 we recall the sequent calculus **G3c** for classical propositional logic. Then, in Sect. 2, we introduce sequent calculi for IMLs. Sect. 3 presents some technical results that will be useful later on. In Sect. 4 we prove that the rules of weakening and contraction are height-preserving admissible. Section 5 is entirely devoted to give a syntactic proof of the admissiblity of the rule of cut. Finally Sect. 6 shows that the admissiblity of the structural rules implies that our treatment of identity by means of left nonlogical rules is equivalent to the standard one by means of nonlogical axioms.

3.1 The Sequent Calculus G3c

The calculi we are going to present are extensions of the **G3**-style calculi for classical logic, see [TS00, NP01]. We take the notion of rooted tree, and the related notions (branch, leaf, etc.) as given in [TS00, NP01]. A *sequent* is an expression

$$\Gamma \Rightarrow \Delta$$

where $\Gamma = A_1, \ldots, A_n$, and $\Delta = B_1, \ldots, B_m$ are (finite, possibly empty) multisets of formulas called *antecedent* and *succedent*, respectively. The intended reading of a sequent is that the conjuction of the formulas in the antecedent implies the disjunction of the formulas in the succedent, thus a sequent $A_1, \ldots, A_n \Rightarrow B_1, \ldots B_m$ has the same meaning as

$$\bigwedge_{i=1}^{n} A_i \to \bigvee_{j=1}^{m} B_j$$

where the empty conjunction has the same value of $\neg\bot$, and the empty disjunction that of $\bot$. Thus the empty sequent $\Rightarrow$ stands for $\bot$.

A one- or two-premiss *rule* of inference between sequents is either a pair or a triple of sequents, to be denoted, respectively, by a figure

$$\frac{\mathcal{S}_1}{\mathcal{S}}\dagger \qquad \frac{\mathcal{S}_1 \qquad \mathcal{S}_2}{\mathcal{S}}\dagger$$

indicating that the sequent S may be inferred from the sequent(s) $\mathcal{S}_1$ (and $\mathcal{S}_2$) by the rule $\dagger$. $\mathcal{S}$, will be also called *lower sequent*, or *conclusion*, of the inference, and $\mathcal{S}_1$ (and $\mathcal{S}_2$) *upper sequent(s)*, or *premiss(es)*.

A sequent calculus is defined by giving its inital sequents and its rules of inference. The sequent calculus **G3c** for classical predicate logic is given in Table 3.1. Given a rule of **G3c**, the formula occurrences in Γ, Δ are called *contexts*; the formula occurrences of the conclusion that are not in the contexts are called *principal formulas*; the formula occurrences in the premiss(es) that are not in the contexts (and that are not principal) are called *active formulas*.

The calculus **G3c** is particularly apt for root-first proof search since the structural rules are absorbed into the logical ones, all rules are context-sharing,and all rules are invertible. This last property holds because there is a single rule for $R\wedge$ and for $L\vee$, and the principal formula is repeated in the upper sequent of rules $L\forall$ and of $R\exists$ (as in Kleene's [Kle52]).

Definition 3.1. Given a sequent calculus **Gx**, we say that a sequent $\mathcal{S}$ is derivable in it, **Gx**$\vdash \mathcal{S}$, whenever there is a tree of sequents of root $\mathcal{S}$, whose leaves are initial sequents or instances or $L\bot$ and whose edges are obtained by some rule in **Gx**. ■

Table 3.1: The sequent calculus **G3c**.

Initial sequents $P, \Gamma \Rightarrow \Delta, P$ where P is an atomic formula

Propositional rules

$$\frac{}{\bot, \Gamma \Rightarrow \Delta}\, L\bot$$

$$\frac{A, B, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta}\, L\wedge \qquad \frac{\Gamma \Rightarrow \Delta, A \qquad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B}\, R\wedge$$

$$\frac{A, \Gamma \Rightarrow \Delta \qquad B, \Gamma \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta}\, L\vee \qquad \frac{\Gamma \Rightarrow \Delta, A, B}{\Gamma \Rightarrow \Delta, A \vee B}\, R\vee$$

$$\frac{\Gamma \Rightarrow \Delta, A \qquad B, \Gamma \Rightarrow \Delta}{A \to B, \Gamma \Rightarrow \Delta}\, L\to \qquad \frac{A, \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \to B}\, R\to$$

Quantifier rules (where, in rules $L\forall$ and $R\exists$, y is an eigenvariable).

$$\frac{A[t/x], \forall x A, \Gamma \Rightarrow \Delta}{\forall x A, \Gamma \Rightarrow \Delta}\, L\forall \qquad \frac{\Gamma \Rightarrow \Delta, A[y/x]}{\Gamma \Rightarrow \forall x A, \Delta}\, R\forall$$

$$\frac{A[y/x], \Gamma \Rightarrow \Delta}{\exists x A, \Gamma \Rightarrow \Delta}\, R\exists \qquad \frac{\Gamma \Rightarrow \Delta, \exists x A, A[t/x]}{\Gamma \Rightarrow \Delta, \exists x A}\, R\exists$$

(Observe that the propositional rules introduce propositional logical symbols, so $L\bot$ is among them even if it has no premiss. Furthermore $\bot, \Gamma \Rightarrow \Delta, \bot$ is an instance of $L\bot$, and not of an initial sequent.)

Definition 3.2. Given a derivation of a sequent $\mathcal{S}$ in **Gx** its *derivation height* is the greatest number of successive application of rules of **Gx** in it, where initial sequents and instances of $L\bot$ have height 0. We write $\mathbf{Gx} \vdash_n \mathcal{S}$ when $\mathcal{S}$ has a derivation in **Gx** of height at most n. ∎

Definition 3.3. Given a sequent calculus **Gx**, we say that

1. a rule is *derivable* in **Gx** iff its conclusion is derivable in it from its premisses;
2. a rule is *admissible* in **Gx** iff its conclusion is derivable in it whenever its premisses are;
3. a rule is *height-preserving admissible* in **Gx** iff whenever its premisses are derivable in it, its conclusion has a derivation in **Gx** whose height is at most the max height of the derivations of the premisses. ∎

3.2 Sequent Calculi for IMLs

3.2.1 Syntax

In order to introduce labelled sequent calculi for IMLs, we have to extend our language in order to reason about notions such as the extension of terms, the accessibility relation, the transition relation and existence. By a *ground term* of a formula A we mean any term that doesn't occur within the scope of an indexed operator (or in its denominator). To illustrate, in the formula $|^t_x|(A \wedge |^s_y|B)$ the only ground term is t, and in $|^t_x|A \wedge P(s)$ the ground terms are t and s. First of all we introduce an infinite set of fresh variables $\{w_1, w_2, \dots\}$ of so-called *world labels*, for which we use the metavariables w, v, z. World labels will allow us to express where (i.e. in which world of a t-model) a term or a formula is instantiated. We introduce three new predicates: the binary $\mathscr{R}$ and $\mathscr{T}$, and the monadic $\mathcal{E}$. Their role is to express information about the accessibility relation, the transition relation and existence, respectively. By a *l-term* —where 'l' stands for 'labelled'– we mean an expression t^w where t is a term and w is a world label; *l-variables*, *l-constants*, and all the relative notions are defined accordingly.

Definition 3.4 (Ext-terms)**.** The set of *ext-terms* is the union of the sets of world-label with that of l-terms. The first are l-terms of sort *worlds* and the second of sort *individuals*. ■

Definition 3.5 (Ext-formulas)**.** The set of *ext-formulas* is the union of the following sets of expressions of the extended language

- For any formula $A(t_1, \ldots, t_n)$, where $t_1, \ldots, t_n$ are all ground (occurrences of) terms in A, and any world label w, the expression $w : A(t_1^w, \ldots, t_n^w)$ is an l-formulas.
- If w and v are world labels, $w\mathscr{R}v$ is an r-formulas.
- If t^w and s^v are l-terms, $t^w\mathscr{T}s^v$ is a t-formula.
- If t^w is an l-term, $\mathcal{E}t^w$ is an e-formula. ■

Observe that in an indexed l-formula such as $w : |^{t_1^w}_{x_1} \ldots {}^{t_n^w}_{x_n}|A$ the ground terms are all and only the terms occurring in its numerator, and therefore the terms occurring in its subformula A are not labelled. Given that each (ground) occurrence of a term t_i in A is replaced in $w : A$ by the l-term t_i^w having same label as the whole formula, we will never write the label of these l-term. With a slight abuse of notation, we will use $t^{\vec{w}}\mathscr{T}s^{\vec{v}}$, where $t^{\vec{w}}$ and $s^{\vec{v}}$ are n-ary vectors of l-terms, as a shorthand for $t_1^w\mathscr{T}s_1^v, \ldots, t_n^w\mathscr{T}s_n^v$. Observe that l-formulas are just indexed modal formulas that are decorated with world labels, and that the ext-formulas of the other three kinds are always atomic and cannot be proper subformulas of other ext-formulas. As a consequence, all the definitions and lemmas regarding indexed formulas, see Chapter 1, can be straightforwardly extended to ext-formulas. E.g., one measure for ext-terms and ext-formulas that we will use in inductive proofs is their height, where the eight of an ext-term is measured just as that of terms with the additional clauses that $h(w) = 0$, for any world label w. For ext-formulas, the height of an l-formula $w : A$ is the same as that of A, and all r-formulas, t-formulas and e-formulas, being atomic, have height 0.

Definition 3.6. Substitution of world labels ($[v/u]$) is defined as follows:

- $(w : \bot)[v/u] = \begin{cases} v : \bot & \text{if } u = w \\ w : \bot & \text{if } u \neq w \end{cases}$;

- $(w : P(t_1, \ldots, t_n))[v/u] = \begin{cases} v : P(t_1 \ldots, t_n) & \text{if } u = w \\ w : P(t_1, \ldots, t_n) & \text{if } u \neq w \end{cases}$;

- $(w_1 \mathscr{R} w_2)[v/u] = \begin{cases} v \mathscr{R} w_2 & \text{if } u = w_1 \text{ and } u \neq w_2 \\ w_1 \mathscr{R} v & \text{if } u \neq w_1 \text{ and } u = w_2 \\ v \mathcal{R} v & \text{if } u = w_1 \text{ and } u = w_2 \\ w_1 \mathscr{R} w_2 & \text{if } u \neq w_1 \text{ and } u \neq w_2 \end{cases}$;

- $(t^{w_1} \mathscr{T} s^{w_2})[v/u] = \begin{cases} t^v \mathscr{T} s^{w_2} & \text{if } u = w_1 \text{ and } u \neq w_2 \\ t^{w_1} \mathscr{T} s^v & \text{if } u \neq w_1 \text{ and } u = w_2 \\ t^v \mathscr{T} s^v & \text{if } u = w_1 \text{ and } u = w_2 \\ t^{w_1} \mathscr{T} s^{w_2} & \text{if } u \neq w_1 \text{ and } u \neq w_2 \end{cases}$;

- $(\mathcal{E} t^w)[v/u] = \begin{cases} \mathcal{E} t^v & \text{if } w = u \\ \mathcal{E} t^w & \text{if } w \neq u \end{cases}$;

- $(w : \dagger A)[v/u] = \begin{cases} v : \dagger A & \text{if } w = u \\ w : \dagger A & \text{if } w \neq u \end{cases}$;[1]

- $(w : A \circ B)[v/u] = \begin{cases} v : A \circ B & \text{if } w = u \\ w : A \circ B & \text{if } w \neq u \end{cases}$.

Substitution of world-labels is extended to multisets, as well as to sequents, componentwise. ■

Definition 3.7. Substitution of an l-term for an l-variable in an l-formula is defined as substitution of terms in formulas, save that now terms are labelled and therefore

- if $w \neq v$, then the substitution has no effect, $(w : A)[s^v/x^v] = w : A$;
- if $w = v$, the substitution is effective, $(w : A)[s^v/x^v] = w : A[s/x]$ where $A[s/x]$ is defined as in Definition 1.14 (but with all terms implicitly labelled by w).

[1] Here $\dagger \in \{\forall x, \exists x, |\frac{\vec{t}}{\vec{x}}|, \langle\frac{\vec{t}}{\vec{x}}\rangle\}$.

Substitution of l-terms has no effect on r-formulas, since no l-term can occur therein, and is defined on t- and e-formulas in a straightforward way. Substitution of l-terms is extended to multisets, as well as to sequents, componentwise. Simultaneous substitution is defined accordingly. ■

Given that substitution will be extremely useful later on, we present also the extended definition:

- $(w : \bot)[s^v/x^v] = w : \bot$;
- $(w : P(t_1, \dots, t_n))[s^v/x^v] = \begin{cases} w : P(t_1[s/x], \dots, t_n[s/x]) & \text{if } v = w \\ w : P(t_1, \dots, t_n) & \text{if } v \neq w \end{cases}$;
- $(w\mathscr{R}z)[s^v/x^v] = w\mathscr{R}z$;
- $(t^w \mathscr{T} r^u)[s^v/x^v] = \begin{cases} s^w \mathscr{T} r^u & \text{if } t^w = x^v \text{ and } r^u \neq x^v \\ t^w \mathscr{T} s^u & \text{if } t^w \neq x^v \text{ and } r^u = x^v \\ s^w \mathscr{T} s^u & \text{if } t^w = x^v \text{ and } r^u = x^v \\ t^w \mathscr{T} r^u & \text{if } t^w \neq x^v \text{ and } r^u \neq x^v \end{cases}$;
- $(\mathcal{E}t^w)[s^v/x^v] = \begin{cases} \mathcal{E}s^v & \text{if } t^w = x^v \\ \mathcal{E}t^w & \text{if } t^w \neq x^v \end{cases}$;
- $(w : A \circ B)[s^v/x^v] = ((w : A)[s^v/x^v]) \circ ((w : B)[s^v/x^v])$;
- $(w : \mathcal{Q}yA)[s^v/x^v] = \begin{cases} w : \mathcal{Q}yA & \text{if } x = y \text{ or } v \neq w \\ w : \mathcal{Q}z((A[z/y])[s/x]) & \text{if } x \neq y \text{ and } s = y \text{ and } v = w \\ & \text{where } z \text{ doesn't occur in } \mathcal{Q}yA \\ w : \mathcal{Q}y(A[s/x]) & \text{if } x \neq y \neq s \text{ and } v = w \end{cases}$;
- $(w : |^{t_1}_{y_1} \dots ^{t_n}_{y_n} |A)[s^v/x^v] = \begin{cases} w : |^{t_1[s/x]}_{y_1} \dots ^{t_n[s/x]}_{y_n} |A & \text{if } v = w \\ w : |^{t_1}_{y_1} \dots ^{t_n}_{y_n} |A & \text{if } v \neq w \end{cases}$;
- $(w : \langle^{t_1}_{y_1} \dots ^{t_n}_{y_n} \rangle A)[s^v/x^v] = \begin{cases} w : \langle^{t_1[s/x]}_{y_1} \dots ^{t_n[s/x]}_{y_n} \rangle A & \text{if } v = w \\ w : \langle^{t_1}_{y_1} \dots ^{t_n}_{y_n} \rangle A & \text{if } v \neq w \end{cases}$.

Substitution of l-terms for l-variables behaves on an l-formula $w : A$ as substitution of terms for variables on the formula A (as long as the l-variable to be substituted and the l-formula have the same label, otherwise the substitution is simply dropped). Thus the properties of Proposition 1.17 hold for the notion of substitution of l-terms too.

Proposition 3.8. *The following properties hold for any ext-formula E:*

1. $(E)[x^w/x^w] = E$.

2. *If* $x^w \notin VAR(E)$, *then* $(E)[s^w/x^w] = E$.

3. *If* $y^w \notin VAR(E)$ *and* s^w *is free for* x^w *in* E, *then* $((E)[y^w/x^w])[s^w/y^w] = (A)[s^w/x^w]$.

4. *If* $x^w \neq y^w$, $x^w \notin VAR(r^w)$, *and* s^w *and* r^w *are free for* x^w *and* y^w *in* E, *then* $((E)[s^w/x^w])[r^w/y^w] = ((E)[r^w/y^w])[s^w[r^w/y^w]/x^w]$.

5. *If* $y^w \neq x^w$, $x^w \notin VAR(r^w)$, $y^w \notin VAR(s^w)$, *and* r^w, s^w *are free respectively for* x^w, y^w *in* E, *then* $((E)[s^w/x^w])[r^w/y^w] = ((E)[r^w/y^w])[s^w/x^w]$.

6. *If* $y^w \notin VAR(E[s^w/x^w])$, *then* $(E[y^w/x^w])[s^w/y^w] = E[s^w/x^w]$.

3.2.2 The basic calculus GIM.K

The sequent calculi for IML are all based on the calculus **G3c**. In general we adopt the following naming conventions: we call GIM.K the basic calculus for the logic defined over the language without identity, which, as we shall see, is sound and complete w.r.t. the class of all t-frames.

The sequent calculus **GIM.K**[2] is given in Table 3.2. Its inital sequents and propositional rules are just like those of **G3c**, save that they are defined over l-formulas. Thus, e.g., initial sequents are all expressions $w : P, \Gamma \Rightarrow \Delta, w : P$ where $w : P$ is an arbitrary atomic l-formula and Γ, Δ are multisets of ext-formulas. A rule like $L\wedge$ says that the sequent $w : A \wedge B, \Gamma \Rightarrow \Delta$ is K-derivable from the sequent $w : A, w : B, \Gamma \Rightarrow \Delta$. Note that the only role of world labels is that of limiting propositional rules in that they are applicable only if the active formulas of that rule have the same world label. We have not allowed r-, t- and e-formulas to be principal in initial sequents

[2] In general the calculi are named **GIM.**$\star$, where $\star$ is the name of the underlying PML and of further rules for $\mathcal{T}$ separated by dots, and **GIM** stands for 'Gentzen indexed modalities'.

because initial sequents of those shapes are irrelevant in deriving indexed modal formulas, they would only a allow to derive properties of t-frames.[3]

The rules for the quantifiers differ more substantially from that of **G3c** because we have taken as basic the theory of quantification of free logic, and not that of classical logic, therefore, omitting the labels, to derive a universal formula $\forall xA$ in the succedent it is not enough to have derived $A[y/x]$ for an arbitrary y, but we need to know also that y is an existing object, where claims of existence are expressed by means of e-formulas. Given that an e-formula $\mathcal{E}y^w$ says that the w-extension of y is an object that exists in w,[4] the addition of e-formulas allows to obtain the rules for the quantifiers from the semantic explanations of their clause of satisfaction. From the clause

$$\sigma \models_w \forall xA \quad iff \quad \text{for all } a \in D_w,\ \sigma^{x \rhd a} \models_w A$$

we obtain the left and right rules for the universal quantifier. From the left-to-right implication we obtain the left rule

$$\frac{w : A[t/x], w : \forall xA, \mathcal{E}t^w, \Gamma \Rightarrow \Delta}{w : \forall xA, \mathcal{E}t^w, \Gamma \Rightarrow \Delta}\ L\forall$$

which says that from the fact that A is true of every object existing in (the domain associated with) w and that t exists in w we can derive that A is true of t in w. From the right-to-left implication we obtain the right rule

$$\frac{\mathcal{E}y^w, \Gamma \Rightarrow \Delta, w : A[y/x]}{\Gamma \Rightarrow \forall xA : w, \Delta}\ R\forall \quad \text{where } y^v \text{ is an eigenvariable}$$

which says that if from the only known fact about y that it exists in w, we can derive that A is true of y in w, then $\forall xA$ holds in w. The rules for $\exists$ are obtained by the same method.

[3] To illustrate, it is well known that reflexive frames are also serial, and an analogous result holds for t-frames. By adding these initial sequents we can give a syntactic derivation of this result in Correspondence Theory, but all indexed formulas that are valid on reflexive t-frames, thus also $\mathcal{D}^t$, are derivable in the sequent calculus for t-reflexive t-frames, which doesn't have these additional initial sequents.

[4] In [NP11, Ch. 11] the role of e-formulas is taken by expressions like $y \in D(w)$. The difference is only notational.

With respect to the labelled sequent calculus for QMLs given in [NP11, Ch. 12], the novelty of **GIM.K** is the introduction of rules for the indexed modal operators. These rules are obtained from semantic explanations that are possible thanks to the introduction of r- and t-formulas. An r-formula $w\mathscr{R}v$ says that the world v is accessible from w, and a t-formula $t^w \mathscr{T} s^v$ says that s is a (w,v)-transition of t. From the left-to-right implication of

$$\sigma \models_w |^{t_1}_{x_1} \dots ^{t_n}_{x_n}|A \quad iff \quad \text{for all } v \in \mathcal{W} \text{ and all assignments } \tau,\ w\mathcal{R}v \text{ and } \sigma_w(t_i)\mathcal{T}_{(w,v)}\tau_v(x_i), \text{ for all } i \text{ s.t. } 1 \leq i \leq n, \text{ imply } \tau \models_v A$$

we obtain the rule

$$\frac{v : A[\vec{s}/\vec{x}], w : |^{\vec{t}}_{\vec{x}}|A, w\mathscr{R}v, \vec{t^w}\mathscr{T}\vec{s^v}, \Gamma \Rightarrow \Delta}{w : |^{\vec{t}}_{\vec{x}}|A, w\mathscr{R}v, \vec{t^w}\mathscr{T}\vec{s^v}, \Gamma \Rightarrow \Delta} \; L|^{\vec{t}}_{\vec{x}}|$$

which says that if $|^{\vec{t}}_{\vec{x}}|A$ holds in a world w and the world v is accessible from w and is such that each s_i is a (w,v)-transition of t_i, then A is true of $s_1, \dots, s_n$ in v. Form the right-to-left implication of the satisfaction clause above we obtain the right rule

$$\frac{w\mathscr{R}v, \vec{t^w}\mathscr{T}\vec{x^v}, \Gamma \Rightarrow \Delta, v : A}{\Gamma \Rightarrow \Delta, w : |^{\vec{t}}_{\vec{x}}|A} \; R|^{\vec{t}}_{\vec{x}}| \quad \text{where } v \text{ is an eigenvariable}$$

which says that if from the only known fact about v that it is accessible form w and that the x_is are (w,v)-transitions of the t_is, we can derive that A is true of the x_is in v, then $|^{\vec{t}}_{\vec{x}}|A$ holds in w. The rules for $\langle^{\vec{t}}_{\vec{x}}\rangle$ are obtained by the same method.

The rules for the minimal calculus **GIM.K** are given in Table 3.2. Observe that the fact that v respects the variable condition in rules $R|^{\vec{t}}_{\vec{x}}|$ and $L\langle^{\vec{t}}_{\vec{x}}\rangle$, i.e. that it doesn't occur (free) in the conclusion of that rule implies that also all members of $\vec{x^v}$ respect the variable condition since they cannot occur (free) in the conclusion whereas their label v doesn't.

Table 3.2: The sequent calculus **GIM.K**.

- **Initial sequents** $w : P, \Gamma \Rightarrow \Delta, w : P$ ($w : P$ is an atomic l-formula)

- **Propositional rules**

$$\frac{}{w : \bot, \Gamma \Rightarrow \Delta} L\bot$$

$$\frac{w : A, w : B, \Gamma \Rightarrow \Delta}{w : A \wedge B, \Gamma \Rightarrow \Delta} L\wedge \qquad \frac{\Gamma \Rightarrow \Delta, w : A \qquad \Gamma \Rightarrow \Delta, w : B}{\Gamma \Rightarrow \Delta, w : A \wedge B} R\wedge$$

$$\frac{w : A, \Gamma \Rightarrow \Delta \qquad w : B, \Gamma \Rightarrow \Delta}{w : A \vee B, \Gamma \Rightarrow \Delta} L\vee \qquad \frac{\Gamma \Rightarrow \Delta, w : A, w : B}{\Gamma \Rightarrow \Delta, w : A \vee B} R\vee$$

$$\frac{\Gamma \Rightarrow \Delta, w : A \qquad w : B, \Gamma \Rightarrow \Delta}{w : A \to B, \Gamma \Rightarrow \Delta} L\to \qquad \frac{w : A, \Gamma \Rightarrow \Delta, w : B}{\Gamma \Rightarrow \Delta, w : A \to B} R\to$$

- **Quantifier rules** (where, in rules $L\forall$ and $R\exists$, y^w is an eigenvariable).

$$\frac{w : A[t/x], w : \forall x A, \mathcal{E}t^w, \Gamma \Rightarrow \Delta}{w : \forall x A, \mathcal{E}t^w, \Gamma \Rightarrow \Delta} L\forall \qquad \frac{\mathcal{E}y^w, \Gamma \Rightarrow \Delta, w : A[y/x]}{\Gamma \Rightarrow \forall x A : w, \Delta} R\forall$$

$$\frac{\mathcal{E}y^w, w : A[y/x], \Gamma \Rightarrow \Delta}{w : \exists x A, \Gamma \Rightarrow \Delta} R\exists \qquad \frac{\mathcal{E}t^w, \Gamma \Rightarrow \Delta, w : \exists x A, w : A[t/x]}{\mathcal{E}t^w, \Gamma \Rightarrow \Delta, w : \exists x A} R\exists$$

- **Modal rules**

$$\frac{v : A[\vec{s}/\vec{x}], w : |\tfrac{\vec{t}}{\vec{x}}|A, w\mathscr{R}v, \vec{t^w}\mathscr{T}\vec{s^v}, \Gamma \Rightarrow \Delta}{w : |\tfrac{\vec{t}}{\vec{x}}|A, w\mathscr{R}v, \vec{t^w}\mathscr{T}\vec{s^v}, \Gamma \Rightarrow \Delta} L|\tfrac{\vec{t}}{\vec{x}}|$$

$$\frac{w\mathscr{R}v, \vec{t^w}\mathscr{T}\vec{x^v}, \Gamma \Rightarrow \Delta, v : A}{\Gamma \Rightarrow \Delta, w : |\tfrac{\vec{t}}{\vec{x}}|A} R|\tfrac{\vec{t}}{\vec{x}}|$$

$$\frac{v : A, w\mathscr{R}v, \vec{t^w}\mathscr{T}\vec{x^v}, \Gamma \Rightarrow \Delta}{w : \langle\tfrac{\vec{t}}{\vec{x}}\rangle A, \Gamma \Rightarrow \Delta} L\langle\tfrac{\vec{t}}{\vec{x}}\rangle$$

$$\frac{w\mathscr{R}v, \vec{t^w}\mathscr{T}\vec{s^v}, \Gamma \Rightarrow \Delta, w : \langle\tfrac{\vec{t}}{\vec{x}}\rangle A, v : A[\vec{s}/\vec{x}]}{w\mathscr{R}v, \vec{t^w}\mathscr{T}\vec{s^v}, \Gamma \Rightarrow \Delta, w : \langle\tfrac{\vec{t}}{\vec{x}}\rangle A} R\langle\tfrac{\vec{t}}{\vec{x}}\rangle$$

Where, in rules $L|\frac{\vec{t}}{\vec{x}}|$ and $L\langle\frac{\vec{t}}{\vec{x}}\rangle$, v is an eigenvariable. Note that it follows that also all $x_i^v \in \vec{x^v}$ are eigenvariables.

3.2.3 Rules for Identity

If the language contains the identity symbol $\doteq$, we have to ensure that it satisfies the properties of equivalence relations and Leibniz's Law of substitution. If we want to do so without impairing the admissibility of the structural rules, we cannot extend the calculus by introducing an axiom A governing identity directly as a new initial sequent $\Rightarrow A$ since this would force us to have cut as a primitive and uneliminable rule, see [Gir87, pp. 125–126]. To illustrate, if we treat identity in **G3c** as a so-called *Post system* by introducing nonlogical initial sequents (axioms) of the form

$$\Rightarrow t \doteq t$$

$$s \doteq t, P[s/x] \Rightarrow P[t/x]$$

with P atomic ext-formula, as in [Gir87], we can show that identity is symmetric by applying cut to two axioms, with $P =_{df} x \doteq s$, as follows

$$\frac{\Rightarrow s \doteq s \qquad s \doteq t, s \doteq s \Rightarrow t \doteq s}{s \doteq t \Rightarrow t \doteq s}\,Cut$$

But there is no cut-free derivation of this fact. Post systems allow to have weakening and contraction admissible, and to reduce all cuts to ones on axioms, but not to eliminate all cuts. Thus, if we want to have the structural rules admissible, we have to express the axioms governing identity in a different manner. This will be done by applying the method introduced in [NP98] of introducing nonlogical universal axioms as left nonlogical rules of inference. This method allows to capture an universal nonlogical axiom that is expressed as a *regular* formula

$$P_1 \wedge \ldots \wedge P_n \to Q_1 \vee \ldots \vee Q_m$$

(where all P_is and Q_js are atomic) by means of a *left nonlogical rule* whose shape is

$$\frac{Q_1, P_1, \ldots, P_n, \Gamma \Rightarrow \Delta \quad \ldots \quad Q_m, P_1, \ldots, P_n, \Gamma \Rightarrow \Delta}{P_1, \ldots, P_n, \Gamma \Rightarrow \Delta} \, Reg$$

where, in order to have contraction admissible, the principal formulas P_1, $\ldots, P_n$ are repeated in the premiss(es) and the following condition holds.

Definition 3.9 (Closure condition)**.** If a substitution instance of a nonlogical rule has two occurrences of an atomic principal formula, as in

$$\frac{Q_1, P_1, \ldots, P_{n-2}, P, P, \Gamma \Rightarrow \Delta \quad \ldots \quad Q_n, P_1, \ldots, P_{n-2}, P, P, \Gamma \Rightarrow \Delta}{P_1, \ldots, P_{n-2}, P, P, \Gamma \Rightarrow \Delta} \, Reg$$

then it contains also the rule

$$\frac{Q_1, P_1, \ldots, P_{n-2}, P, \Gamma \Rightarrow \Delta \quad \ldots \quad Q_m, P_1, \ldots, P_{n-2}, P, \Gamma \Rightarrow \Delta}{P_1, \ldots, P_{n-2}, P, \Gamma \Rightarrow \Delta} \, Reg^C$$

that expresses the contracted version of that instance of rule *Reg*. ■

A key idea behind left nonlogical rules is that the logical structure of a regular formula is expressed by the arboreal structure of a rule of inference, and thus we can capture that formula in a rule where all active and principal formulas are atomic.

Theorem [NP98, Theorem 4.1] says that the structural properties of a sequent calculus are preserved by its extension with left nonlogical rules of this kind. Given that the axioms of identity are expressible as universal regular formulas, this results shows that by expressing the logic of identity in this way it is possible to preserve the admissiblity of cut [NP98, Sect. 4.2]. Thus we define the minimal sequent calculus $\mathbf{GIM.K}_=$ for the language with identity by extending **GIM.K** with the nonlogical rules for identity given in Table 3.3, where, for the sake of clarity, we have preferred to introduce two rules of substitution of identicals instead of the equivalent rule of substitution for atomic ext-formulas E:

$$\frac{(E)[s^w/x^w], w : t \doteq s, (E)[t^w/x^w], \Gamma \Rightarrow \Delta}{w : t \doteq s, (E)[t^w/x^w], \Gamma \Rightarrow \Delta} \, Reg$$

Table 3.3: Identity rules

$$\frac{w : t \doteq t, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\, Ref$$

$$\frac{w : P[s/x], w : t \doteq s, w : P[t/x], \Gamma \Rightarrow \Delta}{w : t \doteq s, w : P[t/x], \Gamma \Rightarrow \Delta}\, Lbz_1$$

$$\frac{(t_1^w \mathscr{T} t_2^v)[r^u/z^u], u : s \doteq r, (t_1^w \mathscr{T} t_2^v)[s^u/z^u], \Gamma \Rightarrow \Delta}{u : s \doteq r, (t_1^w \mathscr{T} t_2^v)[s^u/z^u, \Gamma] \Rightarrow \Delta}\, Lbz_2$$

3.2.4 Correspondence Results as Nonlogical Rules

So far we have presented the sequent calculi **GIM.K** and **GIM.K**$_=$. Now we are interested in the sequent system **GIM.**$\star_{(=)}$ which gives the logic of the class $\mathcal{C}^\star$ of all t-frames respecting the conditions in $\star$, where $\star$ may be (the list of names of) any combination of the correspondence results from Tables 2.1 and 2.2—e.g. **GIM.T.BF** is meant to give the logic of the class of all t-frames that are t-reflexive and $\mathcal{D}$-surjective. The goal is to do so in a modular way and without impairing the admissibility of the structural rules.

By inspecting the correspondence results in Tables 2.1 and 2.2, it is immediate to recognize that many of the properties of the accessibility and of the transition relation there stated are universal regular formulas, and, given that the ext-language allows us to internalize the semantics in the sequent calculus, are expressible as left nonlogical rules. But the properties (corresponding to) $D, 2, BF, GF, SHRT$ involve existential quantifiers, and, therefore, are not expressible as universal regular formulas.

In [Ne03] it has been shown that the method of left nonlogical rules is extendable to axioms expressible as *geometric formulas*, i.e. axioms of the form

$$\forall \vec{x}(A \to B)$$

where neither A nor B contains $\to$ or $\forall$; they can be expressed as conjunctions of formulas of the form

$$\forall \vec{x}(\bigwedge_{i=1}^{n} P_1 \to \exists \vec{y_1} M_1 \vee \ldots \vee \exists \vec{y_m} M_m)$$

where the P_i are atomic, each M_j is a conjunction of atomic formulas that, with an abuse of notation, we can denote as $\vec{Q_j}$, and (each member of any) $\vec{y_j}$ are not free in the P_i. These formulas are expressible as nonlogical rules

$$\frac{\vec{Q_1}[\vec{y_1}/\vec{x_1}], \vec{P}, \Gamma \Rightarrow \Delta \quad \ldots \quad \vec{Q_m}[\vec{y_m}/\vec{x_m}], \vec{P}, \Gamma \Rightarrow \Delta}{\vec{P}, \Gamma \Rightarrow \Delta}$$

where the y_i are all eigenvariables, the principal formulas $\vec{P}$ are repeated in the premiss(es), and the closure condition is satisfied. Note that the existential quantifier is absorbed into the structure of the rule of inference as a variable condition.

