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Existence and approximation results for SKC mappings in CAT(0) spaces

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Abstract

Recently, Karapınar and Tas (Comput. Math. Appl. 61:3370-3380, 2011) extended the class of Suzuki-generalized nonexpansive mappings to the class of SKC mappings. In this paper, we investigate SKC mappings to get a criterion to guarantee a fixed point, via extending the results proved by Karapınar and Tas into the class of CAT(0) spaces. Further, by using Ishikawa-type iteration scheme for two mappings, we derive approximation fixed point sequence. Our results extend, improve and unify some existing results in this direction, such as (Nonlinear Anal. Hybrid Syst. 4:25-31, 2010) by Nanjaras *et al.* or (Comput. Math. Appl. 61:109-116, 2011) by Khan and Abbas.

MSC: 47H09; 47H10; 49M05

Keywords: iterative process; SKC mapping; existence; common fixed point; $\triangle$-convergence; strong convergence; CAT(0) space

1 Introduction

In 2008, Suzuki [1] introduced a class of single valued mappings

$$\frac{1}{2}\|x - Tx\| \leq \|x - y\| \quad \text{implies} \quad \|Tx - Ty\| \leq \|x - y\| \tag{C}$$

which lies between the class of mappings nonexpansiveness and quasi-nonexpansiveness. Later, such mappings were called Suzuki-type nonexpansive. In this interesting paper [1], Suzuki determine the existence of a fixed point of such mappings. In 2009 Dhompongsa *et al.* [2] improved the results of Suzuki [1]. In this distinguished paper [2], the authors obtained a fixed point result for mappings with condition (C) on a Banach space under certain conditions. Afterwards Nanjaras *et al.* [3] gave some characterization of existing fixed point results for mappings with condition (C) in the framework of CAT(0) spaces. Recently, Khan and Abbas [4] derived some fixed point results via different iterative schemes for nonexpansive mappings in CAT(0) spaces (see also [5]). Very recently, Karapınar and Tas [6] proposed some new classes of mappings which substantially generalized the notion of Suzuki-type nonexpansive mappings. The subject of this paper is to extend the mentioned results above for the class of SKC mappings [6] in the framework of CAT(0) spaces. Furthermore, by using Ishikawa-type iteration scheme, we derive some common fixed point results via approximation fixed point sequences. The results we present in this article improve and unify some existing results in this direction, such as [2] and [4].

2 Preliminaries

First of all, we recollect some fundamental definition and results from the report of Dhompongsa and Panyanak [7].

For a metric space (X,d), a map $c\colon [0,l] \to X$ with $c(0) = x$, $c(l) = y$, and

$$d(c(t), c(t')) = |t - t'|, \quad \forall t, t' \in [0,l],$$

is called a *geodesic* from x to y in X. The image of c is said to be a *geodesic segment* joining the points x and y. A geodesic segment is denoted by $[x,y]$, if it is unique.

Let $Y \subseteq X$. The subset Y of X is called *convex* if Y includes each geodesic segment joining for any two points in Y.

Definition 2.1 A metric space (X,d) is called a *geodesic space* if all $x, y \in X$ are joined by a geodesic.

In a geodesic metric space (X,d), the triple $\triangle(x_1,x_2,x_3)$ is said to be a *geodesic triangle* where the points x_1, x_2, x_3 in X are considered as the vertices of $\triangle$ and a geodesic segment between each pair of vertices becomes the edges of $\triangle$. A triangle $\bar{\triangle}(x_1,x_2,x_3) := \triangle(\bar{x}_1,\bar{x}_2,\bar{x}_3)$ in the Euclidean plane $\mathbb{R}^2$ such that $d_{\mathbb{R}^2}(\bar{x}_i,\bar{x}_j) = d(x_i,x_j)$ for $i,j \in \{1,2,3\}$ is called a *comparison triangle* for the geodesic triangle $\triangle(x_1,x_2,x_3)$. A geodesic space is called a CAT(0) *space* [8–12] if all geodesic triangles of appropriate size satisfy the following comparison axiom.

