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Fixed point theorems for weakly T-Chatterjea and weakly T-Kannan contractions in b-metric spaces

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Abstract
In this work, we obtain some fixed point results for generalized weakly T-Chatterjea-contractive and generalized weakly T-Kannan-contractive mappings in the framework of complete b-metric spaces. Examples are provided in order to distinguish these results from the known ones.

MSC: 47H10; 54H25

Keywords: fixed point; complete metric space; b-metric space; weak C-contraction; altering distance function

1 Introduction and preliminaries

The theoretical framework of metric fixed point theory has been an active research field over the last nine decades. Of course, the Banach contraction principle [1] is the first important result on fixed points for contractive-type mappings. So far, there have been a lot of fixed point results dealing with mappings satisfying various types of contractive inequalities. In particular, the concepts of K-contraction and C-contraction were introduced by Kannan [2], respectively, Chatterjea [3] as follows.

Definition 1 Let (X,d) be a metric space and $f : X \to X$.

1. ([2]) The mapping f is said to be a K-contraction if there exists $\alpha \in (0, \frac{1}{2})$ such that for all $x, y \in X$ the following inequality holds:

$$d(fx,fy) \le \alpha\big(d(x,fx) + d(y,fy)\big).$$

2. ([3]) The mapping f is said to be a C-contraction if there exists $\alpha \in (0, \frac{1}{2})$ such that for all $x, y \in X$ the following inequality holds:

$$d(fx,fy) \le \alpha\big(d(x,fy) + d(y,fx)\big).$$

In 1968, Kannan [2] proved that if (X,d) is a complete metric space, then every K-contraction on X has a unique fixed point. In 1972, Chatterjea [3] established a fixed point theorem for C-contractions.

Definition 2 Let (X,d) be a metric space, $f : X \to X$ and $\varphi : [0,\infty)^2 \to [0,\infty)$ be a continuous function such that $\varphi(x,y) = 0$ if and only if $x = y = 0$.

1. ([4]) f is said to be weakly C-contractive (or a weak C-contraction) if for all $x, y \in X$,

$$d(fx,fy) \leq \frac{1}{2}\big(d(x,fy) + d(y,fx)\big) - \varphi\big(d(x,fy), d(y,fx)\big).$$

2. ([5]) f is said to be weakly K-contractive (or a weak K-contraction) if for all $x, y \in X$,

$$d(fx,fy) \leq \frac{1}{2}\big(d(x,fx) + d(y,fy)\big) - \varphi\big(d(x,fx), d(y,fy)\big).$$

In 2009, Choudhury [4] proved the following theorem.

Theorem 1 ([4, Theorem 2.1]) *Every weak C-contraction on a complete metric space has a unique fixed point.*

For more details of weakly C-contractive mappings we refer to [6] and [7].

Definition 3 Let (X, d) be a metric space and $T, f : X \to X$ be two mappings.

1. ([8]) $f : X \to X$ is said to be a T-Kannan-contraction if there exists $\alpha \in (0, \frac{1}{2})$ such that for all $x, y \in X$ the following inequality holds:

$$d(Tfx, Tfy) \leq \alpha\big(d(Tx, Tfx) + d(Ty, Tfy)\big).$$

2. ([5]) $f : X \to X$ is said to be a T-Chatterjea-contraction if there exists $\alpha \in (0, \frac{1}{2})$ such that for all $x, y \in X$ the following inequality holds:

$$d(Tfx, Tfy) \leq \alpha\big(d(Tx, Tfy) + d(Ty, Tfx)\big).$$

T-Kannan-contractions (in short, T-K-contractions) and T-Chatterjea-contractions (in short, T-C-contractions) are special cases of T-Hardy-Rogers contractions [9]. Recently, existence and uniqueness of fixed points for these types of contractions in cone metric spaces have been investigated in [9] and [10].

Definition 4 ([11]) Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be sequentially convergent (respectively, subsequentially convergent) if, for a sequence $\{x_n\}$ in X for which $\{Tx_n\}$ is convergent, $\{x_n\}$ is also convergent (respectively, $\{x_n\}$ has a convergent subsequence).

In [8], Moradi has extended Kannan's theorem [2] as follows.

Theorem 2 (Extended Kannan's theorem [8]) *Let (X, d) be a complete metric space and $T, f : X \to X$ be mappings such that T is continuous, one-to-one and subsequentially convergent. If f is a T-K-contraction then f has a unique fixed point. Moreover, if T is sequentially convergent then, for every $x_0 \in X$, the sequence of iterates $\{f^n x_0\}$ converges to this fixed point.*

The notion of an altering distance function was introduced by Khan *et al.* as follows.

Definition 5 ([12]) The function $\psi : [0, \infty) \to [0, \infty)$ is called an altering distance function, if the following properties are satisfied:

1. ψ is continuous and strictly increasing.
2. $\psi(0) = 0$.

