LABORATORY OF AUTOMATION SYSTEMS
Analytical design of digital controllers

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Analytical design of digital controllers

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Implementation of a digital controller

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- summary on design by discretization
- how to pass from a discrete-time transfer function to a piece of software
- modeling aspects: choice of the model
- 1 dof control schemes ($D(z)$) or 2 dof control schemes (T/R and S/R)

Modeling aspects

The first step in the design of a control law is the definition of a proper mathematical model of the system to be controlled.

$\Longrightarrow$?? $\Longrightarrow$

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t)) \\ y(t) &= h(x(t), u(t)) \end{aligned}$$

...

$$\begin{aligned} \dot{x}(t) &= A\, x(t) + B\, u(t) \\ y(t) &= C\, x(t) + D\, u(t) \end{aligned}$$

...

$$G(s) = \frac{B(s)}{A(s)}, \quad G(z) = \frac{B(z)}{A(z)}$$

This step usually introduces approximations and differences between the behaviours of the real system and the mathematical model.

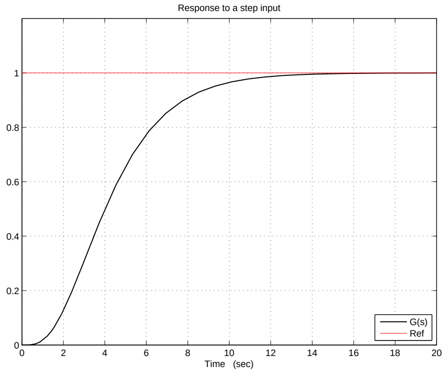
Modeling aspects - Example

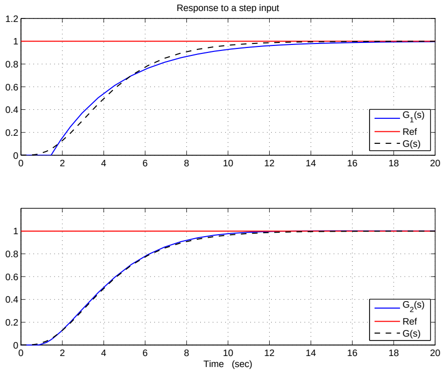
Let us consider the plant described by:

$$G_p(s) = \frac{1}{(0.5\,s+1)(s+1)^2(2\,s+1)}$$

Two simplified models could be considered:

$$G_1(s) = \frac{e^{-1.46\,s}}{3.34\,s+1} \qquad G_2(s) = \frac{e^{-0.78\,s}}{4\,s^2+3.6\,s+1}$$





Note: $G_1(s)$ has a single pole, while $G_2(s)$ has two poles with a smaller time delay.

Modeling aspects - Example

Then, the sampling period T must be defined. Depending on its value, the two simplified models $G_1(s)$ and $G_2(s)$ assume different expressions.
Assuming a zero-order hold, $T = 5\,s$, and using the *modified*[1] Z-transform (with $m = 1 - \frac{1.46}{5} = 0.708$) we have

$$\begin{aligned} G_{p15}(z)| &= \mathcal{Z}\left[\frac{1-e^{-sT}}{s}\,\frac{e^{-1.46\,s}}{3.34\,s+1}\right] \\ &= (1-z^{-1})\mathcal{Z}_m\left[\frac{1}{s\,(3.34\,s+1)}\right] \\ &= \frac{z^{-1}(0.6535+0.1227\,z^{-1})}{1-0.2238\,z^{-1}} \\ &= \frac{0.6535(z+0.1877)}{z(z-0.2238)} \end{aligned}$$

$$G_{p25}(z) = \frac{0.6634(z+0.00434)(z+0.3712)}{z[(z-0.04877)^2+0.0934^2]}$$

[1] It is an extension of the standard Z-transform, to incorporate delays that are not multiples of the sampling time.

