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Distributed Algorithms for the Optimal Design of Wireless Networks

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Abstract
This thesis studies the problem of optimal design of wireless networks whose operating points such as powers, routes and channel capacities are solutions for an optimization problem. Different from existing work that rely on global channel state information (CSI), we focus on distributed algorithms for the optimal wireless networks where terminals only have access to locally available CSI. To begin with, we study random access channels where terminals acquire instantaneous local CSI but do not know the probability distribution of the channel. We develop adaptive scheduling and power control algorithms and show that the proposed algorithm almost surely maximizes a proportional fair utility while adhering to instantaneous and average power constraints. Then, these results are extended to random access multihop wireless networks. In this case, the associated optimization problem is neither convex nor amenable to distributed implementation, so a problem approximation is introduced which allows us to decompose it into local subproblems in the dual domain. The solution method based on stochastic subgradient descent leads to an architecture composed of layers and layer interfaces. With limited amount of message passing among terminals and small computational cost, the proposed algorithm converges almost surely in an ergodic sense. Next, we study the optimal transmission over wireless channels with imperfect CSI available at the transmitter side. To reduce the likelihood of packet losses due to the mismatch between channel estimates and actual channel values, a backoff function is introduced to enforce the selection of more conservative coding modes. Joint determination of optimal power allocations and backoff functions is a nonconvex stochastic optimization problem with infinitely many variables. Exploiting the resulting equivalence between primal and dual problems, we show that optimal power allocations and channel backoff functions are uniquely determined by optimal dual variables and develop algorithms to find the optimal solution. Finally, we study the optimal design of wireless network from a game theoretical perspective. In particular, we formulate the problem as a Bayesian game in which each terminal maximizes the expected utility based on its belief about the network state. We show that optimal solutions for two special cases, namely FDMA and RA, are equilibrium points of the game. Therefore, the proposed game theoretic formulation can be regarded as general framework for optimal design of wireless networks. Furthermore, cognitive access algorithms are developed to find solutions to the game approximately.

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ABSTRACT

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Alejandro Ribeiro

This thesis studies the problem of optimal design of wireless networks whose operating points such as powers, routes and channel capacities are solutions for an optimization problem. Different from existing work that rely on global channel state information (CSI), we focus on distributed algorithms for the optimal wireless networks where terminals only have access to locally available CSI. To begin with, we study random access channels where terminals acquire instantaneous local CSI but do not know the probability distribution of the channel. We develop adaptive scheduling and power control algorithms and show that the proposed algorithm almost surely maximizes a proportional fair utility while adhering to instantaneous and average power constraints. Then, these results are extended to random access multihop wireless networks. In this case, the associated optimization problem is neither convex nor amenable to distributed implementation, so a problem approximation is introduced which allows us to decompose it into local subproblems in the dual domain. The solution method based on stochastic subgradient descent leads to an architecture composed of layers and layer interfaces. With limited amount of message passing among terminals and small computational cost, the proposed algorithm converges almost surely in an ergodic sense. Next, we study the optimal transmission over wireless channels with imperfect CSI available at the transmitter side. To reduce the likelihood of packet losses due to the mismatch between channel estimates and actual channel values, a backoff function is introduced to enforce the selection of more conservative coding modes. Joint determination of optimal power allocations and backoff functions is a nonconvex stochastic optimization problem with infinitely many variables. Exploiting the resulting equivalence between primal and dual

problems, we show that optimal power allocations and channel backoff functions are uniquely determined by optimal dual variables and develop algorithms to find the optimal solution. Finally, we study the optimal design of wireless network from a game theoretical perspective. In particular, we formulate the problem as a Bayesian game in which each terminal maximizes the expected utility based on its belief about the network state. We show that optimal solutions for two special cases, namely FDMA and RA, are equilibrium points of the game. Therefore, the proposed game theoretic formulation can be regarded as general framework for optimal design of wireless networks. Furthermore, cognitive access algorithms are developed to find solutions to the game approximately.

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Chapter 1

Introduction

Optimal design is emerging as the future paradigm for wireless networking. The fundamental idea is to select operating points as solutions of optimization problems, which, inasmuch as optimization criteria are properly chosen, yield the best possible network. Results in this field include architectural insights, e.g., [9], and protocol design, e.g., [13, 22], but a drawback shared by most of these works is that they rely on global channel state information (CSI); i.e., the optimal variables of a terminal depend on the channels between all pairs of terminals in the network. While availability of global CSI is plausible in certain situations, it is unlikely to hold if time varying fading channels are taken into account. In this case, distributed algorithms in which terminals operate based on locally available CSI are more practical. The focus of this thesis is to develop distributed algorithms for the optimal design of wireless networks.

When only local CSI is available, operating variables of each terminal are selected as functions of local CSI. This further leads to the selection of random access as the natural medium access choice. Indeed, if transmission decisions depend on local channels only and these channels are random and independent for different terminals, transmission decisions can be viewed as random and resultant link capacities as limited by collisions. In this chapter, we present an overview of random access channels and networks that will be used in the rest of the thesis.

1.1 Background

1.1.1 Random access wireless channels

Consider a multiple access channel with n terminals contending to communicate with a common AP. Time is divided in slots identified by an index t. We assume a backlogged system, i.e., all terminals always have packets to transmit in each time slot. The time-varying nonnegative channel $h_i(t) \in \mathbf{R}^+$ between terminal i and the AP at time t is modeled as block fading – for this to be true the length of a time slot has to be comparable to the coherence time of the channel. Channel gains $h_i(t_1)$ and $h_i(t_2)$ of terminal i at different time slots $t_1 \neq t_2$ are assumed independent and identically distributed (i.i.d.) with pdf $m_{h_i}(\cdot)$. Channel gains $h_i(t)$ and $h_j(t)$ of different terminals $i \neq j$ are also assumed independent. Channels are assumed to have continuous pdf. This latter assumption holds true for models used in practice, e.g., Rayleigh, Rician and Nakagami [14, Ch. 3]. We assume each terminal i has access to its channel gain $h_i(t)$ at each time slot t. While there are various alternatives to obtain channel state information, the simplest would be for the AP to send a beacon signal at the beginning of each time slot. This beacon signal would serve the double purpose of providing a reference for channel estimation as well as a synchronization signal.

Based on its channel state $h_i(t)$, node i decides whether to transmit or not in time slot t by determining the value of a scheduling function $q_i(t) := Q_i(h_i(t)) : \mathbf{R}^+ \to \{0, 1\}$. Node i transmits in time slot t if $q_i(t) = 1$ and remains silent if $q_i(t) = 0$. Notice that each terminal has a different scheduling function and that schedules $q_i(t)$ are determined based on the CSI of each node independently of other terminals. Although each node has access to its local CSI $h_i(t)$, the underlying pdf $m_{h_i}(\cdot)$ is unknown.

Besides channel access decisions, terminals also adapt transmission power to their channel gains through a power control function $P_i(h_i(t)) : \mathbf{R}^+ \to [0, p_i^{\text{inst}}]$, where $p_i^{\text{inst}} \in \mathbf{R}^+$ is a constant representing the instantaneous power constraint of node i. By using this function, terminal i

adjusts its transmission power $P_i(h_i(t))$ in response to $h_i(t)$. Similar to $q_i(t)$, we define $p_i(t) := P_i(h_i(t))$, representing the power allocated to node i in time slot t. If node i transmits in time slot t, $p_i(t)$ and $h_i(t)$ jointly determine the transmission rate through a function $C_i(h_i(t)p_i(t)) : \mathbf{R}^+ \rightarrow \mathbf{R}^+$. The exact form of $C_i(h_i(t)p_i(t))$ depends on how the signal is modulated and coded at the physical layer. Examples considered later in the thesis include capacity-achieving codes and adaptive modulation and coding (AMC). With capacity-achieving codes, $C_i(h_i(t)p_i(t))$ takes the form

$$C_i(h_i(t)p_i(t)) = B \log \left(1 + \frac{h_i(t)p_i(t)}{BN_0}\right), \tag{1.1}$$

where B and N_0 are the channel bandwidth and the power spectral density of the channel noise, respectively. With AMC, there are M transmission modes available. The mth mode affords communication rate τ_m and is used when the signal to noise ratio (SNR) $h_i(t)p_i(t)/BN_0$ is between η_m and η_{m+1}. The rate function is therefore

$$C_i(h_i(t)p_i(t)) = \sum_{m=1}^{M} \tau_m \mathbb{I}\left(\eta_m \leq \frac{h_i(t)p_i(t)}{BN_0} \leq \eta_{m+1}\right), \tag{1.2}$$

where $\mathbb{I}(\cdot)$ denotes the indicator function. To keep the analysis general we do not restrict $C_i(h_i(t)p_i(t))$ to take either specific form. It is only assumed that $C_i(h_i(t)p_i(t))$ is a nonnegative increasing function of the product of $h_i(t)$ and $p_i(t)$ that takes finite values for finite arguments. These assumptions are satisfied by (1.1) and (1.2) and are likely to hold in practice.

Since terminals contend for channel access, transmission of terminal i in a time slot t is successful if and only if $q_i(t) = 1$ and $q_j(t) = 0$ for all $j \neq i$. If the transmission of terminal i is successful, its transmission rate is determined by $C_i(h_i(t)p_i(t))$. As as consequence, the instantaneous transmission rate for terminal i in time slot t is

$$r_i(t) = C_i\left(h_i(t)p_i(t)\right) q_i(t) \prod_{j=1, j\neq i}^{n} \left[1 - q_j(t)\right]. \tag{1.3}$$

Assuming an ergodic mode of operation, quality of service is determined by the long term behavior of $r_i(t)$. This implies that system performance is determined by the ergodic limits

$$\begin{aligned} r_i &:= \lim_{t\to\infty} \frac{1}{t}\sum_{u=1}^{t} r_i(u) \\ &= \lim_{t\to\infty} \frac{1}{t}\sum_{u=1}^{t} \left[C_i\left(h_i(u)p_i(u)\right) q_i(u) \prod_{j=1,j\neq i}^{n} [1-q_j(u)] \right]. \end{aligned} \tag{1.4}$$

Assuming ergodicity of schedules $q_i(t) = q_i(h_i(t))$ and power allocations $p_i(t) = p_i(h_i(t))$, the limit r_i can be written as a expected value over channel realizations,

$$r_i = \mathbb{E}_{\mathbf{h}} \left[Q_i(h_i) C_i(h_i P_i(h_i)) \prod_{j=1,j\neq i}^{n} [1 - Q_j(h_j)] \right], \tag{1.5}$$

where we have defined the vector $\mathbf{h} = [h_1, \cdots, h_n]^T$ grouping all channels h_i. An important observation here is that since terminals are required to make channel access and power control decisions independently of each other, $Q_i(h_i)$ and $P_i(h_i)$ are independent of $Q_j(h_j)$ and $P_j(h_j)$ for all $i \neq j$. This allows us to rewrite r_i as

$$r_i = \mathbb{E}_{h_i}\left[Q_i(h_i)C_i(h_i P_i(h_i))\right] \prod_{j=1,j\neq i}^{n} \left[1 - \mathbb{E}_{h_j}[Q_j(h_j)]\right]. \tag{1.6}$$

In addition to instantaneous power constraints $p_i(t) \leq p_i^{\text{inst}}$, terminals adhere to average power constraints $p_i^{\text{avg}} \in \mathbf{R}^+$ as in, e.g., [8]. This average power constraint restricts the long term average of transmitted power that we either write as an ergodic limit or as an expectation over channel realizations,

$$p_i := \lim_{t\to\infty} \frac{1}{t}\sum_{u=1}^{t} q_i(u)p_i(u) = \mathbb{E}_{h_i}[Q_i(h_i)P_i(h_i)]. \tag{1.7}$$

1.1.2 Random access wireless networks

Consider an ad-hoc wireless network consisting of J terminals indexed as $i = 1, \ldots J$. Network connectivity is modeled as a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with vertices $v \in \mathcal{V} := \{1, \ldots, J\}$ representing the J terminals and edges $e = (i,j) \in \mathcal{E}$ connecting pairs of terminals that can communicate with each other. Denote the neighborhood of terminal i as $\mathcal{N}(i) := \{j \mid (i,j) \in \mathcal{E}\}$ and define the interference neighborhood of the link (i,j) as the set of nodes $\mathcal{M}_i(j) := \mathcal{N}(j) \cup \{j\} \backslash \{i\}$ whose transmission can interfere with a transmission from i to j. The network supports a set $\mathcal{K} := \{1, \ldots, K\}$ of end-to-end flows through multihop transmission. The average rate at which k-flow packets are generated at i is denoted by a_i^k. Terminal i transmits these packets to neighboring terminals at average rates r_{ij}^k and, consequently, receives k-flow packets from neighbors at average rates r_{ji}^k. To conserve flow, exogenous rates a_i^k and endogenous rates r_{ij}^k at terminal i must satisfy

$$a_i^k \leq \sum_{j \in \mathcal{N}(i)} \left(r_{ij}^k - r_{ji}^k \right), \quad \text{for all } i \in \mathcal{V}, \text{ and } k \in \mathcal{K}. \tag{1.8}$$

Further denote the capacity of the link from $i \to j$ as c_{ij}. Since packets of different flows k are transmitted from i to j at rates r_{ij}^k it must be

$$\sum_{k \in \mathcal{K}} r_{ij}^k \leq c_{ij}, \quad \text{for all } (i,j) \in \mathcal{E}. \tag{1.9}$$

Unlike wireline networks where c_{ij} are fixed, link capacities in wireless networks are dynamic. Similar to what we did in Section 1.1.1, let time be divided into slots indexed by t and denote the channel between i and j at time t as $h_{ij}(t)$. The channel is assumed to be block fading and channel gains $h_{ij}(t)$ of link (i,j) are assumed independent and identically distributed with probability distribution function (pdf) $m_{h_{ij}}(\cdot)$. For reference, define the vector of terminal i outgoing channels $\mathbf{h}_i(t) := \{h_{ij}(t) | j \in \mathcal{N}(i)\}$ and the vector of all channels $\mathbf{h}(t) := \{h_{ij}(t) | (i,j) \in \mathcal{E}\}$. Denote their pdfs as $m_{\mathbf{h}_i}(\cdot)$ and $m_{\mathbf{h}}(\cdot)$, respectively.

Based on the channel state $\mathbf{h}_i(t)$ of his outgoing links, terminal i decides whether to transmit or not on link (i,j) in time slot t by determining the value of a scheduling function $q_{ij}(t) := Q_{ij}(\mathbf{h}_i(t)) \in \{0,1\}$. If $q_{ij}(t) = 1$, terminal i transmits on link (i,j) and remains silent otherwise. Further define $q_i(t) := Q_i(\mathbf{h}_i(t)) := \sum_{j \in \mathcal{N}(i)} Q_{ij}(\mathbf{h}_i(t))$ to indicate a transmission from i to any of his neighbors. We restrict i to communicate with, at most, one neighbor per time slot implying that we must have $q_i(t) \in \{0,1\}$. We emphasize that $q_{ij}(t) := Q_{ij}(\mathbf{h}_i(t))$ depends on local outgoing channels only and not on global CSI. Further note that terminals have access to instantaneous local CSI $\mathbf{h}_i(t)$ but underlying pdfs $m_{\mathbf{h}_i}(\cdot)$ are unknown.

Besides channel access decisions, terminals also adapt transmission power to local CSI through a power control function $p_{ij}(t) := P_{ij}(\mathbf{h}_i(t))$ taking values in $[0, p_{ij}^{\text{inst}}]$. Here, p_{ij}^{inst} represents the maximum allowable instantaneous power on link (i,j). The average power consumed by terminal i is then given as the expected value over channel realizations of the sum of $P_{ij}(\mathbf{h}_i)$ over all $j \in \mathcal{N}(i)$, i.e.,

$$p_i \geq \mathbb{E}_{\mathbf{h}_i}\left[\sum_{j \in \mathcal{N}(i)} P_{ij}(\mathbf{h}_i) Q_{ij}(\mathbf{h}_i)\right], \tag{1.10}$$

where we also relaxed the equality constraint to an inequality, which can be done without loss of optimality. If terminal i transmits to node j in time slot t, $p_{ij}(t)$ and $h_{ij}(t)$ determine the transmission rate through a function $C_{ij}(h_{ij}(t)p_{ij}(t))$ whose form depends on modulation and coding.

Due to contention, a transmission from i to j at time t succeeds if a collision does not occur. In turn, this happens if: (i) Terminal i transmits to j, i.e., $q_{ij}(t) = 1$. (ii) Terminal j is silent, i.e., $q_j(t) = 0$. (iii) No other neighbor of j transmits, i.e. $q_l(t) = 0$ for all $l \in \mathcal{N}(j)$ and $l \neq i$. Recalling the definition of interference neighborhood $\mathcal{M}_i(j)$ and that if a transmission occurs its rate is $C_{ij}(h_{ij}(t)p_{ij}(t))$ we express the instantaneous transmission rate from i to j in time slot t as $c_{ij}(t) := c_{ij}(\mathbf{h}_i(t)) = C_{ij}(h_{ij}(t)p_{ij}(t))q_{ij}(t)\prod_{l \in \mathcal{M}_i(j)}[1 - q_l(t)]$. Assuming an ergodic mode of operation, the capacity of link $i \to j$ can then be written as

$$c_{ij} = \mathbb{E}_{\mathbf{h}}\left[C_{ij}(h_{ij}P_{ij}(\mathbf{h}_i))Q_{ij}(\mathbf{h}_i)\prod_{l \in \mathcal{M}_i(j)}[1 - Q_l(\mathbf{h}_l)]\right]. \tag{1.11}$$

Because terminals are required to make channel access and power control decisions independently of each other, $Q_{ij}(\mathbf{h}_i)$ and $P_{ij}(\mathbf{h}_i)$ are independent of $Q_{lm}(\mathbf{h}_l)$ and $P_{lm}(\mathbf{h}_l)$ for all $i \neq l$. Since $Q_l(\mathbf{h}_l) := \sum_{m \in \mathcal{N}(l)} Q_{lm}(\mathbf{h}_l(t))$ by definition, it follows that $Q_{ij}(\mathbf{h}_i)$ is also independent of $Q_l(\mathbf{h}_l)$ for all $i \neq l$. This allows us to write the expectation of the product on the right hand side of (1.11) as a product of expectations,

$$c_{ij} \leq \mathbb{E}_{\mathbf{h}_i}\left[C_{ij}\Big(h_{ij}P_{ij}(\mathbf{h}_i)\Big)Q_{ij}(\mathbf{h}_i)\right]\prod_{l \in \mathcal{M}_i(j)}\left[1 - \mathbb{E}_{\mathbf{h}_l}\Big[Q_l(\mathbf{h}_l)\Big]\right], \tag{1.12}$$

where we also relaxed the equality constraint to an inequality, which can be done without loss of optimality [1].

The operating point of a wireless network is characterized by variables a_i^k, r_{ij}^k, c_{ij}, p_i and functions $P_{ij}(\mathbf{h}_i)$, $Q_{ij}(\mathbf{h}_i)$. Besides (1.8)-(1.12), these variables are subject to certain box constraints. Admission variables, have lower and upper bounds due to application layer requirements, i.e., $a_i^{\min} \leq a_i^k \leq a_i^{\max}$. Similarly, routing variables, link capacities, and terminal power budgets cannot be negative and are also subject to given upper bounds, i.e., $0 \leq r_{ij}^k \leq r_{ij}^{\max}$, $0 \leq c_{ij} \leq c_{ij}^{\max}$, and $0 \leq p_i \leq p_i^{\max}$. Furthermore, according to definition, $P_{ij}(\mathbf{h}_i)$ and $Q_{ij}(\mathbf{h}_i)$ can only take values from $[0, p_{ij}^{\text{inst}}]$ and $\{0, 1\}$, respectively. For notational simplicity, we define vectors $\mathbf{x}_i := \left\{p_i, a_{ij}^k, r_{ij}^k, c_{ij} : \forall j \in \mathcal{N}(i)\right\}$ and $\mathbf{P}_i(\mathbf{h}_i) := \{P_{ij}(\mathbf{h}_i), Q_{ij}(\mathbf{h}_i) : \forall j \in \mathcal{N}(i)\}$ to group all the variables related to terminal i and summarize these box constraints as $\{\mathbf{x}_i, \mathbf{P}_i(\mathbf{h}_i)\} \in \mathcal{B}_i$ with

$$\begin{aligned}\mathcal{B}_i := \Bigg\{ \mathbf{x}_i, \mathbf{P}_i(\mathbf{h}_i) \,\Bigg|\, & a_i^{\min} \leq a_i^k \leq a_i^{\max},\ 0 \leq r_{ij}^k \leq r_{ij}^{\max}, \\ & 0 \leq c_{ij} \leq c_{ij}^{\max},\ 0 \leq p_i \leq p_i^{\max},\ 0 \leq P_{ij}(\mathbf{h}_i) \leq p_{ij}^{\text{inst}}, \\ & Q_{ij}(\mathbf{h}_i) \in \{0,1\},\ Q_i(\mathbf{h}_i) \in \{0,1\} \Bigg\}. \end{aligned} \tag{1.13}$$

1.2 Roadmap

Our first investigation focuses on random access channel where terminals contend for communicating with a the central AP. This models the physical layer of the wireless random access network we shall study later on. We develop adaptive scheduling and power control algorithms for random access in a multiple access channel where terminals acquire instantaneous channel

[1] If we have channel reciprocity $h_{ij}(t) = h_{ji}(t)$, the derivation of (1.12) from (1.11) is no longer valid since power control and channel access functions of neighboring nodes will have common arguments implying that $Q_{ij}(\mathbf{h}_i)$ and $Q_{ji}(\mathbf{h}_j)$ would not be independent. The general methodology used here seems applicable but is beyond the scope of the present paper.

state information but do not know the probability distribution of the channel [16]. In each time slot, terminals measure the channel to the common access point. Based on the observed channel value, they determine whether to transmit or not and, if they decide to do so, adjust their transmitted power. We show that the proposed algorithm almost surely maximizes a proportional fair utility while adhering to instantaneous and average power constraints. These results are presented in Chapter 2.

We then generalize the algorithm proposed for random access channel to wireless multihop networks where each node determines its operating point using its local CSI distributedly [17]. Since the associated optimization problem is neither convex nor amenable to distributed implementation, a problem approximation is introduced. This approximation is still not convex but it has zero duality gap and can be solved and decomposed into local subproblems in the dual domain. The solution method is through a stochastic subgradient descent algorithm that operates without knowledge of the fading's probability distribution and leads to an architecture composed of layers and layer interfaces. With limited amount of message passing among terminals and small computational cost, we show that the proposed algorithm converges almost surely in an ergodic sense. These results are presented in Chapter 3.

Both above proposed algorithms require terminals to adapt transmission parameters such as power and rate to time-varying channel conditions to improve system's overall performance. Although accurate CSI is essential to achieve this goal, perfect CSI is rarely available in practice due to estimation errors and, perhaps more fundamentally, to feedback delay. Our next topic is to develop algorithms to handle imperfect CSI in the transmission over wireless channels [18]. In particular, we consider three types of wireless channels, namely single user point-to-point block fading channels [15], multiuser downlink orthogonal frequency division multiplexing (OFDM) [38], and multiuser uplink random access (RA) [29], where the transmitter adapt transmitted power and coding mode to imperfect channel estimates in order to maximize expected throughput subject to average power constraints. To reduce the likelihood of packet losses due to the mismatch between channel estimates and actual channel values, a backoff function is further introduced to enforce the selection of more conservative coding modes. Joint determination of optimal power allocations and backoff functions is a nonconvex stochastic optimization problem with infinitely many variables that despite its lack of convexity is part of a class of problems with null duality gap. Exploiting the resulting equivalence between primal and dual problems, we show that optimal power allocations and channel backoff functions are uniquely determined by optimal dual variables. This affords considerable simplification because the dual problem is convex and finite dimensional. We further exploit this reduction in computational complexity to develop iterative algorithms to find optimal operating points. These results are presented in Chapter 4.

So far the distributed algorithms we developed are based on local CSI only (either perfect or imperfect). In practice, terminals may have knowledge about channels of neighboring nodes in addition to local CSI. This motivates us to investigate wireless networks where each terminal has a different belief about the global channel states and adapts its transmission policy to the belief. In this setting, frequency division multiple access (FDMA) and channel aware random access (RA) are two special cases where perfect global and local CSI are available, respectively. To find solutions for general cases, we formulate the problem as a Bayesian game in which each terminal maximizes the expected utility based on its belief. We show that optimal solutions for both FDMA and RA are equilibrium points of the game. Therefore, the proposed game theoretic formulation can be regarded as general framework for multiuser wireless communications. Furthermore, we develop a cognitive access algorithm that solves the problem approximately. These results are presented in Chapter 5.

Chapter 2

Distributed algorithms for optimal random access channels

In this chapter, we consider wireless random access channels in which terminals contend for access to a common access point (AP) as introduced in Section 1.1.1. To exploit favorable channel conditions terminals adapt their transmitted power and access decisions to the state of the random fading channels linking them to the AP. The challenges in developing this adaptive scheme are that terminals have access to their own channel state information (CSI) only, and that the probability distribution function (pdf) of the fading channel is unknown. Our goal is to develop a distributed learning algorithm to determine optimal transmitted power and channel access decisions relying on local CSI only.

The idea of adapting medium access and power control to CSI has been extensively explored in wireless communications. Early references dealing with power adaptation on the uplink of multiuser systems focus on centralized power control schemes where the AP collects channel states for all terminals to select the one to be scheduled. In, e.g., [19], the AP schedules the terminal with the best channel gain with a power adapted to the channel condition. Similar ideas have also been used for scheduling and resource allocation in broadcast downlink channels, see e.g., [3, 11, 23]. Although these centralized schemes exploit multiuser diversity, they require significant information exchange between terminals and the AP; a problem exacerbated when the number of users is large. To avoid this overhead, recent work integrates channel adaptation into random access protocols. Exploiting the idea of aligning schedules to good channel opportunities, [29] develops a distributed channel-aware Aloha protocol in which terminals transmit only when their channel gains exceed pre-defined thresholds. This algorithm is shown to be asymptotically optimal in the sense that the only performance loss compared to a centralized scheme is due to user contention.

Under simple collision models, it has been shown that distributed threshold-based schedulers with properly designed thresholds maximize total throughput of a network with homogeneous users and total logarithmic throughput in the case of heterogeneous users [50]. Similar threshold-based decentralized adaptive random access schemes have been investigated for other types of networks with different packet reception models, see e.g., [1, 6, 25, 27, 30, 46, 51]. To compute the optimal thresholds, however, terminals are assumed to know the probability distribution of their fading channels. This is a restrictive assumption because the channel fading distribution is usually unknown and can only be estimated based on channel observations. Overcoming this limitation motivates the development of adaptive algorithms to learn optimal operating points based on local CSI [4, 37]. The work in [4] proposes a heuristic adaptive algorithm for threshold-based schedulers in which the thresholds are tuned based on local channel realizations in a time window. The work in [37] develops an online learning algorithm for transmission probability and power control under rate constraints using game-theoretic approaches. However, neither [4] nor [37] guarantees global optimality.

The contribution of this chapter is the development of an optimal distributed adaptive algorithm for scheduling and power control given that terminals only have access to local CSI and operate independently of each other. At each time slot, terminals observe their channel states and decide whether to transmit or not. If they decide to transmit, they choose a power for their communication attempt. As time progresses, power budgets are satisfied almost surely, while the network almost surely maximizes a weighted proportional fair utility. We remark that terminals operate independently without access to the channel state of other terminals and that the channel pdf is unknown. The proposed algorithm can handle general non-convex, even discontinuous, rate functions with manageable computational complexity. It is worth noting that under the frame work of network utility maximization (NUM) algorithms for computing optimal channel access probabilities in random access networks are developed (see e.g. [21]). However, neither fading nor power adaptation is considered in these work.

The presentation begins by formulating optimal adaptive random access as a utility maximization problem whose objective is to maximize a weighted sum of throughput logarithms (Section 2.1). The variables to be determined as a solution of this optimization problem are a scheduling function that determines if a terminal should transmit or not based on its CSI, and a power allocation function that maps a terminal CSI to its transmit power. It is important to remark that: (i) because fading takes on a continuum of values, this optimization problem is infinite-dimensional; (ii) the constraints modeling random access are non-convex; (iii) despite the existence of these non-convex constraints optimization problems of this form are known to have null duality gap [33]; and (iv) since the number of constraints turns out to be finite the optimization problem is finite-dimensional in the dual domain. A further complication is that the original problem formulation yields solutions that require access to global CSI.

