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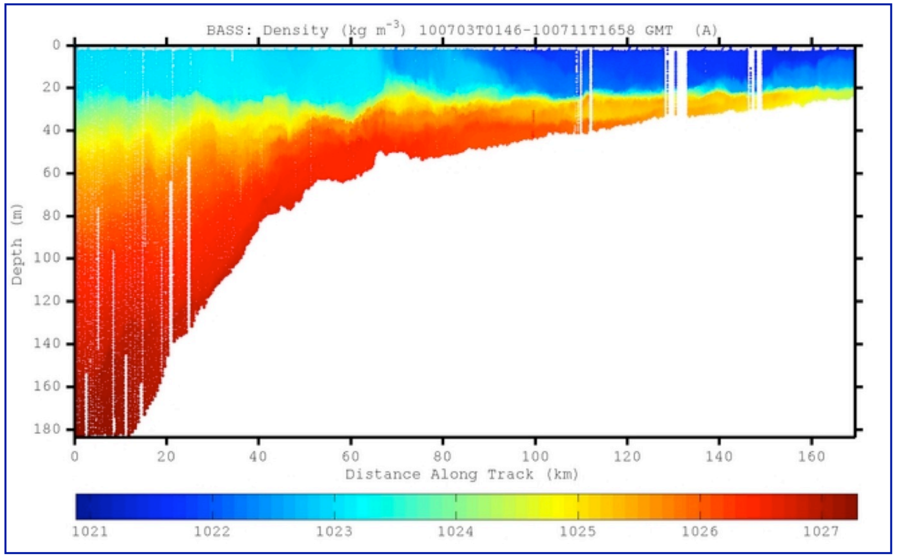
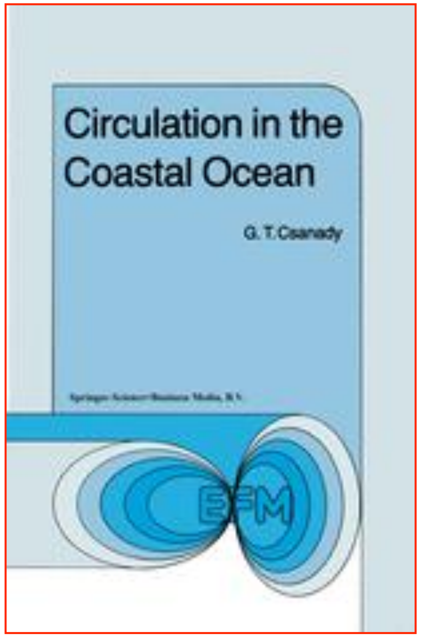
The stratified coastal ocean

Part 2



IL PRESENTE MATERIALE È RISERVATO AL PERSONALE DELL'UNIVERSITÀ DI BOLOGNA E NON PUÒ ESSERE UTILIZZATO AI TERMINI DI LEGGE DA ALTRE PERSONE O PER FINI NON ISTITUZIONALI

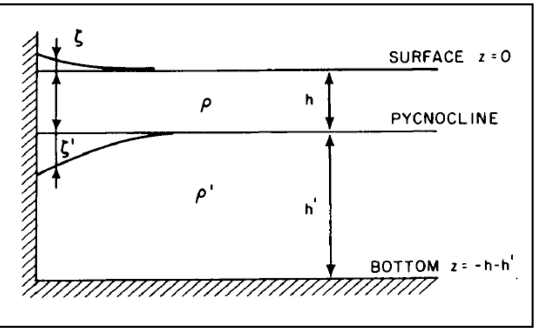
Main reference



G. Csanady
Circulation in the Coastal Ocean.
Chap. 3: The behaviour of the Stratified Sea.
Sections. 3.6, 3.7

Two layer model of the stratified Ocean

The 2 transport modes found correspond to a Barotropic (external) and a Baroclinic (internal) mode.



The Barotropic (external) mode

Chosing $b_1=1$, neglecting quantities $O(\varepsilon)$ where they occur with quantities $O(1)$ and assuming $O(\eta)=O(\eta')$,$O(U,U')=O(V,V')$ we get :

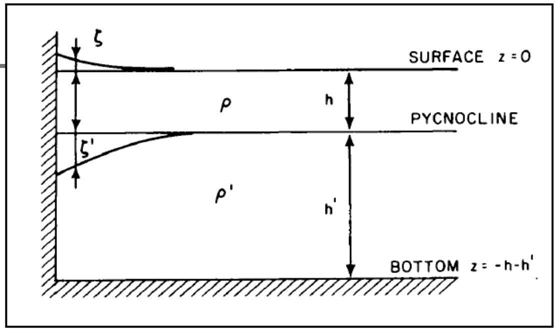
$$U_k = a_k U + b_k U'$$
$$V_k = a_k V + b_k V'$$
$$\eta_k = a_k \eta + (b_k - a_k)\eta'$$
$$\left(\frac{\tau_w^{(x)}}{\rho}\right)_k = a_k \frac{\tau_w^{(x)}}{\rho}$$
$$\left(\frac{\tau_w^{(y)}}{\rho}\right)_k = a_k \frac{\tau_w^{(y)}}{\rho}$$

$$\frac{a_1}{b_1} = 1 - \varepsilon \frac{h'}{h+h'}$$

Assumptions above may be verified noting that if the motion in the 2-layer model is entirely in the 1st mode the 2nd mode transport and elevation vanish.

$$U_1 = U + U'$$
$$V_1 = V + V'$$
$$\eta_1 = \eta$$
$$\left(\frac{\tau_w^{(x)}}{\rho}\right)_1 = \frac{\tau_w^{(x)}}{\rho}$$
$$\left(\frac{\tau_w^{(y)}}{\rho}\right)_1 = \frac{\tau_w^{(y)}}{\rho}$$

Two layer model of the stratified Ocean



Setting:

$$U_2 = V_2 = \eta_2 = 0$$

And neglecting $O(\varepsilon)$ we find:

$$U' = \frac{h'}{h}U \qquad V' = \frac{h'}{h}V \qquad \eta' = \frac{h'}{h+h}\eta$$

Velocities $U'/h = U/h'$ *and* $V'/h\ V/h'$ in this mode are the same and the interface move a little less than the surface.

