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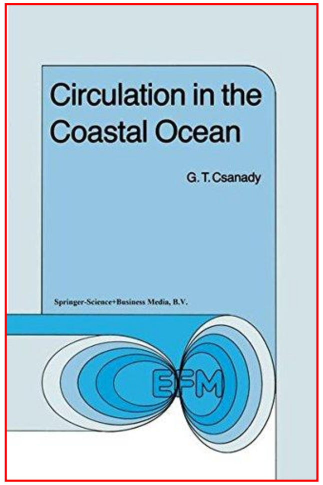
Flow controlled by bottom friction
(Part 1)

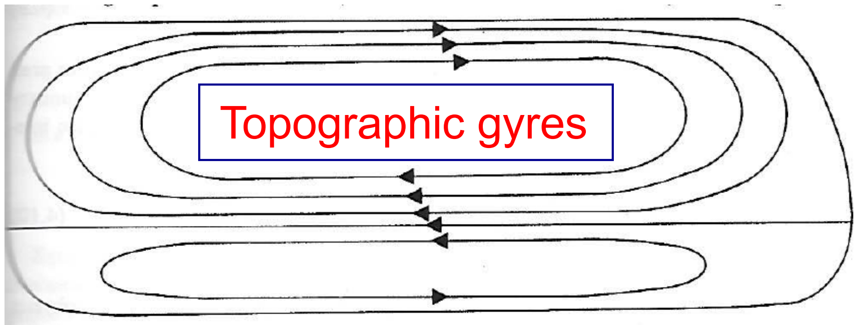
Main references

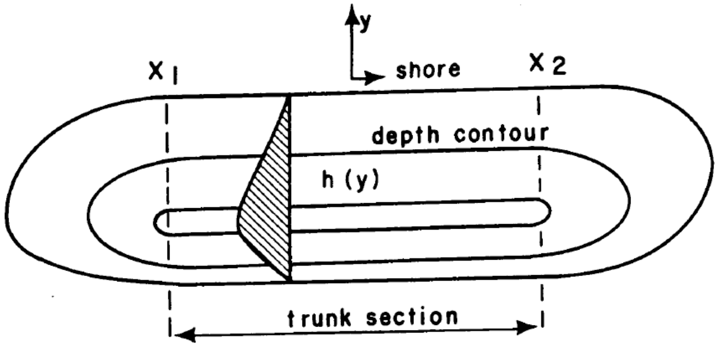


G.T Csanady: Circulation in the coastal ocean.
Chapter 6. Flow controlled by bottom friction
Sections 6.1, 6.2

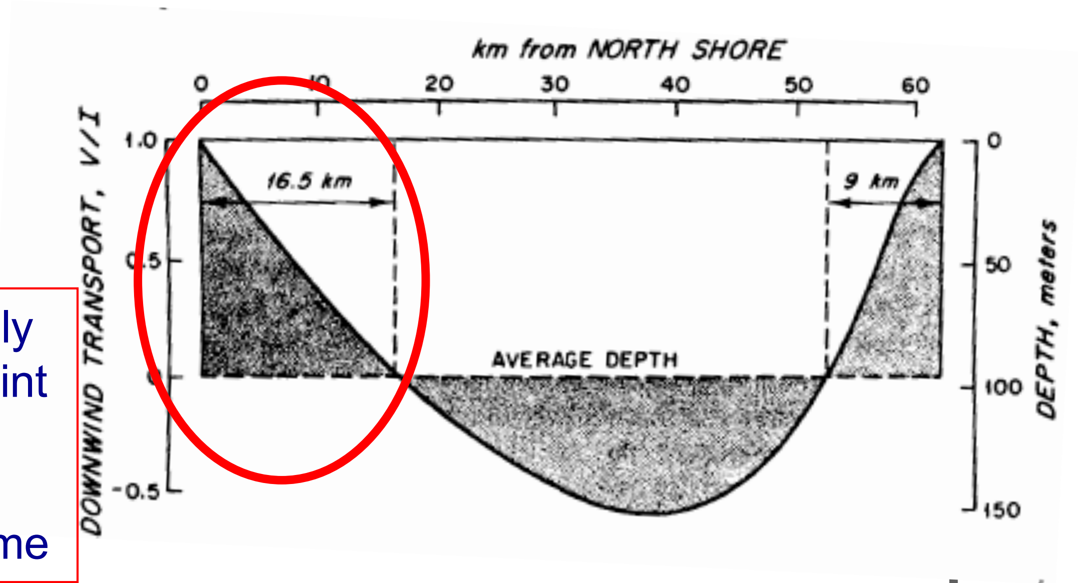
Problems with bottom friction neglected

Revisiting the wind setup over variable depth (long shore wind case with <u>no rotation</u>).





$$U = \frac{\tau_w^{(x)}}{\rho_0} t \left[1 - \frac{H}{\bar{H}} \right]$$



Downwind longshore transport progressively Decreasing to zero from the coast to the point of the section with depth corresponding to average depth. Beyond that point current Reverses (upwind). Transport grows with time

Problems with bottom friction neglected

Revisiting the wind setup over variable depth (long shore wind case with rotation)

$$\eta = -\frac{u_*^2}{gs} f t K_0\left(2\xi^{1/2}\right)$$

$$V = u_*^2 t \xi^{1/2} K_1\left(2\xi^{1/2}\right)$$

$$U = \frac{u_*^2}{f}\left[1 - \xi^{1/2} K_1\left(2\xi^{1/2}\right)\right]$$

$$\xi = \frac{x}{R_s} \quad R_s = \frac{gs}{f^2}$$

$s=H/x$

At $x << R_s$ along shore transport is maximum.

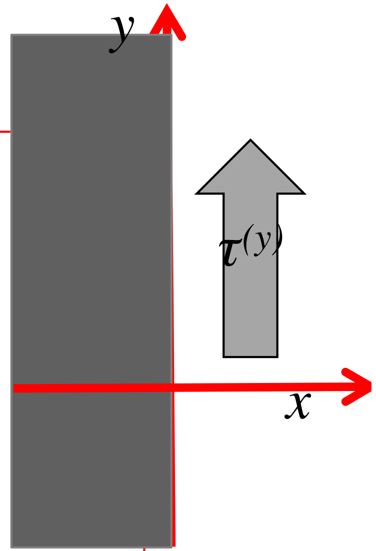
$V = u_*^2 t$ and $U=0$

Also in this case:
long shore transport decrease moving away from the coast.