In the following definitions and theorems, ψ is an altering distance function and $\varphi : [0,\infty)^2 \to [0,\infty)$ is a continuous function such that $\varphi(x,y) = 0$ if and only if $x = y = 0$.

Definition 6 ([5]) Let (X,d) be a metric space and let $T, f : X \to X$ be two mappings.

1. f is said to be a generalized weak T-C-contraction if, for all $x, y \in X$,

$$\psi\bigl(d(Tfx,Tfy)\bigr) \le \psi\left(\frac{d(Tx,Tfy)+d(Ty,Tfx)}{2}\right) - \varphi\bigl(d(Tx,Tfy),d(Ty,Tfx)\bigr).$$

2. f is said to be a generalized weak T-K-contraction if, for all $x, y \in X$,

$$\psi\bigl(d(Tfx,Tfy)\bigr) \le \psi\left(\frac{d(Tx,Tfx)+d(Ty,Tfy)}{2}\right) - \varphi\bigl(d(Tx,Tfx),d(Ty,Tfy)\bigr).$$

Putting $\psi(t) = t$ in the above definition, we obtain the concepts of weak T-C-contraction and weak T-K-contraction.

The following are the main results of [5].

Theorem 3 [5] *Let (X,d) be a complete metric space and let $T, f : X \to X$ be two mappings such that T is one-to-one and continuous. Suppose that*:

1. *f is a generalized weak T-C-contraction, or*
2. *f is a generalized weak T-K-contraction.*

Then we have the following.

(i) *For every $x_0 \in X$ the sequence $\{Tf^n x_0\}$ is convergent.*
(ii) *If T is subsequentially convergent then f has a unique fixed point.*
(iii) *If T is sequentially convergent then for each $x_0 \in X$ the sequence $\{f^n x_0\}$ converges to the fixed point of f.*

The aim of this article is to extend the stated results to the framework of b-metric spaces, introduced in 1993 by Czerwik [13]. These form a nontrivial generalization of metric spaces and several fixed point results for single and multivalued mappings in such spaces have been obtained since then (see, *e.g.*, [14–17] and the references cited therein). We recall the following.

Definition 7 ([13]) Let X be a (nonempty) set and $s \ge 1$ be a given real number. A function $d : X \times X \to [0,\infty)$ is a b-metric if, for all $x, y, z \in X$, the following conditions are satisfied:

(b_1) $d(x,y) = 0$ iff $x = y$,
(b_2) $d(x,y) = d(y,x)$,
(b_3) $d(x,z) \le s[d(x,y) + d(y,z)]$.

The pair (X,d) is called a b-metric space.

It should be noted that the class of b-metric spaces is effectively larger than that of metric spaces, since a b-metric is a metric if (and only if) $s = 1$. We present an easy example to show that in general a b-metric need not be a metric.

Example 1 Let (X,ρ) be a metric space, and $d(x,y)=(\rho(x,y))^p$, where $p\geq 1$ is a real number. Then d is a b-metric with $s=2^{p-1}$.

However, (X,d) is not necessarily a metric space. For example, if $X=\mathbb{R}$ is the set of real numbers and $\rho(x,y)=|x-y|$ is the usual Euclidean metric, then $d(x,y)=(x-y)^2$ is a b-metric on $\mathbb{R}$ with $s=2$, but it is not a metric on $\mathbb{R}$.

Recently, Hussain *et al.* [15] have presented an example of a b-metric which is not continuous (see [15, Example 2]). Thus, while working in b-metric spaces, the following lemma is useful.

Lemma 1 ([14]) *Let (X,d) be a b-metric space with $s\geq 1$, and suppose that the sequences $\{x_n\}$ and $\{y_n\}$ are b-convergent to x, y, respectively. Then we have*

$$\frac{1}{s^2}d(x,y)\leq \liminf_{n\to\infty} d(x_n,y_n)\leq \limsup_{n\to\infty} d(x_n,y_n)\leq s^2 d(x,y).$$

In particular, if $x=y$, then we have $\lim_{n\to\infty} d(x_n,y_n)=0$. Moreover, for each $z\in X$, we have,

$$\frac{1}{s}d(x,z)\leq \liminf_{n\to\infty} d(x_n,z)\leq \limsup_{n\to\infty} d(x_n,z)\leq sd(x,z).$$

2 Fixed points of weakly T-Chatterjea contractions

From now on, we assume:

$$\Psi=\{\psi:[0,\infty)\to[0,\infty)\mid \psi \text{ is an altering distance function}\}$$

and

$$\Phi=\Big\{\varphi:[0,\infty)^2\to[0,\infty)\,\Big|\,\varphi(x,y)=0\Longleftrightarrow x=y=0 \text{ and}$$
$$\varphi\Big(\liminf_{n\to\infty} a_n,\liminf_{n\to\infty} b_n\Big)\leq \liminf_{n\to\infty}\varphi(a_n,b_n)\Big\}.$$

Our first result is the following.

