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Abstract

A dynamic coverage problem for sensor networks that are sufficiently dense but not localized is considered. By maintaining only a small fraction of sensors on at any time, we are aimed to find a decentralized protocol for establishing dynamic, sweeping barriers of awake-state sensors. Network cyclic cellular automata is used

to generate waves. By rigorously analyzing network-based cyclic cellular automata in the context of a system of narrow hallways, it shows that waves of awake-state nodes turn corners and automatically solve pursuit/evasion-type problems without centralized coordination. As a corollary of this work, we unearth some interesting

topological interpretations of features previously observed in cyclic cellular automata (CCA). By considering CCA over networks and completing to simplicial complexes, we induce dynamics on the higher-dimensional complex. In this setting, waves are seen to be generated by topological defects with a nontrivial degree (or winding number). The simplicial complex has the topological type of the underlying map of the workspace (a subset of the plane), and the resulting waves can be classified cohomologically. This allows one to "program" pulses in the sensor network according to cohomology class. We give a realization theorem for such pulse waves.

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Yiqing Cai

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Degree of Doctor of Philosophy

2013

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ABSTRACT

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A dynamic coverage problem for sensor networks that are sufficiently dense but not localized is considered. By maintaining only a small fraction of sensors on at any time, we are aimed to find a decentralized protocol for establishing dynamic, sweeping barriers of awake-state sensors. Network cyclic cellular automata is used to generate waves. By rigorously analyzing network-based cyclic cellular automata in the context of a system of narrow hallways, it shows that waves of awake-state nodes turn corners and automatically solve pursuit/evasion-type problems without centralized coordination. As a corollary of this work, we unearth some interesting topological interpretations of features previously observed in cyclic cellular automata (CCA). By considering CCA over networks and completing to simplicial complexes, we induce dynamics on the higher-dimensional complex. In this setting, waves are seen to be generated by topological defects with a nontrivial degree (or winding number). The simplicial complex has the topological type of the underlying map of the workspace (a subset of the plane), and the resulting waves can be classified cohomologically. This allows one to “program” pulses in the sensor network according to cohomology class. We give a realization theorem for such pulse waves.

Contents

1	Introduction	1
1.1	Wireless Sensor Network and Coverage Problems	1
1.2	Cellular Automata	5
1.3	Cyclic Network Automata and Simulations	16
1.4	Contributions	22
2	Background	24
2.1	Flag Complex, Rips Complex and Nerve	24
2.2	Homology, Cohomology and Degree	30
3	Dynamic Features of Greenberg-Hastings Automata	35
3.1	Degree and Defect	35
3.2	Asymptotic behavior	44
3.3	Controlling the Cohomology	53
4	Evasion game	60

4.1	Limiting Case with 1-d Hallways	61
4.2	Main Theorem and Proofs	64
4.2.1	Assumptions	64
4.2.2	Wave propagation	67
4.2.3	Main theorem	71
4.3	Link Failure Analysis	76
5	Conclusion and Future Plans	78
5.1	GHM on various domains	80
5.2	Various defects	85
5.3	Applications	93

List of Figures

1.1	The 14 colored CCA with random initial state	9
1.2	BZ reaction modeled by cellular automata	11
1.3	BZ reaction experiment shots	12
1.4	Closed loops and intersections with wavefronts	13
1.5	Closed loops' intersections with wavefronts with orientations	15
1.6	Greenberg-Hastings model in a square with blocks	17
1.7	Greenberg-Hastings Model for Narrow hallways space in time	20
2.1	A simplicial complex and its 1-skeleton in black	26
2.2	Open star of the node x in center	26
2.3	Nerve lemma	28
2.4	A counter example of the nerve lemma	29
3.1	Counter example: non-continuous state for all time	38
3.2	A seed and a defect	39
3.3	Discontinuity in region bounded by a defect	43

3.4	GHM with no local defect	54
3.5	Single wave	56
4.1	Product space and S	65
4.2	A barrier	68
4.3	Wavefronts propagation	70
4.4	Propagation around corner	72
4.5	Add nodes to the network while maintaining the same coverage	75
5.1	Floor plan of a building that is not a subspace of 2-d plane	81
5.2	Various 2-manifolds and the possible wavefronts on them	84
5.3	3 shots with different viewpoints from a 3-d GHM in a 3-d cube	86
5.4	A wavefront around a defect in 3-d space	87
5.5	A trefoil knot	87
5.6	Defects in sequence	88
5.7	Wavefronts from a sequence of defects	89
5.8	Defects in a circular sequence	89
5.9	Wavefronts from a sequence of defects	90
5.10	Defects in sequence of a scroll ring	91
5.11	Wavefronts generated by the scroll ring defects	92

Chapter 1

Introduction

1.1 Wireless Sensor Network and Coverage Problems

A **wireless sensor network (WSN)** consists of a collection of sensors networked via wireless communications, with every sensor being a device collecting data of the environment with respect to one or more features, and returning with a signal [36, 2]. There are usually a few components built in a sensor: a signal receiver and/or a transmitter with an internal antenna or connection to an external antenna, a microcontroller, an electronic circuit for interfacing with the sensors and an energy source, usually a battery (today's sensors more often come with energy harvesting equipments, *e.g.*, a solar system). Sensors can read, *inter alia*, temperature, pressure, sound, target presence, range, and identification, by dealing with

all kinds of signals. This makes the sensing function of a sensor. Communications between individual sensors, and between system controller and sensors are carried out by signal receivers and transmitters, making the sensors talking “wireless”. Battery life is usually a big concern when designing sensors for a specific function, because in practice, when sensors are deployed, it is not expected to charge them from time to time, and different functions could have energy consumption differing by millions of times. Besides, once a sensor is out of energy, it immediately loses connection to the system, which may cause problems. Current-generation smart sensors, increasingly smaller in size, can perform data processing and computation, albeit with very limited memory and computation capability. However, constrained by the locality of sensors’ sensing function, networks of sensors have many more applications in monitoring larger domains than a single sensor, which by network we mean the collection of sensors together with the connections among them. Local information is collected and aggregated globally through the communication among sensors and between sensors and the controller. With sensors already chosen, we only worry about the topology and the protocols when designing such WSNs.

However powerful a sensor is, it has limitation in its sensing ability. We say the **coverage region** of a sensor, is the region in which a physical quality could be measured or an event triggered could be detected by the sensor. The **Coverage problem** in sensor networks has been studied for a long time. It considers whether a domain is always fully covered by the union of sensing regions of sensors, static

or mobile. Classical solutions for coverage problems [26, 22, 23] with techniques from graph theory and computational geometry, usually require detailed geometric information of every sensor, which is possible when equipped with GPS, however, is high energy consuming and low fault-tolerant: a global control for the whole system with every sensor's position is needed, which requires information exchange at a high frequency, and brings energy cost; besides, GPS is sensitive to environment conditions, especially when at indoor locations. Other aspects focus on providing a specific degree of coverage, while keeping the connectivity of the network [31]. Current approaches to coverage problems are applications of algebraic topology [11, 10]. Coverage quality is determined automatically without strong (geometric) information requirements, by only requiring local (neighboring) information, by computing homology.

A very common application of coverage problem is intrusion-detection: the network monitors an area, and reports the existence of intruders when they are detected by at least one sensor in the network. Video surveillance provides one such example. Whenever an intruder is detected by at least one of the cameras, system will be alerted. There is considerable activity in this field, focusing on different features and goals, “optimizing” networks in various senses. What concerns us most in the present work is the minimization of energy consumption, keeping in mind that sensors are almost always battery-driven. One of the most intuitive ways is constructing a sleep-wake protocol for the network, allowing sensors to alternate be-

tween higher and lower energy cost states (respectively corresponds with wake and sleep states) [3]. Usually a waking sensor is fully functional, and a sleeping sensor is either totally shut off or minor functioning, such as only receiving signals. For example, PEAS[35] provides a protocol by forcing a node who has an active neighbor to sleep for period according to exponential distribution. It is robust against node failure, however, could not guarantee, or measure the coverage with the rapid change of active sensors. The CDSWS [27] protocol uses a clustering technique to divide the sensors into multiple clusters, and selects a few sensors from each cluster to work, while maintaining nearly full coverage. ASCENT[7] allows sensors to measure their connectivity in the network in order to activate their neighbors based on those measurements. But it never allows working sensors to go back to sleep again, which ends up consuming more energy as time goes by. Compared to those works, our protocol provides the “user” a chance to determine how much energy they would allow to be consumed, as balanced against the probability for the system to succeed. Another advantage over the other protocols is it guarantees the failure of any evader following continuous path in the domain. Although our scheme requires synchronization ahead of time, and has not taken into account node and link failure yet, it provides a new approach to designing distributed sleep-wake WSN with energy constraints.

1.2 Cellular Automata

A **cellular automata**, (**CA**), is a lattice-space and discrete-time dynamical system. Spatial coordinates in the lattice (could be of any dimension) are called **cells** or **nodes**. For each node, a specific collection of nodes in the space is called its **neighborhood**, and every node in its neighborhood is its **neighbor**. Usually, it is the case that neighboring is a symmetric relation, which means if a node x is a neighbor of node y , then y is also a neighbor of x . For example, in \mathbb{Z}^2 lattice, the von Neumann neighborhood of a node with coordinate (i, j) is defined as the set of nodes attached to it, vertically or horizontally, *i.e.*, $\{(i-1, j), (i, j-1), (i+1, j), (i, j+1)\}$. The dynamics generally take values in a finite alphabet \mathcal{A} , which we call a **state space**, with $\mathcal{A} = \mathbb{Z}_2 = \{0, 1\}$ being the most common choice. The dynamics are for local, in that the update rule for a node is a function of its state and the states of its spatial neighbors. An **initial state** (time $t = 0$) is selected by assigning a state for each node, typically at random. A new generation is created (advancing t by 1), according to some fixed rule that universally determines the new state of each node in terms of the current states of the node and its neighborhood. Typically, the updating rule is the same for each node and does not change over time, and is applied to the whole space simultaneously (but see asynchronous cellular automata [3] for one exception).

The history of Cellular automata goes back to 1940s, when John von Neumann was involved in the design of the first digital computers [6, 34]. By trying to design

self-control and self-repair mechanisms for a machine that works like human brain, he came up with a framework of a fully discrete universe made up of cells, following the suggestions of Stanislaw Ulam. Each cell in the discrete space is characterized by an internal state. The system of cells evolves in discrete time steps, and the rule of evolution, which is a function of the internal states of the cell and its neighbors, is universal for all the cells, and updates simultaneously. The first self-replicating CA by von Neumann was on a two dimensional square lattice [28]. The state space contains 29 states. Updating rule for the CA requires each cell to add the states of itself and its all four neighbors (with von Neumann neighborhood).

A very famous but simple example of CA that has been widely studied is named *Game of Life* by John Conway. The set up is on a two dimensional square lattice, with Moore neighborhood (the cells that are attached to the cell vertically, horizontally, and diagonally, which are in total 8 cells). The automata is equipped with the most simple state space, $\{0, 1\}$, with 0 standing for being dead and 1 for being alive. The updating rule is as follows:

1. A dead cell with exact three living neighbors comes back to life.
2. A dead cell with less or greater than three living neighbors keeps dead.
3. A living cell with less than two living neighbors dies of isolation.
4. A living cell with exact two or three living neighbors keeps alive.
5. A living cell with more than three living neighbors dies of overcrowding.

Although the rule is simple, the diversity of behavior turns out to be fairly rich, with enormous initial states. Gliders, one of the most common patterns in game of life, are arrangements of cells that essentially move themselves across the grid. Still life is another kind of pattern which does not change along time. Oscillators, different from those above two patterns, are periodic structures, which are periodic in time.

Our work focuses exclusively on **cyclic** cellular automata (CCA). The alphabet is defined to be $\mathcal{A} = \mathbb{Z}_n = \{0, 1, \dots, n-1\}$ under modular arithmetic. One denotes the (discrete) collection of nodes as X and denotes a **state** at time t as $u_t : X \rightarrow \mathbb{Z}_n$. The updating scheme for a general CCA is increments states in \mathbb{Z}_n , assuming that some excitation threshold is exceeded (typically when there is at least one neighbor in the advanced state). More specifically, $u_{t+1}(x) = u_t(x) + 1$ if certain criteria concerning the states of the immediate neighbors of x , $\mathcal{N}(x)$, are met. Such systems tend to cause periodic or cyclic behavior, spatially distributed and organizing into waves.

In [15], waves (periodic structures) are observed. According to their explanation of dynamics, there are four stages in the evolution of a cyclic cellular automata with 14 states under any initial state, quoted as follows:

1. The vast majority of creatures have nothing to eat initially or quickly run out of food, *i.e.*, within a short time there are only a very few types that have a neighbor they can eat. However, the rare remaining active areas form critical

droplets.

2. These critical droplets remain active and, by generating wave activity, expand at a linear rate until they overrun all of the inactive debris that was present in the initial state.
3. Defects are formed, leading to the emergence of periodic spiral structures. Spirals overtake the Stage 2 wave activity at a linear rate, competing with one another for all of the available space.
4. Certain minimal defects, which we call clocks, give rise to optimally efficient demons. These demons displace less efficient spirals until every state in the lattice is regulated by a local, periodic structure of period N .

and illustrated in Figure 1.1. We notice that the dynamics gradually evolves into a periodic structure, which is evoked and kept by the spiral centers, *i.e.*, the defects (with formal definition in Chapter 3). We also have noticed that one defect is missing by comparing the 3rd and 4th picture. It is because the missing defect originally has period 16, which is larger than the neighboring defect with period 14, and got eaten up and became 14 periodic later. This is our first example of the role a defect is playing in a cyclic cellular automata.

Our second example of a CCA is called Belousov-zhabotinsky reaction, which is a classical example of non-equilibrium thermodynamics, resulting in the establishment of a nonlinear chemical oscillator. The way BZ reactions are modeled is as a

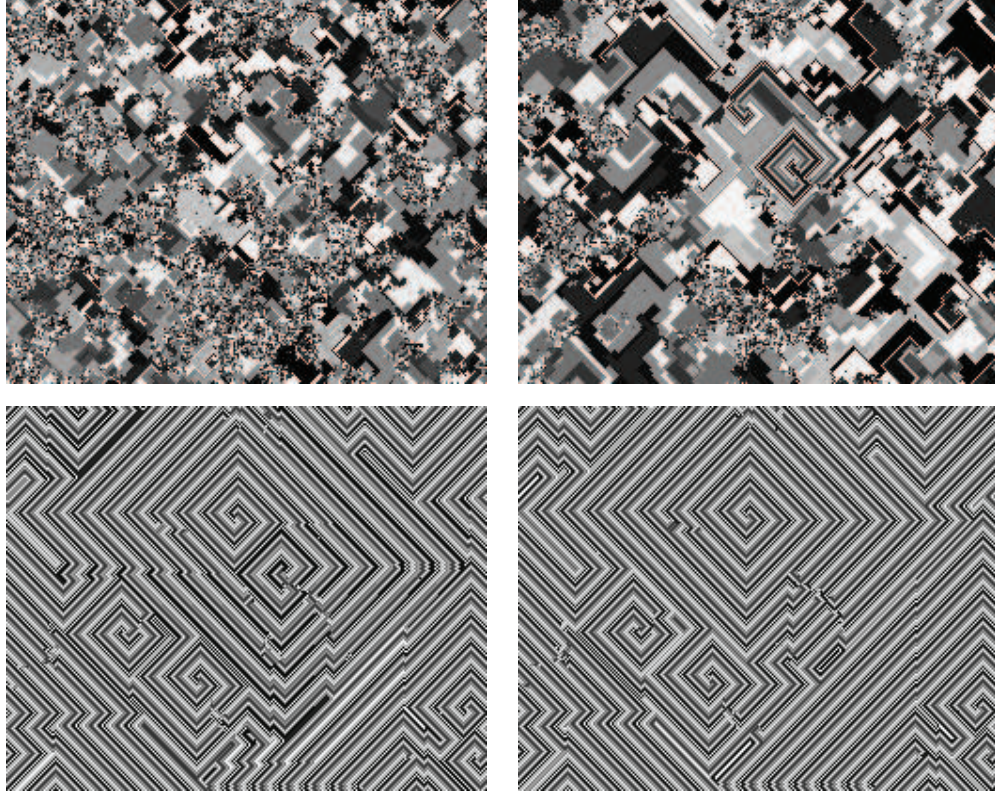


Figure 1.1: The 14 colored CCA with random initial state at time 100, 150, 300, 3000 [15].

cellular automata. The BZ medium can be seen as a massively parallel processor, where every elementary processor is a micro-volume of the medium: each micro-volume takes one of three distinguishable states corresponding to a resting, excited or refractory state of a primary component of the BZ reaction [1]. The state space and updating rule, by A.K. Dewdney in [12] is as follows: the state space is \mathbb{Z}_n . A state of zero is considered healthy (resting); a state of n is considered ill (excited); and a state from 1 to $n - 1$ is considered infected (refractory).

1. If a cell is healthy (0) then the new state is given by: $a/k_1 + b/k_2$
2. If a cell is ill (n) then the new state of the cell is healthy.
3. And if the cell is infected (1 to $n - 1$) the state given by: $s/(a + b + 1) + g$

where a is the number of infected neighbors, b is the number of ill neighbors, k_1, k_2 , and g are constants, and s is the sum of states of the cell and all of its neighbors. This is indeed a variation of CCA, if we reinterpret this mechanism with only three states: ill, healthy and infected. An ill cell automatically becomes healthy at the next step; a healthy cell gets infected to the extend depending on how ill its neighbors are; an infected cell moves towards illness, by some formula depending on its neighbors' states. Therefore, waves propagation in BZ reactions are well explained, modeled and simulated.