We start by overcoming the dependence on global CSI by introducing an equivalent decomposition in per-terminal subproblems whereby nodes maximize local utilities (Section 2.2.A). While this reformulation yields solutions that depend on local CSI only, attempting a solution in the primal domain is difficult because the per-terminal subproblems inherit infinite dimensionality and lack of convexity from the original problem formulation, as well as the need to have access to the channel pdf. We therefore exploit the lack of duality gap to approach their solution through a stochastic subgradient descent algorithm in the dual domain (Section 2.2.B). Based on channel realizations in each time slot, the algorithm computes instantaneous values for the scheduling and power allocation functions and updates Lagrangian multipliers in a direction that can be proven to point towards the set of optimal dual variables in an average sense (Proposition 1). Exploiting this fact we prove that the throughput utility achieved by the algorithm almost surely converges to a value close to the optimal utility. The gap between the optimal and the achieved utility can be made arbitrarily small by reducing a fixed step size (Theorem 1). The chapter closes with a numerical evaluation of the proposed algorithm for a randomly generated heterogeneous network (Section 2.3). To illustrate generality of the proposed approach we consider a system with terminals employing capacity achieving codes (Section 2.3.1) and a more practical scenario with nodes employing adaptive modulation and coding (Section 2.3.2). Concluding remarks are presented in Section 2.4.

2.1 Problem formulation

Consider a random access channel as introduced in Section 1.1.1. With rates r_i given as in (1.6), our objective is to maximize a weighted proportional fair (WPF) utility defined as

$$U(\mathbf{r}) = \sum_{i=1}^{n} w_i \log(r_i), \tag{2.1}$$

where $\mathbf{r} = [r_1, \cdots, r_n]^T$ is the vector of rates and $w_i \in \mathbf{R}^+$ is the weight coefficient for terminal i. Setting $w_i = w_j$ for all $i \neq j$ in a homogenous system with all channels having the same pdf, the WPF utility is equivalent to maximizing the sum of throughputs. In a heterogeneous network where channel pdfs vary among users, maximizing $U(\mathbf{r})$ yields solutions that are fair since it prevents users from having very low transmission rates.

Grouping the objective in (2.1) with the constraints in (1.6) and (1.7), optimal adaptive random access is formulated as the following optimization problem

$$
\begin{aligned}
\mathsf{P} \ &= \max \ U(\mathbf{r}) \\
\text{s.t.} \ & r_i = \mathbb{E}_{h_i}\left[Q_i(h_i)C_i(h_iP_i(h_i))\right] \prod_{j=1, j\neq i}^{n} \left[1 - \mathbb{E}_{h_j}[Q_j(h_j)]\right], \\
& \mathbb{E}_{h_i}\left[Q_i(h_i)P_i(h_i)\right] \leq p_i^{\text{avg}}, \\
& Q_i(h_i) \in \mathcal{Q}, P_i(h_i) \in \mathcal{P}_i, \forall i
\end{aligned} \tag{2.2}
$$

where $\mathcal{Q}$ is the set of functions $\mathbf{R}^+ \to \{0, 1\}$ taking values on $\{0, 1\}$ and $\mathcal{P}_i$ represents the set of functions $\mathbf{R}^+ \to [0, p_i^{\text{inst}}]$ taking values on $[0, p_i^{\text{inst}}]$. Notice that the joint optimization across users required to solve (2.2) introduces *functional* dependence between the actions of different terminals. This is not incongruent with the requirement of *statistically* independent schedules in each time slot. In fact, the notations $Q_i(h_i)$ and $P_i(h_i)$ in (2.2) stipulates that terminals are required to make channel access and power allocation decisions based on local CSI only. Consequently, although problem (9) requires joint optimization across users, it restricts optimization to policies that result in statistically independent operations.

The goal of this chapter is to develop an online algorithm to determine schedules $q_i(t)$ and power assignments $p_i(t)$ having statistics that solve the optimization problem in (2.2). The algorithm is required to: (i) operate without knowledge of the channel distribution; and (ii) yield functions $q_i(t)$ and $p_i(t)$ that depend on the current and past values of the local channel $h_i(t)$ but are independent of other terminal's channels $h_j(t)$ for $j \neq i$.

Remark 1. *In order to allow terminals to know if their transmissions are successful or not, the AP provides feedback on whether the transmission attempt was successful or a collision detected. If a terminal transmits a packet but detects a collision, it can reschedule the packet for retransmission in a subsequent time slot. We remark that feedback does not introduce correlation between the transmission decisions of different terminals. The provided feedback only tells terminals if they should retransmit previous packets or not, but does not enforce them to make channel access or power allocation decisions.*

2.2 Adaptive algorithms for optimal random access channels

The stated goal is to devise scheduling and power control policies based on local CSI that are globally optimal as per (2.2). These two objectives, i.e., global optimality while relying on local CSI, seem to contradict each other. Because r_i depends not only on $Q_i(h_i)$ and $P_i(h_i)$ but on $Q_j(h_j)$ for all $j \neq i$, it seems that optimal $Q_i(h_i)$ and $P_i(h_i)$ solving (2.2) might also be functions of other terminals' CSI. To see that this is not the case, we will show that it is possible to decompose (2.2) in per terminal subproblems. After introducing this decomposition the complicating fact that the channel pdf $f_{h_i}(h_i)$ is unknown remains. To overcome this complication, we will introduce a stochastic subgradient descent algorithm in the dual domain that is optimal in an ergodic sense.

2.2.1 Problem decomposition and its dual

Begin then by separating (2.2) in per terminal subproblems. To do so, we substitute (1.6) into (2.1) and express the logarithm of a product as a sum of logarithms. As a result, the global utility in (2.1) can be rewritten as

$$U(\mathbf{r}) = \sum_{i=1}^{n} w_i \left[\log \mathbb{E}_{h_i}[Q_i(h_i)C_i(h_iP_i(h_i))] + \sum_{j=1, j\neq i}^{n} \log\left[1 - \mathbb{E}_{h_j}[Q_j(h_j)]\right] \right]. \tag{2.3}$$

Note that each summand in (2.3) only involves variables related to a particular node. Thus, we can reorder summands in (2.3) to group all of the terms pertaining to node i. Further defining $\tilde{w}_i := \sum_{j=1, j\neq i}^{n} w_i$, we can rewrite (2.3) as

$$U(\mathbf{r}) = \sum_{i=1}^{n} \left[w_i \log\left[\mathbb{E}_{h_i}[Q_i(h_i)C_i(h_iP_i(h_i))]\right] + \tilde{w}_i \log\left[1 - \mathbb{E}_{h_i}[Q_i(h_i)]\right] \right] := \sum_{i=1}^{n} U_i, \tag{2.4}$$

where we have defined the local utilities U_i. Since U_i only involves variables that are related to terminal i, it can be regarded as a utility function for terminal i. To maximize $U(\mathbf{r})$ for the whole system it suffices to separately maximize U_i for each terminal i. Introducing auxiliary variables $x_i = \mathbb{E}_{h_i}[Q_i(h_i)C_i(h_iP_i(h_i))]$ and $y_i = \mathbb{E}_{h_i}[Q_i(h_i)]$, it follows that (2.2) is equivalent to the following per terminal subproblems

$$
\begin{aligned}
\mathsf{P}_i \quad = \quad & \max w_i \log x_i + \tilde{w}_i \log(1 - y_i) \\
& \text{s.t.} \;\; x_i \leq \mathbb{E}_{h_i}\left[Q_i(h_i) C_i(h_i P_i(h_i))\right], \\
& \qquad y_i \geq \mathbb{E}_{h_i}\left[Q_i(h_i)\right], \\
& \qquad \mathbb{E}_{h_i}\left[Q_i(h_i) P_i(h_i)\right] \leq p_i^{\text{avg}}, \\
& \qquad x_i \geq 0, 0 \leq y_i \leq 1, Q_i(h_i) \in \mathcal{Q}, P_i(h_i) \in \mathcal{P}_i,
\end{aligned} \tag{2.5}
$$

where we relaxed the equality constraints to inequality ones which can be done without loss of optimality. Finding optimal solutions of (2.5) for all terminals i is equivalent to solving (2.2). Different from (2.2), however, there is no coupling between variables of different terminals in (2.5). This property leads naturally to optimal $Q_i(h_i)$ and $P_i(h_i)$ that are independent of other terminals' CSI as required by problem definition. Alas, (2.5) inherits the complex structure of (2.2).

As is the case with (2.2), solving (2.5) is difficult because: (i) The optimization space in (2.5) includes functions $Q_i(h_i)$ and $P_i(h_i)$ that are defined on $\mathbf{R}^+$, implying that the dimension of the problem is infinite. (ii) The rate function $C_i(h_i P_i(h_i))$ is in general non-concave with respect to $h_i P_i(h_i)$, and may be even discontinuous as in (1.2). (iii) The constraints involve expected values over random variables h_i whose pdfs are unknown.

An important observation is that the number of constraints in (2.5) is finite. This implies that while there are infinite variables in the primal domain, there are a finite number of variables in the dual domain. This observation suggests that (2.5) is more tractable in the dual space. Introduce then Lagrange multipliers $\boldsymbol{\lambda}_i = [\lambda_{i1}, \lambda_{i2}, \lambda_{i3}]^T$ associated with the first three inequality constraints in (2.5); define vectors $\mathbf{x}_i := [x_i, y_i]^T$ and $\mathbf{P}_i(h_i) := [Q_i(h_i), P_i(h_i)]^T$; and write the Lagragian of the optimization problem in (2.5) as

$$
\mathcal{L}_i(\mathbf{x}_i, \mathbf{P}_i(h_i), \boldsymbol{\lambda}_i) = w_i \log x_i + \tilde{w}_i \log(1 - y_i) + \lambda_{i1}\left[\mathbb{E}_{h_i}\left[Q_i(h_i) C_i(h_i P_i(h_i))\right] - x_i\right]
$$

$$+ \lambda_{i2} \left[y_i - \mathbb{E}_{h_i}\left[Q_i(h_i)\right]\right] + \lambda_{i3}\left[p_i^{\text{avg}} - \mathbb{E}_{h_i}\left[Q_i(h_i)P_i(h_i)\right]\right]$$

$$= \lambda_{i3} p_i^{\text{avg}} + \left[w_i \log x_i - \lambda_{i1} x_i\right] + \left[\tilde{w}_i \log(1-y_i) + \lambda_{i2} y_i\right]$$

$$+ \mathbb{E}_{h_i}\left[Q_i(h_i)\left[\lambda_{i1} C_i(h_i P_i(h_i)) - \lambda_{i2} - \lambda_{i3} P_i(h_i)\right]\right]. \tag{2.6}$$

where the second equality follows after reordering terms in the first equation. Notice that the first term in the second equality in (2.6) depends on x_i only, the second term on y_i and the third term on $P_i(h_i)$ and $Q_i(h_i)$. This property is exploited later on. The dual function is then defined as the maximum of the Lagrangian over the set of feasible $\mathbf{x}_i$ and $\mathbf{P}_i(h_i)$, i.e.,

$$g_i(\boldsymbol{\lambda}_i) := \max \ \mathcal{L}_i(\mathbf{x}_i, \mathbf{P}_i(h_i), \boldsymbol{\lambda}_i)$$

$$\text{s.t. } x_i \geq 0, 0 \leq y_i \leq 1, Q_i(h_i) \in \mathcal{Q}, P_i(h_i) \in \mathcal{P}_i. \tag{2.7}$$

We now can write the dual problem as the minimum of $g_i(\boldsymbol{\lambda}_i)$ over positive dual variables, i.e.,

$$\mathsf{D}_i = \min_{\boldsymbol{\lambda}_i \geq 0} \quad g_i(\boldsymbol{\lambda}_i). \tag{2.8}$$

In general, the optimal dual value D_i of (2.8) provides an upper bound for the optimal primal value P_i of (2.5), i.e., $\mathsf{D}_i \geq \mathsf{P}_i$. While the inequality is typically strict for non-convex problems, for the problem in (2.5) $\mathsf{P}_i = \mathsf{D}_i$ as long as the fading distribution has no realization with positive probability [33]. Notice that this is true despite the non-convex constraints present in (2.5). This lack of duality gap implies that the finite dimensional convex dual problem is equivalent to the infinite dimensional nonconvex primal problem. While this affords a substantial improvement in computational tractability, it does not necessarily mean that solving the dual problem is easy because evaluation of the dual function's value requires maximization of the Lagrangian. In particular, this maximization includes an expected value over the unknown channel distribution $f_{h_i}(h_i)$. Still, convexity of the dual function allows the use of descent algorithms in the dual domain because any local optimal solution is a global optimal solution $\boldsymbol{\lambda}_i^* = [\lambda_{i1}^*, \lambda_{i2}^*, \lambda_{i3}^*]^T$. This property is exploited next to develop a stochastic subgradient descent algorithm that solves (2.8) using observations of instantaneous channel realizations $h_i(t)$.

2.2.2 Adaptive algorithms using stochastic subgradient descent

Instead of directly finding optimal x_i, y_i, $Q_i(h_i)$ and $P_i(h_i)$ for the primal problem (2.5), the proposed algorithm exploits the lack of duality gap to use a stochastic subgradient descent in the dual domain. Starting from given dual variables $\boldsymbol{\lambda}_i(t)$, the algorithm computes instantaneous primal variables $x_i(t)$, $y_i(t)$, $q_i(t)$ and $p_i(t)$ based on channel realization $h_i(t)$ in time slot t, and uses these values to update dual variables $\boldsymbol{\lambda}_i(t+1)$. Specifically, the algorithm starts finding primal variables that optimize the summands of the Lagrangian in (2.6) (the operator $[\cdot]^+$ denotes projection in the positive orthant)

$$x_i(t) = \underset{x_i \geq 0}{\operatorname{argmax}} \{w_i \log x_i - \lambda_{i1}(t)x_i\} = \frac{w_i}{\lambda_{i1}(t)}, \tag{2.9}$$

$$y_i(t) = \underset{0 \leq y_i \leq 1}{\operatorname{argmax}} \{\tilde{w}_i \log(1-y_i) + \lambda_{i2}(t)y_i\} = \left[1 - \frac{\tilde{w}_i}{\lambda_{i2}(t)}\right]^+, \tag{2.10}$$

$$\{q_i(t), p_i(t)\} = \underset{q_i \in \{0,1\}, p_i \in [0, p_i^{\text{inst}}]}{\operatorname{argmax}} \{q_i \left[\lambda_{i1}(t)C_i(h_i(t)p_i) - \lambda_{i2}(t) - \lambda_{i3}(t)p_i\right]\}, \tag{2.11}$$

The maximization in (2.11) determines schedules and transmitted power associated with current channel realization $h_i(t)$. Since q_i in (2.11) takes values on {0,1} the objective is either 0 when $q_i = 0$ or $\lambda_{i1}(t)C_i(h_i(t)p_i) - \lambda_{i2}(t) - \lambda_{i3}(t)p_i$ when $q_i = 1$. Thus, to solve (2.11) we only need to find the optimal $p_i(t)$ when $q_i(t) = 1$ and see if the resulting objective is greater than 0. Thus, we can rewrite (2.11) as

$$\begin{aligned} p_i(t) &= \underset{p_i \in [0, p_i^{\text{inst}}]}{\operatorname{argmax}} \{\lambda_{i1}(t)C_i(h_i(t)p_i) - \lambda_{i2}(t) - \lambda_{i3}(t)p_i\}, \\ q_i(t) &= H\Big(\lambda_{i1}(t)C_i(h_i(t)p_i(t)) - \lambda_{i2}(t) - \lambda_{i3}(t)p_i(t)\Big), \end{aligned} \tag{2.12}$$

where $H(a)$ denotes Heaviside's step function with $H(a) = 1$ for $a > 0$ and $H(a) = 0$ otherwise.

Based on $x_i(t)$, $y_i(t)$, $q_i(t)$ and $p_i(t)$, define the stochastic subgradient $\mathbf{s}_i(t)$=$[s_{i1}(t), s_{i2}(t)$,$s_{i3}(t)]^T$ with components

$$s_{i1}(t) = q_i(t)C_i(h_i(t)p_i(t)) - x_i(t), \tag{2.13}$$

$$s_{i2}(t) = y_i(t) - q_i(t), \tag{2.14}$$

$$s_{i3}(t) = p_i^{\text{avg}} - q_i(t)p_i(t). \tag{2.15}$$

The algorithm is completed with the introduction of a constant step size ϵ and a descent update in the dual domain along the stochastic subgradient $\mathbf{s}_i(t)$

$$\lambda_{il}(t+1) = \left[\lambda_{il}(t) - \epsilon s_{il}(t)\right]^+, \quad \text{for } l = 1, 2, 3. \tag{2.16}$$

Notice that computation of variables in (2.9)-(2.16) does not require information exchanges between terminals. This guarantees $Q_i(h_i)$ and $P_i(h_i)$ to be independent of $Q_j(h_j)$ and $P_j(h_j)$ for all $i \neq j$ as required by problem formulation. The proposed algorithm is summarized in Algorithm 1.

To analyze convergence of (2.9)-(2.16) let us start by showing that $\mathbf{s}_i(t)$ is indeed a stochastic subgradient of the dual function as stated in the following proposition.

Proposition 1. *Given $\boldsymbol{\lambda}_i(t)$, the expected value of the stochastic subgradient $\mathbf{s}_i(t)$ is a subgradient of the dual function in* (2.7)*, i.e., $\forall \boldsymbol{\lambda}_i \geq 0$,*

$$\mathbb{E}_{h_i}\left[\mathbf{s}_i^T(t)|\boldsymbol{\lambda}_i(t)\right](\boldsymbol{\lambda}_i(t) - \boldsymbol{\lambda}_i) \geq g_i(\boldsymbol{\lambda}_i(t)) - g_i(\boldsymbol{\lambda}_i). \tag{2.17}$$

In particular,

$$\mathbb{E}_{h_i}\left[\mathbf{s}_i^T(t)|\boldsymbol{\lambda}_i(t)\right](\boldsymbol{\lambda}_i(t) - \boldsymbol{\lambda}_i^*) \geq g_i(\boldsymbol{\lambda}_i(t)) - \mathsf{D}_i \geq 0. \tag{2.18}$$

Proof. See Appendix 2.5.1. □

Proposition 1 states that the average of the stochastic subgradient $\mathbf{s}_i(t)$ is a subgradient of the dual function. We can then think of an alternative algorithm by replacing $\mathbb{E}_{h_i}\left[\mathbf{s}_i(t)\middle|\boldsymbol{\lambda}_i(t)\right]$ for $\mathbf{s}_i(t)$ in the dual iteration step (2.16), which would amount to a subgradient descent algorithm for the dual function. Since, $\mathbb{E}_{h_i}\left[\mathbf{s}_i(t)\middle|\boldsymbol{\lambda}_i(t)\right]$ points towards $\boldsymbol{\lambda}^*$ – the angle between $\mathbb{E}_{h_i}\left[\mathbf{s}_i(t)\middle|\boldsymbol{\lambda}_i(t)\right]$ and $\boldsymbol{\lambda}_i(t) - \boldsymbol{\lambda}_i^*$ is positive as indicated by (2.18) –, it is not difficult to prove that $\boldsymbol{\lambda}_i(t)$ eventually approaches $\boldsymbol{\lambda}_i^*$, e.g., [39, Ch. 2]. However, since we assume the pdf of h_i is unknown, the subgradient $\mathbb{E}_{h_i}\left[\mathbf{s}_i(t)\middle|\boldsymbol{\lambda}_i(t)\right]$ can only be approximated using past channel realizations $h_i(1), \ldots, h_i(t)$. While this approach is possible, it is computationally costly.

Algorithm 1: Adaptive scheduling and power control at terminal i

Initialize Lagrangian multipliers $\boldsymbol{\lambda}_i(0)$;

for $t = 0, 1, 2, \cdots$ **do**

  Compute primal variables as per (2.9), (2.10), and (2.12):

  $x_i(t) = \dfrac{w_i}{\lambda_{i1}(t)}$;

  $y_i(t) = \left[1 - \dfrac{\tilde{w}_i}{\lambda_{i2}(t)}\right]^+$;

  $p_i(t) = \underset{p_i \in [0, p_i^{\text{inst}}]}{\operatorname{argmax}} \{\lambda_{i1}(t) C_i(h_i(t) p_i) - \lambda_{i2}(t) - \lambda_{i3}(t) p_i\}$;

  $q_i(t) = H\Big(\lambda_{i1}(t) C_i(h_i(t) p_i(t)) - \lambda_{i2}(t) - \lambda_{i3}(t) p_i(t)\Big)$;

  if $q_i(t) = 1$ **then**

    Transmit with power $p_i(t)$;

  end

  Compute stochastic subgradients as per (2.13)-(2.15):

  $s_{i1}(t) = q_i(t) C_i(h_i(t) p_i(t)) - x_i(t)$;

  $s_{i2}(t) = y_i(t) - q_i(t)$;

  $s_{i3}(t) = p_i^{\text{avg}} - q_i(t) p_i(t)$;

  Update dual variables as per (2.16):

  $\lambda_{il}(t+1) = [\lambda_{il}(t) - \epsilon s_{il}(t)]^+ , \quad$ for $l = 1, 2, 3$;

end

The computation of the stochastic subgradient $\mathbf{s}_i(t)$, on the contrary, is simple because it only depends on the current channel state $h_i(t)$. Furthermore, since $\mathbf{s}_i(t)$ points towards the set of optimal dual variables $\boldsymbol{\lambda}_i^*$ on average [cf. (2.18)] it is reasonable to expect the stochastic subgradient descent iterations in (2.16) to also approach $\boldsymbol{\lambda}_i^*$ in some sense. This can be proved true and leveraged to prove almost sure convergence of primal iterates $x_i(t)$, $y_i(t)$, $p_i(t)$ and $q_i(t)$ to an optimal operating point in an ergodic sense [31]. Specifically, Theorem 1 of [31] assumes as hypotheses that the second moment of the norm of the stochastic subgradient $\mathbf{s}_i(t)$ is finite, i.e., $\mathbb{E}_{h_i}\left[\|\mathbf{s}_i(t)\|^2 \middle| \boldsymbol{\lambda}_i(t)\right] \leq \hat{S}_i^2$, and that there exists a set of strictly feasible primal variables that satisfy the constraints in (2.5) with strict inequality. If these hypotheses are true, primal iterates of dual stochastic subgradient descent are almost surely feasible in an ergodic sense. For the particular case of the problem in (2.5), [31, Theorem 1] implies that

$$\lim_{t\to\infty} \frac{1}{t}\sum_{u=1}^{t} q_i(u)p_i(u) \leq p_i^{\text{avg}} \quad \text{a.s.}, \tag{2.19}$$

$$\lim_{t\to\infty} \frac{1}{t}\sum_{u=1}^{t} x_i(u) \leq \lim_{t\to\infty} \frac{1}{t}\sum_{u=1}^{t} q_i(u)C_i(h_i(u)p_i(u)) \quad \text{a.s.}, \tag{2.20}$$

$$\lim_{t\to\infty} \frac{1}{t}\sum_{u=1}^{t} y_i(u) \geq \lim_{t\to\infty} \frac{1}{t}\sum_{u=1}^{t} q_i(u) \quad \text{a.s.} \tag{2.21}$$

It also follows from [31, Theorem 1] that $x_i(t)$ and $y_i(t)$ yield ergodic utilities that are almost surely within $\epsilon\hat{S}_i^2/2$ of optimal, i.e.,

$$\mathsf{P}_i - \lim_{t\to\infty} \frac{1}{t}\sum_{u=1}^{t} \left[w_i \log x_i(u) + \tilde{w}_i \log(1-y_i(u))\right] \leq \frac{\epsilon\hat{S}_i^2}{2} \text{ a.s.} \tag{2.22}$$

From (2.19) we can conclude that the ergodic limit of the power allocated by the proposed algorithm satisfies the average power constraint. However, (2.22) does not imply that the scheduling and power allocation variables $p_i(t)$ and $q_i(t)$ are optimal. The optimality claim in (2.22) is for the auxiliary variables $x_i(t)$ and $y_i(t)$ but the goal here is to claim optimality of the scheduling and power allocation variables $p_i(t)$ and $q_i(t)$. To prove optimality of the algorithm, we need to show that the ergodic transmission rate r_i of (1.4), achieved by allocations $q_i(t)$ and $p_i(t)$ is optimal in the sense of maximizing the throughput utility $U(\mathbf{r}) = \sum_{i=1}^{n} w_i \log(r_i)$.

If the constraints in (2.5) were satisfied for all times t, i.e., if $x_i(t) \le q_i(t)C_i(h_i(t)p_i(t))$ and $y_i(t) \ge q_i(t)$, transforming (2.22) into an almost sure near optimality claim for the ergodic limit r_i is a simple matter of substitution and algebraic manipulation. However, these inequalities do not necessarily hold for all times t. They hold in an ergodic sense as stated in (2.20) and (2.21). This subtle yet fundamental mismatch is addressed in the proof of the following theorem.

Theorem 1. *Consider a random multiple access channel with n terminals using schedules $q_i(t)$ and power allocations $p_i(t)$ generated by the algorithm defined by* (2.9)-(2.16) *resulting in instantaneous transmission rates $r_i(t)$ as given by* (1.3) *and ergodic rates r_i as defined by* (1.4)*. Define vector $\mathbf{r} := [r_1, \ldots, r_n]^T$, and let $U(\mathbf{r})$ be the weighted proportional fair utility in* (2.1)*. Assume that the second moment of the norm of the stochastic subgradient $\mathbf{s}_i(t)$ with components as in* (2.13)-(2.15) *is finite* [1]*, i.e., $\mathbb{E}_{h_i}\left[\|\mathbf{s}_i(t)\|^2 \middle| \boldsymbol{\lambda}_i(t)\right] \le \hat{S}_i^2$, and that there exists a set of strictly feasible primal variables that satisfy the constraints in* (2.5) *with strict inequality. Then, the average power constraint is almost surely satisfied*

$$\lim_{t\to\infty} \frac{1}{t} \sum_{u=1}^{t} q_i(u)p_i(u) \le p_i^{avg} \quad a.s., \tag{2.23}$$

and the utility of the ergodic limit of the transmission rates almost surely converges to a value within $\epsilon/2 \sum_{i=1}^n \hat{S}_i^2$ of optimality,

$$\mathsf{P} - U(\mathbf{r}) := \mathsf{P} - \sum_{i=1}^{n} w_i \log\left(\lim_{t\to\infty} \frac{1}{t} \sum_{u=1}^{t} r_i(u)\right) \le \frac{\epsilon}{2} \sum_{i=1}^{n} \hat{S}_i^2. \tag{2.24}$$

Proof. The hypotheses of Theorem 1 are chosen to satisfy the hypotheses guaranteeing convergence of ergodic stochastic optimization algorithms [31, Theorem 1]. Thus, almost sure feasibility and almost sure near optimality of iterates $x_i(t)$, $y_i(t)$, $p_i(t)$ and $q_i(t)$ follows in the sense of (2.19)-(2.22). To establish almost sure satisfaction of average power constraints as per (2.23) just notice that this inequality coincides with the one in (2.19). To establish (2.24) start by rearranging terms in (2.22) to conclude that $\mathsf{P}_i - \epsilon\hat{S}_i^2/2 \le \lim_{t\to\infty} \frac{1}{t} \sum_{u=1}^{t} [w_i \log x_i(u) + \tilde{w}_i \log(1 - y_i(u))]$. Due to

[1] The finite assumption of the second moment of the subgradients is necessary for the proof of almost sure near optimality of the ergodic stochastic optimization algorithm [31].

continuity and concavity of the logarithm function we can further bound $\mathsf{P}_i - \epsilon\hat{S}_i^2/2$ as

$$\mathsf{P}_i - \frac{\epsilon\hat{S}_i^2}{2} \leq w_i \log\left[\lim_{t\to\infty}\frac{1}{t}\sum_{u=1}^{t} x_i(u)\right] + \tilde{w}_i \log\left[1 - \lim_{t\to\infty}\frac{1}{t}\sum_{u=1}^{t} y_i(u)\right]. \tag{2.25}$$

The limits in (2.25) are equal to the limits in the left hand sides of the inequalities in (2.20) and (2.21). Thus, using this almost sure ergodic feasibility results $\mathsf{P}_i - \epsilon\hat{S}_i^2/2$ is bounded as

$$\mathsf{P}_i - \frac{\epsilon\hat{S}_i^2}{2} \leq w_i \log\left[\lim_{t\to\infty}\frac{1}{t}\sum_{u=1}^{t} q_i(u)C_i(h_i(u)p_i(u))\right] + \tilde{w}_i \log\left[1 - \lim_{t\to\infty}\frac{1}{t}\sum_{u=1}^{t} q_i(u)\right]. \tag{2.26}$$

Ergodicity, possibly restricted to an ergodic component, allows replacement of the ergodic limits in (2.27) by the corresponding expected values, leading to the bound

$$\mathsf{P}_i - \frac{\epsilon\hat{S}_i^2}{2} \leq w_i \log \mathbb{E}_{h_i}\left[Q_i(h_i)C_i(h_iP_i(h_i))\right] + \tilde{w}_i \log \mathbb{E}_{h_i}[1 - Q_i(h_i)]. \tag{2.27}$$

Recall that $\mathsf{P} = \sum_{i=1}^{n} \mathsf{P}_i$ per definition, and consider the sum of the inequalities in (2.27) for all terminals i so as to write

$$\begin{aligned}
\mathsf{P} - \sum_{i=1}^{n}\frac{\epsilon\hat{S}_i^2}{2} &\leq \sum_{i=1}^{n} w_i \log \mathbb{E}_{h_i}\left[Q_i(h_i)C_i(h_iP_i(h_i))\right] + \tilde{w}_i \log \mathbb{E}_{h_i}[1 - Q_i(h_i)] \\
&\leq \sum_{i=1}^{n} w_i \log\left[\mathbb{E}_{h_i}\left[Q_i(h_i)C_i(h_i(t)P_i(h_i))\right] \prod_{j=1, j\neq i}^{n} \mathbb{E}_{h_j}[1 - Q_j(h_j)]\right],
\end{aligned} \tag{2.28}$$

where the second inequality follows by using the definition $\tilde{w}_i := \sum_{j=1, j\neq i}^{n} w_i$, reordering terms in the sum, and rewriting a sum of logarithms as the logarithm of a product.