This is the same in a homogeneous (barotropic) fluid (true to order ε).
The equation describing this mode substituting the variables in the modal equations are:

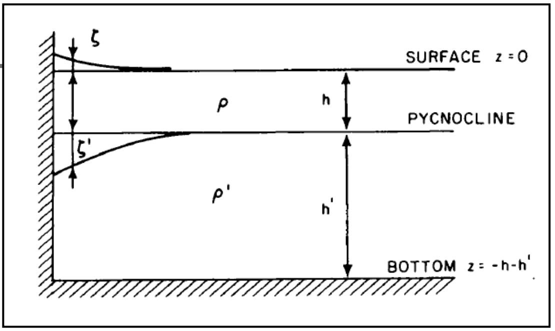
$$\frac{\partial}{\partial t}(U+U') - f(V+V') = -gH\frac{\partial \eta}{\partial x} + \frac{\tau_w^{(x)}}{\rho_0}$$

$$\frac{\partial}{\partial t}(V+V') + f(U+U') = -gH\frac{\partial \eta}{\partial y} + \frac{\tau_w^{(y)}}{\rho_0}$$

$$\frac{\partial}{\partial x}(U+U') + \frac{\partial}{\partial y}(V+V') = -\frac{\partial \eta}{\partial t}$$

and are identical to the homogeneous transport equations

Two layer model of the stratified Ocean



The Baroclinic (internal)mode.
If motion in the 1st mode vanish

$$U_1=V_1=\eta_1=0$$

we get

$$U' = -U \qquad V' = -V \qquad \eta = -\varepsilon \frac{h+h'}{h'} \eta'$$

Surface and interface displacements are now of different order, transport in the two layers Is now equal and opposite.

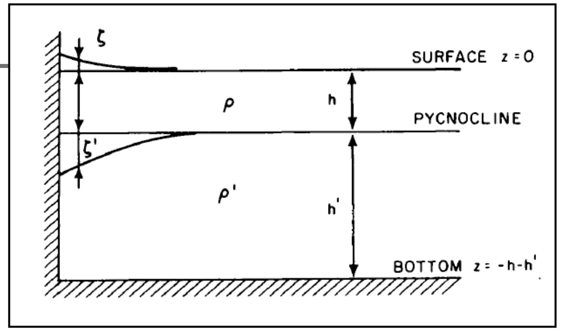
Chosing: $b_2 = \dfrac{h'}{h+h'}$ we get

$$U_2 = U' \qquad \left(\frac{\tau_w^{(x)}}{\rho}\right)_2 = -\frac{h'}{h+h'} \frac{\tau_w^{(x)}}{\rho}$$

$$V_2 = V'$$

$$\eta_2 = \eta' \qquad \left(\frac{\tau_w^{(y)}}{\rho}\right)_2 = -\frac{h'}{h+h'} \frac{\tau_w^{(y)}}{\rho}$$

Two layer model of the stratified Ocean



The equations for this mode are:

$$\frac{\partial U'}{\partial t} - fV' = -\varepsilon g \frac{hh'}{h+h'}\frac{\partial \eta'}{\partial x} - \frac{h'}{h+h'}\frac{\tau_w^{(x)}}{\rho_0}$$

$$\frac{\partial V'}{\partial t} + fU' = -\varepsilon g \frac{hh'}{h+h'}\frac{\partial \eta'}{\partial y} - \frac{h'}{h+h'}\frac{\tau_w^{(y)}}{\rho_0}$$

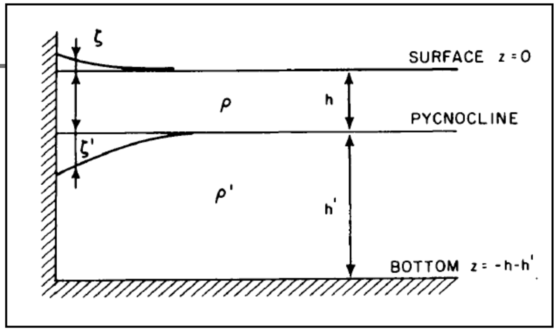
$$\frac{\partial U'}{\partial x} + \frac{\partial V'}{\partial y} = -\frac{\partial \eta'}{\partial t}$$

“Almost” like the equation for an individual bottom layer (interface being the upper boundary) BUT the pressure term and the forcing term are quite different.

•The ε appearing in the pressure term indicates that large displacements η' are needed for this term be significant.

•The negative sign of the forcing term indicate that the interface displacement is opposite to the surface displacement.

Two layer model of the stratified Ocean



Some important notes

The equation for the 2 modes (barotropic and baroclinic) describes the corresponding motions separately (each one computed as if the other be "dormant".

In order to compute the total transport, the transport computed by the two set of equation must be added.