The properties (corresponding to) $D, 2, BF, GF, SHRT$ can be expressed as geometric formulas. In this way we can obtain a system of sequent calculi for the logic of any class of frames $\mathcal{C}^\star$ obtainable by combining the properties of t-frames introduced in Chap. 2.4. We have only to transform the relevant properties of $\mathcal{R}$ and $\mathcal{T}$ into left nonlogical rules whose arboreal structure capture the logical structure of correspondence results. To make a couple of examples, the universal property (corresponding to) $4^{\mathcal{R}}$

$$\forall w\, v\, u(w\mathcal{R}v \,\&\, v\mathcal{R}u \to w\mathcal{R}u)$$

becomes the nonlogical rule

$$\frac{w\mathscr{R}u, w\mathscr{R}v, v\mathscr{R}u, \Gamma \Rightarrow \Delta}{w\mathscr{R}v, v\mathscr{R}u, \Gamma \Rightarrow \Delta}\, 4^{\mathcal{R}}$$

where w, r and u are metavariables for arbitrary world labels, and the principal formulas $w\mathscr{R}v$ and $v\mathscr{R}u$ are repeated in the premiss. Note that this rule, having two ext-atoms principal, is subject to the closure condition. Thus we have to add also the contracted instances of the rule[5]

$$\frac{w\mathscr{R}w, w\mathscr{R}w, \mathcal{G} \Rightarrow \Delta}{w\mathscr{R}w, \Gamma \Rightarrow \Delta}\, 4^{\mathcal{R}c}$$

As a further example, the geometric property (corresponding to) GF

$$\forall w, v, \forall a_w (w\mathcal{R}v \,\&\, a \in D_w \to \exists b_v (a\mathcal{T}b \,\&\, b \in D_v))$$

becomes the nonlogical rule

$$\frac{t^w \mathscr{T} y^v, \mathcal{E}y^v, w\mathscr{R}v, \mathcal{E}t^w, \Gamma \Rightarrow \Delta}{w\mathscr{R}v, \mathcal{E}t^w, \Gamma \Rightarrow \Delta}\, GF$$

where w, v, and t^w are metavariables for arbitrary world-labels (l-terms), and y^v is an l-variable satisfying the variable condition for that rule.

The nonlogical rules for propositional correspondence results are given in Table 3.4 and those for transitional ones are given in Table 3.5. Note that the rules $D^{\mathcal{R}}$ and $2^{\mathcal{R}}$ are redundant since they are limit case of $D^{\mathcal{T}}$ and $2^{\mathcal{T}}$, respectively. We stress that in this way we can determine the indexed extensions, with or without identity, of any PML defined by means of Lemmon-Scott formulas, and their combinations with NI, ND, CBF, BF, GF, and $SHRT$.[6]

3.2.5 Rules for t-Rigidity, Stability, and Domains

In Chaps. 2.5 and 2.6 we have introduced restrictions that we can impose on the denotation of closed terms and the inner domains of existence, respect-

[5] Which become $\star$-derivable, and thus may be omitted, whenever the rule $T^{\mathcal{R}}$ is present, and which, by a result in [HN11], is admissible in the system without it, and therefore may be omitted independently of $T^{\mathcal{R}}$.

[6] By a further generalization [Neg14] of the method of axioms-as-rules we can add express generalized geometric implications by means of systems of nonlogical rules. With systems of nonlogical rules we could define a labelled sequent calculus for each indexed extensions of all PMLs defined by Shalqvist formulas.

Table 3.4: Propositional rules

$$\frac{w\mathscr{R}w, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\, T^{\mathcal{R}} \qquad \frac{t^w \mathscr{T} t^w, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\, T^{\mathcal{T}}$$

$$\frac{w\mathscr{R}u, w\mathscr{R}v, v\mathscr{R}u, \Gamma \Rightarrow \Delta}{w\mathscr{R}v, v\mathscr{R}u, \Gamma \Rightarrow \Delta}\, 4^{\mathcal{R}} \qquad \frac{t^w \mathcal{T} r^u, t^w \mathscr{T} s^v, s^v \mathscr{T} r^u, \Gamma \Rightarrow \Delta}{t^w \mathscr{T} s^v, s^v \mathscr{T} r^u, \Gamma \Rightarrow \Delta}\, 4^{\mathcal{T}}$$

$$\frac{v\mathscr{R}u, w\mathscr{R}v, w\mathscr{R}u, \Gamma \Rightarrow \Delta}{w\mathscr{R}v, w\mathscr{R}u, \Gamma \Rightarrow \Delta}\, 5^{\mathcal{R}} \qquad \frac{s^v \mathcal{T} r^u, t^w \mathscr{T} s^v, t^w \mathscr{T} r^u, \Gamma \Rightarrow \Delta}{t^w \mathscr{T} s^v, t^w \mathscr{T} r^u, \Gamma \Rightarrow \Delta}\, 5^{\mathcal{T}}$$

$$\frac{v\mathscr{R}w, w\mathscr{R}v, \Gamma \Rightarrow \Delta}{w\mathscr{R}v, \Gamma \Rightarrow \Delta}\, B^{\mathcal{R}} \qquad \frac{s^v \mathscr{T} t^w, t^w \mathscr{T} s^v, \Gamma \Rightarrow \Delta}{t^w \mathcal{T} s^v, \Gamma \Rightarrow \Delta}\, B^{\mathcal{T}}$$

$$\frac{w\mathscr{R}v, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\, D^{\mathcal{R}}\ (v \text{ eig.}) \qquad \frac{w\mathscr{R}v, t^{\vec{w}} \mathscr{T} x^{\vec{v}}, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\, D^{\mathcal{T}}\ (v,\ x^{\vec{v}} \text{ eig.})$$

$$\frac{v\mathscr{R}w', u\mathscr{R}w', w\mathscr{R}v, w\mathscr{R}u, \Gamma \Rightarrow \Delta}{w\mathscr{R}v, w\mathscr{R}u, \Gamma \Rightarrow \Delta}\, 2^{\mathcal{R}}\ (w' \text{ eigenvariable})$$

$$\frac{v\mathscr{R}w', u\mathscr{R}w', s^{\vec{v}} \mathscr{T} x^{\vec{w}'}, r^{\vec{u}} \mathscr{T} x^{\vec{w}'}, w\mathscr{R}v, w\mathscr{R}u, t^{\vec{w}} \mathscr{T} s^{\vec{v}}, t^{\vec{w}} \mathscr{T} r^{\vec{u}}, \Gamma \Rightarrow \Delta}{w\mathscr{R}v, w\mathscr{R}u, t^{\vec{w}} \mathscr{T} s^{\vec{v}}, t^{\vec{w}} \mathscr{T} r^{\vec{u}}, \Gamma \Rightarrow \Delta}\, 2^{\mathcal{T}}\ (w',\ x^{\vec{w}'} \text{ eig.})$$

Table 3.5: Transitional rules

$$\frac{\mathcal{E}s^v, t^w \mathscr{T} s^v, \mathcal{E}t^w, \Gamma \Rightarrow \Delta}{t^w \mathscr{T} s^v, \mathcal{E}t^w, \Gamma \Rightarrow \Delta}\, CBF \qquad \frac{v : s \doteq r, t^w \mathscr{T} s^v, t^w \mathscr{T} r^v, \Gamma \Rightarrow \Delta}{t^w \mathscr{T} s^v, t^w \mathscr{T} r^v, \Gamma \Rightarrow \Delta}\, NI$$

$$\frac{w : s \doteq r, s^w \mathscr{T} t^v, r^w \mathscr{T} t^v, \Gamma \Rightarrow \Delta}{s^w \mathscr{T} t^v, r^w \mathscr{T} t^v, \Gamma \Rightarrow \Delta}\, ND \qquad \frac{x^w \mathscr{T} t^v, \mathcal{E}x^w, w\mathscr{R}v, \mathcal{E}t^v, \Gamma \Rightarrow \Delta}{w\mathscr{R}v, \mathcal{E}t^v, \Gamma \Rightarrow \Delta}\, BF$$

$$\frac{t^w \mathscr{T} x^v, \mathcal{E}x^v, w\mathscr{R}v, \mathcal{E}t^w, \Gamma \Rightarrow \Delta}{w\mathscr{R}v, \mathcal{E}t^w, \Gamma \Rightarrow \Delta}\, GF \qquad \frac{t^w \mathscr{T} x^v, w\mathscr{R}v, \Gamma \Rightarrow \Delta}{w\mathscr{R}v, \Gamma \Rightarrow \Delta}\, SHRT$$

In rules GF and $SHRT$ x^v is an eigenvariable, in rule BF x^w is an eigenvariable.

ively. By the method of axioms as rules we can capture also these semantic conditions in our labelled sequent calculi.

The definition of t-rigidity:

- $w\mathcal{R}v$ implies $I_w(c)\mathcal{T}_{(w,v)}I_v(c)$, and
- for all $a_1, \ldots, a_n \in U_w$ and all $b_1, \ldots, b_n \in U_v$, $\bigwedge_{i=1}^{n} a_i\mathcal{T}_{(w,v)}b_i$ implies $I_w(f)(a_1, \ldots, a_n)\mathcal{T}_{(w,v)}I_v(f)(b_1, \ldots, b_n)$.

becomes the following rules

$$\frac{c^w \mathscr{T} c^v, w\mathscr{R}v, \Gamma \Rightarrow \Delta}{w\mathscr{R}v, \Gamma \Rightarrow \Delta}\, Rig^c \qquad \frac{(f(\vec{t}))^w \mathscr{T} (f(\vec{s}))^v, w\mathscr{R}v, t^{\vec{w}} \mathscr{T} s^{\vec{v}}, \Gamma \Rightarrow \Delta}{w\mathscr{R}v, t^{\vec{w}} \mathscr{T} s^{\vec{v}}, \Gamma \Rightarrow \Delta}\, Rig^f$$

In the following we will treat rule Rig^c as a limit case of Rig^f by considering constant as 0-ary function symbols, and therefore we will speak simply of rule Rig and apply the appropriate one.

The definition of stability:

- $I_w(c)\mathcal{T}_{(w,v)}o$ implies $o = I_v(c)$, and
- if $t_1, \ldots, t_n \in U_w$ are closed terms and $I_w(f)(I_w(t_1, \ldots, t_n))\mathcal{T}_{(w,v)}o$, then $o = I_v(f)(I_v(t_1, \ldots, t_n))$.

becomes the following rule (defined over the language with identity)

$$\frac{v : t \doteq f, f^w \mathscr{T} t^v, \Gamma \Rightarrow \Delta}{f^w \mathscr{T} t^v, \Gamma \Rightarrow \Delta}\, Stab$$

where we have the side condition that f^w is a closed l-terms. If the language doesn't contain the identity symbol, it is not possible to express the semantic condition that terms are stable designators as a nonlogical rule.

The rule corresponding to non-empty inner domains is

$$\frac{\mathcal{E}y^w, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\, Exist$$

where y^w is an eigenvariable.

The rule for single domains t-frames is

$$\frac{\mathcal{E}t^w, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\,Class$$

Observe that our rules for the quantifiers together with the rule *Class* are equivalent to the (labelled version of) classical rules for the quantifiers of **G3c**. Thus when working in calculi with the rule *Class* we will omit *e*-formulas and work as if we have the classical rules for the quantifiers.

Given a sequent calculus **GIM.**$\star$ we will denote its extension by rule $Rig, Stab, Exist$ by means of **GIM.**$\star^r$ **GIM.**$\star^s$, respectively. If rule *Exist* *Class* is present, $\star$ is **E.**$\star$ or **C.**$\star$, respectively. These additional rules are recapitulated in Table 3.6.

Table 3.6: Additional rules

$$\frac{(f(\vec{t}))^w \mathscr{T} (f(\vec{s}))^v, w\mathscr{R}v, t^{\vec{w}} \mathscr{T} s^{\vec{v}}, \Gamma \Rightarrow \Delta}{w\mathscr{R}v, t^{\vec{w}} \mathscr{T} s^{\vec{v}}, \Gamma \Rightarrow \Delta}\,Rig \qquad \frac{\mathcal{E}y^w, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\,Exist$$

$$\frac{v : t \doteq f, f^w \mathscr{T} t^v, \Gamma \Rightarrow \Delta}{f^w \mathscr{T} t^v, \Gamma \Rightarrow \Delta}\,Stab\ (f^w \text{ closed}) \qquad \frac{\mathcal{E}t^w, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\,Class$$

3.3 Some Basic Results

We have now defined all the calculi we will be working with, and we can begin to present their properties. But first we have to introduce some definitions. By **GIM.**$\star$ we will denote any of the calculi presented so far. Now we have to introduce the notion of derivation and some related notions.

Definition 3.10. Given a sequent calculus **GIM.**$\star$, a $\star$-*derivation* of a sequent $\mathcal{S}$ is a rooted tree whose nodes are (tagged with) sequents such that

1. the root is $\mathcal{S}$, to be called *endsequent*;
2. the leaves are either *initial sequents* from Table 3.2 or instances of rule $L\bot$;

3. each non-initial sequent follows from the sequent(s) immediately above by one of the rules of inference of **GIM.**$\star$. ■

A $\star$-derivation of a sequent $\mathcal{S}$ will be denoted as $\star \vdash \mathcal{S}$. Given $\star \vdash \mathcal{S}$ its *height* n is the length of the longest branch of its $\star$-derivation tree. We will use δ as a metavariable for $\star$-derivations, and δ_{n1} and δ_{n2} for the left and right sub-derivations of a given $\star$-derivation δ_n. We write $\star \vdash_n \mathcal{S}$ to say that $\mathcal{S}$ has a $\star$-derivation whose height is at most n.

In the following sections we will prove that some additional rules of inference are either derivable, or admissible or height-preserving admissible in **GIM.**$\star$, where these notions are defined as usual, e.g. a rule is (height-preserving) admissible whenever the existence of a $\star$-derivation of its premiss(es) entails the existence of a $\star$-derivation (with equal or lower derivation height) of its conclusion. We will begin by proving some useful preliminary results, and then we will consider the structural rules of weakening, contraction and cut.

3.3.1 Arbitrary Initial Sequents

Initial sequents contain l-atoms, and not arbitrary ext-formulas, as principal formulas. The restriction to atoms is needed to have all rules (of any **GIM.**$\star$) height-preserving invertibles —see Lemma 3.19, and to have the height-preserving admissiblity of contraction —see Lemma 3.20. Having taken initial sequents composed ofh only atomic l-formulas as prinicipal is not a limitation since it can be shown that initial sequents with arbitrary l-formulas are derivable in any **GIM.**$\star$ —see Lemma 3.11.

Observe that we could have introduced also initial sequents with r-, t-, or e-formulas as principal without impairing the height-preserving invertibility of all rules nor the height-preserving admissiblity of contraction or the admissiblity of cut (since these ext-formulas are always atomic). In [Neg05] the labelled sequent calculi for propositional modal logics have also initial sequents with r-formulas principal, however

> no rule removes an atom of the form $w\mathscr{R}v$ from the right-hand side of sequents, and such atoms are never active in the logical

> rules. Moreover, the modal axioms corresponding to the properties of the accessibility relation are derived from their rule presentations alone. As a consequence, initial sequents of the form $w\mathscr{R}v, \Gamma \Rightarrow \Delta, w\mathscr{R}v$ are needed only for deriving properties of the accessibility relation, namely, the axioms corresponding to the rules for $\mathscr{R}$. Thus such initial sequents can as well be left out from the calculus without impairing completeness of the system. [Neg05, p. 513]

The same is true for atoms of the forms $t^w \mathscr{T} s^v$ and $\mathcal{E}t^w$. For the sake of simplicity, we have preferred to left out such initial sequents from our calculi, as we are interested in deriving indexed modal formulas, and not in deriving properties of t-frames. We stress that their addition would not impair any result that we will present in this chapter (with the only exception of cases (2)-(4) of Lemma 3.12).

Observe that, even if only l-formulas can be active (or principal) in the right-hand side of the sequents of a $\star$-derivation δ, we could nonetheless find ext-formulas of the other sorts in it: they can be part of the right context of some leaf of δ. To avoid useless complications with the proofs of admissiblity of the structural rules, we prove that these ext-formulas, as well as $w : \bot$, can be eliminated form the right-hand sides without any effect, see Lemma 3.12. This will allow us to assume that no ext-formula of such shapes occurs in the succedents of the nodes of a $\star$-derivation.

Lemma 3.11. *Sequents of the form $w : A, \Gamma \Rightarrow \Delta, w : A$, with $w : A$ arbitrary l-formulas, are derivable in any* **GIM.**$\star$.

Proof. By induction on the height of $w : A$. If A is atomic the lemma holds trivially. If A is $\bot$, the lemma holds by rule $\bot \Rightarrow$. If A is $\circ B$, where $\circ \in \{\forall x, \exists x, |\frac{\vec{t}}{\vec{x}}|, \langle\frac{\vec{t}}{\vec{x}}\rangle\}$, or $B \circ C$, where $\circ \in \{\wedge, \vee, \rightarrow\}$, we apply, root-first, the rules $L\circ$ and $R\circ$ in some order, and then the lemma follows by the inductive hypothesis (IH). We show the case $A = |\frac{\vec{t}}{\vec{x}}|B$.

$$\dfrac{\dfrac{\overline{v : B, w\mathscr{R}v, \vec{t^w}\mathscr{T}\vec{x^v}, w : |^{\vec{t}}_{\vec{x}}|B, \Gamma \Rightarrow \Delta, v : B}\ IH}{w\mathscr{R}v, \vec{t^w}\mathscr{T}\vec{x^v}, w : |^{\vec{t}}_{\vec{x}}|B, \Gamma \Rightarrow \Delta, v : B}\ L|^{\vec{t}}_{\vec{x}}|}{w : |^{\vec{t}}_{\vec{x}}|B, \Gamma \Rightarrow \Delta, w : |^{\vec{t}}_{\vec{x}}|B}\ R|^{\vec{t}}_{\vec{x}}|$$

■

Lemma 3.12. *The rule of removal of $w : \bot$, of r-formulas, of t-formulas and of e-formulas from the succedent of a sequent is height-preserving admissible in any* **GIM.**$\star$*— i.e.*

(1)	$\star \vdash_n \Gamma \Rightarrow \Delta, w : \bot$	*implies*	$\star \vdash_n \Gamma \Rightarrow \Delta$
(2)	$\star \vdash_n \Gamma \Rightarrow \Delta, w\mathscr{R}v$	*implies*	$\star \vdash_n \Gamma \Rightarrow \Delta$
(3)	$\star \vdash_n \Gamma \Rightarrow \Delta, t^w\mathscr{T}s^v$	*implies*	$\star \vdash_n \Gamma \Rightarrow \Delta$
(4)	$\star \vdash_n \Gamma \Rightarrow \Delta, \mathcal{E}t^w$	*implies*	$\star \vdash_n \Gamma \Rightarrow \Delta$

Proof. The proofs are all by induction on the derivation-height $h(\delta)$ of the premiss. Let E stand for any ext-fomulas of one of those shapes, if $h(\delta) = 0$, then $\Gamma, \Rightarrow \Delta, E$ is either $v : P, \Gamma' \Rightarrow \Delta', E, v : P$ or $v : \bot, \Gamma' \Rightarrow \Delta, E$. After having removed E, we have an initial sequent or a conclusion of $L\bot$, respectively.

If $h(\delta) = n + 1$ then δ is either

$$\dfrac{\begin{matrix}\vdots\ \delta_1\\ \Gamma' \Rightarrow E, \Delta'\end{matrix}}{\Gamma \Rightarrow E, \Delta}\ \dagger \qquad \text{or} \qquad \dfrac{\begin{matrix}\vdots\ \delta_1\\ \Gamma' \Rightarrow E, \Delta'\end{matrix} \qquad \begin{matrix}\vdots\ \delta_2\\ \Gamma'' \Rightarrow E, \Delta''\end{matrix}}{\Gamma \Rightarrow E, \Delta}\ \dagger$$

depending on whether rule † has one or two premiss(es). In both cases, by applying the inductive hypothesis to the derivation(s) of the premiss(es), and then rule †, we obtain a $\star$-derivation of $\Gamma \Rightarrow \Delta$ of the same derivation-height of δ. Observe that for all the possible E as in the statement of the lemma, the rule † remains applicable after having removed E from the premiss(es) because E is never principal nor active when it occurs in the succedent. ■

Given that negation has not been taken as a primitive logical symbol, we have no primitive rules for it, but the usual rules for negation, i.e.

$$\dfrac{\Gamma \Rightarrow \Delta, w : A}{w : \neg A, \Gamma \Rightarrow \Delta}\ L\neg \qquad \dfrac{w : A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, w : \neg A}\ R\neg$$

are easily shown to be derivable in any **GIM.**$\star$ (for the right rule we have to weaken the right-hand side of the sequent with $w : \bot$ and then apply $R \rightarrow$). Thus, whenever convenient, we will make use of the rules for negation.

3.3.2 Substitutions

Definition 3.13. We say that a $\star$-derivation respects

- The *variable convention* if each application of a rule with an eigenvariable has a different eigenvariable.
- The *pure-variable convention* if the free and bound variables occurring in it are kept disjoint.

We shall assume, whenever convenient, that each $\star$-derivation respects the variable and the pure-variable convention, thanks to the following lemmas.

Lemma 3.14. *Let $w : \mathcal{Q}xA$, with $\mathcal{Q} \in \{\forall, \exists\}$, be an l-formula occurring in $\Gamma \Rightarrow \Delta$, possibly as subformula of some l-formula $w : B$. If $\star \vdash_n \Gamma \Rightarrow \Delta$, then $\star \vdash_n \Gamma^r \Rightarrow \Delta^r$, where Γ^r (Δ^r) differs from Γ (Δ) in that every occurrence of $w : \mathcal{Q}xA$ has been replaced by an occurrence of $w : \mathcal{Q}z(A[z/x])$, where z^w is some fresh l-variable.*

Proof. The proof is analogous to the proof of Lemma 4.1.1 of [NP01] (observe that the Lemma has no effect on r-, t- and e-formulas). Let δ be a $\star$-derivation of $\Gamma \Rightarrow \Delta$. We show, by induction on the height n of δ, $h(\delta)$, that there is a $\star$-derivation δ^r, having same derivation height of δ, of the sequent $\Gamma^r \Rightarrow \Delta^r$.

If $h(\delta) = 0$, then $\Gamma \Rightarrow \Delta$ is an initial sequent or an instance of $L\bot$. The sequent $\Gamma^r \Rightarrow \Delta^r$ is an initial sequent or an instance of $L\bot$ because the replacement can alter only the contexts of $\Gamma \Rightarrow \Delta$.

If $h(\delta) = n + 1$, we consider the last rule applied in δ. If it is a propositional, a modal, or a nonlogical rule, we apply the inductive hypothesis to its premiss(es), and then the last rule to transform δ into a $\star$-derivation δ^r, of the same height of δ, of $\Gamma^r \Rightarrow \Delta^r$. Note that δ^r respects all variable conditions occurring in δ.

If the last rule applied is $L\forall$, then, if $w : \forall xA$ is not principal, we apply the inductive hypothesis and then the rule. Else Γ is $w : \forall xA, \mathcal{E}t^w, \Gamma'$, and δ is

$$\dfrac{\begin{array}{c}\vdots\, \delta_1\\ w : A[t/x], w : \forall xA, \mathcal{E}t^w, \Gamma' \Rightarrow \Delta\end{array}}{w : \forall xA, \mathcal{E}t^w, \Gamma' \Rightarrow \Delta}\, L\forall$$

By applying the inductive hypothesis to δ_1, we obtain a $\star$-derivation δ_1^r, having same derivation-height of δ_1, of the sequent

$$w : A[t/x], w : \forall z(A[z/x]), \mathcal{E}t^w, \Gamma'^r \Rightarrow \Delta^r\ ,$$

where z^w is a fresh l-variable. Note that the inductive hypothesis has no effect on $w : A$, nor on its substitution instances, because it is a proper subformula of $w : \forall xA$. By Lemma 3.8.6, we can rewrite $w : A[t/x]$ as $w : (A[z/x])[t/z]$, and then we can apply the rule $L\forall$ to obtain

$$\dfrac{\begin{array}{c}\vdots\, \delta_1^r\\ w : (A[z/x])[t/z], w : \forall z(A[z/x]), \mathcal{E}t^w, \Gamma'^r \Rightarrow \Delta^r\end{array}}{w : \forall z(A[z/x]), \mathcal{E}t^w, \Gamma'^r \Rightarrow \Delta^r}\, L\forall$$

that is a $\star$-derivation δ^r, having same derivation-height of δ, of the sequent $\Gamma^r \Rightarrow \Delta^r$.

If the last rule is $R\forall$, then if $w : \forall xA$ is not principal in it, we apply the inductive hypothesis and then the rule, taking care to avoid problems with the variable condition of that rule. Else Δ is $\Delta', w : \forall xA$, and δ is

$$\dfrac{\begin{array}{c}\vdots\, \delta_1\\ \mathcal{E}y^w, \Gamma \Rightarrow \Delta', w : A[y/x]\end{array}}{\Gamma \Rightarrow \Delta', w : \forall xA}\, R\forall$$

with y^w respecting the variable condition. By inductive hypothesis, there is a $\star$-derivation δ_1^r —having same derivation height of δ_1— of the sequent

$$\mathcal{E}y^w, \Gamma^r \Rightarrow \Delta'^r, w : A[y/x]$$

By Lemma 3.8.6 we can rewrite $w : A[y/x]$ as $w : (A[z/x])[y/z]$ for some fresh z^w, and then we can apply the rule $R\forall$ to obtain

$$\dfrac{\begin{array}{c}\vdots\ \delta_1^r\\ \mathcal{E}y^w, \Gamma^r \Rightarrow \Delta'^r, w : (A[z/x])[y/z]\end{array}}{\Gamma^r \Rightarrow \Delta'^r, w : \forall z(A[z/x])}\ R\forall$$

Note that y^w respects the variable condition.

If the last rule applied is $L\exists$ (resp. $R\exists$), we proceed as for $R\forall$ ($L\forall$). ■

Remark 3.15. Given a $\star$-derivation δ not satisfying the pure-variable convention, we can transform it, thanks to Lemma 3.14, in a $\star$-derivation δ^r of a sequent $\Gamma^r \Rightarrow \Delta^r$ that satisfies the pure variable convention.[7] Henceforth we shall assume that all $\star$-derivations satisfy the pure variable convention, and we shall identify $\star$-derivations modulo any renaming of bound variables that is obtainable by Lemma 3.14. Observe that this identification is motivated both syntactically, given that we don't change any scheme in so doing, and semantically, given that we transform formulas into i-congruent ones (see Definition 1.10) that are co-satisfiable as shown by Theorem 2.12. This identification allows us to make the simplifying assumption that, whenever applying the substitution $[t^w/x^w]$ to some sequent $\Gamma \Rightarrow \Delta$, t^w is free for x^w in all ext-formulas occurring in $\Gamma \Rightarrow \Delta$, .

Lemma 3.16. *The rule of substitution of l-terms for l-variables is height-preserving admissible —i.e.*

$$\textit{if } \star \vdash_n \Gamma \Rightarrow \Delta, \ \textit{then } \star \vdash_n \Gamma[t^w/x^w] \Rightarrow \Delta[t^w/x^w]$$

[7] We can also transform it in a $\star$-derivation such that, in each formula, each quantifier binds a different variable. Thus lemma 3.14 allows us to imitate Gentzen's language where (i) bound variables are distinct form free ones, and (ii) in any formula different quantifiers bind different variables.

Proof. If $h(\delta) = 0$, then $\Gamma \Rightarrow \Delta$ is an instance of an initial sequent or a conclusion of $L\bot$, therefore also $\Gamma[t^w/x^w] \Rightarrow \Delta[t^w/x^w]$ is an instance of an initial sequent or a conclusion of $L\bot$.

If $h(\delta) = n+1$, we consider the last rule applied in δ. Observe that, for all rules, if the substitution $[t^w/x^w]$ is vacuous on the conclusion of δ, then there is nothing to prove; thus we will consider only the cases with a non-vacuous substitution.

If it is a propositional rule or a nonlogical rule with r-atom(s) principal, we apply the inductive hypothesis, and then the rule to obtain a $\star$-derivation δ^s, with $h(\delta^s) = n+1$, of $\Gamma[s^v/x^v] \Rightarrow \Delta[s^v/x^v]$.

If the last rule applied is $L\forall$, then δ is

$$\dfrac{\begin{array}{c}\vdots\,\delta_1\\ v : A[s/y], v : \forall y A, \mathcal{E}s^v, \Gamma' \Rightarrow \Delta\end{array}}{v : \forall y A, \mathcal{E}s^v, \Gamma' \Rightarrow \Delta}\, L\forall$$

We proceed by cases. If $w \neq v$, or if $w = v$ and $x^w = y^v$, then we simply have to apply the inductive hypothesis to δ_1, and then $L\forall$. Else $w = v$ and $x^w \neq y^v$, and, by assumption, t^v is free for x^v and s^v is free for y^v. By IH, we transform δ_1 into δ_1^s, of same derivation-height,

$$w : (A[s/y])[t/x], w : (\forall y A)[t/x], (\mathcal{E}s^w)[t^w/x^w], \Gamma'[t^w/x^w] \Rightarrow \Delta[t^w/x^w]$$

where $(\mathcal{E}s^w)[t^w/x^w]$ can be rewritten as $\mathcal{E}(s[t/x])^w$; $w : (\forall y A)[t/x]$ can be rewritten as $w : \forall y(A[t/x])$; and, by Lemma 3.8.4, $w : (A[s/y])[t/x]$ can be rewritten as $w : (A[t/x])[s[t/x]/y]$. Now, by applying $L\forall$, we get

$$\dfrac{\begin{array}{c}\vdots\,\delta_1^s\\ w : (A[t/x])[s[t/x]/y], w : \forall y(A[t/x]), \mathcal{E}(s[t/x])^w, \Gamma'[t^w/x^w] \Rightarrow \Delta[t^w/x^w]\end{array}}{w : \forall y(A[t/x]), \mathcal{E}(s[t/x])^w, \Gamma'[t^w/x^w] \Rightarrow \Delta[t^w/x^w]}\, L\forall$$

that is, a $\star$-derivation of height $n+1$ of $\Gamma[t^w/x^w] \Rightarrow \Delta[t^w/x^w]$.

If the last rule is $R\forall$, δ is

$$\dfrac{\begin{array}{c}\vdots\ \delta_1\\ \mathcal{E}z^v, \Gamma \Rightarrow \Delta', v : A[z/y]\end{array}}{\Gamma \Rightarrow \Delta', v : \forall y A}\ R\forall$$

where z^v respects the variable condition. Many different cases are possible (we don't consider the case $z^v = x^w$ as it would make the substitution vacuous on the conclusion). If $w \neq v$, or if $w = v$ and $x^w = y^v$, then we simply have to apply the inductive hypothesis to δ_1, and then $L\forall$. The interesting case is when $w = v$ and $x^w \neq y^v$. Observe that, by applying directly the substitution $[t^w/x^w]$ to δ_1, we may then be unable to apply $R\forall$ since, if $z^w \in VAR(t^w)$, it may happen that the variable condition of the rule is not satisfied anymore. To avoid this problem we begin by applying the inductive hypothesis to δ_1 in order to replace z^w with some fresh variable z_1^w. By the variable condition on z^w, this is done without impairing any variable condition in δ and without changing Γ nor Δ'. We have thus transformed δ_1 into δ_1', of same derivation-height, whose conclusion is

$$\mathcal{E}z_1^w, \Gamma \Rightarrow \Delta', w : A[z_1/y]$$

By applying again the inductive hypothesis, we transform δ_1' into a $\star$-derivation $\delta_1^{\text{`}s}$, of same derivation-height of δ_1, of

$$\mathcal{E}z_1^w, \Gamma[t^w/x^w] \Rightarrow \Delta'[t^w/x^w], w : (A[z_1/y])[t/x]$$

We know that $y^w \neq x^w$, $y^w \notin VAR(t^w)$ (by assumption t^w is free for x^w in $w : \forall y A$), $y^w \neq z_1^w$, and z_1^w, t^w are free respectively for x^w, y^w in $w : A$, thus, by Lemma 3.8.4, $w : (A[z_1/y])[t/x]$ can be rewritten as $w : (A[t/x])[z_1/y]$. Now we can apply rule $R\forall$ to obtain

$$\dfrac{\begin{array}{c}\vdots\ \delta_1^{\text{`}s}\\ \mathcal{E}z_1^w, \Gamma[t^w/x^w] \Rightarrow \Delta'[t^w/x^w], w : (A[t/x])[z_1/y]\end{array}}{\Gamma[t^w/x^w] \Rightarrow \Delta'[t^w/x^w], w : \forall y(A[t/x])}\ R\forall$$

that is, a $\star$-derivation of height $n+1$ of $\Gamma[t^w/x^w] \Rightarrow \Delta[t^w/x^w]$.

The cases of $L\exists$ and $R\exists$ are analogous to those of $R\forall$ and $L\forall$, respectively. If the last rule is $L|^{\vec{t}}_{\vec{x}}|$, then Γ is $v : |^{\vec{t}}_{\vec{x}}|A, v\mathscr{R}u, \vec{s^v}\mathscr{T}\vec{y^u}, \Gamma'$, and δ is

$$\dfrac{\begin{array}{c}\vdots\ \delta_1\\ u : A[\vec{r}/\vec{y}], v : |^{\vec{s}}_{\vec{y}}|A, v\mathscr{R}u, \vec{s^v}\mathscr{T}\vec{r^u}, \Gamma' \Rightarrow \Delta\end{array}}{v : |^{\vec{s}}_{\vec{y}}|A, v\mathscr{R}u, \vec{s^v}\mathscr{T}\vec{y^u}, \Gamma' \Rightarrow \Delta}\, L|^{\vec{t}}_{\vec{x}}|$$

By applying the inductive hypothesis to the premiss, we obtain a $\star$-derivation $\delta_1^\star$, of same height of δ_1, of the sequent

$$(u : A[\vec{r}/\vec{x}])[t^w/x^w], (v : |^{\vec{s}}_{\vec{y}}|A)[t^w/x^w], v\mathscr{R}u, (\vec{s^v}\mathscr{T}\vec{r^u})[t^w/x^w], \Gamma'[t^w/x^w] \Rightarrow \Delta[t^w/x^w]$$

that we will call $\mathcal{S}$. Four cases are possible according to whether $w = v$ and/or $w = u$, or not.