By comparing the waves generated by CA (§1.2) and from real experiments (§1.3), we are again confirmed that something happen in the center of spirals (de-

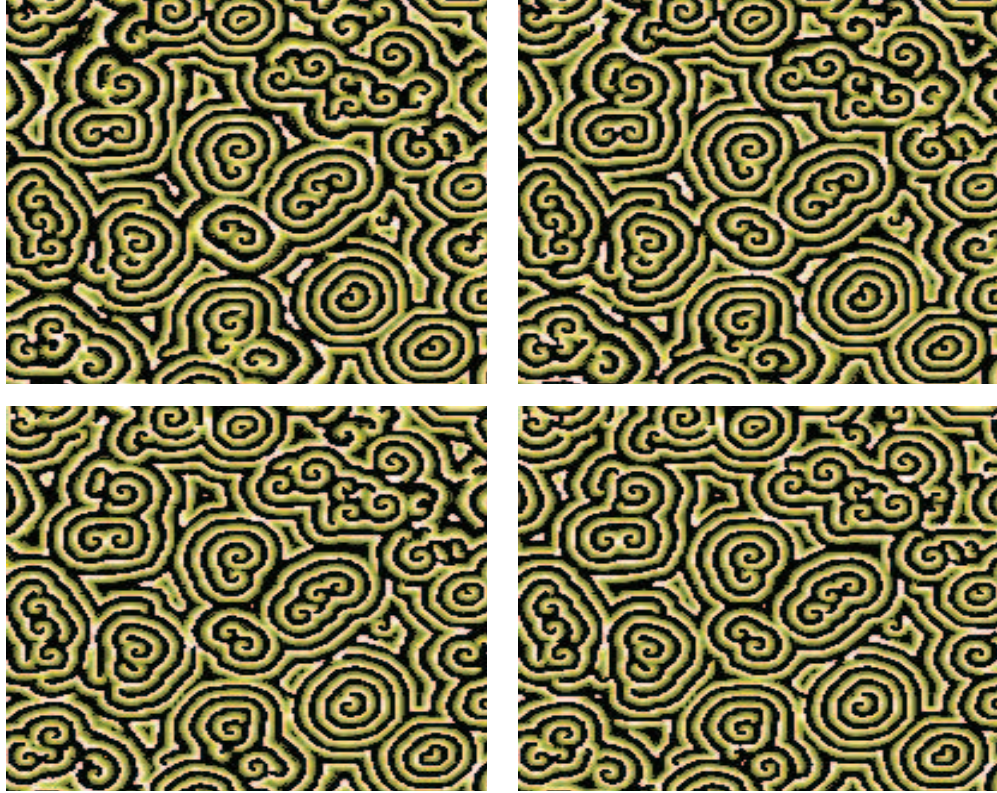


Figure 1.2: BZ reaction modeled by cellular automata with $k_1 = 2, k_2 = 3, g = 35$ [21].

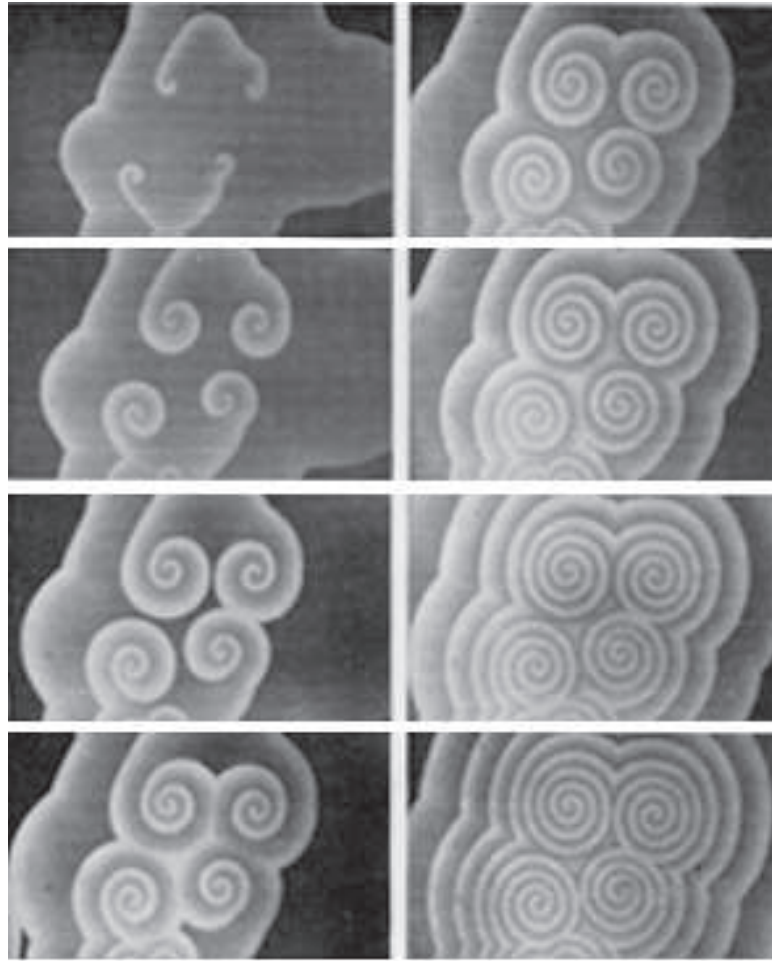


Figure 1.3: BZ reaction experiment shots by Zaikin and Zhabotinsky (1971) [37].



Figure 1.4: A closed loop enclosing a spiral center intersect with wavefronts for odd times (in red), and a closed loop not enclosing any spiral center intersect with wavefronts for even times (in blue).

fect), is deciding the behavior of all the cells (or the chemicals). These concentric waves are inclined to expand in size, unless there is a collision between waves and then annihilate. We also observed the existence of one defect would make the random initial state into an ordered periodic system.

Another observation worth thinking about is that, whenever enclosing a spiral center in a closed loop, the loop always intersects with wavefronts for odd times. However, a closed loop outside any spiral center will intersect with wavefronts for

even times (see Figure 1.4 for illustration). This classification seems not as satisfying since it is not capable of differentiating closed loops enclosing different number of spiral centers. Therefore, we count the directions of intersections in to consideration. As in Figure 1.5, we give each loop a direction. If the wave’s direction agrees with the direction of the loop (within an angle of 90 degrees), we count that as a +1, otherwise it is counted as a −1. By summing up those numbers with signs along the loop, we get a number that identifies the number of defects enclosed in the loop with sign (spirals of different orientations are treated with opposite signs). This is where we get our intuition of “degree”, which is defined in Chapter 3.

In our scenario, we will focus on one specific variation of CCA. **Greenberg-Hastings Model (GHM)** is a CCA first invented to study the spatial patterns in excitable media [19]. In this model, special significance is given to a single state (state 0), interpreted as an excitation state. The update rule for GHM is as follows:

$$u_{t+1}(x) = \begin{cases} u_t(x) + 1 & : u_t(x) \neq 0 \\ 1 & : u_t(x) = 0; u_t(y) = 1 \text{ for some } y \in \mathcal{N}(x) \\ 0 & : \text{else} \end{cases}$$

This updating scheme is interpreted as the result of two mechanisms combined, excitation and diffusion. It is therefore no surprise that there is a strong resemblance between the behavior of GHM on the plane and solutions to reaction-diffusion PDEs on planar domains [18], with both generating spiral-type waves. The states of the nodes are uniquely determined by the initial state, because the GHM is a



Figure 1.5: A closed loop with orientation has intersections with wavefronts being numbered ± 1 . -1 means the loop's orientation agrees with the propagation direction of that wavefront, and $+1$ means they are opposite. Summing these number up along the loop gives the number of spiral centers enclosed with sign.

deterministic model. Denote by \mathcal{G} the evolution operator $\mathcal{G} : \mathbb{Z}_n^X \rightarrow \mathbb{Z}_n^X$.

This Greenberg-Hastings automata on lattices has been frequently investigated in the literature as follows. Some rigorous statistical results have been proved for GHM with state space \mathbb{Z}_3 [14]. For general state space \mathbb{Z}_n , experiments have been carried out in [15], and specific features or patterns that would keep and ergodic behaviors have been studied in [17, 16, 13].

1.3 Cyclic Network Automata and Simulations

For a network graph with nodes set X , we automatically have the definition of neighborhood: two nodes in X are neighbors if and only if they are connected by an edge. Therefore, a network automata taking values in \mathcal{A} is defined by evolution operator $\mathcal{G} : \mathcal{A}^X \rightarrow \mathcal{A}^X$. We applied Greenberg-Hastings automata on a network randomly sampled in a 200×200 square with two blocks as forbidden areas, following [3]. Starting with a random initial state, we have Figure 1.6 after 200 steps.

An interesting and useful fact is, the automata on lattice and network are of no difference in dynamic. Figure 1.6 is a good example for this: spiral wave pattern is the same as them in lattice space in [19], although the underlying structure (cellular or network structure) has changed. Therefore we propose the statement, that the behavior of such an automata is not determined by the cellular structure, but the topology of the underlying space. As a result of the statement, we can extend the concept of cellular automata to **network automata**.

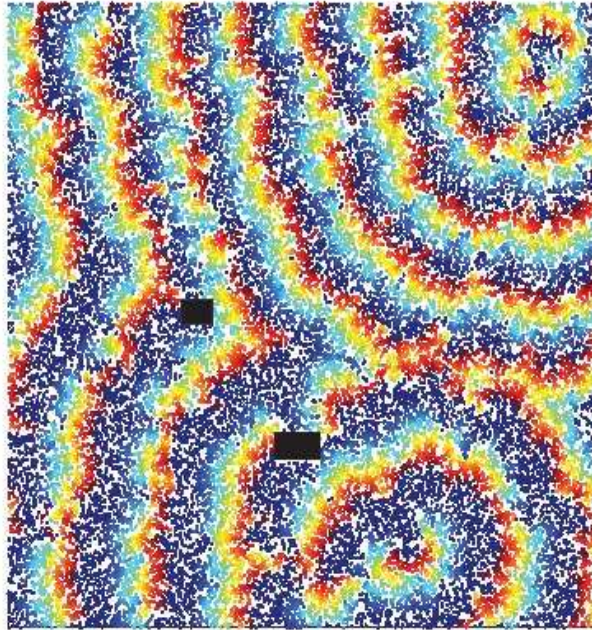


Figure 1.6: Greenberg-Hastings model in a square with blocks: black blocks are regions where no sensors are located, waves behave the same as with no blocks.

The inventive paper [3] applied the Greenberg-Hastings automata on a two dimensional plane to generate “waves” of on-state sensors for intruder detection. In general, this specific automata assigns to each sensor the state space \mathbb{Z}_n (the cyclic group on n elements), and the sensors update their states by advancing one automatically, except in state 0, in which case update to state 1 is induced by contact with at least one neighbor in state 1. A few authors have considered what happens on a graph as opposed to a lattice [25]; this is the starting point for the application in [3] to sensor networks. The main features and results of Baryshnikov-Coffman-Kwak include:

1. The CCA runs on a random graph instead of a lattice, where nodes represent sensors and edges communication links.
2. The network is completely non-localized and coordinate-free.
3. The CCA with random initial conditions generates the familiar spiral-like **wavefronts** that sweep the whole domain with on-state sensors, giving a decentralized scheme for low-power dynamic barrier coverage.
4. Parameters such as wavelengths are controllable.
5. For planar domains with small obstacles, the wavefronts behave as if there are no obstacles at all (see Figure 1.6): waves propagate through, making the problem of undersampling ignorable.

Our research begins where [3] ends, by investigating what happens when this protocol is adapted to an *indoor* network where the geometry and topology are not those of an open plane (perhaps dotted with obstacles), but rather a system of fairly narrow hallways connected with a non-trivial topology: a “fat” planar graph.

Figure 1.7 illustrates the dynamics of the GHM on a specific indoor network. The network is built on **narrow hallways** modeled as a metric space with Euclidean metric; the neighborhood of a node is defined as the collection of nodes within distance r . We parameterize the system of 16250 nodes inside a 200×200 square with $n = 20$ (recall, the size of state space \mathbb{Z}_n) and $r = 1.5$. The colors are representing states, with dark blue representing state 0. At time=0, it is in an unordered initial state. During the first several time steps, generally until time 25, the ratio of nodes with states 0 grows, as a result of the fact that states grow steadily until they reach 0 and wait for a stimulation from its neighborhood. At around time 45, spiral patterns become clearer visually, from top left, bottom and middle right. Those spiral “seeds” propagate waves along hallways. **Wavefronts**, consisting of the nodes with state 0, sweep through the domain, traveling “intelligently,” turning corners, etc. When wavefronts coming from different directions meet, they annihilate. After enough steps (about 250), wavefronts starts periodically sweeping the whole space.

This protocol of “going to sleep by contact and waking up after a fix amount of time” has some properties that make it ideal as a intruder detection sensor network.

1. The system has tunable energy efficiency. Set state 0 to be the waking state,

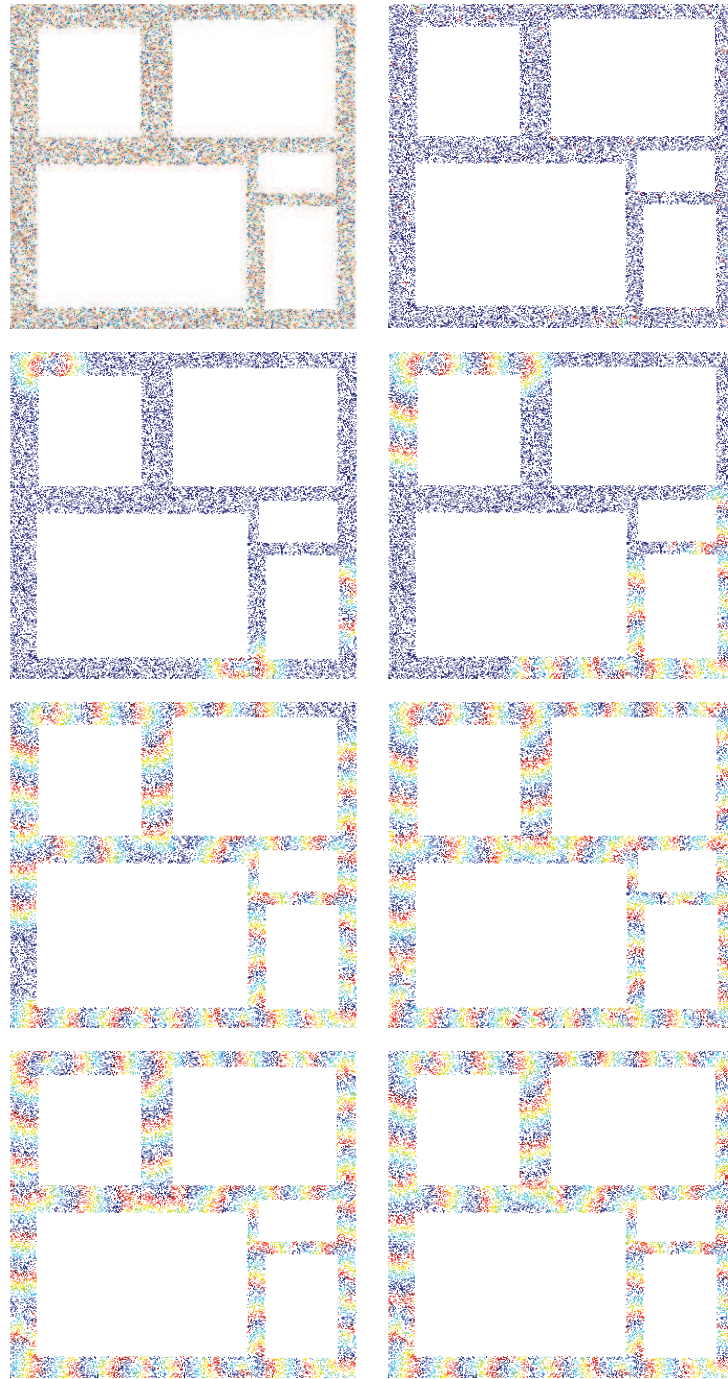


Figure 1.7: Greenberg-Hastings Model for Narrow hallways space at time 0, 20, 45, 90, 150, 200, 250, 350.

with all other states $2, 3, \dots, n - 1$ denoting sleep mode. Those sleep-mode sensors are doing nothing but advancing their states by 1 for every time step. This could be done with very low energy consumption because they only have to follow clock clicks with no computing, sensing, or transmitting. Intrusion-detection is performed by the wake-state 0 nodes. After a sufficient time elapse, only a fraction (about $1/n$) of the total sensors will be in wake mode at any given time: the larger n , the less energy consumed.

2. The wave length is generally fixed no matter which source it is from, assuming the nodes are uniformly distributed: it appears to depend linearly on n . For bigger n , longer wave length is generated, but seeds are generated with smaller probability and longer generation time. This makes a trade-off between energy consumption and system success.
3. If we are given some specific sensors, say, with a big enough sensing radius ϵ , then we will see that nodes in wavefronts form barriers, cutting the hallway into disconnected pieces. We notice that the wavefronts efficiently sweep the corridors. Any intruder between two barriers has to follow the direction wavefronts propagate in order not to be detected immediately, but still is not able to survive for all the time because it will be detected by a coming wavefront in the opposite direction.
4. Since the wave length depends linearly on n , then for a static evader, the

time it takes to catch the evader also depends linearly on n . However, for a mobile evader, the longest time it is possible to survive does not depend on the wavelength, thus the state number n at all. This proposition is concluded from the observations: an evader between two consecutive wavefronts has to follow the propagation of these wavefronts in order not to be detected. The survival could last until an upcoming wavefront annihilate with the wavefront behind the evader. Therefore, when measuring the quality of such systems by considering the longest time an evader could survive, only the network structure need to be investigated, because the wavefront annihilating location only depends on the network. Furthermore, for a randomly generated network on a fixed domain, the annihilating location depends only on the spiral centers' locations, therefore, the surviving time actually only depends on the underlying space and the locations of spiral centers.

1.4 Contributions

Questions arise with the observations, such as: Is there any invariant associated with the initial state, since the system always evolves into a time-periodic scenario? Why are small obstacles ignorable in the wave traveling but large obstacles (walls) changing the directions of wave propagation? What role is the topology, or geometry of the underlying space playing in the system? We will first introduce a few technical terms from algebraic topology in Chapter 2, then use them to find the invariants in

Chapter 3, officially build the mathematical model and study it as well in Chapter 4, and finally conclude in Chapter 5.

We conclude our contributions of this thesis as following:

1. In our model, we analyze a cyclic network automata applied on narrow hallway spaces with nontrivial topology, instead of 2-d squares, with trivial topology.
2. We give a detailed classification of the equilibrium of cyclic network automata, according to the existence of defects in Chapter 3.
3. Defects are classified into two different types, local and global, according to their supports' homological classes in Chapter 3.
4. Continuous states are affiliated with cohomology classes. A theorem explaining every cohomology class is realizable is given in Chapter 3.
5. We build an evasion game model in Chapter 4, and provide a main theorem, Theorem 4.2.13, discussing the conditions for an evader always losing the evasion game.

Chapter 2

Background

In this chapter, preliminaries from algebraic topology will be introduced. Basic geometry and topology concepts are assumed as known, such as manifolds, homotopy, homotopy type, and fundamental group. We will start from all kinds of complices, which is the topological objects we build for networks, and then go to (co)homology group and degree since they are the tools we use later.

2.1 Flag Complex, Rips Complex and Nerve

Complex is one topological object we will use frequently. It has a very simple structure (built combinatorially), therefore easy to count, manipulate, and do computation on. And for the most important reason, network graph is in spirit a 1-complex, with higher dimensional complex structures based on it improving our understanding about the network. Now we get our hands on simplex and complex.