For instance, the equations:

$$\frac{\partial}{\partial t}(U+U')-f(V+V')=-gH\frac{\partial \eta}{\partial x}+\frac{\tau_w^{(x)}}{\rho_0}$$

$$\frac{\partial}{\partial t}(V+V')+f(U+U')=-gH\frac{\partial \eta}{\partial y}+\frac{\tau_w^{(y)}}{\rho_0}$$

$$\frac{\partial}{\partial x}(U+U')+\frac{\partial}{\partial y}(V+V')=-\frac{\partial \eta}{\partial t}$$

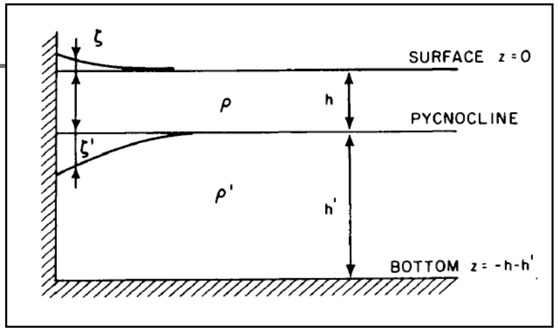
provide the total (barotropic) transport, while

$$U'=\frac{h'}{h}U \qquad V'=\frac{h'}{h}V \qquad \eta'=\frac{h'}{h+h}\eta$$

specify the distribution in the water column.

From these equation sets then the external mode contribution to U, U', V, V' is fully defined

Two layer model of the stratified Ocean



Some important notes

Similarly the equation set:

$$\frac{\partial U'}{\partial t} - fV' = -\varepsilon g \frac{hh'}{h+h'} \frac{\partial \eta'}{\partial x} - \frac{h'}{h+h'} \frac{\tau_w^{(x)}}{\rho_0}$$

$$\frac{\partial V'}{\partial t} + fU' = -\varepsilon g \frac{hh'}{h+h'} \frac{\partial \eta'}{\partial y} - \frac{h'}{h+h'} \frac{\tau_w^{(y)}}{\rho_0}$$

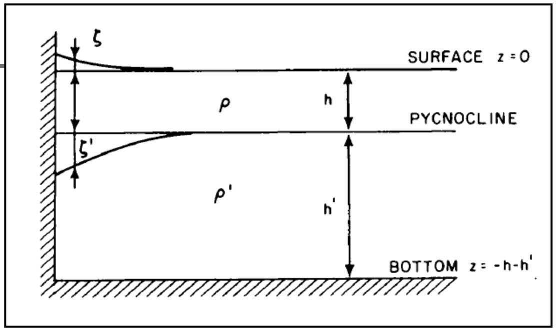
$$\frac{\partial U'}{\partial x} + \frac{\partial V'}{\partial y} = -\frac{\partial \eta'}{\partial t}$$

provides the internal mode (baroclinic) contribution to U', V', While:

$$U' = -U \qquad\qquad V' = -V$$

indicate that the U' and V' transport determined by the excitation of the internal mode are equal and opposite to U, V

Two layer model of the stratified Ocean



Some important notes

The total transport is obtained by simple algebraic addition of the Transport computed by the two modes equatuions,

Since the original 2 layer model equations:

$$\frac{\partial U}{\partial t} - fV = -gh\frac{\partial \eta}{\partial x} + \frac{\tau_w^{(x)}}{\rho_0}$$

$$\frac{\partial V}{\partial t} + fU = -gh\frac{\partial \eta}{\partial y} + \frac{\tau_w^{(y)}}{\rho_0}$$

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = -\frac{\partial}{\partial t}(\eta - \eta')$$

$$\frac{\partial U'}{\partial t} - fV' = -gh'\left(\frac{\rho}{\rho'}\frac{\partial \eta}{\partial x} + \varepsilon\frac{\partial \eta'}{\partial x}\right)$$

$$\frac{\partial V'}{\partial t} + fU' = -gh'\left(\frac{\rho}{\rho'}\frac{\partial \eta}{\partial y} + \varepsilon\frac{\partial \eta'}{\partial x}\right)$$

$$\frac{\partial U'}{\partial x} + \frac{\partial V'}{\partial y} = -\frac{\partial \eta'}{\partial t}$$

Are linear.

Two layer model: Longshore wind case

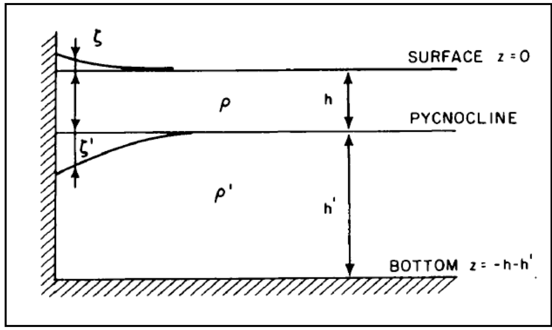
<u>Same configuration seen for the barotropic case.</u>

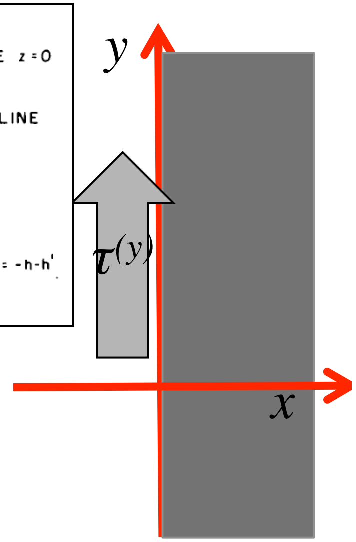
The barotropic mode is governed by:

$$\frac{\partial}{\partial t}\left(U+U'\right)-f\left(V+V'\right)=-gH\frac{\partial \eta}{\partial x}+\frac{\tau_w^{(x)}}{\rho_0}$$

$$\frac{\partial}{\partial t}\left(V+V'\right)+f\left(U+U'\right)=-gH\frac{\partial \eta}{\partial y}+\frac{\tau_w^{(y)}}{\rho_0}$$

$$\frac{\partial}{\partial x}\left(U+U'\right)+\frac{\partial}{\partial y}\left(V+V'\right)=-\frac{\partial \eta}{\partial t}$$