CAT(0): $d(x,y) \le d_{\mathbb{E}^2}(\bar{x},\bar{y})$

for all $x, y \in \triangle$ and all comparison points $\bar{x}, \bar{y} \in \bar{\triangle}$.

Let x, y_1, y_2 be points in a CAT(0) space (X,d). If y_0 is the midpoint of the segment $[y_1,y_2]$, then we have [13]

$$d(x,y_0)^2 \le \frac{1}{2}d(x,y_1)^2 + \frac{1}{2}d(x,y_2)^2 - \frac{1}{4}d(y_1,y_2)^2. \tag{CN}$$

In fact, a geodesic space is a CAT(0) space if and only if it satisfies the (CN) inequality; please, see [8].

Lemma 2.1 ([7]) *Let (X,d) be a CAT(0) space. Then*:

(i) *(X,d) is uniquely geodesic.*

(ii) *Let p, x, y be points of X, let $\alpha \in [0,1]$, and let m_1 and m_2 denote, respectively, the points of $[p,x]$ and $[p,y]$ which satisfy $d(p,m_1) = \alpha d(p,x)$ and $d(p,m_2) = \alpha d(p,y)$. Then*

$$d(m_1,m_2) \le \alpha d(x,y). \tag{2.1}$$

(iii) *Let $x, y \in X$, $x \neq y$ and $z, w \in [x,y]$ such that $d(x,z) = d(x,w)$. Then $z = w$.*

(iv) *Let $x, y \in X$. For each $t \in [0,1]$, there exists a unique point $z \in [x,y]$ such that*

$$d(x,z) = td(x,y) \quad \textit{and} \quad d(y,z) = (1-t)d(x,y). \tag{2.2}$$

Throughout the paper, we will use the notation $(1-t)x \oplus ty$ for the unique point z satisfying (2.2).

Let $\{x_n\}$ be a bounded sequence in a CAT(0) space X. For $x \in X$, we set

$$r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n).$$

The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in X\}$$

and the asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}.$$

In a CAT(0) space, asymptotic center consists of exactly one point [14].

A sequence $\{x_n\}$ in X is called *$\triangle$-convergent* to $x \in X$ [15, 16], denoted by $\triangle\text{-}\lim_n x_n = x$ if x is the unique asymptotic center of $\{u_n\}$, for every subsequence $\{u_n\}$ of $\{x_n\}$.

We set $w_w(x_n) := \bigcup\{A(\{u_n\}),$ where $\{u_n\}$ is a subsequence of $\{x_n\}\}$.

Lemma 2.2 ([7]) *Let X be a* CAT(0) *space. Then*:

1. *Every bounded sequence in X has a $\triangle$-convergent subsequence.*
2. *If C is a closed and convex subset of X, and if $\{x_n\}$ is a bounded sequence in C, then the asymptotic center of $\{x_n\}$ is in C.*
3. *The following inequality*:

   $$d((1-t)x \oplus ty, z) \le (1-t)d(x,z) + td(y,z)$$

   holds, for all $x, y, z \in X$ and $t \in [0,1]$.
4. *The following inequality*:

   $$d((1-t)x \oplus ty, z)^2 \le (1-t)d(x,z)^2 + td(y,z)^2 - t(1-t)d(x,y)^2$$

   holds, for all $x, y, z \in X$ and $t \in [0,1]$.

Let K be a nonempty subset of a CAT(0) space X. A mapping $T : K \to K$ is said to *satisfy condition* (C) if

$$\frac{1}{2}d(x, Tx) \le d(x,y) \quad \Rightarrow \quad d(Tx, Ty) < d(x,y) \quad \text{for all } x, y \in K.$$

Nanjaras *et al.* [3] proved that a self mapping satisfying condition (C), and defined on a nonempty bounded and closed subset of a complete CAT(0) space has a fixed point.

The following definitions are basically due to Karapınar and Tas [6] but here we state them in the framework of CAT(0) spaces.