1. If $w \neq v$ and $w \neq u$, $\mathcal{S}$ is

$$u : A[\vec{r}/\vec{y}], v : |^{\vec{s}}_{\vec{y}}|A, v\mathscr{R}u, \vec{s^v}\mathscr{T}\vec{r^u}, \Gamma'[t^w/x^w] \Rightarrow \Delta[t^w/x^w]$$

   and we can apply rule $L|^{\vec{t}}_{\vec{x}}|$ to conclude

$$u : |^{\vec{s}}_{\vec{y}}|A, v\mathscr{R}u, \vec{s^v}\mathscr{T}\vec{r^u}, \Gamma'[t^w/x^w] \Rightarrow \Delta[t^w/x^w]$$

2. If $w = v$ and $w \neq u$, $\mathcal{S}$ is

$$u : A[\vec{r}/\vec{x}], w : |^{\vec{s}[t^w/x^w]}_{\vec{y}}|A, w\mathscr{R}u, (\vec{s^w}[t^w/x^w])\mathscr{T}\vec{r^u}, \Gamma'[t^w/x^w] \Rightarrow \Delta[t^w/x^w]$$

   and we can apply rule $L|^{\vec{t}}_{\vec{x}}|$ to conclude

$$w : |^{\vec{s}[t^w/x^w]}_{\vec{y}}|A, w\mathscr{R}u, (\vec{s^w}[t^w/x^w])\mathscr{T}\vec{r^u}, \Gamma'[t^w/x^w] \Rightarrow \Delta[t^w/x^w]$$

3. If $w \neq v$ and $w = u$, $\mathcal{S}$ is

$$w : (A[\vec{r}/\vec{y}])[t/x], v : |^{\vec{s}}_{\vec{y}}|A, v\mathscr{R}u, \vec{s^v}\mathscr{T}\overrightarrow{(r^w[t^w/x^w])}, \Gamma'[t^w/x^w] \Rightarrow \Delta[t^w/x^w]$$

   We rewrite $w : (A[\vec{r}/\vec{x}])[t/x]$ as $w : A[\vec{r}[t/x]/\vec{x}]$ (this is feasible because

$FV(A) \subseteq \vec{y}$), and then we can apply rule $L|\frac{\vec{t}}{\vec{x}}|$ to conclude

$$v : |\tfrac{\vec{s}}{\vec{y}}|A, v\mathscr{R}u, \vec{s^v}\mathscr{T}\vec{(r^w}[t^w/x^w]), \Gamma'[t^w/x^w] \Rightarrow \Delta[t^w/x^w]$$

4. If $w = v$ and $w = u$, $\mathcal{S}$ is

$$w : (A[\vec{r}/\vec{y}])[t/x], w : |\tfrac{\vec{s}[t^w/x^w]}{\vec{y}}|A, w\mathscr{R}w, (\vec{s^w}[t^w/x^w])\mathscr{T}(\vec{r^w}[t^w/x^w]), \Gamma'[t^w/x^w] \Rightarrow \Delta[t^w/x^w]$$

As in case 3, we rewrite $w : (A[\vec{r}/\vec{x}])[t/x]$ as $w : A[\vec{r}[s/x]/\vec{x}]$, and then we can apply rule $L|\frac{\vec{t}}{\vec{x}}|$ to conclude

$$w : |\tfrac{\vec{s}[t^w/x^w]}{\vec{y}}|A, w\mathscr{R}w, (\vec{s^w}[t^w/x^w])\mathscr{T}(\vec{r^w}[t^w/x^w]), \Gamma'[t^w/x^w] \Rightarrow \Delta[t^w/x^w]$$

In all four cases we have concluded that $\star \vdash_{n+1} \Gamma[t^w/x^w] \Rightarrow \Delta[t^w/x^w]$.

If the last rule applied is $R|\frac{\vec{t}}{\vec{x}}|$, then Δ is $\Delta', v : |\frac{\vec{s}}{\vec{y}}|A$ and δ is

$$\dfrac{\begin{array}{c}\vdots\ \delta_1\\ v\mathscr{R}u, \vec{s^v}\mathscr{T}\vec{x^u}, \Gamma \Rightarrow \Delta', A : u\end{array}}{\Gamma \Rightarrow \Delta', v : |\frac{\vec{s}}{\vec{y}}|A}\ R|\tfrac{\vec{t}}{\vec{x}}|$$

with u and $\vec{x^u}$ respecting the variable condition. The cases we have to consider are the first two of $L|\frac{\vec{t}}{\vec{x}}|$, since the third is excluded as it makes the substitution ineffective on the conclusion, and the fourth is impossible given that u respects the variable condition. In the first case we have only to apply the inductive hypothesis to the δ_1, and then $R|\frac{\vec{t}}{\vec{x}}|$. In the second case, when $w = v$ and $u \neq v$, we apply the inductive hypothesis to δ_1, and then $R|\frac{\vec{t}}{\vec{x}}|$ to obtain the following $\star$ derivation of height $n + 1$

$$\dfrac{\begin{array}{c}\vdots\ \delta_1^s\\ w\mathscr{R}u, (\vec{s^w}[t^w/x^w])\mathscr{T}\vec{x^u}, \Gamma[t^w/x^w] \Rightarrow \Delta'[t^w/x^w], A : u\end{array}}{\Gamma[t^w/x^w] \Rightarrow \Delta'[t^w/x^w], w : |\frac{\vec{s}[t/w]}{\vec{y}}|A}\ R|\tfrac{\vec{t}}{\vec{x}}|$$

If the last rule is $L\langle\frac{\vec{t}}{\vec{x}}\rangle$ or $R\langle\frac{\vec{t}}{\vec{x}}\rangle$, we proceed as for $R|\frac{\vec{t}}{\vec{x}}|$ and $L|\frac{\vec{t}}{\vec{x}}|$, respectively.

If the last rule is nonlogical rule without a variable condition, we have only to apply the inductive hypothesis and then the rule. Observe that the rule *Stab* is a rule without a variable condition, but it has nevertheless a side condition, i.e. that the terms in its active formula are closed. It is obvious that the application of the inductive hypothesis to its premiss has no effect on this side condition.

If the last rule is a nonlogical rule with a variable condition, if it has a condition on x^w, we apply the inductive hypothesis to the derivation of its premiss(es) to change that eigenvariable with a fresh variable not occurring in t^w. Else we apply the inductive hypothesis to the derivation of the premiss(es) twice, the first time to change the eigenvariable(s) with fresh variable(s) not occurring in t^w, and the second to apply the substitution $[t^w/x^w]$. Finally we apply the nonlogical rule to obtain a $\star$-derivation of $\Gamma[t^w/x^w] \Rightarrow \Delta[t^w/x^w]$ of height $n+1$. ■

Lemma 3.17. *The rule of substitution of world labels is height-preserving admissible —i.e*

$$\textit{if } \star \vdash_n \Gamma \Rightarrow \Delta, \textit{ then } \star \vdash_n \Gamma[v/u] \Rightarrow \Delta[v/u]$$

Proof. The proof, by induction on the height n of the $\star$-derivation of $\Gamma \Rightarrow \Delta$, is similar to that of Lemma 3.16. In general it is simpler in that world labels cannot be bound. The only novelty arises in the cases of quantifier, or nonlogical, rules with eigenvariable y^v, where we have the additional complication that the substitution may clash with the variable condition on y^v. In this case we have to apply Lemma 3.16 —in order to substitute y^v with a fresh l-variable z^v— before applying the inductive hypothesis.

If $h(\delta) = 0$, then $\Gamma \Rightarrow \Delta$ is either an axiom or an instance of $L\bot$, and $\Gamma[v/u] \Rightarrow \Delta[v/u]$ is an instance of an axiom or a conclusion of rule $L\bot$, respectively.

If $h(\delta) = n+1$, we consider the last rule applied. If it is either a propositional rule, or a quantifier, modal, or nonlogical rule without eigenvariables, we apply the inductive hypothesis to the premiss(es) of the rule, and then the rule. For example, if the last rule is $L|^{\vec{t}}_{\vec{x}}|$, $w_1 = u$ and $w_2 \neq u$, then δ ends with

$$\cfrac{\begin{array}{c}\vdots\ \delta_1\\ w_2 : A[\vec{r}/\vec{x}], u : |\tfrac{\vec{t}}{\vec{x}}|A, u\mathscr{R}w_2, \vec{t}^{u}\mathscr{T}\vec{r}^{\vec{w}_2}, \Gamma' \Rightarrow \Delta\end{array}}{u : |\tfrac{\vec{t}}{\vec{x}}|A, u\mathscr{R}w_2, \vec{t}^{u}\mathscr{T}\vec{r}^{\vec{w}_2}, \Gamma' \Rightarrow \Delta}\ L|\tfrac{\vec{t}}{\vec{x}}|$$

We apply the inductive hypothesis to the shorter $\star$-derivation of the premiss, and the rule $L|\frac{\vec{t}}{\vec{x}}|$ in order to obtain the following $\star$-derivation of height $n+1$:

$$\cfrac{\begin{array}{c}\vdots\ \delta_1\\ w_2 : A[\vec{r}/\vec{x}], v : |\tfrac{\vec{t}}{\vec{x}}|A, v\mathscr{R}w_2, \vec{t}^{v}\mathscr{T}\vec{r}^{\vec{w}_2}, \Gamma'[v/u] \Rightarrow \Delta[v/u]\end{array}}{v : |\tfrac{\vec{t}}{\vec{x}}|A, v\mathscr{R}w_2, \vec{t}^{v}\mathscr{T}\vec{r}^{\vec{w}_2}, \Gamma'[v/u] \Rightarrow \Delta[v/u]}\ L|\tfrac{\vec{t}}{\vec{x}}|$$

If the last rule is a quantifier rule with variable condition, say $L\exists$, we have

$$\cfrac{\begin{array}{c}\vdots\ \delta_1\\ \mathcal{E}y^{w}, w : A[y/x], \Gamma' \Rightarrow \Delta\end{array}}{w : \exists xA, \Gamma' \Rightarrow \Delta}\ L\exists$$

with y^w respecting the variable condition. We have two cases according to whether $u = w$ or not. If $u \neq w$, we apply the inductive hypothesis to the premiss, and then the rule. If $u = w$, then we use Lemma 3.16 to apply the height-preserving admissible substitution $[z^u/y^u]$, for some z^u such that both it and z^v are fresh w.r.t. δ_1. Then we apply the inductive hypothesis, and, finally, rule $L\exists$ in order to obtain the following $\star$-derivation (of height $n+1$)

$$\cfrac{\begin{array}{c}\vdots\ \delta_1\\ \mathcal{E}y^{v}, v : A[y/x], \Gamma'[v/u] \Rightarrow \Delta[v/u]\end{array}}{v : \exists xA, \Gamma'[v/u] \Rightarrow \Delta[v/u]}\ L\exists$$

If the last rule is a modal rule with variable condition, say $L\langle\frac{\vec{t}}{\vec{x}}\rangle$ with $\Gamma = w_1 : \langle\frac{\vec{t}}{\vec{x}}\rangle A, \Gamma'$, then δ is

$$\cfrac{\begin{array}{c}\vdots\ \delta_1\\ w_2 : A, w_1\mathscr{R}w_2, \vec{t}^{w_1}\mathscr{T}\vec{x}^{\vec{w}_2}, \Gamma' \Rightarrow \Delta\end{array}}{w_1 : \langle\tfrac{\vec{t}}{\vec{x}}\rangle A, \Gamma' \Rightarrow \Delta}\ R\langle\tfrac{\vec{t}}{\vec{x}}\rangle$$

with w_2 —as well as $x^{\vec{w}_2}$— respecting the variable condition. We have three cases according to whether (i) $u = w_1$, or (ii) $u = w_2$, or (iii) $u \neq w_1$ and $u \neq w_2$. In cases (ii) and (iii) there is nothing to prove as the substitution $[v/u]$ is vacuous on $\Gamma \Rightarrow \Delta$, therefore we consider only case (i), which has two subcases according to whether $v = w_2$ or not. If $v = w_2$ we apply the inductive hypothesis to the premiss twice, the first time to replace the eigenvariable w_2 with a fresh world label w_3, thus obtaining

$$\star \vdash_n w_3 : A, u\mathscr{R}w_3, \vec{t^u}\mathscr{T}x^{\vec{w}_3}, \Gamma' \Rightarrow \Delta$$

the second time to apply the substitution $[v/u]$, thus obtaining

$$\star \vdash_n w_3 : A, v\mathcal{R}w_3, \vec{t^v}\mathscr{T}x^{\vec{w}_3}, \Gamma'[v/u] \Rightarrow \Delta[v/u]$$

Given that w_3 doesn't occur in $v : \langle^{\vec{t}}_{\vec{x}}\rangle A, \Gamma'[v/u] \Rightarrow \Delta[v/u]$ and that $v : \langle^{\vec{t}}_{\vec{x}}\rangle A$ can be rewritten as $(u : \langle^{\vec{t}}_{\vec{x}}\rangle A)[v/u]$, we can apply the rule $L\langle^{\vec{t}}_{\vec{x}}\rangle$ to conclude

$$\star \vdash_{n+1} (u : \langle^{\vec{t}}_{\vec{x}}\rangle A)[v/u], \Gamma'[v/u] \Rightarrow \Delta[v/u]$$

If $v \neq w_2$, we proceed as in before, save that we don't have to replace the eigenvarialbe w_2.

If the last rule is a nonlogical one with an eigenvariable of kind world, let's say δ is

$$\dfrac{\begin{array}{c}\vdots\, \delta_1 \\ w_1\mathscr{R}w_2\, \Gamma' \Rightarrow \Delta\end{array}}{\Gamma \Rightarrow \Delta}\, D^{\mathcal{R}}$$

with w_2 respecting the variable condition, then we have the same three cases of rule $L\langle^{\vec{t}}_{\vec{x}}\rangle$.[8] We consider only the case that $u = w_1$, $u \neq w_2$ and $v = w_2$, where we apply the inductive hypothesis twice, the first time to replace the premiss with $w_1\mathscr{R}w_3, \Gamma \Rightarrow \Delta$ for some fresh world label w_3, and the second to apply the substitution $[v/u]$. Now we can apply the rule D_1 to obtain

[8] Strictly speaking there is also a fourth case since it may happen that $w_1 = w_2$, but it doesn't introduce any novelty in the proof.

$$\dfrac{\dfrac{\vdots\ \delta_1^s}{v\mathscr{R}w_3, \Gamma[v/u] \Rightarrow \Delta[v/u]}}{\Gamma[v/u] \Rightarrow \Delta[v/u]}\,D$$

If the last rule is a nonlogical one with an eigenvariable of kind individual, let's say δ is

$$\dfrac{\dfrac{\vdots\ \delta_1}{t^{w_1}\mathscr{T}y^{w_2}, \mathcal{E}y^{w_2}, w_1\mathscr{R}w_2, \mathcal{E}t^{w_1}, \Gamma' \Rightarrow \Delta}}{w_1\mathscr{R}w_2, \mathcal{E}t^{w_1}, \Gamma' \Rightarrow \Delta}\,GF$$

with y^{w_2} respecting the variable condition, we have once again the same three cases of rule $L\langle^{\vec{t}}_{\vec{x}}\rangle$, and we consider only the case that $u = w_1$, $u \neq w_2$, $v = w_2$. We apply to the premiss the substitution $[z^u/y^u]$ —height-preserving admissible by Lemma 3.16— for some z^u such that both it and z^v are fresh l-variables, and then we apply the inductive hypothesis and the rule GF to obtain

$$\dfrac{\dfrac{\vdots\ \delta_1^s}{t^v\mathscr{T}z^v, v\mathscr{R}v, \mathcal{E}t^v, \Gamma'[v/u] \Rightarrow \Delta[v/u]}}{v\mathscr{R}v, \mathcal{E}t^v, \Gamma'[v/u] \Rightarrow \Delta[v/u]}\,GF$$

thhat is a $\star$-derivation of height $n+1$ of $\Gamma[v/u] \Rightarrow \Delta[v/u]$. ■

3.4 Weakening and Contraction

In this section we will show that the structural rules of weakening and contraction are height-preserving admissible in **GIM.**$\star$. Observe that we have no right rules of weakening and contraction for r-, t- and e-formulas because, by Lemma 3.12, it is not restrictive to assume they don't occur in the right-hand side of a sequent.

Theorem 3.18. *The following rules of weakening are height-preserving admissible in* **GIM.**$\star$

$$\frac{\Gamma \Rightarrow \Delta}{w : A, \Gamma \Rightarrow \Delta} LW_l \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, w : A} RW_l$$

$$\frac{\Gamma \Rightarrow \Delta}{w\mathscr{R}v, \Gamma \Rightarrow \Delta} LW_r \qquad \frac{\Gamma \Rightarrow \Delta}{t^w \mathscr{T} s^v, \Gamma \Rightarrow \Delta} LW_t \qquad \frac{\Gamma \Rightarrow \Delta}{\mathcal{E}t^w, \Gamma \Rightarrow \Delta} LW_e$$

Proof. By induction on the height of the derivation of the premiss(es) $h(\delta)$. If $h(\delta) = 0$ the lemma holds since axioms, and conclusions of the zero premiss rule $L\bot$, have arbitrary contexts Γ and Δ.

If $h(\delta) = n+1$,we consider the last rule applied. If it is a (logical or non-logical) rule without variable condition, we have only to apply the inductive hypothesis to the premiss(es), and then the lemma follows by applying the rule.

If the last rule is $R\forall$ or $L\exists$, we have first to apply Lemma 3.16 to substitute the eigevariable of that rule with a fresh variable not occurring in the weakening formula. Then we apply the inductive hypothesis to the premiss, and finally we can apply the rule as the variable condition is satisfied.

If the last rule is $R|\frac{\vec{t}}{\vec{x}}|$, $L\langle\frac{\vec{t}}{\vec{x}}\rangle$, we proceed as in the previous case, applying Lemma 3.17, instead of 3.16, to avoid problems with the variable condition.

if the last rule is a nonlogical one with a variable condition, we apply either Lemma 3.16 or 3.17, and then we apply the inductive hypothesis and the rule. ■

Lemma 3.19 (Inversion Lemma). *Each rule of* **GIM.**$\star$ *is height-preserving invertible,* i.e.

1. *If* $\star \vdash_n w : A \wedge B, \Gamma \Rightarrow \Delta$*, then* $\star \vdash_n w : A, w : B, \Gamma \Rightarrow \Delta$*;*
2. *If* $\star \vdash_n \Gamma \Rightarrow \Delta, w : A \wedge B$*, then* $\star \vdash_n \Gamma \Rightarrow \Delta, w : A$ *and* $\star \vdash_n \Gamma \Rightarrow \Delta, w : B$*;*
3. *If* $\star \vdash_n w : A \vee B, \Gamma \Rightarrow \Delta$*, then* $\star \vdash_n w : A, \Gamma \Rightarrow \Delta$ *and* $\star \vdash_n w : B, \Gamma \Rightarrow \Delta$*;*
4. *If* $\star \vdash_n \Gamma \Rightarrow \Delta, w : A \vee B$*, then* $\star \vdash_n \Gamma \Rightarrow \Delta, w : A, w : B$*;*
5. *If* $\star \vdash_n w : A \rightarrow B, \Gamma \Rightarrow \Delta$*, then* $\star \vdash_n \Gamma \Rightarrow \Delta, w : A$ *and* $\star \vdash_n w : B, \Gamma \Rightarrow \Delta$*;*
6. *If* $\star \vdash_n \Gamma \Rightarrow \Delta, w : A \rightarrow B$*, then* $\star \vdash_n w : A, \Gamma \Rightarrow \Delta, w : B$*;*

7. *If* $\star \vdash_n w : \forall xA, \mathcal{E}t^w, \Gamma \Rightarrow \Delta$, *then* $\star \vdash_n w : A[t/x], w : \forall xA, \mathcal{E}t^w, \Gamma \Rightarrow \Delta$*;*

8. *If* $\star \vdash_n \Gamma \Rightarrow \Delta, w : \forall xA$, *then, for any* y^w *not occurring in it,* $\star \vdash_n \mathcal{E}y^w, \Gamma \Rightarrow \Delta, w : A[y/x]$*;*

9. *If* $\star \vdash_n w : \exists xA, \Gamma \Rightarrow \Delta$, *then for any* y^w *not occurring in it,* $\star \vdash_n w : A[y/x], \mathcal{E}y^w, \Gamma \Rightarrow \Delta$*;*

10. *If* $\star \vdash_n \mathcal{E}t^w, \Gamma \Rightarrow \Delta, w : \exists xA$, *then* $\star \vdash_n \mathcal{E}t^w, \Gamma \Rightarrow \Delta, w : \exists xA, w : A[t/x]$*;*

11. *If* $\star \vdash_n w : |^{\vec{t}}_{\vec{x}}|A, w\mathscr{R}v, \vec{t^w}\mathscr{T}\vec{s^v}, \Gamma \Rightarrow \Delta$, *then* $\star \vdash_n v : A[\vec{s}/\vec{x}], w : |^{\vec{t}}_{\vec{x}}|A, w\mathscr{R}v, \vec{t^w}\mathscr{T}\vec{s^v}, \Gamma \Rightarrow \Delta$*;*

12. *If* $\star \vdash_n \Gamma \Rightarrow \Delta, w : |^{\vec{t}}_{\vec{x}}|A$, *then, for any* v *not occurring in it,* $\star \vdash_n w\mathscr{R}v, \vec{t^w}\mathscr{T}\vec{x^v}, \Gamma \Rightarrow \Delta, w : |^{\vec{t}}_{\vec{x}}|A, v : A$*;*

13. *If* $\star \vdash_n w : \langle^{\vec{t}}_{\vec{x}}\rangle A, \Gamma \Rightarrow \Delta$, *then, for any* v *not occurring in it,* $\star \vdash_n v : A, w\mathscr{R}v, \vec{t^w}\mathscr{T}\vec{x^v}, w : \langle^{\vec{t}}_{\vec{x}}\rangle A, \Gamma \Rightarrow \Delta$*;*

14. *If* $\star \vdash_n w\mathscr{R}v, \vec{t^w}\mathscr{T}\vec{s^v}, \Gamma \Rightarrow \Delta, w : \langle^{\vec{t}}_{\vec{x}}\rangle A$, *then* $\star \vdash_n w\mathscr{R}v, \vec{t^w}\mathscr{T}\vec{s^v}, \Gamma \Rightarrow \Delta, w : \langle^{\vec{t}}_{\vec{x}}\rangle A, v : A[\vec{s}/\vec{x}]$*;*

15. *If a sequent that can be the conclusion of an instance of a nonlogical rule in* $\star$ *has a* $\star$*-derivation of height* n*, then also any sequent that is a possible premiss of that instance of a nonlogical rule has a* $\star$*-derivation of height* n*.*

Proof. The proofs are by induction on $h(\delta)$. If $h(\delta) = 0$ then, in each case in 1-15, the sequent in the antecedent of the Lemma is an axiom or a conclusion of $L\bot$, therefore also the sequent in its consequent is an axiom or a conclusion of $L\bot$.

If $h(\delta) = n + 1$, we have to consider each case in 1-15 individually.

1. If $w : A \wedge B$ is principal in the last rule, then $w : A, w : B, \Gamma \Rightarrow \Delta$, being the premiss of the last rule, is $\star$-derivable in n-steps. Else the last rule $\dagger$ has premiss(es) $w : A \wedge B, \Gamma' \Rightarrow \Delta'$ (and $w : A \wedge B, \Gamma'' \Rightarrow \Delta''$) with derivation-height $k \leq n$. By applying the inductive hypothesis to such premiss(es) and then rule $\dagger$, we conclude $\star \vdash_{n+1} w : A, w : B, \Gamma \Rightarrow \Delta$.

2. If $w : A \wedge B$ is principal in the last rule, then $\Gamma \Rightarrow \Delta, w : A$ and $\Gamma \Rightarrow \Delta, w : B$, being the premisses of the last rule, are $\star$-derivable in n-steps. Else the last rule $\dagger$ has one (or two) premiss(es) such that $w : A \wedge B$ occurs in their succedent(s). The inductive hypothesis tells us that $\star \vdash_n \Gamma' \Rightarrow \Delta', w : A$, $\star \vdash_n \Gamma' \Rightarrow \Delta', w : B$, and, if $\dagger$ has two premisses, $\star \vdash_n \Gamma'' \Rightarrow \Delta'', w : A$, and $\star \vdash_m \Gamma'' \Rightarrow \Delta'', w : B$. By applying the last rule to the first (and the third), and then to the second (and the fourth), we obtain the following $\star$-derivations of height $n + 1$

$$\dfrac{\begin{matrix}\vdots \\ \Gamma' \Rightarrow \Delta', w : A\end{matrix} \qquad \begin{matrix}\vdots \\ (\Gamma'' \Rightarrow \Delta'', w : A)\end{matrix}}{\Gamma \Rightarrow \Delta, w : A}\dagger \qquad\qquad \dfrac{\begin{matrix}\vdots \\ \Gamma' \Rightarrow \Delta', w : B\end{matrix} \qquad \begin{matrix}\vdots \\ (\Gamma'' \Rightarrow \Delta'', w : B)\end{matrix}}{\Gamma \Rightarrow \Delta, w : b}\dagger$$

3. See case 2.

4. See case 1.

5. See case 2.

6. See case 1.

7. The height-preserving invertibility of rule $L\forall$ follows by the height-preserving admissiblity of rule LW_l (since the rule has the principal formula repeated in the premiss).

8. If $w : \forall x A$ is principal in the last rule, then $\mathcal{E}z^w, \Gamma \Rightarrow \Delta, w : A[z/x]$, for some z^w respecting the variable condition, is $\star$-derivable in n steps. We apply the (height-preserving admissible by Lemma 3.16) substitution $[y^w/z^w]$ to conclude $\star \vdash_m \mathcal{E}y^w, \Gamma \Rightarrow \Delta, w : A[y/x]$. Else the sequent in the antecedent has been derived by some rule $\dagger$ with premiss(es) $\Gamma' \Rightarrow \Delta', w : \forall x A$ (and $\Gamma'' \Rightarrow \Delta'', w : \forall x A$). We apply the inductive

hypothesis to the premiss(es), and then the last rule †[9] to obtain the following $\star$-derivation of height n

$$\dfrac{\begin{matrix}\vdots \\ \mathcal{E}y^w, \Gamma' \Rightarrow \Delta', w : A[y/x]\end{matrix} \qquad \begin{matrix}\vdots \\ (\mathcal{E}y^w, \Gamma'' \Rightarrow \Delta'', w : A[y/x])\end{matrix}}{\mathcal{E}y^w, \Gamma \Rightarrow \Delta, w : A[y/x]} \dagger$$

9. See case 8.

10. See case 7, by height-preserving admissiblity of RW_l.

11. See case 14.

12. See case 13.

13. If $w : \langle^{\vec{t}}_{\vec{x}}\rangle A$ is principal, then $\star \vdash_n u : A, w\mathscr{R}u, t^{\vec{w}}\mathscr{T}x^{\vec{u}}, \Gamma \Rightarrow \Delta$, for some u respecting the variable condition. By Lemma 3.17, we can apply to it the substitution $[v/u]$ to conclude $\star \vdash_n v : A, w\mathscr{R}v, t^{\vec{w}}\mathscr{T}x^{\vec{v}}, \Gamma \Rightarrow \Delta$. Else the last rule † of δ has premiss(es) $w : \langle^{\vec{t}}_{\vec{x}}\rangle A, \Gamma' \Rightarrow \Delta'$ (and $w : \langle^{\vec{t}}_{\vec{x}}\rangle A, \Gamma'' \Rightarrow \Delta''$). By applying to the premiss(es) the inductive hypothesis and then rule †, we get the following $\star$-derivation of height $n + 1$

$$\dfrac{\begin{matrix}\vdots \\ v : A, w\mathscr{R}v, t^{\vec{w}}\mathscr{T}x^{\vec{v}}, \Gamma' \Rightarrow \Delta'\end{matrix} \qquad \begin{matrix}\vdots \\ (v : A, w\mathscr{R}v, t^{\vec{w}}\mathscr{T}x^{\vec{v}}, \Gamma'' \Rightarrow \Delta'')\end{matrix}}{v : A, w\mathscr{R}v, t^{\vec{w}}\mathscr{T}x^{\vec{v}}, \Gamma \Rightarrow \Delta} \dagger$$

Observe that

[9] The side condition that 'y^w does not occur in $\Gamma \Rightarrow \Delta, w : \forall x A$' warrants that any side condition on † is respected. It is possible to prove also the version without side condition: if y^w is an eigenvariable of †, we have to start by applying Lemma 3.16 to replace it with a fresh variable z^w, and then we can proceed as above by applying the inductive hypothesis and †, see [NP01, p. 71]. We opt for the restricted version because, strictly speaking, the unrestricted version is not an case of inversion, as the rule we are inverting has the side condition, but a mixed case of inversion plus substitution. Note that this is not a limitation because we can apply the substitution after inversion to obtain the desired instance. These remarks hold for all cases with a variable condition.

14. By the height-preserving admissibility of the left and right rules of weakening.

15. By the eight-preserving admissibility of the left rules of weakening, taking some care to avoid problems with variable conditions of δ. ■

Theorem 3.20. *The following rules of contraction are height-preserving admissible in* **GIM.**$\star$

$$\frac{w : A, w : A, \Gamma \Rightarrow \Delta}{w : A, \Gamma \Rightarrow \Delta} LC_l \qquad \frac{\Gamma \Rightarrow \Delta, w : A, w : A}{\Gamma \Rightarrow \Delta, w : A} RC_l$$

$$\frac{w\mathscr{R}v, w\mathscr{R}v, \Gamma \Rightarrow \Delta}{w\mathscr{R}v, \Gamma \Rightarrow \Delta} LC_r \qquad \frac{t^w \mathscr{T} s^v, t^w \mathscr{T} s^v, \Gamma \Rightarrow \Delta}{t^w \mathcal{T} s^v, \Gamma \Rightarrow \Delta} LC_t$$

$$\frac{\mathcal{E}t^w, \mathcal{E}t^w, \Gamma \Rightarrow \Delta}{\mathcal{E}t^w, \Gamma \Rightarrow \Delta} LC_e$$

Proof. By simultaneous induction for left and right rules of contraction on the height n of the $\star$-derivation δ of the premiss. If $h(\delta) = 0$, the contracted sequent is an axiom or an instance of $L\bot$ like the non-contracted one.

If $h(\delta) = n + 1$, we assume that the Lemma holds for $\star$-derivations of height $\leq n$. and we consider the last rule $\dagger$ applied in δ. If the contraction formula is not principal in $\dagger$, we apply the inductive hypothesis to the premiss(es), and then rule $\dagger$. Else we consider each rule individually. If it is $L\wedge$, we have the $\star$-derivation

$$\frac{\begin{array}{c}\vdots\\ w : A, w : B, w : A \wedge B, \Gamma \Rightarrow \Delta\end{array}}{w : A \wedge B, w : A \wedge B, \Gamma \Rightarrow \Delta} L\wedge$$

We apply the Inversion Lemma —3.19.1— to the premiss, the inductive hypothesis on LC_l twice,[10] and then $L\wedge$ to obtain the following $\star$-derivation of height $n + 1$

[10] We use a double line to express multiple applications of the same rule.

$$\cfrac{\cfrac{\begin{array}{c}\vdots\\ w:A,w:B,w:A,w:B,\Gamma\Rightarrow\Delta\end{array}}{w:A,w:B,\Gamma\Rightarrow\Delta}\,Ind.}{w:A\wedge B,\Gamma\Rightarrow\Delta}\,L\wedge$$

If the last rule is $R\wedge$ we have

$$\cfrac{\begin{array}{c}\vdots\\ \Gamma\Rightarrow\Delta,w:A\wedge B,w:A\end{array}\qquad\begin{array}{c}\vdots\\ \Gamma\Rightarrow\Delta,w:A\wedge B,w:B\end{array}}{\Gamma\Rightarrow\Delta,w:A\wedge B,w:A\wedge B}\,R\wedge$$

By applying the Inversion Lemma —3.19.2— to its two premisses, we have that $\star\vdash_n\Gamma\Rightarrow\Delta,w:A,w:A$, $\star\vdash_n\Gamma\Rightarrow\Delta,w:B,w:A$, $\star\vdash_n\Gamma\Rightarrow\Delta,w:A,w:B$, and $\star\vdash_m\Gamma\Rightarrow\Delta,w:B,w:B$. By applying the inductive hypothesis on RC_l to the first and the fourth, and then rule $R\wedge$ we obtain the following $\star$-derivation of height $n+1$

$$\cfrac{\begin{array}{c}\vdots\\ \Gamma\Rightarrow\Delta,w:A\end{array}\qquad\begin{array}{c}\vdots\\ \Gamma\Rightarrow\Delta,w:B\end{array}}{\Gamma\Rightarrow\Delta,w:A\wedge B}\,R\wedge$$

If the last rule applied is $L\vee$ or $L\rightarrow$, we proceed as for $R\wedge$; if it is $R\vee$ or $R\rightarrow$, as for $L\wedge$. The only novelty is that for the two rules involving $\rightarrow$ we have to use the inductive hypothesis on LC_l and that on RC_l.