Definition 2.1.1. A k -simplex is a k dimensional polyhedron which is the convex hull of its $k + 1$ vertices. A simplex with orientation is denoted as $[v_0, v_1, \dots, v_k]$, whose vertex set is $\{v_0, v_1, \dots, v_k\}$. Notice that the orientation of a simplex depends on the sign of permutation on vertices, *i.e.*, for a permutation (i_0, i_1, \dots, i_k) of $\{0, 1, \dots, k\}$, $[v_{i_0}, v_{i_1}, \dots, v_{i_k}] = \text{sgn}(v_{i_0}, v_{i_1}, \dots, v_{i_k})[v_0, v_1, \dots, v_k]$.

Definition 2.1.2. A **simplicial complex** is the union of simplices by gluing them together along faces of same dimension. The dimension of a simplicial complex is the highest dimension of its simplices.

Definition 2.1.3. A **subcomplex** of a complex C is a complex whose simplices are all simplices of C . And a simplicial subcomplex of a simplicial complex C is a subcomplex of C which is also a simplicial complex.

For a simplicial complex, its k -**skeleton** is its subcomplex with all its simplices of dimension k . For example, the 1-skeleton of a simplicial complex C is the graph with every vertex and every edge included, as in Figure 2.1.

Definition 2.1.4. For a simplicial complex C , and a vertex x , define the **open star** of x , as the union of x and all open simplices with x as one vertex. Denote this as $\text{St } x$.

As opposed to the operation from a simplicial complex to its 1-skeleton, where the dimension of complex decreases to 1, we have an inverse procedure that makes a 1-complex to higher dimensional.

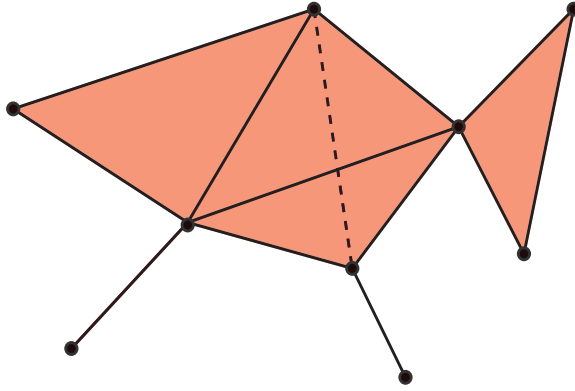


Figure 2.1: A simplicial complex and its 1-skeleton in black

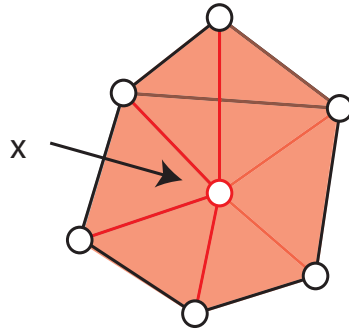


Figure 2.2: Open star of the node x in center: red regions (open faces, open edges, and nodes) is the open star of the node in the center

Definition 2.1.5. Given an undirected graph G , with vertex set V and edge set E , the **flag complex** C_f , of G is the abstract simplicial complex whose k -simplices correspond to unordered $(k+1)$ -tuples of vertices in V which are pairwise connected by edges in E .

Figure 2.1 also illustrated a graph (in black) and its flag complex, which is a 3-complex.

Definition 2.1.6. Given a set of points X in a metric space and a fixed parameter $r > 0$, the **Vietoris-Rips complex** (Rips complex for short) of X , $\mathcal{R}_r(X)$, is the abstract simplicial complex whose k -simplices correspond to unordered $(k+1)$ -tuples of points in X which are pairwise within distance r .

Both flag complex and Rips complex are simplicial complicies. But compared to a flag complex, the Rips complex requires metric information about the space. For a Rips complex of point set X , its 1-skeleton has the flag complex exactly the same as the Rips complex.

These two kinds of simplicial complicies are frequently borrowed in order to study coverage problems for sensor networks. The topological property of them are closely related to the covering quality. However, by themselves, it is not enough to tell whether the coverage is satisfying or not, because the coverage information is not incorporated yet. Therefore, we refer to another simplicial complex, the nerve, for help.

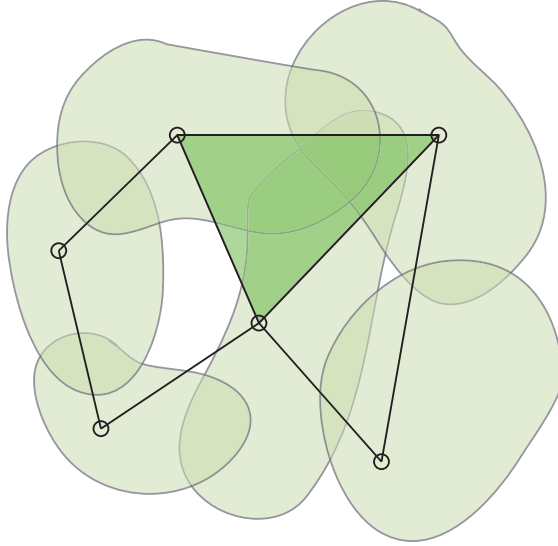


Figure 2.3: Nerve lemma: the union of patches (in light green) has the same homotopy type as the nerve (in dark green).

Definition 2.1.7. For a collection of open sets in topological space X indexed by I , $\mathcal{U} = \{U_i | i \in I\}$, the nerve, $N(\mathcal{U})$, is defined as the simplicial complex whose k -simplices correspond to nonempty intersections of $k + 1$ open sets in \mathcal{U} .

In other words, the nerve of an open covering is the information about how the open sets intersect with each other. We have a lemma about the quality of coverage through investigating the nerve.

Theorem 2.1.8. (Nerve lemma). *For an open cover \mathcal{U} of X , if every nonempty intersection of a sub collection of \mathcal{U} is contractible, then the union of elements of \mathcal{U} has the same homotopy type as $N(\mathcal{U})$.*

See Figure 2.3 as an illustration of the nerve lemma. The nerve lemma has an important condition about the intersections of sub collections of the open sets: each

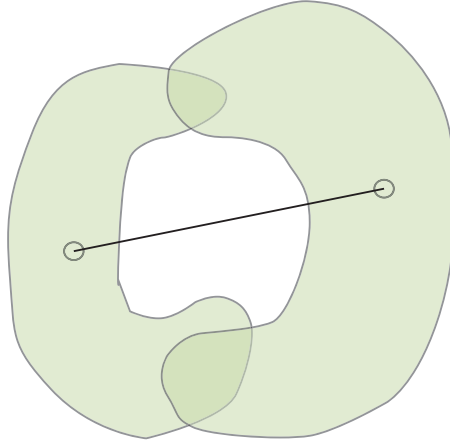


Figure 2.4: A counter example of the nerve lemma: the two open sets (in light green) has a nonempty intersection but not contractible (has two connected componets). Their union (has the homotopy type of a circle) does not has the same homotopy as the nerve, which is contractible.

should be contractible. Refer to the following example in Figure 2.4 as an counter example.

Nerve lemma has extensive applications in coverage problems of sensor network. For a sensor network such that the coverage is a collection of open sets, which satisfying the conditions of nerve lemma, whether the total coverage has a hole or not depends absolutely on the nerve, which could be examined with much less effort.

2.2 Homology, Cohomology and Degree

In last section we have explored some varieties of complices, which is the object we are going to study instead of graphs. And the tools we are going to use is (co)homology.

For here, we will define simplicial (co)homology for convenience. If other forms (singular, cellular, Čech) of (co)homology are desirable, please refer to [20].

For a simplicial complex X , the k -**chain** C_k , with coefficients in \mathbb{Z} (assumed for simplicity), is a vector space over \mathbb{Z} with basis as the collection of all n -simplices of X . The **boundary map** $\partial_k : C_k \rightarrow C_{k-1}$ is a linear transformation taking every k -simplex $[v_0, v_1, \dots, v_k]$ to its boundary $\sum_{i=0}^k (-1)^i [v_0, \dots, \hat{v}_i, \dots, v_k]$, where \hat{v}_i means omitting the vertex v_i in the sequence.

Here we have the following very important lemma:

Lemma 2.2.1. *The composition of two successive boundary maps is zero: $\partial_{k-1} \circ \partial_k = 0$ for all k .*

Remark 2.2.2. The proof of Lemma 2.2.1 is purely combinatorics, by applying the linear map on each simplex. But itself is deep.

Now we have **chain complex** as a sequence of graded chains with boundary maps as transformations:

$$\cdots \rightarrow C_{k+1} \xrightarrow{\partial_{k+1}} C_k \xrightarrow{\partial_k} \cdots \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0 \quad (2.2.1)$$

, where $\partial_k \circ \partial_{k-1} = 0$. We call a k -chain a k -**cycle** if it is in $Z_k = \ker \partial_k$. A k -cycle

is a **boundary** if it is in $B_k = \text{im } \partial_{k+1}$ (keep in mind that $B_k \subset Z_k$ because of Lemma 2.2.1).

Definition 2.2.3. Given a chain complex, the k th **homology group** is defined as $H_k = \ker \partial_k / \text{im } \partial_{k+1}$. Elements of H_k are called **homology classes**. Two cycles representing the same homology class are said to be **homologous**.

Remark 2.2.4. Simplicial homology is defined in Definition 2.2.3. For definition of singular homology, we only need to change the way we define simplex. A **singular simplex** of dimension k in space X is a map $\sigma : \Delta^k \rightarrow X$, where $\Delta^k = \{(t_0, \dots, t_k) \in \mathbb{R}^{k+1} \mid \sum_i t_i = 1, t_i \geq 0 \forall i\}$ is a standard k -simplex. Accordingly the k -chain $C_k(X)$ is defined as vector space over \mathbb{Z} with the collection of singular simplices as basis, and boundary map $\partial_k : C_k(X) \rightarrow C_{k-1}(X)$ is defined as: $\partial_k(\sigma) = \sum_{i=0}^k (-1)^i \sigma|_{[v_0, \dots, \hat{v}_i, \dots, v_k]}$. Homology defined in both way are equivalent.

For a space X , and a subspace $A \subset X$, the **relative chain** of dimension k , denoted as $C_k(X, A)$, is defined as the quotient group $C_k(X)/C_k(A)$. Boundary map ∂_k induces a quotient boundary map from $C_k(X, A)$ to $C_{k-1}(X, A)$. **relative cycle** and **relative boundary** are defined respectively as $\ker \partial$ and $\text{im } \partial$. Accordingly the k th **relative homology group** is the k th homology group of chain complex $\dots \rightarrow C_k(X, A) \xrightarrow{\partial_k} C_{k-1}(X, A) \rightarrow \dots \xrightarrow{\partial_1} C_0(X, A) \rightarrow 0$.

If $f : X \rightarrow Y$ is a map between two spaces, then it would induce a homomorphism from $C_k(X)$ to $C_k(Y)$ for all k . This homomorphism (still denoted as f) is a

chain map, which commutes with boundary map, therefore induces a homomorphism from $H_k(X)$ to $H_k(Y)$, which is called the **induced map** of f , denoted as f_* .

Cohomology is the dual of homology with the homomorphisms going in the opposite direction.

Consider a chain complex (C_*, ∂) , its dual is defined as a **cochain complex**, (C^*, d) , with C^k the **cochain**, which is the dual of C_k , $\text{Hom}(C_k, \mathbb{Z})$, and $d_k : C^k \rightarrow C^{k+1}$ the **coboundary map**, which is the dual of ∂_{k+1} . By Lemma 2.2.1, $d \circ d = 0$. In simplicial case, C^k is the dual vector space of C_k with a basis B , therefore is also a vector space over \mathbb{Z} with a basis $\{\phi_\beta | \phi_\beta(\beta) = 0; \phi_\beta(\alpha) = 1, \forall \alpha \neq \beta, \alpha, \beta \in B\}$. Accordingly, a boundary map applied on a cochain ϕ is as $d_k \phi([v_0, \dots, v_{k+1}]) = \phi(\partial_{k+1}([v_0, \dots, v_{k+1}])) = \sum_{i=1}^{k+1} (-1)^i \phi([v_0, \dots, \hat{v}_i, \dots, v_{k+1}])$. Similar as the definitions for cycle and boundary, a cochain is a **cocycle** if in $\ker d$, and a **coboundary** if in $\text{im } d$.

Definition 2.2.5. Given a cochain complex, the k th cohomology group $H^k = \ker d_k / \text{im } d_{k-1}$. Elements of H^k are called **cohomology classes**, and two cocycle in the same class are said to be **cohomologous**.

The following paragraphs are contributed to the concept of topological degree, as which we are going to develop a similar tool in Chapter 3.

Definition 2.2.6. For a continuous map $f : S^n \rightarrow S^n$, it induces a homomorphism $f_* : H_n(S^n) \rightarrow H_n(S^n)$, where $H_n(S^n) = \mathbb{Z}$. Therefore, f must be of the form

$f_*(\alpha) = d\alpha$ for some fixed integer d only depending on f , this integer d is called the **degree** of f , denoted as $\deg f$.

Remark 2.2.7. In the case of $n = 1$, degree is also called **winding number**, representing how many rounds the image goes around a circle as it goes around the circle once in the domain. For example, a map $f : S^1 \rightarrow S^1$ ($S^1 \subset \mathbb{C}^2$) where a complex number s is mapped to s^2 is a map of degree 2.

Proposition 2.2.8. *Degree is a homotopy invariant, which means if $f \simeq g$ are two homotopic maps, then $\deg f = \deg g$.*

Degree could be generalized to topological spaces whose n th homology is a free abelian group (compact oriented manifolds).

Definition 2.2.9. For a topological space X , and non-negative integer n , if $H_n(X)$ is a free abelian group with a homological class $[\alpha] \in H_n(X)$, then for any continuous map $f : X \rightarrow S^n$, with a chosen generator β for $H_n(S^n)$, the degree of f on subgroup generated by $[\alpha]$ is defined as the integer d such that $f_*(\alpha) = d\beta$. Denote this as $\deg(f, \alpha)$.

Remark 2.2.10. The generalized degree is well-defined: if α_1, α_2 are homologous elements in $H_n(X)$, then $\deg(f, \alpha_1) = \deg(f, \alpha_2)$, because they generate the same subgroup.

Lemma 2.2.11. *Degree is additive: for two homology classes $[\alpha_1]$ and $[\alpha_2]$ in $H_n(X)$, $\deg(f, \alpha_1 + \alpha_2) = \deg(f, \alpha_1) + \deg(f, \alpha_2)$.*

Proof. Since f_* is a homomorphism, $f_*([\alpha_1] + [\alpha_2]) = f_*([\alpha_1]) + f_*([\alpha_2])$. Therefore,
 $\deg(f, \alpha_1 + \alpha_2) = \deg(f, \alpha_1) + \deg(f, \alpha_2)$. □

Chapter 3

Dynamic Features of Greenberg-Hastings Automata

This chapter analyzes the dynamics of GHA on networks, by focusing on states of nontrivial degree, which generate spiral waves patterns. Degree is later proved to have one to one correspondence with the cohomology of the underlying space, which allows us to manipulate the waves.

3.1 Degree and Defect

We reprove certain results from the CCA literature [15, 14] in the more general setting of network (as opposed to lattice) systems. Our perspective is that a CCA is a discrete-time network-based dynamical system. From observation, the interesting dynamical features associated with the GHM are time-periodic. We therefore focus

our efforts on understanding time-periodic states.

First of all, we give notations: \mathcal{D} is the underlying domain. X is the collection of nodes in \mathcal{D} . The network graph is denoted as G .

Definition 3.1.1. An **orbit** of a node $x \in X$ under GHM with an initial state u_0 is the time-sequence of states $(u_t(x))_{t=0}^\infty$. A node x is said to be **K -periodic** if its orbit satisfies $u_{t+K}(x) = u_t(x)$ for some integer $K > 0$ and all t . A node x is said to be **eventually periodic** if its orbit satisfies $u_{t+K}(x) = u_t(x)$ for some $K > 0$ and all sufficiently large t .

We notice from the simulations that after the dynamics are organized into waves, the nodes supporting the waves are with a smooth structure, in that the colors (states) of neighbors are close. Therefore, we bring up with the concept of continuity.

Definition 3.1.2. A state u on a subgraph G' of G is **continuous** if for every pair of neighbors x, y in G' , $|u(x) - u(y)| \leq 1$.

Definition 3.1.3. A node x is **subordinate to** a neighbor y at time t if their states at that time satisfy $u_t(y) = u_t(x) + 1$ (where, recall, all addition is in \mathbb{Z}_n).

Lemma 3.1.4. *Subordinate nodes will remain continuous for all future time.*

Proof. It suffices to assume a subgraph consisting of a single edge with nodes x and y . Assume that $|u_t(x) - u_t(y)| = 1$. Consider the set $S = \{z \in X | u_{t+1}(z) =$

$u_t(z)\} \subset u_t^{-1}(0)$. Depending on membership in S ,

$$(u_{t+1}(x) - u_{t+1}(y)) - (u_t(x) - u_t(y)) = \begin{cases} 0 & x, y \in S \text{ or } x, y \notin S \\ 1 & x \notin S \text{ and } y \in S \\ -1 & x \in S \text{ and } y \notin S \end{cases} \quad (3.1.1)$$

So $|u_{t+1}(x) - u_{t+1}(y)|$ will exceed 1 only if $u_t(x) - u_t(y) = 1$, $((\mathcal{G}(u))(x) - (\mathcal{G}(u))(y)) - (u(x) - u(y)) = 1$ or $u(x) - u(y) = -1$, $((\mathcal{G}(u))(x) - (\mathcal{G}(u))(y)) - (u(x) - u(y)) = -1$. The first case is equivalent to $u(x) = 1, u(y) = 0$ and $x \notin S, y \in S$, which is impossible because y has neighbor x in state 1, and will not stay in state 0 for the next step, thus not in S ; the second case is the symmetric case which by the same argument is not possible either. Then $|(\mathcal{G}(u))(x) - (\mathcal{G}(u))(y)| \leq 1$, which makes $\mathcal{G}(u)$ also continuous. \square

Corollary 3.1.5. *If a node x is subordinate to a neighbor y that is n -periodic at time t , then node x is n -periodic ever since t . Any node subordinate to an eventually periodic node is eventually periodic.*

Proof. Suppose x reaches 0 for the first time (after t) at time t_0 . By the scheme of \mathcal{G} , $u_{t_0}(y) = 1$. Therefore, all we need to prove is for any non-negative integer k , $u_{t_0+kn+1}(x) = 1$. We have already proved $u_{t_0+1}(x) = 1$ because it has a neighbor y in state 1 at that moment. Suppose the statement holds for a particular k_0 , *i.e.*, $u_{t_0+k_0n+1}(x) = 1$, then $u_{t_0+(k_0+1)n}(x) = 0$. By periodicity, $u_{t_0+(k_0+1)n}(y) = 1$, thus $u_{t_0+(k_0+1)n+1}(x) = 1$, which makes the statement hold at $k_0 + 1$. By induction, the statement holds for all k . \square

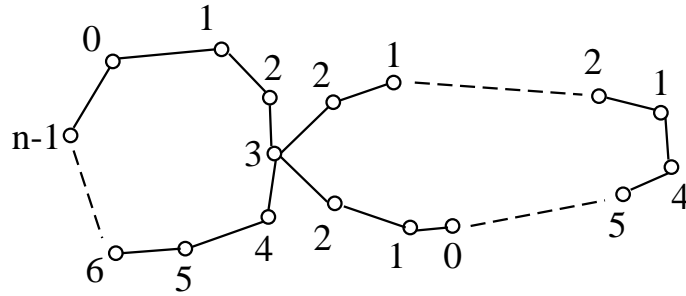


Figure 3.1: Counter example: non-continuous state for all time.