Let K be a nonempty subset of a CAT(0) space X. A mapping $T : K \to K$ is said to be:

(1) a Suzuki-Ćirić conditioned mapping (SCC) if

$$\frac{1}{2}d(x,Tx) \le d(x,y) \quad \text{implies that}$$
$$d(Tx,Ty) \le \max\{d(x,y), d(x,Tx), d(y,Ty), d(y,Tx), d(x,Ty)\} \quad \text{for all } x,y \in K;$$

(2) a Suzuki-Karapınar conditioned mapping (SKC) if

$$\frac{1}{2}d(x,Tx) \le d(x,y) \quad \text{implies that}$$
$$d(Tx,Ty) \le \max\left\{d(x,y), \frac{d(x,Tx)+d(y,Ty)}{2}, \frac{d(y,Tx)+d(x,Ty)}{2}\right\}$$
$$\text{for all } x,y \in K;$$

(3) a Kannan-Suzuki conditioned mapping (KSC) if

$$\frac{1}{2}d(x,Tx) \le d(x,y) \quad \text{implies that}$$
$$d(Tx,Ty) \le \frac{d(x,Tx)+d(y,Ty)}{2} \quad \text{for all } x,y \in K;$$

(4) a Chatterjea-Suzuki conditioned mapping (CSC) if

$$\frac{1}{2}d(x,Tx) \le d(x,y) \quad \text{implies that}$$
$$d(Tx,Ty) \le \frac{d(y,Tx)+d(x,Ty)}{2} \quad \text{for all } x,y \in K.$$

For further details on these mappings and their implications, we refer to [6] and references therein.

The following are some basic properties of SKC mappings whose proofs in the setup of CAT(0) spaces follow the same lines as those of Propositions 11, 14, and 19 in [6], and therefore we omit them.

Proposition 2.1 *Let K be a nonempty subset of a* CAT(0) *space X. An SKC mapping $T: K \to K$ is quasi-nonexpansive provided that the set of fixed point of T is nonempty.*

Proposition 2.2 *Let K be a nonempty closed subset of a* CAT(0) *space X and $T: K \to K$ an SKC mapping then the set of fixed point of T is closed.*

Proposition 2.3 *Let K be a nonempty subset of a* CAT(0) *space X and $T: K \to K$ an SKC mapping, then*

$$d(x,Ty) \le 5d(Tx,x) + d(x,y)$$

holds, for all x, y in K.

Propositions similar to above can be stated for the class of KSC and CSC mappings in the framework of CAT(0) spaces.

An Ishikawa-type iteration process for two mappings S and T is defined by

$$\begin{cases} x_{n+1} = (1-a)x_n \oplus aTy_n, \\ y_n = (1-b)x_n \oplus bSx_n \end{cases} \tag{2.3}$$

for all $n \in \mathbb{N}$, where $\alpha, b \in [\frac{1}{2}, 1)$.

When $S = T$, we have another Ishikawa iteration-type process:

$$\begin{cases} x_{n+1} = (1-a)x_n \oplus aTy_n, \\ y_n = (1-b)x_n \oplus bTx_n \end{cases} \tag{2.4}$$

for all $n \in \mathbb{N}$, where $a, b \in [\frac{1}{2}, 1)$.

When $S = I$, the identity mapping, we have Krasnoselkii-type iteration process:

$$\begin{aligned} &x_1 \in C, \\ &x_{n+1} = (1-a)x_n \oplus aTx_n \end{aligned} \tag{2.5}$$

for all $n \in \mathbb{N}$, where $a \in [\frac{1}{2}, 1)$.

The purpose of this paper is:

(i) To extend existence results given in [6] to the class of SKC mappings in CAT(0) spaces. Consequently, corresponding results for KSC and CSC mappings are also extended to CAT(0) spaces.

(ii) To prove some strong and $\triangle$-convergence results for two SKC mappings using (2.3) in CAT(0) spaces.

3 Main results

In the sequel, $F(T)$ denotes the set of fixed points of T and F the set of common fixed points of T and S. The next two theorems give the existence of fixed points of SKC mappings under different conditions on C.

Theorem 3.1 *Let us consider the nonempty set C be closed, bounded and convex subset of a* CAT(0) *space X, and $T: C \to C$ an SKC mapping. Define a sequence $\{x_n\}$ as in* (2.5). *Then T has a fixed point in C provided that $\{x_n\}$ is an approximate fixed point sequence, that is,* $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.