Theorem 4 *Let (X,d) be a complete b-metric space with parameter $s\geq 1$, $T,f:X\to X$ be such that, for some $\psi\in\Psi$, $\varphi\in\Phi$ and all $x,y\in X$,*

$$\psi\big(sd(Tfx,Tfy)\big)\leq\psi\left(\frac{d(Tx,Tfy)+d(Ty,Tfx)}{s+1}\right)-\varphi\big(d(Tx,Tfy),d(Ty,Tfx)\big),\tag{2.1}$$

and let T be one-to-one and continuous. Then we have the following.

1. *For every $x_0\in X$ the sequence $\{Tf^nx_0\}$ is convergent.*
2. *If T is subsequentially convergent, then f has a unique fixed point.*
3. *If T is sequentially convergent, then for each $x_0\in X$ the sequence $\{f^nx_0\}$ converges to the fixed point of f.*

Proof Let $x_0\in X$ be arbitrary. Consider the sequence $\{x_n\}_{n=0}^{\infty}$ given by $x_{n+1}=fx_n=f^{n+1}x_0$, $n=0,1,2,\ldots$. We will complete the proof in three steps.

Step I. We will prove that $\lim_{n\to\infty} d(Tx_n,Tx_{n+1})=0$.

Using condition (2.1), we obtain

$$\begin{aligned}\psi\big(sd(Tx_{n+1},Tx_n)\big) &= \psi\big(sd(Tfx_n,Tfx_{n-1})\big)\\ &\leq \psi\left(\frac{d(Tx_n,Tfx_{n-1})+d(Tx_{n-1},Tfx_n)}{s+1}\right)\\ &\quad -\varphi\big(d(Tx_n,Tfx_{n-1}),d(Tx_{n-1},Tfx_n)\big)\\ &= \psi\left(\frac{d(Tx_n,Tx_n)+d(Tx_{n-1},Tx_{n+1})}{s+1}\right)\\ &\quad -\varphi\big(d(Tx_n,Tx_n),d(Tx_{n-1},Tx_{n+1})\big). \qquad (2.2)\end{aligned}$$

Therefore, by the triangular inequality and since φ is nonnegative and ψ is an increasing function,

$$\begin{aligned}\psi\big(sd(Tx_{n+1},Tx_n)\big) &\leq \psi\left(\frac{d(Tx_{n-1},Tx_{n+1})}{s+1}\right)\\ &\leq \psi\left(\frac{s}{s+1}\big(d(Tx_{n-1},Tx_n)+d(Tx_n,Tx_{n+1})\big)\right).\end{aligned}$$

Again, since ψ is increasing, we have

$$d(Tx_{n+1},Tx_n) \leq \frac{1}{s+1}\big(d(Tx_{n-1},Tx_n)+d(Tx_n,Tx_{n+1})\big),$$

wherefrom

$$d(Tx_{n+1},Tx_n) \leq \frac{1}{s}d(Tx_n,Tx_{n-1}) \leq d(Tx_n,Tx_{n-1}).$$

Thus, $\{d(Tx_{n+1},Tx_n)\}$ is a decreasing sequence of nonnegative real numbers and hence it is convergent.

Assume that $\lim_{n\to\infty} d(Tx_{n+1},Tx_n) = r \geq 0$. From the above argument we have

$$\begin{aligned}sd(Tx_{n+1},Tx_n) &\leq \frac{1}{s+1}d(Tx_{n-1},Tx_{n+1})\\ &\leq \frac{s}{s+1}\big(d(Tx_{n-1},Tx_n)+d(Tx_n,Tx_{n+1})\big)\\ &\leq \frac{s}{2}\big(d(Tx_{n-1},Tx_n)+d(Tx_n,Tx_{n+1})\big).\end{aligned}$$

Passing to the limit when $n \to \infty$, we obtain

$$\lim_{n\to\infty} d(Tx_{n-1},Tx_{n+1}) = s(s+1)r.$$

We have proved in (2.2) that

$$\psi\big(sd(Tx_{n+1},Tx_n)\big) \leq \psi\left(\frac{0+d(Tx_{n-1},Tx_{n+1})}{s+1}\right) - \varphi\big(0,d(Tx_{n-1},Tx_{n+1})\big).$$

Now, letting $n \to \infty$ and using the continuity of ψ and the properties of φ we obtain

$$\psi(sr) \leq \psi(sr) - \varphi\big(0,s(s+1)r\big),$$

and consequently, $\varphi(0, s(s+1)r) = 0$. This yields

$$r = \lim_{n\to\infty} d(Tx_n, Tx_{n+1}) = 0, \tag{2.3}$$

by our assumptions about φ.

Step II. $\{Tx_n\}$ is a b-Cauchy sequence.