If the last rule is $L\forall$ and the contraction formula is $w:\forall xA$,[11] we have

$$\cfrac{\begin{array}{c}\vdots\\ w:A[t/x],w:\forall xA,\mathcal{E}t^w,w:\forall xA,\Gamma\Rightarrow\Delta\end{array}}{w:\forall xA,\mathcal{E}t^w,w:\forall xA,\Gamma\Rightarrow\Delta}\,L\forall$$

and we apply the inductive hypothesis to its premiss, and then $L\forall$ to transform it into the following $\star$-derivation of same height

[11] If it is $\mathcal{E}t^w$ we proceed analogously.

$$\dfrac{\begin{array}{c}\vdots\\ w : A[t/x], w : \forall x A, \mathcal{E}t^w, \Gamma \Rightarrow \Delta\end{array}}{w : \forall x A, \mathcal{E}t^w, \Gamma \Rightarrow \Delta}\; L\forall$$

If the last rule is $R\forall$, we have

$$\dfrac{\begin{array}{c}\vdots\\ \mathcal{E}y^w, \Gamma \Rightarrow \Delta, w : \forall x A, w : A[y/x]\end{array}}{\Gamma \Rightarrow \Delta, w : \forall x A, w : \forall x A}\; R\forall$$

with y^v respecting the variable condition. By the Inversion Lemma — 3.19.8— we know that $\star \vdash_n \mathcal{E}z^w, \mathcal{E}y^w, \Gamma \Rightarrow \Delta, w : A[y/x], w : A[z/x]$, where z^w does not occur in $\mathcal{E}y^w, \Gamma, A[y/x] : w, \Delta$. We apply the height-preserving admissible substitution $[y^w/z^w]$ to obtain $\star \vdash_n \mathcal{E}y^w, \mathcal{E}y^w, \Gamma \Rightarrow \Delta, w : A[y/x], w : A[y/x]$. By applying to it the inductive hypothesis (on RC_l and LC_e) and then $R\forall$, we obtain the following $\star$-derivation of height $n+1$.

$$\dfrac{\begin{array}{c}\vdots\\ \mathcal{E}y^w, \Gamma \Rightarrow \Delta, w : A[y/x]\end{array}}{\Gamma \Rightarrow \Delta, w : \forall x A}\; R\forall$$

If the last rule is $L\exists$ or $R\exists$, we proceed as for $R\forall$ or $L\forall$, respectively.

If we have $L|^{\vec{t}}_{\vec{x}}|$ as last rule, we have[12]

$$\dfrac{\begin{array}{c}\vdots\\ v : A[\vec{s}/\vec{x}], w : |^{\vec{t}}_{\vec{x}}|A : w, w\mathscr{R}v, t^{\vec{w}}\mathscr{T}s^{\vec{v}}, w : |^{\vec{t}}_{\vec{x}}|A, \Gamma \Rightarrow \Delta\end{array}}{w : |^{\vec{t}}_{\vec{x}}|A, w\mathscr{R}v, t^{\vec{w}}\mathscr{T}s^{\vec{v}}, w : |^{\vec{t}}_{\vec{x}}|A, \Gamma \Rightarrow \Delta}\; L|^{\vec{t}}_{\vec{x}}|$$

Given that $L|^{\vec{t}}_{\vec{x}}|$ is a rule with repetition of the principal formula, we have simply to apply the inductive hypothesis to the premiss, and then the rule to transform it into the following $\star$-derivation of same height

[12] The cases where the contraction formula is a principal r- or t-formula are treated in the same way.

$$\dfrac{\vdots \atop v : A[\vec{s}/\vec{x}], w : |^{\vec{t}}_{\vec{x}}|A, w\mathscr{R}v, t^{\vec{w}}\mathscr{T}s^{\vec{v}}, \Gamma \Rightarrow \Delta}{w : |^{\vec{t}}_{\vec{x}}|A, w\mathscr{R}v, t^{\vec{w}}\mathscr{T}s^{\vec{v}}, \Gamma \Rightarrow \Delta} \; L|^{\vec{t}}_{\vec{x}}|$$

If $R|^{\vec{t}}_{\vec{x}}|$ is the last rule, we have

$$\dfrac{\vdots \atop w\mathscr{R}v, t^{\vec{w}}\mathscr{T}x^{\vec{v}}, \Gamma \Rightarrow \Delta, w : |^{\vec{t}}_{\vec{x}}|A, v : A}{\Gamma \Rightarrow \Delta, w : |^{\vec{t}}_{\vec{x}}|A, w : |^{\vec{t}}_{\vec{x}}|A} \; R|^{\vec{t}}_{\vec{x}}|$$

with v respecting the variable condition. By Inversion Lemma 3.19.12,

$$\star \vdash_n w\mathscr{R}v, t^{\vec{w}}\mathscr{T}x^{\vec{v}}, w\mathscr{R}u, t^{\vec{w}}\mathscr{T}x^{\vec{u}}, \Gamma \Rightarrow \Delta, v : A, u : A$$

where u is a fresh world label. By the the height-preserving admissible substitution $[v/u]$, we get

$$\star \vdash_n w\mathscr{R}v, t^{\vec{w}}\mathscr{T}x^{\vec{v}}, w\mathscr{R}v, t^{\vec{w}}\mathscr{T}x^{\vec{v}}, \Gamma \Rightarrow \Delta, v : A, v : A$$

We apply the inductive hypothesis for LC_r, LC_t[13] and RC_l, and then rule $R|^{\vec{t}}_{\vec{x}}|$, to obtain the following $\star$-derivation of height $n + 1$

$$\dfrac{\vdots \atop w\mathscr{R}v, t^{\vec{w}}\mathscr{T}x^{\vec{v}}, \Gamma \Rightarrow \Delta, v : A}{\Gamma \Rightarrow \Delta, w : |^{\vec{t}}_{\vec{x}}|A} \; R|^{\vec{t}}_{\vec{x}}|$$

If the last rule is $L\langle^{\vec{t}}_{\vec{x}}\rangle$ or $R\langle^{\vec{t}}_{\vec{x}}\rangle$ we proceed as for $L|^{\vec{t}}_{\vec{x}}|$ and $R|^{\vec{t}}_{\vec{x}}|$, respectively (replacing the application of the inductive hypothesis on RC_l with one on LC_l and vice versa).

If the last rule is a nonlogical one with no variable condition, we can have one or two occurrences of the principal formula in the conclusion. In the first case the lemma holds as the principal formula is repeated in the premiss. Suppose, e.g., that we have

[13] Once for each t-formula in $t^{\vec{w}}\mathscr{T}x^{\vec{v}}$.

$$\dfrac{\begin{array}{c}\vdots\\ w : P[r/x], w : s \doteq r, w : P[s/x], w : P[s/x], \Gamma' \Rightarrow \Delta\end{array}}{w : s \doteq r, w : P[s/x], w : P[s/x], \Gamma' \Rightarrow \Delta} Lbz_1$$

By applying the inductive hypothesis to the premiss, and then the rule, we transform it into

$$\dfrac{\begin{array}{c}\vdots\\ w : P[r/x], w : s \doteq r, w : P[s/x], \Gamma' \Rightarrow \Delta\end{array}}{w : s \doteq r, w : P[s/x], \Gamma' \Rightarrow \Delta} Lbz_1$$

In the second case the Lemma holds by the closure condition. Suppose, e.g., that we have

$$\dfrac{\begin{array}{c}\vdots\\ v\mathscr{R}v, w\mathscr{R}v, w\mathscr{R}v, \Gamma \Rightarrow \Delta\end{array}}{w\mathscr{R}v, w\mathscr{R}v, \Gamma \Rightarrow \Delta} 5^{\mathcal{R}}$$

We have simply to apply the inductive hypothesis (IH) to the premiss, and then the contracted instance of the rule to obtain the following $\star$-derivation of height $n+1$

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ v\mathscr{R}v, w\mathscr{R}v, w\mathscr{R}v, \Gamma \Rightarrow \Delta\end{array}}{v\mathscr{R}v, w\mathscr{R}v, \Gamma \Rightarrow \Delta} IH}{w\mathscr{R}v, \Gamma \Rightarrow \Delta} 5^{1c}$$

If the last rule is a nonlogical one with a variable condition, say BF with LC_r, we have

$$\dfrac{x^w \mathscr{T} t^v, \mathcal{E}y^v, w\mathscr{R}v, w\mathscr{R}v, \mathcal{E}t^v, \Gamma \Rightarrow \Delta}{w\mathscr{R}v, w\mathscr{R}v, \mathcal{E}t^v, \Gamma \Rightarrow \Delta} BF$$

and we have simply to apply the inductive hypothesis to the premiss, and then the rule. ∎

Observe that we need contracted instances only of the rules for the t-euclidean

t-structures, and that those contracted instances are dispensable whenever the rules for t-reflexive t-frames are present. In fact for the rules $4^{\mathcal{R}}$ and $4^{\mathcal{T}}$ (see Table 3.4) for t-transitive t-frames we can apply the the inductive hypothesis twice[14] instead of the inductive hypothesis and then the contracted instance of $4^{\mathcal{R}}$ or $4^{\mathcal{T}}$, therefore we don't need to add the contracted instances 4^{iC}. For rules NI and ND, Lbz_1 and Lbz_2 the contracted instances are instances of rule Ref —see [NP01, p. 138]— and, thus, are already in **GIM.**$\star_=$. All other nonlogical rules that we have introduced have no contracted instance. Furthermore we have that, despite appearance to the contrary, the need to include contracted instances of some nonlogical rule doesn't imply that we have a primitive rule of contraction (on atoms) in our calculi, this result follows from (the generalization to r- and t-formulas of) [HN11, Prop. 3]: 'Let R be a frame rule, R^C the contracted instance that arises from the closure condition. If R^C is an instance of contraction, it is hp-admissible in the system extended with those rules arising from the closure condition that are not instances of contraction'.

3.5 Cut

In order to show that the rule of cut is admissible in **GIM.**$\star$, we need some preliminary definitions.

Definition 3.21. Given an instance of Cut:

$$\dfrac{\begin{array}{c}\vdots\,\delta_1\\ \Gamma \Rightarrow \Delta, w : A\end{array} \qquad \begin{array}{c}\vdots\,\delta_2\\ w : A, \Pi \Rightarrow \Sigma\end{array}}{\Gamma, \Pi \Rightarrow \Delta, \Sigma}\,Cut$$

$w : A$ will be called *cut formula*. We define the *rank* (of an instance of Cut) a pair $\langle n, i\rangle$ where n is the height of its cut formula, and i is the sum of the heights of the two $\star$-derivations of its premisses. Ranks are ordered by the

[14] Given that it is hp-admissible, we can apply it as many times as we want.

following well-founded relation of lexicographical order

$$\langle n, i\rangle \prec \langle m, j\rangle \qquad \Longleftrightarrow \qquad n < m, \text{ or } n = m \text{ and } i < j$$

We write $Cut\langle n, i\rangle$ for an application of Cut the rank of which is at most $\langle n, i\rangle$. ■

We remind the reader that, thanks to lemma 3.12, we have assumed that all ext-formulas occurring in the succedents of an arbitrary derivation are l-formulas that differs from $w : \bot$ (for any w), thus we have only cut formulas of such shapes. Furthermore, in all cases of the proof where we have a duplication of a right context Δ or Σ, we need only the eight-preserving admissible rule of contraction RC_l, because all ext-formulas in Δ are l-formulas by the same assumption.

Theorem 3.22. *The rule of cut is admissible in* **GIM.**$\star$

$$\dfrac{\begin{matrix}\vdots\,\delta_1 \\ \Gamma \Rightarrow \Delta, w : A\end{matrix} \qquad \begin{matrix}\vdots\,\delta_2 \\ w : A, \Pi \Rightarrow \Sigma\end{matrix}}{\Gamma, \Pi \Rightarrow \Delta, \Sigma}\,Cut$$

Proof. The proof is by induction on the rank $\langle n, i\rangle$ of an uppermost application of Cut. The inductive hypothesis (IH) is that every Cut of lower rank is admissible in **GIM.**$\star$.

If $\mathbf{n} = \mathbf{0}$, the cut formula is an atomic l-formula $w : P$. If $\mathbf{i} = \mathbf{0}$, either $w : P$ is principal in both premisses of the application of cut or not. If it is principal in both premisses we have

$$\dfrac{\begin{matrix}\vdots\,\delta_1 \\ w : P, \Gamma' \Rightarrow \Delta, w : P\end{matrix} \qquad \begin{matrix}\vdots\,\delta_2 \\ w : P, \Pi \Rightarrow \Sigma', w : P\end{matrix}}{w : P, \Gamma', \Pi \Rightarrow \Delta, \Sigma', w : P}\,Cut$$

with $w : P, \Gamma' = \Gamma$ and $\Sigma', w : P = \Sigma$. The conclusion $\Gamma, \Delta \Rightarrow \Pi, \Sigma$ is then an initial sequent. Else the cut formula is not principal in at least one of the two premisses, say the left one, which is an initial sequent $v : Q, w : P, \Gamma' \Rightarrow$

$\Delta', v : Q$ and then $v : Q, \Gamma', \Pi \Rightarrow \Delta', \Sigma, v : Q$ is an initial sequent, or it is an instance of the zero premiss rule $L\bot$, say $v : \bot, w : P, \Gamma' \Rightarrow \Delta$, and then also $v : \bot, \Gamma', \Pi \Rightarrow \Delta, \Sigma$ is an instance of $L\bot$. Thus all cases of a cut of rank $\langle 0, 0\rangle$ are admissible in **GIM.**$\star$.

If $\mathbf{i} = \mathbf{j} + \mathbf{1}$, then if the $\star$-derivation of one of the two premiss has derivation-height 0, then either the cut formula is not principal in the other premiss and we can proceed as in the previous case, or the cut formula is $w : P[t/x]$ (or $w : t \doteq s$), it is the principal formula of the right premiss that is an instance of Lbz_1 (it is the only rule with atomic l-formulas principal), and it is either principal or not in the left premiss, which is an initial sequent. If the cut formula is not principal in the left premiss, then the conclusion of Cut is derivable by weakening the left premiss. If it is principal in the left premiss, we have

$$\dfrac{w : P[t/x], \Gamma' \Rightarrow \Delta, w : P[t/x] \qquad \dfrac{\begin{array}{c}\vdots\,\delta_{21}\\ w : P[s/x], w : t \doteq s, w : P[t/x], \Pi' \Rightarrow \Sigma\end{array}}{w : P[t/x], w : t \doteq s, \Pi' \Rightarrow \Sigma}}{w : P[t/x], \Gamma', w : t \doteq s, \Pi' \Rightarrow \Delta, \Sigma}\; Cut\langle 0, j+1\rangle$$

and can be transformed into

$$\dfrac{\dfrac{w : P[t/x], \Gamma' \Rightarrow \Delta, w : P[t/x] \qquad \begin{array}{c}\vdots\,\delta_{21}\\ w : P[t/x], w : P[s/x], w : t \doteq s, \Pi' \Rightarrow \Sigma\end{array}}{w : P[t/x], \Gamma', w : P[s/x], w : t \doteq s, \Pi' \Rightarrow \Delta, \Sigma}\; Cut\langle 0, j\rangle}{w : P[t/x], \Gamma', w : t \doteq s, \Pi' \Rightarrow \Delta, \Sigma}\; Lbz_1$$

where we have by IH an admissible cut of less cut-height. If the cut formula is $w : t \doteq s$, we proceed in the same way. If, instead, both premisses have a $\star$-derivation of height ≥ 1, δ is

$$\dfrac{\begin{array}{c}\vdots\,\delta_1\\ \Gamma \Rightarrow \Delta, w : P\end{array} \qquad \begin{array}{c}\vdots\,\delta_2\\ w : P, \Pi \Rightarrow \Sigma\end{array}}{\Gamma, \Pi \Rightarrow \Delta, \Sigma}\; Cut\langle 0, j+1\rangle$$

Given that $w : P$ is an atomic l-formula, it cannot be principal in the last step of δ_1 (because atomic l-formulas cannot be principal in right rules), which is

$$\dfrac{\begin{array}{c}\delta_{11}\\ \vdots\\ \Gamma' \Rightarrow \Delta', w : P\end{array} \qquad \begin{array}{c}\delta_{12}\\ \vdots\\ (\Gamma'' \Rightarrow \Delta'', w : P)\end{array}}{\Gamma \Rightarrow \Delta, w : P}\ \dagger$$

If $\dagger$ has a variable condition on some ext-variable occurring in Π or in Σ, we apply the height-preserving admissible substitution Lemmas 3.16 and/or 3.17 in order to substitute it with a fresh ext-variable. Now we can permute the cut upward in δ_{11} (and in δ_{12}), and then apply rule $\dagger$, to transform our original $\star$-derivation into

$$\dfrac{\dfrac{\begin{array}{c}\vdots\ \delta_{11}\\ \Gamma' \Rightarrow \Delta', w : P\end{array} \quad \begin{array}{c}\vdots\ \delta_{2}\\ w : P, \Pi \Rightarrow \Sigma\end{array}}{\Gamma', \Pi \Rightarrow \Delta', \Sigma}\ Cut\langle 0, j\rangle \qquad \dfrac{\begin{array}{c}\vdots\ \delta_{12}\\ \Gamma'' \Rightarrow \Delta'', w : P\end{array} \quad \begin{array}{c}\vdots\ \delta_{2}\\ w : P, \Pi \Rightarrow \Sigma\end{array}}{\Gamma'', \Pi \Rightarrow \Delta'', \Sigma}\ Cut\langle 0, j\rangle}{\Gamma, \Pi \Rightarrow \Delta, \Sigma}\ \dagger$$

with one (or two) cut(s) of less rank that are admissible by IH.

If $\mathbf{n = m + 1}$, the cut formula is not atomic. We proceed, once again, by induction on i to show that an uppermost application of cut can be eliminated or reduced to an admissible cut of less rank. If $\mathbf{i = 0}$, the cut is on two initial sequents/conclusions of $L\bot$, and the cut formula is in the context of both premisses, respectively in the right-context and in the left-context; therefore either some atomic ext-formula occurs both in Γ and Δ (Π and Σ), or $\bot$ is in Γ (Π). In all cases the cut is admissible since its conclusion is an initial sequent and/or a conclusion of $L\bot$.

If $\mathbf{i = j + 1}$, then if the $\star$-derivation of one of the two premiss has derivation-height 0, we can proceed as in the previous case; therefore we have to consider only the case where both premisses have a $\star$-derivation of height ≥ 1 —i.e. we have the following derivation δ

$$\dfrac{\dfrac{\vdots\ \delta_1}{\Gamma \Rightarrow \Delta, w : A}\ \dagger_1 \qquad \dfrac{\vdots\ \delta_2}{w : A, \Pi \Rightarrow \Sigma}\ \dagger_2}{\Gamma, \Pi \Rightarrow \Delta, \Sigma}\ Cut\langle m + 1, j + 1\rangle$$

and three cases are possible w.r.t. the cut formula $w : A$: (i) it is not principal in $\dagger_1$, or (ii) it is principal $\dagger_1$ but not principal in $\dagger_2$, or (ii) it is principal in both premisses.

(i) If $w : A$ is not principal in $\dagger_1$, we have to consider the last step of δ_1 (i.e. of the derivation of the left premiss of Cut) to show that we can permute the application of Cut with $\dagger_1$. The lemma follows by the inductive hypothesis that all cuts with rank less than $\langle m+1, j+1\rangle$ are admissible.

If $\dagger_1$ is $L\wedge$, we have $\Gamma = v : B \wedge C, \Gamma'$ and δ_1 is

$$\dfrac{\overset{\vdots\, \delta_{11}}{v : B, v : C, \Gamma' \Rightarrow \Delta, w : A}}{v : B \wedge C, \Gamma' \Rightarrow \Delta, w : A}\, L\wedge$$

We can permute Cut with $L\wedge$ to transform δ into

$$\dfrac{\dfrac{\overset{\vdots\, \delta_{11}}{v : B, v : C, \Gamma' \Rightarrow \Delta, w : A} \qquad \overset{\vdots\, \delta_{2}}{w : A, \Pi \Rightarrow \Sigma}}{v : B, v : C, \Gamma', \Pi \Rightarrow \Delta, \Sigma}\, Cut\langle m+1, j\rangle}{v : B \wedge C, \Gamma', \Pi \Rightarrow \Delta, \Sigma}\, L\wedge$$

If $\dagger_1$ is $R\wedge$, we have $\Delta = \Delta', v : B \wedge C$ and the last step of δ_1 is

$$\dfrac{\overset{\vdots\, \delta_{11}}{\Gamma \Rightarrow \Delta', w : A, v : B} \qquad \overset{\vdots\, \delta_{12}}{\Gamma \Rightarrow \Delta', w : A, c : C}}{\Gamma \Rightarrow \Delta', w : A, v : B \wedge C}\, R\wedge$$

We can transform δ into

$$\dfrac{\dfrac{\overset{\vdots\, \delta_{11}}{\Gamma \Rightarrow \Delta', v : B, w : A} \qquad \overset{\vdots\, \delta_{2}}{w : A, \Pi \Rightarrow \Sigma}}{\Gamma, \Pi \Rightarrow \Delta', v : B, \Sigma}\, Cut\langle m+1, j\rangle \qquad \dfrac{\overset{\vdots\, \delta_{12}}{\Gamma \Rightarrow \Delta', v : C, w : A} \qquad \overset{\vdots\, \delta_{2}}{w : A, \Pi \Rightarrow \Sigma}}{\Gamma, \Pi \Rightarrow \Delta', v : C, \Sigma}\, Cut\langle m+1, j\rangle}{\Gamma, \Pi \Rightarrow \Delta', v : B \wedge C, \Sigma}\, R\wedge$$

Where we have two admissible cuts of less rank.

The cases of left and right rules for $\vee$ and $\rightarrow$ are analogous to the previous ones.

If $\dagger_1$ is $L\forall$, we have $\Gamma = v : \forall x B, \mathcal{E}t^v, \Gamma'$, and δ_1 is

$$\dfrac{\begin{array}{c}\vdots\ \delta_{11}\\ v : B[t/x], v : \forall x B, \mathcal{E}t^v, \Gamma' \Rightarrow \Delta, w : A\end{array}}{v : \forall x B, \mathcal{E}t^v, \Gamma' \Rightarrow \Delta, w : A}\ L\forall$$

We can permute Cut with $L\forall$ to transform δ into

$$\dfrac{\dfrac{\begin{array}{c}\vdots\ \delta_{11}\\ v : B[t/x], v : \forall x B, \mathcal{E}t^v, \Gamma' \Rightarrow \Delta, w : A\end{array} \qquad \begin{array}{c}\vdots\ \delta_2\\ w : A, \Pi \Rightarrow \Sigma\end{array}}{v : B[t/x], v : \forall x B, \mathcal{E}t^v, \Gamma', \Pi \Rightarrow \Delta, \Sigma}\ Cut\langle m+1, j\rangle}{v : \forall x B, \mathcal{E}t^v, \Gamma', \Pi \Rightarrow \Delta, \Sigma}\ L\forall$$

If $\dagger_1$ is $R\forall$ we have $\Delta = \Delta', v : \forall x B$, and δ_1 is

$$\dfrac{\begin{array}{c}\vdots\ \delta_{11}\\ v : \mathcal{E}y, \Gamma \Rightarrow \Delta', w : A, v : B[y/x]\end{array}}{\Gamma \Rightarrow \Delta', w : A, v : \forall x B}\ R\forall$$

Where y^v respects the variable condition. If y^v occurs in Π or in Σ, we apply Lemma 3.16 to replace it with a fresh l-variable z^w, otherwise we take z^w to be y^w. Now we can permute Cut with $R\forall$ to obtain

$$\dfrac{\dfrac{\begin{array}{c}\vdots\ \delta'_{11}\\ \mathcal{E}z^v, \Gamma \Rightarrow \Delta', v : B[z/x], w : A\end{array} \qquad \begin{array}{c}\vdots\ \delta_2\\ w : A, \Pi \Rightarrow \Sigma\end{array}}{\mathcal{E}z^v, \Gamma, \Pi \Rightarrow \Delta', v : A[z/x]\Sigma}\ Cut\langle m+1, j\rangle}{\Gamma, \Pi \Rightarrow \Delta', v : \forall x B, \Sigma}\ L\forall$$

If $\dagger_1$ is $L\exists$ or $R\exists$, we proceed as for $R\forall$ or $L\forall$.

If $\dagger_1$ is $L|^{\vec{t}}_{\vec{x}}|$, we have $\Gamma = |^{\vec{t}}_{\vec{x}}|B : v, v\mathscr{R}u, \vec{t^v}\mathscr{T}\vec{s^u}, \Gamma'$, and δ_1 is

$$\dfrac{\begin{array}{c}\vdots\ \delta_{11}\\ u : B[\vec{s}/\vec{x}], v : |^{\vec{t}}_{\vec{x}}|B, v\mathscr{R}u, \vec{t^v}\mathscr{T}\vec{s^u}, \Gamma' \Rightarrow \Delta, w : A\end{array}}{v : |^{\vec{t}}_{\vec{x}}|B, v\mathscr{R}u, \vec{t^v}\mathscr{T}\vec{s^u}, \Gamma' \Rightarrow \Delta, w : A}\ L|^{\vec{t}}_{\vec{x}}|$$

We can permute Cut with $L|\frac{\vec{t}}{\vec{x}}|$ to transform δ into

$$\dfrac{\dfrac{\begin{array}{c}\vdots\ \delta_{11}\\ u : B[\vec{s}/\vec{x}], v : |\frac{\vec{t}}{\vec{x}}|B, v\mathscr{R}u, \vec{t^v}\mathscr{T}\vec{s^u}, \Gamma' \Rightarrow \Delta, w : A\end{array} \qquad \begin{array}{c}\vdots\ \delta_2\\ w : A, \Pi \Rightarrow \Sigma\end{array}}{u : B[\vec{s}/\vec{x}], v : |\frac{\vec{t}}{\vec{x}}|B, v\mathscr{R}u, \vec{t^v}\mathscr{T}\vec{s^u}, \Gamma', \Pi \Rightarrow \Delta, \Sigma}\,Cut\langle m+1, j\rangle}{v : |\frac{\vec{t}}{\vec{x}}|B, v\mathscr{R}u, \vec{t^v}\mathscr{T}\vec{s^u}, \Gamma', \Pi \Rightarrow \Delta, \Sigma}\,L|\tfrac{\vec{t}}{\vec{x}}|$$

If $\dagger_1$ is $R|\frac{\vec{t}}{\vec{x}}|$, we have $\Delta = \Delta', v : |\frac{\vec{t}}{\vec{x}}|B$, and δ_1 is

$$\dfrac{\begin{array}{c}\vdots\ \delta_{11}\\ v\mathscr{R}u, \vec{t^v}\mathscr{T}\vec{x^u}, \Gamma \Rightarrow \Delta', w : A, u : B\end{array}}{\Gamma \Rightarrow \Delta', w : A, v : |\frac{\vec{t}}{\vec{x}}|B}\,R|\tfrac{\vec{t}}{\vec{x}}|$$

where u (and each member of $\vec{x^u}$) respects the variable condition. For simplicity, we assume that u doesn't occur in Π nor in Σ (otherwise we would have to apply the substitution $[u_1/u]$ —height-preserving admissible by Lemma 3.17— for some fresh u_1). We can transform δ into

$$\dfrac{\dfrac{\begin{array}{c}\vdots\ \delta_{11}\\ v\mathscr{R}u, \vec{t^v}\mathscr{T}\vec{x^u}, \Gamma \Rightarrow \Delta', u : B, w : A\end{array} \qquad \begin{array}{c}\vdots\ \delta_2\\ w : A, \Pi \Rightarrow \Sigma\end{array}}{v\mathscr{R}u, \vec{t^v}\mathscr{T}\vec{x^u}, \Gamma, \Pi \Rightarrow \Delta', u : B, \Sigma}\,Cut\langle m+1, j\rangle}{\Gamma, \Pi \Rightarrow \Delta', v : |\frac{\vec{t}}{\vec{x}}|A, \Sigma}\,R|\tfrac{\vec{t}}{\vec{x}}|$$

If $\dagger_1$ is $L\langle\frac{\vec{t}}{\vec{x}}\rangle$ or $R\langle\frac{\vec{t}}{\vec{x}}\rangle$, we proceed as for $R|\frac{\vec{t}}{\vec{x}}|$ or $L|\frac{\vec{t}}{\vec{x}}|$, respectively.

If the last step of δ_1 is a nonlogical rule without variable condition, we can permute Cut with it. If it is a nonlogical rule with a variable condition, we apply either Lemma 3.16 or 3.17, depending on the sort of the eigenvariable(s), to replace δ_1 with a derivation of same height δ_1' where the eigenvariable of the last step of δ_1' is new to δ. Now we can permute Cut with the nonlogical rule, and then apply the nonlogical rule as its variable condition is respected.

(ii) If $w : A$ is principal in $\dagger_1$, but not principal in $\dagger_2$, the application of Cut can be permuted with $\dagger_2$ —i.e. with the last step of δ_2. Each case proceeds as the corresponding one with cut formula not principal in both premisses. We show a couple of cases.

If $\dagger_2$ is $L\rightarrow$, we have $\Pi = v : B \rightarrow C, \Pi'$, and δ_2 is

$$\dfrac{\dfrac{\vdots\ \delta_{21}}{w : A, \Pi' \Rightarrow \Sigma, v : B} \qquad \dfrac{\vdots\ \delta_{22}}{v : C, w : A, \Pi' \Rightarrow \sigma}}{v : B \rightarrow C, w : A, \Pi' \Rightarrow \Sigma}\ L\rightarrow$$

and we can transform δ into

$$\dfrac{\dfrac{\dfrac{\vdots\ \delta_1}{\Gamma \Rightarrow \Delta, w : A} \quad \dfrac{\vdots\ \delta_{21}}{w : A, \Pi' \Rightarrow \Sigma, v : B}}{\Gamma, \Pi' \Rightarrow \Delta, \Sigma, v : B}\ Cut\langle n, j\rangle \quad \dfrac{\dfrac{\vdots\ \delta_1}{\Gamma \Rightarrow \Delta, w : A} \quad \dfrac{\vdots\ \delta_{22}}{v : C, w : A, \Pi' \Rightarrow \sigma}}{\Gamma, v : C, \Pi' \Rightarrow \Delta, \Sigma}\ Cut\langle m+1, j\rangle}{\Gamma, v : B \rightarrow C, \Pi' \Rightarrow \Delta, \Sigma}\ L\rightarrow$$

Where we have two cuts of less rank.

If $\dagger_2$ is $R\rightarrow$, we have $\Sigma = \Sigma, v : B \rightarrow C'$ and δ_2 is

$$\dfrac{\dfrac{\vdots\ \delta_{21}}{v : B, w : A, \Pi \Rightarrow \Sigma', v : C}}{w : A, \Pi \Rightarrow \Sigma', v : B \rightarrow C}\ R\rightarrow$$

We can permute Cut with $R\rightarrow$ to transform δ into

$$\dfrac{\dfrac{\dfrac{\vdots\ \delta_1}{\Gamma \Rightarrow \Delta, w : A} \qquad \dfrac{\vdots\ \delta_{21}}{w : A, v : B, \Pi \Rightarrow \Sigma', v : C}}{\Gamma, v : B, \Pi \Rightarrow \Delta, \Sigma', v : C}\ Cut\langle m+1, j\rangle}{\Gamma, \Pi \Rightarrow \Delta, \Sigma', v : B \rightarrow C}\ R\rightarrow$$

If $\dagger_2$ is $L\langle\frac{\vec{t}}{\vec{x}}\rangle$, we have $\Pi = v : \langle\frac{\vec{t}}{\vec{x}}\rangle B, \Pi'$, and δ_2 is

$$\dfrac{\dfrac{\vdots\ \delta_{21}}{u : B, v\mathscr{R}u, \vec{t^v}\mathscr{T}\vec{x^u}, w : A, \Pi' \Rightarrow \Sigma}}{v : \langle\frac{\vec{t}}{\vec{x}}\rangle B, w : A, \Pi' \Rightarrow \Sigma}\ L\langle\tfrac{\vec{t}}{\vec{x}}\rangle$$

where u respects the variable condition. We assume that u doesn't occur in Γ nor in Δ (otherwise we would have to apply Lemma 3.17), and we permute

Cut with $L\langle^{\vec{t}}_{\vec{x}}\rangle$ to transform δ into

$$\dfrac{\dfrac{\begin{matrix}\vdots\ \delta_1 \\ \Gamma \Rightarrow \Delta, w : A\end{matrix} \qquad \begin{matrix}\vdots\ \delta_{21} \\ w : A, u : B, v\mathscr{R}u, \vec{t^v}\mathscr{T}\vec{x^u}, \Pi' \Rightarrow \Sigma\end{matrix}}{\Gamma, u : B, v\mathscr{R}u, \vec{t^v}\mathscr{T}\vec{x^u}, \Pi' \Rightarrow \Delta, \Sigma}\ Cut\langle m+1, j\rangle}{\Gamma, v : \langle^{\vec{t}}_{\vec{x}}\rangle A, \Pi' \Rightarrow \Delta, \Sigma}\ L\langle^{\vec{t}}_{\vec{x}}\rangle$$

If $\dagger_2$ is $R\langle^{\vec{t}}_{\vec{x}}\rangle$, we have $\Pi = v\mathscr{R}u, \vec{t^v}\mathscr{T}\vec{s^u}, \Pi'$, and $\Sigma = \Sigma', v : \langle^{\vec{t}}_{\vec{x}}\rangle B$. δ_2 is

$$\dfrac{\begin{matrix}\vdots\ \delta_{21} \\ v\mathscr{R}u, \vec{t^v}\mathscr{T}\vec{s^u}, w : A, \Pi' \Rightarrow \Sigma', v : \langle^{\vec{t}}_{\vec{x}}\rangle B, u : B[\vec{s}/\vec{x}]\end{matrix}}{v\mathscr{R}u, \vec{t^v}\mathscr{T}\vec{s^u}, w : A, \Pi' \Rightarrow \Sigma', v : \langle^{\vec{t}}_{\vec{x}}\rangle B}\ R\langle^{\vec{t}}_{\vec{x}}\rangle$$

and we can transform δ into

$$\dfrac{\dfrac{\begin{matrix}\vdots\ \delta_1 \\ \Gamma \Rightarrow \Delta, w : A\end{matrix} \qquad \begin{matrix}\vdots\ \delta_{21} \\ w : A, v\mathscr{R}u, \vec{t^v}\mathscr{T}\vec{s^u}, \Pi' \Rightarrow \Sigma', v : \langle^{\vec{t}}_{\vec{x}}\rangle B, u : B[\vec{s}/\vec{x}]\end{matrix}}{\Gamma, v\mathscr{R}u, \vec{t^v}\mathscr{T}\vec{s^u}, \Pi' \Rightarrow \Delta, \Sigma', v : \langle^{\vec{t}}_{\vec{x}}\rangle B, u : B[\vec{s}/\vec{x}]}\ Cut\langle m+1, j\rangle}{\Gamma, v\mathscr{R}u, \vec{t^v}\mathscr{T}\vec{s^u}, \Pi' \Rightarrow \Delta, \Sigma', v : \langle^{\vec{t}}_{\vec{x}}\rangle B}\ R\langle^{\vec{t}}_{\vec{x}}\rangle$$

If the last step of δ_2 is a nonlogical rule without variable condition, we can permute Cut with it. If it is a nonlogical rule with a variable condition, we apply either Lemma 3.16 or 3.17, depending on the sort of the eigenvariable, to replace δ_2 with a derivation of same height δ_2' where the eigenvariable of the last step is new to δ. Now we can permute Cut with the nonlogical rule, and then apply the rule to obtain a $\star$-derivation of $\Gamma, \Pi \Rightarrow \Delta, \Sigma$ with a cut of less rank that is admissible by IH.