Proof Since $\{x_n\}$ is a bounded sequence in C, $A(\{x_n\})$ consists of exactly one point by ([14], Proposition 7). Suppose that $A(\{x_n\}) = \{p\}$. Using Lemma 2.2, we obtain $\{p\} \subset C$. Since T is an SKC mapping, therefore

$$d(x_n, Tp) \le 5d(Tx_n, x_n) + d(x_n, p)$$

which on taking lim sup on both sides implies that

$$\limsup_{n\to\infty} d(x_n, Tp) \le \limsup_{n\to\infty} d(x_n, p).$$

Hence, we have

$$r(Tp,\{x_n\}) \leq r(p,\{x_n\}).$$

Uniqueness of asymptotic centers now implies that $p = Tp$. □

Theorem 3.2 *Let C be a nonempty compact convex subset of a* CAT(0) *space X, $T: C \to C$ a SKC mapping, then $F(T) \neq \emptyset$ and $\{x_n\}$ given by* (2.5) *converge strongly to a fixed point of T provided that $\{x_n\}$ is an approximate fixed point sequence.*

Proof Since C is compact, we obtain a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and p in C such that $d(x_{n_k}, p) \to 0$ as $k \to \infty$. By Proposition 2.3, we have

$$d(x_{n_k}, Tp) \leq 5d(Tx_{n_k}, x_{n_k}) + d(x_{n_k}, p)$$

for all $k \in \mathbb{N}$. Now taking the limit as $k \to \infty$, we obtain $d(x_{n_k}, Tp) \to 0$, which implies that $Tp = p$. Now for such p,

$$\begin{aligned} d(x_{n+1}, p) &= d\big((1-a)x_n \oplus aTx_n, p\big) \\ &\leq (1-a)d(x_n, p) + ad(x_n, p) \\ &= d(x_n, p). \end{aligned}$$

Thus $\lim_{n\to\infty} d(x_n, p)$ exists and hence $\{x_n\}$ converges strongly to p. □

To prepare for our approximation results, we start with the following useful lemma.

Lemma 3.1 (See [4]) *Let C be a nonempty closed convex subset of a* CAT(0) *space X, $T, S : C \to C$ be two SKC mappings. Define a sequence $\{x_n\}$ as in* (2.3). *If $F \neq \emptyset$, then*:

(i) $\lim_{n\to\infty} d(x_n, q)$ *exists, for all $q \in F$.*
(ii) $\lim_{n\to\infty} d(x_n, Tx_n) = 0 = \lim_{n\to\infty} d(x_n, Sx_n)$.

We now give our $\triangle$-convergence results.

Theorem 3.3 *Let X, C, T, S and $\{x_n\}$ be as in Lemma* 3.1. *If $F \neq \emptyset$, then $\{x_n\}$ $\triangle$-converges to a common fixed point of T and S.*

Proof Let $q \in F$. Then by Lemma 3.1, $\lim_{n\to\infty} d(x_n, q)$ exists, for all $q \in F$. Thus sequence $\{x_n\}$ is bounded. Also, Lemma 3.1 gives

$$\lim_{n\to\infty} d(x_n, Tx_n) = \lim_{n\to\infty} d(x_n, Sx_n) = 0.$$

First, we show that $w_w(\{x_n\}) \subseteq F$.

Let $u \in w_w(\{x_n\})$, then there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$. Since $\{u_n\}$ being a subsequence of $\{x_n\}$ is bounded, by Lemma 2.2 there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\triangle$-$\lim_n v_n = v$, for some $v \in C$. Since $\lim_{n\to\infty} d(v_n, Tv_n) = 0$ and T is SKC mapping, therefore

$$d(v_n, Tv) \leq 5d(Tv_n, v_n) + d(v_n, v)$$

which on taking lim sup on both sides implies that

$$\limsup_{n\to\infty} d(v_n, Tv) \le \limsup_{n\to\infty} d(v_n, v).$$

Hence, we have

$$r\bigl(Tv, \{v_n\}\bigr) \le r\bigl(v, \{v_n\}\bigr).$$

Since $\{v_n\}$ is $\triangle$-convergent to v, thus v is unique asymptotic center for every subsequence of $\{v_n\}$. Hence uniqueness of asymptotic centers implies that $v = Tv$. That is, $v \in F(T)$.