Longshore transport grows infinitely with time

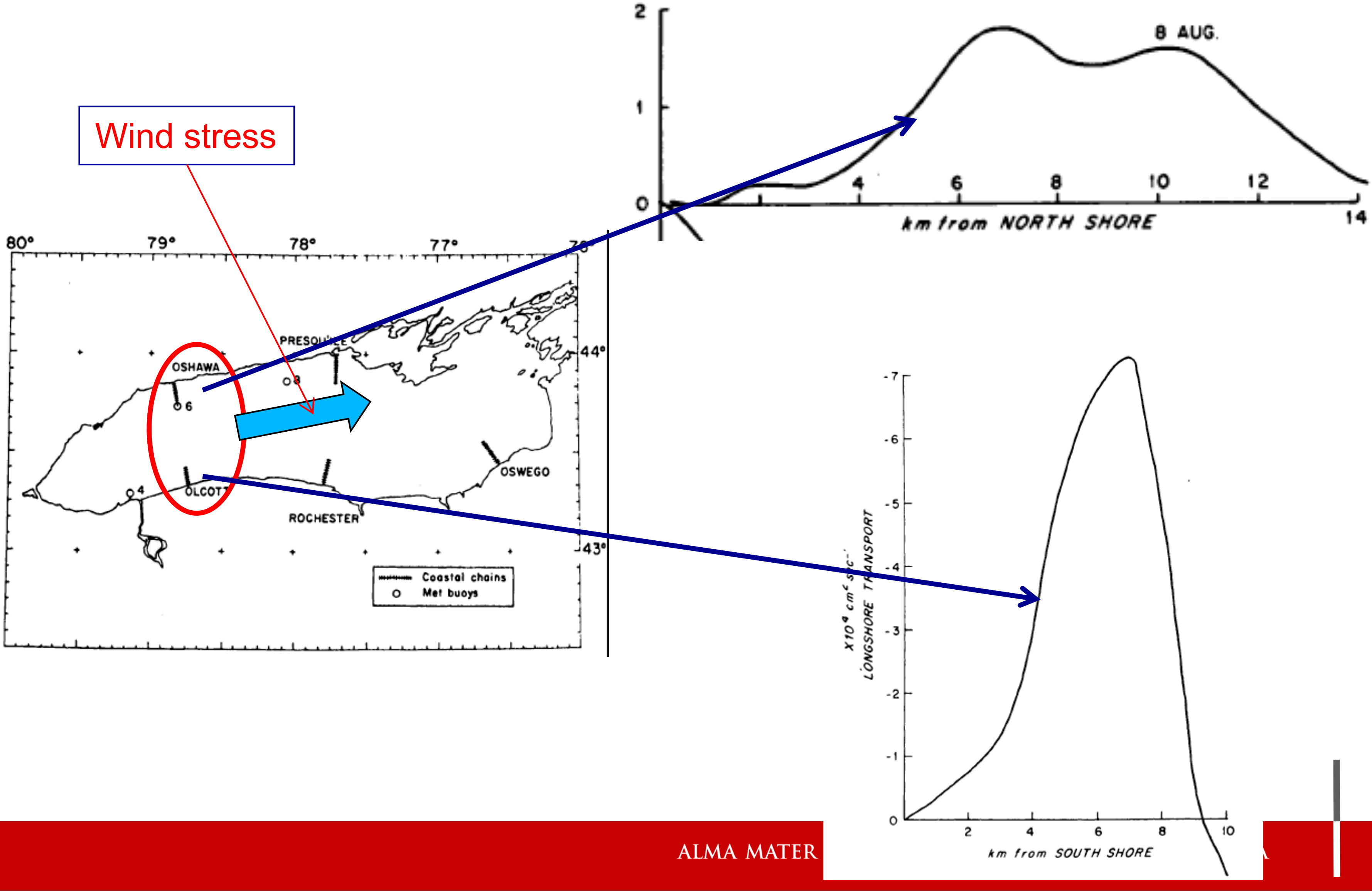
NB: remember also that at $x=0$ there is a singularity (along shore transport becomes infinite)

HOWEVER……………………………………….



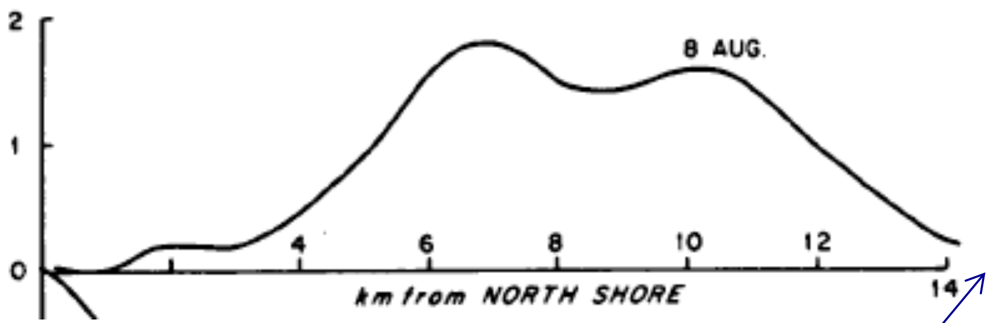
Problems with bottom friction neglected

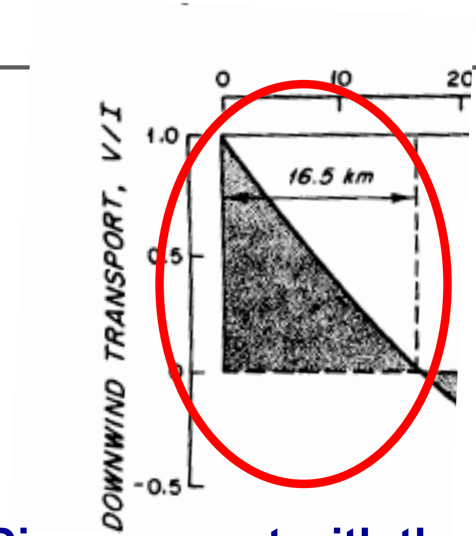
Observations at two sections in Lake Ontario



Problems with bottom friction neglected

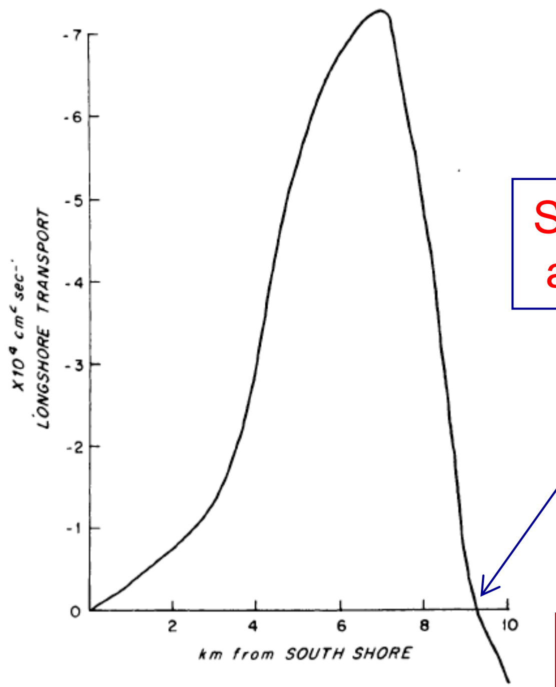
Observations at two sections in Lake Ontario





Disagreement with theory:
At the coast Transport is 0
And grows leaving the coast
Theory predicts opposite
Behaviour.

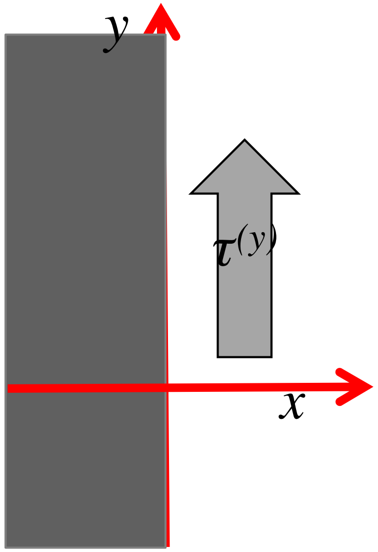
Site of section
average depth



Agreement with theory:
At a certain distance from
Coast transport starts to decrease
To reach 0 at approx. the site of
Section average depth.
(as predicted by theory).

Long shore wind, Variable depth and bottom friction

The general setup is the same seen previously, but
The Transport equations to be considered now are:



$$\frac{\partial U}{\partial t} - fV = -gH\frac{\partial \eta}{\partial x} + \frac{1}{\rho_0}\left(\tau_w^{(x)} - \tau_B^{(x)}\right)$$

$$\frac{\partial V}{\partial t} + fU = +\frac{1}{\rho_0}\left(\tau_w^{(y)} - \tau_B^{(y)}\right)$$

$$\frac{\partial U}{\partial x} = -\frac{\partial \eta}{\partial t}$$

$x \geq 0$.