(iii) If the cut formula is principal in both premisses, it is either $w : B \wedge C, w : B \vee C, w : B \rightarrow C, w : \forall x B, w : \exists x B, w : |^{\vec{t}}_{\vec{x}}| B$, or $w : \langle^{\vec{t}}_{\vec{x}}\rangle B$.

If the cut formula is $w : B \wedge C$, we have

$$\dfrac{\dfrac{\begin{array}{c}\vdots\,\delta_{11}\\ \Gamma \Rightarrow \Delta, w : B\end{array} \quad \begin{array}{c}\vdots\,\delta_{12}\\ \Gamma \Rightarrow \Delta, w : C\end{array}}{\Gamma \Rightarrow \Delta, w : B \wedge C}\,R\wedge \quad \dfrac{\begin{array}{c}\vdots\,\delta_{21}\\ w : B, w : C, \Pi \Rightarrow \Sigma\end{array}}{w : B \wedge C, \Pi \Rightarrow \Sigma}\,L\wedge}{\Gamma, \Pi \Rightarrow \Delta, \Sigma}\,Cut\langle m+1, j+1\rangle$$

and we transform it into

$$\cfrac{\cfrac{\begin{array}{c}\vdots\,\delta_{11}\\ \Gamma \Rightarrow \Delta, w : A\end{array} \quad \cfrac{\begin{array}{c}\vdots\,\delta_{12}\\ \Gamma \Rightarrow \Delta, w : B\end{array} \quad \begin{array}{c}\vdots\,\delta_{21}\\ w : B, \Pi, A : w \Rightarrow \Sigma\end{array}}{w : A, \Gamma, \Pi \Rightarrow \Delta, \Sigma}\,Cut\langle m, j\rangle}{\Gamma, \Gamma, \Pi \Rightarrow \Delta, \Delta, \Sigma}\,Cut\langle m, j+k\rangle}{\Gamma, \Pi \Rightarrow \Delta, \Sigma}\,C_{\dagger}$$

Where $C_{\dagger}$ stands for some applications of the left and right rules of contraction. We have thus found a $\star$-derivation of $\Gamma, \Pi \Rightarrow \Delta, \Sigma$ with two admissible (by IH) cuts of less rank. Note that the height $j + k$ of the second cut is bounded by $h(\delta_{11}) + max\{h(\delta_{12}), h(\delta_{21})\} + 1$ (see [NP01, p. 39]), thus in the transformation we can introduce a cut with a cut-height that is not less than that of the original one.

If the cut formula is $w : A \vee B$, we have

$$\dfrac{\dfrac{\begin{array}{c}\vdots\,\delta_{11}\\ \Gamma \Rightarrow \Delta, w : B, w : C\end{array}}{\Gamma \Rightarrow \Delta, w : B \vee C}\,R\vee \quad \dfrac{\begin{array}{c}\vdots\,\delta_{21}\\ w : B, \Pi \Rightarrow \Sigma\end{array} \quad \begin{array}{c}\vdots\,\delta_{22}\\ w : C, \Pi \Rightarrow \Sigma\end{array}}{w : B \vee C, \Pi \Rightarrow \Sigma}\,L\vee}{\Gamma, \Pi \Rightarrow \Delta, \Sigma}\,Cut\langle m+1, j+1\rangle$$

and we transform it into

$$\cfrac{\cfrac{\cfrac{\begin{array}{c}\vdots\,\delta_{11}\\ \Gamma \Rightarrow \Delta, w : B, w : C\end{array} \quad \begin{array}{c}\vdots\,\delta_{22}\\ w : C, \Pi \Rightarrow \Sigma\end{array}}{\Gamma, \Pi \Rightarrow \Delta, \Sigma, w : B}\,Cut_l\langle m, j\rangle \quad \begin{array}{c}\vdots\,\delta_{21}\\ w : B, \Pi \Rightarrow \Sigma\end{array}}{\Gamma, \Pi, \Pi \Rightarrow \Delta, \Sigma, \Sigma}\,Cut\langle m, j+k\rangle}{\Gamma, \Pi \Rightarrow \Delta, \Sigma}\,C_{\dagger}$$

where we have two cuts of lesser rank that are admissible by IH.

If the cut formula is $w : B \to C$, we have

$$\dfrac{\dfrac{\begin{array}{c}\vdots\,\delta_{11}\\ w:B,\Gamma\Rightarrow\Delta,w:C\end{array}}{\Gamma\Rightarrow\Delta,w:B\to C}R\to \quad \dfrac{\begin{array}{c}\vdots\,\delta_{21}\\ \Pi\Rightarrow\Sigma,w:B\end{array}\quad\begin{array}{c}\vdots\,\delta_{22}\\ w:C,\Pi\Rightarrow\Sigma\end{array}}{w:B\to C,\Pi\Rightarrow\Sigma}L\to}{\Gamma,\Pi\Rightarrow\Delta,\Sigma}Cut\langle m+1,j+1\rangle$$

and we transform it into

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\,\delta_{21}\\ \Pi\Rightarrow\Sigma,w:B\end{array}\quad\begin{array}{c}\vdots\,\delta_{11}\\ w:B,\Gamma\Rightarrow\Delta,w:C\end{array}}{\Pi,\Gamma\Rightarrow,\Sigma,\Delta,w:C}Cut\langle m,j\rangle\quad\begin{array}{c}\vdots\,\delta_{22}\\ w:C,\Pi\Rightarrow\Sigma\end{array}}{\Pi,\Gamma,\Pi\Rightarrow\Sigma,\Delta,\Sigma}Cut\langle m,j+k\rangle}{\Gamma,\Pi\Rightarrow\Delta,\Sigma}C_\dagger$$

where we have two cuts of lesser rank that are admissible by IH.

If the cut formula is $w : \forall xB$, we have

$$\dfrac{\dfrac{\begin{array}{c}\vdots\,\delta_{11}\\ \mathcal{E}y^w,\Gamma\Rightarrow\Delta,w:B[y/x]\end{array}}{\Gamma\Rightarrow\Delta,w:\forall xB}R\forall\quad\dfrac{\begin{array}{c}\vdots\,\delta_{21}\\ w:B[t/x],w:\forall xB,\mathcal{E}t^w,\Pi'\Rightarrow\Sigma\end{array}}{w:\forall xB,\mathcal{E}t^w,\Pi'\Rightarrow\Sigma}L\forall}{\Gamma,\mathcal{E}t^w,\Pi'\Rightarrow\Delta,\Sigma}Cut\langle m+1,j+1\rangle$$

with y^w respecting the variable condition in δ_1. By the substitution Lemma 3.16, we know that there is a $\star$-derivation δ'_{11} of $\mathcal{E}t^w,\Gamma\Rightarrow\Delta,w:B[t/x]$ of the same height of δ_{11}. By using δ'_{11}, we transform δ into

$$\dfrac{\dfrac{\begin{array}{c}\vdots\,\delta'_{11}\\ \mathcal{E}t^w,\Gamma\Rightarrow\Delta,w:B[t/x]\end{array}\quad\dfrac{\begin{array}{c}\vdots\,\delta_{1}\\ \Gamma\Rightarrow\Delta,w:\forall xB\end{array}\quad\begin{array}{c}\vdots\,\delta_{21}\\ w:\forall xB,w:B[t/x],\mathcal{E}t^w,\Pi'\Rightarrow\Sigma\end{array}}{w:B[t/x],\Gamma,\mathcal{E}t^w,\Pi'\Rightarrow\Delta,\Sigma}Cut\langle m+1,j\rangle}{\mathcal{E}t^w,\Gamma,\Gamma,\mathcal{E}t^w,\Pi'\Rightarrow\Delta,\Delta,\Sigma}Cut\langle m,j+k\rangle}{\Gamma,\mathcal{E}t^w,\Pi'\Rightarrow\Delta,\Sigma}C_\dagger$$

where we have two cuts of less rank that are admissible by IH.

If the cut formula is is $w : \exists x B$, we have

$$\dfrac{\dfrac{\begin{matrix}\vdots\,\delta_{11}\\ \mathcal{E}t^w, \Gamma' \Rightarrow \Delta, w : \exists x B, w : B[t/x]\end{matrix}}{\mathcal{E}t^w, \Gamma' \Rightarrow \exists x A : w, \Delta}R\exists \qquad \dfrac{\begin{matrix}\vdots\,\delta_{21}\\ w : B[y/x], \mathcal{E}y^w, \Pi \Rightarrow \Sigma\end{matrix}}{w : \exists x B, \Pi \Rightarrow \Sigma}L\exists}{\mathcal{E}t^w, \Gamma', \Pi \Rightarrow \Delta, \Sigma}Cut\langle m+1, j+1\rangle$$

with y^w respecting the variable condition in δ_2. As for the previous case, we apply Lemma 3.16, and then we transform δ into

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\vdots\,\delta_{11}\\ \mathcal{E}t^w, \Gamma' \Rightarrow \Delta, w : \exists x B, w : B[t/x]\end{matrix} \quad \begin{matrix}\vdots\,\delta_{2}\\ w : \exists x B, \Pi \Rightarrow \Sigma\end{matrix}}{\mathcal{E}t^w, \Gamma', \Pi \Rightarrow \Delta, \Sigma, w : B[t/x]}Cut\langle m+1, j\rangle \quad \begin{matrix}\vdots\,\delta'_{21}\\ w : B[t/x], \mathcal{E}t^w, \Pi \Rightarrow \Sigma\end{matrix}}{\mathcal{E}t^w, \Gamma', \Pi, \mathcal{E}t^w, \Pi \Rightarrow \Delta, \Sigma, \Sigma}Cut\langle m, j+k\rangle}{\mathcal{E}t^w, \Gamma', \Pi \Rightarrow \Delta, \Sigma}C_{\dagger}$$

where we have two cuts of less rank that are admissible by IH.

If the cut formula is $w : |^{\vec{t}}_{\vec{x}}|B$, we have

$$\dfrac{\dfrac{\begin{matrix}\vdots\,\delta_{11}\\ w\mathscr{R}u, t^{\vec{w}}\mathscr{T}\vec{x}^{\vec{u}}, \Gamma \Rightarrow \Delta, u : B\end{matrix}}{\Gamma \Rightarrow \Delta, w : |^{\vec{t}}_{\vec{x}}|B}R|^{\vec{t}}_{\vec{x}}| \qquad \dfrac{\begin{matrix}\vdots\,\delta_{21}\\ v : B[\vec{s}/\vec{x}], w : |^{\vec{t}}_{\vec{x}}|B, w\mathscr{R}v, t^{\vec{w}}\mathscr{T}\vec{s}^{\vec{v}}, \Pi' \Rightarrow \Sigma\end{matrix}}{w : |^{\vec{t}}_{\vec{x}}|B, w\mathscr{R}v, t^{\vec{w}}\mathscr{T}\vec{s}^{\vec{v}}, \Pi' \Rightarrow \Sigma}L|^{\vec{t}}_{\vec{x}}|}{\Gamma, w\mathscr{R}v, t^{\vec{w}}\mathscr{T}\vec{y}^{\vec{v}}, \Pi' \Rightarrow \Delta, \Sigma}Cut\langle m+1, j+1\rangle$$

where $\Pi = w\mathscr{R}v, t^{\vec{w}}\mathscr{T}\vec{s}^{\vec{v}}, \Pi'$, and with u respecting the variable condition in δ_1. Observe that also the $x_i \in \vec{x}^{\vec{u}}$ are eigenvariables, and that they are pairwise disjoint (since otherwise the expression $w : |^{\vec{t}}_{\vec{x}}|B$ wouldn't be an ext-formula). We apply to δ_{11} the height-preserving admissible Lemmas of substitution 3.16 and 3.17, in this precise order, to substitute $\vec{x}^{\vec{u}}$ with $\vec{s}^{\vec{u}}$ and u with v. By the observations made above, these substitutions have no effect on Γ nor on Δ. We obtain a $\star$-derivation δ'_{11} of $w\mathscr{R}v, t^{\vec{w}}\mathscr{T}\vec{s}^{\vec{v}}, \Gamma \Rightarrow \Delta, v : B[\vec{s}/\vec{x}]$. We can now transform δ into

$$
\dfrac{\dfrac{\vdots\ \delta'_{11}}{w\mathscr{R}v, t^{\vec{w}}\mathscr{T}s^{\vec{v}}, \Gamma \Rightarrow \Delta, v : B[\vec{s}/\vec{x}]} \qquad \dfrac{\dfrac{\vdots\ \delta_{1}}{\Gamma \Rightarrow \Delta, w : |{}^{\vec{t}}_{\vec{x}}|B} \qquad \dfrac{\vdots\ \delta_{21}}{w : |{}^{\vec{t}}_{\vec{x}}|B, v : B[\vec{s}/\vec{x}], w\mathscr{R}v, t^{\vec{w}}\mathscr{T}s^{\vec{v}}, \Pi' \Rightarrow \Sigma}}{v : B[\vec{s}/\vec{x}], \Gamma, w\mathscr{R}v, t^{\vec{w}}\mathscr{T}s^{\vec{v}}, \Pi' \Rightarrow \Delta, \Sigma}\, Cut\langle m+1, j\rangle}{\dfrac{\Gamma, w\mathscr{R}v, t^{\vec{w}}\mathscr{T}s^{\vec{v}}, \Gamma, w\mathscr{R}v, t^{\vec{w}}\mathscr{T}s^{\vec{v}}, \Pi' \Rightarrow \Delta, \Delta, \Sigma}{\Gamma, w\mathscr{R}v, t^{\vec{w}}\mathscr{T}s^{\vec{v}}, \Pi' \Rightarrow \Delta, \Sigma}\, C_{\dagger}}\, Cut < m, j+k\rangle
$$

with two admissible cuts of less rank.

If the cut formula is $w : \langle{}^{\vec{t}}_{\vec{x}}\rangle B$, we have

$$
\dfrac{\dfrac{\dfrac{\vdots\ \delta_{11}}{w\mathscr{R}v, t^{\vec{w}}\mathscr{T}s^{\vec{v}}, \Gamma' \Rightarrow \Delta, w : \langle{}^{\vec{t}}_{\vec{x}}\rangle B, v : B[\vec{s}/\vec{x}]}}{w\mathscr{R}v, t^{\vec{w}}\mathscr{T}s^{\vec{v}}, \Gamma' \Rightarrow \Delta, w : \langle{}^{\vec{t}}_{\vec{x}}\rangle B}\, R\langle{}^{\vec{t}}_{\vec{x}}\rangle \qquad \dfrac{\dfrac{\vdots\ \delta_{21}}{u : B, w\mathscr{R}u, t^{\vec{w}}\mathscr{T}x^{\vec{u}}, \Pi \Rightarrow \Sigma}}{w : \langle{}^{\vec{t}}_{\vec{x}}\rangle B, \Pi \Rightarrow \Sigma}\, L\langle{}^{\vec{t}}_{\vec{x}}\rangle}{w\mathscr{R}v, t^{\vec{w}}\mathscr{T}s^{\vec{v}}, \Gamma', \Pi \Rightarrow \Delta, \Sigma}\, Cut\langle m+1, j+1\rangle
$$

with u respecting the variable condition in δ_2. By Lemmas 3.16 and 3.17, there is a $\star$-derivation δ'_{21}, having same derivation-height of δ_{21}, of the sequent $v : B[\vec{s}/\vec{x}], w\mathscr{R}v, t^{\vec{w}}\mathscr{T}s^{\vec{v}}, \Pi \Rightarrow \Sigma$. We transform δ into

$$
\dfrac{\dfrac{\dfrac{\vdots\ \delta_{11}}{w\mathscr{R}v, t^{\vec{w}}\mathscr{T}s^{\vec{v}}, \Gamma' \Rightarrow \Delta, v : B[\vec{s}/\vec{x}], w : \langle{}^{\vec{t}}_{\vec{x}}\rangle B} \qquad \dfrac{\vdots\ \delta_{2}}{w : \langle{}^{\vec{t}}_{\vec{x}}\rangle B, \Pi \Rightarrow \Sigma}}{w\mathscr{R}v, t^{\vec{w}}\mathscr{T}s^{\vec{v}}, \Gamma', \Pi \Rightarrow \Delta, \Sigma, v : B[\vec{s}/\vec{x}]}\, Cut\langle m+1, j\rangle \qquad \dfrac{\vdots\ \delta'_{21}}{v : B[\vec{s}/\vec{x}], w\mathscr{R}v, t^{\vec{w}}\mathscr{T}s^{\vec{v}}, \Pi \Rightarrow \Sigma}}{\dfrac{w\mathscr{R}v, t^{\vec{w}}\mathscr{T}s^{\vec{v}}, \Gamma', \Pi, w\mathscr{R}v, t^{\vec{w}}\mathscr{T}s^{\vec{v}}, \Pi \Rightarrow \Delta, \Sigma, \Sigma}{w\mathscr{R}v, t^{\vec{w}}\mathscr{T}s^{\vec{v}}, \Gamma', \Pi \Rightarrow \Delta, \Sigma}\, C_{\dagger}}\, Cut\langle m, j+k\rangle
$$

with two cuts of less rank that are admissible by IH. ■

Observe that nonlogical rules have to be considered only in the case of a cut of rank $\langle 0, j+1\rangle$ because their principal formulas are all atomic. The presence of nonlogical rules doesn't impair the admissibility of cut because their active and principal formulas cannot be principal in the left premiss (unless they are the principal formulas of an initial sequent).

3.6 Structural Rules and Identity

In axiomatic systems the logic of identity is expressed by the axioms $t \doteq t$ and $t \doteq s \wedge A[t/x] \to A[s/x]$, whereas in sequent calculi it is often expressed as a Post system by extending the basic logical calculus with the nonlogical initial sequents $\Rightarrow t \doteq t$ and $t \doteq s, P[t/x] \Rightarrow P[s/x]$ (P atomic) [Gir87, p. 123]. In this section we show that the rules Ref, Lbz_1 (and Lbz_2) are adequate to capture the logic of the identity relation '$\doteq$' in the sense that the nonlogical initial sequents for for identity are derivable from our nonlogical rules for identity. We will prove also that the rule of replacement of identicals in arbitrary l-formulas is admissible in **GIM.**$\star_=$. We begin by proving that the rules of transitivity and symmetry of the identity symbol are derivable.

Lemma 3.23. *The following rules are derivable in* **GIM.**$\star_=$

1.
$$\frac{w : t \doteq r, w : t \doteq s, w : s \doteq r, \Gamma \Rightarrow \Delta}{w : t \doteq s, w : s \doteq r, \Gamma \Rightarrow \Delta}\, Trans$$

2.
$$\frac{w : s \doteq t, w : t \doteq s, \Gamma \Rightarrow \Delta}{w : t \doteq s, \Gamma \Rightarrow \Delta}\, Sym$$

Proof. In both cases we show how to obtain a $\star$-derivation of the conclusion from one of the premiss.

1. For rule $Trans$ we have only to apply Lbz_1 w.r.t. the atomic l-formula $w : x \doteq r$

$$\frac{w : t \doteq r, w : t \doteq s, w : s \doteq r, \Gamma \Rightarrow \Delta}{w : t \doteq s, w : s \doteq r, \Gamma \Rightarrow \Delta}\, Lbz_1$$

2. For rule Sym we proceed as follows, applying Lbz_1 w.r.t. $w : x \doteq t$

$$\dfrac{\dfrac{\dfrac{w : s \doteq t, w : t \doteq s, \Gamma \Rightarrow \Delta}{w : t \doteq t, w : t \doteq s, w : s \doteq t, \Gamma \Rightarrow \Delta}\, W_l \Rightarrow}{w : t \doteq t, w : t \doteq s, \Rightarrow \Delta}\, Lbz_1}{w : t \doteq s, \Gamma \Rightarrow \Delta}\, Ref$$

∎

Lemma 3.24. *The following sequents are derivable in* **GIM.**$\star_=$

1. $\Rightarrow w : t \doteq t$;

2. $w : t \doteq s, w : A[t/x] \Rightarrow w : A[t/x]$.

Proof. To prove 1, we apply Ref to the initial sequent $w : t \doteq t \Rightarrow w : t \doteq t$.

The proof of 2 is by induction on the height $h(w : A)$ of the l-formula $w : A$. For the base case, if $A = \bot$, then we have an instance of the zero premiss rule $L\bot$; whereas, if $A = P$, we have only to apply Lbz_1 to the sequent $w : A[s/x], w : t \doteq s, w : A[t/x] \Rightarrow w : A[s/x]$, which is $\star$-derivable by Lemma 3.11.

If $h(w : A) = n + 1$, we argue by cases. If $A = B \circ C$, the only complex case is $A = B \to C$, where we proceed as follows

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{IH}
\UnaryInfC{$w : s \doteq t, w : B[s/x] \Rightarrow w : B[t/x]$}
\RightLabel{RW_l}
\UnaryInfC{$w : s \doteq t, w : B[s/x] \Rightarrow w : C[s/x], w : B[t/x]$}
\RightLabel{LW_l}
\UnaryInfC{$w : s \doteq t, w : B[s/x], w : t \doteq s \Rightarrow w : C[s/x], w : B[t/x]$}
\RightLabel{Sym}
\UnaryInfC{$w : B[s/x], w : t \doteq s \Rightarrow w : C[s/x], w : B[t/x]$}
\AxiomC{}
\RightLabel{IH}
\UnaryInfC{$w : C[t/x], w : t \doteq s \Rightarrow w : C[s/x]$}
\RightLabel{LW_l}
\UnaryInfC{$w : C[t/x], w : B[s/x], w : t \doteq s \Rightarrow w : C[s/x]$}
\RightLabel{$L \to$}
\BinaryInfC{$w : B[s/x], w : t \doteq s, w : B[t/x] \to C[t/x] \Rightarrow C[s/x]$}
\RightLabel{$R \to$}
\UnaryInfC{$w : t \doteq s, w : B[t/x] \to C[t/x] \Rightarrow w : B[s/x] \to C[s/x]$}
\end{prooftree}
$$

If $A = \mathcal{Q}yB$, if $x^w = y^w$ there is nothing to prove as the substitution is vacuous. Else, by the pure-variable condition, we know that s^w and t^w are free for x^w in $w : \mathcal{Q}yA$, therefore we have only to apply the inductive hypothesis to B and then the rules $L\mathcal{Q}$ and $R\mathcal{Q}$.

If $A = \dagger B$ with $\dagger \in \{|\frac{\vec{t}}{\vec{x}}|, \langle\frac{\vec{t}}{\vec{x}}\rangle\}$, we don't have to use the inductive hypothesis, but only the rules for $\dagger$, Lbz_2, the derivable rule Sym, and Lemma 3.11. Observe that we don't need to rely on the inductive hypothesis because indexed operators behave as atomic formulas w.r.t. substitutions. If, e.g., A is $w : |\frac{\vec{r}}{\vec{y}}|B$, we have

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{u : B, (\vec{r}[t/x])^w \mathscr{T} \vec{y}^{\vec{u}}, w : s \doteq t, w\mathscr{R}u, (\vec{r}[s/x])^w \mathscr{T} \vec{y}^{\vec{u}}, w : t \doteq s, w : |^{\vec{r}[t/x]}_{\vec{y}}|B \Rightarrow u : B}\ 3.11}{(\vec{r}[t/x])^w \mathscr{T} \vec{y}^{\vec{u}}, w : s \doteq t, w\mathscr{R}u, (\vec{r}[s/x])^w \mathscr{T} \vec{y}^{\vec{u}}, w : t \doteq s, w : |^{\vec{r}[t/x]}_{\vec{y}}|B \Rightarrow u : B}\ L|^{\vec{t}}_{\vec{x}}|}{w : s \doteq t, w\mathscr{R}u, (\vec{r}[s/x])^w \mathscr{T} \vec{y}^{\vec{u}}, w : t \doteq s, w : |^{\vec{r}[t/x]}_{\vec{y}}|B \Rightarrow u : B}\ Lbz_2}{w\mathscr{R}u, (\vec{r}[s/x])^w \mathscr{T} \vec{y}^{\vec{u}}, w : t \doteq s, w : |^{\vec{r}[t/x]}_{\vec{y}}|B \Rightarrow u : B}\ Sym}{w : t \doteq s, w : |^{\vec{r}[t/x]}_{\vec{y}}|B \Rightarrow w : |^{\vec{r}[s/x]}_{\vec{y}}|B}\ R|^{\vec{t}}_{\vec{x}}|$$

■

Before ending this section we show that the restriction of rule Lbz_1 to atomic predicate is not a real limitation since the rule that generalizes it to arbitrary l-formulas is admissible in any calculus with identity. The generalized rule will be useful later on in proving completeness of $\mathbf{GIM.}\star_=$. Note that the situation with Lbz_1 resembles that with initial sequents: in both cases we had to take as primitive only the instances with atomic l-formulas as principal in order to prove the admissibility of the structural rules, but then we can prove that we can drop the restriction to atomic formulas by showing that their generalization to arbitrary l-formulas are admissible in $\mathbf{GIM.}\star_{(=)}$.

Lemma 3.25. *The replacement rule*

$$\dfrac{w : A[s/x], w : t \doteq s, w : A[t/x], \Gamma \Rightarrow \Delta}{w : t \doteq s, w : A[t/x], \Gamma \Rightarrow \Delta}\ Repl$$

is admissible in $\mathbf{GIM.}\star_=$.

Proof. The proof depends on Lemma 3.24.2, and on the admissibility of cut and contraction: we have, simply, to apply a cut with an instance of the derivable 3.24.2, and then several instances of contraction as follows

$$\dfrac{\dfrac{\dfrac{}{w : t \doteq s, w : A[t/x] \Rightarrow w : A[s/x]}\ 3.24 \qquad w : A[s/x], w : t \doteq s, w : A[t/x], \Gamma \Rightarrow \Delta}{w : t \doteq s, w : A[t/x], w : t \doteq s, w : A[t/x], \Gamma \Rightarrow \Delta}\ Cut}{w : t \doteq s, w : A[t/x], \Gamma \Rightarrow \Delta}\ LC_l$$

■

CHAPTER 4

CHARACTERIZATION AND RELATED ISSUES

In this chapter we will show that each of the labelled sequent calculi we have defined is sound and complete w.r.t. the appropriate class of t-frames, and therefore that we are able to characterize the indexed extensions of a PML in the Lemmon-Scott fragment obtained by adding any combination of the properties corresponding to $CBF, NI, ND, BF, GF, SHRT$ as well as those corresponding to t-rigid and stable terms and to single domains. In this way we will be able to give completeness results in a modular way that encompasses all the cases mentioned above. Note that this modularity is not usually possible for completeness results given in the Henkin-style.

The chapter proceeds as follows. In Section 1 we prove soundness and completeness results. Then, in Section 2, we show that if a formula A is a theorem of an indexed axiomatic system, it is derivable in the corresponding labelled sequent calculus. Finally, in Section 3, we show that Kripke frames are a particular case of transition frames. Then we consider some well-known quantified modal logics valid on particular classes of Kripke fames and we highlight that our labelled calculi give a complete axiomatization even in cases where we have no complete axiomatic system for them. As an example, the axiomatic system **Q.2.BF** is sound w.r.t. the class $\mathcal{C}^{2,BF}$ of all convergent Kripke frames with constant domains, but incomplete w.r.t. it, see [Cre95], whereas the labelled calculus **GIM.C.2.SHRT.NI.ND.BF**^{r} is both sound and complete w.r.t. $\mathcal{C}^{2,BF}$.

4.1 Soundness and Completeness

The proof of soundness is structured as that in [Neg09]; we proceed, roughly, by defining what it means for a sequent to be valid on a class of t-frames, and then we show that initial sequents are valid in any class of t-frames and that each rule of **GIM.**$\star$ preserves validity over the appropriate class of t-frames.

Definition 4.1. Let $W^\star$ be the set of all world labels occurring in a sequent $\mathcal{S}$, $\mathcal{M}^t =< \mathcal{W}, \mathcal{R}, \mathcal{U}, \mathcal{D}, \mathcal{T}, \mathcal{I}\rangle$ a t-model, f a mapping from $W^\star$ to $\mathcal{W}$, and σ_f a function associating to each $f(w)$ some $f(w)$-assignment $\sigma_{f(w)}$. We say that:

- $\langle f, \sigma_f, \mathcal{M}\rangle$ satisfies $w : A(\in \mathcal{S})$ iff $\sigma_{f(w)} \models^{\mathcal{M}}_{f(w)} A$;
- $\langle f, \sigma_f, \mathcal{M}\rangle$ satisfies $w\mathscr{R}v(\in \mathcal{S})$ iff $f(w)\mathcal{R}f(v)$;
- $\langle f, \sigma_f, \mathcal{M}\rangle$ satisfies $t^w \mathscr{T} s^v(\in \mathcal{S})$ iff $I^{\sigma_{f(w)}}_{f(w)}(t)\mathcal{T}_{(f(w),f(v))}I^{\sigma_{f(v)}}_{f(v)}(s)$;
- $\langle f, \sigma_f, \mathcal{M}\rangle$ satisfies $\mathcal{E}t^w(\in \mathcal{S})$ iff $I^{\sigma_{f(w)}}_{f(w)}(t) \in D_w$. ■

Definition 4.2 ($\mathcal{C}^\star$-validity)**.** A sequent $\Gamma \Rightarrow \Delta$ is said to be:

- $\mathcal{C}^\star$*-valid* iff for every triple $\langle s, \sigma_f, \mathcal{M}\rangle$ where $\mathcal{M}$ is based on a t-frame in $\mathcal{C}^\star$, if all ext-formulas in Γ are satisfied by $\langle s, \sigma_f, \mathcal{M}\rangle$, then some l-formula in Δ is satisfied by $\langle s, \sigma_f, \mathcal{M}\rangle$.

The notions of $\mathcal{C}^\star$-r-validity, $\mathcal{C}^\star$-s-validity, and of $\mathcal{C}^\star$-r-s-validity are defined analogously, save that we restrict the universal quantification to the triples $\langle f, \sigma_f, \mathcal{M}\rangle$ such that $\mathcal{M}$ is based on a t-frame in $\mathcal{C}^\star$ and it is t-rigid, stable, both t-rigid and stable, respectively. ■

Theorem 4.3 (Soundness)**.** *If* $\Gamma \Rightarrow \Delta$ *is* $\star$*-derivable, then*

- *if neither r nor s is (a superscript) in $\star$, $\Gamma \Rightarrow \Delta$ is $\mathcal{C}^\star$-valid;*
- *if r, but not s, is (a superscript) in $\star$, $\Gamma \Rightarrow \Delta$ is $\mathcal{C}^\star$-r-valid;*
- *if s, but not r, is (a superscript) in $\star$, $\Gamma \Rightarrow \Delta$ is $\mathcal{C}^\star$-s-valid;*
- *if r and s, are (superscripts) in $\star$, $\Gamma \Rightarrow \Delta$ is $\mathcal{C}^\star$-r-s-valid.* ■

Proof. The proof is by induction on the height of the derivation δ of $\Gamma \Rightarrow \Delta$. We don't give a separate proof for each case, and we use 'valid' as a placeholder for the appropriate notion of validity of a sequent. If $\Gamma \Rightarrow \Delta$ is an initial sequent then some atomic l-formula $w : P$ occurs both in Γ and in Δ, and therefore $\Gamma \Rightarrow \Delta$ is valid; if it is a conclusion of $L\bot$, the theorem holds vacuously since $w : \bot$ is satisfied by no triple $\langle f, \sigma_f, \mathcal{M}\rangle$.

If the last step in δ is by a propositional rule, the theorem holds thanks to the inductive hypothesis (IH) and the definition of satisfaction. If, e.g., the last step is by $L\wedge$ with premiss $w : A, w : B, \Gamma' \Rightarrow \Delta$, then, by IH, we know that the premiss is valid, and thus also $w : A \wedge B, \Gamma' \Rightarrow \Delta$ is valid.

If the last step is by a quantifier rule without variable condition, say $L\forall$ with premiss $w : A[t/x], \mathcal{E}t^w, w : \forall x A, \Gamma' \Rightarrow \Delta$, then, by IH, every triple $\langle f, \sigma_f, \mathcal{M}\rangle$ that satisfies all ext-formulas in $w : A[t/x], \mathcal{E}t^w, w : \forall x A, \Gamma'$ satisfies also some l-formula in Δ. By the satisfaction clause for $\forall$ this entails that every triple $\langle f, \sigma_f, \mathcal{M}\rangle$ that satisfies all ext-formulas in $\mathcal{E}t^w, w : \forall x A, \Gamma'$ satisfies also some l-formula in Δ.

If the last step is by a quantifier rule with a variable condition, say $R\forall$ with premiss $\mathcal{E}y^w, \Gamma' \Rightarrow \Delta', w : A[y/x]$, then, by IH, every triple $\langle f, \sigma_f, \mathcal{M}\rangle$ that satisfies all ext-formulas in $\mathcal{E}y^w, \Gamma'$ satisfies also some l-formula in $\Delta', w : A[y/x]$. We have to prove that if $\langle f, \sigma_f, \mathcal{M}\rangle$ satisfies all ext-formulas in Γ', it satisfies also some l-formula in $\Delta', w : \forall x A$. Take a generic $\langle f, \sigma_f, \mathcal{M}\rangle$ that satisfies all ext-formulas in $\mathcal{E}y^w, \Gamma'$, and any $a \in D_w$, we can define a triple $\langle g, \sigma_g, \mathcal{M}\rangle$ where $\mathcal{M}$ is as before, g is identical to f and σ_g is like σ_f save that $\sigma_{g(w)}(y) = a$. $\langle g, \sigma_g, \mathcal{M}\rangle$, which is a generic triple that satisfies all ext-formulas in Γ' (since y^v is an eigenvariable of that rule), is such that if it doesn't satisfy some l-formula in Δ', then it satisfies $w : \forall x A$. Thus the sequent $\Gamma' \Rightarrow \Delta', w : \forall x A$ is valid.