Corollary 3.1.6. *Continuity is forward-invariant: continuous states remain continuous in time.*

Proof. According to Corollary 3.1.5, two neighbors that are subordination will remain continuous. For one step forward, two neighbors that are of the same state will either remain the same state, or be offset by state 1, which means subordination, thus also continuous. □

However, it is *not* necessarily the case that all initial conditions converge to a continuous state (even in a connected compact network). See, for example, Figure 3.1: every node has period n , and the two nodes on the right end have states always differ by 3. Thus this is never a continuous network.

In order to define the network version of degree, we have to first of all give a definition of cycle. The concept of 1-cycle is borrowed from topology to describe an end connecting 1-d path.

Definition 3.1.7. We call a formal linear combination α of edges $\alpha_i = [a_i, b_i], i =$

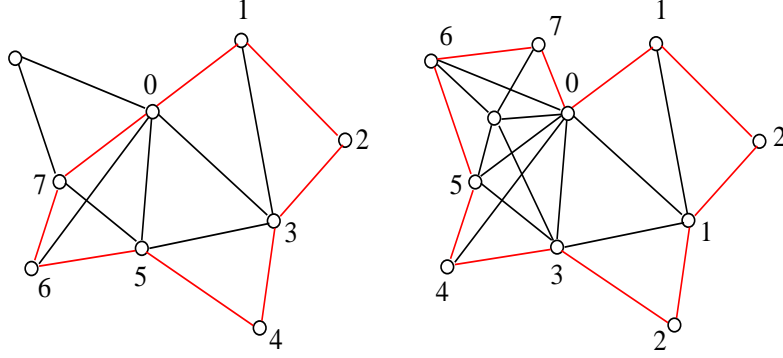


Figure 3.2: A seed (left) and a defect (right) for $n = 8$ on cycles in light red.

$1, \dots, K$ a **cycle** if the boundary of α , $\partial\alpha = \sum_{i=1}^K (b_i - a_i)$ is 0. A cycle is called a **loop** if $b_i = a_i + 1$ for $i = 1, \dots, K - 1$, and $b_K = a_1$.

As a remark, a loop is a cycle, and a cycle is the sum of one or more loops. We also remark that the set of cycles Z has the structure of an abelian group: one can add cycles and scale them by (integer) coefficients.

The following definition is a network-theoretic version of the lattice-based analogue from, *e.g.*, [15].

Definition 3.1.8. A state $u : X \rightarrow \mathbb{Z}_n$ contains a **seed** if the network contains a loop $\sum_{i=0}^{K-1} [x_i, x_{i+1}]$ ($x_0 = x_K$), for which $u(x_i) = i \pmod n$.

By definition, the length K of a loop that makes a seed has to be a nonzero multiple of n , because $u(x_0) = u(x_K), K = 0 \pmod n$. Every node on a seed has period n , because it always has a neighbor of state 1 on the seed when it reaches state 0.

Lemma 3.1.9. *If an initial condition u_0 on a connected compact network X con-*

tains at least one seed, then all nodes are eventually n -periodic.

Proof. Let the set of n -periodic nodes in X be P_t , and state at time t be u_t . Suppose the loop $\sum_{i=0}^{K-1} [x_i, x_{i+1}], x_0 = x_K$ makes a seed in initial condition, then P_0 is nonempty, with $(x_i)_0^{k-1}$ as a subset. P_t is non-descending with respect to time t , $P_0 \subset P_1 \subset \dots \subset P_t \subset P_{t+1} \subset \dots$. For a node x that is not n -periodic that has at least one neighbor that is n -periodic at time t_0 , if x never gets to be n -periodic, it means for any t positive, there exists some $s \geq t$ such that $u_{s+1}(x) \neq u_s(x) + 1 \pmod n$. It will induce that $u_{s+1}(x) = u_s(x) = 0$, which is saying x gets to stay in state 0 for a while from time to time. But the neighbors of x that are n -periodic are advancing their states by 1 at every time step, this will make the face difference between them and x bigger and bigger until it reaches $1 \pmod n$. When such an offset by 1 appears, a subordination between x and one of its n -periodic neighbors is built up, which makes x periodic ever since as a result of corollary 3.1.5. Therefore every node which has at least one neighbor that is periodic n will be periodic n after a finite amount of time (no longer than n). By the above argument and the fact that the network is connected, for any t , if $P_t \neq X$, then $P_t \subsetneq P_{t+n}$. On the other hand side, since X is compact, there exists a time T , such that

$$\bigcup_{t=0}^{\infty} P_t = P_T$$

Therefore, the whole system is in n -periodic state since time T . □

We see in the above arguments that a loop which makes a seed at one moment will support a seed forever with the dynamics. The key feature that is invariant

under the dynamics is the concept of “winding number”, which records how many rounds it goes through while chasing continuously on a loop. We will define this as degree and extend the concept to all cycles.

Definition 3.1.10. For a given network X and a state $u \in \mathbb{Z}_n^X$, if u is continuous on a cycle $\alpha = \sum_{i=1}^K [a_i, b_i]$, then the **degree** of u on this cycle is defined as

$$\deg(u, \alpha) := 1/n \sum_{i=0}^{k-1} (u(b_i) - u(a_i))$$

where the summands are forced to be -1 , 0 , or 1 , and the sum is ordinary addition (not $\pmod n$).

Definition 3.1.11. We call a cycle $\alpha = \sum_{i=1}^K [a_i, b_i]$ in the network X a **defect** for some state $u \in \mathbb{Z}_n^X$ if the degree of u on this cycle is nonzero.

An example of a defect is as in Figure 3.2. The concept of a defect is a generalization of a seed, in the sense that it has nonzero degree. The term “degree” defined here is consistent with the use of degree in topology, which is a homotopy invariant [20]. Here, it is the discrete version of “winding number” for continuous self-maps of the circle S^1 [14], describing how many times it wraps around with direction. Similar to Lemma 5 in [14], we will prove the \mathbb{R}^2 version instead of the lattice \mathbb{Z}^2 version, presenting a necessary and sufficient condition for a continuous system not dying out.

Lemma 3.1.12. *For two cycles α and β , if a state u is continuous on both cycles, then it is also continuous on their sum $\alpha + \beta$, and $\deg(u, \alpha + \beta) = \deg(u, \alpha) +$*

$\deg(u, \beta)$.

Proof. Let $\alpha = \sum_{i=1}^K [a_i, b_i]$ and $\beta = \sum_{i=1}^L [c_i, d_i]$, then

$$\deg(u, \alpha + \beta) \tag{3.1.2}$$

$$= 1/n \left(\sum_{i=0}^{k-1} (u(b_i) - u(a_i)) + \sum_{i=0}^{k-1} (u(d_i) - u(c_i)) \right) \tag{3.1.3}$$

$$= 1/n \sum_{i=0}^{k-1} (u(b_i) - u(a_i)) + 1/n \sum_{i=0}^{k-1} (u(d_i) - u(c_i)) \tag{3.1.4}$$

$$= \deg(u, \alpha) + \deg(u, \beta) \tag{3.1.5}$$

□

Lemma 3.1.13. *For a cycle α and a continuous state u , the degree of u on this cycle is invariant under the GHM updating rule \mathcal{G} , i.e.,*

$$1/n \sum_{i=0}^{k-1} ((\mathcal{G}(u))(x_{i+1}) - (\mathcal{G}(u))(x_i)) = 1/n \sum_{i=0}^{k-1} (u(x_{i+1}) - u(x_i)) \tag{3.1.6}$$

Proof. We first prove that degree on a loop $\sum_{i=0}^{K-1} [x_i, x_{i+1}]$, $x_0 = x_K$ is invariant. As before, equation 3.1.1 holds for every pair of neighbors x_{i+1} and x_i . Since the number of pairs (x_{i+1}, x_i) with $x_i \in S, x_{i+1} \notin S$ is the same as the number of pairs with $x_i \notin S, x_{i+1} \in S$, the summation of $((\mathcal{G}(u))(x_{i+1}) - (\mathcal{G}(u))(x_i)) - (u(x_{i+1}) - u(x_i))$ is 0, which makes Equation 3.1.6 hold. Since every cycle is the sum of one or more loops, and degree is additive by Lemma 3.1.12, then it is also invariant on cycles. □

For a state u on a loop that forms a defect, if the loop bounds a region V in \mathbb{R}^2 that belongs to \mathcal{D} , we can discuss the continuity of the subnetwork in V . If the

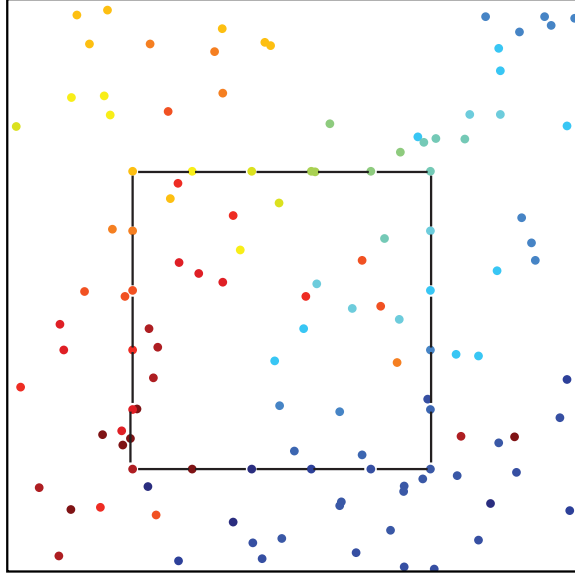


Figure 3.3: In the region bounded by a defect, the state is discontinuous.

subnetwork in V is sufficiently dense (*e.g.*, Rips complex has shadow containing V , for definition of shadow, please refer to Definition 4.2.1), we observe that the subnetwork could never reach continuity, with at least one singularity (a discontinuity) forced, as in Figure 3.3. This could be understood intuitively as a discrete version of the theorem in complex analysis, which says a holomorphic function on a domain always has integration 0 on the boundary. It would also contradict the fact that a continuous map from a contractible space to S^1 has degree 0 restricted on any loop.

Lemma 3.1.14. *Consider a state u on X with $n > 3$, and a loop $l = \sum_{i=0}^{K-1} [x_i, x_{i+1}]$, $x_0 = x_K$ in X . If the loop l is null homologous in the 2-complex built on X , and u on l makes a defect, then u is discontinuous on X . For $n \leq 3$ any state on X is continuous.*

Proof. Since l is null homologous in 2-complex built on X , $l = \partial\beta$ where $\beta = \sum_{i=1}^m \beta_i$ is a 2-chain in the 2-complex and β_i are 2-simplices. l can be deformed to a single 2-simplex through a sequence of homologous loops in X , $\sum_{i=1}^{m-j} \partial\beta_i$, $j = 1, \dots, m-1$, while the successive two loops only differ by the boundary of one 2-simplex. Suppose u is continuous on X , such operation could not change the degree at all, since at most three of the summands $u(x_{i+1}) - u(x_i)$ have been changed value up to 1. Thus the summation is at most changed by 3, which makes the degree changed by at most $3/n$, which has to be zero when $n > 3$. Thus the nonzero degree remains the same for the sequence of homologous loops, which can not be true because the degree on the boundary of a single 2-simplex has to be zero. Therefore the state on X could not be continuous.

It is trivial to see the continuity when $n \leq 3$, because any two elements in \mathbb{Z}_n differ by at most 1. □

3.2 Asymptotic behavior

Since we have defined degree and defect, we are now ready to reason the correspondence between defects and the dynamics.

Theorem 3.2.1. *For a continuous state u on a connected compact network X , the system eventually turns to all-0 state (dies out) if and only if u does not contain a defect.*

Proof. Suppose u contains a defect on cycle $\alpha = \sum_{i=1}^K [a_i, b_i]$. By lemma 3.1.13, the degree is invariant under \mathcal{G} , so it will never be 0, thus the system will never turn to all-0 state.

For the converse, we need to show for a continuous state u not dying out eventually, it has to contain a defect at the beginning of time. Firstly, it is obvious that after long enough time, in such system u_t , every node in state i must have a neighbor in state $i + 1$, for all $i \neq 0$, since $u_t = \mathcal{G}^i(u_{t-i})$. So if we start from a node x_0 , such that $u_t(x_0) = 1$, we can find a neighbor of x_0 , x_1 , such that $u_t(x_1) = 2$. Following the process, we get a sequence of nodes x_0, x_1, \dots, x_{n-1} , such that x_i and x_{i+1} are neighbors and $u_t(x_i) = i + 1 \pmod n$. If from every state 0 node, a state 1 node could be reached by jumping along neighbors which are in state 0, then following the process, we will finally reach a node has been visited before, as X is compact. In this way, we have obtained a defect in u_t , which is also a defect in u , by Lemma 3.1.13 and the fact that u is continuous.

To see that a node with state 1 could always be reached from a node with state 0 by jumping along neighbors in state 0, all we need to prove is there is no such set A of nodes with state 0, that their neighbors not in A could only be in state $n - 1$. If such A exists in u_t , then there is a proper subset of A with state 0, and state $n - 1$ on the complement in u_{t-1} . Following these procedure, we should finally obtain a set A_0 of state 0 nodes, each has at least one neighbor with state $n - 1$ and other neighbors with state 0 in u_s . Then in u_{s-1} , nodes in A_0 have to be in state $n - 1$

(by continuity), and their neighbors not in A_0 must all be in state $n - 2$, and for n steps back, in u_{s-n} , nodes in A_0 have to be in state 0, and their neighbors not in A_0 must all be in state $n - 1$. But such a u_{s-n} could not produce u_{s-n+1} under \mathcal{G} , because those state 0 nodes have no neighbors in state 1. Therefore such a set A does not exist, which makes the statement in the beginning true. \square

Since degree is invariant under the update rule \mathcal{G} , Theorem 3.2.1 can be interpreted as saying that a continuous state dies out eventually if and only if it is cohomologically trivial (see §3.3 for details on how to define the cohomology class of a state).

However, Theorem 3.2.1 itself is not enough for deciding the long term dynamics, especially when we are given a random initial state, for most of the time is not continuous. Therefore, we have the following theorem, with weaker conditions.

Theorem 3.2.2. *For an initial state u_0 on a connected compact network X , the system eventually turns to all-0 state (die out) if and only if there exist a moment t such that u_t contains at least one defect.*

Proof. Suppose there exists a time t , such that u_t contains a defect on cycle α . By Lemma 3.1.13, the degree is invariant under \mathcal{G} , so it will never be 0, thus the system will never turn to all-0 state.

For the converse, we need to show for a system not dying out eventually, it has to contain a defect at some moment t . Following the argument used in proof of Theorem 3.2.1, we could find a moment t_0 (big enough) and a node x_0 , such

that $u_{t_0}(x_0) = 1 \pmod n$. Since 1 step ago, $u_{t_0-1}(x_0) = 0 \pmod n$, there has to be a neighbor of x_0 , x_1 , such that $u_{t_0-1}(x_1) = 1 \pmod n$. And following that argument, we could find a sequence of nodes, x_0, x_1, \dots, x_{n-1} , such that x_i and x_{i+1} are neighbors and $u_{t_0}(x_i) = i + 1 \pmod n$. For node x_{n-1} , it has a neighbor was in state 1 when it was in state 0 for the last round, and this neighbor is currently in state 0 or 1. Let this neighbor be x_n . Suppose $u_{t_0}(x_n) = 1$, the above arguments automatically finds another sequence $x_n, x_{n+1}, \dots, x_{2n-1}$ of length n that is starting from x_n and have states in sequence $1, 2, \dots, n-1, 0$. Otherwise, $u_{t_0}(x_n) = 0$ and x_n has a neighbor x_{n+1} which was in state 1 when x_n was in state 0 for last round, and is currently in state 0 or 1 (continuity is forward invariant by Corollary 3.1.6), and we are back at the previous argument. Therefore, following this construction, we can have a long sequence of nodes x_0, x_1, \dots with states on neighbors are continuous, and states along the sequence non-decreasing. Because this network is compact, it has to end at some node in this sequence, which makes a defect. \square

Although we now have a classification of the dynamic equilibrium according to the existence of defects, it is still not crystal clear how the system evolves into equilibrium. Therefore, we develop the following theorem.

Theorem 3.2.3. *There exists a directed subgraph \mathcal{F} of the network that is a spanning forest rooted at seeds, with directed edges in the direction of subordination.*

Proof. Every node that is not originally a seed node will become periodic by building up a subordination with some periodic neighbor. For every non-seed node, choose

one from its neighbors via subordination and use a directed edge with itself as head to represent the relationship. This forms a directed subgraph of the network. From any non-seed node, following those directed edges with inverse directions, it has to end in a seed node, because it is a compact network. We argue that the subgraph is a tree because it contains no loop: if it did contain a loop, then the loop is comprised of non-seed nodes, but for any directed edge, the head node becomes periodic later than the tail node, which is a contradiction with being in a loop. And furthermore, we can treat the forests as rooted at seeds, which makes every edge in the direction that goes deeper in a branch to the leaves. Such structure gives the nodes a hierarchy, and since for every edge, the two end nodes have states offset by 1, we can induce the state after the system reaches equilibrium. \square

According to the above proof, we have made a point in that the growth of the forest is at most one level at a time, which means in every time step, there could not be more than one node from a same branch that becomes subordinated.

Definition 3.2.4. The depth of a node in a tree is the number of hops between the node and the root of the tree.

Starting from a uniformly randomly generated initial condition (a reasonable if idealized statistical model), the system is not guaranteed to converge to a periodic system, not including all-zero states. One sufficient condition is the existence of a seed, which we prove to be of high probability with certain reasonable assumptions (Lemma 3.2.5). It is possible that the system became messy with no wavefront ob-

servable (too many defects all around in the space, for instance). We would require those nodes that are far away (in the hop-metric) from the defects to be in state 0 at one moment (in our case, larger than the number of states is already enough). This assumption is proved later to be of high probability (Lemma 3.2.6). Under the above two assumptions, continuity in the acquired region will be guaranteed. Therefore, from now on, we limit discussion to the region far away from defects.