Modeling aspects - Example

Discretizing $G_1(s)$ with $T = 1\,s$ ($m = 1 - 0.46 = 0.54$) we have

$$
\begin{aligned}
G_{p11}(z)|_{T=1} &= z^{-1}(1 - z^{-1})\mathcal{Z}\left[\frac{e^{-0.46\,s}}{s\,(3.34\,s + 1)}\right] \\
&= \frac{z^{-2}(0.1493 + 0.1095\,z^{-1})}{1 - 0.7413\,z^{-1}} \\
&= \frac{0.1493(z + 0.7334)}{z^2(z - 0.7413)}
\end{aligned}
$$

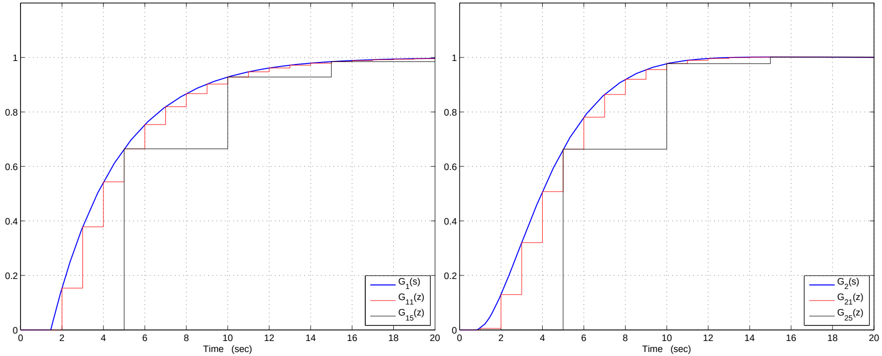
and ($G_2(s)$ with $T = 1\,s$)

$$
G_{p21}(z) = \frac{0.005664(z + 0.3407)(z + 20.26)}{z[(z - 0.6225)^2 + 0.1379^2]}
$$

Modeling aspects - Example

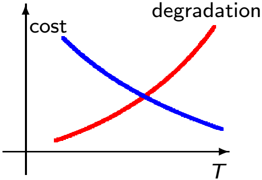
Step responses of

$$G_1(s),\ G_{11}(z),\ G_{15}(z) \qquad \text{and of} \qquad G_2(s),\ G_{21}(z),\ G_{25}(z)$$



Choice of the sampling period

The choice of a proper sampling period T is fundamental for any digital control system. and is the tradeoff among several factors related on one side to the cost and on the other to the degradation of performance.



Performance (control quality) refer to:

- disturbante rejection;
- set-point tracking;
- control energy;
- delay and stability;
- robustness wrt varying parameter;

The (computational) cost refer to

- exploitation of the computational power;
- AD/DA conversion speed;
- computational speed;
- numerical precision (data storage).

Choice of the sampling period

1) *Loss of information*: $\omega_s > 2\omega_b$
where ω_b is the closed-loop bandwidth (usually $\omega_b > \omega_c$, the bandwidth of the process).

2) *Smoot dynamics and low time-delays*: $6 < \dfrac{\omega_s}{\omega_b} < 20$

3) *Compensation of disturbances*: $\omega_s > 2\omega_r$
where ω_r is the highest disturbance frequency that should be compensated (to compensate for disturbances, these must be "known", as the process).

4) *Effects of anti-aliasing filters*: the transfer function of the process to be considered in the control design becomes: $G'_p(z) = \mathcal{Z}[G_p(s)G_f(s)]$
where $G_f(j\omega)$ represents the filter, with a bandwidth given by a frequency ω_p such that $\omega_p/\omega_b = 2$. Therefore

$$\frac{\omega_s}{\omega_b} \geq 20$$

Choice of the sampling period

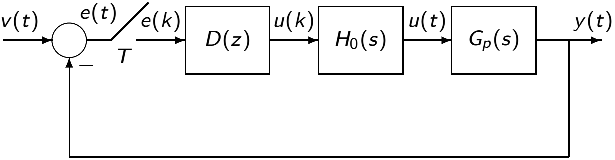
Other *practical rules* often used in practice are:

a)
$$T \le \frac{\tau_{dom}}{10}$$
where τ_{dom} is the dominant time constant of the open-loop system. Note that this condition must be accurately verified, since the open loop bandwidth must be significantly different from the closed loop one.

b)
$$T \le \frac{\theta}{4}$$
where θ is the process time-delay. As in the previous point, this condition must be verified also for the closed-loop system.

c)
$$T < \frac{T_a}{10}, \qquad \omega_s > 10\omega_n$$
where T_a is the settling time and ω_n the natural frequency of the open loop system, that is in this case characterised by a (dominant) pair of complex conjugated poles.