A similar argument shows that $v \in F(S)$ and hence $v \in F$.

We now claim that $u = v$.

By *reductio ad absurdum*, assume that $u \neq v$. Then, by the uniqueness of asymptotic centers, we have

$$\begin{aligned}
\limsup_{n\to\infty} d(v_n, v) &< \limsup_{n\to\infty} d(v_n, u) \\
&\le \limsup_{n\to\infty} d(u_n, u) \\
&< \limsup_{n\to\infty} d(u_n, v) \\
&= \limsup_{n\to\infty} d(x_n, v) \\
&= \limsup_{n\to\infty} d(v_n, v),
\end{aligned}$$

a contradiction. Thus, $u = v \in F$ and hence $w_w(\{x_n\}) \subseteq F$.

To show that $\{x_n\}$ is $\triangle$-convergent to a common fixed point of T and S, it suffices to show that $w_w(\{x_n\})$ consists of exactly one point.

Let $\{u_n\}$ be a subsequence of $\{x_n\}$. By Lemma 2.2, there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\triangle\text{-}\lim_n v_n = v$, for some $v \in C$. Let $A(\{u_n\}) = \{u\}$ and $A(\{x_n\}) = \{x\}$. We have already seen that $u = v$ and $v \in F$. Finally, we claim that $x = v$. If not, then by existence of $\lim_{n\to\infty} d(x_n, v)$ and uniqueness of asymptotic centers, we have

$$\begin{aligned}
\limsup_{n\to\infty} d(v_n, v) &< \limsup_{n\to\infty} d(v_n, x) \\
&\le \limsup_{n\to\infty} d(x_n, x) \\
&< \limsup_{n\to\infty} d(x_n, v) \\
&= \limsup_{n\to\infty} d(v_n, v),
\end{aligned}$$

a contradiction and hence $x = v \in F$. Therefore, $w_w(\{x_n\}) = \{x\}$. □

Remark 3.1 The above theorem extends Theorem 4 of Khan and Abbas [4] to SKC mappings.

Although the following is a corollary to our above theorem, yet it is new in itself.

Corollary 3.1 *Let C be a nonempty, closed and convex subset of a* CAT(0) *space X, $T\colon C \to C$ an SKC mapping. Let $\{x_n\}$ be as in* (2.4). *If $F(T) \neq \emptyset$, then the sequence $\{x_n\}$ is $\triangle$-convergent to a fixed point of T.*

Proof Take $S = T$ in Theorem 3.3. □

The following corollary extends Theorem 30 of Karapınar and Tas [6] to the setting of a CAT(0) space.

Corollary 3.2 *Let C be a nonempty, closed and convex subset of a* CAT(0) *space X, $T\colon C \to C$ an SKC mapping. If $F(T) \neq \emptyset$, then the sequence $\{x_n\}$ defined in* (2.5) *$\triangle$-converges to a fixed point of T.*

Proof Take $S = I$, the identity mapping, in Theorem 3.3. □

Following Senter and Dotson [17], Khan and Fukhar-ud-din [18] introduced the so-called *condition* (A') for two mappings and gave an improved version of it in [19] as in the following.

Two mappings $S, T\colon C \to C$ are said to satisfy the *condition* (A') if there exists a non-decreasing function $f\colon [0,\infty) \to [0,\infty)$ with $f(0) = 0$, $f(r) > 0$ for all $r \in (0,\infty)$ such that either $d(x, Tx) \geq f(d(x,F))$ or $d(x, Sx) \geq f(d(x,F))$ for all $x \in C$.

This condition becomes condition (A) of Senter and Dotson [17] whenever $S = T$.

Nanjaras *et al.* [3] obtained a strong convergence result for a Suzuki-generalized nonexpansive mappings employing condition (A).