Lemma 3.2.5. *For a uniformly sampled network with communication radius r and fixed n on a domain consisting of fixed narrow hallways, the probability of at least one seed existing in the initial condition generated according to uniform distribution approaches 1 as the number of nodes grows.*

Proof. Divide the space into square shaped pieces \mathcal{D}_i indexed by I , with side length smaller or equal to $r/\sqrt{2}$. As the network size $|X|$ approaches infinity, the probability of there to be no less than n nodes in each \mathcal{D}_i approaches 1. For a \mathcal{D}_i with n or more nodes, the sub-network in this sub domain makes a complete graph. Therefore, there is no seed in the initial condition, if and only if the nodes do not cover all the states, which means there is at least one state missing in the initial

condition. Thus

$$\begin{aligned}
& P(\text{no seed in initial condition in } \mathcal{D}_i \text{ with } m_i \text{ nodes}) \\
& \leq \frac{n(n-1)^{m_i}}{n^{m_i}} \\
& = n(1-1/n)^{m_i}
\end{aligned} \tag{3.2.1}$$

and

$$\begin{aligned}
& P(\text{no seed in initial condition in } \mathcal{D}) \\
& \leq \prod_{i=1}^{|I|} P(\text{no seed in initial condition in } \mathcal{D}_i \text{ with } m_i \text{ nodes}) \\
& \leq \prod_{i=1}^{|I|} n(1-1/n)^{m_i} \\
& = n^{|I|}(1-1/n)^{|X|}
\end{aligned} \tag{3.2.2}$$

which approaches 0 as $|X|$ approaches infinity. \square

Lemma 3.2.6. *Starting with a fixed network and uniformly distributed initial conditions, with probability approaching 1 as the state number n grows, nodes with hop distance to all defects bigger than $2n$ will turn to state 0 after $2n - 2$ time steps.*

Proof. Suppose there is no state 1 node in u_{n-1} in the region n hops away from any defect, then there could be no state 1 or 2 node in u_n in the region $n + 1$ hops away from any defect, and with the same reason, there could only be state 0 node

in u_{2n-2} in the region $2n$ hops away from any defect. Therefore the probability that every node at least $2n$ hops away from defects are state 0 in u_{2n-2} is no smaller than that of no state 1 node at least n hops away from defects in u_{n-1} .

Now suppose there is a node x at least n hops away from any defect, and $u_{n-1}(x_1) = 1$. Such x_1 must have at least one neighbor of state 2, named x_2 , otherwise in one step before, it would not be able to update from 0 to 1. Via the same argument, there exists a sequence of nodes: $x_j, j = 1, 2, \dots, n-1$, such that x_j and x_{j+1} are neighbors, and $u_{n-1}(x_j) = j$. For one step ago, $u_{n-2}(x_j) = j-1$ for $j \neq n$, and two steps ago, $u_{n-3}(x_j) = j-2$ for $j = 2, \dots, n-1$ and $u_{n-3}(x_1) \in \{0, n-1\}$. Following such argument, back at time 0, $u_0(x_j) \in \{0, n-1, \dots, j+1\}$ for $j = 1, \dots, n-2$ and $u_0(x_{n-1}) = 0$ with at least one neighbor of state 1.

Let $I_j = \{0, n-1, \dots, j+1\}$. For a fixed node x ,

$$\begin{aligned}
 &P(\text{at least one of } x\text{'s neighbor have a state in } I_j \text{ at time 0}) \\
 &= 1 - (1 - j/n)^{|\mathcal{N}(x)|}
 \end{aligned}
 \tag{3.2.3}$$

where $|\mathcal{N}(x)|$ is the number of neighbors of node x . Suppose \tilde{N} is a universal upper

bound on $|\mathcal{N}(x)|$, then

$$\begin{aligned}
& P(u_{n-1}(x) = 1 \text{ for some } x \text{ at least } n \text{ hops away from any seed}) \\
& \leq |X| \prod_{j=1}^{n-1} (1 - (1 - j/n)^{|\mathcal{N}(x)|}) \\
& \leq |X| \prod_{j=1}^{n-1} (1 - (1 - j/n)^{\tilde{N}}) \\
& \leq |X| (1 - (1/2)^{\tilde{N}})^{(n-1)/2}
\end{aligned} \tag{3.2.4}$$

which approaches 0 as n approaches infinity. \square

For example, in a 40000 nodes network, where every node could have up to 6 neighbors and $n = 20$, the probability of a seed existing is bounded below by 0.9656, which validates the observation that in previous simulations, most of the time at least one seed was observed. As per the above two lemmas, we will always assume that at least one seed exists in initial condition, and the nodes at least $2n$ hops away from any defect will turn to state 0 after $2n - 2$ step. These two assumptions guarantee not only the system not dying out (turn into an all-zero state), but also the continuity of the system in acquired region, *e.g.*, the region far away from seeds that are once all in state 0.

3.3 Controlling the Cohomology

We have observed in simulation that sometimes there is no “local defect” continuously generating wavefronts, but the system still reaches a nonzero equilibrium, with the remaining wavefronts propagating along hallways in periodic way (see Figure 3.4 as an example). This phenomenon contributes to the existence of a “global defect”, which differs from the “local defect” in that the cycle on which the defect is supported is in a non-zero class in first homology of the Rips complex $\mathcal{R}_r(X)$, instead of a trivial one. Such an equilibrium presents a much higher portion of nodes in state 0 than those with local defects. We will try to manually generate such patterns in GHM by turning off local defects.

Such protocol is not energy efficient unless we shift state 0 to sleep state as follows: the new interpolation lets state 1 be waking state, state 2 be broadcasting state, and state 3 till 0 be sleeping state. Then most of the nodes will be sleeping after they are eventually periodic.

Recall from Definition 3.1.10, the degree (or winding number) of a continuous state on a cycle is an index measuring how many times the states cycle through the alphabet on the cycle. Therefore, after local defects are turned off by breaking the links between state 0 and state 1 nodes, the degree of a cycle which makes a nonzero class in first homology of the Rips complex $\mathcal{R}_r(X)$ is determined by the number of wavefronts already generated and their directions of propagation. In other words, degree for all cycles is determined absolutely by local defects’ location

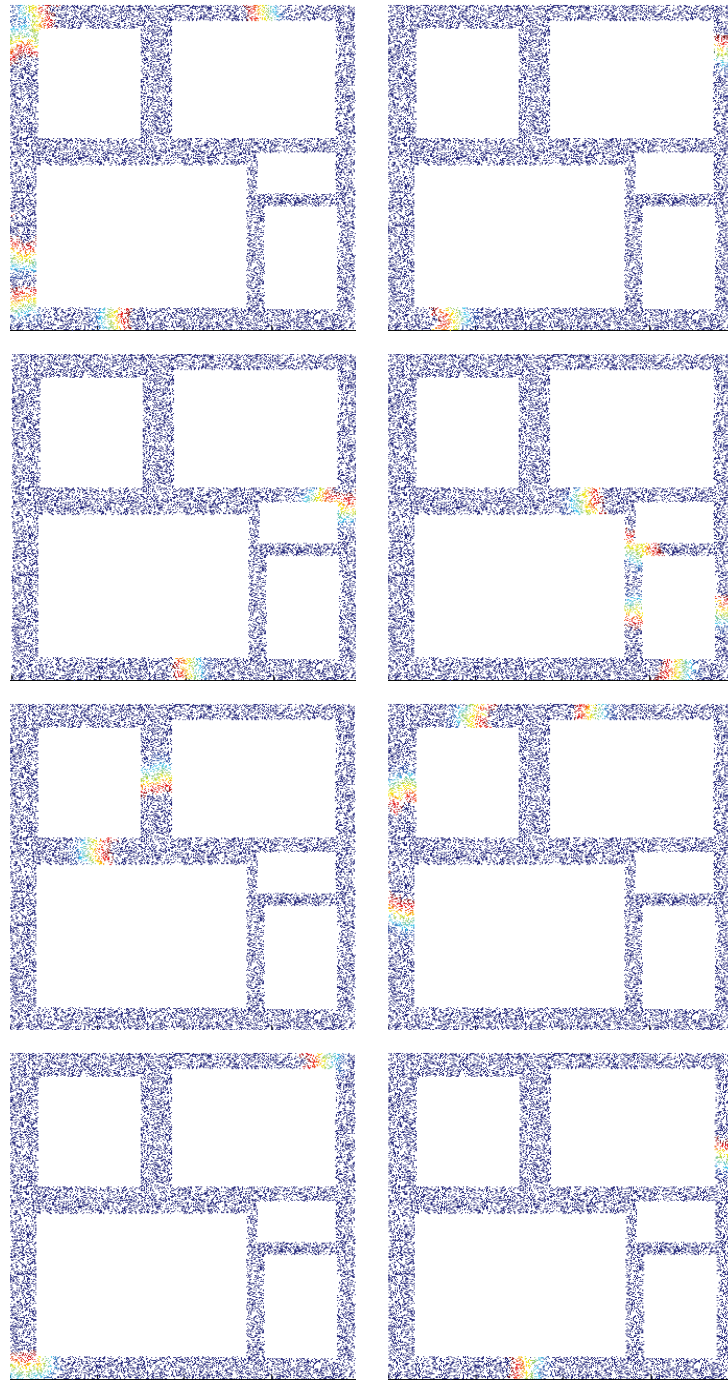


Figure 3.4: An example of GHM in equilibrium with no local defect at time 250, 300, 350, 400, 450, 500, 550, 600.

and the number of wavefronts they have sent out in the hallways. Note that the degree is invariant in time for a continuous state. Thus counting the degree for a cycle after local defects are turned off is not a difficult problem: following the direction of this cycle, the number of wavefronts in the same direction minus the number of wavefronts in the opposite direction determines the degree.

Definition 3.3.1. For a class $[\alpha]$ in $H_1(\mathcal{R}_r(X))$ (or $H_1(C_2(X))$, the first homology of the 2-complex), define the degree of a continuous state u on $[\alpha]$ as the degree of u on cycle α . If the projection $\pi : \mathcal{R}_r(X) \rightarrow \mathcal{D}$ induces an isomorphism $\pi_* : H_1(\mathcal{R}_r(X)) \rightarrow H_1(\mathcal{D})$, then define the degree of u on $\pi_*([\alpha])$ as the degree of u on α .

As a remark, the degree on a first homology class is well-defined, if for homologous cycles α and β , degree of u restricted on both are the same. By Lemma 3.1.14, degree of u restricted on $\alpha - \beta$, which is a null-homologous cycle, has to be zero. Thus, the degrees on α and β have to be the same. By abuse of notation, $\deg(u, [\alpha])$ will be used for $[\alpha]$ as a first homology class in either $\mathcal{R}_r(X)$ or \mathcal{D} .

Definition 3.3.2. Let $Cont(X)$ represent the set of continuous states on X . Define a **cohomologizing map** $h : Cont(X) \rightarrow H^1(\mathcal{R}_r(X)) = Hom(H_1(\mathcal{R}_r(X)), \mathbb{Z})$, such that $h(u)([\alpha]) = \deg(u, [\alpha])$.

As a remark, the first cohomology $H^1(\mathcal{R}_r(X))$ defined here is a simplicial cohomology. The homology of $\mathcal{R}_r(X)$ is torsion free and therefore $H^1(\mathcal{R}_r(X))$ can be treated as $Hom(H_1(\mathcal{R}_r(X)), \mathbb{Z})$.

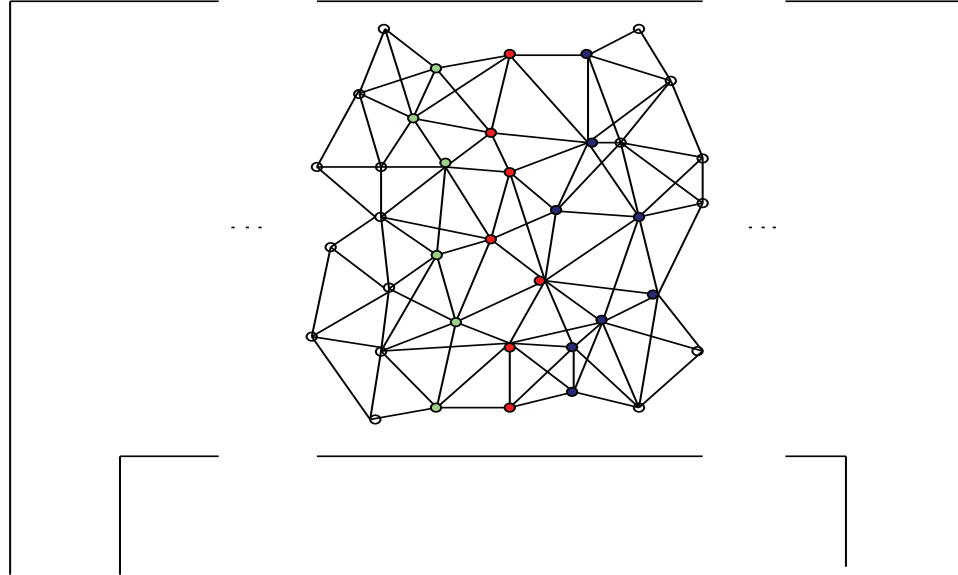


Figure 3.5: Single wave: state space is \mathbb{Z}_4 , with white, light green, red, dark blue respectively representing states 0,3,2,1, other nodes not shown are all in state 0.

Definition 3.3.3. A **single wave** is a continuous state u on X , such that

1. there exists a barrier (see Definition 4.2.6) on which u is supported,
2. there exists a cycle α on which the degree of u is 1.

Refer to Figure 3.5 for an example.

By the definition of a wave, the degree is zero on those cycles that do not intersect the wave's support. The waves move (changing supports in time) in a way that degrees are invariant. They are even additive under some circumstances, by next lemma, which allows for algebraic manipulations.

Lemma 3.3.4. *Let ϕ_1 and ϕ_2 be two continuous states on X with supports X_1 and*

X_2 , with no two nodes from X_1 and X_2 being neighbors. Let $\phi = \phi_1 + \phi_2$ be a state on X , then ϕ is a continuous state on X , which satisfies $h(\phi) = h(\phi_1) + h(\phi_2)$.

Proof. The continuity of ϕ inside X_1 and X_2 is inherited from the continuity of ϕ_1 and ϕ_2 . If $x_1 \in X_1$ and $x_2 \notin X_1$ are neighbors, then $x_2 \notin X_2$. This means $\phi(x_2) = 0$, which makes ϕ on the pair (x_1, x_2) continuous. The same argument works for a pair of neighbors in and out of X_2 . For two neighbors both outside X_1 and X_2 , on which ϕ is 0, the continuity also holds since the values have to be both 0. Let $\alpha = \sum_{i=1}^K [a_i, b_i]$ be a cycle in X , then $\alpha \cap X_1$ and $\alpha \cap X_2$ are two non neighboring subsets, and:

$$h(\phi) = 1/n \sum_{i=1}^K (\phi(b_i) - \phi(a_i))$$

For these pairs of neighbors a_i and b_i , there could be at least one in X_1 , which sum up to be $h(\phi_1)$, or at least one in X_2 , which sum up to be $h(\phi_2)$, otherwise, both are in neither X_1 or X_2 , which sum up to 0. Therefore, $h(\phi) = h(\phi_1) + h(\phi_2)$. \square

Corollary 3.3.5. *If states ϕ_1, \dots, ϕ_k have distinct and non-neighboring supports, then $h(\sum \phi_i) = \sum h(\phi_i)$.*

An important property of the narrow hallways \mathcal{D} is it has the topological type of a planar graph G ; specifically, G is a deformation retraction of \mathcal{D} , with retraction map $r : \mathcal{D} \rightarrow G$ and injection map $i : G \rightarrow \mathcal{D}$. Suppose $H_1(\mathcal{R}_r(X)) = H_1(\mathcal{D}) = \bigoplus_g \mathbb{Z}$ (a graph's first homology is always a free abelian group), and $\{[\alpha_1], \dots, [\alpha_g]\}$ is a basis for $H_1(\mathcal{R}_r(X))$, accordingly, $\{\pi_*([\alpha_1]), \dots, \pi_*([\alpha_g])\}$ is a

basis for $H_1(\mathcal{D})$. Since the degree of u on a cycle in $\mathcal{R}_r(X)$ is totally determined by integers $\deg(u, \alpha_1), \dots, \deg(u, \alpha_g)$ by Lemma 3.1.12, we only need to focus on controlling the degree on a basis.

One problem we care about is whether one can realize every possible degree. In other words, the question could be reformed as whether the map h is surjective. Specifically, is it possible to realize a continuous state u , such that $\deg(u, [\alpha]) = f([\alpha])$, where $f : H_1(\mathcal{R}_r(X)) \rightarrow \mathbb{R}$ is any integer valued linear map satisfying $f([\alpha + \beta]) = f([\alpha]) + f([\beta])$?

Our last theorem in this chapter concerns this ability to *program pulses* in the network for customizing the response.

Theorem 3.3.6. *The map h is surjective: if $[f] \in H^1(\mathcal{R}_r(X))$, then there exist a continuous state u on X , such that $h(u) = [f]$.*

Proof. We start by selecting a specific basis for $H_1(G)$, using the standard basis of the complement of a spanning tree T : each remaining edge corresponds with an element in a basis of $H_1(G)$. Let this basis be $\{[\alpha'_1], \dots, [\alpha'_g]\}$, and the edges in corresponding sequence be e_1, \dots, e_g , where each e_i is contained in only one element α'_i . For each i , there exists at least one single wave ϕ_i that is supported only on a subnetwork in $r^{-1}(e_i)$, and satisfies $h(\phi_i)([\alpha_j]) = \delta_{ij}$. From the density assumption on the network X , those waves can be supported on non-neighboring subnetworks and therefore we can sum $[f]([\alpha])$ times of them up to obtain a continuous state ϕ'_i such that $h(\phi'_i)([\alpha]) = [f]([\alpha])\delta_{ij}$ by Corollary 3.3.5. From the same argument,

$\sum \phi'_i$ is a continuous state which maps to $[f]$ under h .

□

Chapter 4

Evasion game

We propose a sensor-network based “Evasion Game” formally, and then use the model to verify the system: why the wavefronts sweep the entire domain; how to interpret the phenomenon that wavefronts are dividing their neighborhoods and how to prove that an intruder will always fail to evade detection; what are the parameters that control the system and how they are changing the behaviors of those wavefronts.