Implementation of a digital controller



$$G_p(s) = \frac{1}{s(s+1)}$$

$$T = 0.8\ s$$

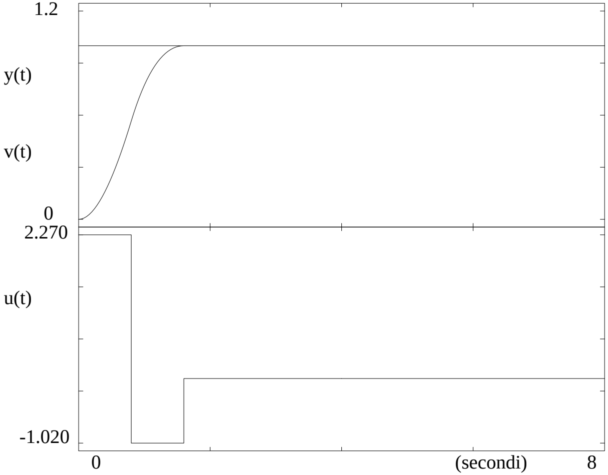
$$G_p(z) = \mathcal{Z}\left[\frac{1-e^{-sT}}{s}\frac{1}{s(s+1)}\right] = \frac{K(z-b)}{(z-1)(z-a)} = \frac{K(1-bz^{-1})z^{-1}}{(1-z^{-1})(1-az^{-1})}$$

$$= \frac{0.2493(1+0.7669z^{-1})z^{-1}}{(1-z^{-1})(1-0.4493z^{-1})}$$

Deadbeat controller:

$$D(z) = \frac{2.27-1.02z^{-1}}{1+0.434z^{-1}}$$

Implementation of a digital controller



Implementation of a digital controller

Deadbeat controller:

$$D(z) = \frac{2.27 - 1.02z^{-1}}{1 + 0.434z^{-1}} = \frac{U(z)}{E(z)}$$

Therefore

$$U(z)(1 + 0.434z^{-1}) = E(z)(2.27 - 1.02z^{-1})$$

or (z^{-1} is a time delay of a sample period)

$$u(k) + 0.434u(k-1) = 2.27e(k) - 1.02e(k-1)$$

Finally, the control action should be implemented with the following difference equation

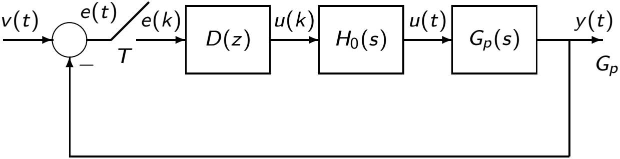
$$\begin{aligned} u(k) &= -0.434u(k-1) + 2.27e(k) - 1.02e(k-1) \\ u(k) &= -d_1 u(k-1) + n_0 e(k) - n_1 e(k-1) \end{aligned}$$

Implementation of a digital controller

```
global ek1, yk1, n0, n1, d1
. . .
n0 = 2.27;   n1 = 1.02;   d1 = 0.434;
. . .
while true,                          % wait interrupt
        [yk, vk] = AcquireData();
        [uk] = ComputeControl(yk, vk);
        [error] = OutputControl(uk);
end
function [uk] = ComputeControl(yk, vk);
        ek = vk - yk;
        uk = - d1 * uk1 + n0 * ek - n1 * ek1;
        ek1 = ek;
        uk1 = uk
end
```

1-dof and 2-dof control schemes

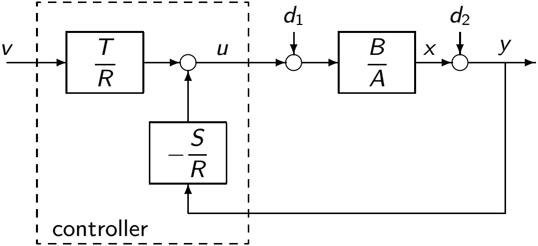
The classical feedback control scheme is based on the following diagram (1 dof control)



$$G_p(z) = \frac{B(z)}{A(z)} \quad D(z) = \frac{T(z)}{R(z)}$$

$$\frac{Y(z)}{V(z)} = \frac{B\,T}{A\,R + B\,T}$$

Alternatively, a 2 -dof controller can be implemented, with more choices to the control design:



$$\frac{Y(z)}{V(z)} = \frac{B\,T}{A\,R + B\,S}$$