Suppose that $\{Tx_n\}$ is not a b-Cauchy sequence. Then there exists $\varepsilon > 0$ for which we can find subsequences $\{Tx_{m(k)}\}$ and $\{Tx_{n(k)}\}$ of $\{Tx_n\}$ such that $n(k)$ is the smallest index for which $n(k) > m(k) > k$ and

$$d(Tx_{m(k)}, Tx_{n(k)}) \geq \varepsilon. \tag{2.4}$$

This means that

$$d(Tx_{m(k)}, Tx_{n(k)-1}) < \varepsilon. \tag{2.5}$$

From (2.4), (2.5) and the triangular inequality,

$$\begin{aligned}\varepsilon &\leq d(Tx_{m(k)}, Tx_{n(k)}) \leq s\bigl[d(Tx_{m(k)}, Tx_{n(k)-1}) + d(Tx_{n(k)-1}, Tx_{n(k)})\bigr] \\ &< s\varepsilon + sd(Tx_{n(k)-1}, Tx_{n(k)}).\end{aligned}$$

Letting $k \to \infty$, and taking into account (2.3), we can conclude that

$$\varepsilon \leq \limsup_{k\to\infty} d(Tx_{m(k)}, Tx_{n(k)}) \leq s\varepsilon. \tag{2.6}$$

Further, from

$$d(Tx_{m(k)}, Tx_{n(k)}) \leq s\bigl[d(Tx_{m(k)}, Tx_{n(k)-1}) + d(Tx_{n(k)-1}, Tx_{n(k)})\bigr]$$

and (2.5), and using (2.3), we get

$$\frac{\varepsilon}{s} \leq \limsup_{k\to\infty} d(Tx_{n(k)-1}, Tx_{m(k)}) \leq \varepsilon. \tag{2.7}$$

Moreover, from

$$d(Tx_{m(k)}, Tx_{n(k)}) \leq s\bigl[d(Tx_{m(k)}, Tx_{m(k)-1}) + d(Tx_{m(k)-1}, Tx_{n(k)})\bigr]$$

and

$$d(Tx_{m(k)-1}, Tx_{n(k)}) \leq s\bigl[d(Tx_{m(k)-1}, Tx_{m(k)}) + d(Tx_{m(k)}, Tx_{n(k)})\bigr],$$

and using (2.3) and (2.6), we get

$$\frac{\varepsilon}{s} \leq \limsup_{k\to\infty} d(Tx_{m(k)-1}, Tx_{n(k)}) \leq s^2\varepsilon. \tag{2.8}$$

Similarly, we can show that

$$\frac{\varepsilon}{s} \leq \liminf_{k\to\infty} d(Tx_{n(k)-1}, Tx_{m(k)}) \leq \varepsilon \tag{2.9}$$

and

$$\frac{\varepsilon}{s} \leq \liminf_{k\to\infty} d(Tx_{m(k)-1}, Tx_{n(k)}) \leq s^2\varepsilon. \tag{2.10}$$

Using (2.1) and (2.7)-(2.10) we have

$$\begin{aligned}
\psi(s\varepsilon) &\leq \psi\Big(s\limsup_{k\to\infty} d(Tx_{m(k)}, Tx_{n(k)})\Big) \\
&= \psi\Big(s\limsup_{k\to\infty} d(Tfx_{m(k)-1}, Tfx_{n(k)-1})\Big) \\
&\leq \limsup_{k\to\infty} \psi\left(\frac{d(Tx_{m(k)-1}, Tfx_{n(k)-1}) + d(Tx_{n(k)-1}, Tfx_{m(k)-1})}{s+1}\right) \\
&\quad - \liminf_{k\to\infty} \varphi\big(d(Tx_{m(k)-1}, Tfx_{n(k)-1}), d(Tx_{n(k)-1}, Tfx_{m(k)-1})\big) \\
&\leq \psi\left(\limsup_{k\to\infty} \frac{d(Tx_{m(k)-1}, Tx_{n(k)}) + d(Tx_{n(k)-1}, Tx_{m(k)})}{s+1}\right) \\
&\quad - \varphi\Big(\liminf_{k\to\infty} d(Tx_{m(k)-1}, Tx_{n(k)}), \liminf_{k\to\infty} d(Tx_{n(k)-1}, Tx_{m(k)})\Big) \\
&\leq \psi\left(\frac{s^2\varepsilon+\varepsilon}{s+1}\right) - \varphi\Big(\liminf_{k\to\infty} d(Tx_{m(k)-1}, Tx_{n(k)}), \liminf_{k\to\infty} d(Tx_{n(k)-1}, Tx_{m(k)})\Big) \\
&\leq \psi(s\varepsilon) - \varphi\Big(\liminf_{k\to\infty} d(Tx_{m(k)-1}, Tx_{n(k)}), \liminf_{k\to\infty} d(Tx_{n(k)-1}, Tx_{m(k)})\Big)
\end{aligned}$$

since $\frac{s^2+1}{s+1} \leq s$. Hence, we have

$$\varphi\Big(\liminf_{k\to\infty} d(Tx_{m(k)-1}, Tx_{n(k)}), \liminf_{k\to\infty} d(Tx_{n(k)-1}, Tx_{m(k)})\Big) \leq 0.$$

By our assumption about φ, we have

$$\liminf_{k\to\infty} d(Tx_{m(k)-1}, Tx_{n(k)}) = \liminf_{k\to\infty} d(Tx_{n(k)-1}, Tx_{m(k)}) = 0,$$

which contradicts (2.9) and (2.10).