The fundamental observation in this proof is that the scheduling function $Q_i(h_i)$ and the power allocation function $P_i(h_i)$ are independent of the corresponding $Q_j(h_j)$ and $P_j(h_j)$ of other terminals. This is not a coincidence, but the intended goal of reformulating (2.2) as (2.5). Using this independence, the product of expectations in (2.28) can be written as single expectation over the vector channel $\mathbf{h}$ to yield

$$\mathsf{P} - \sum_{i=1}^{n}\frac{\epsilon\hat{S}_i^2}{2} \leq \sum_{i=1}^{n} w_i \log\left[\mathbb{E}_{\mathbf{h}}\left(Q_i(h_i)C_i(h_iP_i(h_i)) \prod_{j=1, j\neq i}^{n} (1 - Q_j(h_j))\right)\right]. \tag{2.29}$$

To finalize the proof use ergodicity, possibly restricted to an ergodic component, to substitute the expectation in (2.29) by an ergodic limit to yield

$$\mathsf{P} - \sum_{i=1}^{n} \frac{\epsilon \hat{S}_i^2}{2} \leq \sum_{i=1}^{n} w_i \log \left[\lim_{t \to \infty} \frac{1}{t} \sum_{u=1}^{t} q_i(u) C_i(h_i(u) p_i(u)) \prod_{j=1, j \neq i}^{n} (1 - q_j(u)) \right] := U(\mathbf{r}), \tag{2.30}$$

where we have used the definitions of the ergodic rate in (1.4) and of the utility in (2.1). The result in (2.24) follows after reordering terms in (2.30). □

Theorem 1 states that the stochastic dual descent algorithm in (2.9)-(2.16) computes schedules $q_i(t)$ and power allocations $p_i(t)$ yielding rates $r_i(t)$ that are almost surely near optimal in an ergodic sense [cf. (2.24)]. It also states that $p_i(t)$ satisfies the average power constraint with probability 1. Notice that the stochastic dual descent algorithm in (2.9)-(2.16) does not compute the optimal scheduling and power control functions for each terminal. Rather, it draws schedules $q_i(t)$ and power allocations $p_i(t)$ that are close to the optimal functions. This is not a drawback because the latter property is sufficient for a practical implementation. Further note that the use of constant step sizes ϵ endows the algorithm with adaptability to time-varying channel distributions. This is important in practice because wireless channels are non-stationary due to user mobility and environmental dynamics. The gap between $U(\mathbf{r})$ and P can be made arbitrarily small by reducing ϵ.

Remark 2. *The desired optimal schedules $Q^*(h(t))$ and power allocations $P^*(h(t))$ as prescribed in Section 2.1 are functions of the current channel realizations only. The proposed online policy, however, computes schedules $q_i(t)$ and power allocations $p_i(t)$ based on the current channel $h_i(t)$ and dual variables $\boldsymbol{\lambda}_i(t)$. In each time slot the iterative policy updates $\boldsymbol{\lambda}_i(t)$ using $\boldsymbol{\lambda}_i(t-1)$ and stochastic subgradients $\mathbf{s}_i(t)$ which depend on $q_i(t)$, $p_i(t)$ and $h_i(t)$. As a result, the dual variable $\boldsymbol{\lambda}_i(t)$ depends on all previous channel gains from $h_i(0)$ up to $h_i(t)$. Since $q_i(t)$ and $p_i(t)$ are functions of $\boldsymbol{\lambda}_i(t)$, they depend on all previous channel gains as well. This is not a contradiction because as the algorithm progresses, $\boldsymbol{\lambda}_i(t)$ approaches the optimal multiplier $\boldsymbol{\lambda}_i^*$, implying that the time-dependent variables $q_i(t), p_i(t)$ converge towards the optimal policy $P^*(h(t)), Q^*(h(t))$. As a matter of fact, $\boldsymbol{\lambda}_i(t)$ does not converge to $\boldsymbol{\lambda}_i^*$, but to a neighborhood of $\boldsymbol{\lambda}_i^*$. This results in some residual time dependence in the variables $q_i(t), p_i(t)$ that accounts for the* *algorithm's (arbitrarily small) optimality penalty as stated in Theorem 1.*

2.2.3 Structure of the optimal primal solution

While the algorithm in (2.9)-(2.16) provides a method to find the optimal operating point for the random multiple access channel, it does not provide intuition on the properties of this operating point. This section studies structural properties of the optimal primal solution.

In convex optimization problems optimal primal variables are obtained as the Lagrangian maximizers for optimal dual variables. The optimization problem in (2.5) is not convex. This is not a hindrance because the recovery of optimal primals from optimal duals through Lagrangian maximization follows from the lack of duality gap, which is a property that (2.5) does possess [33]. Let us then begin by showing that the optimal primal variables $\mathbf{x}_i^* = [x_i^*, y_i^*]^T$ and $\mathbf{P}_i^*(h_i) = [Q_i^*(h_i), P_i^*(h_i)]^T$ of the primal problem in (2.5) can be obtained from the maximizers of the Lagrangian $\mathcal{L}_i(\mathbf{x}_i, \mathbf{P}_i(h_i), \boldsymbol{\lambda}_i^*)$. From the definition of the dual function in (2.7), the optimal dual value can be written as

$$\begin{aligned}\mathsf{D}_i =& g_i(\boldsymbol{\lambda}_i^*) = \max \mathcal{L}_i(\mathbf{x}_i, \mathbf{P}_i(h_i), \boldsymbol{\lambda}_i^*) \\ & \text{s.t. } x_i \geq 0, 0 \leq y_i \leq 1, Q_i(h_i) \in \mathcal{Q}, P_i(h_i) \in \mathcal{P}_i.\end{aligned} \tag{2.31}$$

Since the maximization in (2.31) is with respect to all primal variables satisfying the stated constraints and the optimal variables $\mathbf{x}_i^*$ and $\mathbf{P}_i^*(h_i)$ satisfy these constraints, it must be

$$\mathsf{D}_i \geq \mathcal{L}_i(\mathbf{x}_i^*, \mathbf{P}_i^*(h_i), \boldsymbol{\lambda}_i^*). \tag{2.32}$$

Consider now the explicit expression of $\mathcal{L}_i(\mathbf{x}_i^*, \mathbf{P}_i^*(h_i), \boldsymbol{\lambda}_i^*)$ as it follows from the definition in (2.6)

$$\begin{aligned}\mathcal{L}_i(\mathbf{x}_i^*, \mathbf{P}_i^*(h_i), \boldsymbol{\lambda}_i^*) =& \; w_i \log x_i^* + \tilde{w}_i \log(1 - y_i^*) + \lambda_{i1}^* \left[\mathbb{E}_{h_i}\left[Q_i^*(h_i) C_i(h_i P_i^*(h_i))\right] - x_i^*\right] \\ &+ \lambda_{i2}^* \left[y_i^* - \mathbb{E}_{h_i}\left[Q_i^*(h_i)\right]\right] + \lambda_{i3}^* \left[p_i^{\text{avg}} - \mathbb{E}_{h_i}\left[Q_i^*(h_i) P_i^*(h_i)\right]\right].\end{aligned} \tag{2.33}$$

Since $\mathbf{x}_i^*$ and $\mathbf{P}_i^*(h_i)$ are solutions of (2.5), they are feasible, i.e., they satisfy the inequalities in (2.5). Thus, the terms $\mathbb{E}_{h_i}\left[Q_i^*(h_i) C_i(h_i P_i^*(h_i))\right] - x_i^* \geq 0$, $y_i^* - \mathbb{E}_{h_i}\left[Q_i^*(h_i)\right] \geq 0$, and $p_i^{\text{avg}} -$

$\mathbb{E}_{h_i}\left[Q_i^*(h_i)P_i^*(h_i)\right] \geq 0$ are all nonnegative. Since the Lagrange multipliers $\lambda_{i1} \geq 0$, $\lambda_{i2} \geq 0$, and $\lambda_{i3} \geq 0$, are also nonnegative, it holds

$$\mathsf{D}_i \geq \mathcal{L}_i(\mathbf{x}_i^*, \mathbf{P}_i^*(h_i), \boldsymbol{\lambda}_i^*) \geq w_i \log x_i^* + \tilde{w}_i \log(1 - y_i^*) = \mathsf{P}_i, \tag{2.34}$$

where the first inequality follows from (2.32) and the last equality from the fact that $\mathbf{x}_i^*$ is optimal. Since the duality gap is null, i.e., $\mathsf{D}_i = \mathsf{P}_i$, the inequalities in (2.34) must hold with equality. It then must be that $\mathbf{x}_i^*$ and $\mathbf{P}_i^*(h_i)$ are a solution to the maximization in (2.31). Further note that because $\mathbf{x}_i$ and $\mathbf{P}_i(h_i)$ appear in different terms in $\mathcal{L}_i(\mathbf{x}_i, \mathbf{P}_i(h_i), \boldsymbol{\lambda}_i^*)$, the joint maximization with respect to $\mathbf{x}_i$ and $\mathbf{P}_i(h_i)$ can be carried out as separate maximizations with respect to $\mathbf{x}_i$ and $\mathbf{P}_i(h_i)$ [cf. (2.42)]. In particular, for $\mathbf{P}_i^*(h_i)$ we have

$$\{Q_i^*(h_i), P_i^*(h_i)\} \in \underset{Q_i(h_i), P_i(h_i)}{\operatorname{argmax}} \ \mathbb{E}_{h_i}\left[Q_i(h_i)\left[\lambda_{i1}^* C_i(h_i P_i(h_i)) - \lambda_{i2}^* - \lambda_{i3}^* P_i(h_i)\right]\right]. \tag{2.35}$$

where the relation is belong to ($\in$) rather than equality (=) because there might be more than one argument that maximizes the expression in (2.35).

Due to linearity of the expectation operator $\mathbb{E}_{h_i}[\cdot]$, to maximize the expected value with respect to the functions $Q_i(h_i) \in \mathcal{Q}$ and $P_i(h_i) \in \mathcal{P}_i$ it is equivalent to maximize with respect to individual values. Therefore, it must be for all $h_i > 0$,

$$\{Q_i^*(h_i), P_i^*(h_i)\} \in \underset{q_i \in \{0,1\}, p_i \in [0, p_i^{\text{inst}}]}{\operatorname{argmax}} \left\{q_i\left[\lambda_{i1}^* C_i(h_i p_i) - \lambda_{i2}^* - \lambda_{i3}^* p_i\right]\right\}. \tag{2.36}$$

Using the expression in (2.36) it is possible to infer that the optimal scheduling function $Q_i^*(h_i)$ is a threshold rule as stated in the following theorem.

Theorem 2. *The optimal scheduling function $Q_i^*(h_i)$ solving* (2.5) *is a threshold rule. I.e., there exists a constant h_0 such that $Q_i^*(h_i) = H(h_i - h_0)$.*

Proof. Let us start by elaborating on the implications of (2.36). Define $u_i(p_i, h_i) := \lambda_{i1}^* C_i(h_i p_i) - \lambda_{i2}^* - \lambda_{i3}^* p_i$ as the part of the maximand of (2.36) that depends on p_i and let $v_i(h_i) := \max_{p_i \in [0, p_i^{\text{inst}}]} \{u_i(p_i, h_i)\}$ be the maximum of $u_i(p_i, h_i)$ over allowed p_i. If for given h_i, we have

$v_i(h_i) > 0$ it then must be $Q_i^*(h_i) = 1$ because $q_i = 1$ is the sole argument maximizing the expression in (2.36). Likewise, if $v_i(h_i) < 0$ it must be $Q_i^*(h_i) = 0$. When $v_i(h_i) = 0$ the value of $Q_i^*(h_i)$ cannot be inferred from (2.36) because both $q_i = 0$ and $q_i = 1$ are maximizing arguments. We then conclude the following two implications pertaining to $Q_i^*(h_i) = 1$: (i) if $v_i(h_i) > 0$ then $Q_i^*(h_i) = 1$; and (ii) if $Q_i^*(h_i) = 1$ then $v_i(h_i) \geq 0$.

To prove that the optimal schedule is a threshold rule it suffices to prove that if $Q_i^*(h_i) = 1$ for some given h_i then $Q_i^*(h_i') = 1$ for any $h_i' > h_i$. We will prove that for h_i' it must be $v_i(h_i') > 0$ from where $Q_i^*(h_i') = 1$ follows as per implication (i) of the previous paragraph. To prove that $v_i(h_i') > 0$ let p_0 denote a maximizer of $u_i(p_i, h_i)$ so that $v_i(h_i) = u_i(p_0, h_i)$. Since $Q_i^*(h_i) = 1$ it follows from implication (ii) in the previous paragraph that $u_i(p_0, h_i) = v_i(h_i) \geq 0$. Observing that for $p_i = 0$ we have $u_i(0, h_i) = -\lambda_{i2}^* < 0$ it follows that it must be $p_0 > 0$. Define now power $p_0' = (h_i/h_i')p_0$. With this selection it follows $h_i p_0 = h_i' p_0'$ and as a consequence $C(h_i p_0) = C(h_i' p_0')$. We can then write the difference $u_i(p_0', h_i') - u_i(p_0, h_i)$ as

$$\begin{aligned} u_i(p_0', h_i') - u_i(p_0, h_i) &= \left[C(h_i' p_0') - \lambda_{i2}^* - \lambda_{i3}^* p_0' \right] - \left[C(h_i p_0) - \lambda_{i2}^* - \lambda_{i3}^* p_0 \right] \\ &= \lambda_{i3}^* p_0 \left(1 - \frac{h_i}{h_i'} \right) > 0 \end{aligned} \tag{2.37}$$

where the inequality indicating a strictly positive difference follows from the fact that $h_i' > h_i$ and that $p_0 \neq 0$. Since $u_i(p_0, h_i) \geq 0$ it follows from (2.37) that $u_i(p_0', h_i') > 0$ and as a consequence that the maximum value $v_i(h_i') \geq u_i(p_0', h_i') > 0$. From implication (i) it then follows that $Q_i^*(h_i') = 1$ and that the optimal schedule is a threshold rule as already argued. $\square$

When there is no power control function and the rate function is continuous, the optimality of threshold-based schedulers has been proved in [50]. This result is extended here to general cases allowing for power control and the use of discontinuous rate functions. It is worth emphasizing that the optimality of a threshold-based scheduler is independent of the specific form of the rate function $C_i(h_i P_i(h_i))$. Recall that the sole constraint on the function $C_i(h_i P_i(h_i))$ is that is must be finite for finite argument.

If the form of the transmission rate function $C_i(h_iP_i(h_i))$ is known, it is also possible to infer functional forms for the optimal power control functions $P_i^*(h_i)$. If AMC is used at the physical layer the rate function takes the form in (1.2). In this case it is possible to find unique maximizers of (2.36) that as a consequence determine the form of the optimal power allocation $P_i^*(h_i)$. The corresponding functional form requires finding the AMC mode $m^* = \operatorname{argmax}_{m=\{1,\dots,M\}}\{\lambda_1^*\tau_m - \lambda_2^* - \lambda_3^*\frac{\eta_m N_0 B}{h_i}\}$ and setting the transmitted power to

$$P_i^*(h_i) = \frac{\eta_{m^*} N_0 B}{h_i} Q_i(h_i), \tag{2.38}$$

With capacity achieving codes used at the physical layer the rate function takes the form in (1.1). The optimal power control function then takes the form

$$P_i^*(h_i) = \left(\frac{\lambda_{i1}^*}{\lambda_{i3}^*} - \frac{N_0}{h_i}\right) BQ(h_i), \tag{2.39}$$

because the $P_i^*(h_i)$ in (2.39) are the unique arguments maximizing (2.36). The expression in (2.39) implies the optimality of power waterfilling across fading states.

Remark 3. *Since the optimal policy is a function of the channels' probability distribution, it seems that these distributions have to be estimated in order to design the optimal policy. However, the proposed Algorithm 1 only maintains three Lagrange multipliers $\lambda_{i1}(t)$, $\lambda_{i2}(t)$ and $\lambda_{i3}(t)$. The reason for this is that as can be seen in* (2.36) *the optimal solution can be uniquely determined by the optimal Lagrange multipliers λ_{i1}^*, λ_{i2}^* and λ_{i3}^*. Thus, instead of learning the channels' probability distribution it suffices to learn the optimal dual variables $\boldsymbol{\lambda}_i^*$. Learning $\boldsymbol{\lambda}_i^*$ is, in effect, the purpose of Algorithm 1. This is an important simplification. Whereas the unknown channel distributions are infinite-dimensional, the dual variables $\boldsymbol{\lambda}_i^*$ are 3-dimensional.*

Remark 4. *It is possible to interpret* (2.36) *in economic terms. Consider λ_{i1}^* as the reward for transmitting a unit of information, while regarding λ_{i2}^* and λ_{i3}^* as the prices for accessing the channel once and for consuming a unit of transmit power, respectively. With these interpretations, $u_i(p_i, h_i)$ represents the profit generated by transmitting with power p_i when the channel state is h_i, and $v_i(h_i)$ is the maximum*

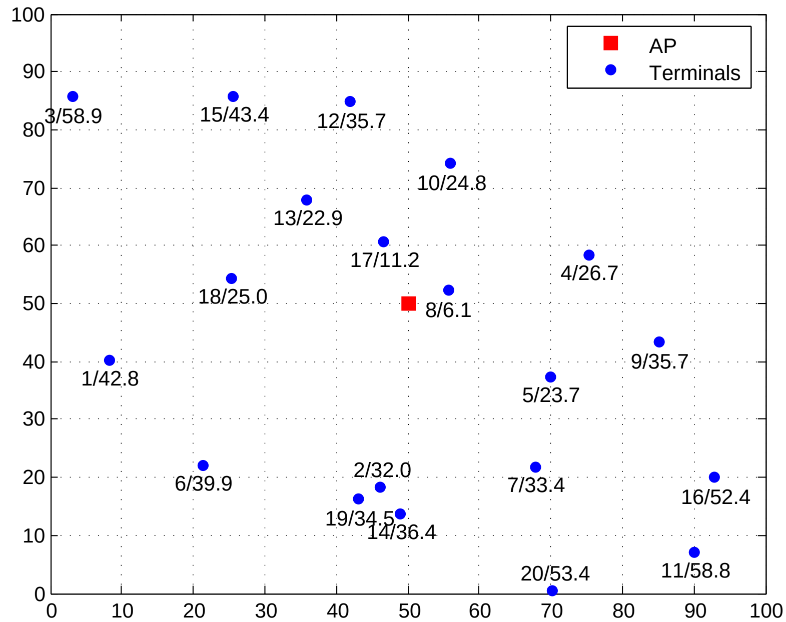


Figure 2.1: An example multiple access channel with $n = 20$ nodes communicating with a common access point (AP). Nodes are randomly placed in a 100 m $\times$ 100 m square and the AP is located at the center of the square. Nodes' labels represent indexes and distances to the AP. Subsequent numerical experiments use this realization of the random placement.

profit that can be obtained while satisfying the instantaneous power constraint. Consequently, (2.36) *can be interpreted as stating that terminals are allowed to transmit if and only if their maximum possible profits are positive.*

2.3 Numerical results

To illustrate performance of the proposed algorithms, we conduct numerical experiments on a network with $n = 20$ terminals randomly placed in a square with side $L = 100$ m and a common AP located at the center of the square. Numerical experiments here utilize the realization of this random placement shown in Fig. 2.1. Communication between terminals and the AP is over a bandlimited Gaussian channel with bandwidth B and noise power spectral density N_0. We set $B = 1$ so that capacities are measured in bits per second per Hertz (b/s/Hz) and $N_0 = 10^{-10}$ W. Channel gains $h_i(t)$ are Rayleigh distributed with mean $\bar{h}_i$ and are independent across terminals and time. The average channel gain $\bar{h}_i := \mathbb{E}[h_i]$ follows an exponential pathloss law, $\bar{h}_i = \alpha d_i^{-\beta}$ with $\alpha = 10^{-6}\text{m}^{-1}$ and $\beta = 2$ constants and d_i denoting the distance in meters between terminal i and the AP. All weights in the proportional fair utility in (2.1) are set to $w_i = 1$. Throughout, the performance metric of interest is the average transmission rate $\bar{r}_i(t)$ of terminal i at time t defined as

$$\bar{r}_i(t) = \frac{1}{t}\sum_{u=1}^{t} r_i(u), \tag{2.40}$$

where $r_i(u)$ is normalized so that it represents bits/s/Hz. The system's throughput utility by time t is then defined in terms of $\bar{r}_i(t)$ as $\bar{U}(t) := \sum_{i=1}^{n} w_i \log(\bar{r}_i(t))$.

The algorithm in (2.9)-(2.16) is first tested in a network where nodes use capacity achieving codes and have instantaneous power constraints but do not have average power constraints; see Section 2.3.1. We then consider nodes that have average as well as instantaneous power constraints using AMC; see Section 2.3.2.

2.3.1 System with instantaneous power constraint

Assume the use of capacity achieving codes so that the rate function for terminal i takes the form in (1.1). Further assume that there is an instantaneous power constraint p_i^{inst} = 100 mW for each terminal, but that there is no average power constraint. Since the rate function is a nonnegative increasing function of power it is optimal for each terminal to transmit with its maximum allowed instantaneous power every time it decides to transmit. Therefore, the power control function is a constant $p_i(t) = p_i^{\text{inst}}$ and the system's performance depends solely on the terminals' scheduling functions $q_i(t)$. In this simplified setting, a closed form solution for $q_i(t)$ is

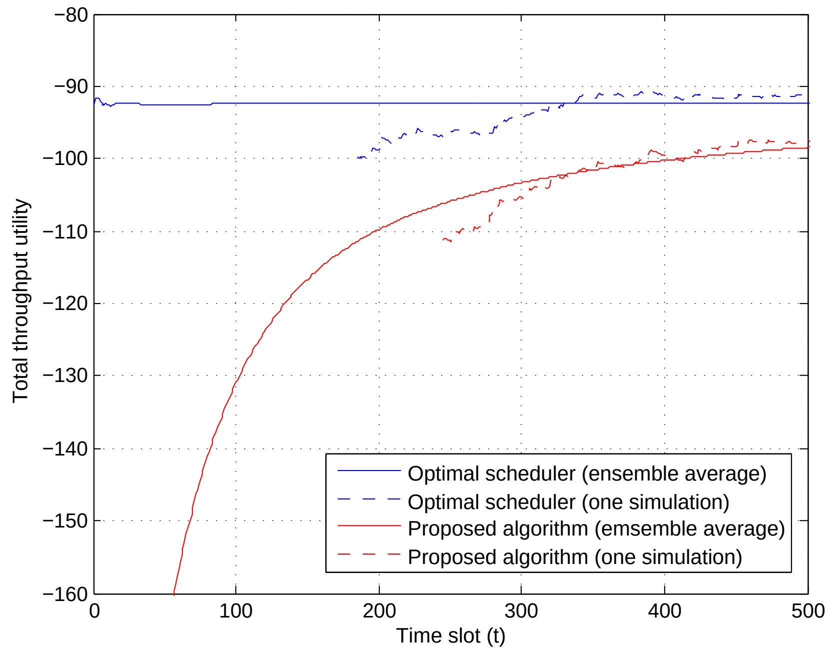


Figure 2.2: Convergence of the proposed algorithm to near optimal utility with instantaneous power constrains but no average power constraints. Throughput utility of the proposed adaptive algorithm and of the optimal offline scheduler are shown as functions of time for one realization and for the ensemble average of realizations. In steady state the adaptive algorithm operates with minimal performance loss with respect to the optimal offline scheduler. A utility gap smaller than 10 is achieved in about 350 iterations. Power constraint $p_i^{\text{inst}} = 100$ mW, step size $\epsilon = 0.1$, capacity achieving codes.

known if the channel pdf is available [50]. Our interest in this simplified problem is that it allows a performance comparison between the schedules yielded by (2.9)-(2.16) and those of the optimal offline scheduler.

Convergence of (2.9)-(2.16) to a near optimal operating point is illustrated in Fig. 2.2 for step size $\epsilon = 0.1$. The ergodic utility $\bar{U}(t)$ is shown through 500 iterations and is compared with the utility of the optimal offline scheduler. When using (2.9)-(2.16) the total throughput utility converges to a value with negligible optimality gap with respect to the offline scheduler. Observe that convergence is fast as it takes less than 180 iterations to reach a utility with optimality gap smaller than 20 and 360 iterations to get an optimality gap smaller than 10. Figs. 2.3 and 2.4 respectively show average rates and transmission probabilities after 500 iterations for each terminal. Observe in Fig. 2.3 that all terminals achieve average rates that are very close to the optimal ones. Further observe that even though terminals experience different channel conditions, fair schedules are obtained as a consequence of the use of a logarithmic utility. Indeed, as seen in Fig. 2.4, average transmission probabilities are close for all terminals. Note, however, that the achieved rates shown in Fig. 2.3 are different because terminals have different average channels.

To test how the optimality gap changes as the step size ϵ varies, we ran the algorithm (2.9)-(2.16) with different step sizes. Fig. 2.5 shows the optimality gap when the step size ϵ varies between 10^{-2} to 10^{-1}. The optimality gap indeed decreases as the step size ϵ is reduced. This corroborates the result of Theorem 1 that ensures a vanishing optimality gap as $\epsilon \to 0$. Using smaller step size, however, leads to slower convergence. This tradeoff between convergence speed and optimality gap determines the choice of ϵ for practical implementations.

2.3.2 System with average power constraint

For the same network in Fig. 2.1, consider now the case in which each terminal adheres to both, instantaneous and average power constraints. We also deviate from Section 2.3.1 in the use of AMC instead of capacity achieving codes at the physcial layer, so that the rate function for terminal i takes the form in (1.2). Each terminal has $M = 4$ AMC modes with respective rates $\tau_1 = 1$ bits/s/Hz, $\tau_2 = 2$ bits/s/Hz, $\tau_3 = 3$ bits/s/Hz, and $\tau_4 = 4$ bits/s/Hz. The transitions between AMC modes are at SNRs $\eta_1 = 1$, $\eta_2 = 4$, $\eta_3 = 8$, and $\eta_4 = 16$. The instantaneous power constraint is set to $p_i^{\text{inst}} = 100$ mW and the average power constraint to $p_i^{\text{avg}} = 5$ mW for all terminals i.

To demonstrate optimality of the proposed algorithm, we compute the primal objective $\bar{U}(t)$, the dual value $D(t) = \sum_{i=1}^{n} g_i(\boldsymbol{\lambda}_i(t))$, and examine the duality gap between them. Fig. 2.6 shows $\bar{U}(t)$ and $D(t)$ for 10^3 time slots. As time grows, the duality gap decreases and eventually

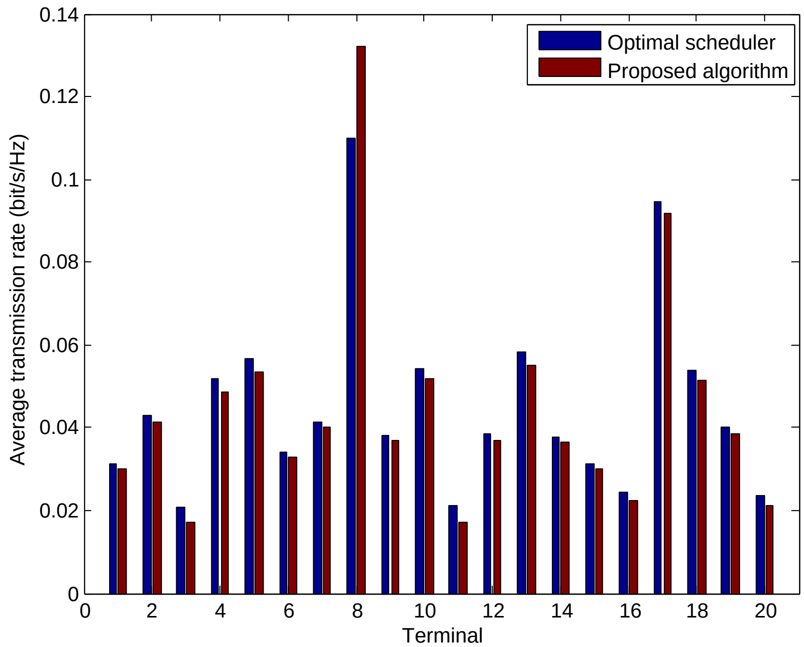


Figure 2.3: Average transmission rates (bits/s/Hz) in 500 time slots, i.e., $\bar{r}_i(500)$ as defined in (2.40), for all terminals. The optimal offline scheduler and the proposed adaptive algorithm yield similar close to optimal average rates. The variation in achieved rates is commensurate with the variation in average signal to noise ratios (SNRs) due to different distances to the access point. For the network in Fig.2.1 and the pathloss and power parameters used here, average signal to noise ratios vary between 0.4 and 10. Instantaneous power constraint $p_i^{\text{inst}} = 100$ mW, step size $\epsilon = 0.1$, capacity achieving codes.

approaches a small positive constant, implying near optimality of the proposed algorithm.