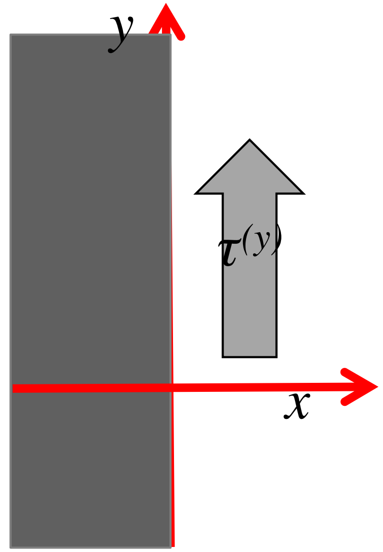
Depth distribution function of x only→no gradients along y exists.
Only alongshore wind
Discussion initially limited to a nearshore region where the the coastal constraints hold to a satisfactory approximation: $U=0$.
Bottom stress along y defined as (justified later);
C_{da}: drag coeff.

$$\frac{\tau_B^{(y)}}{\rho_0} = c_{da}\left(\frac{V}{H}\right)^2$$

Long shore wind, Variable depth and bottom friction

The equation

$$\frac{\partial V}{\partial t} + fU = -gH\frac{\partial \eta}{\partial y} + \frac{1}{\rho_0}\left(\tau_w^{(y)} - \tau_B^{(y)}\right)$$



Becomes:

$$\frac{\partial V}{\partial t} = u_*^2 - c_{da}\left(\frac{V}{H}\right)^2$$

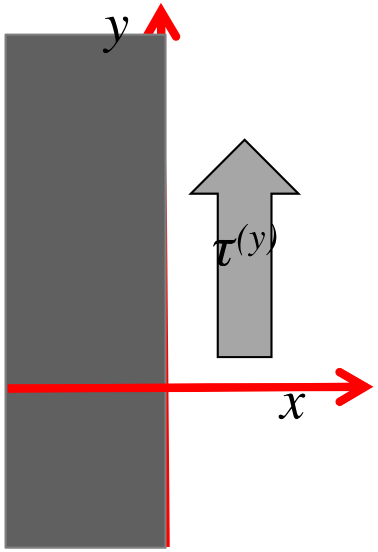
And its solution is

$$V = \frac{u_* H}{\sqrt{c_{da}}}\left[\frac{1-\exp\left(-2u_* t\frac{\sqrt{c_{da}}}{H}\right)}{1+\exp\left(-2u_* t\frac{\sqrt{c_{da}}}{H}\right)}\right]$$

Where the x-dependence is entirely contained in the variable depth $H(x)$

Long shore wind, Variable depth and bottom friction

$$V = \frac{u_* H}{\sqrt{c_{da}}} \left[\frac{1 - \exp\left(-2u_* t \frac{\sqrt{c_{da}}}{H}\right)}{1 + \exp\left(-2u_* t \frac{\sqrt{c_{da}}}{H}\right)} \right]$$



For a given depth and for $t \ll t_f$, with t_f (a frictional adjustment time scale) given by:

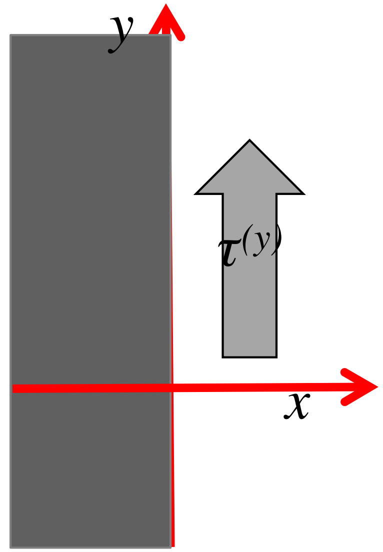
$$t_f = \frac{H}{2u_* \sqrt{c_{da}}} \qquad (t \ll t_f)$$

We obtain the inertial response result seen previously:

$$V = u_*^2 t$$

Long shore wind, Variable depth and bottom friction

$$V = \frac{u_*^2 H}{\sqrt{c_{da}}}\left[\frac{1-\exp\left(-2u_*^2 t\,\frac{\sqrt{c_{da}}}{H}\right)}{1+\exp\left(-2u_*^2 t\,\frac{\sqrt{c_{da}}}{H}\right)}\right]$$



for $t >> t_f$:

$$V = \frac{u_* H}{\sqrt{c_{da}}} \qquad (t >> t_f)$$

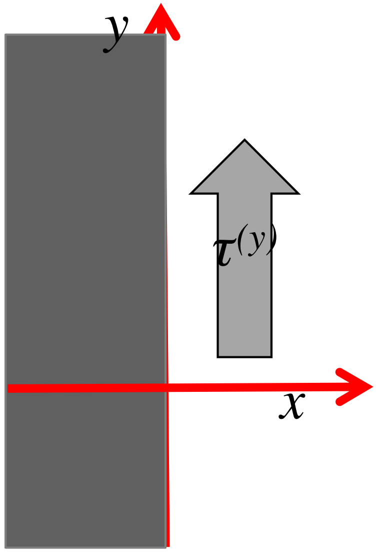
The transport reach a constant value (in time) and is defined by a depth averaged velocity:

$$\frac{u_*}{\sqrt{c_{da}}}$$

Long shore wind, Variable depth and bottom friction

$$V = \frac{u_*^2 H}{\sqrt{c_{da}}} \left[\frac{1 - \exp\left(-2u_*^2 t \frac{\sqrt{c_{da}}}{H} \right)}{1 + \exp\left(-2u_*^2 t \frac{\sqrt{c_{da}}}{H} \right)} \right]$$

$$V = \frac{u_* H}{\sqrt{c_{da}}} \qquad (t >> t_f)$$



Recalling the definition of the bottom stress: $\frac{\tau_B^{(y)}}{\rho_0} = c_{da} \left(\frac{V}{H} \right)^2$

$\frac{\tau_B^{(y)}}{\rho_0} = u_*^2$ or also, since $\frac{\tau_B^{(x)}}{\rho_0} = 0$ (see also lecture about BBL): $u_{*B}^2 = u_*^2$

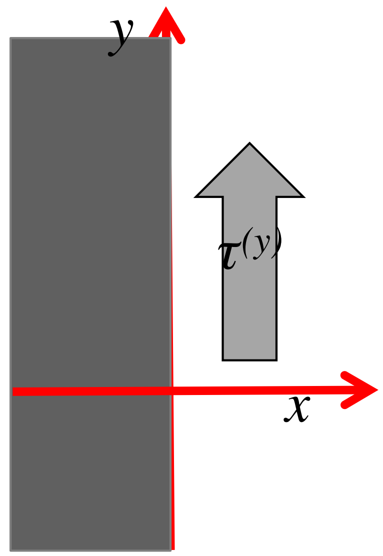
Constant along shore transport is achieved when wind and bottom stress balance each other.

Long shore wind, Variable depth and bottom friction

$$V = \frac{u_*^2 H}{\sqrt{c_{da}}}\left[\frac{1-\exp\left(-2u_*^2 t\frac{\sqrt{c_{da}}}{H}\right)}{1+\exp\left(-2u_*^2 t\frac{\sqrt{c_{da}}}{H}\right)}\right]$$

$$V = u_*^2 t \qquad (t \ll t_f)$$

$$V = \frac{u_* H}{\sqrt{c_{da}}} \qquad (t \gg t_f)$$



Given a $\tau_w^{(y)} = 0.1Pa$, corresponding to a $u_* = 0.01ms^{-1}$, $H=100m$,

$c_{da} = 2 \cdot 10^{-3}$

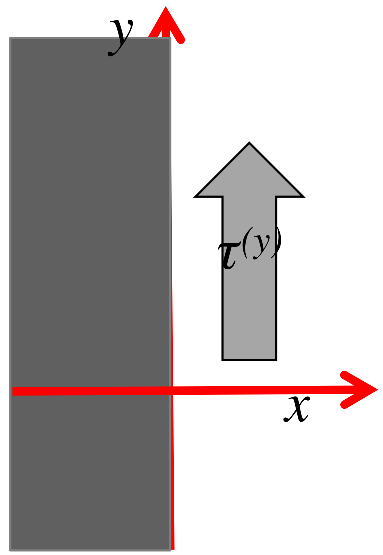
One finds t_f slightly over *30 hrs*

t_f is obviously directly prportional to H and inversely to the friction velocity u_* and to c_{da}
The larger the wind stress (al so c_{da} increase) the shorter t_f .