In the following, we will use condition (A') to study the strong convergence of sequence $\{x_n\}$ defined in Lemma 3.1.

Theorem 3.4 *Let C be a nonempty closed and convex subset of a* CAT(0) *space X, $T, S\colon C \to C$ be two SKC mappings satisfying condition (A'). If $F \neq \emptyset$, then the sequence $\{x_n\}$ given in* (2.3) *converges strongly to a common fixed point of S and T.*

Proof By Lemma 3.1, it follows that $\lim_{n\to\infty} d(x_n, x^*)$ exists for all $x^* \in F$. Let this limit be c, where $c \geq 0$.

If $c = 0$, there is nothing to prove.

Suppose that $c > 0$. Now, $d(x_{n+1}, x^*) \leq d(x_n, x^*)$ gives that

$$\inf_{x^*\in F} d\bigl(x_{n+1}, x^*\bigr) \leq \inf_{x^*\in F} d\bigl(x_n, x^*\bigr),$$

which means that $d(x_{n+1}, F) \leq d(x_n, F)$ and so $\lim_{n\to\infty} d(x_n, F)$ exists.

By using the condition (A'), either

$$\lim_{n\to\infty} f\bigl(d(x_n, F)\bigr) \leq \lim_{n\to\infty} d(x_n, Tx_n) = 0$$

or

$$\lim_{n\to\infty} f\bigl(d(x_n, F)\bigr) \leq \lim_{n\to\infty} d(x_n, Sx_n) = 0.$$

In both cases, we have

$$\lim_{n\to\infty} f\bigl(d(x_n,F)\bigr) = 0.$$

Since f is a nondecreasing function and $f(0) = 0$, it follows that $\lim_{n\to\infty} d(x_n,F) = 0$.

Next, we show that $\{x_n\}$ is a Cauchy sequence in C.

Let $\epsilon > 0$ be arbitrarily chosen. Since $\lim_{n\to\infty} d(x_n,F) = 0$, there exists a positive integer n_0 such that

$$d(x_n,F) < \frac{\epsilon}{4}, \quad \forall n \geq n_0.$$

In particular, $\inf\{d(x_{n_0},p) : p \in F\} < \frac{\epsilon}{4}$. Thus there must exist $p^* \in F$ such that

$$d\bigl(x_{n_0},p^*\bigr) < \frac{\epsilon}{2}.$$

Now, for all $m,n \geq n_0$, we have

$$\begin{aligned} d(x_{n+m},x_n) &\leq d\bigl(x_{n+m},p^*\bigr) + d\bigl(p^*,x_n\bigr) \\ &\leq 2d\bigl(x_{n_0},p^*\bigr) \\ &< 2\left(\frac{\epsilon}{2}\right) = \epsilon. \end{aligned}$$

Hence $\{x_n\}$ is a Cauchy sequence in a closed subset C of a complete CAT(0) space and so it must converge to a point p in C.

Now, $\lim_{n\to\infty} d(x_n,F) = 0$ gives $d(p,F) = 0$ and closedness of F forces p to be in F. □

Remark 3.2 The above theorem extends Theorem 6 of Khan and Abbas [4] to SKC mappings.

Although the following is a corollary to Theorem 3.4, yet it is new in itself.

Corollary 3.3 *Let C be a nonempty, closed and convex subset of a* CAT(0) *space X, $T: C \to C$ an SKC mapping satisfying condition* (A). *Let $\{x_n\}$ be as in* (2.4). *If $F(T) \neq \emptyset$, then $\{x_n\}$ converge strongly to a fixed point of T.*

Proof Take $S = T$ in Theorem 3.4. □

The following corollary extends Theorem 5.5 of Nanjaras *et al.* [3] to SKC mappings and, in turn, the results involving KSC and CSC mappings.

Corollary 3.4 *Let C be a nonempty, closed and convex subset of a* CAT(0) *space X, $T: C \to C$ an SKC mapping satisfying condition* (A). *Let $\{x_n\}$ be as in* (2.5). *If $F(T) \neq \emptyset$, then $\{x_n\}$ converge strongly to a fixed point of T.*

Proof Take $S = I$, the identity mapping, in Theorem 3.4. □

Remark 3.3 (1) Theorem 4.4 of Nanjaras *et al.* [3] about the existence of common fixed point of a countable family of commuting maps can now be extended to a countable family of SKC mappings.