If the last step is by a modal rule without variable conditions, say $R\langle{\vec{t} \atop \vec{x}}\rangle$ with premiss $w\mathscr{R}v, \vec{t^w}\mathscr{T}\vec{s^v}, \Gamma' \Rightarrow \Delta', w : \langle{\vec{t} \atop \vec{x}}\rangle A, v : A[\vec{s}/\vec{x}]$, every triple $\langle f, \sigma_f, \mathcal{M}\rangle$ that satisfies all ext-formulas in $w\mathscr{R}v, \vec{t^w}\mathscr{T}\vec{s^v}, \Gamma'$ satisfies also some l-formula in the succedent $\Delta', w : \langle{\vec{t} \atop \vec{x}}\rangle A, v : A[\vec{s}/\vec{x}]$. We prove that any triple that satisfies the antecedent (of the premiss) satisfies also $\Delta', w : \langle{\vec{t} \atop \vec{x}}\rangle A$. For any triple, if the l-formula it satisfies in the succedent (of the premiss) is in Δ' or it is $w : \langle{\vec{t} \atop \vec{x}}\rangle A$, there is nothing to prove, else it is $v : A[\vec{s}/\vec{x}]$, and, by the semantic clause for $\langle{\vec{t} \atop \vec{x}}\rangle$, that triple satisfies also $w : \langle{\vec{t} \atop \vec{x}}\rangle A$.

If the last step is by a modal rule with a variable condition, say $L\langle{\vec{t} \atop \vec{x}}\rangle$ with premiss $v : A, w\mathscr{R}v, \vec{t^w}\mathscr{T}\vec{x^v}, \Gamma' \Rightarrow \Delta$ (v and $\vec{x^v}$ eigenvariables), every triple $\langle f, \sigma_f, \mathcal{M}\rangle$ that satisfies all ext-formulas in $v : A, w\mathscr{R}v, \vec{t^w}\mathscr{T}\vec{x^v}, \Gamma'$ satisfies also some l-formula in the succedent Δ. We have to prove that every triple that satisfies all ext-formulas in $w : \langle{\vec{t} \atop \vec{x}}\rangle A, \Gamma'$ satisfies some l–formula in Δ. Take a generic triple $\langle f, \sigma_f, \mathcal{M}\rangle$ satisfying all ext-formulas in the antecedent of the premiss, given the semantic clause for $\langle{\vec{t} \atop \vec{x}}\rangle$ and given that neither v nor any variable in $\vec{x^v}$ occurs in $w : \langle{\vec{t} \atop \vec{x}}\rangle A, \Gamma'$, it is also a generic triple that satisfies $w : \langle{\vec{t} \atop \vec{x}}\rangle A, \Gamma'$, and we know that it satisfies some ext-formula in Δ.

If the language contains identity and the last rule is a nonlogical rule for identity, it is immediate to see that the theorem holds.

If the last rule is a propositional or transitional correspondence rule without eigenvariables or rule *Class*, the proof is straightforward. For example, if the last rule is $T^{\mathcal{R}}$ with premiss $w\mathscr{R}w, \Gamma \Rightarrow \Delta$, we have to check that rule $T^{\mathcal{R}}$ preserves validity over t-reflexive t-frames. Take any triple $\langle f, \sigma_f, \mathcal{M}\rangle$ satisfying all ext-formulas in $w\mathscr{R}w, \Gamma$ and some l-formula in Δ, the theorem holds because $\mathcal{M}$ is based on a t-reflexive t-frame.

If the last rule is a propositional or transitional correspondence rule with a variable condition or rule *Exist*, we proceed as in the previous case, but we have to make use of the variable condition. For example, if the last rule is BF with premiss $x^w\mathscr{T}t^v, \mathcal{E}x^v, w\mathscr{R}v, \mathcal{E}t^v, \Gamma' \Rightarrow \Delta$ (x^v eigenvariable), we have to check if that rule preserves validity over $\mathcal{D}$-surjective t-frames. Take any triple $\langle f, \sigma_f, \mathcal{M}\rangle$ that satisfies all ext-formulas in the antecedent of the premiss and some l-formula in Δ. Given that $\mathcal{M}$ is based on a $\mathcal{D}$-surjective t-frame, we know that there is an $o \in D_w$ such that $o\mathcal{T}_{(w,v)}I_{f(v)}^{\sigma_{f(v)}}(t)$. Let's define a triple $\langle g, \sigma_g, \mathcal{M}\rangle$ where g is identical to f, $\mathcal{M}$ is as before, and σ_g is like σ_f save that $\sigma_{g(w)}(x) = o$. $\langle g, \sigma_g, \mathcal{M}\rangle$ is a generic triple (with $\mathcal{M}$ based on a $\mathcal{D}$-surjective t-frame) that satisfies all ext-formulas in $w\mathscr{R}v, \mathcal{E}t^v, \Gamma'$; by construction it satisfies also $x^w \mathscr{T} t^v$ and $\mathcal{E}x^v$, therefore it satisfies some l-formula in Δ.

If the last rule is *Rig*, we have to show that it preserves $\mathcal{C}^\star$-r-validity (and $\mathcal{C}^\star$-r-s-validity). Let's suppose the last step of the $\star$-derivation is

$$\frac{(f(c,x))^w \mathscr{T} (f(c,x))^v, w\mathscr{R}v, c^w \mathscr{T} c^v, x^v \mathscr{T} x^v, \Gamma' \Rightarrow \Delta}{w\mathscr{R}v, c^w \mathscr{T} c^v, x^v \mathscr{T} x^v, \Gamma' \Rightarrow \Delta}\, Rig$$

Let $\langle f, \sigma_f, \mathcal{M}^r\rangle$ be any triple such that $\mathcal{M}^r$ is a t-rigid t-model based on a t-frame in $\mathcal{C}^\star$, and such that it satisfies all ext-formulas in $w\mathscr{R}v, c^w \mathscr{T} c^v, x^v \mathscr{T} x^v, \Gamma'$. Given that $\mathcal{M}^r$ is t-rigid, that triple satisfies also $(f(c,x))^w \mathscr{T} (f(c,x))^v$, i.e. that triple satisfies the antecedent of the premiss. By IH, we conclude that $\langle f, \sigma_f, \mathcal{M}^r\rangle$ satisfies some l-formula in Δ.

If the last rule is *Stab*, we have to show that it preserves $\mathcal{C}^\star$-s-validity (and $\mathcal{C}^\star$-r-s-validity). We proceed as for *Rig*: let's suppose the last step of the $\star$-derivation is

$$\frac{v : t \doteq f(c_1, c_2), (f(c_1, c_2))^w \mathscr{T} t^v, \Gamma' \Rightarrow \Delta}{(f(c_1, c_2))^w \mathscr{T} t^v, \Gamma' \Rightarrow \Delta}\, Stab$$

Let $\langle f, \sigma_f, \mathcal{M}^s\rangle$ be any triple such that $\mathcal{M}^s$ is a stable t-model based on a t-frame in $\mathcal{C}^\star$, and such that it satisfies all ext-formulas in $(f(c_1, c_2))^w \mathscr{T} t^v, \Gamma'$. Given that $\mathcal{M}^s$ is stable, this entails that $\sigma_{f(v)}(t) = \sigma_{f(v)}(f)(\sigma_{f(v)}(c_1), \sigma_{f(v)}(c_2))$, and therefore that that triple satisfies also $v : t \doteq f(c_1, c_2)$. Since $\langle f, \sigma_f, \mathcal{M}^s\rangle$ satisfies the antecedent of the premiss, we conclude, by IH, that it satisfies some l-formula in Δ. ■

4.1.1 Completeness

The proof of completeness is structured as that in [Neg09], which, in its turn, is based on that in [NP01] for the calculus **G3c** for classical fist-order logic. The proof proceeds as follows: we begin by defining a procedure for root-first proof search in an arbitrary calculus **GIM.**$\star$ that is *fair* in the sense that every rule of **GIM.**$\star$ that is applicable at some step is applied after a finite amount of steps. Then, if the proof search fails, we define a t-model and an assignment based on an open branch of the failed proof search. Finally, we show that the t-model $\mathcal{M}^{\mathcal{S}}$ and the assignment thus defined are such that they provide a countermodel to the root of the tree, and that $\mathcal{M}^{\mathcal{S}}$ is based on a t-frame in the class of all t-frames for the relevant IML.

Definition 4.4 ($\star$-reduction tree)**.** Let $\Gamma \Rightarrow \Delta$ be a sequent in the language of **GIM.**$\star$. We define inductively the following procedure for constructing a $\star$-reduction tree of $\Gamma \Rightarrow \Delta$:

Stage 0 has $\Gamma \Rightarrow \Delta$ at the root of the tree. **Stage** $\mathbf{n+1}$ has two cases:

Case 1. If each topmost sequent of the tree constructed at stage n is an initial sequent or a conclusion of $L\bot$, the construction of the tree ends.

Case 2. Else, we continue the construction of the tree by writing above the topmost sequents that don't satisfy the antecedent of case 1 the sequents that are obtainable by applying root-first the rules of **GIM.**$\star$. There are $14 + k + 1$ substages, where k is the number of nonlogical rules of GIM.$\star$, that have to be applied to each such topmost sequent $\Gamma' \Rightarrow \Delta'$ obtained at stage n.

Substage 1. The topmost sequent is such that m instances of rule $L\wedge$ are applicable (root-first): if the topmost sequent $\Gamma' \Rightarrow \Delta'$ is

$$w_1 : A_1 \wedge B_1, \ldots, w_m : A_m \wedge B_m, \Gamma'' \Rightarrow \Delta'$$

where no l-formula in Γ'' has $\wedge$ as principal operator. We write over it the new topmost sequent

$$w_1 : A_1, w_1 : B_1, \ldots, w_m : A_m, w_m : B_m, \Gamma'' \Rightarrow \Delta'$$

Substage 2. The topmost sequent is such that m instances of rule $R\wedge$ are applicable: if the topmost sequent is

$$\Gamma' \Rightarrow \Delta'', w_1 : A_1 \wedge B_1, \ldots, w_m : A_m \wedge B_m$$

where no l-formula in Δ'' has $\wedge$ as principal operator, we add the 2^m topmost sequent

$$\Gamma' \Rightarrow \Delta'', w_1 : C_1, \ldots, w_m : C_m$$

where each C_i is either A_i or B_i, and where all possible combinations of the C_is are made.

Substage 3 for $L\vee$ and *substage 4* for $R\vee$ are symmetrical to substage 2 and to substage 1, respectively.

Substage 5. We reduce all implications in the antecedent, if the topmost sequent is

$$w_1 : A_1 \to B_1, \ldots, w_m : A_m \to B_m, \Gamma'' \Rightarrow \Delta'$$

where no l-formula in Γ'' has $\to$ as principal operator, we write over it the new topmost sequents

$$w_{j_1} : B_{j_1}, \ldots, w_{j_k} : B_{j_k}, \Gamma'' \Rightarrow \Delta', w_{j_{k+1}} : A_{j_{k+1}}, \ldots, w_{j_m} : A_{j_m}$$

where $j_1, \ldots, j_k \in \{1, \ldots, m\}$ and $j_{k+1}, \ldots j_m \in \{1, \ldots, m\} - \{j_1, \ldots, j_k\}$. Note that this rule is equivalent to m root-first applications of $L \to$.

Substage 6. We reduce all implications in the succedent, if the topmost sequent is

$$\Gamma' \Rightarrow \Delta'', w_1 : A_1 \to B_1, \ldots, w_m : A_m \to B_m$$

where no l-formula in Γ'' has $\to$ as principal operator, we write over it the new topmost sequent

$$w_1 : A_1, \ldots, w_m : A_m, \Gamma' \Rightarrow \Delta'', w_1 : B_1, \ldots, w_m : B_m$$

Substage 7. We reduce all universally quantified l-formulas in the antecedent of the topmost sequent

$$w_1 : \forall x_1 A_1, \ldots, w_m : \forall x_m A_m, \Gamma'' \Rightarrow \Delta'$$

We write on top of it the sequent

$$w_{j_1} : A_{j_1}[t_{j_1}/x_{j_1}], \ldots, w_{j_k} : A_{j_k}[t_{j_k}/x_{j_k}], w_1 : \forall x_1 A_1, \ldots, w_m : \forall x_m A_m, \Gamma'' \Rightarrow \Delta'$$

where $j_1, \dots, j_k \in \{1, \dots, m\}$ and the $t_{j_i}^{w_{j_i}}$s $(1 \leq i \leq k)$ are all l-terms such that $\mathcal{E}t_{j_i}^{w_{j_i}}$ occurs in Γ''.

Substage 8. We reduce all universally quantified l-formulas in the succedent of the topmost sequent

$$\Gamma' \Rightarrow \Delta'', w_1 : \forall x_1 A_1, \dots, w_m : \forall x_m A_m$$

We write on top of it the sequent

$$\mathcal{E}y_1^{w_1}, \dots, \mathcal{E}y_m^{w_m}, \Gamma \Rightarrow \Delta', w_1 : A_1[y_1/x_1], \dots, w_m : A_m[y_m/x_m]$$

where the $y_i^{w_i}$s are l-variables not occurring in the sequent $\Gamma' \Rightarrow \Delta'$ that we are reducing.

Substage 9 for $L\exists$ and *substage 10* for $R\exists$ are symmetrical (w.r.t l-formulas) to substage 8 and to substage 7, respectively.

Substage 11. We reduce all 'boxed' l-formulas in the antecedent of the topmost sequent

$$w_1 : |_{\vec{x_1}}^{\vec{t_1}}|A_1, \dots, w_m : |_{\vec{x_m}}^{\vec{t_m}}|A_m, \Gamma'' \Rightarrow \Delta'$$

We write on top of it the sequent

$$v_{j_1} : A_{j_1}[\vec{s_{j_1}}/\vec{x_{j_1}}], \dots, v_{j_k} : A_{j_k}[\vec{s_{j_k}}/\vec{x_{j_k}}], w_1 : |_{\vec{x_1}}^{\vec{t_1}}|A_1, \dots, w_m : |_{\vec{x_m}}^{\vec{t_m}}|A_m, \Gamma'' \Rightarrow \Delta'$$

where $j_1, \dots, j_k \in \{1, \dots, m\}$; and where the v_{j_i}s and the $\vec{s}_{j_i}^{w_{j_i}}$s $(1 \leq i \leq k)$ are, respectively all world labels such that $w_{j_i}\mathscr{R}v_{j_i}$ is in Γ'', and all tuples of l-terms such that $\vec{t}_{j_i}^{w_{j_i}} \mathscr{T} \vec{s}_{j_i}^{v_{j_i}}$ occurs in Γ''.

Substage 12. We reduce all 'boxed' l-formulas in the succedent of the topmost sequent

$$\Gamma' \Rightarrow \Delta'', w_1 : |_{\vec{x_1}}^{\vec{t_1}}|A_1, \dots, w_m : |_{\vec{x_m}}^{\vec{t_m}}|A_m$$

We write on top of it the sequent

$$w_1\mathscr{R}v_1, \dots w_m\mathscr{R}v_m, \vec{t}_1^{w_1} \mathscr{T} \vec{x}_1^{v_1}, \dots, \vec{t}_m^{w_m} \mathscr{T} \vec{x}_m^{v_m}, \Gamma' \Rightarrow \Delta'', v_1 : A_1, \dots, v_m : A_m$$

where the v_is are world labels not occurring in the original topmost sequent.

Substage 13 for $L\langle^{\vec{t}}_{\vec{x}}\rangle$ and *substage 14* for $R\langle^{\vec{t}}_{\vec{x}}\rangle$ are symmetrical (w.r.t l-formulas) to substage 12 and to substage 11, respectively.

Substage 14+j. We apply root-first all applicable instances of some nonlogical rule of **GIM.**$\star$ as follows. If the rule has no variable condition, we apply it for all (pairs of) ext-terms occurring in the original topmost sequent for which it is applicable. To illustrate, if the rule is $T^{\mathcal{R}}$ and the topmost sequent is $\Gamma' \Rightarrow \Delta'$, we write on top of it the sequent $w_1\mathscr{R}w_1, \ldots, w_m\mathscr{R}w_m, \Gamma' \Rightarrow \Delta'$ where $w_1, \ldots, w_m$ are all the world labels occurring in $\Gamma' \Rightarrow \Delta'$. If the rule has a variable condition, we apply it root first for all sets of its principal formulas for which it has not already been applied at some previous step, keeping attention to respect the variable condition. To illustrate, if we are applying GF to the topmost sequent

$$w_1\mathscr{R}v_1, \mathcal{E}t_1^{w_1}, w_2\mathscr{R}v_2, \mathcal{E}t_2^{w_2}, \Gamma'' \Rightarrow \Delta'$$

we write on top of it the sequent

$$t_1^{w_1}\mathscr{T}x^{v_1}, \mathcal{E}x^{v_1}, t_2^{w_2}\mathscr{T}x^{v_2}, \mathcal{E}x^{v_2}, w_1\mathscr{R}v_1, \mathcal{E}t_1^{w_1}, w_2\mathscr{R}v_2, \mathcal{E}t_2^{w_2}, \Gamma'' \Rightarrow \Delta'$$

where x^{v_1} and x^{v_2} are l-variables new to $\Gamma' \Rightarrow \Delta'$. As noted in [NP11, p. 217] 'because of height-preserving substitution and height-preserving admissibility of contraction, once a rule with eigenvariables has been considered, it need not be instantiated again on the same principal formulas'. This observation is important for nonlogical rules with eigenvariables since it allows us to avoid infinite countermodels whenever possible.

Substage 14+j+1. If at no previous substage we have written some new topmost sequent, we rewrite the original topmost sequent on top of itself. ■

The $\star$-reduction tree of a sequent $\Gamma \Rightarrow \Delta$ is such that if it produces a finite tree, then the sequent $\Gamma \Rightarrow \Delta$ is $\star$-derivable. Else it produces an infinite tree. In this case the root-first proof search has failed and by König's Lemma the $\star$-reduction tree has an infinite branch from which it is possible to construct a countermodel to $\Gamma \Rightarrow \Delta$ that is based on a t-frame in the appropriate class.

Observe that the substage $14 + j + 1$

> is needed to treat uniformly the failure of proof search in the following two cases: the case in which the search goes on forever because new rules always become applicable and the case in which a sequent is reached which is not a conclusion of any rule nor an initial sequent.
> [NP11, p. 217]

Note that the procedure defined for constructing the $\star$-reduction stage is fair because if an instance of a rule of **GIM.**$\star$ becomes applicable at stage k, then it is applied at stage $k + 1$.[1] In most cases the procedure writes a topmost sequent with more than one instance of the same formula in its antecedent or in its succedent, but we can erase all instances except one by the height-preserving admissibility of contraction.

Next, we are going to use an infinite branch of a $\star$-reduction tree of a sequent $\Gamma \Rightarrow \Delta$ to construct a t-model and an assignment that satisfy every formula in Γ and no formula in Δ —i.e. it is a countermodel for $\Gamma \Rightarrow \Delta$.

Definition 4.5 ($\mathcal{M}^{\mathcal{S}}$ and $\sigma_{\mathcal{S}}$)**.** Let $\mathcal{S}$ be the union of all sequents occurring in an infinite branch of a $\star$-reduction tree. $\mathcal{M}^{\mathcal{S}} = \langle \mathcal{W}^{\mathcal{S}}, \mathcal{R}^{\mathcal{S}}, \mathcal{U}^{\mathcal{S}}, \mathcal{D}^{\mathcal{S}}, \mathcal{T}^{\mathcal{S}}, \mathcal{I}^{\mathcal{S}} \rangle$ and $\sigma_{\mathcal{S}}$ are defined as follows:[2]

- $\mathcal{W}^{\mathcal{S}}$ is the set of all world labels occurring in $\mathcal{S}$;
- $\mathcal{U}^{\mathcal{S}}$ maps each $w \in \mathcal{W}^{\mathcal{S}}$ to the set $U^{\mathcal{S}}_w$ of all equivalence classes $[t^w]$ under w-identities occurring in the antecedent of $\mathcal{S}$ of ground l-terms t^w occurring in $\mathcal{S}$;
- $\mathcal{D}^{\mathcal{S}}$ maps each $w \in \mathcal{W}^{\mathcal{S}}$ to the set $D^{\mathcal{S}}_w$ of all equivalence classes $[t^w]$ of l-terms t^w such that $\mathcal{E}t^w$ occurs in $\mathcal{S}$;
- $\mathcal{R}^{\mathcal{S}}$ is such that for all $w, v \in \mathcal{W}^{\mathcal{S}}$, $w\mathcal{R}^{\mathcal{S}}v$ iff $w\mathscr{R}v$ occurs in $\mathcal{S}$;

[1] Or, if it is a redundant instance of a nonlogical rule with a variable condition, it is never applied because there is no need to do so.
[2] Where by 'ground term' we mean any term that is ground and such that no l-variable occurring in it is bound by a quantifier.

- $\mathcal{T}^{\mathcal{S}}$ is such that for every $t^w \in U_w$ and every $s^v \in U_v$, $[t^w]\mathcal{T}^{\mathcal{S}}_{(w,v)}[s^v]$ iff $t^w \mathscr{T} s^v$ occurs in $\mathcal{S}$;[3]

- $\mathcal{I}^{\mathcal{S}}$ maps each $w \in \mathcal{W}^{\mathcal{S}}$ to a local interpretation I_w such that
  - $I_w(c) = [c^w]$, for every ground l-constant c^w occurring in $\mathcal{S}$;
  - $I_w(f(\vec{t})) = [(f(\vec{t}))^w]$ for every ground functional l-term $(f(\vec{t}))^w$ occurring in $\mathcal{S}$;
  - $\mathcal{I}_w(R^n) = \{\langle [t_1^w], \ldots, [t_n^w] \rangle : \quad w : R^n(t_1, \ldots, t_n)$ occurs in the antecedent of $\mathcal{S}\}$.

- $\sigma^{\mathcal{S}}$ maps each $w \in \mathcal{W}^{\mathcal{S}}$ to a w-assignment $\sigma_{\mathcal{S},w}$ such that if x^w is an l-variable occurring free in $\mathcal{S}$, then $\sigma_{\mathcal{S},w}(x) = [x^w]$. ■

Observe that $\mathcal{M}^{\mathcal{S}}$ is well defined because $\langle \mathcal{W}^{\mathcal{S}}, \mathcal{R}^{\mathcal{S}}, \mathcal{U}^{\mathcal{S}}, \mathcal{D}^{\mathcal{S}}, \mathcal{T}^{\mathcal{S}} \rangle$ is a t-frame, and because no atomic l-formulas can occur both in the antecedent and in the succedent of $\mathcal{S}$ since no rule applied root-first allows to delete an atomic l-formulas. It is also immediate to recognize that the equivalence classes $[t^w]$ are well defined. In order to simplify the notation, in the following lemmas we will allow an l-term to stand for its equivalence class.[4]

Lemma 4.6 (Truth lemma). *Let $\Gamma \Rightarrow \Delta$ be some sequent occurring in some node of the infinite branch of a $\star$-reduction tree that we have used to define $\mathcal{M}^{\mathcal{S}}$ and $\sigma^{\mathcal{S}}$. For any l-formula $w : A$ we have that*

1. *if $w : A$ occurs in Γ, then* $\quad \sigma_{\mathcal{S},w} \models^{\mathcal{M}^{\mathcal{S}}}_{w} A$;
2. *if $w : A$ occurs in Δ, then* $\quad \sigma_{\mathcal{S},w} \not\models^{\mathcal{M}^{\mathcal{S}}}_{w} A$.

Proof. The proof is by induction on the formula-height of $w : A$. If $w : A$ is an atomic formula $w : R^n(t_1, \ldots, t_n)$, the lemma holds by construction of $\mathcal{M}^{\mathcal{S}}$ and $\sigma^{\mathcal{S}}$, and if $w : A$ is $w : t \doteq s$, it holds because the l-terms t^w and

[3] Note that if a member of an equivalence class is a counterpart of some object, then also all other members of that class are counterparts of that object because we have added the relevant t-formulas at some stage.

[4] Thus a term stands for itself when it occurs in a formula of a sequent, and for its equivalence class when it occurs as an object of a domain.

s^w, being in the same equivalence class, have the same w-extension. The l-formula $w : \bot$ cannot occur in Γ.

If $w : A$ is $w : B \wedge C$ and it occurs in the antecedent of $\Gamma \Rightarrow \Delta$, then it has been reduced at some stage k of the construction of the $\star$-reduction tree. Thus $w : B$ and $w : C$ occur in the antecedent of some sequent in $\mathcal{S}$ (namely in that introduced at the 1^{st} substage of stage k), and the lemma holds by IH and by definition of satisfaction.

If $w : A$ is $w : B \wedge C$ and it occurs in the succedent of $\Gamma \Rightarrow \Delta$, then it has been reduced at some stage k of the construction of the $\star$-reduction tree. Thus $w : B$ or $w : C$ occur in the succedent of some sequent in $\mathcal{S}$, and the lemma holds by IH and by the definition of satisfaction.

If $w : A$ is $w : B \vee C$ or $w : B \rightarrow C$, we proceed as in one of the two cases above.

If $w : A$ is $w : \forall x B$ and it occurs in the antecedent of $\Gamma \Rightarrow \Delta$, two cases are possible: either D_w is empty or not. If D_w is empty, $\forall x B$ is vacuously true at w. If D_w is not empty, then for every $t_w \in D_w$, the e-formula $\mathcal{E}t^w$ occurs in the antecedent of the member of $\mathcal{S}$ introduced at stake k of the construction, and therefore at stage $k+1$ we have introduced a sequent such that $w : B[t/x]$ occurs in its antecedent. By IH, this implies that $\sigma_{\mathcal{S},w} \models_w^{\mathcal{M}^{\mathcal{S}}} B[t/x]$, and therefore, by Lemma 2.10, $\sigma_{\mathcal{S},w}^{x \triangleright \sigma_{\mathcal{S},w}(t)} \models_w^{\mathcal{M}^{\mathcal{S}}} B$. We conclude that $\sigma_{\mathcal{S},w} \models_w^{\mathcal{M}^{\mathcal{S}}} \forall x B$.

If $w : A$ is $w : \forall x B$ and it occurs in the succedent of $\Gamma \Rightarrow \Delta$, then it has been reduced at some stage k of the construction of the $\star$-reduction tree. Thus $\mathcal{E}y^w$ is in the antecedent of the sequent written at stage $k+1$ and $w : B[y/x]$ is in its succedent. By construction $y^w \in D_w$, and, by IH, $\sigma_{\mathcal{S},w} \not\models_w^{\mathcal{M}^{\mathcal{S}}} B[y/x]$. We conclude that $\sigma_{\mathcal{S},w} \not\models_w^{\mathcal{M}^{\mathcal{S}}} \forall x B$.

The two cases for $w : \exists x B$ are symmetrical to those for $w : \forall x B$ in the succedent and in the antecedent, respectively.

If $w : A$ is $w : |\frac{\vec{t}}{\vec{x}}|B$ and it occurs in the antecedent of $\Gamma \Rightarrow \Delta$, two cases are possible: either for some v we have that $w\mathcal{R}^{\mathcal{S}}v$ and $\vec{t^w}\mathcal{T}^{\mathcal{S}}_{(w,v)}\vec{s^v}$ (for some $\vec{s^v}$) or not. In the latter case $|\frac{\vec{t}}{\vec{x}}|B$ is vacuously satisfied at w. In the former case, for every v and every $\vec{s^v}$ such that $w\mathcal{R}^{\mathcal{S}}v$ and $\vec{t^w}\mathcal{T}^{\mathcal{S}}_{(w,v)}\vec{s^v}$, we know that at some stage k of the construction we encountered a sequent whose antecedent included $w : |^{\vec{t}}_{\vec{x}}|B$, $w\mathscr{R}v$ and each member of $\vec{t^w}\mathscr{T}\vec{s^v}$, therefore at stage $k+1$ we have written a sequent containing in the antecedent also $v : B[\vec{s}/\vec{x}]$, which, by IH, entails that $\sigma_{\mathcal{S},v} \models^{\mathcal{M}^{\mathcal{S}}}_{w} B[\vec{s}/\vec{x}]$. By Lemma 2.10, this implies that $\sigma^{\vec{x} \triangleright \vec{s}}_{\mathcal{S},v} \models^{\mathcal{M}^{\mathcal{S}}}_{w} B$. We have shown that for all $v \in \mathcal{W}^{\mathcal{S}}$ and for all v-assignment σ_v, $w\mathcal{R}^{\mathcal{S}}v$ and $\sigma_{\mathcal{S},w}(\vec{t})\mathcal{T}^{\mathcal{S}}_{(w,v)}\sigma_v(\vec{x})$ implies $\sigma_v \models^{\mathcal{M}^{\mathcal{S}}}_{v} B$, therefore $\sigma_{\mathcal{S},w} \models^{\mathcal{M}^{\mathcal{S}}}_{w} |^{\vec{t}}_{\vec{x}}|B$.

If $w : A$ is $w : |^{\vec{t}}_{\vec{x}}|B$ and it occurs in the succedent of $\Gamma \Rightarrow \Delta$, then it has been reduced at some stage k of the construction of the $\star$-reduction tree. Thus $w\mathscr{R}v$ and all members of $\vec{t^w}\mathscr{T}\vec{x^v}$ are in the antecedent of the sequent written at stage $k+1$ and $v : B$ is in its succedent. By construction of $\mathcal{M}^{\mathcal{S}}$, we have that $w\mathcal{R}^{\mathcal{S}}v$ and $\sigma_{\mathcal{S},w}(\vec{t})\mathcal{T}^{\mathcal{S}}_{(w,v)}\sigma_{\mathcal{S},v}(\vec{x})$. By IH, $\sigma_{\mathcal{S},v} \not\models^{\mathcal{M}^{\mathcal{S}}}_{v} B$. We conclude that $\sigma_{\mathcal{S},w} \not\models^{\mathcal{M}^{\mathcal{S}}}_{w} |^{\vec{t}}_{\vec{x}}|B$.

The two cases for $w : \langle^{\vec{t}}_{\vec{x}}\rangle B$ are symmetrical to those for $w : |^{\vec{t}}_{\vec{x}}|B$ in the succedent and in the antecedent, respectively. ■

Theorem 4.7 (Completeness)**.** *Each sequent calculus we have defined is complete with respect to the appropriate notion of validity. More precisely*

- *If a sequent is valid in the set of all t-rigid and stable t-models that are based on a t-frame in $\mathcal{C}^\star$, then it is derivable in* **GIM.**$\star^{r,s}$*;*
- *If a sequent is valid in the set of all t-rigid t-models that are based on a t-frame in $\mathcal{C}^\star$, then it is derivable in* **GIM.**$\star^{r}$*;*
- *If a sequent is valid in the set of all stable t-models that are based on a t-frame in $\mathcal{C}^\star$, then it is derivable in* **GIM.**$\star^{s}$*;*
- *If a sequent is valid in the set of all t-models that are based on a t-frame in $\mathcal{C}^\star$, then it is derivable in* **GIM.**$\star$.

Proof. For **GIM.K** the theorem is an immediate corollary of Lemma 4.6: if a sequent is not **K**-derivable, then it is not **K**-valid since $\mathcal{M}^{\mathcal{S}}$ satisfies (under $\sigma^{\mathcal{S}}$) all formulas occurring in its antecedent and no formula occurring in its succedent.

All other cases hold by Lemma 4.6 since the construction of the model $\mathcal{M}^{\mathcal{S}}$ is such that it is based on a t-frame in the class $\mathcal{C}^\star$ of all t-frames for the calculus **GIM.**$\star$, and, if needed, $\mathcal{M}^{\mathcal{S}}$ is a t-rigid and/or stable t-model. In general this fact holds because the nonlogical rules are obtained as semantical explanations of the semanic conditions we impose on t.frames and/or on t-models. We show some cases. If *Rig* and/or *Stab* are rules of **GIM.**$\star$ then $\mathcal{M}^{\mathcal{S}}$ is t-rigid and/or stable. For example, if for some $t_1^w, \dots t_n^w \in U_w$ and some $s_1^v, \dots s_n^v \in U_w$ such that $\bigwedge_{i=1}^n t_i^w \mathcal{T}_{(w,v)}^{\mathcal{S}} s_i^v$, then, for any ground l-term $(f(t_1, \dots, t_n))^w$ occurring in $\mathcal{S}$, rule *Rig* entails that $I_w^{\mathcal{S}}(f)(\mathcal{I}_w^{\mathcal{S}}(t_1, \dots, t_n))\mathcal{T}_{(w,v)}^{\mathcal{S}} I_v^s(f)(I_v^{\mathcal{S}}(s_1, \dots, s_n))$ because at some stage of the construction we encountered the topmost sequent

$$w\mathscr{R}v, t_1^w \mathscr{T} s_1^v, \dots, t_n^w \mathscr{T} s_n^v, \Gamma' \Rightarrow \Delta'$$

and at its substage $14 + j$ we have written on top of it the sequent

$$(f(t_1, \dots, t_n))^w \mathscr{T} (f(s_1, \dots, s_n))^v, w\mathscr{R}v, t_1^w \mathscr{T} s_1^v, \dots, t_n^w \mathscr{T} s_n^v, \Gamma' \Rightarrow \Delta'$$

which entails that $\mathcal{M}^{\mathcal{S}}$ is t-rigid.[5]

If the calculus **GIM.**$\star$ is defined over the language with identity, then $\doteq$ is treated as real identity because objects are (representative of) equivalence classes of terms under l-identities occurring in some sequent in the $\star$-reduction tree.