And the system has a non-oscillatory solution identical to the one seen for the barotropic case:

$$\eta=\frac{u_*^2 t}{c_1}e^{x/R_1}$$

$$U+U'=\frac{u_*^2}{f}\left(1-e^{x/R_1}\right)$$

$$V+V'=u_*^2 t e^{x/R_1}$$

$$c_1=\sqrt{gH}$$

$$R_1=\frac{c_1}{f}$$

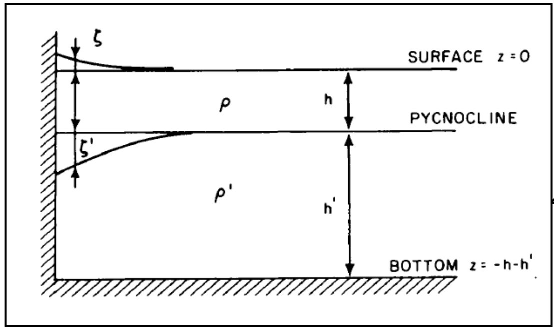
Two layer model: Longshore wind case

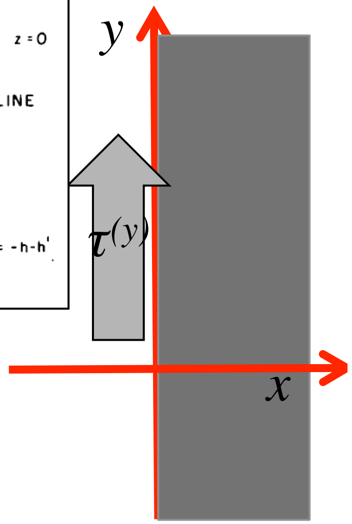
The baroclinic mode is governed by:

$$\frac{\partial U'}{\partial t} - fV' = -\varepsilon g \frac{hh'}{h+h'}\frac{\partial \eta'}{\partial x} - \frac{h'}{h+h'}\frac{\tau_w^{(x)}}{\rho_0}$$

$$\frac{\partial V'}{\partial t} + fU' = -\varepsilon g \frac{hh'}{h+h'}\frac{\partial \eta'}{\partial y} - \frac{h'}{h+h'}\frac{\tau_w^{(y)}}{\rho_0}$$

$$\frac{\partial U'}{\partial x} + \frac{\partial V'}{\partial y} = -\frac{\partial \eta'}{\partial t}$$





And the non-oscillatory solution for interface displacement and bottom transport are:

$$\eta' = -\frac{u_*^2 t}{c_2}\frac{h'}{h+h'}e^{x/R_2}$$

$$U' = -\frac{u_*^2}{f}\frac{h'}{h+h'}\left(1-e^{x/R_2}\right)$$

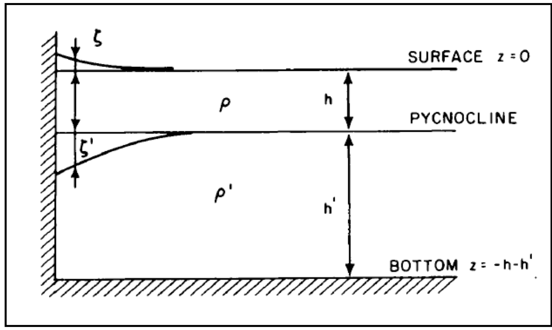
$$V' = -u_*^2 t\frac{h'}{h+h'}e^{x/R_1}$$

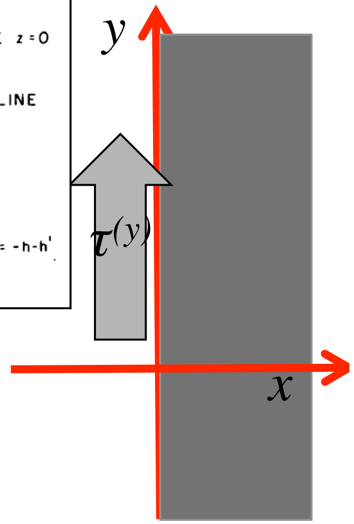
$$c_2 = \sqrt{\varepsilon g \frac{hh'}{h+h'}}$$

$$R_2 = \frac{c_2}{f}$$

Two layer model: Longshore wind case

Using the rules given before about relationships
Among the 2 layers surface and interface
Displacement in the two modes, the combinerd
Response is given by:





$$\eta = \frac{u_*^2 t}{c_1}\left[e^{x/R_1} + \left(\frac{\varepsilon H^2}{hh'}\right)^{1/2} e^{x/R_2}\right]$$

$$U = \frac{u_*^2}{f}\left[1 - \frac{h}{H}e^{x/R_1} - \frac{h'}{H}e^{x/R_2}\right]$$

$$V = u_*^2 t \frac{h}{H}\left[e^{x/R_2} - e^{x/R_1}\right]$$

$$\eta' = -\frac{u_*^2 t}{c_2}\left(\frac{h'}{H}\right)\left[e^{R_1}\left(\varepsilon\frac{h'}{h}\right)^{1/2} e^{x/R_1}\right]$$

$$U' = \frac{u_*^2 h'}{fH}\left[e^{x/R_2} - e^{x/R_1}\right]$$

$$V' = u_*^2 t \frac{h}{H}\left[e^{x/R_1} - e^{x/R_2}\right]$$

Two layer model: Longshore wind case

$$\eta = \frac{u_*^2 t}{c_1}\left[e^{x/R_1} + \left(\frac{\varepsilon H^2}{hh'}\right)^{1/2} e^{x/R_2}\right]$$