Since (X,d) is b-complete and $\{Tx_n\} = \{Tf^n x_0\}$ is a b-Cauchy sequence, there exists $v \in X$ such that

$$\lim_{n\to\infty} Tf^n x_0 = v. \tag{2.11}$$

Step III. f has a unique fixed point, assuming that T is subsequentially convergent.

As T is subsequentially convergent, $\{f^n x_0\}$ has a b-convergent subsequence. Hence, there exist $u \in X$ and a subsequence $\{n_i\}$ such that

$$\lim_{i\to\infty} f^{n_i} x_0 = u. \tag{2.12}$$

Since T is continuous, by (2.12) we obtain

$$\lim_{i\to\infty} Tf^{n_i}x_0 = Tu, \tag{2.13}$$

and by (2.11) and (2.13) we conclude that $Tu = v$.

From Lemma 1 and (2.1) we have

$$\begin{aligned}
\psi\left(s\cdot\frac{1}{s}d(Tfu,Tu)\right) &\le \psi\left(\limsup_{n\to\infty} sd\big(Tfu, Tf^{n+1}x_0\big)\right)\\
&= \psi\left(\limsup_{n\to\infty} sd(Tfu, Tfx_n)\right)\\
&\le \psi\left(\limsup_{n\to\infty}\frac{d(Tu,Tfx_n)+d(Tx_n,Tfu)}{s+1}\right)\\
&\quad - \liminf_{n\to\infty}\varphi\big(d(Tu,Tfx_n), d(Tx_n,Tfu)\big)\\
&\le \psi\left(\frac{sd(Tu,Tu)+sd(Tu,Tfu)}{s+1}\right)\\
&\quad - \varphi\Big(\liminf_{n\to\infty} d(Tu,Tfx_n), \liminf_{n\to\infty} d(Tx_n,Tfu)\Big)\\
&\le \psi\big(d(Tu,Tfu)\big) - \varphi\Big(0, \liminf_{n\to\infty} d(Tx_n,Tfu)\Big),
\end{aligned}$$

since ψ is increasing. By the properties of $\varphi\in\Phi$, it follows that $\liminf_{n\to\infty} d(Tx_n,Tfu) = 0$. By the triangular inequality we have

$$d(Tfu,Tu) \le s\big[d(Tfu,Tx_n) + d(Tx_n,Tu)\big].$$

Letting $n\to\infty$ we can conclude that $d(Tfu,Tu) = 0$. Hence, $Tfu = Tu$. As T is one-to-one, $fu = u$. Consequently, f has a fixed point.

If we assume that w is another fixed point of f, because of (2.1), we have

$$\begin{aligned}
\psi\big(sd(Tu,Tw)\big) &= \psi\big(sd(Tfu,Tfw)\big)\\
&\le \psi\left(\frac{d(Tu,Tfw)+d(Tw,Tfu)}{s+1}\right) - \varphi\big(d(Tu,Tfw), d(Tw,Tfu)\big)\\
&= \psi\left(\frac{d(Tu,Tw)+d(Tw,Tu)}{s+1}\right) - \varphi\big(d(Tu,Tw), d(Tw,Tu)\big)\\
&\le \psi\big(sd(Tu,Tw)\big) - \varphi\big(d(Tu,Tw), d(Tw,Tu)\big),
\end{aligned}$$

since $\frac{2}{s+1}\le s$ and ψ is increasing. Hence, $d(Tu,Tw) = 0$. Since T is one-to-one, it follows that $u = w$. Consequently, f has a unique fixed point.

Finally, if T is sequentially convergent, replacing $\{n\}$ with $\{n_i\}$ we conclude that $\lim_{n\to\infty} f^n x_0 = u$. □

Taking $\psi(t) = t$ and $\varphi(t,u) = (\frac{1}{s+1} - \alpha)(t+u)$, where $\alpha\in[0,\frac{1}{s+1})$ in Theorem 4, the extended Chatterjea's theorem in the setting of b-metric spaces is obtained.

Corollary 1 *Let (X,d) be a complete b-metric space and $T,f : X \to X$ be mappings such that T is continuous, one-to-one and subsequentially convergent. If $\alpha \in [0, \frac{1}{s+1})$ and*

$$d(Tfx, Tfy) \leq \frac{\alpha}{s}\big(d(Tx, Tfy) + d(Ty, Tfx)\big),$$

for all $x, y \in X$, then f has a unique fixed point. Moreover, if T is sequentially convergent, then for every $x_0 \in X$ the sequence of iterates $f^n x_0$ converges to this fixed point.