Definition 4.0.7. Let the domain where the evader and sensors are located be denoted $\mathcal{D} \subset \mathbb{R}^2$, and the sensor network with node set X . For each sensor $x \in X$, its coverage is a subset $U_x \subset \mathcal{D}$. Denote by $X(t)$ the set of sensors in wake-state (0) at time t . We define the *Evasion Game* as follows: the strategy for the pursuer is to control the network following GHM, and the strategy for the evader is to pick a moment t_0 to come into the domain, and follow a continuous path in \mathcal{D} :

$f : [t_0, \infty) \rightarrow \mathcal{D}$. The pursuer wins if and only if $\exists \tau \in [t_0, \infty)$, such that

$$f(\tau) \in \bigcup_{x \in X(\lfloor \tau \rfloor)} U_x.$$

Otherwise, the evader wins.

We note that the only requirement on the evader is its trajectory being continuous: there are no constraints on the velocity or acceleration. Even with such minimal constraints, the evader is not able to win.

4.1 Limiting Case with 1-d Hallways

We begin our analysis with the limiting case where every hallway is sufficiently narrow compared to the walls, so that the domain \mathcal{D} can be approximated as a (topologically equivalent) one-dimensional space. We assume those sensors are located in \mathcal{D} with each node having a coverage which is a one dimensional convex set around itself, and the convex hull of two neighbors is covered by the union of their coverage. We also assume the union of convex hulls of neighbors (subspace of \mathcal{D}) is good enough to cover \mathcal{D} , in which case the whole space is covered when every sensor is turned on. If we run GHM on this network, with at least one defect in initial condition, then every evader (not near the defects) loses the evasion game.

Theorem 4.1.1. *For GHM on network X with communication distance r in a compact and connected 1-d complex \mathcal{D} , if the initial condition contains at least one defect, and there exists a subnetwork X' covering a sub-domain \mathcal{D}' , such that the*

state on X' is eventually continuous and contains no defect, an evader will always lose the evasion game on \mathcal{D}' .

Proof. For any time t_0 when the evader comes into the domain, consider the product space $\mathcal{D}' \times [t_0, \infty)$ with the second coordinate representing time. Treat the coverage of the sensors also as a subspace P_c of $\mathcal{D}' \times [t_0, \infty)$, which is

$$\bigcup_{t=\lceil t_0 \rceil, t \in \mathbb{Z}} \bigcup_{x \in X'(t)} U_x \times [t, t+1) .$$

Let p be the projection map: $p : \mathcal{D}' \times [t_0, \infty) \rightarrow \mathcal{D}'$, $p(a, t) = a$. Then $p : P_c \rightarrow \mathcal{D}'$ is onto, because \mathcal{D}' is fully covered when every node in X' is on. If we could prove that there exists a subspace in P_c homeomorphic to \mathcal{D}' , with map p as homeomorphism, then $\mathcal{D}' \times [t_0, \infty) \setminus P_c$ contains no continuous path from top $\mathcal{D}' \times \{t_0\}$ to bottom $\mathcal{D}' \times \{T\}$ for T big enough, because they are dual to each other. Therefore no evader could survive forever. The construction of the subspace in P_c is as follows in Lemma 4.1.2, below. □

As a remark, a good example of an eventually continuous state on subnetwork X' which contains no defect, is an all-0 state at one moment, which is observed most of the time in simulations.

Lemma 4.1.2. *Under the conditions of Theorem 4.1.1, there exists a subspace $S \in P_c$, such that p induces a homeomorphism from S to \mathcal{D}' .*

Proof. First, reduce to a subnetwork X'' of X' such that within X'' the convex hulls of neighbors is still enough to cover \mathcal{D}' , but any two distinct convex hulls intersect

in at most one node. We then construct S inductively from the empty set as follows:

1. Select a big enough integer time t which is no earlier than t_0 , such that every node in X'' has already been periodic for a long enough time. Pick a node $x \in X''(t)$ and add (x, t) to S .
2. For any neighbor of x in X'' , say y , there exists a continuous path lying in P_c , between (x, t) and (y, t_y) , where $y \in X''(t_y)$ and $|t - t_y| \leq 1$ (t_y is an integer time), which is mapped homeomorphically to the convex hull of x and y in \mathcal{D}' , because continuity holds on edge $[x, y]$, and $U_x \times [t, t+1) \cup U_y \times [t_y, t_y+1)$ is a path connected set. For any x 's neighbors y that has not been visited, add (y, t_y) with the continuous paths between (x, t) and (y, t_y) to S .
3. Repeat step 2 for every newly visited node, until every node in a connected component of X'' has been visited.

Such procedure could not be realized only if there is a cycle in X'' , such that the continuous lift of the path to P_c is not a loop, which means the state restricted on the cycle is a defect. However there is no defect in $u_t(X'')$ when t is big enough, by Lemma 3.1.13. Therefore the procedure is well-defined. Start above procedures until every node in X'' has been visited.

Such an S is mapped onto \mathcal{D}' by p , because every convex hull of two neighbors, say x and y , is mapped onto from the path between (x, t_x) and (y, t_y) . The restriction of p to S is also injective, because every node and edge is only visited once, and

p restricted on every continuous path between (x, t_x) and (y, t_y) is homeomorphism.

The only thing left to be proved is that there exists a continuous inverse of $p|_S$. Let f be a map from \mathcal{D}' to S , such that f maps every node x in X'' to (x, t_x) in S , and maps every edge between node x and y to the continuous path between (x, t_x) and (y, t_y) . Such f is an inverse of $p|_S$, and is continuous: for a point in \mathcal{D}' that is not a node, it's covered by a convex hull of two neighboring nodes in X'' , thus its small neighborhood maps to the lift of the the convex hall in S continuously; for a node point x in X'' , its neighborhood maps to a neighborhood of the lift (x, t_x) homeomorphically, by the procedure of constructing S . Thus f is a continuous inverse of p restricted on S , which validates that p induces a homeomorphism from S to \mathcal{D}' . □

4.2 Main Theorem and Proofs

4.2.1 Assumptions

We restate the goal here: we want to prove the connected and compact network X following the GHA to be able to win the evasion game on narrow hallways under certain assumptions about the density of the network, the coverage of each sensor, and the cohomological class of the states.

The main theorem, Theorem 4.2.13, shows that any evader in the evasion game on a narrow hallway space $\mathcal{D} \subset \mathbb{R}^2$ will lose, given the following assumptions:

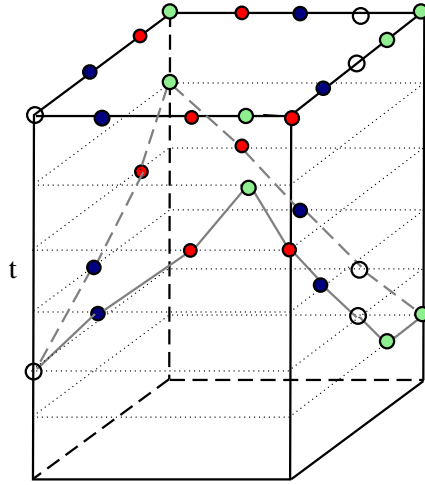


Figure 4.1: Product space and S : space \mathcal{D}' on top, time increases from top to bottom. State space is \mathbb{Z}_4 , with white for state 0, light green for state 3, red for state 2, and dark blue for state 1. Gray curve represent S . $S \cong \mathcal{D}'$ by p . Continuous curves connecting top and bottom without intersecting \bar{S} do not exist.

1. the projection from Rips complex $\mathcal{R}_r(X)$ to \mathcal{D} preserves homotopy type;
2. each sensor $x \in X$ covers a convex set $U_x \subset \mathcal{D}$ around its location;
3. the convex hull of sensors that are pairwise neighbors is covered by the union of coverage of those sensors;
4. the union of all sensors' coverage is able to cover the whole domain;
5. there is one moment such that the state contains at least one defect (this is the same as saying the system never die out by Theorem 3.2.2).

The following definition explains the shadow and projection map.

Definition 4.2.1. For an abstract simplicial complex C whose 0-simplices are located in a d -dimensional Euclidean space \mathbb{E}^d , the **shadow** of C in \mathbb{E}^d , $S(C)$, is the union of convex hulls of 0-simplices forming a simplex in C . The projection map $p : C \rightarrow \mathbb{E}^d$ maps a simplex $[v_0, v_1, \dots, v_K]$ in C to the convex hull of $\{v_0, v_1, \dots, v_K\}$ in \mathbb{E}^d .

According to a theorem of [9], we will be able to build the correspondence between the Rips complex $\mathcal{R}_r(X)$ of a planar point set and its shadow $S(\mathcal{R}_r(X))$ in \mathbb{R}^2 .

Theorem 4.2.2. [9] *For any set of points in \mathbb{R}^2 , $\pi_1(\mathcal{R}_r(X)) \rightarrow \pi_1(S(\mathcal{R}_r(X)))$ is an isomorphism.*

Definition 4.2.3. A **local hole** in the Rips complex is a non zero element of $\pi_1(\mathcal{R}_r(X))$ that has trivial image under the projection map in $\pi_1(\mathcal{D})$.

In the sense of local holes, Theorem 4.2.2 is saying that in our case, the Rips complex $\mathcal{R}_r(X)$ has no local hole if and only if its shadow $S(\mathcal{R}_r(X))$ has no local hole.

Another useful fact is that with very high probability, when the network is dense enough, the Rips complex $\mathcal{R}_r(X)$ has no local holes [29]. Therefore, with enough sensors uniformly distributed in the domain and with high probability, the Rips complex $\mathcal{R}_r(X)$, and its shadow $S(\mathcal{R}_r(X))$ both have no local hole.

4.2.2 Wave propagation

Definition 4.2.4. A **boundary path** along a boundary component, is defined as a simple path such that every node on the path has a coverage that intersects with the corresponding boundary, and the intersection of the coverage of every two neighbors, x and y on the path, also intersects the boundary nontrivially. A **boundary of a network** X , ∂X on \mathcal{D} is a collection of boundary paths, one with each component of $\partial\mathcal{D}$. Refer to Figure 4.3 for illustration.

Definition 4.2.5. For positive integer set A , define the **depth** A node set, X_A , as the set of all the nodes that are with depth $k \in A$ in the directed forest \mathcal{F} built on the network.

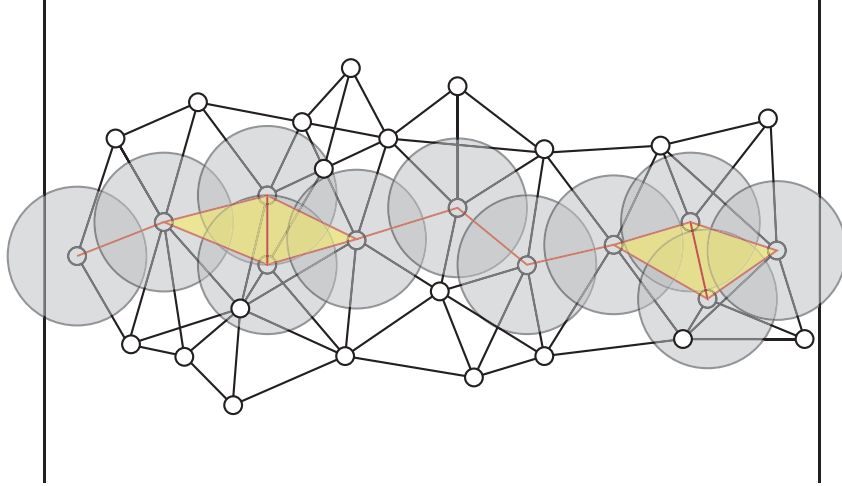


Figure 4.2: A piece of hallway with a subnetwork that is a barrier.

Definition 4.2.6. A connected sub network X' of X makes a **barrier**, if there exists a piece of hallway $\tilde{\mathcal{D}}$, which intersects $\partial\mathcal{D}$ at $\partial\tilde{\mathcal{D}}$, and the composition $\partial \circ i_* : H_1(\tilde{\mathcal{D}}, \partial\tilde{\mathcal{D}}) \rightarrow H_0(\partial\mathcal{D})$ of $i_* : H_1(\tilde{\mathcal{D}}, \partial\tilde{\mathcal{D}}) \rightarrow H_1(\mathcal{D}, \partial\mathcal{D})$ and $\partial : H_1(\mathcal{D}, \partial\mathcal{D}) \rightarrow H_0(\partial\mathcal{D})$ is an injection, such that X' 's coverage contains at least one element in a nonzero class of $H_1(\tilde{\mathcal{D}}, \partial\tilde{\mathcal{D}})$. In other words, it covers a region that divides the hallway locally and transversally as in Figure 4.2.

Theorem 4.2.7. *Let X be a connected and compact network on a narrow hallway space \mathcal{D} , running under GHM, which never die out; if there is a moment after equilibrium that the state in subnetwork X' in a sub domain \mathcal{D}' is continuous, and boundary paths $\partial X'$ exists, then if at time t , there is a wavefront that makes a barrier, which is not supported on any end leaves of the forest \mathcal{F} , then there would be a wavefront also makes a barrier at time $t + 1$.*

Proof. If there is a wavefront of nodes with depth k at that makes a barrier, then we

want to prove that a wavefront of nodes with depth $k + 1$ exists which also makes a barrier. Let A denote the sub complex on subnetwork $X_{\leq k+1}$, and let B denote the sub complex on subnetwork $X_{\geq k+1}$. Then $A \cap B$ is precisely the sub complex with nodes with depth $k + 1$. On the other hand, $A \cup B$ is the whole complex, because for every simplex in the whole complex, their vertices are pairwise neighbors, so by continuity of states on X' , their depths could only differ at most by 1, by Corollary 3.1.6, which means the simplex is either in A or in B . The Mayer-Vietoris sequence for A and B gives:

$$\begin{aligned} H_1(A \cap B, \partial X') &\xrightarrow{\phi} H_1(A, \partial X') \oplus H_1(B, \partial X') \\ &\xrightarrow{\psi} H_1(A \cup B, \partial X') \end{aligned} \tag{4.2.1}$$

Let $[\alpha] \in H_1(A, \partial X')$, $[\beta] \in H_1(B, \partial X')$, where α and β are both connecting boundary nodes of different sides. Such an α exists because of the existence of a previous wavefront of depth k , and β exists because the network is sufficiently dense in \mathcal{D} , and α is not supported on any end leaves of \mathcal{F} . Then $\psi([\alpha], [\beta]) = 0$ (if not, let β be of opposite orientation) in $H_1(A \cup B, \partial X')$, because first homology of $A \cup B$ is trivial. Therefore $\psi([\alpha], [0])$ and $\psi([0], [\beta])$ are homologous. Thus $([\alpha], [\beta]) \in \ker \psi$. By exactness, $\ker \psi = \text{im } \phi$, thus there exists a γ , such that $\phi([\gamma]) = ([\alpha], [\beta])$. As ϕ is induced by inclusion maps, γ has to be a path connecting boundaries of two sides, which is a wavefront of depth $k + 1$, *cf.* Figure 4.3. Therefore, by induction, barrier-inducing wavefronts of every depth exist. \square

The above wave propagation theorem presents how waves travel along the hall-

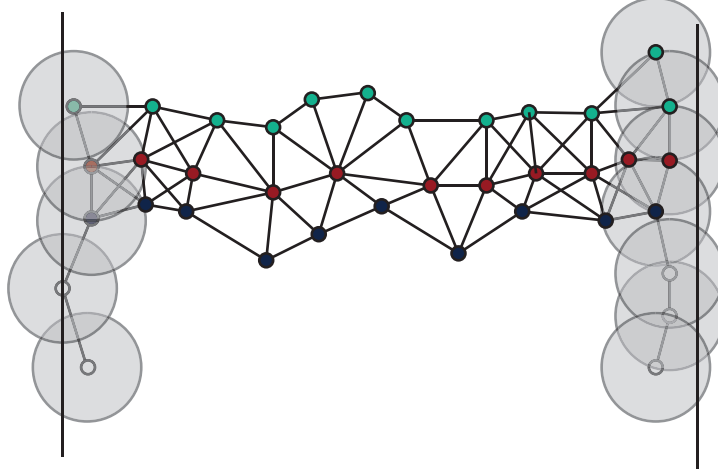


Figure 4.3: Wavefronts propagation: light green nodes are with depth $k - 1$, red nodes are with depth k , dark blue nodes are with depth $k + 1$; Boundaries of the domain are covered by boundary paths.

ways, but did not mention the generation of waves. The following proposition would explain how a first wave is generated under same assumptions.

Proposition 4.2.8. *With the assumptions of Theorem 4.2.7, at least one wavefront that is a barrier and intersects with boundary paths on both sides must be generated.*

Proof. Let $X_{\leq k}$ be the set of nodes with depth less than or equal to k . Then there is a filtration of Rips complexes:

$$\mathcal{R}_r(X_{\{0\}}) \subset \mathcal{R}_r(X_{\leq 1}) \subset \dots \subset \mathcal{R}_r(X_{\leq k}) \subset \dots \mathcal{R}_r(X)$$

Since they grow by attaching nodes within communication distance as k increases, therefore, as $\mathcal{R}_r(X)$ is connected, there has to be a k_0 , such that $\mathcal{R}_r(X_{\leq k})$ are

all connected for $k \geq k_0$. For two boundary nodes of $X_{\{k_0\}}$, if they belong to boundary paths near different boundaries, since they are connected, and for the same argument from Theorem 4.2.7 by using the Mayer-Vietoris sequence, they are connected by a path with nodes from $X_{\{k_0\}}$. This path generates a wavefront that is a barrier. \square

The above results explain well the behaviors of the wavefronts seen in simulations. After the first several steps, the nodes far away from the seed are all turned on, until wavefronts generated by the seeds reach them. The movements of wavefronts are verified to be moving away from local defects, and they provide locally separating barriers, as observed. Another significant property we observe from simulations is that the wavefronts make turns when reaching a corner, as shown in Figure 4.4. This reminds again that the behavior of the system does only depend on topology, not geometry, of the underlying space.

4.2.3 Main theorem

For now, we will start arguing that under assumptions made in §4.2.1, evader will always lose the evasion game.