Long shore wind, Variable depth and bottom friction

Equation

$$t_f = \frac{H}{2u_* \sqrt{c_{da}}}$$



can be written in function of the Ekman depth D_E:

$$D_E = 0.1\frac{u_*}{f}$$

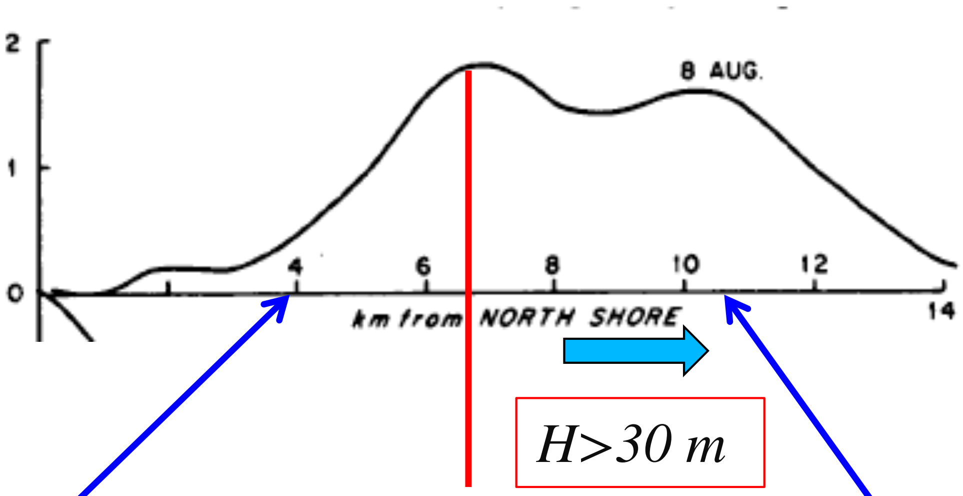
Upon substitution:

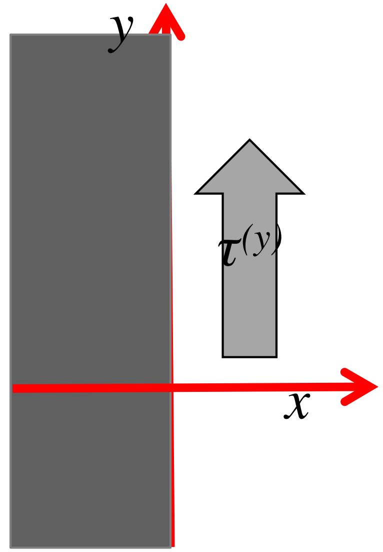
$$t_f = f^{-1}\frac{H}{20\sqrt{c_{da}}D_E}$$

Since $c_{da}=O(10^{-3})$ the "typical t_f is: $t_f \cong 2f^{-1}\frac{H}{D_E}$

For large H/D_E, ft_f is also large, so often the frictional adjustments are confined to depths of order D_E, given that wind (storm) events have a time scale of few f^{-1}

Long shore wind, Variable depth and bottom friction





$$V = \frac{u_*^2 H}{\sqrt{c_{da}}} \left[\frac{1 - \exp\left(-2u_*^2 t \frac{\sqrt{c_{da}}}{H} \right)}{1 + \exp\left(-2u_*^2 t \frac{\sqrt{c_{da}}}{H} \right)} \right]$$

$$V = u_*^2 t \xi^{1/2} K_1\left(2\xi^{1/2}\right)$$

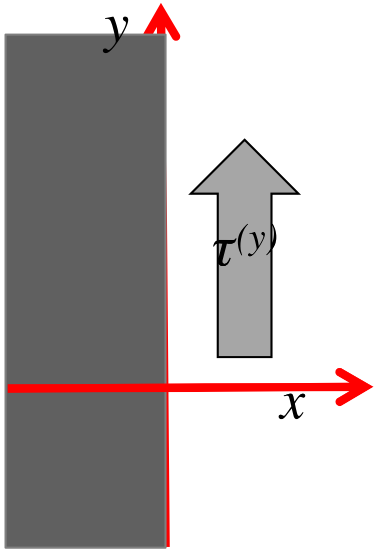
The theory depicted above applies to the example above for the coastal domain embedded between the coast and the depth of 30 m. Infact the wind event generating the pattern was active for about 16 hrs

With $u_* = 0.01 ms^{-1}$ $c_{da} = 2 \cdot 10^{-3}$ the relation $t_f = f^{-1} \frac{H}{10\sqrt{c_{da} D_E}}$ in fact yields $H \cong 30\ m$

Long shore wind, Variable depth and bottom friction (local problem)

The local problem

In absence of a longshore pressure gradient the equation system for The local problem is:



$$\frac{\partial u}{\partial t} - fv = -g\frac{\partial \eta}{\partial x} + \frac{A_v}{\rho_0}\frac{\partial^2 u}{\partial z^2}$$

$$\frac{\partial v}{\partial t} + fu = +\frac{A_v}{\rho_0}\frac{\partial^2 v}{\partial z^2}$$

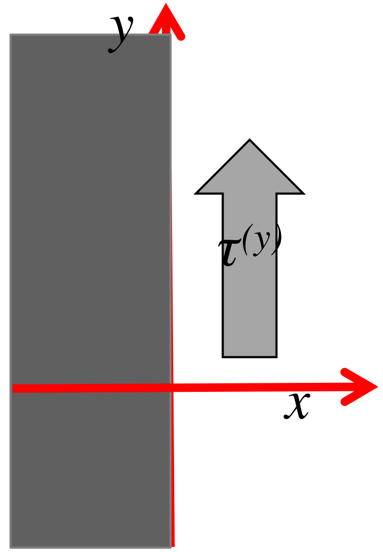
As usual the local solution is broken into two components

$(u,v)=(u_1,v_1)+(u_2,v_2)$

With $u_1=u_1(z)$, $v_1=v_1(z)$ being the pressure induced field and $u_2=u_2(z,t)$, $v_2=v_2(z,t)$ being the frictional component

Long shore wind, Variable depth and bottom friction (local problem)

Loking for a non-oscillatory solutions the equation system becomes:



PRESSURE INDUCED

$$-fv_1 = -g\frac{\partial \eta}{\partial x}$$

$$\frac{\partial v_1}{\partial t} + fu_1 = 0$$

FRICTIONALLY INDUCED

$$-fv_2 = \frac{A_v}{\rho_0}\frac{\partial^2 u}{\partial z^2}$$

$$fu_2 = \frac{A_v}{\rho_0}\frac{\partial^2 v}{\partial z^2}$$

The top and bottom boundary condition for the frictional velocity (u_2 and v_2) are:

$$\left.\frac{A_v}{\rho_0}\frac{\partial u_2}{\partial z}\right|_{z=0} = 0$$

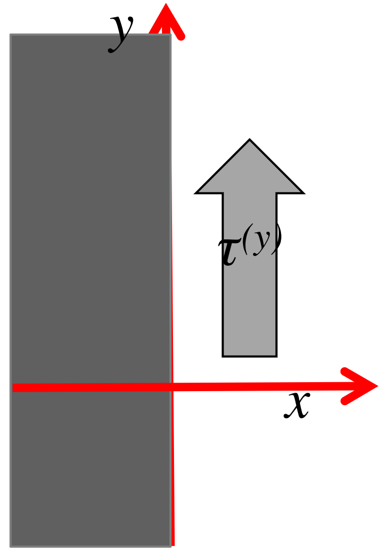
$$\left.\frac{A_v}{\rho_0}\frac{\partial u_2}{\partial z}\right|_{z=-H} = \frac{\tau_B^{(x)}}{\rho_0}$$

$$\left.\frac{A_v}{\rho_0}\frac{\partial v_2}{\partial z}\right|_{z=0} = u_*^2$$

$$\left.\frac{A_v}{\rho_0}\frac{\partial v_2}{\partial z}\right|_{z=-H} = \frac{\tau_B^{(y)}}{\rho_0}$$

Long shore wind, Variable depth and bottom friction (local problem)

$$\frac{A_v}{\rho_0}\frac{\partial u_2}{\partial z}\bigg|_{z=-H} = \frac{\tau_B^{(x)}}{\rho_0} \qquad \frac{A_v}{\rho_0}\frac{\partial v_2}{\partial z}\bigg|_{z=-H} = \frac{\tau_B^{(y)}}{\rho_0}$$



The bottom stress is given by a drag law (see previous lecture on the BBL):

$$\tau_B^{(x)} = c_d\left(u_1 + u_2\right)q \qquad \tau_B^{(y)} = c_d\left(v_1 + v_2\right)q$$

Where: $$q = \left[\left(u_1 + u_2\right)^2 + \left(v_1 + v_2\right)^2\right]^{0.5}$$

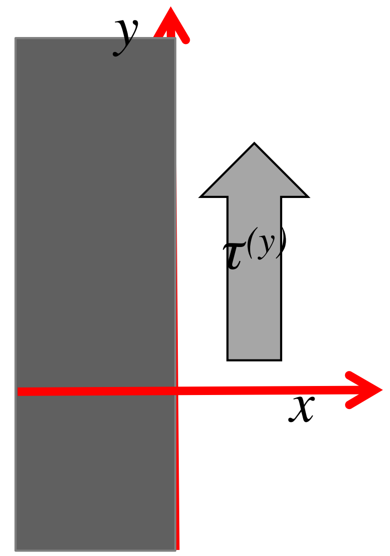
Is the total velocity magnitude .