Remark 1 In the case when $Tx = x$, this corollary reduces to [18, Corollary 3.8.3°] (the case $g = f$), which is Chatterjea's theorem [3] in the framework of b-metric spaces.

By taking $Tx = x$ and $\psi(t) = t$ in Theorem 4, we derive an extension of Choudhury's theorem (Theorem 1) to the setup of b-metric spaces.

If $s = 1$, Theorem 4 reduces to Theorem 3 (case (1)).

We demonstrate the use of the obtained results by the following.

Example 2 (Inspired by [8]) Let $X = \{0\} \cup \{1/n \mid n \in \mathbb{N}\}$, and let $d(x, y) = (x - y)^2$ for $x, y \in X$. Then d is a b-metric with the parameter $s = 2$ and (X, d) is a complete b-metric space. Consider the mappings $f, T : X \to X$ given by

$$f(0) = 0, \quad f\left(\frac{1}{n}\right) = \frac{1}{n+1}, \qquad T(0) = 0, \quad T\left(\frac{1}{n}\right) = \frac{1}{n^n}, \quad n \in \mathbb{N}.$$

We will show that the mappings f, T satisfy the conditions of Corollary 1 with $\alpha = \frac{2}{9} < \frac{1}{3} = \frac{1}{s+1}$. Indeed, for $m, n \in \mathbb{N}$, $m > n$, we have

$$d\left(Tf\frac{1}{n}, Tf\frac{1}{m}\right) = \left[\frac{1}{(n+1)^{n+1}} - \frac{1}{(m+1)^{m+1}}\right]^2 < \left[\frac{1}{(n+1)^{n+1}}\right]^2.$$

It is easy to prove that, for $n \in \mathbb{N}$,

$$\frac{1}{(n+1)^{n+1}} < \frac{1}{3}\left[\frac{1}{n^n} - \frac{1}{(n+2)^{n+2}}\right].$$

It follows that

$$d\left(Tf\frac{1}{n}, Tf\frac{1}{m}\right) < \frac{1}{9}\left[\frac{1}{n^n} - \frac{1}{(n+2)^{n+2}}\right]^2.$$

Now, $m > n$ implies that $m \geq n + 1$ and $n + 2 \leq m + 1$. It follows that $1/(n+2)^{n+2} \geq 1/(m+1)^{m+1}$, and hence

$$\begin{aligned} d\left(Tf\frac{1}{n}, Tf\frac{1}{m}\right) &< \frac{1}{9}\left[\frac{1}{n^n} - \frac{1}{(m+1)^{m+1}}\right]^2 \\ &\leq \frac{\alpha}{s}\left[d\left(T\frac{1}{n}, Tf\frac{1}{m}\right) + d\left(T\frac{1}{m}, TF\frac{1}{n}\right)\right]. \end{aligned}$$

If one of the points is equal to 0, the proof is even simpler.

By Corollary 1, it follows that f has a unique fixed point (which is $u = 0$).

3 Fixed points of weakly *T*-Kannan contractions

Our second main result is the following.

Theorem 5 *Let (X,d) be a complete b-metric space with the parameter $s \geq 1$, $T, f : X \to X$ be such that for some $\psi \in \Psi$, $\varphi \in \Phi$ and all $x, y \in X$,*

$$\psi\big(d(Tfx,Tfy)\big) \leq \psi\left(\frac{d(Tx,Tfx)+d(Ty,Tfy)}{s+1}\right) - \varphi\big(d(Tx,Tfx),d(Ty,Tfy)\big). \tag{3.1}$$

and let T be one-to-one and continuous. Then:

(1) *For every $x_0 \in X$ the sequence $\{Tf^n x_0\}$ is convergent.*
(2) *If T is subsequentially convergent, then f has a unique fixed point.*
(3) *If T is sequentially convergent then, for each $x_0 \in X$, the sequence $\{f^n x_0\}$ converges to the fixed point of f.*

Proof Let $x_0 \in X$ be arbitrary. Consider the sequence $\{x_n\}_{n=0}^{\infty}$ given by $x_{n+1} = fx_n = f^{n+1}x_0$, $n = 0, 1, 2, \ldots$. At first, we will prove that

$$\lim_{n\to\infty} d(Tx_n, Tx_{n+1}) = 0.$$

Using condition (3.1), we obtain

$$\begin{aligned}
\psi\big(d(Tx_{n+1},Tx_n)\big) &= \psi\big(d(Tfx_n,Tfx_{n-1})\big) \\
&\leq \psi\left(\frac{d(Tx_n,Tfx_n)+d(Tx_{n-1},Tfx_{n-1})}{s+1}\right) \\
&\quad - \varphi\big(d(Tx_n,Tfx_n),d(Tx_{n-1},Tfx_{n-1})\big) \\
&= \psi\left(\frac{d(Tx_n,Tx_{n+1})+d(Tx_{n-1},Tx_n)}{s+1}\right) \\
&\quad - \varphi\big(d(Tx_n,Tfx_n),d(Tx_{n-1},Tfx_{n-1})\big).
\end{aligned} \tag{3.2}$$