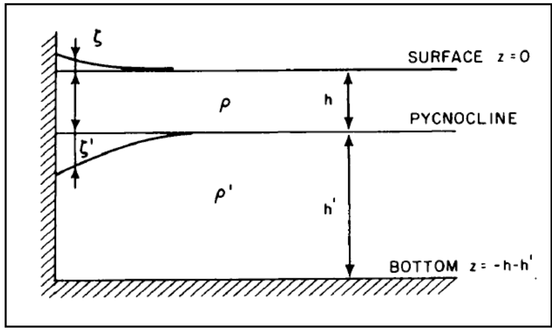
$$U = \frac{u_*^2}{f}\left[1 - \frac{h}{H}e^{x/R_1} - \frac{h'}{H}e^{x/R_2}\right]$$

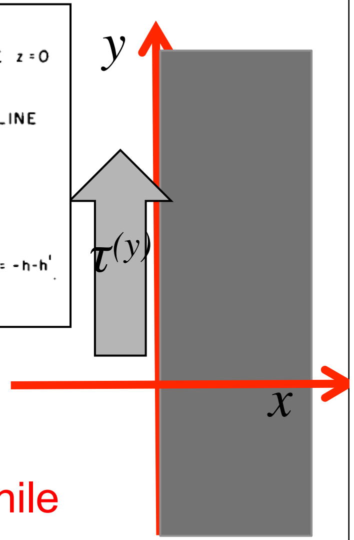
$$V = u_*^2 t \frac{h}{H}\left[e^{x/R_2} - e^{x/R_1}\right]$$

$$\eta' = -\frac{u_*^2 t}{c_2}\left(\frac{h'}{H}\right)\left[e^{R_1}\left(\varepsilon\frac{h'}{h}\right)^{1/2} e^{x/R_1}\right]$$

$$U' = \frac{u_*^2 h'}{fH}\left[e^{x/R_2} - e^{x/R_1}\right]$$

$$V' = u_*^2 t \frac{h}{H}\left[e^{x/R_1} - e^{x/R_2}\right]$$





Far from the coast ($-x >> R_1$) ,5 of the 6 variables (η, V, η', U', V') vanishes, while

$$U = \frac{u_*^2}{f}$$

Ekman transport

At intermediate distances: $R_2 << -x << R_1$ we have Barotropic mode contribution ONLY For η, η', V, V' , while the cross-shore transport is :

$$U = \frac{u_*^2}{f}\frac{h}{H}$$

$$U' = -\frac{u_*^2}{f}\frac{h}{H}$$

Ekman transport in the top layer EXACTLY compensated in the bottom layer by an adjustement drift at velocity:

$$\frac{u_*^2}{f}\frac{1}{H}$$

as in homogeneous fluid case with $-x << R_1$

Two layer model: Longshore wind case

$$\eta = \frac{u_*^2 t}{c_1}\left[e^{x/R_1} + \left(\frac{\varepsilon H^2}{hh'}\right)^{1/2} e^{x/R_2} \right]$$

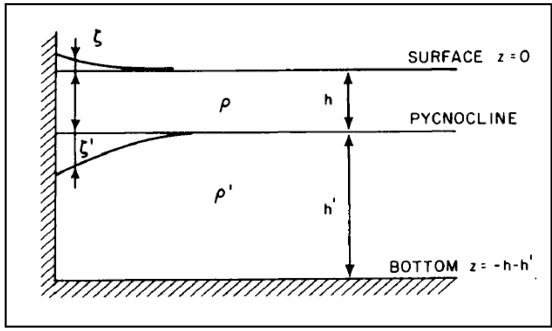
$$U = \frac{u_*^2}{f}\left[1 - \frac{h}{H} e^{x/R_1} - \frac{h'}{H} e^{x/R_2}\right]$$

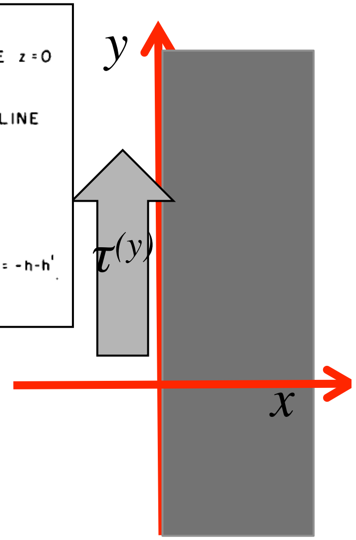
$$V = u_*^2 t \frac{h}{H}\left[e^{x/R_2} - e^{x/R_1}\right]$$

$$\eta' = -\frac{u_*^2 t}{c_2}\left(\frac{h'}{H}\right)\left[e^{R_1}\left(\varepsilon \frac{h'}{h}\right)^{1/2} e^{x/R_1}\right]$$

$$U' = \frac{u_*^2 h'}{fH}\left[e^{x/R_2} - e^{x/R_1}\right]$$

$$V' = u_*^2 t \frac{h}{H}\left[e^{x/R_1} - e^{x/R_2}\right]$$





$$c_1 = \sqrt{gH} \qquad c_2 = \sqrt{\varepsilon g \frac{hh'}{H}}$$

$$R_1 = \frac{c_1}{f} \qquad R_2 = \frac{c_2}{f}$$

"Sufficiently" close to the coast:

$$-x << R_2$$

$$U = U' = V' = 0$$

$$V = -u_*^2 t$$

As in the homogeneous fluid case.(fluid accelerates alongshore)