If some propositional or transitional correspondence rule or one of the two rules for existence is in **GIM.**$\star$, then the definition of $\mathcal{M}^{\mathcal{S}}$ is such that it is based on a t-frame that has the property corresponding to that rule. For example, if rule *BF* is in **GIM.**$\star$, whenever at some stage we have encountered the topmost sequent

$$w\mathscr{R}v, \mathcal{E}t^v, \Gamma' \Rightarrow \Delta'$$

so that t^v is a member of D_v and $w\mathcal{R}^{\mathcal{S}}v$, then, at some substage, we have

[5] We assume, w.l.o.g., that if $(f(t_1, \dots, t_n))^w$ does not occur in $\mathcal{S}$, it is interpreted in such a way that it is both t-rigid and stable.

written as new topmost sequent

$$x^w \mathscr{T} t^v, \mathcal{E}x^v, w\mathscr{R}v, \mathcal{E}t^v, \Gamma' \Rightarrow \Delta'$$

so that x^w is a member of D_w and t^w is one of its (w, v)-transition. Given that we have been able to prove this for an arbitrary object t^w that exists in some world that is accessible from another world, we conclude that $\mathcal{M}^{\mathcal{S}}$ is based on a $\mathcal{D}$-surjective t-frame. ■

4.2 Sequent Calculi and Axiomatic Systems

In this section we consider axiomatic systems **Q°.⋆im** for IMLs, see Appendix B. We will show that if an $\mathcal{L}^{\boxdot}$-formula is a theorem of the axiomatic system **Q°.⋆im**, then the sequent $\Rightarrow w : A$ is $\star$-derivable. As shown in [Cor09], the basic axiomatic systems (with classical quantification) **Q.Kim** and **R.Kim** are sound and complete w.r.t. the class of all single domain t-models and w.r.t. the class of all single domain t-rigid t-models, respectively. Thus, by the completeness theorem, we already know that if A is a theorem of **Q.Kim** (**R.Kim**), then $\Rightarrow w : A$ is derivable in **GIM.K.C** (**GIM.K.C**^{r}) and vice versa, but we cannot use this semantical argument for others IMLs we have considered because we don't know whether their respective axiomatic systems are complete.

4.2.1 The Basic System

We begin by proving that **GIM.K** allows us to derive all theorems of the axiomatic system **Q°.Kim** given in Appendix B. To prove this result, we have simply to show that the axioms of **Q°.Kim** are **K**-derivable, and that its rules are admissible in **GIM.K**.

Proposition 4.8. *Let A, B be formulas whose free variables are among $\vec{x} = x_1, \ldots, x_n$, and w be an arbitrary world label. The following sequents are* **K***-derivable*

TAUT) $\Rightarrow w : A$,
where A is any (indexed instance of some propositional) tautology;

UI°) $\Rightarrow w : \forall y(\forall x A \to A[y/x])$;

CQ) $\Rightarrow w : \forall x \forall y A \leftrightarrow \forall y \forall x A$;

UD) $\Rightarrow w : \forall x(A \to B) \to (\forall x A \to \forall x B)$;

VQ) $\Rightarrow w : A \to \forall x A$ *where x is not free in A.*

K) $\Rightarrow w : |\vec{x}|(A \to B) \to (|\vec{x}|A \to |\vec{x}|B)$;

LNGT) $\Rightarrow w : |\vec{x}|A \to |\vec{x}\, y|A$;

PRM) $\Rightarrow w : |x_1 \ldots x_n|A \to |x_{p_1} \ldots x_{p_n}|A$,
for any permutation $x_{p_1}, \ldots, x_{p_n}$ *of* $x_1, \ldots, x_n$;

R^v) $\Rightarrow w : |^{y_1}_{x_1} \ldots ^{y_n}_{x_n}|A \to |y_1 \ldots y_k|(A[y_1/x_1 \ldots, y_n/x_n])$,
where $y_1, \ldots, y_k$ *include all different variables among* $y_1, \ldots, y_n$.

Proof. For *TAUT* it is enough to observe that, for indexed instances of propositional formulas, **GIM.K.** is nothing but the sequent system **G3cp** decorated with inessential world label.

The other cases are by applying, root-first, the rules of **GIM.K** and Lemma 3.11. The axioms for the quantifiers, *K* and *LNGT* are straightforward, and thus we omit the proofs. Observe that the **K**-derivability of *LNGT* depends on the fact that the *t*-formulas that are needed for rule $L|^{\vec{t}}_{\vec{x}}|$ are a proper subset of the ones we have introduced (root-first) with $R|^{\vec{t}}_{\vec{x}}|$. For *PRM* we have

$$\dfrac{\dfrac{\dfrac{\dfrac{}{v : A, w : |x_1 \ldots x_n|A, w\mathscr{R}v, x^w_{p_1}\mathscr{T}x^v_{p_1}, \ldots, x^w_{p_n}\mathscr{T}x^v_{p_n} \Rightarrow v : A}\,3.11}{v : |x_1 \ldots x_n|A, w\mathscr{R}v, x^w_{p_1}\mathscr{T}x^v_{p_1}, \ldots, x^w_{p_n}\mathscr{T}x^v_{p_n} \Rightarrow v : A}\,L|^{\vec{t}}_{\vec{x}}|}{w : |x_1 \ldots x_n|A \Rightarrow w : |x_{p_1} \ldots x_{p_n}|A}\,R|^{\vec{t}}_{\vec{x}}|}{\Rightarrow w : |x_1 \ldots x_n|A \to |x_{p_1} \ldots x_{p_n}|A}\,R\to$$

where the step by $L|^{\vec{t}}_{\vec{x}}|$ is feasible because $x_{p_1}, \ldots, x_{p_n}$ is a permutation of $x_1, \ldots x_n$. For R^v we have

$$\dfrac{\dfrac{\dfrac{\overline{v : A[y_n/x_n,\ldots,y_n/x_n], w\mathscr{R}v, y_1^w \mathscr{T} y_1^v,\ldots,y_k^w \mathscr{T} y_k^v, w : |^{y_1}_{x_1}\ldots^{y_n}_{x_n}|A \Rightarrow v : A[y_n/x_n,\ldots,y_n/x_n]}\ 3.11}{w\mathscr{R}v, y_1^w \mathscr{T} y_1^v,\ldots,y_k^w \mathscr{T} y_k^v, w : |^{y_1}_{x_1}\ldots^{y_n}_{x_n}|A \Rightarrow v : A[y_n/x_n,\ldots,y_n/x_n]}\ L|^{\vec{t}}_{\vec{x}}|}{w : |^{y_1}_{x_1}\ldots^{y_n}_{x_n}|A \Rightarrow w : |y_1\ldots y_k|(A[y_1/x_1,\ldots,y_n/x_n])}\ R|^{\vec{t}}_{\vec{x}}|}{\Rightarrow w : |^{y_1}_{x_1}\ldots^{y_n}_{x_n}|A \to |y_1\ldots y_k|(A[y_1/x_1,\ldots,yn/x_n])}\ R\to$$

where the step by $L|^{\vec{t}}_{\vec{x}}|$ is feasible because $y_1,\ldots,y_k$ include all different variables among $y_1,\ldots,y_n$. ■

Proposition 4.9. *The following rules are admissible in* **GIM.K**.

$MP)$
$$\dfrac{\Rightarrow w : A \qquad \Rightarrow w : A \to B}{\Rightarrow w : B}$$

$UG)$
$$\dfrac{\Rightarrow w : A \to B}{\Rightarrow w : A \to \forall x B} \quad \textit{if } x^w \textit{ doesn't occur free in } A$$

$N)$
$$\dfrac{\Rightarrow w : A}{\Rightarrow w : |\vec{x}|A} \quad \textit{if all free l-variables of } w : A \textit{ are in } \vec{x}$$

$S)$
$$\dfrac{\Rightarrow w : A}{\Rightarrow w : A[s/x]}$$

Proof. The **K**-admissiblity of MP is a corollary of the **K**-admissiblity of Cut and of RW_l, i.e. it is provable as follows:

$$\dfrac{\Rightarrow w : A \to B \qquad \dfrac{\dfrac{\Rightarrow w : A}{\Rightarrow w : B, w : A}\ RW_l \qquad \dfrac{}{w : B \Rightarrow w : B}\ 3.11}{w : A \to B \Rightarrow w : B}\ L\to}{\Rightarrow w : B}\ Cut$$

To show that UG is **K**-admissible, we proceed as follows:

$$
\dfrac{\Rightarrow w : A \to B \qquad \dfrac{\dfrac{}{\mathcal{E}x^w, w : A \Rightarrow w : B, w : A}\,3.11 \qquad \dfrac{}{w : B, \mathcal{E}x^w, w : A \Rightarrow w : B}\,3.11}{w : A \to B, \mathcal{E}x^w, w : A \Rightarrow w : B}\,L\to}{\dfrac{\dfrac{\mathcal{E}x^w, w : A \Rightarrow w : B[x/x]}{w : A \Rightarrow w : \forall x B}\,R\forall}{\Rightarrow w : A \to \forall x B}\,R\to}\,Cut
$$

Observe that the step by $R\forall$ is feasible because $w : B[x/x]$ is $w : B$ and —thanks to the side condition on UG— x^w satisfies the variable condition.

To show that N is **K**-admissible, we rely on the **K**-admissibility of the left rules of weakening, 3.18, and of substitution for world labels, 3.17, as follows (where the side condition $FV(w : A) \subseteq \vec{x}$ ensures that $w : |\vec{x}|A$ is well-formed)

$$
\dfrac{\dfrac{\dfrac{\Rightarrow w : A}{\Rightarrow v : A}\,3.17}{w\mathscr{R}v, \vec{x^w}\mathscr{T}\vec{x^v} \Rightarrow v : A}\,LW_\circ}{\Rightarrow w : |\vec{x}|A}\,R|^{\vec{t}}_{\vec{x}}|
$$

The **K**-admissibility of SFV is a corollary of Lemma 3.16. ■

Corollary 4.10. *For any formula A and any world label w, if A is a theorem of* **Q°.Kim**, *then the sequent $\Rightarrow w : A$ is* **K**-*derivable, i.e.*

$$
\text{if } \mathbf{Q^\circ.Kim} \vdash A, \text{ then } \mathbf{GIM.K} \vdash \Rightarrow w : A
$$

Proof. It follows from Proposition 4.8 and Lemma 4.9. ■

4.2.2 Derivability of Additional Axioms

We show that the characteristic axiom of an IML is derivable in the labelled sequent calculus for that IML. We begin by proving that rule *Class* allow us to derive every theorem of **Q.Kim**, and that the axioms for identity are derivable from the rules for identity.

Proposition 4.11. *The following sequent is $\star$-derivable in any calculus that includes rule Class*

$UI)\ \Rightarrow w : \forall x A \rightarrow A[y/x]$

Proof. We apply, root-first, the rules of **GIM.K** and *Class*

$$\dfrac{\dfrac{\dfrac{\dfrac{}{w : A[y/x], \mathcal{E}y^w, w : \forall x A \Rightarrow w : A[y/x]}\ 3.11}{\mathcal{E}y^w, w : \forall x A \Rightarrow w : A[y/x]}\ L\forall}{w : \forall x A \Rightarrow w : A[y/x]}\ Class}{\Rightarrow w : \forall x A \rightarrow A[y/x]}\ R\rightarrow$$

■

Proposition 4.12. *The following sequent are $\star$-derivable in any sequent calculus defined over the language with identity*

$ID)\qquad \Rightarrow w : x \doteq x\ ;$

$LBZ)\quad \Rightarrow w : t \doteq s \rightarrow (A[t/x] \rightarrow A[s/x])\ .$

Proof. See Lemma 3.24. ■

Next we show that rules *Rig* and *Stab* allows us to derive the axioms that characterize the assumptions that terms are t-rigid and stable, respectively.

Proposition 4.13. *We have that:*

$R)$ $\quad \Rightarrow w : |^{t_1}_{x_1} \dots {}^{t_n}_{x_n}|A \rightarrow |y_1 \dots y_k|(A[t_1, \dots, t_n/x_1, \dots x_n])$ *(where $y_1, \dots y_k$ are all the variables occurring in $t_1, \dots, t_n$) is $\star$-derivable whenever $Rig \in \star$;*

$S)$ $\quad \Rightarrow w : |\vec{y}|(A[\vec{t}/\vec{x}]) \rightarrow |\vec{y}\,{}^{\vec{t}}_{\vec{x}}|A$ *(where $\vec{t}$ is a tuple of closed terms) is $\star$-derivable whenever $Stab \in \star$.*

Proof. As far as r is concerned the root-first proof search procedure gives the following derivation

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{}{v : A[t_1, \dots, t_n/x_1, \dots, x_n], t_1^w \mathscr{T} t_1^v, \dots, t_n^v \mathscr{T} t_n^v, w\mathscr{R}v, y_1^w \mathscr{T} y_1^v, \dots, y_k^w \mathscr{T} y_k^v, w : |{}^{t_1}_{x_1} \dots {}^{t_n}_{x_n}|A \Rightarrow v : A[t_1, \dots, t_n/x_1, \dots, t_n]}\;3.11}{t_1^w \mathscr{T} t_1^v, \dots, t_n^v \mathscr{T} t_n^v, w\mathscr{R}v, y_1^w \mathscr{T} y_1^v, \dots, y_k^w \mathscr{T} y_k^v, w : |{}^{t_1}_{x_1} \dots {}^{t_n}_{x_n}|A \Rightarrow v : A[t_1, \dots, t_n/x_1, \dots, t_n]}\;L|{}^{\vec{t}}_{\vec{x}}|}{w\mathscr{R}v, y_1^w \mathscr{T} y_1^v, \dots, y_k^w \mathscr{T} y_k^v, w : |{}^{t_1}_{x_1} \dots {}^{t_n}_{x_n}|A \Rightarrow v : A[t_1, \dots, t_n/x_1, \dots, t_n]}\;\dagger}{w : |{}^{t_1}_{x_1} \dots {}^{t_n}_{x_n}|A \Rightarrow w : |y_1 \dots y_k|(A[t_1, \dots, t_n/x_1, \dots, t_n])}\;R|{}^{\vec{t}}_{\vec{x}}|}{\Rightarrow w : |{}^{t_1}_{x_1} \dots {}^{t_n}_{x_n}|A \to |y_1 \dots y_k|(A[t_1, \dots, t_n/x_1, \dots, t_n])}\;R\to$$

Where the step by † is feasible thanks to rule *Rig*, which, reasoning root-first, allows us to prove, by induction on term-height, that for every $t_i \in \{t_1, \dots, t_n\}$, $t_i^w \mathscr{T} t_i^v$ can be added to the left-hand side of the sequent. If $t_i = y_i$ there is nothing to prove and if $t_i = c$ we apply rule *Rig*. If $t_i = f(s_1, \dots, s_m)$ we know, by inductive hypothesis, that $s_1^w \mathscr{T} s_1^v, \dots, s_m^w \mathscr{T} s_m^v$, and we can apply n instances of *Rig*.

The sequent S is provable by applying root-first the rules of **GIM.K** and rule *Stab*. ■

Finally we prove that the (labelled version of the) formula that correspond to a condition on a t-frame is $\star$-derivable from the nonlogical rules for that class of t-frames.

Proposition 4.14. *We have that:*

T^t) $\Rightarrow w : |\vec{x}|A \to A$ *is $\star$-derivable whenever $T \in \star$;*

4^t) $\Rightarrow w : |\vec{x}|A \to |\vec{x}||\vec{x}|A$ *is $\star$-derivable whenever $4 \in \star$;*

5^t) $\Rightarrow w : \langle\vec{x}\rangle A \to |\vec{x}|\langle\vec{x}\rangle A$ *is $\star$-derivable whenever $5 \in \star$;*

B^t) $\Rightarrow w : A \to |\vec{x}|\langle\vec{x}\rangle A$ *is $\star$-derivable whenever $B \in \star$;*

D^t) $\Rightarrow w : |\vec{x}|A \to \langle\vec{x}\rangle A$ *is $\star$-derivable whenever $D \in \star$;*

2^t) $\Rightarrow w : \langle\vec{x}\rangle|\vec{x}|A \to |\vec{x}|\langle\vec{x}\rangle A$ *is $\star$-derivable whenever $2 \in \star$;*

CBF) $\Rightarrow w : |\vec{x}|\forall yA \to \forall y|\vec{x}\,y|A$ *is $\star$-derivable whenever $CBF \in \star$;*

NI) $\Rightarrow w : x = y \to |x\,y|x = y$ *is $\star$-derivable whenever $NI \in \star$;*

ND) $\Rightarrow w : x \neq y \to |x\,y|x \neq y$ *is $\star$-derivable whenever $ND \in \star$;*

$BF)\quad \Rightarrow w : \forall y|\vec{x}\,y|A \to |\vec{x}|\forall yA$ *is $\star$-derivable whenever $BF \in \star$;*

$GF)\quad \Rightarrow w : \exists y|\vec{x}\,y|A \to |\vec{x}|\exists yA$ *is $\star$-derivable whenever $GF \in \star$;*

$SHRT)\quad \Rightarrow w : |\vec{x}\,y|A \to |\vec{x}|A$ *is $\star$-derivable whenever $SHRT \in \star$.*

Proof. All derivability results are obtained by applying root-first the rule of the relevant sequent calculus —i.e. of **GIM.K**+ the rule(s) stated in the proposition. We give some examples.

4ᵗ. By applying, root-first, the rules of **GIM.4**

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{u : A, w\mathscr{R}u, \vec{x}^w\mathscr{T}\vec{x}^u, v\mathscr{R}u, \vec{x}^v\mathscr{T}\vec{x}^u, w\mathscr{R}v, \vec{x}^w\mathscr{T}\vec{x}^v, w : |\vec{x}|A \Rightarrow u : A}\,3.11}{w\mathscr{R}u, \vec{x}^w\mathscr{T}\vec{x}^u, v\mathscr{R}u, \vec{x}^v\mathscr{T}\vec{x}^u, w\mathscr{R}v, \vec{x}^w\mathscr{T}\vec{x}^v, w : |\vec{x}|A \Rightarrow u : A}\,L|\tfrac{t}{\vec{x}}|}{\vec{x}^w\mathscr{T}\vec{x}^u, v\mathscr{R}u, \vec{x}^v\mathscr{T}\vec{x}^u, w\mathscr{R}v, \vec{x}^w\mathscr{T}\vec{x}^v, w : |\vec{x}|A \Rightarrow u : A}\,4^1}{v\mathscr{R}u, \vec{x}^v\mathscr{T}\vec{x}^u, w\mathscr{R}v, \vec{x}^w\mathscr{T}\vec{x}^v, w : |\vec{x}|A \Rightarrow u : A}\,4^2}{w\mathscr{R}v, \vec{x}^w\mathscr{T}\vec{x}^v, w : |\vec{x}|A \Rightarrow v : |\vec{x}|A}\,R|\tfrac{t}{\vec{x}}|}{w : |\vec{x}|A \Rightarrow w : |\vec{x}||\vec{x}|A}\,R|\tfrac{t}{\vec{x}}|}{\Rightarrow w : |\vec{x}|A \to |\vec{x}||\vec{x}|A}\,R\to$$

Dᵗ. By applying, root-first, the rules of **GIM.D**

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{v : A, w\mathscr{R}v, \vec{x}^w\mathscr{T}\vec{x}^v, w : |\vec{x}|A \Rightarrow w : \langle\vec{x}\rangle A, v : A}\,3.11}{w\mathscr{R}v, \vec{x}^w\mathscr{T}\vec{x}^v, w : |\vec{x}|A \Rightarrow w : \langle\vec{x}\rangle A, v : A}\,L|\tfrac{t}{\vec{x}}|}{w\mathscr{R}v, \vec{x}^w\mathscr{T}\vec{x}^v, w : |\vec{x}|A \Rightarrow w : \langle\vec{x}\rangle A}\,R\langle\tfrac{t}{\vec{x}}\rangle}{w : |\vec{x}|A \Rightarrow w : \langle\vec{x}\rangle A}\,D^{\mathcal{T}}}{\Rightarrow w : |\vec{x}|A \to \langle\vec{x}\rangle A}\,R\to$$

Observe that by applying root-first rule $D^{\mathcal{T}}$ we don't have to apply also rule $D^{\mathcal{R}}$, which is inapplicable whenever rule $D^{\mathcal{T}}$ is applicable. For this reason we haven't been able to give two separate rules for t-serial t-frames: one for the r-formulas ans one for the t-formulas, but we had to give a single rule $D^{\mathcal{T}}$ working on ext-formulas of both kinds, and we have introduced rule $D^{\mathcal{R}}$ has a limit case of $D^{\mathcal{T}}$. The same holds for the rules for 2^t.

NI. By applying, root-first, the rules of **GIM.K.NI**

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{v : x \doteq y, y^w \mathscr{T} x^v, w\mathscr{R}v, x^w \mathscr{T} x^v, y^w \mathcal{T} y^v, w : x \doteq y \Rightarrow v : x \doteq y}\;3.11}{y^w \mathscr{T} x^v, w\mathscr{R}v, x^w \mathscr{T} x^v, y^w \mathcal{T} y^v, w : x \doteq y \Rightarrow v : x \doteq y}\;NI}{w\mathscr{R}v, x^w \mathscr{T} x^v, y^w \mathcal{T} y^v, w : x \doteq y \Rightarrow v : x \doteq y}\;Lbz_2}{w : x \doteq y \Rightarrow w : |x\, y| x \doteq y}\;R|^{\vec{t}}_{\vec{x}}|}{\Rightarrow w : x \doteq y \to |x\, y| x \doteq y}\;R\to$$

BF. By applying, root-first, the rules of **GIM.K.BF**

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{v : A[z/y], |\vec{x}^{\,y[z/y]}_{y}|A : w, z^w \mathscr{T} z^v, \mathcal{E}z^w, \mathcal{E}z^v, w : \forall y|\vec{x}\, y|A, w\mathcal{R}v, \vec{x}^{\vec{w}} \mathscr{T} \vec{x}^{\vec{v}} \Rightarrow v : A[z/y]}\;3.11}{|\vec{x}^{\,y[z/y]}_{y}|A : w, z^w \mathscr{T} z^v, \mathcal{E}z^w, \mathcal{E}z^v, w : \forall y|\vec{x}\, y|A, w\mathcal{R}v, \vec{x}^{\vec{w}} \mathscr{T} \vec{x}^{\vec{v}} \Rightarrow v : A[z/y]}\;L|^{\vec{t}}_{\vec{x}}|}{z^w \mathscr{T} z^v, \mathcal{E}z^w, \mathcal{E}z^v, w : \forall y|\vec{x}\, y|A, w\mathcal{R}v, \vec{x}^{\vec{w}} \mathscr{T} \vec{x}^{\vec{v}} \Rightarrow v : A[z/y]}\;L\forall}{\mathcal{E}z^v, w : \forall y|\vec{x}\, y|A, w\mathcal{R}v, \vec{x}^{\vec{w}} \mathscr{T} \vec{x}^{\vec{v}} \Rightarrow v : A[z/y]}\;BF}{w\mathscr{R}v, \vec{x}^{\vec{w}} \mathscr{T} \vec{x}^{\vec{v}}, w : \forall y|\vec{x}\, y|A \Rightarrow v : \forall y A}\;R\forall}{w : \forall y|\vec{x}\, y|A \Rightarrow w : |\vec{x}|\forall y A}\;R|^{\vec{t}}_{\vec{x}}|}{\Rightarrow w : \forall y|\vec{x}\, y|A \to |\vec{x}|\forall y A}\;R\to$$

SHRT. By applying, root-first, the rules of **GIM.K.SHRT**

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{v : A, y^w \mathscr{T} z^v, w\mathscr{R}v, \vec{x}^{\vec{w}} \mathscr{T} \vec{x}^{\vec{v}}, w : |\vec{x}\, y|A \Rightarrow v : A}\;3.11}{y^w \mathscr{T} z^v, w\mathscr{R}v, \vec{x}^{\vec{w}} \mathscr{T} \vec{x}^{\vec{v}}, w : |\vec{x}\, y|A \Rightarrow v : A}\;L|^{\vec{t}}_{\vec{x}}|}{w\mathscr{R}v, \vec{x}^{\vec{w}} \mathscr{T} \vec{x}^{\vec{v}}, w : |\vec{x}\, y|A \Rightarrow v : A}\;SHRT}{w : \exists y|\vec{x}\, y|A \Rightarrow w : |\vec{x}|A}\;R|^{\vec{t}}_{\vec{x}}|}{\Rightarrow w : |\vec{x}\, y|A \to |\vec{x}|A}\;R\to$$

Observe that the well-formedness of $|\vec{x}|A$ and of $|\vec{x}\, y|A$ entails that y^v doesn't occur (free) in A, therefore in the premiss of $L|^{\vec{t}}_{\vec{x}}|$ we have written $v : A$ instead of $v : A[z/y]$ as active formula. ■

Corollary 4.15. *For any indexed modal formula A*

$$\mathbf{Q}^{(\circ)}.\star^{(r)(s)}_{(=)} \vdash A \quad \textit{implies} \quad \mathbf{GIM}.\star^{(r)(s)}_{(=)} \vdash\ \Rightarrow w : A$$

Proof. It is a corollary of (the appropriate cases of) Propositions 4.8, 4.9, 4.11, 4.13, and 4.14. ■

4.3 Sequent Calculi for IMLs and QMLs

In this section we show the relations between labelled sequent calculi and axiomatic systems for quantified modal logics based on the languages $\mathcal{L}^{\Box}$ and $\mathcal{L}^{\lambda}$. First of all we will show how to define a class of t-frames that is isomorphic (w.r.t. validity) to a class of (T)K-frames both with and without rigid designators, see Appendix A, or to a class of van Benthem's frames, see Chapter 2.2.3, let us call these collectively as *Kripke-type frames*. By the completeness Theorem 4.7 this will show that our labelled sequent calculi allow to give a (modular) proof-theoretic characterization of the formulas that are valid on classes of frames of these kinds. Then, we will prove some results about axiomatic systems for QMLs.

4.3.1 Transition Semantics and Kripkean Semantics

The main novelty of transition semantics w.r.t. Kripke-type semantics is that the relation of trans-world identity, used to evaluate modal open formulas in the latter, is replaced by an arbitrary transition relation between objects inhabiting possible worlds. This has allowed us to have a more fine-grained Correspondence Theory: many important formulas that are valid on every Kripke-type frame correspond to particular classes of transition-frames, witness NI and ND. As a matter of fact it turns out that all Kripke-type frames can be seen as particular kinds of t-frames, and therefore the notion of validity in a class of Kripke-type frames is isomorphic to the notion of validity in classes of t-frames such that the transition relation respects some constraint.

Kripke Semantics

We begin by showing how to construct a t-frame from a (T)K-frame and vice-versa. it is immediate to recognize that the totally defined trans-world identity relation of (T)K-frames can be captured in transition semantics by a transition relation that is a totally defined injective function. Given a K-frame $\mathcal{F}^K = \langle \mathcal{W}, \mathcal{R}, \mathcal{U}, \mathcal{D} \rangle$ we can define the t-frame $\mathcal{F}^{\mathcal{F}^K} = \langle \mathcal{W}, \mathcal{R}, \mathcal{U}, \mathcal{D}, \mathcal{T} \rangle$

where $\mathcal{W}$, $\mathcal{R}$, $\mathcal{U}$, and $\mathcal{D}$ are as before, and where $\mathcal{T}$ is the transition function defined as:

$$\text{for all } w, v \in \mathcal{W} \text{ s.t. } w\mathcal{R}v, \text{ and for all } a \in U_w, b \in U_v, \quad a\mathcal{T}_{(w,v)}b \iff a = b$$

Conversely, given a the t-frame $\mathcal{F} = \langle \mathcal{W}, \mathcal{R}, \mathcal{U}, \mathcal{D}, \mathcal{T} \rangle$ where $\mathcal{T}$ is a totally defined injective function, we can define the (T)K-frame $\mathcal{F}^{\mathcal{K}^{\mathcal{F}}} = \langle \mathcal{W}, \mathcal{R}, \mathcal{U}^\star, \mathcal{D}^\star \rangle$ where $\mathcal{W}$ and $\mathcal{R}$ are as before, and, for all $w \in \mathcal{W}$, $U^\star_w$ ($D^\star_w$) is defined as follows:[6]

$$\text{if for no } v \in \mathcal{W}\ a\mathcal{T}_{(v,w)}b, \text{ then } b \in U^\star_w\ (b \in D^\star_w \text{ if } b \in D_w);$$

$$\text{if for some } v \in \mathcal{W}\ a\mathcal{T}_{(v,w)}b, \text{ then } a \in U^\star_w\ (a \in D^\star_w \text{ if } b \in D_w).$$

Observe that $\mathcal{U}^\star$ ($\mathcal{D}^\star$) is well defined —i.e. it is such that $w\mathcal{R}v$ implies $U^\star_w \subseteq U^\star_v$, because $\mathcal{T}$ is a (totally defined) injective function that determines equivalence classes of members of $\mathcal{U}$ ($\mathcal{D}$) that have been shrunk to a representative element thereof.

For t-rigid t-models defined over t-frames where $\mathcal{T}$ is a totally defined function (and thus for t-frames of kind $\mathcal{F}^{\mathcal{F}^{\mathcal{K}}}$)[7] we have that

$$\mathcal{F}^{\mathcal{F}^{\mathcal{K}}} \models |y_1 \dots y_k|(A[t_1, \dots, t_n/x_1, \dots, t_n]) \leftrightarrow |^{t_1}_{x_1} \dots {}^{t_n}_{x_n}|A$$
$$\text{where } \{y_1, \dots, y_k\} \text{ are all variables in } \{y_1, \dots, y_k\}$$

which means that substitution of terms for variables commutes with indexed operators, and therefore indexed operators are replaceable by standard modal operators. We can easily define an invertible translation function $\tau_\Box$ from $\mathcal{L}^\Box$-formulas to $\mathcal{L}^{\boxdot}$-formulas[8] such that for any (T)K-frame $\mathcal{F}^K$with rigid designators,

$$\mathcal{F}^{\mathcal{K}} \models^r A(\in \mathcal{L}^\Box) \quad \iff \quad \mathcal{F}^{\mathcal{F}^{\mathcal{K}}} \models^r \tau_\Box(A)$$

[6] The definition says that if an object b in a domain of a world v of the t-frame is a counterpart of some other object a, then we add to the other object a to the domain of w in the Kripke frame, else we add b.

[7] Remember that the notions of validity in t-rigid, in stable, and in t-rigid and stable t-models defined over a t-frame such that $\mathcal{T}$ is a totally defined function coincide.

[8] For any $A \in \mathcal{L}_\Box$, $\tau_\Box(A)$ is A for atomic formulas and $\bot$, $\tau_\Box$ commutes with non-modal operators, and $\tau_\Box(\Box A) = |y_1 \dots y_k|A$, where $y_1, \dots, y_k$ are all variables free in A; analogously for $\tau_\Box(\Diamond A)$.

and

$$\mathcal{F}^{\mathcal{F}^{\mathcal{K}}} \models^{r} A(\in \mathcal{L}^{\boxdot}) \quad \Longleftrightarrow \quad \mathcal{F}^{\mathcal{K}} \models \tau_{\Box}^{-1}(A)$$

Thus validity in the two classes of quantified frames coincides, and therefore we have that **GIM.SHRT.NI.ND.**$\star^r$ determines the set of $\mathcal{L}^{\Box}$-formulas that are valid over the class $\mathcal{C}^{\star}$ of (T)K-frames with rigid designators that satisfy the properties in $\star$ (if each of these properties corresponds to a Lemmon-Scott $\mathcal{L}^{\Box}$-formula).

If, instead, we consider validity of $\mathcal{L}^{\lambda}$-formulas over a class $\mathcal{C}^{\star}$ of (T)K-frames with non-rigid designators, we proceed analogously, but we consider validity over all t-models based on an $\mathcal{F}^{\mathcal{F}^{K}}$ in $\mathcal{C}^{\star}$, and not only over the t-rigid t-models therein. In this case we have that substitution of variables —but not of terms— for variables commutes with indexed operators because constant and function are not rigid designators. In this case indexed operators have exactly the same role of the λ operator for quantified modal logics with non-rigid designators, and therefore we can easily define an invertible translation function τ_{λ} from $\mathcal{L}^{\lambda}$-formulas to $\mathcal{L}^{\boxdot}$-formulas[9] that preserves validity. This allows us to conclude that **GIM.SHRT.NI.ND.**$\star$ determines the set of $\mathcal{L}^{\lambda}$-formulas that are valid over the class $\mathcal{C}^{\star}$ of (T)K-frames with non-rigid designators that satisfies the properties in $\star$ (if each of these properties corresponds to a Lemmon-Scott $\mathcal{L}^{\Box}$-formula).

van Benthem's Semantics

To construct a t-frame from a van Benthem's one (with or without double domains) and vice-versa, we use the same construction used for (T)K-frames, but we drop the assumption that $\mathcal{T}$ is $\mathcal{U}$- and $\mathcal{D}$-totally defined. In this case we need indexed operators to avoid problems with the validity of some instances of distribution of modal operators over implications. Arguing as before we can show that the set of $\mathcal{L}^{\boxdot}$-formulas that are valid over a class $\mathcal{C}^{\star}$ of van Benthem's frames is determined by the labelled sequent calcu-

[9] It is $\tau_{\Box}$ with the additional clause $\tau_{\lambda}(\langle \lambda x.A \rangle(t)) = A[t/x]$. Note that the inverse translation has to maps $P(t_1, \ldots, t_n)$ to $\langle \lambda x_1, \ldots, x_n.P(x_1, \ldots, x_n)\rangle(t_1, \ldots, t_n)$, which is equivalent to $P(t_1, \ldots, t_n)$ only if $t_1, \ldots, t_n$ are variables.

lus **GIM.NI.ND.**$\star^r$ (if each property in $\star$ corresponds to a Lemmon-Scott $\mathcal{L}^{\boxdot}$-formula). In principle we can do the same for models with non-rigid designators defined over van Benthem's frame. Observe that we have thus given the first proof-theoretic presentation of the formulas valid over (some classes of) van Benthem's frames.