Since φ is nonnegative and ψ is increasing, it follows that

$$d(Tx_{n+1},Tx_n) \leq \frac{d(Tx_n,Tx_{n+1})+d(Tx_{n-1},Tx_n)}{s+1},$$

that is,

$$d(Tx_{n+1},Tx_n) \leq \frac{1}{s} d(Tx_n,Tx_{n-1}) \leq d(Tx_n,Tx_{n-1}).$$

Thus, $\{d(Tx_{n+1},Tx_n)\}$ is a decreasing sequence of nonnegative real numbers and hence it is convergent.

Assume that $\lim_{n\to\infty} d(Tx_{n+1},Tx_n) = r$. If in (3.2) $n \to \infty$, using the properties of ψ and φ we obtain

$$\psi(r) \leq \psi\left(\frac{2r}{s+1}\right) - \varphi(r,r) \leq \psi(r) - \varphi(r,r),$$

which is possible only if

$$r = \lim_{n\to\infty} d(Tx_n, Tx_{n+1}) = 0.$$

Now, we will show that $\{Tx_n\}$ is a b-Cauchy sequence.

Suppose that this is not true. Then there exists $\varepsilon > 0$ for which we can find subsequences $\{Tx_{m(k)}\}$ and $\{Tx_{n(k)}\}$ of $\{Tx_n\}$ such that $n(k)$ is the smallest index for which $n(k) > m(k) > k$ and $d(Tx_{m(k)}, Tx_{n(k)}) \geq \varepsilon$. This means that

$$d(Tx_{m(k)}, Tx_{n(k)-1}) < \varepsilon.$$

Again, as in Step II of Theorem 4 one can prove that

$$\varepsilon \leq \limsup_{k\to\infty} d(Tx_{m(k)}, Tx_{n(k)}) \leq s\varepsilon. \tag{3.3}$$

Using (3.1) we have

$$\begin{aligned} \psi\big(d(Tx_{m(k)}, Tx_{n(k)})\big) &= \psi\big(d(Tfx_{m(k)-1}, Tfx_{n(k)-1})\big) \\ &\leq \psi\left(\frac{d(Tx_{m(k)-1}, Tfx_{m(k)-1}) + d(Tx_{n(k)-1}, Tfx_{n(k)-1})}{s+1}\right) \\ &\quad - \varphi\big(d(Tx_{m(k)-1}, Tfx_{m(k)-1}), d(Tx_{n(k)-1}, Tfx_{n(k)-1})\big) \\ &= \psi\left(\frac{d(Tx_{m(k)-1}, Tx_{m(k)}) + d(Tx_{n(k)-1}, Tx_{n(k)})}{s+1}\right) \\ &\quad - \varphi\big(d(Tx_{m(k)-1}, Tx_{m(k)}), d(Tx_{n(k)-1}, Tx_{n(k)})\big). \end{aligned}$$

Passing to the upper limit as $k \to \infty$ in the above inequality and taking into account (3.3), we have

$$\psi(\varepsilon) \leq \psi(0) - \varphi(0, 0) = 0,$$

and so $\psi(\varepsilon) = 0$. By our assumptions about ψ, we have $\varepsilon = 0$, which is a contradiction.

Since (X, d) is b-complete and $\{Tx_n\} = \{Tf^n x_0\}$ is a b-Cauchy sequence, there exists $v \in X$ such that

$$\lim_{n\to\infty} Tf^n x_0 = v. \tag{3.4}$$

Now, if T is subsequentially convergent, then $\{f^n x_0\}$ has a convergent subsequence. Hence, there exist a point $u \in X$ and a sequence $\{n_i\}$ such that

$$\lim_{i\to\infty} f^{n_i} x_0 = u. \tag{3.5}$$

Since T is continuous, by (3.5) we obtain

$$\lim_{i\to\infty} Tf^{n_i} x_0 = Tu, \tag{3.6}$$

and by (3.4) and (3.6) we conclude that $Tu = v$.