The pycnocline displacement, neglecting quantities $O(\sqrt{\varepsilon})$ is :

$$\eta' = -\frac{u_*^2 t}{c_2}\frac{h'}{H}$$

Two layer model: Longshore wind case

$$\eta = \frac{u_*^2 t}{c_1}\left[e^{x/R_1} + \left(\frac{\varepsilon H^2}{hh'}\right)^{1/2} e^{x/R_2}\right]$$

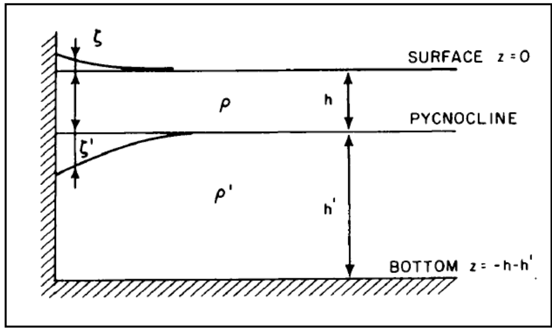
$$U = \frac{u_*^2}{f}\left[1 - \frac{h}{H}e^{x/R_1} - \frac{h'}{H}e^{x/R_2}\right]$$

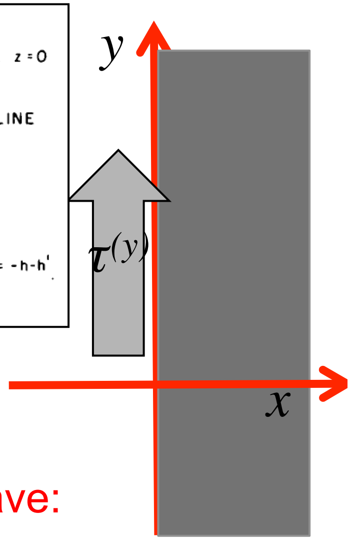
$$V = u_*^2 t \frac{h}{H}\left[e^{x/R_2} - e^{x/R_1}\right]$$

$$\eta' = -\frac{u_*^2 t}{c_2}\left(\frac{h'}{H}\right)\left[e^{R_1}\left(\varepsilon\frac{h'}{h}\right)^{1/2} e^{x/R_1}\right]$$

$$U' = \frac{u_*^2 h'}{fH}\left[e^{x/R_2} - e^{x/R_1}\right]$$

$$V' = u_*^2 t \frac{h}{H}\left[e^{x/R_1} - e^{x/R_2}\right]$$





$-x << R_2$

Example: for a wind impulse of $0.1\ Pa$ acting for $10hr$ and $c_2=0.5\ ms^{-1}$ we have:

$$u_*^2 t = 3m^2 s^{-1}$$

And the pycnocline displacement is of the order of 5m:

Downward for $\tau_w^{(y)}$

Upward for $-\tau_w^{(y)}$

Two layer model: Longshore wind case

$$\eta = \frac{u_*^2 t}{c_1}\left[e^{x/R_1} + \left(\frac{\varepsilon H^2}{hh'}\right)^{1/2} e^{x/R_2}\right]$$

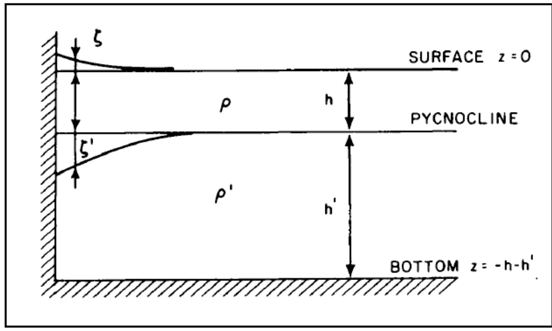
$$U = \frac{u_*^2}{f}\left[1 - \frac{h}{H}e^{x/R_1} - \frac{h'}{H}e^{x/R_2}\right]$$

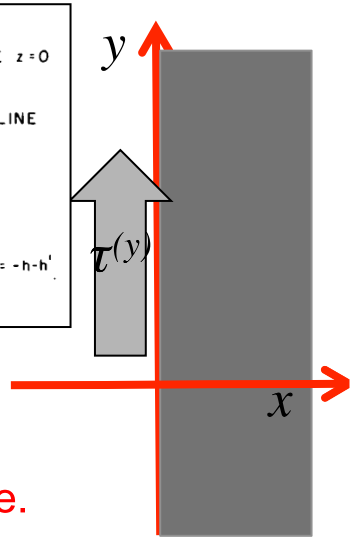
$$V = u_*^2 t \frac{h}{H}\left[e^{x/R_2} - e^{x/R_1}\right]$$

$$\eta' = -\frac{u_*^2 t}{c_2}\left(\frac{h'}{H}\right)\left[e^{R_1}\left(\varepsilon\frac{h'}{h}\right)^{1/2} e^{x/R_1}\right]$$

$$U' = \frac{u_*^2 h'}{fH}\left[e^{x/R_2} - e^{x/R_1}\right]$$

$$V' = u_*^2 t \frac{h}{H}\left[e^{x/R_1} - e^{x/R_2}\right]$$





$-x << R_2$

The surface level displacement η is what is required for the geostrophic balance.
In the homogeneous fluid case it was:

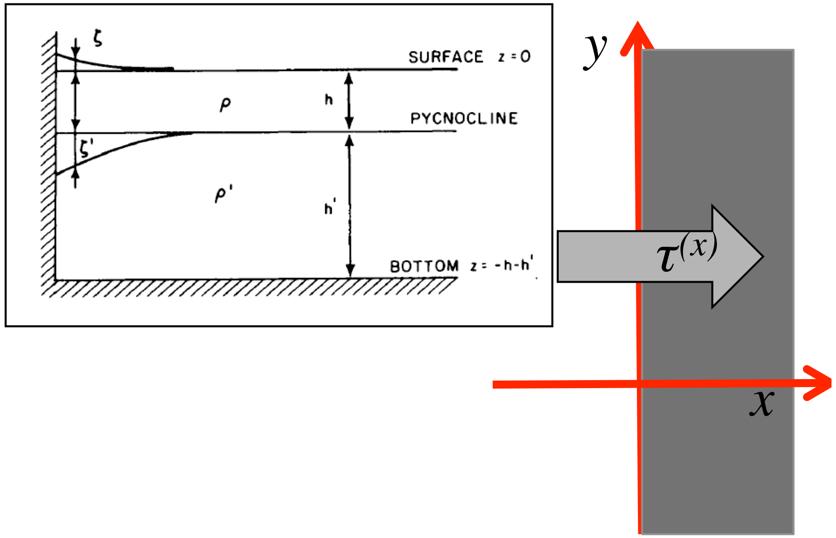
$$\eta = \frac{u_*^2 t}{c_1}$$

non oscillatory solution

In the two layer model it differ from the above by a quantity of $O(\sqrt{\varepsilon})$ (a few percent)

Two layer model: Cross shore wind case

The non oscillatory solution for the **barotropic** mode Are (see the homogeneous cross shore wind case)



$$\eta = \frac{u_*^2}{fc_1} e^{x/R_1}$$

$$U + U' = 0$$

$$V + V' = -\frac{u_*^2}{f}\left(1 - e^{x/R_1}\right)$$

$$c_1 = \sqrt{gH}$$

$$R_1 = \frac{c_1}{f}$$

The baroclinic mode contributions are:

$$\eta' = -\frac{h'}{H}\frac{u_*^2}{fc_2} e^{x/R_2}$$

$$U' = 0$$

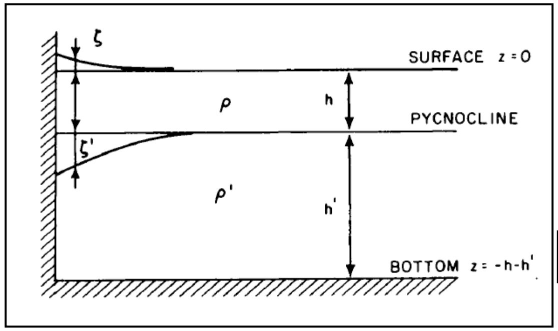
$$V' = \frac{h'}{H}\frac{u_*^2}{f}\left(1 - e^{x/R_2}\right)$$

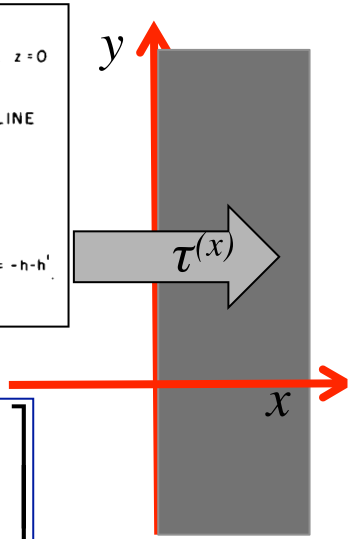
$$c_2 = \sqrt{\varepsilon g \frac{hh'}{H}}$$

$$R_2 = \frac{c_2}{f}$$

Two layer model: Cross shore wind case

The combined solutions are:





$$\eta = \frac{u_*^2}{fc_1}\left[e^{x/R_1} + \frac{h}{H}\left(\varepsilon \frac{h'}{h}\right)^{1/2} e^{x/R_2} \right]$$

$$U = 0$$

$$V = -\frac{u_*^2}{f} + \frac{u_*^2 h}{fH} e^{x/R_1} + \frac{u_*^2 h'}{fH} e^{x/R2}$$

$$\eta' = -\frac{h'}{H}\frac{u_*^2}{fc_2}\left[e^{x/R_2} - \frac{c_2}{c_1} e^{x/R_1} \right]$$

$$U' = 0$$

$$V' = \frac{h'}{H}\frac{u_*^2}{f}\left(e^{x/R_1} - e^{x/R_2} \right)$$

Two layer model: Cross shore wind case

$$\eta = \frac{u_*^2}{fc_1}\left[e^{x/R_1} + \frac{h}{H}\left(\varepsilon\frac{h'}{h}\right)^{1/2} e^{x/R_2}\right]$$

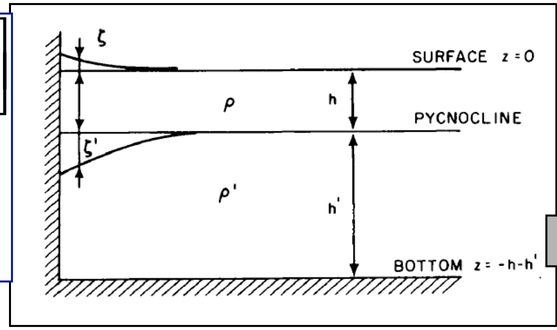
$$U = 0$$

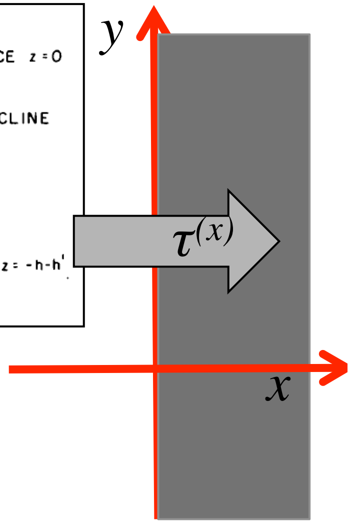
$$V = -\frac{u_*^2}{f} + \frac{u_*^2 h}{fH} e^{x/R_1} + \frac{u_*^2 h'}{fH} e^{x/R2}$$