(2) Theorem 5 of Khan and Abbas [4] can also be extended to SKC mappings.

(3) Theorem 25 and Theorem 32 of Karapınar and Tas [6] and their corollaries can now be extended to the setting of a CAT(0) space.

(4) Results for KSC and CSC mappings or for mappings given in [6] satisfying the so-called conditions (A_1) and (A_2) in the setup of CAT(0) spaces can also be obtained from corresponding results proved in this paper. As a matter of fact, these results are special cases of our results presented here.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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References

1. Suzuki, T: Fixed point theorems and convergence theorems for some generalized nonexpansive mappings. J. Math. Anal. Appl. **340**, 1088-1095 (2008)
2. Dhompongsa, S, Inthakon, W, Kaewkhao, A: Edelstein's method and fixed point theorems for some generalized nonexpansive mappings. J. Math. Anal. Appl. **350**, 12-17 (2009)
3. Nanjaras, B, Panyanak, B, Phuengrattana, W: Fixed point theorems and convergence theorems for Suzuki-generalized nonexpansive mappings in CAT(0) spaces. Nonlinear Anal. Hybrid Syst. **4**, 25-31 (2010)
4. Khan, SH, Abbas, M: Strong and Δ-convergence of some iterative schemes in CAT(0) spaces. Comput. Math. Appl. **61**, 109-116 (2011)
5. Fukhar-ud-din, H, Domolo, A, Khan, AR: Strong convergence of an implicit algorithm in CAT(0) spaces. Fixed Point Theory Appl. **2011**, Article ID 173621 (2011). doi:10.1155/2011/173621
6. Karapınar, E, Tas, K: Generalized (C)-conditions and related fixed point theorems. Comput. Math. Appl. **61**, 3370-3380 (2011)
7. Dhompongsa, S, Panyanak, B: On Δ-convergence theorems in CAT(0) spaces. Comput. Math. Appl. **56**, 2572-2579 (2008)
8. Bridson, M, Haefliger, A: Metric Spaces of Non-Positive Curvature. Springer, Berlin (1999)
9. Burago, D, Burago, Y, Ivanov, S: A Course in Metric Geometry. Graduate Studies in Math., vol. 33. Am. Math. Soc., Providence (2001)
10. Goebel, K, Reich, S: Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings. Dekker, New York (1984)
11. Gromov, M: Metric Structure for Riemannian and Non-Riemannian Spaces. Progr. Math., vol. 152. Birkhäuser, Boston (1984)
12. Reich, S, Shafrir, I: Nonexpansive iterations in hyperbolic space. Nonlinear Anal. **15**, 537-558 (1990)
13. Bruhat, F, Tits, J: Groupes réductifs sur un corps local. I. Données radicielles valuées. Publ. Math. Inst. Hautes Études Sci. **41**, 5-251 (1972)
14. Dhompongsa, S, Kirk, WA, Sims, B: Fixed points of uniformly Lipschitzian mappings. Nonlinear Anal. **65**, 762-772 (2006)
15. Lim, TC: Remarks on some fixed point theorems. Proc. Am. Math. Soc. **60**, 179-182 (1976)
16. Kirk, WA, Panyanak, B: A concept of convergence in geodesic spaces. Nonlinear Anal. **68**, 3689-3696 (2008)
17. Senter, HF, Dotson, WG: Approximating fixed points of nonexpansive mappings. Proc. Am. Math. Soc. **44**, 375-380 (1974)
18. Khan, SH, Fukhar-ud-din, H: Weak and strong convergence of a scheme with errors for two nonexpansive mappings. Nonlinear Anal. **8**, 1295-1301 (2005)
19. Fukhar-ud-din, H, Khan, SH: Convergence of iterates with errors of asymptotically quasi-nonexpansive mappings and applications. J. Math. Anal. Appl. **328**, 821-829 (2007)

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