To test the satisfaction of the average power constraint, define the average power consumption of terminal i by time t as

$$\bar{p}_i(t) = \frac{1}{t}\sum_{u=1}^{t} p_i(u). \tag{2.41}$$

Average power consumptions $\bar{p}_3(t)$ and $\bar{p}_{13}(t)$ for terminals 3 and 13 are shown in Fig. 2.7. Observe that in both cases the average power constraints are satisfied as time increases. For Terminal 3, $\bar{p}_3(t)$ is always smaller than p_3^{avg} since channel conditions are unfavorable, resulting in Termi-

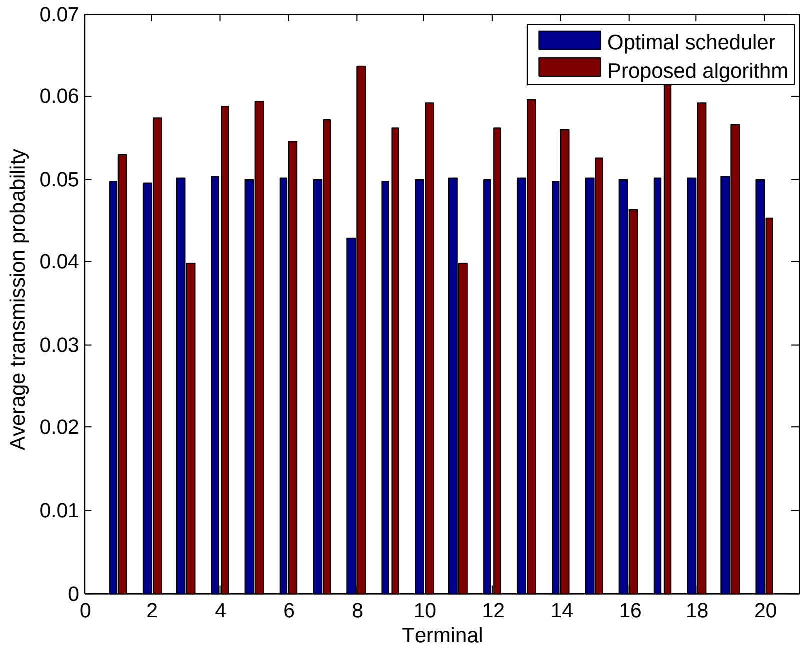


Figure 2.4: Average transmission probabilities in 500 time slots for all terminals. Offline and adaptive optimal schedulers shown. Despite different channel conditions all terminals transmit with a similar probability close to $1/n = 0.05$. This is consistent with the use of a logarithmic, i.e., proportional fair, utility. Instantaneous power constraint $p_i^{\text{inst}} = 100$ mW, step size $\epsilon = 0.1$, capacity achieving codes.

nal 3 utilizing only mode 1 for communication to the AP. Finally, notice that the average power consumed by Terminal 3 is smaller than the available budget $p_3^{\text{avg}} = 5$ mW. For Terminal 13, $\bar{p}_{13}(t)$ falls below p_{13}^{avg} after 600 iterations. This is as expected due to the almost sure feasibility result of Theorem 1.

Fig. 2.8 illustrates the relationship between instantaneous power allocations $p_i(t)$ and instantaneous channel gains $h_i(t)$ for terminals 3 and 13. Consistent with the fact that the optimal power allocation is a threshold rule, no power is allocated when channel realizations are bad. Further note that Terminal 3 only uses the AMC mode with the lowest rate $\tau_1 = 1$ bits/s/Hz while Terminal 13 uses two modes with rates $\tau_2 = 2$ bits/s/Hz and $\tau_3 = 3$ bits/s/Hz. This

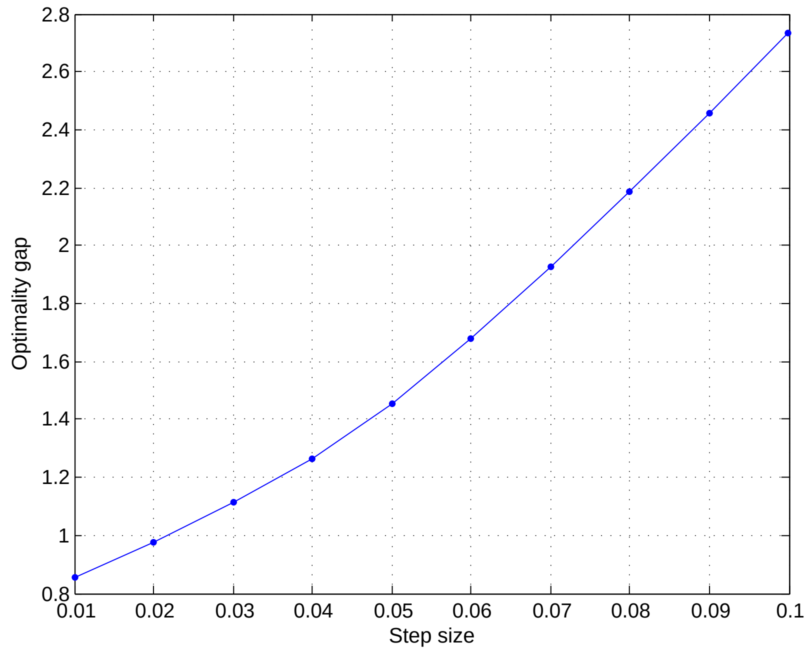


Figure 2.5: Steady state optimality gap between proposed adaptive algorithm and optimal offline scheduler as a function of step size ϵ. Values of ϵ between 10^{-2} and 10^{-3} shown. As the step size decreases, the optimality gap decreases. The optimality gap can be made arbitrarily small by reducing ϵ. Instantaneous power constraint $p_i^{\text{inst}} = 100$ mW, capacity achieving codes.

happens because terminal 13, being closer to the AP, has a better average channel than terminal 3.

2.4 Summary

We developed optimal adaptive scheduling and power control algorithms for random multiple access channels. Terminals are assumed to know their local channel state information but have no access to the probability distribution of the channel or the channel state of other terminals. In this setting, the proposed online algorithm determines schedules and transmitted powers that maxi-

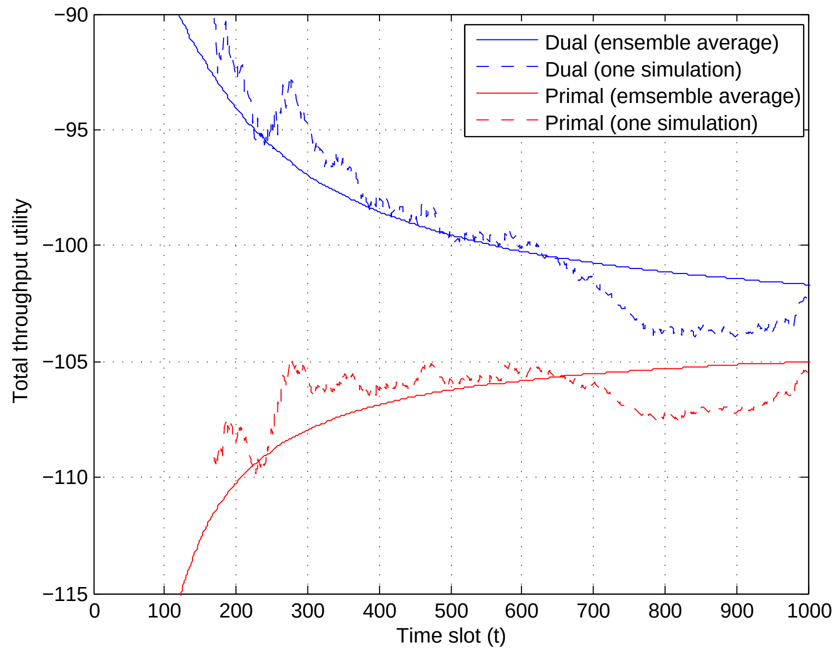


Figure 2.6: Primal and dual objectives when instantaneous and average power constraints are in effect. One realization and ensemble average of realizations shown. As time grows the duality gap decreases, eventually approaching a small positive constant and implying near optimality of the achieved rates. Instantaneous power constraint $p_i^{\text{inst}} = 100$ mW, average power constraint $p_i^{\text{avg}} = 5$ mW, step size $\epsilon = 0.1$, adaptive modulation and coding with $M = 4$ modes with rates $\tau_1 = 1$ bits/s/Hz, $\tau_2 = 2$ bits/s/Hz, $\tau_3 = 3$ bits/s/Hz, and $\tau_4 = 4$ bits/s/Hz and transitions at SNRs $\eta_1 = 1$, $\eta_2 = 4$, $\eta_3 = 8$, and $\eta_4 = 16$.

mize a global proportional fair utility. The global utility maximization problem was decomposed in per-terminal utility maximization subproblems. Adaptive algorithms using stochastic subgradient descent in the dual domain were then used to solve these local optimizations. Almost sure convergence and almost sure near optimality of the proposed algorithm was established. Important properties of the algorithm are low computational complexity and the ability to handle non-convex rate functions. Numerical results for a randomly generated network under different physical layer settings corroborated theoretical results.

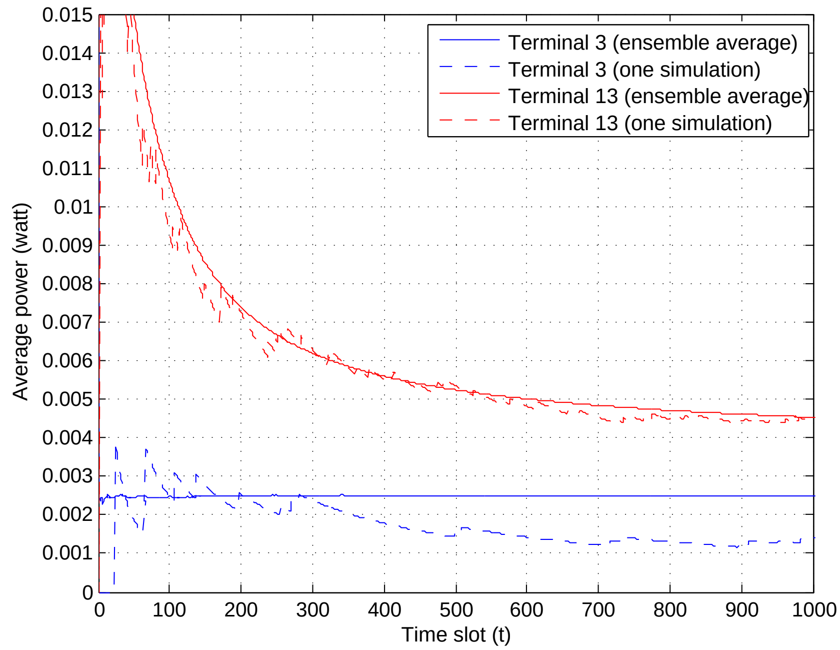
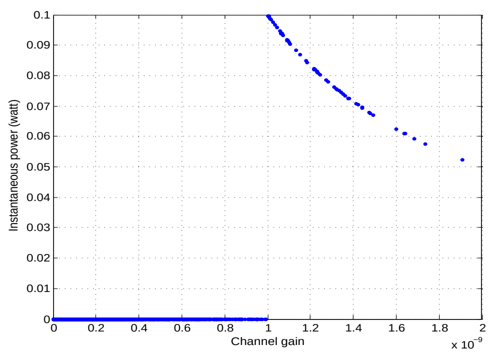


Figure 2.7: Average power consumption for terminals 3 and 13, i.e., $\bar{p}_3(t)$ and $\bar{p}_{13}(t)$ as defined in (2.41). Average power constraints $p_i^{\text{avg}} = 5$ mW are satisfied as time grows. Power $\bar{p}_3(t)$ consumed by Terminal 3 is smaller than the allowed budget p_3^{avg} due to unfavorable channel conditions. Terminal 13 adheres to its power budget after approximately 600 iterations. Parameters as in Fig. 2.6

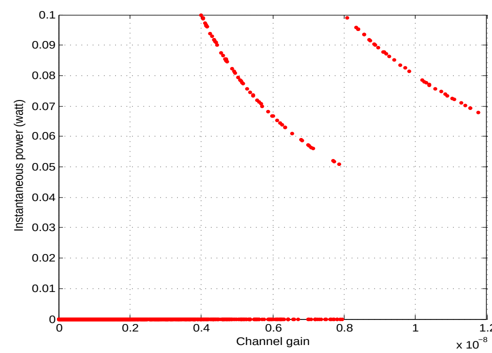
2.5 Appendices

2.5.1 Proof of Proposition 1

Proof. To show that the expected value of the stochastic subgradient $\mathbf{s}_i(t)$ given $\boldsymbol{\lambda}_i(t)$ is a subgradient of the dual function $g_i(\boldsymbol{\lambda}_i)$, we have to establish the validity of the relationship in (2.17). To do so start noticing that in the Lagrangian $\mathcal{L}_i(\mathbf{x}_i, \mathbf{P}_i(h_i), \boldsymbol{\lambda}_i(t))$ the terms involving $\mathbf{x}_i$ and $\mathbf{P}_i(h_i)$ are decoupled [cf. (2.6)]. Consequently, the maximization of $\mathcal{L}_i(\mathbf{x}_i, \mathbf{P}_i(h_i), \boldsymbol{\lambda}_i(t))$ in (2.7) required to evaluate the dual function's value $g_i(\boldsymbol{\lambda}_i(t))$ can be undertaken as maximizations of separate



(a) Furthest terminal



(b) Closest terminal

Figure 2.8: Instantaneous power allocations $p_i(t)$ for terminals $i = 3$ and $i = 13$ plotted against the channel realization $h_i(t)$. Notice that the channel axes scales are different in (a) and (b). In both cases, no power is allocated when channel realizations are bad. Terminal 3 uses only the AMC mode with the lowest rate $\tau_1 = 1$ bits/s/Hz, while Terminal 13 uses two modes with rates $\tau_2 = 2$ bits/s/Hz and $\tau_3 = 3$ bits/s/Hz. This happens because Terminal 13, being closer to the AP, has a better average channel than Terminal 3. Parameters as in Fig. 2.6.

terms with respect to $\mathbf{x}_i$ and $\mathbf{P}_i(h_i)$. Therefore, $g_i(\boldsymbol{\lambda}_i(t))$ can be written as

$$\begin{aligned} g_i(\boldsymbol{\lambda}_i(t)) =& \lambda_{i3}(t)p_i^{\text{avg}} + \max_{x_i \geq 0} \{w_i \log x_i - \lambda_{i1}(t)x_i\} + \max_{0 \leq y_i \leq 1} \{\tilde{w}_i \log(1-y_i) + \lambda_{i2}(t)y_i\} \\ &+ \max_{Q(h_i),P(h_i)} \mathbb{E}_{h_i(t)} \left[\Gamma_i(Q_i(h_i), P_i(h_i), h_i, \boldsymbol{\lambda}_i(t)) \middle| \boldsymbol{\lambda}_i(t) \right], \end{aligned} \tag{2.42}$$

where for notational simplicity we defined $\Gamma_i(q_i, p_i, h_i, \boldsymbol{\lambda}_i) := q_i \left[\lambda_{i1} C_i(p_i, h_i) - \lambda_{i2} - \lambda_{i3} p_i\right]$. The expected value is conditional with respect to $\boldsymbol{\lambda}_i(t)$ because $\boldsymbol{\lambda}_i$ is deterministic in (2.7) but random in (2.42).

The last summand on the right hand side of (2.42) is the maximum over the set of functions taking values $Q(h_i) \in \mathcal{Q}$ and $P(h_i) \in \mathcal{P}_i$. Due to linearity of the expectation operator $\mathbb{E}_{h_i(t)}[\cdot]$, this maximum over functions is equal to the expected value of maxima with respect to individual function values. This allows rewriting of (2.42) as

$$\begin{aligned} g_i(\boldsymbol{\lambda}_i(t)) =& \lambda_{i3}(t)p_i^{\text{avg}} + \max_{x_i \geq 0} \{w_i \log x_i - \lambda_{i1}(t)x_i\} + \max_{0 \leq y_i \leq 1} \{\tilde{w}_i \log(1-y_i) + \lambda_{i2}(t)y_i\} \\ &+ \mathbb{E}_{h_i(t)} \left[\max_{q_i \in \{0,1\}, p_i \in [0, p_i^{\text{inst}}]} \Gamma_i(q_i, p_i, h_i(t), \boldsymbol{\lambda}_i(t)) \middle| \boldsymbol{\lambda}_i(t) \right]. \end{aligned} \tag{2.43}$$

Notice that the maximizations over x_i, y_i, and $\{q_i, p_i\}$ in (2.43) coincide with the primal iteration maximizations in (2.9)-(2.11). Therefore, $x_i(t)$, $y_i(t)$, $q_i(t)$, and $p_i(t)$ obtained from (2.9)-(2.11) maximize the right hand side of (2.43) implying that (2.43) is equivalent to

$$\begin{aligned} g_i(\boldsymbol{\lambda}_i(t)) =& \lambda_{i3}(t)p_i^{\text{avg}} + [w_i \log x_i(t) - \lambda_{i1}(t)x_i(t)] + [\tilde{w}_i \log(1-y_i(t)) + \lambda_{i2}(t)y_i(t)] \\ &+ \mathbb{E}_{h_i(t)} \left[\Gamma_i(q_i(t), p_i(t), h_i(t), \boldsymbol{\lambda}_i(t)) \middle| \boldsymbol{\lambda}_i(t) \right]. \end{aligned} \tag{2.44}$$

Because $x_i(t)$ and $y_i(t)$ are deterministic functions of $\boldsymbol{\lambda}_i(t)$ it follows that $x_i(t) = \mathbb{E}_{h_i(t)}[x_i(t) | \boldsymbol{\lambda}_i(t)]$ and $y_i(t) = \mathbb{E}_{h_i(t)}[y_i(t) | \boldsymbol{\lambda}_i(t)]$. Use this fact and rearrange terms in (2.44) to obtain

$$\begin{aligned} g_i(\boldsymbol{\lambda}_i(t)) = [w_i \log x_i(t) + \tilde{w}_i \log(1-y_i(t))] + \lambda_{i1}(t)\mathbb{E}_{h_i(t)} \left[q_i(t) C_i(h_i(t) p_i(t)) - x_i(t) \middle| \boldsymbol{\lambda}_i(t) \right] \\ + \lambda_{i2}(t)\mathbb{E}_{h_i(t)} \left[y_i(t) - q_i(t) \middle| \boldsymbol{\lambda}_i(t) \right] + \lambda_{i3}(t)\mathbb{E}_{h_i(t)} \left[p_i^{\text{avg}} - q_i(t) p_i(t) \middle| \boldsymbol{\lambda}_i(t) \right]. \end{aligned} \tag{2.45}$$

According to the definitions in (2.13)-(2.15) the terms inside the expectations in (2.45) are the components $s_i(t)$ of the stochastic subgradient. It then follows

$$g_i(\boldsymbol{\lambda}_i(t)) = w_i \log x_i(t) + \tilde{w}_i \log(1 - y_i(t)) + \mathbb{E}_{h_i(t)} \left[\mathbf{s}_i^T(t)|\boldsymbol{\lambda}_i(t)\right] \boldsymbol{\lambda}_i(t). \tag{2.46}$$

Consider now an arbitrary dual variable $\boldsymbol{\lambda}_i \geq 0$ and the corresponding value of the dual function $g(\boldsymbol{\lambda}_i)$ given by the maximum of the Lagrangian $\mathcal{L}_i(\mathbf{x}_i, \mathbf{P}_i(h_i), \boldsymbol{\lambda}_i)$ [cf. 2.7]. As was done for $\boldsymbol{\lambda}_i = \boldsymbol{\lambda}(t)$ repeat the steps in (2.42) and (2.43) to express $g_i(\boldsymbol{\lambda}_i)$ as

$$\begin{aligned} g_i(\boldsymbol{\lambda}_i) =& \lambda_{i3} p_i^{\text{avg}} + \max_{x_i \geq 0} \{w_i \log x_i - \lambda_{i1} x_i\} + \max_{0 \leq y_i \leq 1} \{\tilde{w}_i \log(1 - y_i) + \lambda_{i2} y_i\} \\ &+ \mathbb{E}_{h_i(t)} \left[\max_{q_i \in \{0,1\}, p \in [0, p_i^{\text{inst}}]} \Gamma_i(q, p, h_i(t), \boldsymbol{\lambda}_i)] \middle| \boldsymbol{\lambda}_i(t) \right], \end{aligned} \tag{2.47}$$

where the conditioning on $\boldsymbol{\lambda}_i(t)$ is irrelevant because all variables are independent of $\boldsymbol{\lambda}_i(t)$ but will be exploited later on. Since the expression in (2.47) involves maximizations with respect to x_i, y_i, and $\{q_i, p_i\}$ a lower bound of $g_i(\boldsymbol{\lambda}_i)$ is obtained by evaluating the maximands at $x_i = x_i(t)$, $y_i = y_i(t)$ and $\{q_i, p_i\} = \{q_i(t), p_i(t)\}$. Thus

$$\begin{aligned} g_i(\boldsymbol{\lambda}_i) \geq& \lambda_{i3} p_i^{\text{avg}} + [w_i \log x_i(t) - \lambda_{i1} x_i(t)] + [\tilde{w}_i \log(1 - y_i(t)) + \lambda_{i2} y_i(t)] \\ &+ \mathbb{E}_{h_i(t)} \left[\Gamma_i(q_i(t), p_i(t), h_i(t), \boldsymbol{\lambda}_i) \middle| \boldsymbol{\lambda}_i(t) \right]. \end{aligned} \tag{2.48}$$

Reordering terms as when obtaining (2.45) from (2.44) we rewrite the bound in (2.48) as

$$\begin{aligned} g_i(\boldsymbol{\lambda}_i) \geq& [w_i \log x_i(t) + \tilde{w}_i \log(1 - y_i(t))] + \lambda_{i1} \mathbb{E}_{h_i(t)} \left[q_i(t) C_i(h_i(t) p_i(t)) - x_i(t) \middle| \boldsymbol{\lambda}_i(t) \right] \\ &+ \lambda_{i2} \mathbb{E}_{h_i(t)} \left[y_i(t) - q_i(t) \middle| \boldsymbol{\lambda}_i(t) \right] + \lambda_{i3} \mathbb{E}_{h_i(t)} \left[p_i^{\text{avg}} - q_i(t) p_i(t) \middle| \boldsymbol{\lambda}_i(t) \right]. \end{aligned} \tag{2.49}$$

Using the definition of the stochastic subgradient as when going from (2.45) to (2.46) it finally follows

$$g_i(\boldsymbol{\lambda}_i) \geq w_i \log x_i(t) + \tilde{w}_i \log(1 - y_i(t)) + \mathbb{E}_{h_i(t)} \left[\mathbf{s}_i^T(t)|\boldsymbol{\lambda}_i(t)\right] \boldsymbol{\lambda}_i. \tag{2.50}$$

Subtracting (2.50) from (2.46) yields (2.17). Eq. (2.18) is a particular case of (2.17) with $\boldsymbol{\lambda}_i = \boldsymbol{\lambda}_i^*$ and $g(\boldsymbol{\lambda}_i) = g(\boldsymbol{\lambda}_i^*) = \mathsf{D}_i$. $\square$

Chapter 3

Distributed algorithms for optimal random access networks

In this chapter, we focus on random access networks as introduced in Section 1.1.2 where terminals only have access to local CSI and operate without cooperating with each other. Our goal is to develop distributed algorithms that allow terminals to operate optimally according to certain criteria. Due to additional variables and constraints in random access networks, the distributed algorithm for optimal random access channel developed in Chapter 2 cannot be directly applied. However, we can still leverage the property of null duality gap of the optimization problem and develop distributed algorithms in the dual domain. To do so, we begin by introducing an optimization problem that defines the optimal random access network (Section 3.1). Since this problem is not amenable to distributed implementation we proceed to a suboptimal approximation through a problem that while still not convex has zero duality gap [34] (Section 3.1.2). We further observe that solution is simpler in the dual domain – and equivalent because of the lack of duality gap – and proceed to develop stochastic dual descent algorithms that converge to the optimal operating point (Section 3.2). The resultant algorithm decomposes in a layered architecture and is computationally tractable in that iterations require a few simple algebraic operations (Section 3.2.2). We also explain a decentralized implementation based on information exchanges with neighboring terminals (Section 3.2.3). Results on ergodic stochastic optimization from [32] are finally leveraged to show that the proposed algorithm yields operating points that are almost surely close to optimal (Section 3.3). Numerical results and concluding remarks are presented in Sections 3.4 and 3.5.

3.1 Problem formulation

3.1.1 Optimal operating point

Consider a random access wireless network as introduced in Section 1.1.2. As network designers, we wish to find the optimal operating point of the wireless network defined as a set of variables a_i^k, r_{ij}^k, c_{ij}, p_i and functions $Q_{ij}(\mathbf{h}_i)$, $P_{ij}(\mathbf{h}_i)$ that satisfy constraints (1.8)-(1.10), (1.12), and (1.13) and are optimal according to certain criteria. In particular, we are interested in large rates a_i^k and low power consumptions p_i. Define then increasing concave functions $U_i^k(\cdot)$ representing rewards for accepting a_i^k units of information for flow k at terminal i and increasing convex functions $V_i(\cdot)$ typifying penalties for consuming p_i units of power at i. The optimal network based on local CSI is then defined as the solution of

$$\mathsf{P} = \max_{\{\mathbf{x}_i, \mathbf{P}_i(\mathbf{h}_i)\} \in \mathcal{B}_i} \sum_{i \in \mathcal{V}, k \in \mathcal{K}} U_i^k \left(a_i^k\right) - \sum_{i \in \mathcal{V}} V_i(p_i) \tag{3.1}$$

$$\text{s.t. constraints (1.8), (1.9), (1.10), (1.12).}$$

Our goal is to develop a distributed algorithm to solve (3.1) without accessing the channel pdf $m_{\mathbf{h}}(\cdot)$. This is challenging because: (i) The optimization space in (3.1) includes functions $Q_{ij}(\mathbf{h}_i)$ and $P_{ij}(\mathbf{h}_i)$ implying that the dimension of the problem is infinite. (ii) Since the capacity constraint (1.12) is non-convex and the capacity function may be even discontinuous, (3.1) is a non-convex optimization problem. (iii) Constraints (1.10) and (1.12) involve expectations over channel states $\mathbf{h}$ whose pdf is unknown. (iv) The fact that the transmission rate c_{ij} is determined not only by the transmitter but also by the receiver and his neighbors [cf. (1.12)] hinders the development of distributed optimization algorithms.

Notice that the number of constraints in (3.1) is finite. This implies that while there are infinite number of variables in the primal domain, there are a finite number of variables in the dual domain. Thus, while working in the dual domain may entail some loss of optimality due the non-convex constraints in (3.1), it does overcome challenge (i) because the dual function is finite dimensional. It also overcomes challenge (ii) since the dual function is always convex, while challenge (iii) can be solved by using stochastic subgradient descent algorithms on the dual function; see e.g., [16] and [32]. However, working with the dual problem of (3.1) does not conduce to a distributed optimization algorithm due to the coupling introduced by constraint (1.12). This prompts the introduction of a decomposable approximation that we pursue in the next section.