From Lemma 1 and (3.1) we have

$$\begin{aligned}
\psi\left(\frac{1}{s}d(Tfu,Tu)\right) &\le \psi\Bigl(\limsup_{n\to\infty} d\bigl(Tfu, Tf^{n+1}x_0\bigr)\Bigr)\\
&= \psi\Bigl(\limsup_{n\to\infty} d\bigl(Tfu, Tfx_n\bigr)\Bigr)\\
&\le \psi\left(\limsup_{n\to\infty} \frac{d(Tu,Tfu)+d(Tx_n,Tfx_n)}{s+1}\right)\\
&\quad - \liminf_{n\to\infty}\varphi\bigl(d(Tu,Tfu), d(Tx_n,Tfx_n)\bigr)\\
&= \psi\left(\frac{d(Tu,Tfu)+0}{s+1}\right) - \varphi\bigl(d(Tu,Tfu),0\bigr)\\
&\le \psi\left(\frac{d(Tu,Tfu)}{s}\right) - \varphi\bigl(d(Tu,Tfu),0\bigr).
\end{aligned}$$

By the properties of $\varphi \in \Phi$, it follows that

$$d(Tu,Tfu) = 0.$$

Since T is one-to-one, we obtain $fu = u$. Consequently, f has a fixed point.

Uniqueness of the fixed point can be proved in the same manner as in Theorem 4.

Finally, if T is sequentially convergent, replacing $\{n\}$ with $\{n_i\}$ we conclude that $\lim_{n\to\infty} f^n x_0 = u$. □

Taking $\psi(t) = t$ and $\varphi(t,u) = (\frac{1}{s+1} - \alpha)(t+u)$, where $\alpha \in [0, \frac{1}{s+1})$ in Theorem 5, the extended Kannan's theorem in the setting of b-metric spaces is obtained.

Corollary 2 *Let (X,d) be a complete b-metric space with the parameter $s \ge 1$, $T, f : X \to X$ be such that for some $\alpha < \frac{1}{s+1}$ and all $x, y \in X$,*

$$d(Tfx, Tfy) \le \alpha\bigl(d(Tx,Tfx) + d(Ty,Tfy)\bigr) \tag{3.7}$$

and let T be one-to-one and continuous. Then we have the following.

1. *For every $x_0 \in X$ the sequence $\{Tf^n x_0\}$ is convergent.*
2. *If T is subsequentially convergent then f has a unique fixed point.*
3. *If T is sequentially convergent then, for each $x_0 \in X$, the sequence $\{f^n x_0\}$ converges to the fixed point of f.*

Remark 2 In the case when $Tx = x$, this corollary reduces to [18, Corollary 3.8.2°] (the case $g = f$). If $s = 1$, Corollary 2 reduces to Theorem 2 (*i.e.*, [8, Theorem 2.1]). Of course, if both of these conditions are fulfilled, we get just the classical Kannan's theorem [2].

The following example distinguishes our results from the previously known ones.

Example 3 Let $X = \{a,b,c\}$ and $d : X \times X \to \mathbb{R}$ be defined by $d(x,x) = 0$ for $x \in X$, $d(a,b) = d(b,c) = 1$, $d(a,c) = \frac{9}{4}$, $d(x,y) = d(y,x)$ for $x, y \in X$. It is easy to check that (X,d) is a b-metric space (with $s = \frac{9}{8} > 1$) which is not a metric space. Consider the mapping $f : X \to X$ given by

$$f = \begin{pmatrix} a & b & c \\ a & a & b \end{pmatrix}.$$

We first note that the b-metric version of classical weak Kannan's theorem is not satisfied in this example. Indeed, for $x = b$, $y = c$, we have $d(fx, fy) = d(a, b) = 1$ and $d(x, fx) + d(y, fy) = d(b, a) + d(c, b) = 2$, hence the inequality

$$\psi\big(d(fx, fy)\big) \le \psi\left(\frac{d(x, fx) + d(y, fy)}{s + 1}\right) - \varphi\big(d(x, fx), d(y, fy)\big)$$

cannot hold, whatever $\psi \in \Psi$ and $\varphi \in \Phi$ are chosen.

Take now $T : X \to X$ defined by

$$T = \begin{pmatrix} a & b & c \\ b & c & a \end{pmatrix}.$$

Obviously, all the properties of T given in Corollary 2 are fulfilled. We will check that the contractive condition (3.7) holds true if α is chosen such that

$$\frac{4}{9} < \alpha < \frac{8}{17} = \frac{1}{s + 1}.$$

Only the following cases are nontrivial:

1° $x = a$, $y = c$. Then (3.7) reduces to

$$d(Tfa, Tfc) = d(b, c) = 1 = \frac{4}{9} \cdot \frac{9}{4} < \alpha\big(d(b, b) + d(a, c)\big) = \alpha\big(d(Ta, Tfa) + d(Tc, Tfc)\big).$$

2° $x = b$, $y = c$. Then (3.7) reduces to

$$d(Tfb, Tfc) = d(b, c) = 1 < \frac{4}{9} \cdot \frac{13}{4} < \alpha\big(d(c, b) + d(a, c)\big) = \alpha\big(d(Tb, Tfb) + d(Tc, Tfc)\big).$$

All the conditions of Corollary 2 are satisfied and f has a unique fixed point ($u = a$).

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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