4.3.2 Labelled Sequent Calculi and Axiomatic Systems

We have shown that our labelled sequent calculi characterize the set of valid formulas of many interesting classes of t-frames, of (T)K-frames and of van Benthem's frames. Observe that for (T)K-frames with rigid designators this result is equivalent to that obtained in [NP11, Ch. 12.1], where labelled sequent calculi **G3Kq*** for $\mathcal{L}^\Box$-formulas (without identity) are introduced, and where it has been shown that **G3Kq**$^\star \vdash\ \Rightarrow w : A$ iff A is valid over the class $\mathcal{C}^\star$ of (T)K-frames (if the properties in $\star$ are defined by universal and/or geometric first-order formulas). Given that for t-rigid t-models where $\mathcal{T}$ is a totally defined function the indexes of indexed operators become replaceable by standard modal operators, it is easy to notice that **GIM.SHRT.NI.**$\star^r$ and **G3Kq*** coincide (for formulas in the respective languages without identity). If we impose that $\mathcal{T}$ is also injective and add rule ND, then if $w\mathcal{R}v$, the only (w, v)-transition of an object a can be taken to be as a itself, therefore **GIM.SHRT.NI.ND**$\star^r_=$ allows us to introduce the labelled sequent calculus **G3Kq**$^\star_=$ for the language with identity.

We have also shown that if a formula A is a theorem of an axiomatic calculus that is sound w.r.t. a class $\mathcal{C}^\star$ of frames (of the appropriate kind), then $\Rightarrow w : A$ is derivable in the labelled sequent calculus for that class of frames. Thus a question naturally arises as to whether the converse implication holds or not. The answer in general is no because labelled sequent calculi for quantified modal logics are stronger than axiomatic systems, to wit the axiomatic system **Q.2.BF** is not complete w.r.t. the class of all convergent TK-frames with constant domains, whereas **GIM.C.2.SHRT.NI.ND.BF**^{r} proves all indexed version of $\mathcal{L}^\Box$-formulas valid over that class of TK-frames.

This incompleteness result for axiomatic calculi has been proved in [Cre95],[10] where it is shown that the $\mathcal{L}^{\Box}$-formula

$$\Diamond(\forall x(A(x) \to \Box A(x)) \wedge \Box\neg\forall x A(x)) \wedge \Diamond\forall x(A(x) \vee \Box A(x)) \wedge \forall x(\Diamond A(x) \to \Box A(x)) \quad (3.1)$$

is consistent in **Q.2.BF**, but it cannot be satisfied on any convergent TK-frame with constant domains. In labelled sequent calculi, the negation of the formula in 3.1 is derivable in **GIM.C.2.SHRT.NI.ND.BF**^{r} as shown in Figure 4.1. In the derivation of ¬3.1, t-atoms of kind $y^w \mathscr{T} y^v$ are left implicit; ext-formulas are not reported in the premiss(es) if unnecessary; and in the steps by BF we work as if we have trans-world identity as transition relation. Observe that this derivation is essentially a derivation in the calculus **G3Kq**^{2}**+Decr** presented in [NP11]: it is the methodology of labelled sequents that allows d to supersede this incompleteness results, and not the introduction of IML, which, instead, allows to widen sensibly the class of QMLs that we can define.

Given that our completeness theorem covers various quantified extensions (over both the indexed language and the standard one) of all PMLs defined by Lemmon-Scott formulas, if an axiomatic system defined over the standard modal language or over the indexed modal language is incomplete w.r.t. the appropriate class of frames $\mathcal{C}^\star$ —where $\mathcal{C}^\star$ is defined by universal and geometric formulas, we can give a proof-theoretic presentation of the formulas valid on $\mathcal{C}^\star$ by means of the labelled sequent calculus for that class of frames. We can thus conclude that the methodology of labelled sequent calculi permits us to have not only a proof-theoretic presentation of QMLs with well-behaved structural properties where we can easily find proofs, but also to circumvent many incompleteness results that are one of the main problems of axiomatic systems for quantified modal logics.

[10] Where **Q.2.BF** is called **KG1+BF**.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{u : A[y/x] \Rightarrow u : A[y/x]}\,3.11}{u : A[y/x], w\mathscr{R}u \Rightarrow w : \Diamond A[y/x]} \quad \dfrac{\dfrac{}{v : A[y/x] \Rightarrow v : A[y/x]}\,3.11}{w : \Box A[y/x], w\mathscr{R}v \Rightarrow v : A[y/x]}}{u : A[y/x], w : \Diamond A[y/x] \to \Box A[y/x], w\mathscr{R}v, w\mathscr{R}u \Rightarrow v : A[y/x]}\,L\to \quad \dfrac{\dfrac{}{w_1 : A[y/x] \Rightarrow w_1 : A[y/x]}\,3.11}{v : \Box A[y/x], v\mathscr{R}w_1 \Rightarrow w_1 : A[y/x]}}{u : A[y/x], v\mathscr{R}w_1, v : A[y/x] \to \Box A[y/x], w : \Diamond A[y/x] \to \Box A[y/x], w\mathscr{R}v, w\mathscr{R}u \Rightarrow w_1 : A[y/x]}\,L\to \quad \dfrac{\dfrac{}{w_1 : A[y/x] \Rightarrow w_1 : A[y/x]}\,3.11}{u : \Box A[y/x], u\mathscr{R}w_1 \Rightarrow w_1 : A[y/x]}}{\mathcal{E}y^v, \mathcal{E}y^u, \mathcal{E}y^w, v\mathscr{R}w_1, u\mathscr{R}w_1, v : A[y/x] \to \Box A[y/x], w\mathscr{R}v, w\mathscr{R}u, u : A[y/x] \vee \Box A[y/x], w : \Diamond A[y/x] \to \Box A[y/x] \Rightarrow w_1 : A[y/x]}\,L\vee$$

$$\dfrac{\mathcal{E}y^v, \mathcal{E}y^u, \mathcal{E}y^w, v\mathscr{R}w_1, u\mathscr{R}w_1, v : A[y/x] \to \Box A[y/x], w\mathscr{R}v, w\mathscr{R}u, u : A[y/x] \vee \Box A[y/x], w : \Diamond A[y/x] \to \Box A[y/x] \Rightarrow w_1 : A[y/x]}{\mathcal{E}y^v, \mathcal{E}y^u, \mathcal{E}y^w, v\mathscr{R}w_1, u\mathscr{R}w_1, v : \forall x(A \to \Box A), w\mathscr{R}v, w\mathscr{R}u, u : \forall x(A \vee \Box A), w : \forall x(\Diamond A \to \Box A) \Rightarrow w_1 : A[y/x]}\,L\forall$$

$$\dfrac{}{\mathcal{E}y^{w_1}, v\mathscr{R}w_1, u\mathscr{R}w_1, v : \forall x(A \to \Box A), w\mathscr{R}v, w\mathscr{R}u, u : \forall x(A \vee \Box A), w : \forall x(\Diamond A \to \Box A) \Rightarrow w_1 : A[y/x]}\,BF$$

$$\dfrac{}{v\mathscr{R}w_1, u\mathscr{R}w_1, v : \forall x(A \to \Box A), w\mathscr{R}v, w\mathscr{R}u, u : \forall x(A \vee \Box A), w : \forall x(\Diamond A \to \Box A) \Rightarrow w_1 : \forall x A}\,R\forall$$

$$\dfrac{}{w_1 : \neg\forall x A, v\mathscr{R}w_1, u\mathscr{R}w_1, v : \forall x(A \to \Box A), w\mathscr{R}v, w\mathscr{R}u, u : \forall x(A \vee \Box A), w : \forall x(\Diamond A \to \Box A) \Rightarrow}\,L\neg$$

$$\dfrac{}{v\mathscr{R}w_1, u\mathscr{R}w_1, v : \Box\neg\forall x A, v : \forall x(A \to \Box A), w\mathscr{R}v, w\mathscr{R}u, u : \forall x(A \vee \Box A), w : \forall x(\Diamond A \to \Box A) \Rightarrow}\,L\Box$$

$$\dfrac{}{v : \Box\neg\forall x A, v : \forall x(A \to \Box A), w\mathscr{R}v, w\mathscr{R}u, u : \forall x(A \vee \Box A), w : \forall x(\Diamond A \to \Box A) \Rightarrow}\,2$$

$$\dfrac{}{w\mathscr{R}v, w\mathscr{R}u, v : \forall x(A \to \Box A) \wedge \Box\neg\forall x A, u : \forall x(A \vee \Box A), w : \forall x(\Diamond A \to \Box A) \Rightarrow}\,L\wedge$$

$$\dfrac{}{w : \Diamond(\forall x(A \to \Box A) \wedge \Box\neg\forall x A), w : \Diamond\forall x(A \vee \Box A), w : \forall x(\Diamond A \to \Box A) \Rightarrow}\,L\Diamond$$

$$\dfrac{}{w : \Diamond(\forall x(A \to \Box A) \wedge \Box\neg\forall x A) \;\wedge\; \Diamond\forall x(A \vee \Box A) \;\wedge\; \forall x(\Diamond A \to \Box A) \Rightarrow}\,L\wedge$$

$$\dfrac{}{\Rightarrow w : \neg(\Diamond(\forall x(A \to \Box A) \wedge \Box\neg\forall x A) \;\wedge\; \Diamond\forall x(A \vee \Box A) \;\wedge\; \forall x(\Diamond A \to \Box A))}\,R\neg$$

Figure 4.1: **GIM.C.2.SHRT.NI.ND.BF^r** $\vdash$ $\neg$3.1.

CONCLUSIONS

Main results. Proof theory for indexed modal logics has found its natural setting in the method of labelled sequent calculi. Calculi of this kind were introduced in [Neg05] for propositional modal logics and extended in [NP11, Chap. 12.1] for quantified modal logics based on the standard $\mathcal{L}^{\Box}$-language without identity. Characteristic of these calculi is that the rules internalize the semantic conditions of the modal operators. A feature that has revealed itself to be of great utility and importance is that general properties of t-frames can be internalized via the rules: not only properties of $\mathcal{R}$, as in [Neg05, NP11], but also properties of the transition relation. Take the Barcan Formula BF, whose validity corresponds to the fact that the transition relation is surjective, in order to axiomatize a logic with BF, we can add a rule that expresses surjectivity. Of course not all properties of t-frames can be expressed by rules, but the universal and geometric ones can, and these properties cover a wide range of classes of t-frames. We have been able to define rules for the properties corresponding to $CBF, NI, ND, BF, GF, SHRT$, for those corresponding to non-empty inner domains and to single domains, and for those corresponding to t-rigid and stable terms. Thus for any propositional modal logic in the Lemmon-Scott fragment, we have introduced a sequent calculus for its indexed extensions that are defined by any combination of the properties above.

From the proof-theoretic point of view, our calculi are extremely well-behaved insofar as they have the same structural properties of the underlying calculus **G3cp**: the rules of weakening and contraction are height-preserving admissible, all rules are height-preserving invertible, and the rule of cut is admissible. As a consequence it becomes possible to find derivations by applying a (semi-decidable) root-first proof search procedure.

From the semantical point of view, we have shown that each of our calculi characterizes validity in the appropriate class of transition frames. This result is of particular interest insofar as we are thus able to characterize the valid formulas of many classes of frames for which there is no known complete quantified axiomatic systems. For example we have defined a sequent calculus that characterizes validity in the class of all directed, reflexive and transitive Kripke frames with decreasing domains, whereas the axiomatic system **Q.S4.2.BF** is incomplete w.r.t. that class, and it is still unknown which additional axiom could complete it [Cre00].

Related works. The idea of internalizing frame semantics into proof systems for propositional and quantified modal logics is not new. For quantified modal logics based on Kripke semantics it has been applied, e.g., to tableau systems [Fit83, FM98, Fit06], natural deduction systems [Gab96, Rus96, Vig00, Gar05], and sequent calculi [Cas05, NP11]. But none of these works consider generalizations of Kripke semantics. The only other presentation of a labelled proof system for a more general semantics than Kripke's one is, to our knowledge, that in [Kup12], where tableau systems for modal metaframes are introduced. A difference is that in [Kup12] the focus is on presenting the logic of a particular class of metaframes, and thus only the basic system with rigid designators and its extensions $T, 4$ and B are considered.

An approach that is somehow related to labelled proof systems is that of hybrid logic [BS95], where the semantics is internalized directly in the logic, and not only in the proof systems. Hybrid logics are more expressive than modal logics, and, therefore, are beyond the scope of this work. We refer the reader to [Bra11] for natural deduction systems for propositional and quantified hybrid logics.

Other generalizations of Kripke semantics are sheaves semantics, bundles semantics [SS90], functor semantics [GM88], and metaframes semantics [SS93]. All these approaches share with transition semantics the basic idea that we have to interpret modal open formulas by means of functions or relations between objects of the different world-bound domains, and not by trans-world identity. For a detailed model-theoretic study of generalizations of Kripke semantics, and of their logics, the reader is referred to [BG06, GSS09]. Here we simply note that indexed modal logics are equivalent to the hyperdoctrinal modal logics of [GM88] since transition semantics is a set-theoretic version of the functor semantics used therein and the indexed language is equivalent to the typed one.

A central problem of our work has been that of finding general completeness results in quantified modal logics. Some general completeness theorems for quantified modal logics with respect to Kripke semantics (and w.r.t. some of its generalizations) are [GSS09, Thm. 6.1.29] for quantified modal logics $\mathbf{Q}^{\circ}.\mathbf{S}$, where $\mathbf{S}$ is a propositional modal logic that extends $\mathbf{K}$ with an axiom of shape $\Box^m A \to \Box^k A$, and [GSS09, Thm. 7.4.7] for logics $\mathbf{Q.S.BF}$, where $\mathbf{S}$ is a propositional modal logic whose frame $\mathcal{F}^p$ is defined by universal properties of $\mathcal{R}$.[11] For an overview of completeness and incompleteness results in quantified modal logics the reader is referred to [She06].

A different strategy for dealing with incompleteness of quantified extensions of canonical propositional modal logics is followed in [GM06, Gol11], where it is introduced a general-frame-style semantics called admissible semantics where Kripke frames are augmented with a set of admissible propositions, and where a universal quantifier is interpreted as the greatest lower bound in the lattice of admissible propositions. This allows to give a completeness theorem (w.r.t. to this general-frame-style semantics) for the quantified extensions with and without BF of all canonical propositional modal logics. In a sense, the approach to incompleteness results in quantified modal logics of [GM06, Gol11] is the dual of the one we have adopted here because we have circumvented incompleteness results by adopting proof systems that are stronger (i.e. which allow to prove more theorems) than axiomatic sys-

[11] This results is taken from [TO00].

tems, whereas they do so by adopting a semantics that is weaker (i.e. which makes less formulas valid) than the intended one, and where validity matches with theoremhood in axiomatic systems.

Future works. We have introduced a general framework for quantified modal logics, but we have not discussed its possible applications. One immediate extension of the labelled sequent calculi presented here are contraction- and cut-free calculi for the indexed epistemic logics presented in [CO13]. This would allow us to answer positively the question whether it is possible to find a cut-free sequent calculus for indexed epistemic logics with GF, question posed in [CO13]. Another possible application of our labelled calculi is in multi-modal quantified logics such as the temporal ones, see [Cas05].

The method of axioms-as-rules has allowed us to define a labelled sequent calculus for many indexed extensions of each propositional modal logic in the Lemmon-Scott fragment. By a result in [Neg14], it becomes possible to express so-called generalized geometric implication by means of *systems of rules*, and therefore to define a labelled sequent calculus for many indexed extensions of each propositional modal logic in the Sahlqvist fragment. In this way we can define sequent calculi for a wider class of indexed modal logics. To wit, it becomes possible to give a complete labelled sequent calculus for the quantified extension of the propositional modal logic **S4M**,[12] whose incomplete axiomatic systems **Q.S4M**, with and without BF, has been completed in [Cre00] by adding the following axiom

$$\Diamond\forall\vec{x}(A(\vec{x}) \to \Box A(\vec{x})) \tag{3.2}$$

We have focused our work on labelled sequent systems for indexed modal logics, but there is still much work to be done in order to understand completely indexed modal logics both from the proof-theoretic point of view and from the model-theoretic one. To make an example, it has been shown that our calculi are complete for validity, but not for logical consequence.

[12] Observe that **S4M** is not in the Sahlqvist fragment, but the semantic conditions corresponding to it, which are first-order conditions, may nevertheless be expressed by a system of rules.

APPENDIX A

KRIPKE-TYPE SEMANTICS

We sketch QMLs over the ordinary modal language $\mathcal{L}^{\Box}$ as they are presented in [Cor02], and QMLs over the language $\mathcal{L}^{\lambda}$ presented in [FM98]. For detailed introductions to these logics and Kripke semantics the reader is referred to [HC96, FM98].

Syntax. The language $\mathcal{L}^{\Box}$ is made of the binary connective $\vee$; the unary connectives $\neg$, $\Box$; the existential quantifier $\exists$; the logical constant $\bot$; three countable sets: one of variables $x_1, x_2, \ldots$, one of individual constants $c_1, c_2, \ldots$, and one of relational symbols $P_1, P_2, \ldots$, all of arity $n \in \mathbb{N}$; and, possibly, the symbol $\doteq$ for identity. A *term* is either a variable or an individual constant. The set of $\mathcal{L}^{\Box}$*-formulas* is the smallest set containing:

1. $\bot$;
2. all expressions $P(t_1, \ldots, t_n)$ where P is an n-ary relational symbol and $t_1, \ldots, t_n$ are terms;
3. all expressions $(\neg A)$, $(\Box A)$, $(\exists x A)$, and $(A \vee B)$ where A and B are $\mathcal{L}^{\Box}$-formulas and x is a variable;
4. if the language contains $\doteq$, all expressiosn $t \doteq s$ where t and s are terms.

We follow the usual conventions for parentheses; the symbols $\wedge, \rightarrow, \forall, \Diamond, \not\doteq$, and free and bound occurrences of variables are defined as usual.

We use the following metavariables (all possibly with numerical subscripts): t, s, q for terms; x, y, z for variables; A, B, C for $\mathcal{L}^{\Box}$-formulas. The $L^{\Box}$-formula $A[t/x]$ is the $\mathcal{L}^{\Box}$-formula A with all free occurrences of x replaced by occurrences of t, possibly renaming bound variables to avoid the capture of free variables.

Kripke-Semantics. A *Kripke-frame* (K-frame for shortness) is a quadruple $\mathcal{F}^{\mathcal{K}} = \langle \mathcal{W}, \mathcal{R}, \mathcal{U}, \mathcal{D} \rangle$ such that:

- $\mathcal{W} \neq \emptyset$ is a non-empty set of *possible worlds* $w, v, u, \ldots$;
- $\mathcal{R} \subseteq \mathcal{W} \times \mathcal{W}$ is a binary *accessibility relation* between worlds;
- $\mathcal{U} = \bigcup\{U_w : w \in \mathcal{W}\}$ is the union of a non-empty *outer domain* U_w for each possible world w such that $w\mathcal{R}v$ implies $U_w \subseteq U_v$;
- $\mathcal{D} = \bigcup\{D_w : D_w \subseteq U_w\}$ is the union of a possibly empty subset of each member of $\mathcal{U}$, D_w is the *inner domain* of the objects existing in w.

Note that $w\mathcal{R}v$ implies that $U_w \subseteq U_v$, but not that $D_w \subseteq D_v$. In this case $\mathcal{F}^K$ is said to have *varying domains*. If we impose that $w\mathcal{R}v$ implies $D_w \subseteq D_v$ ($D_w = D_v$), $\mathcal{F}^K$ is said to have *increasing* (*constant*) *domains*. Another widely studied class of K-frames are the so called *TK-frames* $\mathcal{F}^{TK}$ (for 'Tarski-Kripke-frames'), which are K-frames such that, for all $w \in \mathcal{W}$, $D_w = U_w$. We stress that TK-frames can have either increasing or constant domains, but they cannot have varying domains.

A *model* $\mathcal{M}^K$ over a K-frame $\mathcal{F}^k = \langle \mathcal{W}, \mathcal{R}, \mathcal{U}, \mathcal{D} \rangle$ ($\mathcal{M}^{TK}$ over $\mathcal{F}^{TK}$) is $\mathcal{F}^K$ ($\mathcal{F}^{TK}$, respectively) augmented with a function mapping every $w \in \mathcal{W}$ to an interpretation function I_w of the descriptive symbols of the language that is defined over $\mathcal{U}_w$ and such that $w\mathcal{R}v$ implies that $I_w(c) = I_v(c)$ for all individual constant c —i.e. it is a *rigid interpretation function*.[1]

For any $w \in \mathcal{W}$, a w-*assignment* is a function σ_w mapping the variables to objects in U_w. By $\sigma_w^{x \triangleright a}$ we mean the w-assignment that is like σ_w, but maps x to $a \in U_w$. $I_w^{\sigma}(t)$ denotes the extension of an arbitrary term t —i.e. if

[1] In [Cor02] the assumption of rigidity is made only for languages containing identity, because for a language without identity it is unnecessary; here we have generalized it for the sake of uniformity since nothing essential relies on this difference.

t is a variable it stands for $\sigma_w(t)$ and if it is a constant for $I_w(t)$. A key fact of K-semantics is that $w\mathcal{R}v$ implies that a w-assignment is also a v-assignment (since $U_w \subseteq U_v$). This ensures that in moving to a world v accessible from w to evaluate a modal formula we can continue to use a w-assignment without leaving some free variable of that formula without a value (see Section 2.2.3).

The notion of *satisfaction* of a $\mathcal{L}^\Box$-formula A at a world w of a model $\mathcal{M}^K$ ($\mathcal{M}^{TK}$) under a w-assignment σ ($\sigma \models_w A$) is defined as:

$$
\begin{array}{lcl}
\sigma_w \not\models_w \bot & & \\
\sigma_w \models_w P(t_1, \ldots, t_n) & \iff & \langle I^\sigma_w(t_1), \ldots, I^\sigma_w(t_n)\rangle \in I_w(P^n) \\
\sigma_w \models_w \neg B & \iff & \sigma \not\models_w B \\
\sigma_w \models_w B \vee C & \iff & \sigma \models^{\mathcal{M}^{(T)K}}_w B \quad \text{or} \quad \sigma \models_w C \\
\sigma_w \models_w \exists x B & \iff & \text{for some } a \in D_w,\ \sigma^{x \triangleright a}_w \models_w B \\
\sigma_w \models_w \Box B & \iff & \text{for all } v \in \mathcal{W},\ w\mathcal{R}v \text{ implies } \sigma_w \models_v B
\end{array}
$$

If the language contains the identity symbol $\doteq$, we add the clause:

$$\sigma_w \models_w t \doteq s \qquad \iff \qquad I^\sigma_w(t) = I^\sigma_w(s)$$

Finally we say that A is *true in* $\mathcal{M}$ *at* w ($\models^{\mathcal{M}}_w A$) iff for all w-assignment σ_w, $\sigma_w \models_w A$; that it is *true in* $\mathcal{M}$ ($\models^{\mathcal{M}} A$) iff for all $w \in \mathcal{W}$, $\models^{\mathcal{M}}_w A$; that it is *valid over a K- or TK-frame* $\mathcal{F}^{(T)K}$ ($\mathcal{F}^{(T)K} \models A$), iff for every model $\mathcal{M}^{(T)K}$ based on it $\models^{M^{(T)K}} A$; and that it *valid in a class* $\mathcal{C}$ of K- or TK-frames ($\mathcal{C} \models A$) iff it is valid in every member of that class.

Observe that the main difference between K-frames and TK-frames is that the first are double domain frames which validate the theory of quantification of free logic and the latter are single domain frames that validate that of classical logic. The difference between varying, increasing and constant domain frames is that they validate different formulas ($\in \{BF, CBF, GF\}$) governing the interaction of quantifiers and modal operators. Finally note that both the necessity of identity and the necessity of distinctness are valid on every (T)K-frmame.

QMLs with Non-Rigid Terms. In the previous section we have presented QMLs based on the language $\mathcal{L}^\Box$ and Kripke-semantics. One feature of these logics is that we had to assume that terms are rigid designators in order to avoid problems in axiomatizing the identity relation $\doteq$. In fact $s \doteq t \rightarrow \Box(s \doteq t)$ is a theorem of all axiomatic systems over the $\mathcal{L}^{\Box}$-language with identity, but it is not valid if s or t is not a rigid designator. Furthermore, if we had introduced non-rigid designators, we would have had problems in evaluating a formula such as $\Box A(c)$ in a world w because this formula doesn't say where we have to determine the extension of c: in the world w or in those accessible from w, see [Fit91, FM98]. One well-known way to overcome this limitation is by enriching the language with the abstraction operator λ in order to distinguish the two possible readings of a formula like $\Box A(c)$. We are now going to sketch this solution as it has been presented in [FM98].[2]

The language $\mathcal{L}^{\lambda}$ is that of $\mathcal{L}^{\Box}$ enriched with λ. The set of $\mathcal{L}^{\lambda}$-formulas is like that of $\mathcal{L}^{\Box}$ where clause 2 is replaced by

2^{λ}. all expressions$P(x_1, \ldots, x_n)$ where P is an n-ary relational symbol and $x_1, \ldots, x_n$ are variables;

and with the additional clause

5. all expressions $\langle \lambda x.A \rangle(t)$ where A is an $\mathcal{L}^{\lambda}$-formula, x a variable and t a term.

The semantics for the QMLs over this language is that of (T)K-frames presented before, but where we don't impose anymore that $w\mathcal{R}v$ implies $I_w(c) = I_v(c)$ —i.e. where individual constants are not rigid designators, and where the notion of satisfaction has the following additional clause for λ-formulas:

$$\sigma_w \models_w^{\mathcal{M}^{\lambda}} \langle \lambda x.A \rangle(t) \quad \Longleftrightarrow \quad \sigma_w^{x \triangleright I_w^{\sigma}(t)} \models_w^{\mathcal{M}^{\lambda}} A$$

Having imposed that variables are the only terms that can occur in atomic formulas, we can distinguish the two possible readings of the $\mathcal{L}^{\Box}$-formula $\Box A(c)$: they are expressed by the formulas $\langle \lambda x.\Box A(x) \rangle(c)$ (c's extension has to be determined in w) and $\Box \langle \lambda x.A(x) \rangle(c)$ (c's extension has to be determined in the worlds accessible from w).

[2] We are not going to talk about **FOIL**s and conceptual quantification, see [Fit04].

APPENDIX B

AXIOMATIC SYSTEMS

Axiomatic systems over the language $\mathcal{L}^\Box$. We are now going to present axiomatic systems for quantified modal logics defined over the language $\mathcal{L}^\Box$.[1] In general we adopt the standard definitions of derivation and theorem. Given an axiomatic system $\mathbf{X}$, we write $\mathbf{X} \vdash A$ whenever A is a theorem of $\mathbf{X}$. We start with axiomatic systems for the language without identity. The basic system $\mathbf{Q}^\circ.\mathbf{K}$ is obtained by conjoining the axiomatic system $\mathbf{K}$ of the minimal normal propositional modal logic with the axioms and rules of positive free quantification. Here we present axiomatizations with some redundancies for the sake of uniformity. For detailed presentations of axiomatic systems for QMLs and for completeness results the reader is referred to [HC96, Cor02]; for incompleteness results see [SS90, Gar91, Cre95, Cre00, Cor02, Gol11]. The axioms and rules of $\mathbf{Q}^\circ.\mathbf{K}$ are given in Table B.1 and B.2.

The axiomatic system $\mathbf{Q}.\mathbf{K}$ is $\mathbf{Q}^\circ.\mathbf{K}$ with axiom UI° replaced by[2]

UI) $\forall x A \to A[t/x]$

Note that in $\mathbf{Q}^\circ.\mathbf{K}$ none of BF, CBF, GF (and, obviously, of NI, ND) is a

[1] We don't present axiomatizations of logics defined over the language $\mathcal{L}^\lambda$, see [FM98, Chap. 10] where tableaux for them are introduced.

[2] We give redundant axiomatizations since the axioms CQ, UD and VQ become derivable when UI is present.

theorem, whereas in **Q.K** CBF and GF, but not BF, are theorems.

All other axiomatic system we will consider are obtained by adding some additional axiom to $\mathbf{Q}^{\circ}.\mathbf{K}$ or to **Q.K** as follows:

- Whenever the language contains identity we have to add the axioms of Table B.3. In this case we talk of $\mathbf{Q}^{(\circ)}.\mathbf{K}_{=}$.
- We can add the quantified instance of some propositional modal formulas as those in Table B.4, in this case we will talk of the axiomatic system $\mathbf{Q}^{(\circ)}.\mathbf{S}_{(=)}$, where **S** is the name of the underlying PML.[3]
- Finally we can add BF and/or CBF, and we talk of $\mathbf{Q}^{(\circ)}.\mathbf{S}_{(=)}.\mathbf{(C)BF}$.

Table B.1: Axioms of $\mathbf{Q}^{\circ}.\mathbf{K}$

PC) All $\mathcal{L}^{\Box}$-instances of propositional tautologies;

UI°) $\forall y(\forall x A \to A[y/x])$;

CQ) $\forall x \forall y A \leftrightarrow \forall y \forall x A$;

UD) $\forall x(A \to B) \to (\forall x A \to \forall x B)$;

VQ) $A \to \forall x A$, if x is not free in A;

K) $\Box(A \to B) \to (\Box A \to \Box B)$.

Table B.2: Rules of $\mathbf{Q}^{\circ}.\mathbf{K}$

$$\frac{A \quad A \to B}{B}\, MP \qquad \frac{A}{\Box A}\, Nec \qquad \frac{A \to B}{A \to \forall x B}\, UG\ (x \notin FV(A))$$

Table B.3: Axioms for identity

Ref) $t \doteq t$ $\qquad$ Lbz) $s \doteq t \to (A[s/x] \to A[t/x])$ $\qquad$ ND) $s \not\doteq t \to \Box(s \not\doteq t)$

[3] We follow the usual conventions for names of PMLs.

Table B.4: Additional propositional axioms

$T)$	$\Box A \to A$	$4)$	$\Box A \to \Box\Box A$	$5)$	$\Diamond A \to \Box\Diamond A$
$B)$	$A \to \Box\Diamond A$	$D)$	$\Box A \to \Diamond A$	$2)$	$\Diamond\Box A \to \Box\Diamond A$

Axiomatic systems over the language $\mathcal{L}^{\boxdot}$. The minimal axiomatic system over the indexed language is called $\mathbf{Q}^{\circ}$**.Kim** is determined by the axioms and rules of Tables B.5 and B.6. This calculus is analogous to that introduced in [Cor09], save that here we have free quantification. The minimal system with classical quantification **Q.Kim** is obtained by replacing UI° with UI. Also in this case we give a redundant axiomatization.

Given an axiomatic system $\mathbf{Q}^{(\circ)}.\mathbf{S}_{(=)}$ over the standard language, the axiomatic system $\mathbf{Q}^{(\circ)}.\mathbf{Sim}_{(=)}$ over the indexed language is obtained by extending $\mathbf{Q}^{(\circ)}$**.Kim** with the indexed instances of the same propositional axioms (see Table 2.1), with the only difference that if the language contains identity we have to add axioms Ref and Lbz, but not ND. Given a system $\mathbf{Q}^{(\circ)}.\mathbf{Sim}_{=}$, we can extend it with some *de re* axioms taken for Table 2.2. In this case we denote that system by adding after **S** a list containing the names of the additional axioms separated by dot, e.g. $\mathbf{Q}^{\circ}.\mathbf{S4.NIim}_{=}$ is the calculus defined over the language with identity that is obtained by adding the axioms T, 4, and NI. Observe that we have followed [Cor09] in giving all 'modal' axioms indexed by variables, and not by arbitrary terms. This can be done w.l.o.g. because we have an explicit rule of substitution of terms for variables (SFV).

Finally, given an axiomatic system **Q.Xim**, we call **R.Xim**, **S.Xim** and **RS.Xim** the system obtained by extending **Q.Xim** with axioms R, S or R and S from Table B.7, where R is the axiom characteristic of t-rigidity and S of stability.

In [Cor09] the systems **Q.Kim** and **R.Kim** are shown to be complete w.r.t., respectively, the class of all t-models and that of all t-rigid t-models.

Table B.5: Axioms of $\mathbf{Q}^\circ$**.Kim**

PC) All $\mathcal{L}^{\boxdot}$-instances of propositional tautologies;

UI°) $\forall y(\forall x A \to A[y/x])$;

CQ) $\forall x \forall y A \leftrightarrow \forall y \forall x A$;

UD) $\forall x(A \to B) \to (\forall x A \to \forall x B)$;

VQ) $A \to \forall x A$, if x is not free in A;

K) $|\vec{x}|(A \to B) \to (|\vec{x}|A \to |\vec{x}|B)$;

$LNGT$) $|\vec{x}|A \to |\vec{x}\, y|A$;

PRM) $|x_1 \dots x_n|A \to |x_{p_1} \dots x_{p_n}|A$,
for any permutation $x_{p_1}, \dots, x_{p_n}$ of $x_1, \dots, x_n$;

R^v) $|^{y_1}_{x_1} \dots ^{y_n}_{x_n}|A \to |y_1 \dots y_k|(A[y_1/x_1 \dots, y_n/x_n])$,
where $y_1, \dots, y_k$ include all different variables among $y_1, \dots, y_n$.

Table B.6: Rules of $\mathbf{Q}^\circ$.**K**

$$\frac{A \quad A \to B}{B}\, MP \qquad \frac{A}{|\vec{x}|A}\, Nec \qquad \frac{A \to B}{A \to \forall x B}\, UG\ (x \notin FV(A)) \qquad \frac{A}{A[t/x]}\, SFV$$

Table B.7: Axioms for t-rigid and stable terms

R) $|^{t_1}_{x_1} \dots ^{t_n}_{x_n}|A \to |y_1 \dots y_k|(A[t_1, \dots, t_n/x_1, \dots x_n])$
where $y_1, \dots y_k$ are all the variables occurring in $t_1, \dots, t_n$,

S) $|\vec{y}|(A[\vec{t}/\vec{x}]) \to |\vec{y}\,^{\vec{t}}_{\vec{x}}|A$ where $\vec{t}$ is a tuple of closed terms.

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