3.1.2 Problem approximation

For reasons that will become clear in Section 3.2 a distributed solution of the problem in (3.1) is not possible because scheduling functions $Q_{ij}(\mathbf{h}_i)$ and $Q_l(\mathbf{h}_l)$ are coupled as a product in constraint (1.12). If we reformulate this constraint into an expression in which the terms $C_{ij}(h_{ij}P_{ij}(\mathbf{h}_i))$ $Q_{ij}(\mathbf{h}_i)$ and $1 - Q_l(\mathbf{h}_l)$ appear as summands instead of as factors of a product the problem will become decomposable in the dual domain. This reformulation can be accomplished by taking logarithms on both sides of (1.12), yielding

$$\tilde{c}_{ij} := \log c_{ij} \leq \log \mathbb{E}_{\mathbf{h}_i}\left[C_{ij}\Big(h_{ij}P_{ij}(\mathbf{h}_i)\Big)Q_{ij}(\mathbf{h}_i)\right] + \sum_{l\in\mathcal{M}_i(j)} \log\left[1 - \mathbb{E}_{\mathbf{h}_l}[Q_l(\mathbf{h}_l)]\right], \tag{3.2}$$

where we defined $\tilde{c}_{ij} := \log c_{ij}$. While scheduling functions of different terminals now appear as summands on the right hand side of (3.2), the link capacity constraint (1.9) mutates into the non-convex constraint $\sum_{k\in\mathcal{K}} r_{ij}^k \leq e^{\tilde{c}_{ij}}$. To avoid this issue we use the linear lower bound $1+\tilde{c}_{ij} \leq e^{\tilde{c}_{ij}}$ and approximate this constraint as $\sum_{k\in\mathcal{K}} r_{ij}^k \leq 1 + \tilde{c}_{ij}$. Upon defining the average attempted transmission rate of link (i, j) as

$$x_{ij} := \mathbb{E}_{\mathbf{h}_i}\left[C_{ij}\left(h_{ij}P_{ij}\left(\mathbf{h}_i\right)\right)Q_{ij}\left(\mathbf{h}_i\right)\right], \tag{3.3}$$

and the transmission probability of terminal i as

$$y_i := \mathbb{E}_{\mathbf{h}_i}\left[Q_i\left(\mathbf{h}_i\right)\right], \tag{3.4}$$

the original optimization problem P is approximated by

$$\begin{aligned}
\mathsf{P} \geq \tilde{\mathsf{P}} &= \max_{\{\tilde{\mathbf{x}}_i, \mathbf{P}_i(\mathbf{h}_i)\} \in \mathcal{B}_i} \sum_{i \in \mathcal{V}, k \in \mathcal{K}} U_i^k\left(a_i^k\right) - \sum_{i \in \mathcal{V}} V_i\left(p_i\right) \qquad (3.5)\\
&\text{s.t. } a_i^k \leq \sum_{j \in \mathcal{N}(i)} \left(r_{ij}^k - r_{ji}^k\right), \quad \sum_{k \in \mathcal{K}} r_{ij}^k \leq 1 + \tilde{c}_{ij},\\
&\tilde{c}_{ij} \leq \log x_{ij} + \sum_{l \in \mathcal{M}_i(j)} \log\left(1 - y_l\right),\\
&x_{ij} \leq \mathbb{E}_{\mathbf{h}_i}\left[C_{ij}\Big(h_{ij}P_{ij}\left(\mathbf{h}_i\right)\Big)Q_{ij}\Big(\mathbf{h}_i\Big)\right], y_i \geq \mathbb{E}_{\mathbf{h}_i}\left[Q_i\left(\mathbf{h}_i\right)\right],\\
&p_i \geq \mathbb{E}_{\mathbf{h}_i}\left[\sum_{j \in \mathcal{N}(i)} P_{ij}\left(\mathbf{h}_i\right)Q_{ij}\left(\mathbf{h}_i\right)\right],
\end{aligned}$$

where we defined $\tilde{\mathbf{x}}_i := [\mathbf{x}_i, x_{ij}, y_i]$ and relaxed the definitions of attempted transmission rate and transmission probability, which we can do without loss of optimality. Problems (3.1) and (3.5) are *not* equivalent because of the linear approximation to the link capacity constraint. However, since $1+\tilde{c}_{ij}$ is a lower bound on $e^{\tilde{c}_{ij}}$, any operating point that satisfies the constraints in (3.5) also satisfies the constraints in (3.1). In particular, the solution of (3.5) is feasible in (3.1), although possibly suboptimal. Further note that variables associated with different terminals appear as different summands of the objective and constraints in (3.5). This is the signature of optimization problems amenable to distributed implementations as we explain in the next section.

3.2 Distributed stochastic learning algorithm

To define the dual of the optimization problem in (3.5) introduce Lagrange multipliers $\mathbf{\Lambda}_i$, associated with terminal i where $\mathbf{\Lambda}_i := \left\{\lambda_i^k, \mu_{ij}, \nu_{ij}, \alpha_{ij}, \beta_i, \xi_i : \forall j \in \mathcal{N}(i)\right\}$. The dual variable λ_i^k

is associated with the flow conservation constraint in (1.8), the multiplier μ_{ij} with the reformulated rate constraint $\sum_{k\in\mathcal{K}} r_{ij}^k \leq 1+\tilde{c}_{ij}$, the variable ν_{ij} with the link capacity constraint $\tilde{c}_{ij} \leq \log x_{ij} + \sum_{l\in\mathcal{M}_i(j)} \log(1-y_l)$, multiplier α_{ij} with the attempted transmission rate constraint in (3.3), β_i with the transmission probability constraint in (3.4), and ξ_i with the average power constraint in (1.10). The Lagrangian for the optimization problem in (3.5) is given by the sum of the objective and the products of the constraints with their respective multipliers,

$$
\begin{aligned}
\mathcal{L}\left(\tilde{\mathbf{x}}, \mathbf{P}(\mathbf{h}), \boldsymbol{\Lambda}\right) &= \sum_{i\in\mathcal{V},k\in\mathcal{K}} U_i^k\left(a_i^k\right) - \sum_{i\in\mathcal{V}} V_i\left(p_i\right) + \sum_{i\in\mathcal{V},k\in\mathcal{K}} \lambda_i^k \left[\sum_{j\in\mathcal{N}(i)} \left(r_{ij}^k - r_{ji}^k\right) - a_i^k\right] \\
&+ \sum_{(i,j)\in\mathcal{E}} \mu_{ij}\left[(1+\tilde{c}_{ij}) - \sum_{k\in\mathcal{K}} r_{ij}^k\right] + \sum_{(i,j)\in\mathcal{E}} \nu_{ij}\left[\log x_{ij} + \sum_{l\in\mathcal{M}_i(j)} \log(1-y_l) - \tilde{c}_{ij}\right] \\
&+ \sum_{(i,j)\in\mathcal{E}} \alpha_{ij}\left[\mathbb{E}_{\mathbf{h}_i}\left[C_{ij}\left(h_{ij}P_{ij}\left(\mathbf{h}_i\right)\right)Q_{ij}\left(\mathbf{h}_i\right)\right] - x_{ij}\right] + \sum_{i\in\mathcal{V}} \beta_i\left[y_i - \mathbb{E}_{\mathbf{h}_i}\left[Q_i\left(\mathbf{h}_i\right)\right]\right] \\
&+ \sum_{i\in\mathcal{V}} \xi_i\left[p_i - \mathbb{E}_{\mathbf{h}_i}\left[\sum_{j\in\mathcal{N}(i)} P_{ij}\left(\mathbf{h}_i\right)Q_{ij}\left(\mathbf{h}_i\right)\right]\right].
\end{aligned} \tag{3.6}
$$

where we introduced vectors $\tilde{\mathbf{x}}$, $\mathbf{P}(\mathbf{h})$, and $\boldsymbol{\Lambda}$ grouping $\tilde{\mathbf{x}}_i$, $\mathbf{P}_i(\mathbf{h}_i)$, and $\boldsymbol{\Lambda}_i$ for all nodes $i\in\mathcal{V}$. The dual function is now defined as the maximum of the Lagrangian in (3.6) over the set of feasible $\tilde{\mathbf{x}}_i$ and $\mathbf{P}_i(\mathbf{h}_i)$ and the dual problem as the minimum of $g(\boldsymbol{\Lambda})$ over positive dual variables, i.e.,

$$
\tilde{\mathsf{D}} = \min_{\boldsymbol{\Lambda}\geq 0} g(\boldsymbol{\Lambda}) = \min_{\boldsymbol{\Lambda}\geq 0} \max_{\{\tilde{\mathbf{x}}_i, \mathbf{P}_i(\mathbf{h}_i)\}\in\mathcal{B}_i} \mathcal{L}\left(\tilde{\mathbf{x}}, \mathbf{P}(\mathbf{h}), \boldsymbol{\Lambda}\right). \tag{3.7}
$$

Despite being non-convex, the structure of the problem in (3.5) is such that $\tilde{\mathsf{P}} = \tilde{\mathsf{D}}$ as long as the fading distribution has no realization of nonzero probability; see [34]. This lack of duality gap implies that the finite dimensional and convex dual problem is equivalent to the infinite dimensional and nonconvex primal problem.

Further note that the Lagrangian in (3.6) exhibits a separable structure because all summands involve a single primal variable. Consider all summands of (3.6) that involve network variables associated with terminal i and define the local Lagrangian at terminal i as

$$
\mathcal{L}_i^{(1)}\left(\tilde{\mathbf{x}}_i, \boldsymbol{\Lambda}\right) := \sum_k U_i^k\left(a_i^k\right) - \lambda_i^k a_i^k + \sum_{j\in\mathcal{N}(i)} \left(\lambda_i^k - \lambda_j^k - \mu_{ij}\right) r_{ij}^k + \sum_{j\in\mathcal{N}(i)} \Big(\mu_{ij} - \nu_{ij}\Big)\tilde{c}_{ij} + \Big(\xi_i p_i - V_i(p_i)\Big)
$$

$$+\sum_{j\in\mathcal{N}_i}[\nu_{ij}\log x_{ij}-\alpha_{ij}x_{ij}]+\beta_i y_i+\left[\sum_{k\in\mathcal{N}(i)}\left(\nu_{ki}+\sum_{l\in\mathcal{M}_i(j)}\nu_{lk}\right)\right]\log(1-y_i). \tag{3.8}$$

Define also the local per channel Lagrangian $\mathcal{L}_i^{(2)}\Big(\mathbf{P}_i(\mathbf{h}_i),\mathbf{h}_i,\mathbf{\Lambda}\Big)$ grouping all summands of (3.6) that involve resource allocations of a given terminal i and a given channel realization $\mathbf{h}_i$, i.e.,

$$\mathcal{L}_i^{(2)}(\mathbf{P}_i(\mathbf{h}_i),\mathbf{h}_i,\mathbf{\Lambda}):=\sum_{j\in\mathcal{N}(i)}Q_{ij}\left(\mathbf{h}_i\right)\left[\alpha_{ij}C_{ij}\left(h_{ij}P_{ij}\left(\mathbf{h}_i\right)\right)-\beta_i-\xi_i P_{ij}\left(\mathbf{h}_i\right)\right]. \tag{3.9}$$

It is easy to see by reordering summands in (3.6) that we can rewrite the Lagrangian as a sum of the local terms $\mathcal{L}_i^{(1)}(\tilde{\mathbf{x}}_i,\mathbf{\Lambda})$ and an expectation of the local per channel components $\mathcal{L}_i^{(2)}(\mathbf{P}_i(\mathbf{h}_i),\mathbf{h}_i,\mathbf{\Lambda})$,

$$\mathcal{L}\left(\tilde{\mathbf{x}},\mathbf{P}(\mathbf{h}),\mathbf{\Lambda}\right)=\sum_{i\in\mathcal{V}}\mathcal{L}_i^{(1)}\left(\tilde{\mathbf{x}}_i,\mathbf{\Lambda}\right)+\mathbb{E}_{\mathbf{h}_i}\left[\mathcal{L}_i^{(2)}\left(\mathbf{P}_i(\mathbf{h}_i),\mathbf{h}_i,\mathbf{\Lambda}\right)\right]. \tag{3.10}$$

This separability on per-terminal terms $\mathcal{L}_i^{(1)}\Big(\tilde{\mathbf{x}}_i,\mathbf{\Lambda}\Big)$ and per-terminal and per-channel elements $\mathcal{L}_i^{(2)}\Big(\mathbf{P}_i(\mathbf{h}_i),\mathbf{h}_i,\mathbf{\Lambda}\Big)$ is exploited in the next section to develop a distributed stochastic subgradient descent algorithm on the dual domain that solves the dual problem (3.7) and, indirectly, the primal problem (3.5).

3.2.1 Stochastic subgradient descent

The dual stochastic subgradient descent algorithm consists of recursive updates of dual variables along stochastic subgradient directions $\mathbf{s}(t)$ moderated by a constant stepsize ϵ,

$$\mathbf{\Lambda}(t+1)=\Big[\mathbf{\Lambda}(t)-\epsilon\mathbf{s}(t)\Big]^+, \tag{3.11}$$

where the operator $[\cdot]^+$ denotes projection to the nonnegative quadrant. The stochastic subgradient $\mathbf{s}(t)$ in (3.11) is a vector whose expectation is a descent direction of the dual function.

The important observation is that a stochastic subgradient $\mathbf{s}(t)$ can be computed from primal maximizers of the Lagrangian $\mathcal{L}\left(\tilde{\mathbf{x}},\mathbf{P}(\mathbf{h}),\mathbf{\Lambda}(t)\right)$. At time t terminal i proceeds to compute primal variables $\tilde{\mathbf{x}}_i(t)=[a_i^k(t),r_{ij}^k(t),\tilde{c}_{ij}(t),p_i(t),x_{ij}(t),y_i(t)]$ that maximize the local Lagrangian

$\mathcal{L}_i^{(1)}\left(\tilde{\mathbf{x}}_i, \mathbf{\Lambda}\right)$,

$$\tilde{\mathbf{x}}_i(t) = \underset{\tilde{\mathbf{x}}_i}{\operatorname{argmax}}\, \mathcal{L}_i^{(1)}\left(\tilde{\mathbf{x}}_i, \mathbf{\Lambda}(t)\right). \tag{3.12}$$

It then observes local channel realizations $\mathbf{h}_i(t)$ and determines instantaneous resource allocation variables $\mathbf{P}_i(t) = [p_{ij}(t), q_{ij}(t)]$ that optimize the local per-channel Lagrangian $\mathcal{L}_i^{(2)}(\mathbf{P}_i(\mathbf{h}_i(t)), \mathbf{h}_i(t), \mathbf{\Lambda})$ associated with the observed channel realization $\mathbf{h}_i(t)$, i.e.,

$$\mathbf{P}_i(t) = \underset{\mathbf{P}_i}{\operatorname{argmax}}\, \mathcal{L}_i^{(2)}\left(\mathbf{P}_i, \mathbf{h}_i(t), \mathbf{\Lambda}(t)\right). \tag{3.13}$$

Based on the primal Lagrangian maximizers $\tilde{\mathbf{x}}_i(t)$ and $\mathbf{P}_i(t)$ defined in (3.12)-(3.13), a stochastic subgradient $\mathbf{s}(t)$ is obtained by evaluating the resultant constraint slack; see e.g., [32]. E.g., the multiplier λ_i^k is associated with the flow conservation constraint $\sum_{j\in\mathcal{N}(i)}\left(r_{ij}^k - r_{ji}^k\right) - a_i^k$. Consequently, the stochastic subgradient component $s_{\lambda_i^k}(t)$ along the λ_i^k direction is given by the constraint slack

$$s_{\lambda_i^k}(t) = \sum_{j\in\mathcal{N}(i)}\left(r_{ij}^k(t) - r_{ji}^k(t)\right) - a_i^k(t). \tag{3.14}$$

Likewise, components $s_{\mu_{ij}}(t)$ along the μ_{ij} direction and $s_{\nu_{ij}}(t)$ along the ν_{ij} direction can be obtained as

$$\begin{aligned} s_{\mu_{ij}}(t) &= (1 + \tilde{c}_{ij}(t)) - \sum_{k\in\mathcal{K}} r_{ij}^k(t), \\ s_{\nu_{ij}}(t) &= \log x_{ij}(t) + \sum_{l\in\mathcal{M}_i(j)} \log(1 - y_l(t)) - \tilde{c}_{ij}(t). \end{aligned} \tag{3.15}$$

For the components $s_{\alpha_{ij}}(t)$, $s_{\beta_i}(t)$, and $s_{\xi_i}(t)$ along the α_{ij}, β_i, and ξ_i directions the corresponding constraints involve expectation with respect to the channel distribution. Since we implement stochastic subgradient descent algorithm, we compute instantaneous constraint slacks where the expectation is replaced by the values associated with the current channel realizations $\mathbf{h}_i(t)$

$$s_{\alpha_{ij}}(t) = C_{ij}\left(h_{ij}(t)p_{ij}(t)\right)q_{ij}(t) - x_{ij}(t),$$

$$s_{\beta_i}(t) = y_i(t) - q_i(t),$$

$$s_{\xi_i}(t) \;\; = p_i(t) - \sum_{j\in\mathcal{N}(i)} p_{ij}(t) q_{ij}(t). \tag{3.16}$$

Further note that since network variables $\tilde{\mathbf{x}}_i = [a_i^k, r_{ij}^k, \tilde{c}_{ij}, p_i, x_{ij}, y_i]$ appear as separate summands in $\mathcal{L}_i^{(1)}(\tilde{\mathbf{x}}_i, \mathbf{\Lambda}(t))$ [cf. (3.10)], the maximization in (3.12) can be carried out separately with respect to individual variables. Specifically, $r_{ij}^k(t)$ and $\tilde{c}_{ij}(t)$ are obtained by solving the following maximization problems

$$\begin{aligned} r_{ij}^k(t) &= \operatorname*{argmax}_{0\le r_{ij}^k \le r_{ij}^{\max}} \left(\lambda_i^k(t) - \lambda_j^k(t) - \mu_{ij}(t)\right) r_{ij}^k, \\ \tilde{c}_{ij}(t) &= \operatorname*{argmax}_{0\le \tilde{c}_{ij} \le \tilde{c}_{ij}^{\max}} \left(\mu_{ij}(t) - \nu_{ij}(t)\right) \tilde{c}_{ij}. \end{aligned} \tag{3.17}$$

Notice that the maximands in (3.17) are linear functions of bounded variables which therefore have trivial solutions. E.g., $r_{ij}^k(t) = r_{ij}^{\max}$ if $\lambda_i^k(t) - \lambda_j^k(t) - \mu_{ij}(t) > 0$ and $r_{ij}^k(t) = 0$ otherwise. Solving for $a_i^k(t)$, $p_i(t)$, $x_{ij}(t)$ and $y_i(t)$ is also easy as it involves maximizing concave functions over convex sets of variables,

$$\begin{aligned} a_i^k(t) &= \operatorname*{argmax}_{a_i^{\min}\le a_i^k \le a_i^{\max}} \;\; U_i^k\left(a_i^k\right) - \lambda_i^k(t) a_i^k, \\ p_i(t) &= \operatorname*{argmax}_{0\le p_i \le p_i^{\max}} \xi_i(t) p_i - V_i\left(p_i\right), \\ x_{ij}(t) &= \operatorname*{argmax}_{x_{ij}\ge 0} \nu_{ij}(t) \log x_{ij} - \alpha_{ij}(t) x_{ij}, \\ y_i(t) &= \operatorname*{argmax}_{0\le y_i\le 1} \beta_i(t) y_i + \left[\sum_{j\in\mathcal{N}(i)} \left(\nu_{ji}(t) + \sum_{l\in\mathcal{M}_i(j)} \nu_{lj}(t)\right)\right] \log\left(1-y_i\right) \end{aligned} \tag{3.18}$$

Closed-form solutions for the maximizations in (3.18) can be easily obtained by solving for the zero of the derivative with respect to the optimization variable, and projecting the result on the feasible set. E.g., the solution for the attempted transmission rate is $x_{ij}(t) = \nu_{ij}(t)/\alpha_{ij}(t)$.

The maximization in (3.13) can be written explicitly as

$$\begin{aligned} \{p_{ij}(t), q_{ij}(t)\} = &\operatorname{argmax} \sum_{j\in\mathcal{N}(i)} q_{ij}\left[\alpha_{ij}(t) C_{ij}(h_{ij}(t) p_{ij}) - \beta_i(t) - \xi_i(t) p_{ij}\right] \\ &\text{s.t. } p_{ij} \in \left[0, p_{ij}^{\text{inst}}\right],\; q_{ij}\in\{0,1\},\; \sum_{j\in\mathcal{N}(i)} q_{ij} \in \{0,1\}. \end{aligned} \tag{3.19}$$

Different from the maximizations in (3.17)-(3.18), the one in (3.19) is a non-convex problem because $C_{ij}(h_{ij}p_{ij})$ may be a non-convex function of p_{ij} and in any event the channel access indicator q_{ij} is an integer variable. Solving (3.19) is still simple, however, as it involves just two variables; see Remark 5.

To complete the definition of the stochastic subgradient descent algorithm we need an expression for $c_{ij}(t)$. Recall that in formulating (3.5) we made $c_{ij} = e^{\tilde{c}_{ij}} \geq 1 + \tilde{c}_{ij}$, which implies that at time t we should set

$$c_{ij}(t) = 1 + \tilde{c}_{ij}(t). \tag{3.20}$$

While the sequence of primal variables $\tilde{\mathbf{x}}_i(\mathbb{N})$ and $\mathbf{P}_i(\mathbb{N})$ is a byproduct of the dual stochastic subgradient descent algorithm, it is the optimality of these sequences, not $\mathbf{\Lambda}(\mathbb{N})$, that we want to study. In general, individual primal iterates $\tilde{\mathbf{x}}_i(t)$ and $\mathbf{P}_i(t)$ may not be optimal but sequences $\tilde{\mathbf{x}}_i(\mathbb{N})$ and $\mathbf{P}_i(\mathbb{N})$ have ergodic limits that are almost surely feasible and give a utility yield close to $\tilde{\mathsf{P}}$; see Section 3.3. In order to simplify upcoming discussions, define the ergodic limit of the sequence of operating points $\mathbf{x}_i(\mathbb{N})$ as

$$\bar{\bar{\mathbf{x}}}_i := \lim_{t\to\infty} \frac{1}{t} \sum_{u=1}^{t} \mathbf{x}_i(u). \tag{3.21}$$

Note that subsumed in the definition in (3.21) are corresponding definitions for each of the individual sequences of admission rates $\bar{a}_i^k := \lim_{t\to\infty} \frac{1}{t}\sum_{u=1}^{t} a_i^k(u)$, routes, $\bar{r}_{ij}^k := \lim_{t\to\infty} \frac{1}{t}\sum_{u=1}^{t} r_{ij}^k(u)$, link capacities $\bar{c}_{ij} := \lim_{t\to\infty} \frac{1}{t}\sum_{u=1}^{t} c_{ij}(u)$, powers $\bar{p}_i := \lim_{t\to\infty} \frac{1}{t}\sum_{u=1}^{t} p_i(u)$, attempted transmission rates $\bar{x}_{ij} := \lim_{t\to\infty} \frac{1}{t}\sum_{u=1}^{t} x_{ij}(u)$, and transmission probabilities $\bar{y}_i := \lim_{t\to\infty} \frac{1}{t}\sum_{u=1}^{t} y_i(u)$.

Remark 5. To find $p_{ij}(t)$ and $q_{ij}(t)$ that solve (3.19) observe that since $q_{ij} \geq 0$ and the constraints on p_{ij} are separate for different j, the optimal selection for p_{ij} is

$$p_{ij}(t) = \operatorname*{argmax}_{p_{ij} \in \left[0, p_{ij}^{\text{inst}}\right]} \alpha_{ij}(t) C_{ij}(h_{ij}(t) p_{ij}) - \beta_i(t) - \xi_i(t) p_{ij}. \tag{3.22}$$

Also note that q_{ij} can only take values from $\{0, 1\}$ and that only one of the q_{ij} variables can be set to 1. If all the optimal objectives computed by (3.22) are negative, i.e., $\alpha_{ij}(t)C_{ij}(h_{ij}(t)p_{ij}(t)) -$

$\beta_i(t) - \xi_i(t)p_{ij}(t) \leq 0$, the optimal solution for (3.19) is $q_{ij}(t) = 0$ for all neighbors. Otherwise, the optimal solution for (3.19) is obtained by setting $q_{ij}(t) = 1$ for the neighbor with the largest objective in (3.22). In summary, we determine

$$j^*(t) = \underset{j \in \mathcal{N}(i)}{\operatorname{argmax}} \; \alpha_{ij}(t)C_{ij}(h_{ij}(t)p_{ij}(t)) - \beta_i(t) - \xi_i(t)p_{ij}(t) \tag{3.23}$$

and set $q_{ij}(t) = 0$ for $j \neq j^*(t)$. For $j = j^*(t)$ we set $q_{ij}(t) = q_{ij^*(t)}(t) = 1$ as long as $\alpha_{ij}(t)C_{ij}(h_{ij}(t)p_{ij}(t)) - \beta_i(t) - \xi_i(t)p_{ij}(t) > 0$ or we make $q_{ij^*(t)}(t) = 0$ otherwise.

Remark 6. If the channel probability distribution is known we can compute powers corresponding not only to $\mathbf{h}(t)$ as in (3.13), but to generic channel realization $\mathbf{h}$

$$\mathbf{P}_i(\mathbf{h}, \mathbf{\Lambda}(t)) = \underset{\mathbf{P}_i}{\operatorname{argmax}} \, \mathcal{L}_i^{(2)}\left(\mathbf{P}_i, \mathbf{h}_i, \mathbf{\Lambda}(t)\right). \tag{3.24}$$

We can then use knowledge of the channel distribution to compute not instantaneous constraint slacks as in (3.16) but actual (average) constraint slacks

$$\begin{aligned}
\tilde{s}_{\alpha_{ij}}(t) &= \mathbb{E}\left[C_{ij}\left(h_{ij}p_{ij}(\mathbf{h}, \mathbf{\Lambda}(t))\right) q_{ij}(\mathbf{h}, \mathbf{\Lambda}(t))\right] - x_{ij}(t), \\
\tilde{s}_{\beta_i}(t) &= y_i(t) - \mathbb{E}\left[q_i(\mathbf{h}, \mathbf{\Lambda}(t))\right], \\
\tilde{s}_{\xi_i}(t) &= p_i(t) - \mathbb{E}\left[\sum_{j \in \mathcal{N}(i)} p_{ij}(\mathbf{h}, \mathbf{\Lambda}(t)) q_{ij}(\mathbf{h}, \mathbf{\Lambda}(t))\right].
\end{aligned} \tag{3.25}$$

The constraint slacks $\tilde{s}_{\alpha_{ij}}(t)$, $\tilde{s}_{\beta_i}(t)$, and $\tilde{s}_{\xi_i}(t)$ are gradients of the dual function and can be used in the descent equation (3.11) in lieu of the stochastic subgradients $s_{\alpha_{ij}}(t)$, $s_{\beta_i}(t)$, and $s_{\xi_i}(t)$. This will result in faster convergence but necessitates estimation of the channel probability distribution. The use of stochastic subgradients not only avoids this estimation problem but is also less computationally demanding and makes it easier to adapt to changes in channel statistics.