Lemma 4.2.9. *Let σ be a d -simplex in $\mathcal{R}_r(X)$. If there is a continuous function $f : [t_1, t_2) \rightarrow S(\sigma)$, such that $f(t) \notin \bigcup_{x \in X(t) \cap \sigma} U_x$, $\forall t \in [t_1, t_2)$, then there exists a continuous function $\tilde{f} : [t_1, t_2) \rightarrow \sigma$, such that $\tilde{f}(t) \notin \bigcup_{x \in X(t) \cap \sigma} \text{St } x$, $\forall t \in [t_1, t_2)$.*

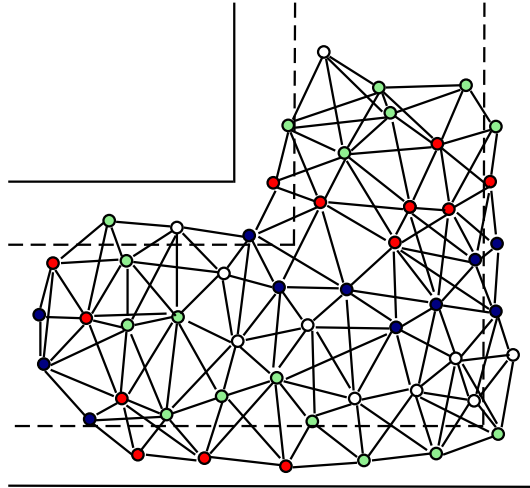


Figure 4.4: Corner of a hallway: with state space \mathbb{Z}_4 , white for state 0, light green for state 3, red for state 2, and dark blue for state 1. The outer side boundary path have more nodes than the inner boundary path, but more nodes stay in the same states: four in light green and three in red. Thus the wavefronts propagate from vertically to horizontally.

Proof. It suffices to prove the lemma for a 2-simplex, because the shadow of $\mathcal{R}_r(X)$ in \mathcal{D} is the shadow of the 2-skeleton of $\mathcal{R}_r(X)$. For a 2-simplex $\sigma = [x_0, x_1, x_2]$. There exists a homotopy equivalence h from $S(\sigma)$ to itself, such that the interior of $\bigcap_{i=0,1,2} U_{x_i} \cap S(\sigma)$ is mapped onto the interior of $S(\sigma)$, which is $\bigcap_{i=0,1,2} \text{St } x_i$, and the inverse image of every open edge $S([x_i, x_j])$ belongs to $U_{x_i} \cap U_{x_j} \cap S(\sigma)$. Therefore, we can construct \tilde{f} as $h \circ f$, with the property that $h^{-1}(\bigcap_{i \in A} \text{St } x_i) \subset \bigcap_{i \in A} U_{x_i}$, which induces $\tilde{f}(t) \notin \bigcup_{x \in X(t) \cap \sigma} \text{St } x$, $\forall t \in [t_1, t_2]$. \square

Lemma 4.2.10. *For the Rips complex $\mathcal{R}_r(X)$, if there exists a continuous function $f : [t_0, \infty) \rightarrow S(\mathcal{R}_r(X))$, such that $f(t) \notin \bigcup_{x \in X(t)} U_x$, $\forall t \in [t_0, \infty)$, then there exists a continuous function $\tilde{f} : [t_0, \infty) \rightarrow \mathcal{R}_r(X)$, such that $\tilde{f}(t) \notin \bigcup_{x \in X(t)} \text{St } x$, $\forall t \in [t_0, \infty)$.*

Proof. The 2-complex of $\mathcal{R}_r(X)$, $C_2(X)$, as a sub complex, has the same shadow as the Rips complex $\mathcal{R}_r(X)$. Lift the path f from the shadow $S(C_2(X))$ to the complex $C_2(X)$, then apply Lemma 4.2.9 on every simplex it goes through. This will give a lift $\tilde{f} : [t_0, \infty) \rightarrow C_2(X) \subset \mathcal{R}_r(X)$, such that $\tilde{f}(t) \notin \bigcup_{x \in X(t)} \text{St } x$, $\forall t \in [t_0, \infty)$. \square

Theorem 4.2.11. *For a network X with coverage regions \mathcal{U} and Rips complex $\mathcal{R}_r(X)$ with the shadow the whole \mathcal{D} . Then if the state on X is eventually continuous and contains no defect, then the evader loses the evasion game.*

Proof. Suppose there is a continuous path f for the evader to follow in order to win the evasion game, $f : [t_0, \infty) \rightarrow S(\mathcal{R}_r(X)) = \mathcal{D}$, then by Lemma 4.2.10, there

exists a lift of f , $\tilde{f} : [t_0, \infty) \rightarrow \mathcal{R}_r(X)$, such that following \tilde{f} , the evader could win the evasion game with coverage regions $\{\text{St } x | x \in X\}$. Furthermore, since $\tilde{f}(t) \notin \bigcup_{x \in X(t)} \text{St } x$, $\forall t \in [0, \infty)$, and by the fact that a 1-simplex is covered by a subset of sensors that covers the simplex containing it, we can construct a continuous path f' that travels only on the 1-skeleton of $\mathcal{R}_r(X)$ and still is safe, never being detected. However, by the same argument as in Theorem 4.1.1, since there is no defect in initial condition, such a strategy does not exist: any such evader would lose the game. \square

It may not be the case that $S(\mathcal{R}_r(X)) \supset \mathcal{D}$. Our approach for solving this problem is by adding sensors to the network without changing the coverage, but enlarge the Rips complex such that it projects onto the whole domain.

Lemma 4.2.12. *If a boundary path exists within distance $\sqrt{3}/2r$ to each boundary component of $\partial\mathcal{D}$, then there is a new sensor network \tilde{X} by adding sensors to X , with the same coverage at every moment, such that the shadow of $\mathcal{R}_r(\tilde{X})$ is \mathcal{D} .*

Proof. For every node x in the boundary path, add a node x' in $U_x \cap \partial\mathcal{D}$ to the new network \tilde{X} , and for every edge on the path $[x, y]$, add a node z' in $U_x \cap U_y \cap \partial\mathcal{D}$ to \tilde{X} . For a quadrangle with vertices x, y, x', y' , it is covered by union of U_x and U_y . Let x' and z' have same coverage and states as x , and y' has the same as y after equilibrium, then the coverage of \tilde{X} is exactly the same as that of X at every moment. Another property worth noticing is $\mathcal{R}_r(\tilde{X})$ now has its shadow same as \mathcal{D} , because $[x, x'], [x', z'], [x, z'], [y, z'], [z', y'], [y', y]$ are all 1-simplices in $\mathcal{R}_r(\tilde{X})$, which

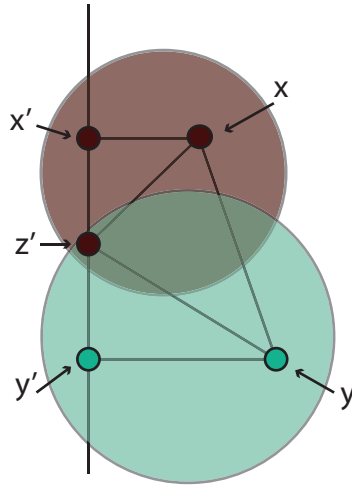


Figure 4.5:]

Add new nodes x', y', z' to the network, with x' and z' have same state and coverage as x , and y' has same state and coverage as y

makes the shadow exactly \mathcal{D} . □

Theorem 4.2.13 (Main Theorem). *With all the assumptions from §4.2.1, and the existence of boundary paths within distance $\sqrt{3}/2r$ to boundary of hallways, the evader will always lose the evasion game in the sub domain $\mathcal{D}' \subset \mathcal{D}$ on which the states is eventually continuous and contains no defect.*

Proof. By Theorem 4.2.11 and Lemma 4.2.12. □

4.3 Link Failure Analysis

Reliability of links is a serious issue for achieving stability of WSN [2, 36]; in practice, stability is not guaranteed, as wireless communication quality is unpredictable under different environmental and other physical conditions [38]. For our GHM system, it is important to keep communication stable, especially the links between sensors of state 0 and state 1, since they will determine those nodes' state at the next time step.

In this section, we will assume that every link works well with a fixed probability p_s , as a more practical GHM system. By modifying our simulation accordingly, we observe that most of the nodes goes to state 0 after the first several steps, as before. Afterwards, either the system dies out if there is no defect (either local or global defects), or wavefronts are generated around local defects. But these defects do not guarantee the system's periodicity, since link failure might result in their dying, with a probability associated with p_s .

For a fixed network X , if given an initial state u with at least one local defect, the probability that one local defect dies after T time steps is a function f of X , u , T and p_s . The smaller p_s is, the bigger the probability of defect dying. Meanwhile, $f(X, u, T, p_s)$ is an increasing function of T , which approaches 1 as T goes to infinity.

Although local defects die eventually almost surely, it does not affect pattern propagation. For a continuous state of waves with no local defect, which are what remain in the network after all local defects die, it could either be in a trivial co-

homology class, which will die out after a while, or has at least one global defect. As in the latter case, wave propagation is not necessarily the same as in the deterministic model, since a state 0 wavefront may not turn into a state 1 wavefront. However, even this wavefront does not update to state 1 as a whole piece, it is of great chance that at least one of the nodes on the wavefront successfully updates to state 1 (which still makes a global defect), and therefore will gradually correct the neighbors states by contact.

Chapter 5

Conclusion and Future Plans

We provide a decentralized, coordinate-free, energy-efficient intruder-detection protocol based on the Greenberg-Hastings cyclic cellular automata. The system could easily be adapted to real indoor environment if using sensing devices functioned with communication and proper sensing ranges. It displays coherence in the sense that it is a self-assembling system with random initial conditions; its efficiency comes from low power-consuming property inherited from the scheme of the CCA. Demonstrations in §1.3 are evidence that the system behaves as intended, and this thesis gives both intuition and rigor about how and why the system works:

- Wave patterns are explained as a topological phenomenon, determined and described by the existence of defects with nonzero degree.
- Assigning to waves a cohomology class reveals the qualitative structure of the wave patterns, greatly clarifying certain classical results about CCA on

lattices.

- A non zero restriction of a cohomology class to a sub domain corresponds with a set of strategies with which the evader could win the evasion game; meanwhile, a zero restriction stands for the failure of the evader: the cohomology class is the *obstruction* for the pursuer to win.

We also formally answer those questions arose at the beginning:

- Is there any invariant associated with the states?

Yes, degree is such an invariant.

- What role is the topology, or geometry of the underlying space playing in the system?

For a dense network built on the underlying domain, the only thing that affects the wave propagation is the topology, not the geometry. (There is an exception for homology class of very small geometrical scale, which are ignorable when computing the topology).

- Why are small obstacles ignorable in the wave traveling but large obstacles changing directions of wave propagation?

The scale of obstacles matters because the length of a cycle that supports a defect could be no smaller than n . Small scaled obstacles are ignorable (*e.g.*, example in Figure 1.6) because its scale is even smaller compared to the smallest cycle that a defect could be built on. Therefore two cycles differed

by the boundary of such a small obstacle (they are not homological cycles) have to be of the same degree, and thus observed as they belong to the same homology class.

5.1 GHM on various domains

GHM on narrow hallways is studied in previous chapters, one natural following up question to ask is, what if the underlying space is some other topological space. In this section, we would explore various possible domains as the underlying space for the network.

Recall that narrow hallways space is a subspace of 2-d Euclidean plane. For other subspaces of \mathbb{E}^2 , we have concluded that GHM mostly depends on the topology (except the cases with homology classes of small scale). A natural generalization is a 2-manifold (with or without boundary). Intuitively, a 2-manifold is a topological space such that near each point, it locally looks like a 2-d Euclidean space. Therefore, instead of embedding network in 2-d plane, we will sample the network in a 2-manifold.

A good example of network in 2-manifold is given by indoor network within a building. Floor plan of a multi-storied building with staircases connecting different levels is a 2-manifold with boundary, but not a subspace of 2-d plane (see Figure 5.1).

We find that the theory applies to the narrow hallways case also applies to the

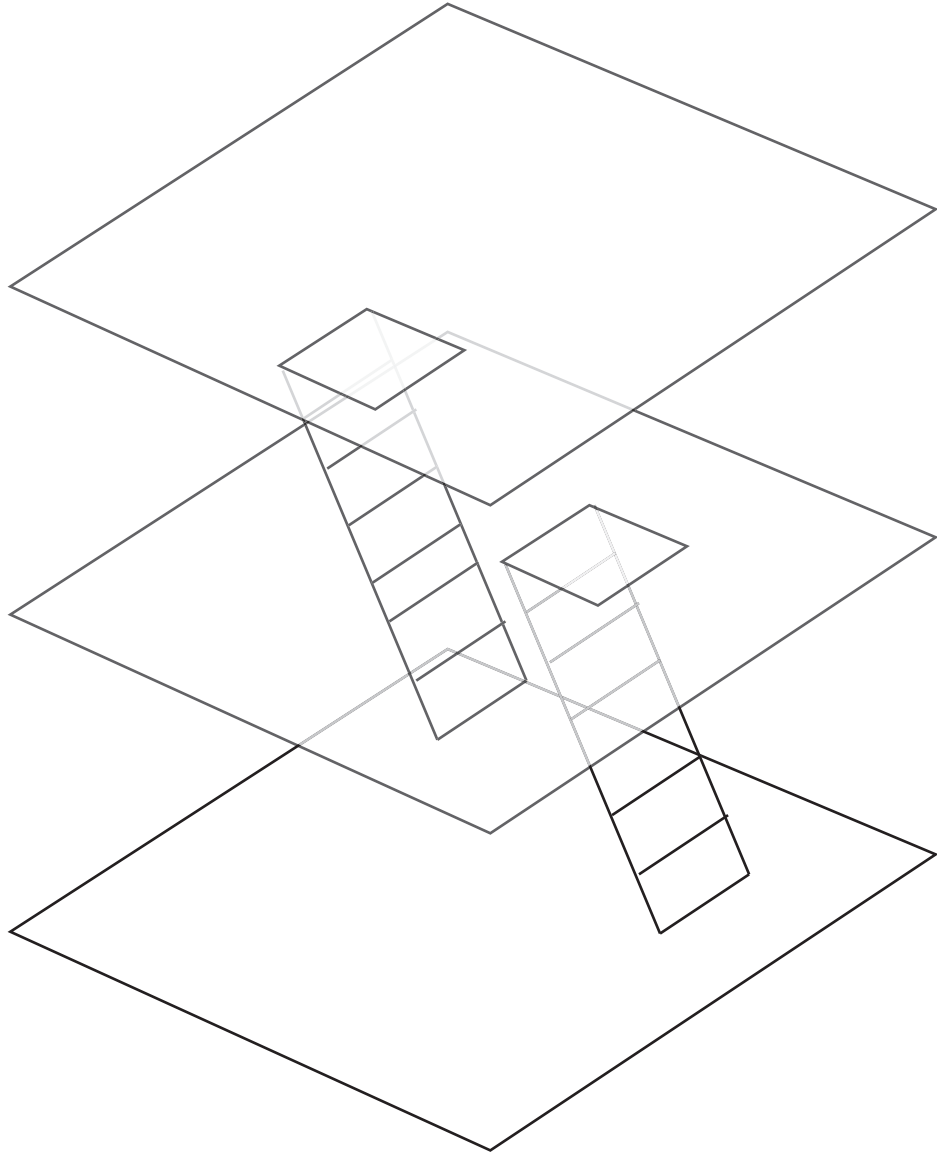


Figure 5.1: Floor plan of a building that is not a subspace of 2-d plane

2-manifold case. The main theorem is rewritten as follows.

Theorem 5.1.1. *(As a corollary of Theorem 4.2.13) With the same assumptions from §4.2.1, and the existence of boundary paths within distance $\sqrt{3}/2r$ to boundary ∂M of a 2-manifold with boundary M , the evader will always lose the evasion game in the sub domain $M' \subset M$ on which the states is eventually continuous and contains no defect.*

We look back at the previous example of multi-storied building. Assume the network embedded in this 2-manifold with boundary M_B meets all the assumptions about density and coverage. If starting with an initial state with at least one defect, the system will evolve to a none dying out periodic state with great probability as before. Wavefronts propagation has the exact same properties as the 2-d case: they make turns when coming to a corner, and two opposite wavefronts annihilate when meeting each other. Take a subspace M'_B of M_B such that the state restricted on M'_B is continuous, then the intruder could win the evasion game in M'_B if and only if there exists a nontrivial first homology class of M'_B , and for any cycle in this class, the degree is non zero. More specifically, we could even write down the path following which the intruder could survive for all the time, which is a cycle restricted on which the state is has a nontrivial image under the cohomologizing map.

Other 2-manifolds such as 2-sphere and torus follow the same arguments. See Figure 5.2 for illustrations. We notice that for a 2-sphere S^2 , since $H_1(S^2) = H^1(S^2) = \{0\}$, therefore, there is no nontrivial first homology class of the space

that could allow a defect to live in, and in the meanwhile keep the state continuous. In other words, there is no nontrivial first cohomology class of the space that is the image of cohomologizing map h . To classify the possible defects on S^2 , there could only be local defects, but no global defects. With exact one local defect around north pole of a 2-sphere, wavefronts propagate downwards to the bottom and finally annihilate. Because of the symmetry of S^2 , we can figure out all the cases in which only one local defect exists.

In a torus $T = S^1 \times S^1$ case, the classification of defects is more complicated. Besides local defects, there could be global defects living in the 2-d vector space $H_1(T) = \mathbb{Z} \oplus \mathbb{Z}$. Suppose α and β are generators for a basis of $H_1(T)$, where α is a cycle representing one S^1 , and β representing the other S^1 . Then wavefronts propagating along α and β respectively are dual to α and β , the single waves respectively propagating along α and β are two generators of $H^1(T)$.

The further question we are about to ask is, what if the underlying space is not a 2-manifold, but a higher dimensional structure. For example, a natural extension is a 3-d subspace in \mathbb{R}^3 . Although no full view has been made available in simulations of our 3-d case, because the nodes are not transparent, we could still take a look at the wavefronts from different angles, and try to imagine what they look like in 3-d. Figure 5.3 is a moment of GHM in a 3-d cube. We take a viewpoint from different angles, and observe that the wavefronts propagation is like a growing bubble around the defect. We could also imagine the wavefronts near the defect,

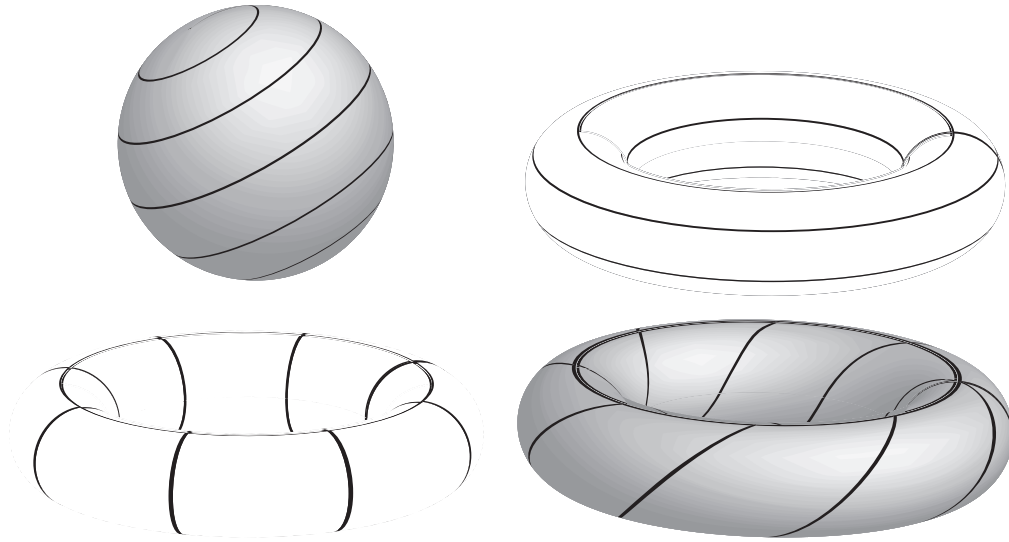


Figure 5.2: The top left figure is a two sphere with wavefronts propagating vertically. Following figures are various wavefronts propagation in a torus generated by revolving a small circle by a big circle. The top right figure is a torus space with wavefronts propagating vertically, all cycles going around the small circle none zero times make defects. The bottom left figure has circular propagating wavefronts, all cycles go around the big circle none zero times make defects. The bottom right figure has tilted propagating wavefronts, it has none zero degree on both two generator of the first homology of the torus, the small circle and the large circle.

which are spirals on every slice of 2-d plane. Furthermore, as we go around the defect in direction with positive degree, the nodes of state 0 are farther away from the defect, which make the wavefronts looking like a revolution of a circle which grows bigger. Therefore, we have the picture of the wavefronts around defect as in Figure 5.4. Even though, this is still the least complicated case in our minds, we could have many more various domains for the GHM worth looking into.