Now the system can be solved.

Long shore wind, Variable depth and bottom friction (local problem)

The equations for the frictionally induced velociy field

$$-fv_2 = \frac{A_v}{\rho_0}\frac{\partial^2 u}{\partial z^2} \qquad fu_2 = \frac{A_v}{\rho_0}\frac{\partial^2 v}{\partial z^2}$$



with the above boundary conditions yield:

$$v_2 = -\frac{1}{fH}\left(0 - \frac{\tau_B^{(x)}}{\rho_0}\right) = \frac{1}{fH}\frac{\tau_B^{(x)}}{\rho_0} \qquad u_2 = \frac{1}{fH}\left(u_*^2 - \frac{\tau_B^{(y)}}{\rho_0}\right)$$

Recall that:

$$v_1 = \frac{V}{H} - v_2 \qquad u_1 = \frac{U}{H} - u_1$$

And that for the longshore wind case $U=0$. Then the equations for the pressure induced velocity components:

$$-fv_1 = -g\frac{\partial \eta}{\partial x} \qquad \frac{\partial v_1}{\partial t} + fu_1 = 0$$

upon substitution become:

$$v_1 = \frac{g}{f}\frac{\partial \eta}{\partial x} \qquad \frac{\partial v_1}{\partial t} = -fu_1$$

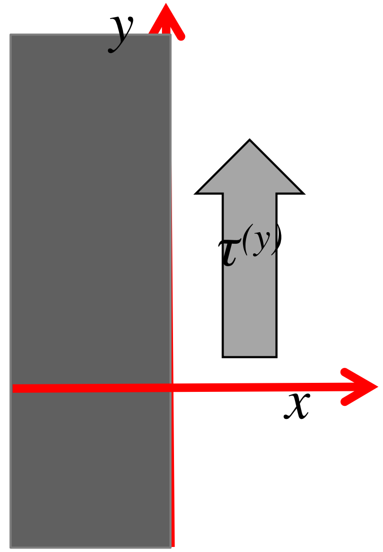
$$\Rightarrow \quad v_1 = \left(\frac{V}{H} - v_2\right) \qquad \frac{\partial v_1}{\partial t} = -f\left(0 - u_2\right)$$

$$\Rightarrow \quad v_1 = \left(\frac{V}{H} - \frac{1}{fH}\frac{\tau_B^{(x)}}{\rho_0}\right) \qquad \frac{\partial v_1}{\partial t} = \frac{1}{H}\left(u_*^2 - \frac{\tau_B^{(y)}}{\rho_0}\right)$$

Long shore wind, Variable depth and bottom friction (local problem)

$$v_1 = \left(\frac{V}{H} - \frac{1}{fH} \frac{\tau_B^{(x)}}{\rho_0} \right) \qquad v_2 = -\frac{1}{fH}\left(0 - \frac{\tau_B^{(x)}}{\rho_0}\right) = \frac{1}{fH}\frac{\tau_B^{(x)}}{\rho_0}$$

$$\frac{\partial v_1}{\partial t} = \frac{1}{H}\left(u_*^2 - \frac{\tau_B^{(y)}}{\rho_0} \right) \qquad u_2 = \frac{1}{fH}\frac{A_v}{\rho_0}\left(u_*^2 - \frac{\tau_B^{(y)}}{\rho_0} \right)$$



The set of equations above is discussed for some limit cases.

Shallow water limit.

It is assumed that

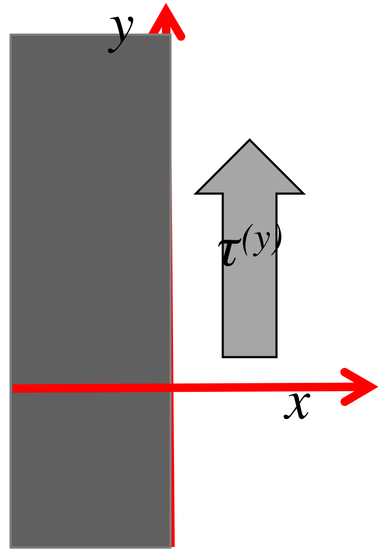
$$H << D_E$$

Under such condition the shear stress changes (from surface to bottom) by an order H/D_E quantity. Therefore it can be assumed

$$\frac{\tau_B^{(y)}}{\rho_0} = u_*^2 \qquad \frac{\tau_B^{(x)}}{\rho_0} = 0$$

Long shore wind, Variable depth and bottom friction (local problem)

Shallow water limit. $\dfrac{\tau_B^{(y)}}{\rho_0} = u_*^2$ $\dfrac{\tau_B^{(x)}}{\rho_0} = 0$



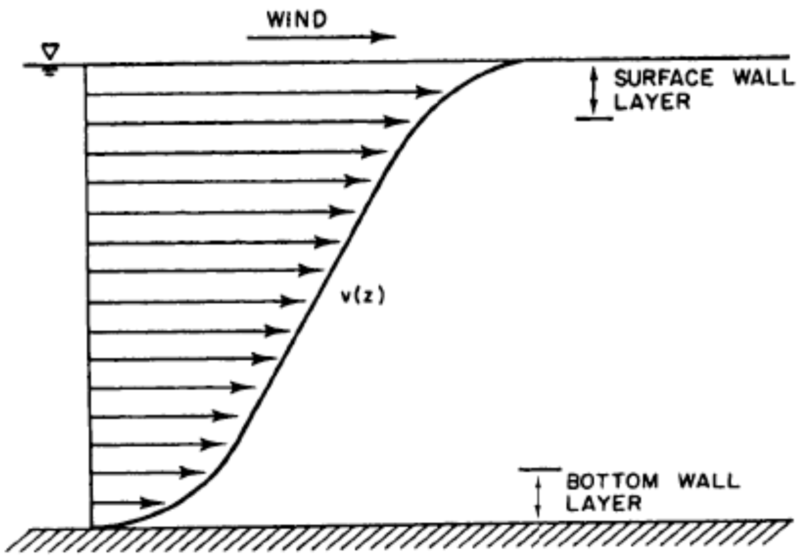
Under such assumptions the equations

$$v_1 = \left(\frac{V}{H} - \frac{1}{fH} \frac{\tau_B^{(x)}}{\rho_0} \right)$$

$$\frac{\partial v_1}{\partial t} = \frac{1}{H} \left(u_*^2 - \frac{\tau_B^{(y)}}{\rho_0} \right)$$