Transport Layer

$$a_i^k(t) = \operatorname{argmax}_{a_i^{\min} \le a_i^k \le a_i^{\max}} U_i^k\left(a_i^k\right) - \lambda_i^k(t) a_i^k$$

$$\lambda_i^k(t+1) = \left[\lambda_i^k(t) - \epsilon\left[\sum_{j \in \mathcal{N}(i)} \left(r_{ij}^k(t) - r_{ji}^k(t)\right) - a_i^k(t)\right]\right]^+$$

Network Layer

$$r_{ij}^k(t) = \operatorname{argmax}_{0 \le r_{ij}^k \le r_{ij}^{\max}} \left(\lambda_i^k(t) - \lambda_j^k(t) - \mu_{ij}(t)\right) r_{ij}^k$$

$$\mu_{ij}(t+1) = \left[\mu_{ij}(t) - \epsilon\left[1 + \tilde{c}_{ij}(t) - \sum_{k \in \mathcal{K}} r_{ij}^k(t)\right]\right]^+$$

Link Layer

$$\tilde{c}_{ij}(t) = \operatorname{argmax}_{0 \le \tilde{c}_{ij} \le \tilde{c}_{ij}^{\max}} \mu_{ij}(t)\tilde{c}_{ij} - \nu_{ij}(t)\tilde{c}_{ij}, \qquad c_{ij}(t) = 1 + \tilde{c}_{ij}(t)$$

$$\nu_{ij}(t+1) = \left[\nu_{ij}(t) - \epsilon\left[\log x_{ij}(t) + \sum_{l \in \mathcal{M}_i(j)} \log\left(1 - y_l(t)\right) - \tilde{c}_{ij}(t)\right]\right]^+$$

MAC Layer

$$p_i(t) = \operatorname{argmax}_{0 \le p_i \le p_i^{\max}} \xi_i(t) p_i - V_i\left(p_i\right)$$

$$x_{ij}(t) = \operatorname{argmax}_{x_{ij} \ge 0} \nu_{ij}(t) \log x_{ij} - \alpha_{ij}(t) x_{ij}$$

$$y_i(t) = \operatorname{argmax}_{0 \le y_i \le 1} \beta_i(t) y_i + \left[\sum_{j \in \mathcal{N}(i)} \left[\nu_{ji}(t) + \sum_{l \in \mathcal{M}_i(j)} \nu_{lj}(t)\right]\right] \log\left(1 - y_i\right)$$

$$\alpha_{ij}(t+1) = \left[\alpha_{ij}(t) - \epsilon\left[C_{ij}\left(h_{ij}(t) p_{ij}(t)\right) q_{ij}(t) - x_{ij}(t)\right]\right]^+$$

$$\beta_i(t+1) = \left[\beta_i(t) - \epsilon\left[y_i(t) - q_i(t)\right]\right]^+$$

$$\xi_i(t+1) = \left[\xi_i(t) - \epsilon\left[p_i(t) - \sum_{j \in \mathcal{N}(i)} q_{ij}(t) p_{ij}(t)\right]\right]^+$$

Physical Layer

$$\left\{p_{ij}(t), q_{ij}(t)\right\} = \operatorname{argmax}_{p_{ij} \in \left[0, p_{ij}^{\max}\right], q_{ij} \in \{0,1\}} \sum_{j \in \mathcal{N}(i)} q_{ij}\left[\alpha_{ij}(t) C_{ij}\left(h_{ij}(t) p_{ij}\right) - \beta_i(t) - \xi_i(t) p_{ij}\right]$$

Figure 3.1: Layers and layer interfaces. The stochastic subgradient descent algorithm in terms of layers and layer interfaces. Layers maintain primal variables $a_i^k(t)$, $r_{ij}^k(t)$, $\tilde{c}_{ij}(t)$, $p_{ij}(t)$, $q_{ij}(t)$ as well as auxiliary variables $p_i(t)$, $x_{ij}(t)$, and $y_i(t)$ while multipliers $\lambda_i^k(t)$, $\mu_{ij}(t)$, $\nu_{ij}(t)$, $\alpha_{ij}(t)$, $\beta_i(t)$ and $\xi_i(t)$ are associated with interfaces between adjacent layers. Primal variables can be easily computed based on multipliers from interfaces to adjacent layers and dual variables are updated using information from adjacent layers.

3.2.2 Network operation, layers, and layer interfaces

To describe the role of different variables as computed in (3.17)-(3.20) in the network's operation it is convenient to think in terms of a layered architecture with $a_i^k(t)$ associated with the transport layer, $r_{ij}^k(t)$ with the network layer, $c_{ij}(t)$ with the link layer, $x_{ij}(t)$, $y_i(t)$, and $p_i(t)$ with the medium access (MAC) layer, and $p_{ij}(t)$ and $q_{ij}(t)$ with the physical layer; see figs. 3.1 and 3.2.

Variables $a_i^k(t)$, $r_{ij}^k(t)$, $c_{ij}(t)$, $p_{ij}(t)$ and $q_{ij}(t)$ determine network operation by controlling the flow of packets through queues associated with their corresponding layers; see Fig. 3.2. In the transport and network layers there are queues associated with each of the $|\mathcal{K}|$ flows. In the link and physical layers, queues for each of the $|\mathcal{N}(i)|$ outgoing links (i, j) are maintained. The value of $a_i^k(t)$ determines how many packets are moved from the k-flow queue in the transport layer to the k-flow queue at the network layer at time t. The number of packets transferred at time t from the k-flow network layer queue to the (i, j) queue at the link layer is determined by $r_{ij}^k(t)$. Notice that packets of a particular queue in the network layer may be distributed to different queues in the link layer. Conversely, packets in a particular queue in the link layer may come from different network layer queues, i.e., they may belong to different flows. At time t there are $c_{ij}(t)$ packets moved from the (i, j) queue at the link layer to the (i, j) queue at the physical layer.

At the physical layer queues are emptied through transmission to neighboring terminals. Resource allocation variables $q_{ij}(t)$ and $p_{ij}(t)$ determine the scheduling and transmitted power of link (i, j). If a transmission is scheduled and successful, i.e., a collision does not occur, $C_{ij}(h_{ij}(t)p_{ij}(t))$ units of information are transferred to terminal j from the (i, j) physical layer queue at terminal i. If a collision occurs, they stay at the same queue awaiting retransmission in a future time slot. When a packet is successfully decoded by terminal j it determines which flow they belong to and what destination they are heading for. If the terminal happens to be the destination, packets are forwarded to the application layer. If the terminal is not the designated destination, packets are put into a network layer queue according to their flow identifications.

Besides administering queues, layers are also responsible for updating the values of their corresponding primal variables according to (3.17)-(3.20); see Fig. 3.1. The transport layer updates $a_i^k(t)$ as in (3.18), the network layer keeps track of $r_{ij}^k(t)$ as per (3.17), while the link layer computes $\tilde{c}_{ij}(t)$ as in (3.17) and $c_{ij}(t)$ using (3.20). The MAC layer updates $p_i(t)$, $x_{ij}(t)$, and $y_i(t)$ according to the expressions in (3.18), while the physical layer determines $p_{ij}(t)$ and $q_{ij}(t)$ as dictated by (3.19).

Computation of these primal per layer updates necessitates access to Lagrange multipliers

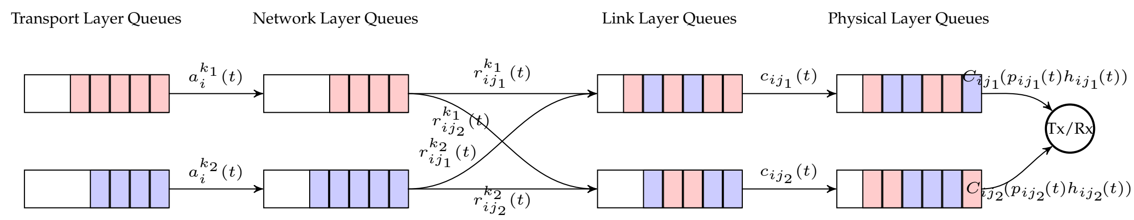


Figure 3.2: Queue dynamics. Terminal i operates by controlling queues in different layers based on operating points $a_i^k(t)$, $r_{ij}^k(t)$, $c_{ij}(t)$, $p_{ij}(t)$ and $q_{ij}(t)$. In the transport layer and the network layer, each flow k has a queue. In the link layer and the physical layer, each outgoing link (i, j) maintains a queue. In this particular example, there are two flows k_1 and k_2 and there are two neighboring nodes j_1 and j_2. Packets for flow k_1 are marked red while packets for k_2 are in blue.

motivating the introduction of layer interfaces to maintain and update their values. E.g., since $\lambda_{ij}^k(t)$ is associated with the flow conservation constraint that relates transport variables $a_i^k(t)$ and network variables $r_{ij}^k(t)$ it provides a natural interface between the transport and network layers. Thus, we introduce a transport-network interface tasked with computing the dual stochastic subgradient component $s_{\lambda_i^k}(t)$ in (3.14) and executing the update $\lambda_i^k(t+1) = [\lambda_i^k(t) - \epsilon s_{\lambda_i^k}(t)]^+$. Similarly, a network-link interface is introduced to keep track of multipliers $\mu_{ij}(t)$, compute the dual stochastic subgradient component $s_{\mu_{ij}}(t)$ in (3.15), and execute the corresponding update. A link-MAC interface does the proper for multipliers $\nu_{ij}(t)$ and dual stochastic subgradient components $s_{\nu_{ij}}(t)$ in (3.15). The remaining multipliers $\alpha_{ij}(t)$, $\beta_i(t)$, and $\xi_i(t)$ provide a MAC-physical interface with stochastic subgradient components $s_{\alpha_{ij}}(t)$, $s_{\beta_i}(t)$, and $s_{\xi_i}(t)$ as given in (3.16). Observe that primal variables are updated with information available at adjacent interfaces, while dual variable updates are undertaken with information available at adjacent layers. Their definition is thereby justified, because information is exchanged only between adjacent layers and interfaces.

We remark that MAC layer variables $x_{ij}(t)$, $y_i(t)$, and $p_i(t)$ do not affect network operation, i.e., queue dynamics, at time t. The role of these variables is to record average behaviors of the

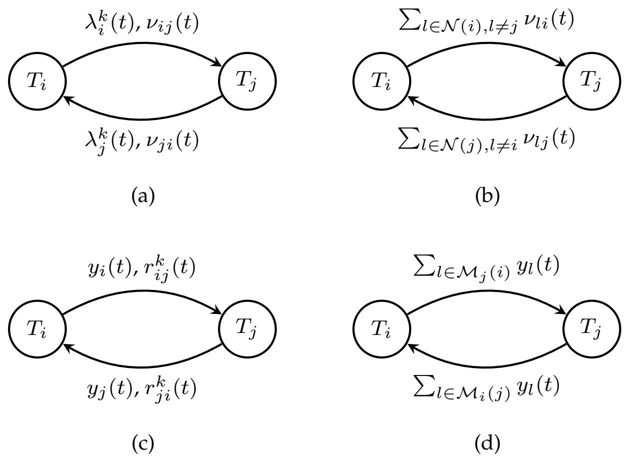


Figure 3.3: Message passing. (a) Terminal i begins by transmitting dual variables $\lambda_i^k(t)$ and $\nu_{ij}(t)$ to all neighbors $j \in \mathcal{N}(i)$. (b) It then computes and shares $\sum_{k\in\mathcal{N}(i)}\nu_{ki}(t)$ with all $j \in \mathcal{N}(i)$. This information, along with locally available multipliers, is then used to perform the primal iterations associated with all the layers in Fig.3.1. (c) Terminal i passes primal variables $y_i(t)$ and $r_{ij}^k(t)$ to all neighbors $j \in \mathcal{N}(i)$. (d) It then evaluates and broadcasts $\sum_{k\in\mathcal{N}(i)} y_k(t)$ to $j \in \mathcal{N}(i)$. Dual updates associated with the layer interfaces in Fig.3.1 are now performed using these and locally accessible primal variables. We proceed to (a) for the next iteration.

terminal to affect determination of $c_{ij}(t)$, $p_{ij}(t)$, and $q_{ij}(t)$ in subsequent time slots. This role is consistent with the definitions of p_i as the the average transmitted power [cf. (1.10)], x_{ij} as the average attempted transmission rate [cf. (3.3)], and y_i as the (average) transmission probability [cf. (3.4)].

3.2.3 Message passing

Most primal and dual variable updates in Fig. 3.1 can be done locally at terminal i. E.g., the physical layer update at terminal i requires access to multipliers $\alpha_{ij}(t)$, $\beta_i(t)$, and $\xi_i(t)$ which are available at the physical-MAC interface of terminal i. The updates for primal variables $r_{ij}^k(t)$ and $y_i(t)$, as well as duals $\lambda_{ij}^k(t)$ and $\nu_{ij}(t)$, however, necessitate access to variables of other terminals. The update of multiplier $\lambda_i^k(t)$ at the network-transport interface depends on network variables $r_{ij}^k(t)$ and $a_i^k(t)$ which are available at terminal i, but also on the variable $r_{ji}^k(t)$ available at (neighboring) terminal j. Similarly, the $r_{ij}^k(t)$ update at the network layer depends on locally available multipliers $\lambda_i^k(t)$ and $\mu_{ij}(t)$, but also on the neighboring multiplier $\lambda_j^k(t)$. The update of multiplier $\nu_{ij}(t)$ is somewhat more complex as it depends on local variables $x_{ij}(t)$ and $\tilde{c}_{ij}(t)$, 1-hop neighborhood variables $y_j(t)$, and 2-hop neighborhood variables $y_l(t)$ for all $l \in \mathcal{N}(j)$. Likewise, the update for $y_i(t)$ at the MAC layer depends on local dual variables $\beta_i(t)$, 1-hop neighborhood variables $\nu_{ji}(t)$ for all $j \in \mathcal{N}(i)$, and 2-hop neighboring variables $\nu_{lj}(t)$ for all $l \in \mathcal{N}(j)$ in the neighborhood of j for some $j \in \mathcal{N}(i)$ in the neighborhood of i. Therefore, implementation of these four updates requires sharing appropriate variables with 1-hop and 2-hop neighbors.

Given that these four updates depend on quantities available at 1-hop and 2-hop neighbors it is necessary to devise a message passing mechanism among terminals to share the necessary values. For doing so we use the 4-step message passing mechanism illustrated in Fig. 3.3. At the beginning of primal iteration, terminal i transmits $\lambda_i^k(t)$ and $\nu_{ij}(t)$ to all his neighbors $j \in \mathcal{N}(i)$; Fig. 3.3(a). As a result, terminal i receives multipliers $\lambda_j^k(t)$ and $\nu_{ji}(t)$ from all of their neighbors $j \in \mathcal{N}(i)$. Terminal i follows by computing and broadcasting the term $\sum_{l \in \mathcal{N}(i)} \nu_{li}(t)$ to all his neighbors $j \in \mathcal{N}(i)$; Fig. 3.3(b). Upon receiving this information, terminal j subtracts $\nu_{ji}(t)$ from the received value to evaluate the expression $\sum_{l \in \mathcal{N}(i), l \neq j} \nu_{li}(t)$. The terms required for computing primal variables $r_{ij}^k(t)$ and $y_i(t)$ are now available at i. Since the variables necessary for the remaining primal updates are locally accessible the primal iterations associated with all the layers in Fig.3.1 are performed at each terminal.

After completing the layer updates, primal iterates $r_{ij}^k(t)$ and $y_i(t)$ need to be exchanged between neighbors to perform the dual updates associated with the layer interfaces in Fig.3.1. Terminal i starts passing variables $y_i(t)$ and $r_{ij}^k(t)$ to all his neighbors; Fig. 3.3(c). Having received $y_j(t)$ from all $j \in \mathcal{N}(i)$ terminal i computes and broadcasts the sum $\sum_{l \in \mathcal{N}(i)} y_l(t)$ to all his neighbors; Fig. 3.3(d). With this information in hand terminal j adds $y_i(t)$ and subtracts $y_j(t)$ from this value to evaluate $\sum_{l\in\mathcal{M}_j(i)} y_l(t) = \sum_{l\in\mathcal{N}(i)} y_l(t) + y_i(t) - y_j(t)$. Quantities necessary to update $\lambda_i^k(t)$ and $\nu_{ij}(t)$ are now available along with the terms necessary for the remaining dual updates that were locally available. The dual updates associated with the layer interfaces in Fig.3.1 are now performed and we proceed to the next primal iteration.

We remark that $r_{ij}^k(t)$ and $\lambda_i^k(t)$ are transmitted to 1-hop neighbors, whereas $y_i(t)$ and $\nu_{ij}(t)$ are sent to 2-hop neighbors. This latter fact holds because transmissions of a given terminal can interfere with neighbors two hops away from her.

3.2.4 Successive convex approximation

As mentioned in the problem reformulation in Sec. 3.1.2, we a use linear lower bound to approximate the capacity constraint. In general, we can use a concave function $f_{ij}(\tilde{c}_{ij})$ which is smaller than $e^{\tilde{c}_{ij}}$ to approximate $e^{\tilde{c}_{ij}}$. As a result, instead of directly computing link capacity variable $c_{ij}(t)$, an approximated version $\tilde{c}_{ij}(t)$ is calculated in the primal iteration. In the network operation, the link capacity $c_{ij}(t) = f_{ij}(\tilde{c}_{ij}(t))$ is used in the link layer. While this approximation convexifies the capacity constraint and provides a feasible solution to the original problem, it reduces the size of the feasible set of primal variables. This implies that this obtained link capacity $c_{ij}(t)$ may not be optimal to the original problem. To reduce its impact on optimality, we use different $f_{ij}(\tilde{c}_{ij})$ at different time slots and hope the approximations become better as time grows. Define then $\bar{\tilde{c}}_{ij}(t) := 1/t\sum_{u=1}^{t}\tilde{c}_{ij}(u)$ and lower bound $e^{\tilde{c}_{ij}(t+1)}$ with the first order approximation

$$e^{\tilde{c}_{ij}(t+1)} \geq e^{\bar{\tilde{c}}_{ij}(t)}\tilde{c}_{ij}(t+1) + e^{\bar{\tilde{c}}_{ij}(t)}\left[1 - \bar{\tilde{c}}_{ij}(t)\right]. \tag{3.26}$$

Notice that the right hand side of (3.26) is a linear function of $\tilde{c}_{ij}(t+1)$ and thus concave. We can then choose $f_{ij}^{(t+1)}(\tilde{c}_{ij}) = e^{\bar{\tilde{c}}_{ij}(t)}\tilde{c}_{ij} + e^{\bar{\tilde{c}}_{ij}(t)}\left[1 - \bar{\tilde{c}}_{ij}(t)\right]$ to approximate $e^{\tilde{c}_{ij}}$ at time slot $t+1$.

3.3 Feasibility and optimality

Solving the optimization problem in (3.1) entails finding optimal variables $\mathbf{x}_i^*$, and power allocations $\mathbf{P}_i^*(\mathbf{h}_i)$ that satisfy problem constraints and offer optimal yield P. This would require knowledge of the channels' probability distributions and a joint optimization among terminals. To overcome these restrictions and develop an adaptive distributed solution, we reformulated the problem as in (3.5) entailing a performance degradation to $\tilde{\mathsf{P}} \leq \mathsf{P}$. This reformulation permits introduction of the dual stochastic subgradient descent algorithm, defined by recursive application of (3.11) - (3.19), that produces a sequence of network operating points $\mathbf{x}_i(\mathbb{N})$ and $\mathbf{P}_i(\mathbb{N})$ – as well as sequences of auxiliary variables $x_{ij}(\mathbb{N})$ and $y_i(\mathbb{N})$ – which given results in [32] are expected to be almost surely feasible and give a utility yield close to $\tilde{\mathsf{P}}$ in an ergodic sense. Notice however, that since (3.11) - (3.19) descends on the dual function of the reformulated problem, feasibility holds with respect to the constraints in (3.5). Our main intent here is to show that sequences of operating points $\mathbf{x}_i(\mathbb{N})$ and $\mathbf{P}_i(\mathbb{N})$ generated by (3.11) - (3.19) are also feasible for the optimization problem in (3.1). Specifically, our goal is to prove the following theorem.

Theorem 3. *Consider a wireless network $\mathcal{G}(\mathcal{V}, \mathcal{E})$ using random access at the physical layer so that ergodic link capacities are as given in* (1.12). *Let $a_i^k(\mathbb{N})$, $r_{ij}^k(\mathbb{N})$, $c_{ij}(\mathbb{N})$, $p_i(\mathbb{N})$, $q_{ij}(\mathbb{N})$ and $p_{ij}(\mathbb{N})$ be sequences of network operating points generated by the stochastic descent algorithm in* (3.11) - (3.19) *and denote as $\bar{a}_i^k$, $\bar{r}_{ij}^k$, $\bar{c}_{ij}$, and $\bar{p}_i$ the corresponding ergodic limits of $a_i^k(\mathbb{N})$, $r_{ij}^k(\mathbb{N})$, $c_{ij}(\mathbb{N})$, and $p_i(\mathbb{N})$. Assume the following hypotheses: (h1) The second moment of the norm of the stochastic subgradient $\mathbf{s}(t)$ is finite, i.e., $\mathbb{E}_{\mathbf{h}}\left[\|\mathbf{s}(t)\|^2 \middle| \mathbf{\Lambda}(t)\right] \leq \hat{S}^2$. (h2) There exists a set of strictly feasible primal variables that satisfy the constraints of the reformulated optimization problem in* (3.5) *with strict inequality. (h3) The dual function $g(\mathbf{\Lambda})$ of the reformulated problem as defined in* (3.7) *has a unique minimizer $\mathbf{\Lambda}^*$. It then holds:*

(i) Near feasibility of physical layer constraints. *There exists a function $M(\epsilon)$ with $\lim_{\epsilon\to 0} M(\epsilon) = 0$ such that the average transmission rate constraint in* (1.12) *is almost surely satisfied with feasibility gap* *smaller than $M(\epsilon)$ in an ergodic sense, i.e.,*

$$\bar{c}_{ij} \leq \lim_{t\to\infty} \frac{1}{t} \sum_{u=1}^{t} \left[C_{ij}(h_{ij}(u)p_{ij}(u))q_{ij}(u) \prod_{k\in\mathcal{M}_i(j)} [1 - q_k(u)] \right] M(\epsilon), \quad \textit{a.s.} \tag{3.27}$$

(ii) Feasibility of upper layer constraints. *The flow conservation constraint in* (1.8)*, the link capacity constraint in* (1.9) *and the average power constraint in* (1.10) *are almost surely satisfied in an ergodic sense, i.e.,*

$$\bar{a}_i^k \leq \sum_{j\in\mathcal{N}(i)} \left[\bar{r}_{ij}^k - \bar{r}_{ji}^k\right], \qquad \sum_{k\in\mathcal{K}} \bar{r}_{ij}^k \leq \bar{c}_{ij}, \qquad \textit{a.s.}, \tag{3.28}$$

$$\bar{p}_i \geq \lim_{t\to\infty} \frac{1}{t} \sum_{u=1}^{t} \sum_{j\in\mathcal{N}(i)} p_{ij}(u)q_{ij}(u), \qquad \textit{a.s.} \tag{3.29}$$

(iii) Utility yield. *The utility yield of the ergodic averages of sequences $a_i^k(\mathbb{N})$ and $p_i(\mathbb{N})$ converges to a value within $\epsilon\hat{S}^2/2$ of $\tilde{\mathsf{P}}$, i.e.,*

$$\tilde{\mathsf{P}} - \left[\sum_{i\in\mathcal{V},k\in\mathcal{K}} U_i^k\left(\bar{a}_i^k\right) - \sum_{i\in\mathcal{V}} V_i\left(\bar{p}_i\right) \right] \leq \frac{\epsilon\hat{S}^2}{2}, \quad \textit{a.s.} \tag{3.30}$$

The feasibility results in (3.28) for the flow conservation and rate constraints are identical to (1.8) and (1.9). As such they imply that the ergodic limits $\bar{a}_i^k$, $\bar{r}_{ij}^k$, $\bar{c}_{ij}$ obtained from recursive application of (3.11) - (3.19) satisfy these constraints with probability 1. Notice that these limits may be different for different realizations of the algorithm's run. Nonetheless, constraints (1.8) and (1.9) are satisfied for almost all runs. The feasibility result in (3.27) for the link capacity constraint, however, is not identical to (1.12). The difference is not only the presence of the $M(\epsilon)$ feasibility gap, but the fact that (1.12) involves an expectation over channel realizations whereas (3.27) does not. In fact, asides from the $M(\epsilon)$ constant, (3.27) is stronger than (1.12). The feasibility result in (3.27) states that even though sequences $\mathbf{x}_i(\mathbb{N})$ and $\mathbf{P}_i(\mathbb{N})$ may not be ergodic, the possibly different ergodic limits in the right and left hand sides of (3.27) satisfy the stated inequality. This implies that operating the network using variables $\mathbf{x}_i(t)$ and $\mathbf{P}_i(t)$ as generated by (3.11) - (3.19) results in long-term feasibility in that all packets are (almost surely) delivered to their corresponding destinations. Further notice that the power feasibility result in (3.29) is not identical to the corresponding power constraint in (1.10) because (1.10) involves an expected value whereas (3.29) does not. The same comments stated for the comparison of (3.27) and (1.12) extend naturally.

The utility yield result in (3.30) states that the long term performance of the network, as determined by average end-to-end rates $\bar{a}_i^k$ and powers $\bar{p}_i$, is close to the optimal yield $\tilde{\mathsf{P}}$ of the reformulated problem. The gap between $\tilde{\mathsf{P}}$ and the attained yield can be controlled by reducing ϵ. Notice that reducing the step size ϵ also reduces the feasibility gap $M(\epsilon)$ in (3.27). We also remark that the use of constant step sizes ϵ endows the algorithm with adaptability to time-varying channel distributions. This is important in practice because wireless channels are non-stationary due to user mobility and environmental dynamics.

3.3.1 Proof of Theorem 3

Hypotheses (h1) and (h2) are sufficient for Theorem 1 of [32] to hold. The utility yield result in (3.30) is a direct consequence of [32, Theorem 1]. It also follows that all constraints in problem (3.5) are almost surely satisfied in an ergodic sense. Since the flow conservation constraint in (1.8) and the power constraint in (1.10) are part of (3.5) the first inequality in (3.28) and the inequality in (3.29) follow from direct application of [32, Theorem 1]. In addition, considering the constraint $\sum_{k\in\mathcal{K}} r_{ij}^k \leq 1 + \tilde{c}_{ij}$ Theorem 1 of [32] gives us

$$\sum_{k\in\mathcal{K}} \bar{r}_{ij}^k(u) \leq \lim_{t\to\infty} \frac{1}{t} \sum_{u=1}^{t} \left[1 + \tilde{c}_{ij}(u)\right], \quad \text{a.s..} \tag{3.31}$$

Recall now that at every iteration we set the link capacity to $c_{ij}(u) = 1 + \tilde{c}_{ij}(u)$. Substituting this equality into (3.31) the second inequality in (3.28) follows from the definition $\bar{c}_{ij} := \lim_{t\to\infty} \frac{1}{t} \sum_{u=1}^{t} c_{ij}(u)$.

The result that does not follow as a simple application of [32, Theorem 1] is the almost sure near feasibility of the average transmission rate constraint as shown in (3.27). Since we introduced auxiliary variables x_{ij} and y_i and decomposed the average transmission rate constraint in two separate constraints [32, Theorem 1] does not make a claim on the feasibility of (1.12). Instead, the claim is for the last three constraints in (3.5), i.e.,

$$\lim_{t\to\infty} \frac{1}{t}\sum_{u=1}^{t} \tilde{c}_{ij}(u) \le \log\left[\bar{x}_{ij}(u)\right] + \sum_{k\in\mathcal{M}_i(j)} \log\left[1-\bar{y}_k(u)\right], \text{a.s.}, \tag{3.32}$$

$$\bar{x}_{ij} \le \lim_{t\to\infty} \frac{1}{t}\sum_{u=1}^{t} C_{ij}(h_{ij}(u)p_{ij}(u))q_{ij}(u), \text{a.s.}, \tag{3.33}$$

$$\bar{y}_i \le \lim_{t\to\infty} \frac{1}{t}\sum_{u=1}^{t} q_i(u), \text{a.s.}, \tag{3.34}$$

Since link capacity iterates are set to $c_{ij}(u) = 1 + \tilde{c}_{ij}(u)$ we use the fact that $1 + x \le e^x$ for all x to write

$$\bar{c}_{ij} := \lim_{t\to\infty} \frac{1}{t}\sum_{u=1}^{t} c_{ij}(u) = 1 + \lim_{t\to\infty} \frac{1}{t}\sum_{u=1}^{t} \tilde{c}_{ij}(u) \le \exp\left[\lim_{t\to\infty} \frac{1}{t}\sum_{u=1}^{t} \tilde{c}_{ij}(u)\right]. \tag{3.35}$$

Substitute now the inequality in (3.32) into the exponent in (3.35) to obtain

$$\bar{c}_{ij} \le \exp\left[\log\left[\bar{x}_{ij}(u)\right] + \sum_{l\in\mathcal{M}_i(j)} \log\left[1-\bar{y}_l(u)\right]\right] = \bar{x}_{ij} \prod_{l\in\mathcal{M}_i(j)} \left[1-\bar{y}_l(u)\right], \tag{3.36}$$

where in the equality we cancelled out the exponential and logarithm functions. Further substituting (3.33) and (3.34) into the right hand side of (3.36) yields

$$\bar{c}_{ij} \le \left[\lim_{t\to\infty} \frac{1}{t}\sum_{u=1}^{t} C_{ij}(h_{ij}(u)p_{ij}(u))q_{ij}(u)\right] \prod_{l\in\mathcal{M}_i(j)} \left[\lim_{t\to\infty} \frac{1}{t}\sum_{u=1}^{t} \left[1-q_l(u)\right]\right]. \tag{3.37}$$

While similar, (3.37) is substantially different from the statement in (3.27) that we want to prove. To see the difference exploit ergodicity, possibly restricted to an ergodic component, to replace the ergodic limit in (3.27) by the corresponding expected value so as to write

$$\lim_{t\to\infty} \frac{1}{t}\sum_{u=1}^{t} \left[C_{ij}(h_{ij}(u)p_{ij}(u))q_{ij}(u) \prod_{l\in\mathcal{M}_i(j)} \left[1-q_l(u)\right]\right] = \lim_{t\to\infty} \mathbb{E}\left[C_{ij}(h_{ij}(t)p_{ij}(t))q_{ij}(t) \prod_{l\in\mathcal{M}_i(j)} \left[1-q_l(t)\right]\right]. \tag{3.38}$$

Similarly, consider the product of ergodic limits in (3.37) and use ergodicity, also possibly restricted to an ergodic component, to write each *individual* limit as an expectation,

$$\bar{c}_{ij} \le \lim_{t\to\infty} \mathbb{E}\left[C_{ij}(h_{ij}(t)p_{ij}(t))q_{ij}(t)\right] \prod_{l\in\mathcal{M}_i(j)} \mathbb{E}\left[\left[1-q_l(t)\right]\right]. \tag{3.39}$$

If schedules of different terminals were independent, the expectation in (3.38) would coincide with the product of expectations in (3.39) yielding the result in (3.27) with $M(\epsilon) = 0$ after substituting (3.38) into (3.39). However, due to the message passing between neighboring terminals correlation in transmission decisions is introduced, independence is violated, and the expectation in (3.38) may not coincide with the product of expectations in (3.39). It follows from this discussion that the key point in establishing (3.27) is to show that the correlation between schedules introduced by message passing is small so that the expectation in (3.38) equals the product of expectations in (3.39) except for the vanishingly small difference $M(\epsilon)$.