5.2 Various defects

One variation of the system is as in §5.1, by considering domains of various topology (or geometry). In this section, we will focus on exploring different kinds of defects.

Recall from Definition 3.1.11, a state u makes a defect on a cycle α if $\deg(u, \alpha) \neq 0$. Since this has no requirement on the shape of the cycle, we could make the problem more interesting by using a complicated knot instead of a simple cycle to generate a defect, *e.g.*, a trefoil knot in Figure 5.5.

Starting in a 3-d domain with a defect on a trefoil knot of degree 1, it is not hard to imagine the wavefronts around this defect, by knotting the wavefronts in Figure 5.4, the same way as knotting a unknot to a trefoil knot. However, imagining how those wavefronts propagate to the whole space is not that easy. As they move away from the cycle, it is inevitable that they will intersect with each other, and according to the rule, annihilate. There are multiple wavefronts with various directions of propagation, therefore, after they have traveled long, and have interacted with each

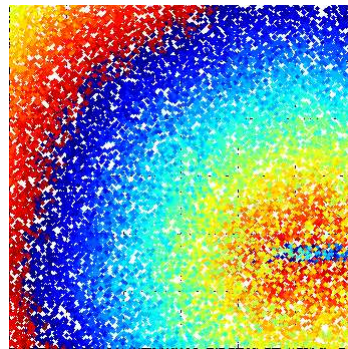
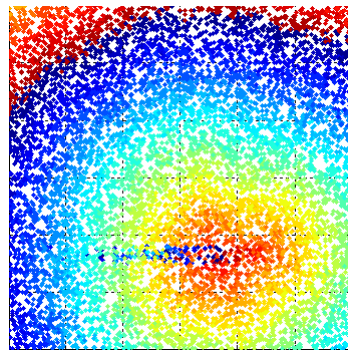
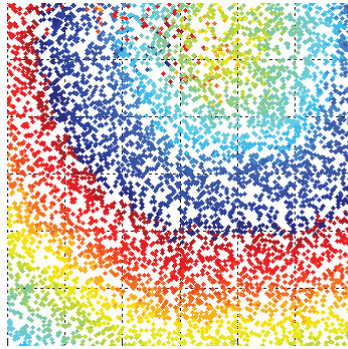


Figure 5.3: 3 shots with different viewpoints from a 3-d GHM in a 3-d cube: the first one takes the picture from above, second taken from the front, and last one taken from the left.

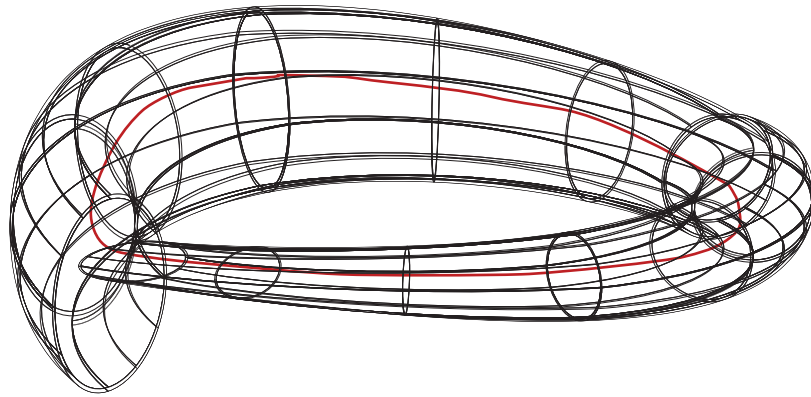


Figure 5.4: A wavefront around a defect in 3-d space: the defect is a cycle in red, and the wavefront is represented by the surface wrapping around it.

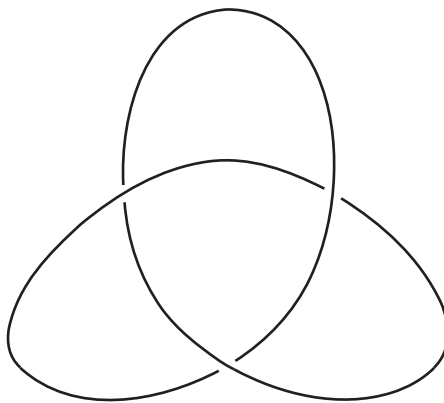


Figure 5.5: A trefoil knot

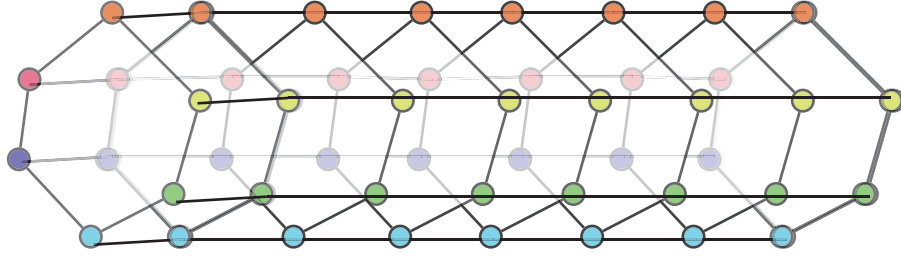


Figure 5.6: Defects in sequence: different colors represent different states: orange, yellow, green, blue, purple, pink are respectively 0,1,2,3,4,6, in a GHM with alphabet \mathbb{Z}_7 . Duplicated defects are shifted and arranged in sequence, and connected as neighbors

other for multiple times, it is not easy to tell what the wavefronts look like.

Another variation could be made by coupling multiple defects' behavior. A simple example is arranging multiple defects in a line, with the same orientation and states, as in Figure 5.6. This defect structure does not make too much sense in a 2-d space, because the group of defects itself is a 2-d structure. Therefore, we consider embedding this into a 3-d space. Since each defect in sequence can produce spiral waves, we may combine them together and believe that by looking in the direction of the axis of the defects, we will observe spirals around each defect, and those spirals together make the wavefronts in Figure 5.7.

Consider manipulating the defects in the above example, by identifying the two cycles at two ends of the sequence, without tilting the arrangement of states (see Figure 5.8). And consequently, the wavefronts as in Figure 5.7 are also identified at both ends, therefore have the shape in Figure 5.9.



Figure 5.7: Wavefronts from a sequence of defects in Figure 5.6.

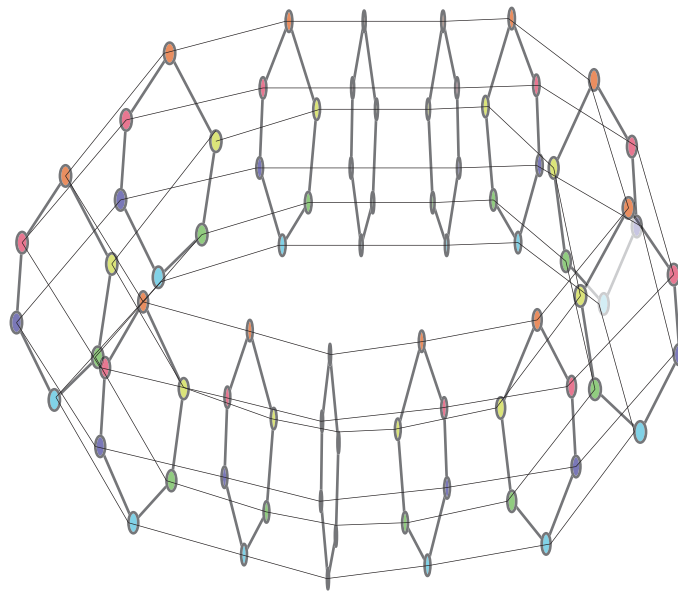


Figure 5.8: Defects in a circular sequence: different colors represent different states: orange, yellow, green, blue, purple, pink are respectively 0,1,2,3,4,6, in a GHM with alphabet \mathbb{Z}_7 . Duplicated defects are shifted and arranged in sequence, and connected as neighbors.

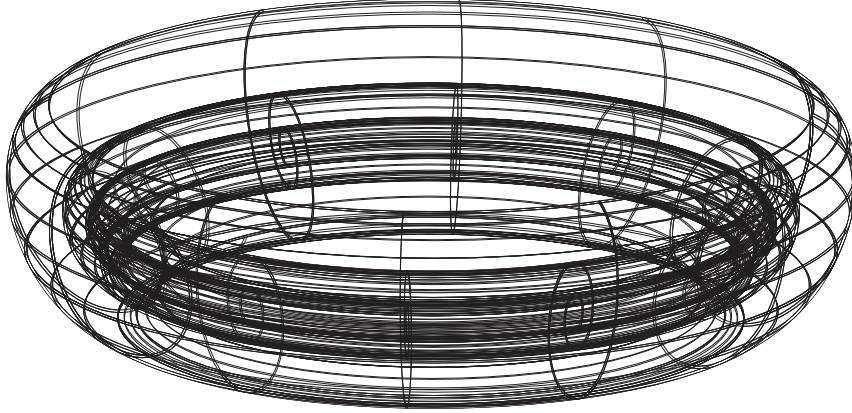


Figure 5.9: Wavefronts from a sequence of defects in Figure 5.8.

What is a more interesting case is when identifying the two ends of the defects sequence, the arrangement in sequence is tilted as well, such that following the sequence, the nodes are not of the same states, but advancing 1 in the sequence, and therefore making a defect itself (see Figure 5.10). We call such structure a scroll ring, following the naming in [33]. We believe a scroll ring would produce wavefronts in the way that attach spirals at both ends as in last example, but with a degree of multiple of 2π tilted, as in Figure 5.11. Although, it is hard to observe this with a random initial state, because it is fairly rare for a scroll ring type of defects to be generated randomly.

All those above defect groups seem to have a 2-d structure, but they are indeed collection of 1-d defects. We wonder what a real 2-d defect looks like. Since a defect is a continuous state with nontrivial degree, we need to look back into the definition of degree.

A topological degree is a a homotopy invariant describing a continuous map

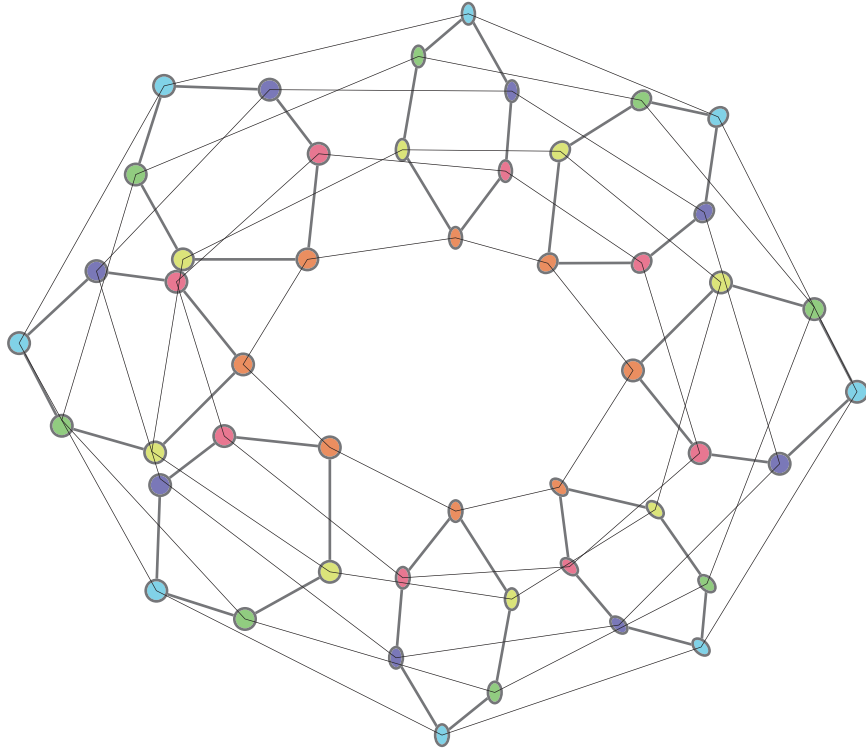


Figure 5.10: Defects in sequence of a scroll ring: different colors represent different states: orange, yellow, green, blue, purple, pink are respectively 0,1,2,3,4,6, in a GHM with alphabet \mathbb{Z}_7 . Duplicated defects are shifted, rotated and arranged in sequence (along the cycle, the defects is rotated by 2π in total), and connected as neighbors.

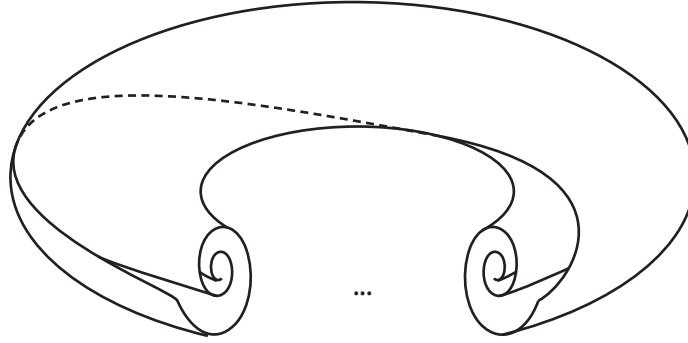


Figure 5.11: Wavefronts generated by the scroll ring defects in 5.10.

between spheres of the same dimension. In cellular automata, degree is borrowed to describe the map from a sub network homeomorphic to S^1 to the image \mathbb{Z}_n . The reason why this discretization works seems mysterious. One reasonable guess is that it is a result of the automata with directions on group \mathbb{Z}_n has the same structure as S^1 with orientation. One thing worth thinking about is how the directions between elements in \mathbb{Z}_n are affecting the whole structure (for example, if there exists a state in \mathbb{Z}_n only has arrows towards it, the automata will not allow any defect).

A difficulty comes in when trying to extend the discretized defect to a higher dimension: from S^2 to S^2 . A sub network with a flag complex homeomorphic to S^2 seems a good choice for the discretization. But we still need to come up with a good group structure as the image and the automata as the directions of arrows.

5.3 Applications

There are extensive applications of CCA beyond sensor networks, even beyond engineering. Some of the applications could be extended to the context of network automata, instead of cellular automata, with more freedom of the space and cells.

One famous CCA application in biology is the model of cardiac pacemaker [24]. One could program, by controlling of the parameters, the rhythmic of pacemaker, which is periodically pulse generating.

Some other applications in biology involve modeling bacterial colony pattern. Observations have been made that some varieties of bacteria have the colony grow as nested rings, with the size of rings enlarging along time. One guess is this could be modeled as a CCA, where within each site the bacteria grow and die. We try to understand the dynamic happening in each site, as an empty site is taken if a neighboring site has a productive bacteria; A productive bacteria will lose its productivity after a while, and then dies. This cyclic structure in each cell gives the whole space a wave-like structure.

The above models could be improved to three dimensional, if CCA and defects in 3-d are fully understood.

It is also interesting to look into models with various types of networks, such as the neural network.

People have been trying to understand the neural networks in visual cortex for decades. It is already known that visual perception of orientation of edges and

their directions of motions is through neurons, each with a preferred orientation. The preferred orientation is changing along time, by excitatory connections (with positive feedback) and inhibitory connections (with negative feedback) to its neighbors, and stimuli from retinal receptors. Neurons in adjacent columns in cortical surface inclined to have similar but slightly shifted preferred orientations. We have observed some cyclic structure in the orientations in measured experimentally in animals [32, 30], and in computational simulations [4], which are nontrivial in degree. Specifically, if we treat the orientations as elements in S^1 , then for a cycle of neurons, each taking a specific orientation, we could induce a map $f : S^1 \rightarrow S^1$. Such maps with nontrivial degrees are observed. The question one naturally would ask is what roles those cyclic structures with nontrivial degrees are playing in the dynamics over the neural network. Are these phenomena just happen by chance, or do they persist some patterns for the network? Is the degree also invariant under some constraints? What is the connection between the self-organization pattern of neural networks and the topology of cortical surface where the neurons live? If these questions could be answered, it helps a lot to understand to what extend of neuron failures in visual cortex are associated with vision problems, because in the cyclic network automata context, small scales of nodes failure does not infect the wave propagation.

Network automata is also used in epidemic models [5]. For example, the SIRS model has three possible states for each individual, namely, the susceptible state, the

infective state, and the removed state. The state space is cyclic in that a removed (recovered) individual will be susceptible again, since the immunity does not go on forever. The state space, again, could be modeled as cyclic group \mathbb{Z}_3 , where each individual has to be in one state among the three, and could only stay in the state or move forward by 1, as in a discrete time dynamic. This is quite similar to the sensor network case. The only difference is that an excitement (from susceptible to infective) is triggered with probability. Therefore, this is a probabilistic network automata model instead of a deterministic one. We would like to study this as our first attempt in probabilistic CCA. We guess the disease propagation is again in the waves, and the propagation has some connection with the underlying space, which in our case is the earth. Since the continents where human beings live is connected, mostly by airplanes between major airports, it would well explain why diseases usually explode in metropolitan areas, because the airlines between them are the “narrow hallways” connecting continents. It is also a possible explanation to the fact that some kind of diseases attack at a fixed pace, because the propagation of the disease may go around a cycle and come back to its origin, as in a “global defect”.

In social network studies, people are trying to understand how innovations spread among individuals. More focus is put on individuals with a lot of connections, in other words, those nodes in the social network that have more neighbors than others. It makes a lot of senses to study those greatly influential nodes, be-

cause the network is much more dense around them. But our research infer some extra attention to “narrow hallways”, which in social networks, are the people in a sequence, connecting big components of the network.

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