indicate: $v_1 = \dfrac{V}{H}$, $u_1 = 0$

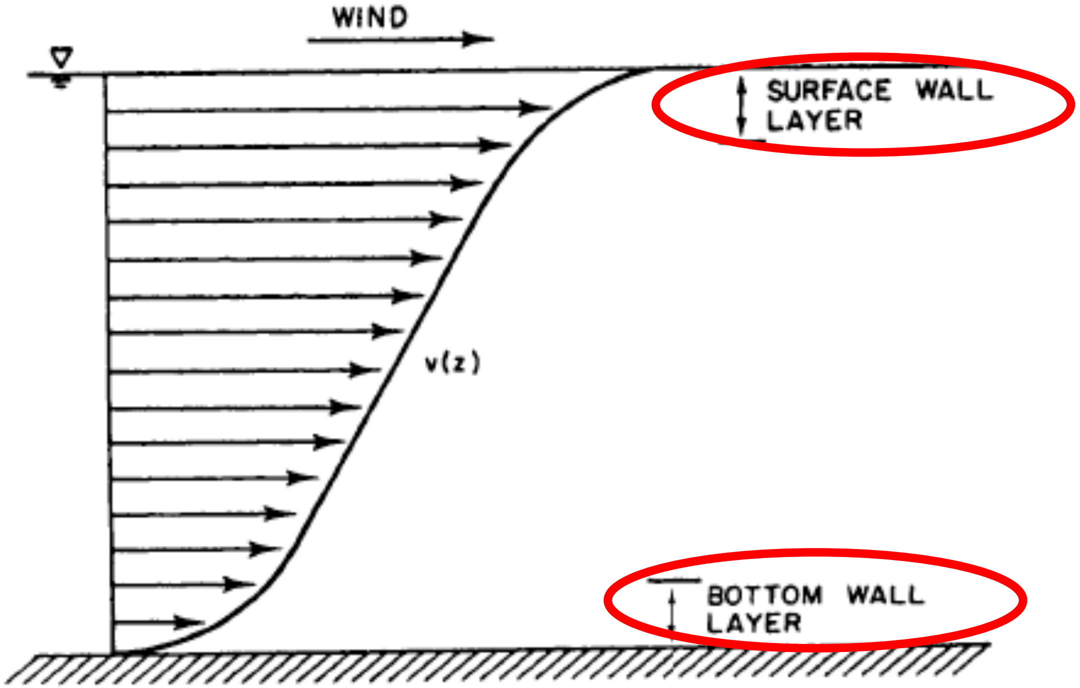
A flow without significant Rotational effect

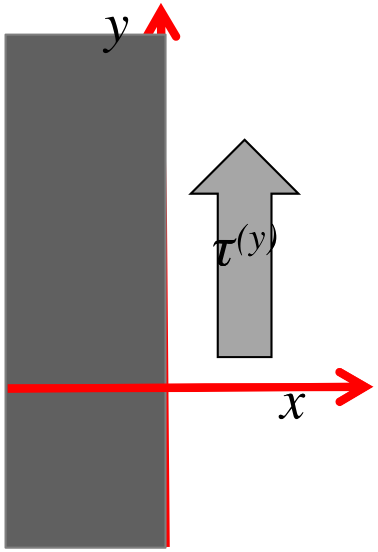


With bottom stress equalling the wind stress

Long shore wind, Variable depth and bottom friction (local problem)

Shallow water limit.

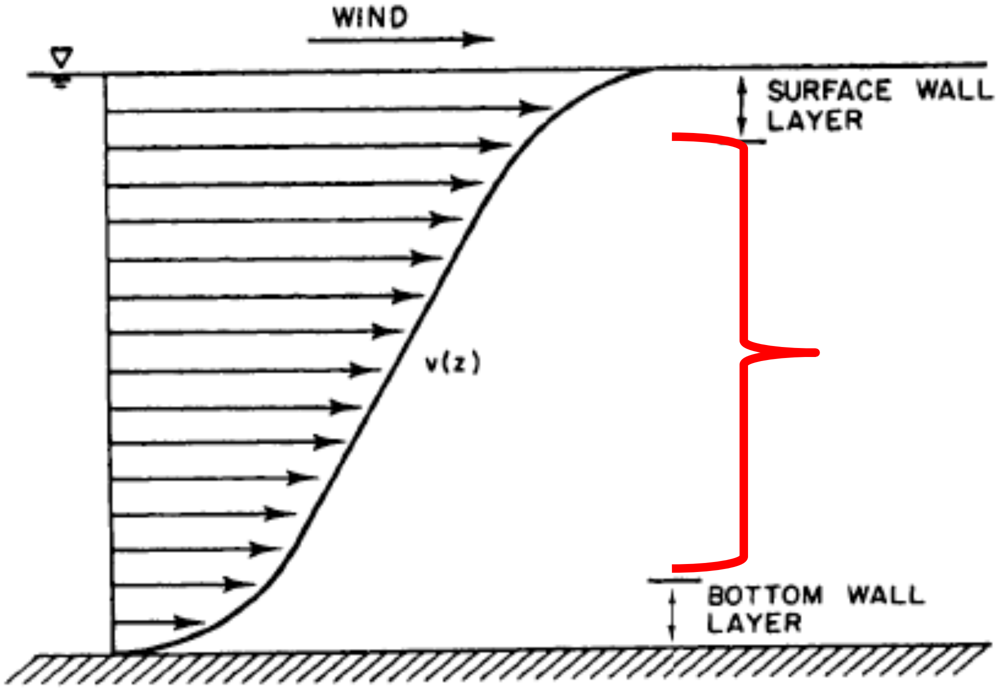


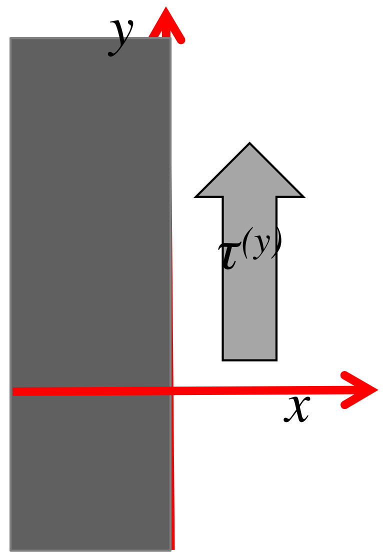


The two thin highly sheared layers developing at the surface and at the bottom are the wall layers (see previous lecture on the BBL and or Pinardi notes on the atmospheric boundary layer) within which the length scale of eddies varies rapidly in direct proportion with the distance From the surface or bottom.

Long shore wind, Variable depth and bottom friction (local problem)

Shallow water limit.



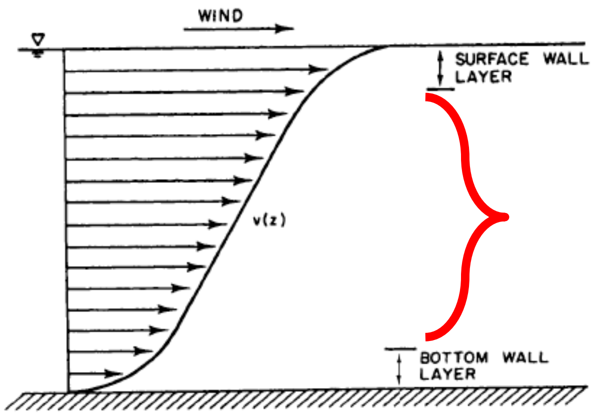


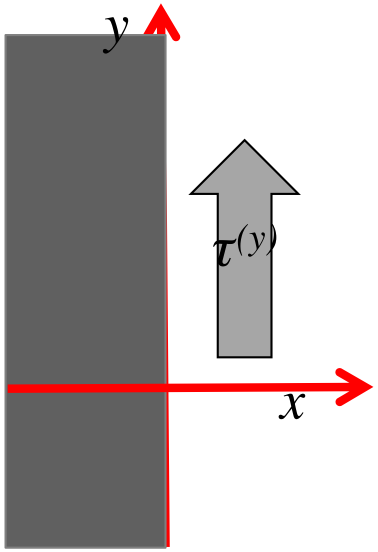
Between the two "wall Layers" there is the "turbulent" (outer) layer with:

- eddies length scales constant and approximately proportional to H
- eddy velocity proportional to the friction velocity u_*
- constant eddy viscosity.

Long shore wind, Variable depth and bottom friction (local problem)

Shallow water limit.