To prove so start noting that while $C_{ij}(h_{ij}(t)p_{ij}(t))q_{ij}(t)$ and $q_l(t)$ for $l \in \mathcal{M}_i(j)$ correlate through message passing, they are conditionally uncorrelated if multipliers $\mathbf{\Lambda}(t)$ are given. This is true because for given $\mathbf{\Lambda}(t)$ schedules and power allocations depend only on local channel realizations, which are assumed independent for different channels. We can therefore write

$$\begin{aligned}
&\mathbb{E}_{\mathbf{h}}\left[C_{ij}(h_{ij}(t)p_{ij}(t))q_{ij}(t)\prod_{l\in\mathcal{M}_i(j)}[1-q_l(t)]\,\middle|\,\mathbf{\Lambda}(t)\right] \\
&= \mathbb{E}_{\mathbf{h}_i}\left[C_{ij}(h_{ij}(t)p_{ij}(t))q_{ij}(t)\,\middle|\,\mathbf{\Lambda}(t)\right]\prod_{l\in\mathcal{M}_i(j)}\left[1-\mathbb{E}_{\mathbf{h}_l}\left[q_l(t)\,\middle|\,\mathbf{\Lambda}(t)\right]\right].
\end{aligned} \tag{3.40}$$

The conditional expectations in (3.40) and the (unconditional) ones in (3.38) and (3.39) can be related through double integration, e.g.,

$$\mathbb{E}\left[q_i(t)\right] = \int \mathbb{E}_{\mathbf{h}_i}\left[q_i(t)|\mathbf{\Lambda}(t)\right] d\mathbf{\Lambda}(t). \tag{3.41}$$

The crucial observation is that since (3.11) - (3.19) descends in the dual domain, $\mathbf{\Lambda}(t)$ approaches the optimal multiplier $\mathbf{\Lambda}^*$ as t grows; see e.g., [32, Theorem 2]. This motivates the introduction of a set $\mathcal{A}$ containing all multipliers $\mathbf{\Lambda}$ within a given small distance $\sqrt{\delta}$ of $\mathbf{\Lambda}^*$, i.e., $\mathcal{A} = \{\mathbf{\Lambda} \,\big|\, ||\mathbf{\Lambda} - \mathbf{\Lambda}^*||^2 \leq \delta\}$. We can then separate the integration with respect to $\mathbf{\Lambda}(t)$ in (3.41) into terms that contain multipliers inside and outside $\mathcal{A}$,

$$\mathbb{E}\left[q_i(t)\right] = \int_{\mathbf{\Lambda}(t)\in\mathcal{A}} \mathbb{E}_{\mathbf{h}_i}\left[q_i(t)|\mathbf{\Lambda}(t)\right] d\mathbf{\Lambda}(t) + \int_{\mathbf{\Lambda}(t)\in\mathcal{A}^c} \mathbb{E}_{\mathbf{h}_i}\left[q_i(t)|\mathbf{\Lambda}(t)\right] d\mathbf{\Lambda}(t). \tag{3.42}$$

By making δ small enough the first integral in (3.42) can be made arbitrarily close to $\mathbb{E}_{\mathbf{h}_i}[q_i(t)|\mathbf{\Lambda}(t) = \mathbf{\Lambda}^*]$. Since $\mathbf{\Lambda}(t)$ gets close to $\mathbf{\Lambda}^*$ as t increases, the second integral can be made small for sufficiently large t.

While we have exemplified the argument for the expectation $\mathbb{E}[q_i(t)]$ the same is true for the other expectations in (3.38) and (3.39). The idea to complete the proof is to show that for sufficiently large t all expectations can be written as conditional expectations given $\mathbf{\Lambda}^*$ plus small error terms. Conditional independence is then used to claim (3.40) from the equivalence of the right hand sides of (3.38) and (3.39). In summary we need to make the following arguments in order to conclude the proof:

(A1) For sufficiently large t, the probability of $\mathbf{\Lambda}(t)$ staying within a small distance of $\mathbf{\Lambda}^*$ is close to 1. The distance can be made arbitrarily small and the probability arbitrarily close to 1 by reducing ϵ. This argument is formalized and proved in Lemma 1.

(A2) All of the expectations in (3.38) and (3.39) can be written as integrals of conditional expectations of the form shown in (3.42) for $\mathbb{E}[q_i(t)]$. By making the ball $\mathcal{A}$ sufficiently small the (first) integral with respect to multipliers $\mathbf{\Lambda}(t) \in \mathcal{A}$ can be made arbitrarily close to the expectation conditional on $\mathbf{\Lambda}(t) = \mathbf{\Lambda}^*$. From (A1) it follows that for any small ball $\mathcal{A}$ the (second) integral with respect to $\mathbf{\Lambda}(t)$ for multipliers $\mathbf{\Lambda}(t) \notin \mathcal{A}$ can be made close to 0 by reducing ϵ. Therefore, it follows that unconditional, e.g., $\mathbb{E}[q_i(t)]$, and conditional, e.g., $\mathbb{E}_{\mathbf{h}_i}[q_i(t)|\mathbf{\Lambda}(t) = \mathbf{\Lambda}^*]$, expectations get arbitrarily close as $\epsilon \to 0$. This argument is formalized and proved in Lemma 2.

(A3) From Argument (A2), it follows that the unconditional expectation in (3.38) can be expressed as an expectation conditioned on $\mathbf{\Lambda}(t) = \mathbf{\Lambda}^*$ plus an arbitrarily small error term. Recalling the fact that given $\mathbf{\Lambda}(t)$ schedules and power allocations for different terminals are uncorrelated we can write the resulting conditional expectation as a product of conditional expectations [cf. (3.40)]. In turn, Argument (A2) implies that each of these expectations is close to the unconditional expectation plus an small error term. The result in (3.27) follows from ergodicity. This argument is formalized after Lemma 2 to conclude the proof.

Let us start by formalizing argument (A1) in the following lemma. The proof of is technical and relegated to Appendix A.

Lemma 1. *Consider the the stochastic descent algorithm in* (3.11) - (3.19) *with the same hypotheses and definitions of Theorem 3. Let the dual variable $\mathbf{\Lambda}(T_0)$ at given time T_0 be given. Then, there exists time $T_1 > T_0$ such that for all $t > T_1$ it holds*

$$Pr\left[||\mathbf{\Lambda}(t) - \mathbf{\Lambda}^*||^2 \geq L(\epsilon)\middle|\mathbf{\Lambda}(T_0)\right] \leq L(\epsilon), \tag{3.43}$$

where $L(\epsilon)$ is a function of the step size ϵ such that $\lim_{\epsilon\to 0} L(\epsilon) = 0$.

Proof. See Appendix 3.6.1. □

Lemma 1 states, as required by argument (A1), that the probability of $\mathbf{\Lambda}(t)$ being outside arbitrarily small distance $\sqrt{L(\epsilon)}$ of $\mathbf{\Lambda}^*$ is the arbitrarily small factor $L(\epsilon)$. To formalize (A2) we introduce a bounded function $D(\mathbf{h}(t), \mathbf{P}(t))$ to stand in for the functions inside the expectations in (3.38) and (3.39). We show that for arbitrary bounded function $D(\mathbf{h}(t), \mathbf{P}(t))$, its unconditional mean is within a small $N(\epsilon)$ constant of its expectation conditional on $\mathbf{\Lambda}(t) = \mathbf{\Lambda}^*$ as long as the conditional expectation is a continuous function of $\mathbf{\Lambda}(t)$.

Lemma 2. *Consider the stochastic descent algorithm in* (3.11) - (3.19) *with the same hypotheses and definitions of Theorem 3. Let $0 \leq D(\mathbf{h}(t), \mathbf{P}(t)) \leq D_{\max}$ be a nonnegative continuous function of $\mathbf{h}(t)$, $\mathbf{p}(t)$ and $\mathbf{q}(t)$ upper bounded by $D_{\max}$. Assume the dual variable $\mathbf{\Lambda}(T_0)$ at given time T_0 is given and that the conditional expectation $\mathbb{E}_{\mathbf{h}}\left[D(\mathbf{h}(t), \mathbf{P}(t))\middle|\mathbf{\Lambda}(t)\right]$ is continuous in $\mathbf{\Lambda}(t)$. Then almost surely there exists $T_1 > T_0$ such that for all $t > T_1$ it holds*

$$\left|\mathbb{E}\left[D(\mathbf{h}(t), \mathbf{P}(t))\middle|\mathbf{\Lambda}(T_0)\right] - \mathbb{E}_{\mathbf{h}}\left[D(\mathbf{h}(t), \mathbf{P}(t))\middle|\mathbf{\Lambda}(t) = \mathbf{\Lambda}^*\right]\right| \leq N(\epsilon), \tag{3.44}$$

where the first and the second expectations are with respect to $\mathbf{h}(T_0), \cdots, \mathbf{h}(t)$ and $\mathbf{h}(t)$, respectively, and $N(\epsilon)$ is a function of the step size ϵ such that $\lim_{\epsilon\to 0} N(\epsilon) = 0$.

Proof. Start noting that we can write $\mathbb{E}_{\mathbf{h}}\left[D(\mathbf{h}(t),\mathbf{P}(t))\middle|\mathbf{\Lambda}(T_0)\right]$ as an integral of conditional expectations [cf. (3.41)],

$$\mathbb{E}\left[D(\mathbf{h}(t),\mathbf{P}(t))\middle|\mathbf{\Lambda}(T_0)\right] = \int \mathbb{E}_{\mathbf{h}}\left[D(\mathbf{h}(t),\mathbf{P}(t))\middle|\mathbf{\Lambda}(t),\mathbf{\Lambda}(T_0)\right] d\mathbf{\Lambda}(t)$$
$$= \int \mathbb{E}_{\mathbf{h}}\left[D(\mathbf{h}(t),\mathbf{P}(t))\middle|\mathbf{\Lambda}(t)\right] d\mathbf{\Lambda}(t), \tag{3.45}$$

where the second equality follows because $\mathbf{\Lambda}(\mathbb{N})$ is a Markov process. Partitioning the integration space into the sets $\mathcal{A}_\epsilon = \{\mathbf{\Lambda} | \|\mathbf{\Lambda}-\mathbf{\Lambda}^*\|^2 < L(\epsilon)\}$ and $\mathcal{A}_\epsilon^c = \{\mathbf{\Lambda} | \|\mathbf{\Lambda}-\mathbf{\Lambda}^*\|^2 \geq L(\epsilon)\}$ allows us to rewrite (3.45) as [cf. (3.42)]

$$\mathbb{E}\left[D(\mathbf{h}(t),\mathbf{P}(t))\middle|\mathbf{\Lambda}(T_0)\right] = \int_{\mathbf{\Lambda}(t)\in\mathcal{A}_\epsilon} \mathbb{E}_{\mathbf{h}}\left[D(\mathbf{h}(t),\mathbf{P}(t))\middle|\mathbf{\Lambda}(t)\right] d\mathbf{\Lambda}(t)$$
$$+ \int_{\mathbf{\Lambda}(t)\in\mathcal{A}_\epsilon^c} \mathbb{E}_{\mathbf{h}}\left[D(\mathbf{h}(t),\mathbf{P}(t))\middle|\mathbf{\Lambda}(t)\right] d\mathbf{\Lambda}(t). \tag{3.46}$$

Since we are assuming that $0 \leq D(\mathbf{h}(t),\mathbf{P}(t)) \leq D_{\max}$ we can bound the second integral on the right hand side of (3.46) by

$$0 \leq \int_{\mathbf{\Lambda}(t)\in\mathcal{A}_\epsilon^c} \mathbb{E}_{\mathbf{h}}\left[D(\mathbf{h}(t),\mathbf{P}(t))\middle|\mathbf{\Lambda}(t)\right] d\mathbf{\Lambda}(t) \leq D_{\max}\Pr\left[\mathbf{\Lambda}(t)\in\mathcal{A}_\epsilon^c\middle|\mathbf{\Lambda}(T_0)\right]. \tag{3.47}$$

According to Lemma 1, we know that there exists time $T_1 > T_0$ such that for all $t > T_1$ we have $\Pr\left[\mathbf{\Lambda}(t)\in\mathcal{A}_\epsilon^c\middle|\mathbf{\Lambda}(T_0)\right] = \Pr\left[||\mathbf{\Lambda}(t)-\mathbf{\Lambda}^*||^2 \geq L(\epsilon)\middle|\mathbf{\Lambda}(T_0)\right] \leq L(\epsilon)$. Substituting this bound into (3.47) yields

$$0 \leq \int_{\mathbf{\Lambda}(t)\in\mathcal{A}_\epsilon^c} \mathbb{E}_{\mathbf{h}}\left[D(\mathbf{h}(t),\mathbf{P}(t))\middle|\mathbf{\Lambda}(t)\right] d\mathbf{\Lambda}(t) \leq D_{\max}L(\epsilon) \tag{3.48}$$

for all times $t > T_1$. For the first integral on the right hand side of (3.46), observe that since $\mathbb{E}_{\mathbf{h}}\left[D(\mathbf{h}(t),\mathbf{P}(t))\middle|\mathbf{\Lambda}(t)\right]$ is continuous in $\mathbf{\Lambda}(t)$ we can use the mean value theorem to write the integral as

$$\int_{\mathbf{\Lambda}(t)\in\mathcal{A}_\epsilon} \mathbb{E}_{\mathbf{h}}\left[D(\mathbf{h}(t),\mathbf{P}(t))\middle|\mathbf{\Lambda}(t)\right] d\mathbf{\Lambda}(t) = \mathbb{E}_{\mathbf{h}}\left[D(\mathbf{h}(t),\mathbf{P}(t))\middle|\mathbf{\Lambda}(t)=\mathbf{\Lambda}_0\right] \Pr\left[\mathbf{\Lambda}(t)\in\mathcal{A}_\epsilon\middle|\mathbf{\Lambda}(T_0)\right], \tag{3.49}$$

for a certain $\mathbf{\Lambda}_0 \in \mathcal{A}_\epsilon$. Since for any $t > T_1$ we have $0 \leq \Pr\left[\mathbf{\Lambda}(t) \in \mathcal{A}_\epsilon^c \middle| \mathbf{\Lambda}(T_0)\right] \leq L(\epsilon)$, it follows that $1 - L(\epsilon) \leq \Pr\left[\mathbf{\Lambda}(t) \in \mathcal{A}_\epsilon \middle| \mathbf{\Lambda}(T_0)\right] \leq 1$. Substituting this into (3.49) we have

$$
\begin{aligned}
[1 - L(\epsilon)]\, \mathbb{E}_{\mathbf{h}}\left[D(\mathbf{h}(t), \mathbf{P}(t)) \middle| \mathbf{\Lambda}(t) = \mathbf{\Lambda}_0\right] &\leq \int_{\mathbf{\Lambda}(t) \in \mathcal{A}_\epsilon} \mathbb{E}_{\mathbf{h}}\left[D(\mathbf{h}(t), \mathbf{P}(t)) \middle| \mathbf{\Lambda}(t)\right] d\mathbf{\Lambda}(t) \\
&\leq \mathbb{E}_{\mathbf{h}}\left[D(\mathbf{h}(t), \mathbf{P}(t)) \middle| \mathbf{\Lambda}(t) = \mathbf{\Lambda}_0\right], \qquad (3.50)
\end{aligned}
$$

Substituting (3.48) and (3.50) into (3.46) yields

$$
\begin{aligned}
[1 - L(\epsilon)]\, \mathbb{E}_{\mathbf{h}}\left[D(\mathbf{h}(t), \mathbf{P}(t)) \middle| \mathbf{\Lambda}(t) = \mathbf{\Lambda}_0\right] &\leq \mathbb{E}\left[D(\mathbf{h}(t), \mathbf{P}(t)) \middle| \mathbf{\Lambda}(T_0)\right] \\
&\leq D_{\max} L(\epsilon) + \mathbb{E}_{\mathbf{h}}\left[D(\mathbf{h}(t), \mathbf{P}(t)) \middle| \mathbf{\Lambda}(t) = \mathbf{\Lambda}_0\right]. \qquad (3.51)
\end{aligned}
$$

To show that (3.44) is true we find upper bounds for

$$
\mathbb{E}\left[D(\mathbf{h}(t), \mathbf{P}(t)) \middle| \mathbf{\Lambda}(T_0)\right] - \mathbb{E}_{\mathbf{h}}\left[D(\mathbf{h}(t), \mathbf{P}(t)) \middle| \mathbf{\Lambda}(t) = \mathbf{\Lambda}^*\right]
$$

and its opposite

$$
\mathbb{E}_{\mathbf{h}}\left[D(\mathbf{h}(t), \mathbf{P}(t)) \middle| \mathbf{\Lambda}(t) = \mathbf{\Lambda}^*\right] - \mathbb{E}\left[D(\mathbf{h}(t), \mathbf{P}(t)) \middle| \mathbf{\Lambda}(T_0)\right].
$$

Define $L'(\epsilon) := \left|\mathbb{E}_{\mathbf{h}}\left[D(\mathbf{h}(t), \mathbf{P}(t)) \middle| \mathbf{\Lambda}(t) = \mathbf{\Lambda}_0\right] - \mathbb{E}_{\mathbf{h}}\left[D(\mathbf{h}(t), \mathbf{P}(t)) \middle| \mathbf{\Lambda}(t) = \mathbf{\Lambda}^*\right]\right|$ and observe that since $\mathbb{E}_{\mathbf{h}}\left[D(\mathbf{h}(t), \mathbf{P}(t)) \middle| \mathbf{\Lambda}(t)\right]$ is continuous in $\mathbf{\Lambda}(t)$ and $\mathbf{\Lambda}_0 \in \mathcal{A}_\epsilon$, it follows that $\lim_{\epsilon \to 0} L'(\epsilon) = 0$. Using this definition for $L'(\epsilon)$ and the upper bound in (3.51) we obtain

$$
\begin{aligned}
&\mathbb{E}\left[D(\mathbf{h}(t), \mathbf{P}(t)) \middle| \mathbf{\Lambda}(T_0)\right] - \mathbb{E}_{\mathbf{h}}\left[D(\mathbf{h}(t), \mathbf{P}(t)) \middle| \mathbf{\Lambda}(t) = \mathbf{\Lambda}^*\right] \qquad (3.52) \\
&\leq D_{\max} L(\epsilon) + \mathbb{E}_{\mathbf{h}}\left[D(\mathbf{h}(t), \mathbf{P}(t)) \middle| \mathbf{\Lambda}(t) = \mathbf{\Lambda}_0\right] - \mathbb{E}_{\mathbf{h}}\left[D(\mathbf{h}(t), \mathbf{P}(t)) \middle| \mathbf{\Lambda}(t) = \mathbf{\Lambda}^*\right] \\
&\leq D_{\max} L(\epsilon) + L'(\epsilon). \qquad (3.53)
\end{aligned}
$$

Similarly, using the definition of $L'(\epsilon)$ and the lower bound in (3.51) we have

$$
\begin{aligned}
&\mathbb{E}_{\mathbf{h}}\left[D(\mathbf{h}(t), \mathbf{P}(t)) \middle| \mathbf{\Lambda}(t) = \mathbf{\Lambda}^*\right] - \mathbb{E}\left[D(\mathbf{h}(t), \mathbf{P}(t)) \middle| \mathbf{\Lambda}(T_0)\right] \\
&\leq \mathbb{E}_{\mathbf{h}}\left[D(\mathbf{h}(t), \mathbf{P}(t)) \middle| \mathbf{\Lambda}(t) = \mathbf{\Lambda}^*\right] - [1 - L(\epsilon)]\, \mathbb{E}_{\mathbf{h}}\left[D(\mathbf{h}(t), \mathbf{P}(t)) \middle| \mathbf{\Lambda}(t) = \mathbf{\Lambda}_0\right]
\end{aligned}
$$

$$
\begin{aligned}
&= L(\epsilon)\mathbb{E}_{\mathbf{h}}\left[D(\mathbf{h}(t),\mathbf{P}(t))\middle|\mathbf{\Lambda}(t)=\mathbf{\Lambda}_0\right] + \mathbb{E}_{\mathbf{h}}\left[D(\mathbf{h}(t),\mathbf{P}(t))\middle|\mathbf{\Lambda}(t)=\mathbf{\Lambda}^*\right] \\
&\quad - \mathbb{E}_{\mathbf{h}}\left[D(\mathbf{h}(t),\mathbf{P}(t))\middle|\mathbf{\Lambda}(t)=\mathbf{\Lambda}_0\right] \\
&\le D_{\max}L(\epsilon) + L'(\epsilon).
\end{aligned}
\tag{3.54}
$$

where the last inequality follows from the fact that $D(\mathbf{h}(t),\mathbf{P}(t)) \le D_{\max}$. From (3.52) and (3.54) we conclude

$$
\left|\mathbb{E}\left[D(\mathbf{h}(t),\mathbf{P}(t))\middle|\mathbf{\Lambda}(T_0)\right] - \mathbb{E}_{\mathbf{h}}\left[D(\mathbf{h}(t),\mathbf{P}(t))\middle|\mathbf{\Lambda}(t)=\mathbf{\Lambda}^*\right]\right| \le D_{\max}L(\epsilon) + L'(\epsilon). \tag{3.55}
$$

Making $N(\epsilon) := D_{\max}L(\epsilon) + L'(\epsilon)$ in (3.55) yields (3.44). Since both $L(\epsilon)$ and $L'(\epsilon)$ approach 0 as ϵ goes to 0, it follows $\lim_{\epsilon\to 0} N(\epsilon) = 0$. ☐

In Lemma 2, continuity of $\mathbb{E}_{\mathbf{h}}\left[D(\mathbf{h}(t),\mathbf{P}(t))\middle|\mathbf{\Lambda}(t)\right]$ is assumed. Specifically, we need continuity of $\mathbb{E}_{\mathbf{h}_i}\left[q_i(t)\middle|\mathbf{\Lambda}(t)\right]$ and $\mathbb{E}_{h_{ij}}\left[q_{ij}(t)C(h_{ij}(t)p_{ij}(t))\middle|\mathbf{\Lambda}(t)\right]$. This is indeed true as claimed by the following lemma.

Lemma 3. *Consider the calculation of primal variables $p_{ij}(t)$ and $q_{ij}(t)$ as shown in* (3.19)*, $\mathbb{E}_{\mathbf{h}_i}\left[q_i(t)\middle|\mathbf{\Lambda}(t)\right]$ and $\mathbb{E}_{h_{ij}}\left[q_{ij}(t)C(h_{ij}(t)p_{ij}(t))\middle|\mathbf{\Lambda}(t)\right]$ are continuous functions of $\mathbf{\Lambda}(t)$.*

Proof. See Appendix 3.6.2. ☐

Using Lemma 3 we conclude that the hypotheses of Lemma 2 are satisfied. Applying the result in Lemma 2 we then have that for sufficiently large time index t we can rewrite (3.39) as

$$
\bar{c}_{ij} \le \mathbb{E}\left[C_{ij}(h_{ij}(t)p_{ij}(t))q_{ij}(t)\middle|\mathbf{\Lambda}(t)=\mathbf{\Lambda}^*\right]\prod_{l\in\mathcal{M}_i(j)}\mathbb{E}\left[1-q_l(t)|\mathbf{\Lambda}(t)=\mathbf{\Lambda}^*\right] + N_1(\epsilon), \tag{3.56}
$$

where $\lim_{\epsilon\to 0} N_1(\epsilon) = 0$. Given $\mathbf{\Lambda}(t)$, $C_{ij}(h_{ij}(t)p_{ij}(t))q_{ij}(t)$ and $q_l(t)$ are uncorrelated [cf. (3.40)]. This allows us to write the product of expectations on the right hand side of (3.56) as an expectation of products, i.e.,

$$
\bar{c}_{ij} \le \mathbb{E}_{\mathbf{h}_i}\left[C_{ij}(h_{ij}(t)p_{ij}(t))q_{ij}(t)\prod_{l\in\mathcal{M}_i(j)}[1-q_l(t)]\middle|\mathbf{\Lambda}(t)=\mathbf{\Lambda}^*\right] + N_1(\epsilon). \tag{3.57}
$$

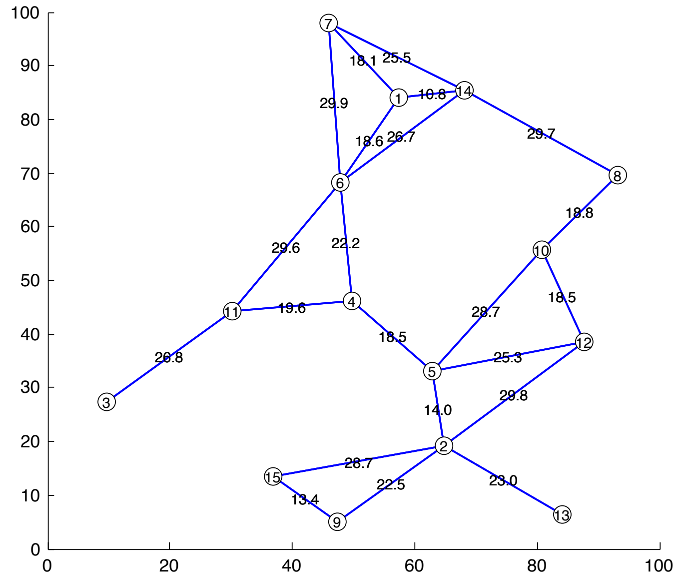


Figure 3.4: Connectivity graph of a network with $n = 15$ terminals randomly placed in a square with side $L = 100$ meters. Terminals can communicate with neighbors whose distances are within 30 meters. The numbers on each edge shows the distance (in meters) between two communicating terminals.

Using Lemma 2 again, the conditional expectation on the right hand side of (3.57) can be expressed as an unconditional expectation plus a small term $N_2(\epsilon)$, leading us to

$$\bar{c}_{ij} \leq \mathbb{E}_{\mathbf{h}_i}\left[C_{ij}(h_{ij}(t)p_{ij}(t))q_{ij}(t)\prod_{l\in\mathcal{M}_i(j)}[1-q_l(t)]\right] + N_2(\epsilon) + N_1(\epsilon), \tag{3.58}$$

where $\lim_{\epsilon\to 0} N_2(\epsilon) = 0$. Define $M(\epsilon) = N_1(\epsilon) + N_2(\epsilon)$ and substitute (3.38) into (3.58) to obtain (3.27). □

3.4 Numerical results

We illustrate performance of the proposed algorithm by implementing and simulating it over a network with $n = 15$ terminals randomly placed in a square with side $L = 100$ meters. Terminals can communicate with neighbors whose distances are within 30 meters. Numerical experiments here utilize the realization of this random placement shown in Fig. 3.4. Channel gains $h_{ij}(t)$

are Rayleigh distributed with mean $\bar{h}_{ij}$ and are independent across links and time. The average channel gain $\bar{h}_{ij} := \mathbb{E}\left[h_{ij}\right]$ follows an exponential pathloss law, $\bar{h}_{ij} = \alpha d_{ij}^{-\beta}$ with d_{ij} denoting the distance in meters between T_i and T_j and constants $\alpha = 10^{-1}\mathrm{m}^{-1}$ and $\beta = 2.5$. Assume the use of capacity achieving codes so that the instantaneous transmission rate takes the form

$$C_{ij}(h_{ij}(t)p_{ij}(t)) = \log\left(1 + \frac{h_{ij}(t)p_{ij}(t)}{N_0}\right), \tag{3.59}$$

where N_0 is the channel noise set to $N_0 = 10^{-4}$ for all links. Fading channels are generated as i.i.d. There are two flows supported by the network, one from T_1 to T_2 and the other from T_8 to T_{11}. For each flow the minimum and maximum amount of information to be delivered are constrained by $a_i^{\text{min}} = 0.1$ bits/s/Hz and $a_i^{\max} = 1$ bits/s/Hz for all nodes i. The routing and link capacity variables are bounded by $r_{ij}^{\text{min}} = c_{ij}^{\text{min}} = 0$ bits/s/Hz and $r_{ij}^{\max} = c_{ij}^{\max} = 1$ bits/s/Hz. The maximum average power consumption per terminal and maximum instantaneous power consumption per terminal are set to 2, i.e., $p_i^{\max} = p_{ij}^{\text{inst}} = 2$. Our objective is to maximize total amount of information delivered by the network, i.e., $U_i^k(a_i^k) = a_i^k$ and $V_i(p_i) = 0$. We set $\epsilon = 0.02$ and the simulation is conducted for 10^4 time slots. Successive convex approximation is used.