$$\eta' = -\frac{h'}{H}\frac{u_*^2}{fc_2}\left[e^{x/R_2} - \frac{c_2}{c_1} e^{x/R_1}\right]$$

$$U' = 0$$

$$V' = \frac{h'}{H}\frac{u_*^2}{f}\left(e^{x/R_1} - e^{x/R_2}\right)$$





In the far field $-x >> R_1$ surface, pycnocline displacement and bottom transport η', U', V' vanish.
Surface layer transport is Ekman transport

$$V = -\frac{u_*^2}{f}$$

$$U = 0$$

At intermediate distances $R_2 << -x << R_1$ a constant longshore velocity u_*^2 / fH is added in both layers, whose associated depth-integrated transport cancels Ekman transport.

Velocity distribution is then the same obtained in the homogeneous case (no stratification close to the shore)

Two layer model: Cross shore wind case

$$\eta = \frac{u_*^2}{fc_1}\left[e^{x/R_1} + \frac{h}{H}\left(\varepsilon\frac{h'}{h}\right)^{1/2} e^{x/R_2}\right]$$

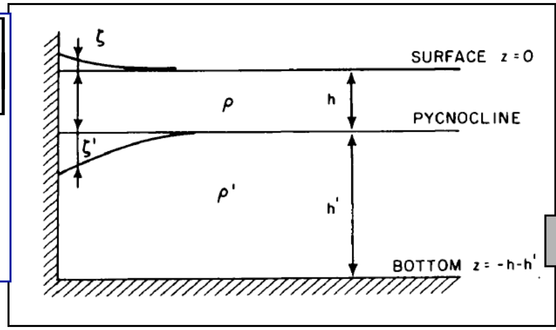
$$U = 0$$

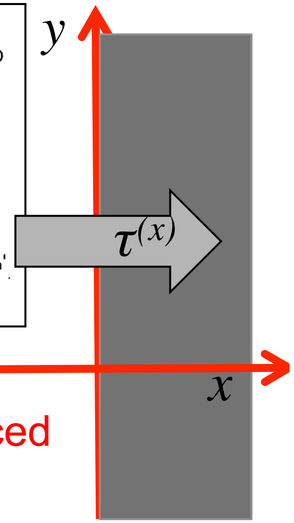
$$V = -\frac{u_*^2}{f} + \frac{u_*^2 h}{fH}e^{x/R_1} + \frac{u_*^2 h'}{fH}e^{x/R2}$$

$$\eta' = -\frac{h'}{H}\frac{u_*^2}{fc_2}\left[e^{x/R_2} - \frac{c_2}{c_1}e^{x/R_1}\right]$$

$$U' = 0$$

$$V' = \frac{h'}{H}\frac{u_*^2}{f}\left(e^2 - e^{x/R_2}\right)$$





Close to the coast $-x << R_2$ Ekman transport is is canceled by a pressure field induced ONLY in the top layer.

Surface elevation η rises a little more (with respect to the Homogeneous case (by an order of $O(\sqrt{\varepsilon})$)

Pycnocline is depressed/raised for an onshore/offshore windstress by a quantity of order

$$\frac{u_*^2}{fc_2}$$

The pycnocline tilt becomes negligible at distances of $O(R_2)$

Two layer model: Cross shore wind case

$$\eta = \frac{u_*^2}{fc_1}\left[e^{x/R_1} + \frac{h}{H}\left(\varepsilon \frac{h'}{h}\right)^{1/2} e^{x/R_2} \right]$$

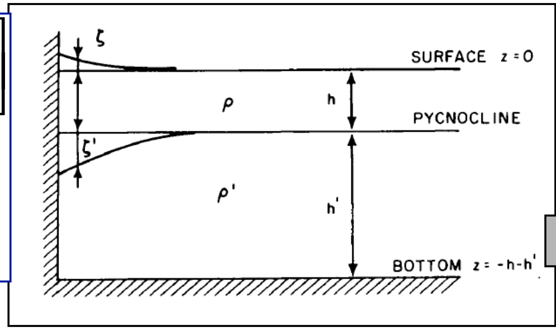
$$U = 0$$

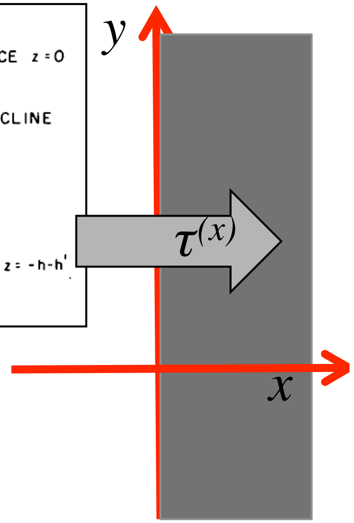
$$V = -\frac{u_*^2}{f} + \frac{u_*^2 h}{fH} e^{x/R_1} + \frac{u_*^2 h'}{fH} e^{x/R2}$$

$$\eta' = -\frac{h'}{H}\frac{u_*^2}{fc_2}\left[e^{x/R_2} - \frac{c_2}{c_1} e^{x/R_1} \right]$$

$$U' = 0$$

$$V' = \frac{h'}{H}\frac{u_*^2}{f}\left(e^2 - e^{x/R_2} \right)$$





Surface elevation gradient becomes negligible at distances of $O(R_2)$

At the coast:
Balance of forces (as usual) between wind stress and pressure gradient, but pressure Gradient Acting only in the top layer and vanishing in the bottom layer

Depth integrated Coriolis force also vanishes in each layer (no longshore and cross-shore Transport) in each layer