In the turbulent layer, the eddy Reynolds number:

$$\mathrm{Re} = \frac{\rho_0 u_* h}{A_v} \cong 16$$

(h=mixed layer depth)

And the shear is

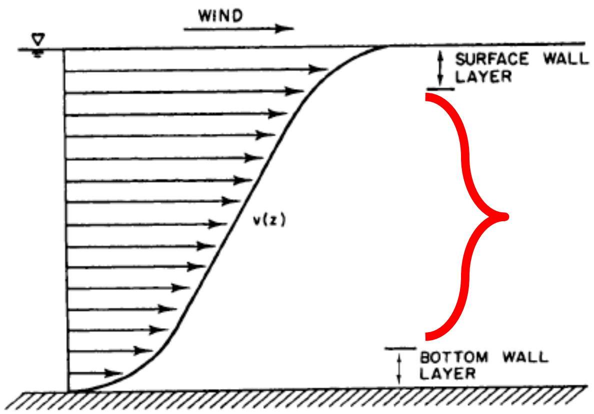
$$\frac{dv_2}{dz} = 16\frac{u_*^2}{H}$$

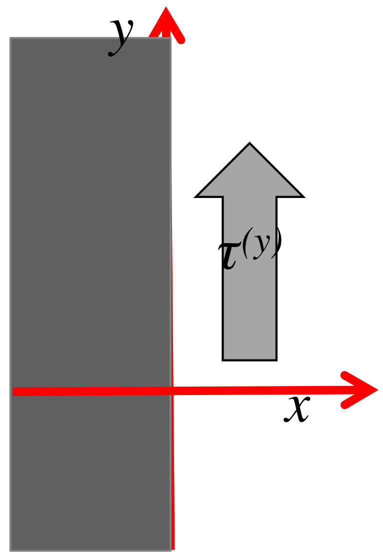
If the turbulent layer is (for instance) 80% of the total depth then the difference in velocity at the two extreme of the turbulent layer is about 13 u_*

The average Velocity V/H coincide with the velocity at mid-depth.

Long shore wind, Variable depth and bottom friction (local problem)

Shallow water limit.





Extrapolating the constant velocity gradient $\dfrac{dv_2}{dz} = 16\dfrac{u_*^2}{H}$ to the bottom it is Found:

$$v_B = \frac{V}{H} - 8u_*^2$$

Applying the bottom drag law we find: $$\frac{V}{H} - 8u_*^2 = u_*^2\sqrt{c_d}$$

Or also: $$\frac{\tau_B^y}{\rho_0} = u_*^* = \frac{(V/H)^2}{\left(8 + c_d^{-1/2}\right)^2}$$

c_d=drag coefficient referred to the extrapolated velocity

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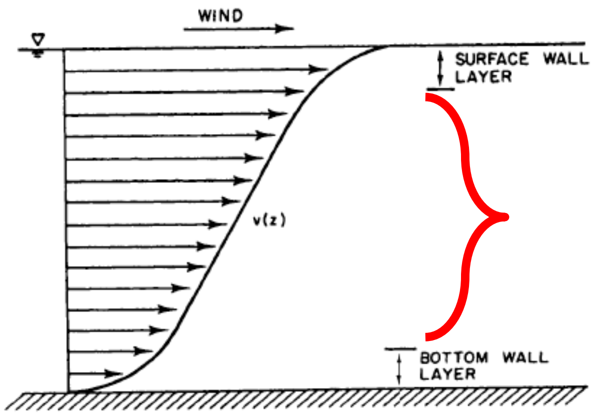
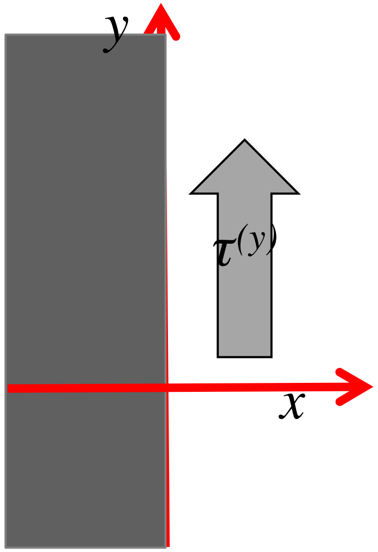
Long shore wind, Variable depth and bottom friction (local problem)

Shallow water limit.





$$\frac{\tau_B^y}{\rho_0} = u_{*B}^2 = \frac{(V/H)^2}{\left(8 + c_d^{-1/2}\right)^2}$$

The above is equivalent to the bottom stress definition adopted to solve the global problem:

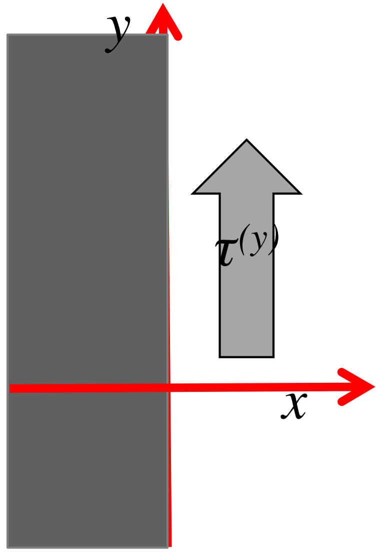
$$\frac{\tau_B^{(y)}}{\rho_0} = c_{da}\left(\frac{V}{H}\right)^2$$

Where: $c_{da} = \left(8 + c_d^{-1/2}\right)^{-2}$

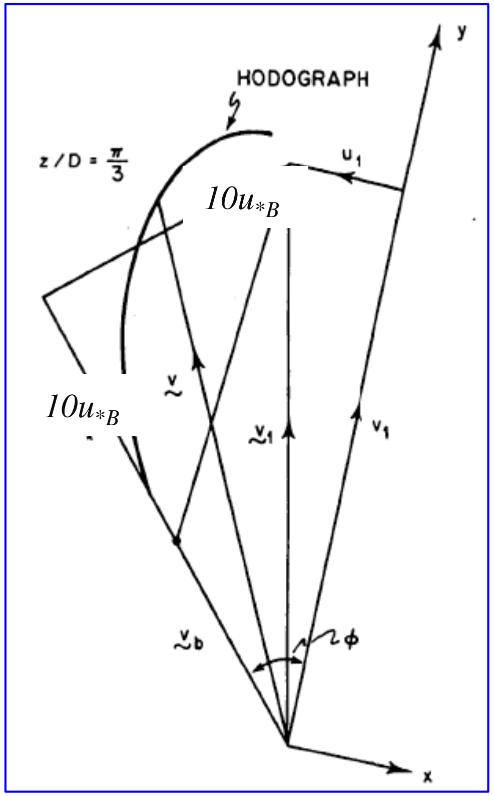
Long shore wind, Variable depth and bottom friction (local problem)

Deep water limit.
The condition is now

$$H >> D_E$$



Then there are two Ekman layers (surface and bottom with their own wall layer)
And an interior region which is frictionless.



The velocity distribution in the Bottom Ekman layer has the following characteristics:

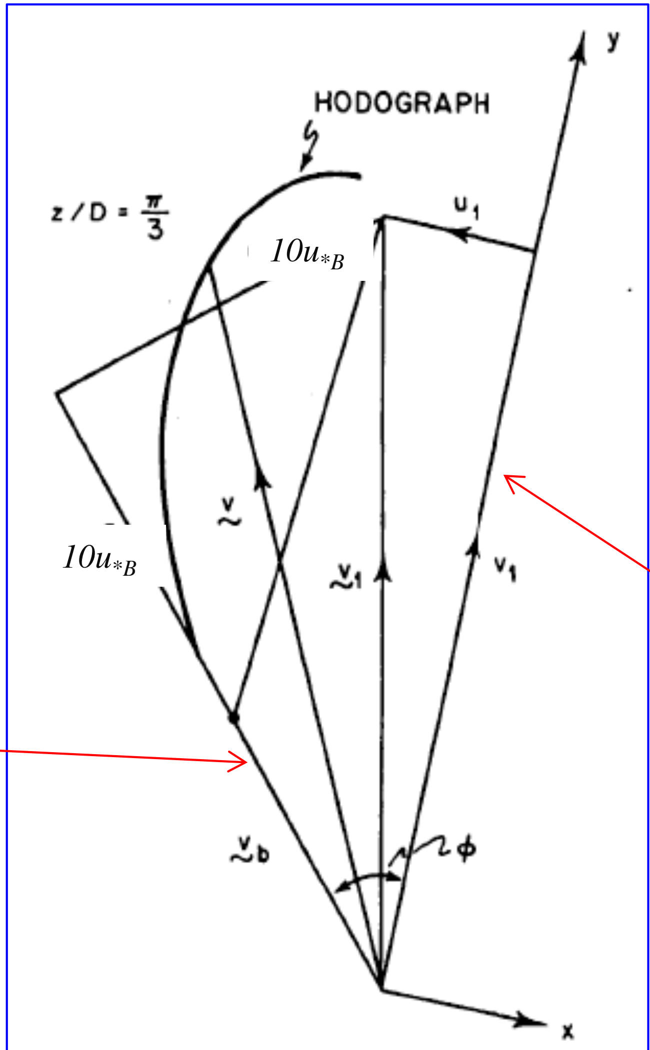
- across the wall layer velocity increase in magnitude from zero with increasing distance from bottom, but It maintains the same direction of the Bottom stress (deviated by some angle ϕ to the LEFT of the wind stress direction.

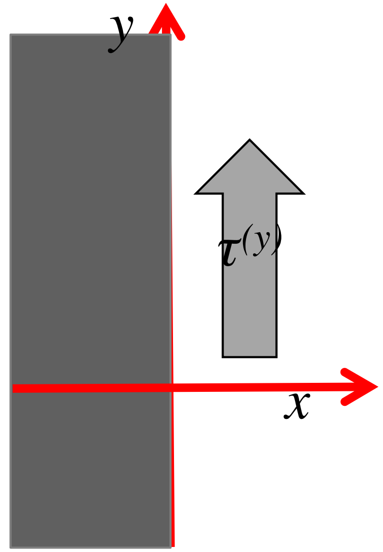
- in the turbulent layer velocity increase further and rotate forming

Ekman spiral rotating (in the northern emisphere) to the right/left of the bottom/wind stress

Long shore wind, Variable depth and bottom friction (local problem)

Deep water limit. $H >> D_E$





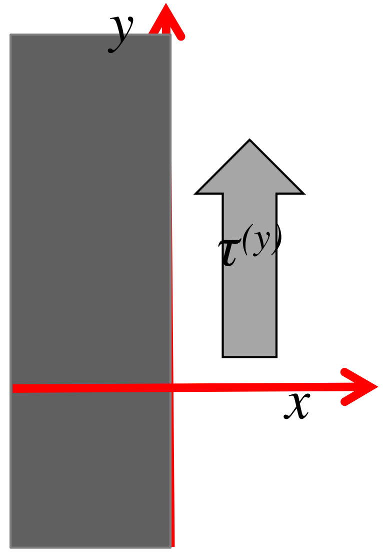
velocity above the Wall layer

Geostrophic velocity in the Frictionless region.
Also wind stress direction

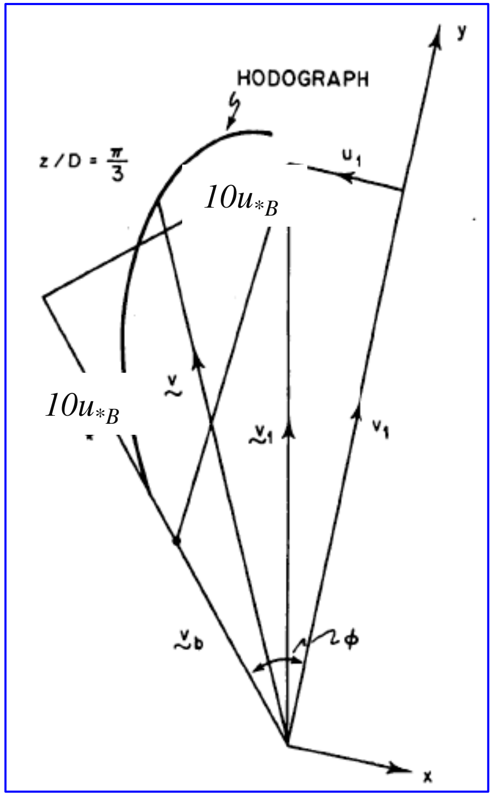
Long shore wind, Variable depth and bottom friction (local problem)

Deep water limit. $H >> D_E$

The velocity increase from bottom upward is $\dfrac{u_*}{fD_e}$



Given the empirical value for $D_E = 0.1 u_* f^{-1}$ one gets:



$$\frac{u_*}{fD_e} = 10$$

for the increment of both velocity components across the turbulent Ekman layer.

the magnitude of the extrapolated bottom velocity is

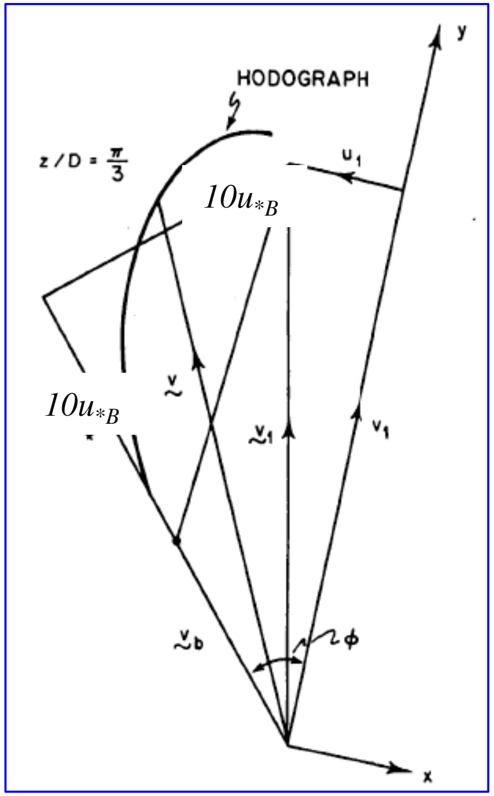
$$|v_b| = c_d^{-1/2} u_{*B}$$

distribution in the Bottom Ekman layer

Long shore wind, Variable depth and bottom friction (local problem)

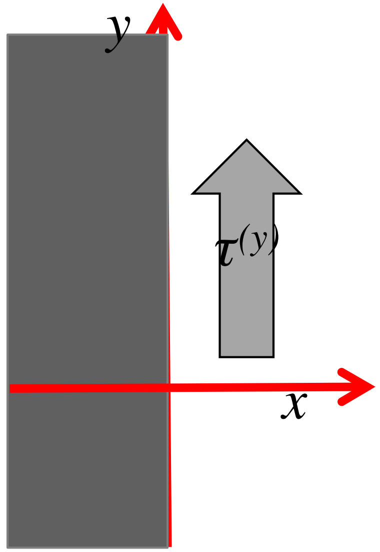
Deep water limit. $H >> D_E$

Velocities above the Ekman layer are the pressure field induced velocities u_1 and v_1. That can be computed geometrically



$$u_1 = -10u_{*B}\sin\phi + 10u_{*B}\cos\phi - c_d^{-1/2}u_{*B}\sin\phi$$

$$v_1 = 10u_{*B}\sin\phi + 10u_{*B}\cos\phi + c_d^{-1/2}u_{*B}\cos\phi$$



Physically the cross shore main balance is between coriolis Force and cross shore pressure gradient, with the cross shore bottom stress playing a minor role, so that equation

$$v_1 = \left(\frac{V}{H} - \frac{1}{fH}\frac{\tau_B^{(x)}}{\rho_0}\right)$$

can be simplified to $v_1 = \dfrac{V}{H}$

with this approximation the bottom stress may be described by

$$\frac{\tau_B^y}{\rho_0} = u_{*B}^2 = c_d\cos\phi\left(\frac{V}{H}\right)^2$$

again as the bottom stress used to solve the global problem