Fig. 3.5 shows feasibility of the proposed algorithm in terms of constraint violations. Specifically, $\bar{V}_{\lambda_i^k}(t)$, $\bar{V}_{\mu_{ij}}(t)$, $\bar{V}_{\nu_{ij}}(t)$ and $\bar{V}_{\xi_i}(t)$, representing average violations of the flow conservation, link capacity, average rate and average power constraints, respectively, are presented in the figure. At each time t, we compute

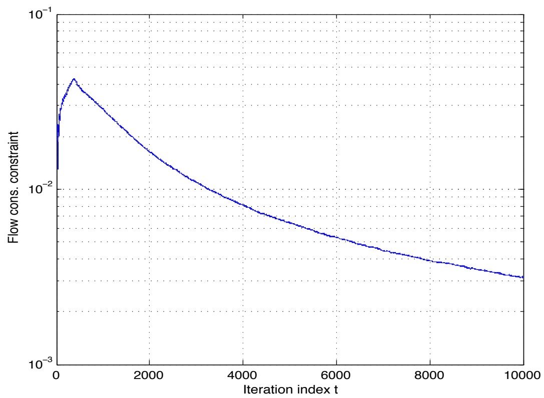
$$\bar{V}_{\lambda_i^k}(t) = \frac{1}{t}\sum_{u=1}^{t}\left[\sum_{j\in\mathcal{N}(i)}\left(r_{ij}^k(u) - r_{ji}^k(u)\right) - a_i^k(u)\right], \tag{3.60}$$

$$\bar{V}_{\mu_{ij}}(t) = \frac{1}{t}\sum_{u=1}^{t}\left[c_{ij}(u) - \sum_{k\in\mathcal{K}} r_{ij}^k(u)\right], \tag{3.61}$$

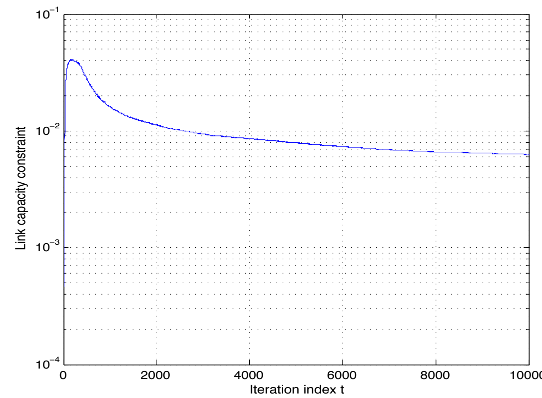
$$\bar{V}_{\nu_{ij}}(t) = \frac{1}{t}\sum_{u=1}^{t}\left[C_{ij}\Big(h_{ij}(u)p_{ij}(u)\Big)q_{ij}(u)\prod_{k\in\mathcal{M}_i(j)}\left[1 - q_k(u)\right] - c_{ij}(u)\right], \tag{3.62}$$

$$\bar{V}_{\xi_i}(t) = \frac{1}{t}\sum_{u=1}^{t}\left[p_i(u) - \sum_{j\in\mathcal{N}(i)} p_{ij}(u)q_{ij}(u)\right]. \tag{3.63}$$

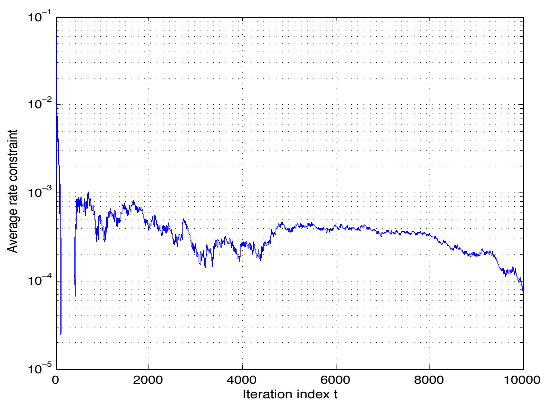
If the above values are nonnegative, it means the corresponding constraints are satisfied in an



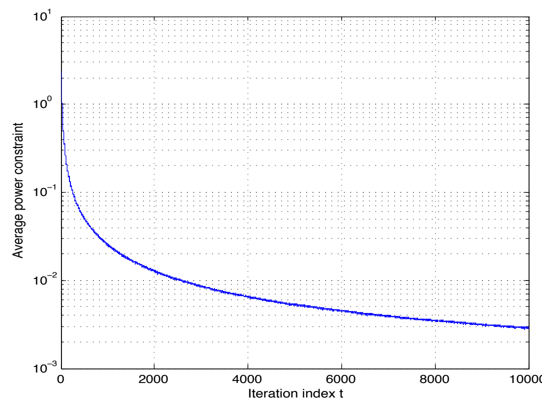
(a) Flow conservation constraint



(b) Link capacity constraint



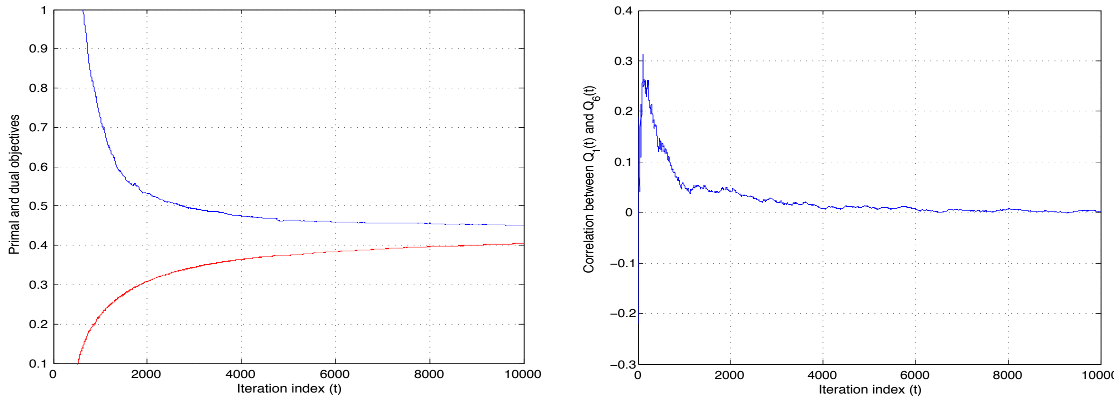
(c) Average rate constraint



(d) Average power constraint

Figure 3.5: Feasibility. After about 500 steps, all constraints are satisfied in an ergodic sense within 10^{-2} tolerance. The average rate constraint takes the longest time to be satisfied. This is because the transmission rate on link $T_i \to T_j$ depends not only on schedules and powers of T_i but also on those of T_j and neighbors of T_j. This requires information to be received from, and propagated to, 2-hop neighbors.

average sense. As we can see, after about 500 steps all constraints are satisfied within 10^{-2} tolerance. The average rate constraint takes the longest time to be satisfied (see Fig. 3.5 (c)). This is because the transmission rate on link $T_i \to T_j$ depends not only on schedules and powers of T_i but also on those of T_j and his neighbors. This requires information to be received from, and propagated to, 2-hop networks.



(a) (b)

Figure 3.6: (a) Optimality. As time grows, primal and dual objectives approach each other. (b) Correlation between $Q_1(t)$ and $Q_6(t)$. At the beginning, there is significant correlation between $Q_1(t)$ and $Q_6(t)$. But as time grows, the correlation vanishes and becomes negligible.

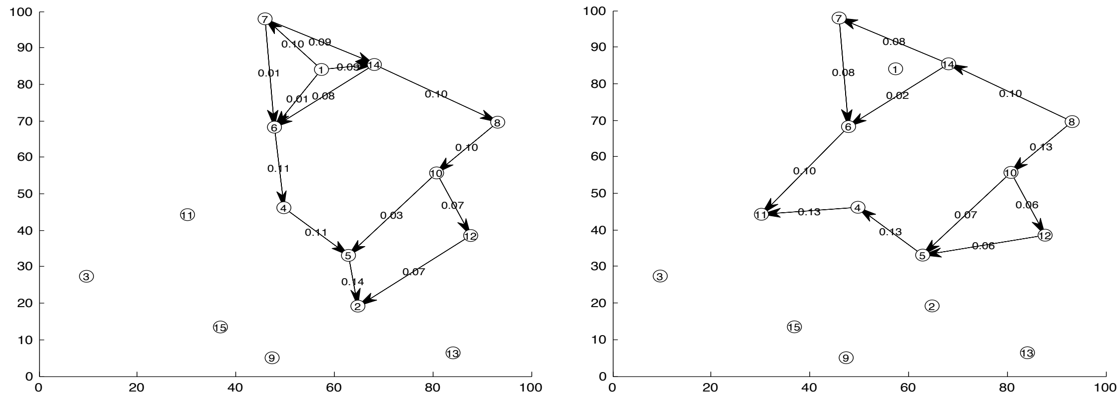
To show optimality of the algorithm we compare ergodic primal and dual objectives. Since we are maximizing total admission control variables, the ergodic primal objective is

$$\mathsf{P}(t) = 1/t \sum_{u=1}^{t} \sum_{k\in\mathcal{K}} a_i^k(u). \tag{3.64}$$

Furthermore, upon defining average Lagrange multipliers as $\bar{\lambda}_i^k(t) = 1/t\sum_{u=1}^t \lambda_i^k(u)$, $\bar{\mu}_{ij}(t) = 1/t\sum_{u=1}^t \mu_{ij}(u)$, $\bar{\nu}_{ij}(t) = 1/t\sum_{u=1}^t \nu_{ij}(u)$ and $\bar{\xi}_i(t) = 1/t\sum_{u=1}^t \xi_i(u)$, we can compute the ergodic dual objective as

$$\mathsf{D}(t) = \mathsf{P}(t) + \sum_{i\in\mathcal{V}}\sum_{k\in\mathcal{K}} \bar{V}_{\lambda_i^k}(t)\bar{\lambda}_i^k(t) + \sum_{(i,j)\in\mathcal{E}} \bar{V}_{\mu_{ij}}(t)\bar{\mu}_{ij}(t) + \sum_{(i,j)\in\mathcal{E}} \bar{V}_{\nu_{ij}}(t)\bar{\nu}_{ij}(t) + \sum_{i\in\mathcal{V}} \bar{V}_{\xi_i}(t)\bar{\xi}_i(t). \tag{3.65}$$

Fig. 3.6 (a) compares the ergodic primal and dual objectives. As time grows, the convergence of the proposed algorithm is observed as the primal and dual values approach each other. By Theorem 3, the algorithm is almost surely near optimal in the sense that the ergodic average of the utility almost surely converges to a value with optimality gap smaller than $\epsilon\hat{S}^2/2$ with respect to the optimal objective. Indeed, this is true as shown in Fig. 3.6 (a) that the gap between primal



(a) Flow 1: from T_1 to T_2

(b) Flow 2: from T_8 to T_{11}

Figure 3.7: Optimal routes for flow 1 (from T_1 to T_2) and flow 2 (from T_8 to T_{11}).

and dual values becomes a small constant (about 0.05) as t increases. Moreover, we compute the correlation between $Q_1(t)$ and $Q_6(t)$ using samples from time 1 to t. The result is shown in Fig. 3.6 (b). At the beginning, there is significant correlation between $Q_1(t)$ and $Q_6(t)$. But as time grows, the correlation vanishes and becomes negligible.

Optimal routes for flow 1 and 2 are shown in Fig. 3.7 (a) and (b). In addition to the shortest path from source to destination, other longer paths are used to deliver information for both flows. For example, the shortest path for flow 2 is $T_8 \to T_{14} \to T_6 \to T_{11}$, but a longer path $T_8 \to T_{10} \to T_5 \to T_4 \to T_{11}$ is utilized as well. It is interesting to note that the longer path delivers more information than the shorter path does. This is because the shorter path goes through T_{14} and T_6 which interfere with the source node of flow 1 (T_1). To limit interference with flow 1, some packets in flow 2 are transmitted via other longer paths.

3.5 Summary

We developed algorithms for optimal design of wireless networks using local channel state information. Due to the time-varying nature of fading states, random access is the natural medium access choice leading to the formulation of an optimization problem for random access networks. To obtain a distributed solution, we approximated the problem so that it can be decomposed in the dual domain and developed a stochastic subgradient descent algorithm. Based on instantaneous local channel conditions, the algorithm finds network operating points that are almost surely feasible and optimal in an ergodic sense. The solution exhibits a layered architecture in which variables in each layer are computed using information from interfaces to adjacent layers. The algorithm is fully distributed in that all operations necessary to achieve optimal operation are based on local information and information exchanges between neighboring terminals. The computational cost per iteration is minimal. In the proposed algorithm, all terminals act independently of each other. Algorithms that consider collaboration among terminals will be a future research direction.

3.6 Appendices

3.6.1 Proof of Lemma 1

Define $g(t) := g(\mathbf{\Lambda}(t))$. According to Thereom 2 in [32], for arbitrary $\delta > 0$, $g(t) - \tilde{\mathsf{D}}$ falls below $\epsilon \hat{S}^2/2 + \delta$ at least once almost surely as t grows. If $g(t) - \tilde{\mathsf{D}}$ falls below $\epsilon \hat{S}^2/2 + \delta$, it may stay below or jump above $\epsilon \hat{S}^2/2 + \delta$. The key idea in this proof is to show that if $g(t)$ exceeds $\tilde{\mathsf{D}} + \epsilon \hat{S}^2/2 + \delta$ the probability that it gets even bigger is very small. Let us then define T_1 as a time at which $g(T_1)$ stays below $\tilde{\mathsf{D}} + \epsilon \hat{S}^2/2 + \delta$ but jumps above it at time $T_1 + 1$, i.e., $g(T_1 + 1) - \tilde{\mathsf{D}} > \epsilon \hat{S}^2/2 + \delta$. The rest of the proof relies on the following chain of arguments:

(A1) The expected value of the distance between $\mathbf{\Lambda}(T_1 + 1)$ and the optimal dual variable $\mathbf{\Lambda}^*$ is bounded by a function $L_0(\epsilon)$ where $\lim_{\epsilon \to 0} L_0(\epsilon) = 0$, i.e.,

$$\mathbb{E}\left[||\mathbf{\Lambda}(T_1 + 1) - \mathbf{\Lambda}^*||^2 \right] \leq L_0(\epsilon). \tag{3.66}$$

(A2) Define $g_{\text{best}}(t) = \min_{u \in [0,t]} g(u)$ and $\psi(t) = ||\mathbf{\Lambda}(T_1 + t) - \mathbf{\Lambda}^*||^2 \mathbb{I}\left\{ g_{\text{best}}(T_1 + t) - \tilde{\mathsf{D}} \geq \epsilon \hat{S}^2/2 \right\}$

for $t = 1, 2, \cdots$ and $\mathbb{I}\{\cdot\}$ denotes the indicator function. Then, $\psi(t)$ is a supermartingale, i.e.,

$$\mathbb{E}\left[\psi(t+1)|\psi(1:t)\right] \leq \psi(t). \tag{3.67}$$

(A3) Assume $L_0(\epsilon)$ is small enough such that $L_0(\epsilon) < \sqrt{L_0(\epsilon)}$. Define then a stopping rule $\psi(t) \geq \sqrt{L_0(\epsilon)}$ or $\psi(t) = 0$. Let T be a stopping time, by the optional stopping theorem [44, Theorem 10.10] we have

$$\mathbb{E}[\psi(T)] \leq \mathbb{E}[\psi(1)]. \tag{3.68}$$

Using the fact that $\psi(1) = ||\mathbf{\Lambda}(T_1+1) - \mathbf{\Lambda}^*||^2$ and results in (3.66) we can further bound (3.68) by

$$\mathbb{E}[\psi(T)] \leq \mathbb{E}\left[||\mathbf{\Lambda}(T_1+1) - \mathbf{\Lambda}^*||^2\right] \leq L_0(\epsilon), \tag{3.69}$$

According to the stopping rule, either $\psi(T) \geq \sqrt{L_0(\epsilon)}$ or $\psi(T) = 0$. As a result, we can lower bound $\mathbb{E}[\psi(T)]$ by

$$\mathbb{E}[\psi(T)] \geq \sqrt{L_0(\epsilon)} \Pr\left[||\mathbf{\Lambda}(T_1+T) - \mathbf{\Lambda}^*||^2 \geq \sqrt{L_0(\epsilon)} \middle| \mathbf{\Lambda}(T_0)\right]. \tag{3.70}$$

Substituting (3.70) into (3.69) and dividing both sides by $\sqrt{L_0(\epsilon)}$ yields

$$\Pr\left[||\mathbf{\Lambda}(T_1+T) - \mathbf{\Lambda}^*||^2 \geq \sqrt{L_0(\epsilon)} \middle| \mathbf{\Lambda}(T_0)\right] \leq \sqrt{L_0(\epsilon)}. \tag{3.71}$$

(A4) For any $t > 0$, the event $||\mathbf{\Lambda}(T_1+t) - \mathbf{\Lambda}^*||^2 \geq \sqrt{L_0(\epsilon)}$ happens only when there exists $T \leq t$ such that T is a stopping time and $||\mathbf{\Lambda}(T_1+T) - \mathbf{\Lambda}^*||^2 \geq \sqrt{L_0(\epsilon)}$. Then, we have

$$\begin{aligned}
&\Pr\left[||\mathbf{\Lambda}(T_1+t) - \mathbf{\Lambda}^*||^2 \geq \sqrt{L_1(\epsilon,\delta)} \middle| \mathbf{\Lambda}(T_0)\right] \\
&\leq \Pr\left[||\mathbf{\Lambda}(T_1+T) - \mathbf{\Lambda}^*||^2 \geq \sqrt{L_0(\epsilon)} \middle| \mathbf{\Lambda}(T_0)\right] \leq \sqrt{L_0(\epsilon)}
\end{aligned} \tag{3.72}$$

where the second inequality follows from (3.71). Substituting $L(\epsilon) = \sqrt{L_0(\epsilon)}$ into (3.72) completes the proof. In the following, we provide detailed proofs for (A1) and (A2).

First, we show that (3.66) is true, i.e., $\mathbb{E}\left[||\mathbf{\Lambda}(T_1+1) - \mathbf{\Lambda}^*||^2\right] \leq L_0(\epsilon)$. Start by noting that $g(t)$ is a convex function of $\mathbf{\Lambda}(t)$ with a unique minimizer $\mathbf{\Lambda}^*$, then $g(T_1) - \tilde{\mathsf{D}} \leq \epsilon\hat{S}^2/2 + \delta$ is equivalent to

$$||\mathbf{\Lambda}(T_1) - \mathbf{\Lambda}^*||^2 \leq L_1(\epsilon\hat{S}^2/2 + \delta), \tag{3.73}$$

where $L_1(\cdot)$ is a nonnegative function such that $\lim_{x\to 0} L_1(x) = 0$. According to the dual update (3.11), we can write $||\mathbf{\Lambda}(T_1+1) - \mathbf{\Lambda}^*||^2$ as

$$||\mathbf{\Lambda}(T_1+1) - \mathbf{\Lambda}^*||^2 = ||\left[\mathbf{\Lambda}(T_1) - \epsilon\mathbf{s}(T_1)\right]^+ - \mathbf{\Lambda}^*||^2 \tag{3.74}$$

$$\leq ||\mathbf{\Lambda}(T_1) - \mathbf{\Lambda}^* - \epsilon\mathbf{s}(T_1)||^2 \tag{3.75}$$

$$= ||\mathbf{\Lambda}(T_1) - \mathbf{\Lambda}^*||^2 + \epsilon^2||\mathbf{s}(T_1)||^2 - 2\epsilon\mathbf{s}^T(T_1)\left[\mathbf{\Lambda}(T_1) - \mathbf{\Lambda}^*\right], \tag{3.76}$$

where inequality (3.75) follows because setting negative elements in $\mathbf{\Lambda}(T_1) - \epsilon\mathbf{s}(T_1)$ to zero reduces its distance to $\mathbf{\Lambda}^*$. Expanding (3.75) yields (3.76). Taking expectation conditioned on $\mathbf{\Lambda}(T_1)$ for both sides of (3.76) yields

$$\begin{aligned}\mathbb{E}\left[||\mathbf{\Lambda}(T_1+1) - \mathbf{\Lambda}^*||^2 \middle| \mathbf{\Lambda}(T_1)\right] \leq ||\mathbf{\Lambda}(T_1) - \mathbf{\Lambda}^*||^2 + \epsilon^2\mathbb{E}\left[||\mathbf{s}(T_1)||^2 \middle| \mathbf{\Lambda}(T_1)\right] \\ - 2\epsilon\mathbb{E}\left[\mathbf{s}^T(T_1) \middle| \mathbf{\Lambda}(T_1)\right]\left[\mathbf{\Lambda}(T_1) - \mathbf{\Lambda}^*\right].\end{aligned} \tag{3.77}$$

Note that the first term on the right hand side of (3.77) is upper bounded by $L_1(\epsilon\hat{S}^2/2 + \delta)$ [cf. (3.73)]. As per the hypothesis, $\mathbb{E}\left[||\mathbf{s}(T_1)||^2|\mathbf{\Lambda}(T_1)\right]$ is upper bounded by $\hat{S}^2$. The third term is lower bounded by 0 because $\mathbb{E}\left[\mathbf{s}(T_1)|\mathbf{\Lambda}(T_1)\right]$ is subgradient of $g(\mathbf{\Lambda}(T_1))$ [32, Proposition 1]. Plugging these bounds into (3.77) yields

$$\mathbb{E}\left[||\mathbf{\Lambda}(T_1+1) - \mathbf{\Lambda}^*||^2 \middle| \mathbf{\Lambda}(T_1)\right] \leq L_1(\epsilon\hat{S}^2/2 + \delta) + \epsilon^2\hat{S}^2 := L_2(\epsilon, \delta). \tag{3.78}$$

where we defined function $L_2(\epsilon, \delta)$. Taking expectation with respect to $\mathbf{\Lambda}(T_1)$ on both sides of (3.78) and defining $L_0(\epsilon) = \lim_{\delta\to 0} L_2(\epsilon, \delta)$ lead us to (3.66).

We then show $\psi(t)$ is a supermartingale. We discuss two cases $\psi(t) = 0$ and $\psi(t) > 0$ separately. If $\psi(t) = 0$, it implies either $\mathbf{\Lambda}(T_1 + t) = \mathbf{\Lambda}^*$ or $g_{\text{best}}(T_1 + t) - \tilde{\mathsf{D}} < \epsilon\hat{S}^2/2$. If $\mathbf{\Lambda}(T_1 + t) = \mathbf{\Lambda}^*$, then it must be $g(T_1 + t) = \tilde{\mathsf{D}}$. Since the dual function is lower bounded by $\tilde{\mathsf{D}}$, it implies $g_{\text{best}}(T_1 + t + 1) = g_{\text{best}}(T_1 + t) = \tilde{\mathsf{D}}$. If $g_{\text{best}}(T_1 + t) - \tilde{\mathsf{D}} < \epsilon\hat{S}^2/2$, it follows that

$g_{\text{best}}(T_1+t+1) - \tilde{\mathsf{D}} < \epsilon\hat{S}^2/2$ since $g_{\text{best}}(T_1+t+1) \le g_{\text{best}}(T_0+t)$. In either case, $\psi(t+1) = 0$ and (3.67) holds for equality. If $\psi(t) \neq 0$, it must be $g_{\text{best}}(T_1+t) - \tilde{D} \ge \epsilon\hat{S}^2/2$, which implies $g(T_1+t) - \tilde{\mathsf{D}} \ge \epsilon\hat{S}^2/2$ and $\psi(t) = ||\mathbf{\Lambda}(T_1+t) - \mathbf{\Lambda}^*||^2$. Since $\psi(t)$ is completely determined by $\mathbf{\Lambda}(t)$, we can write following relationship

$$
\begin{aligned}
\mathbb{E}\left[\psi(t+1)|\psi(1:t)\right] &= \mathbb{E}\left[\psi(t+1)\middle|\mathbf{\Lambda}(T_1+1:T_1+t)\right] && (3.79)\\
&\le \mathbb{E}\left[||\mathbf{\Lambda}(T_1+t+1) - \mathbf{\Lambda}^*||^2\middle|\mathbf{\Lambda}(T_1+1:T_1+t)\right] && (3.80)\\
&= \mathbb{E}\left[||\mathbf{\Lambda}(T_1+t+1) - \mathbf{\Lambda}^*||^2\middle|\mathbf{\Lambda}(T_1+t)\right], && (3.81)
\end{aligned}
$$

where inequality (3.80) follows since $\psi(t+1) = ||\mathbf{\Lambda}(T_1+t+1) - \mathbf{\Lambda}^*||^2\mathbb{I}\left\{g_{\text{best}}(T_1+t+1) - \tilde{\mathsf{D}} \ge \epsilon\hat{S}^2/2\right\}$ $\le ||\mathbf{\Lambda}(T_1+t+1) - \mathbf{\Lambda}^*||^2$ and equality (3.81) is true since $\mathbf{\Lambda}(\mathbb{N})$ is a Markov process. Using the dual update rule (3.11) we can bound (3.81) by

$$
\begin{aligned}
\mathbb{E}\left[\psi(t+1)|\psi(1:t)\right] &\le ||\mathbf{\Lambda}(T_1+t) - \mathbf{\Lambda}^*||^2 + \epsilon^2\mathbb{E}\left[||\mathbf{s}(T_1+t)||^2\middle|\mathbf{\Lambda}(T_1+t)\right]\\
&\quad - 2\epsilon\mathbb{E}\left[\mathbf{s}^T(T_1+t)\middle|\mathbf{\Lambda}(T_1+t)\right]\left[\mathbf{\Lambda}(T_1+t) - \mathbf{\Lambda}^*\right] && (3.82)\\
&\le ||\mathbf{\Lambda}(T_1+t) - \mathbf{\Lambda}^*||^2 + \epsilon\hat{S}^2 - 2\epsilon\left[g(T_1+t) - \tilde{\mathsf{D}}\right] && (3.83)\\
&\le ||\mathbf{\Lambda}(T_1+t) - \mathbf{\Lambda}^*||^2 = \psi(t). && (3.84)
\end{aligned}
$$

where (3.83) follows because $\mathbb{E}\left[||\mathbf{s}(T_1+t)||^2|\mathbf{\Lambda}(T_1+t)\right] \le \hat{S}^2$ and $\mathbb{E}\left[\mathbf{s}^T(T_1+t)|\mathbf{\Lambda}(T_1+t)\right]$ $\left[\mathbf{\Lambda}(T_1+t) - \mathbf{\Lambda}^*\right]$ is lower bounded by $g(T_1+t) - \tilde{\mathsf{D}}$ and (3.84) follows from the fact that $g(T_1+t) - \tilde{\mathsf{D}} \ge \epsilon\hat{S}^2/2$. Therefore, for both cases $\psi(t) = 0$ and $\psi(t) > 0$ (3.67) holds true.

3.6.2 Proof of Lemma 3

For notational simplicity, we ignore time index t in this proof. Recall that q_i is uniquely determined by $\mathbf{h}_i$ and $\mathbf{\Lambda}_i$. Thus, we can write q_i as a function of $\mathbf{h}_i$ and $\mathbf{\Lambda}_i$, i.e. $q_i(\mathbf{h}_i, \mathbf{\Lambda}_i)$. To show $\mathbb{E}_{\mathbf{h}_i}\left[q_i|\mathbf{\Lambda}_i\right]$ is continuous in $\mathbf{\Lambda}_i$, we have to establish that for any sequence $\mathbf{\Lambda}_i(n)$ that converges to $\mathbf{\Lambda}_i$ as $n \to \infty$, $\mathbb{E}_{\mathbf{h}_i}[q_i|\mathbf{\Lambda}_i(n)]$ converges to $\mathbb{E}_{\mathbf{h}_i}[q_i|\mathbf{\Lambda}_i]$, i.e.,

$$\lim_{n\to\infty} \int q_i(\mathbf{h}_i, \mathbf{\Lambda}_i(n)) d\mathbf{h}_i = \int q_i(\mathbf{h}_i, \mathbf{\Lambda}_i) d\mathbf{h}_i. \tag{3.85}$$

To show (3.85) is true, define

$$W_{ij}(\mathbf{h}_i, \mathbf{\Lambda}_i) = \max_{p\in[0, p_{ij}^{\text{inst}}]} \left\{\alpha_{ij} C_{ij}(h_{ij} p) - \beta_i - \xi_i p\right\}, \tag{3.86}$$

and

$$W_i(\mathbf{h}_i, \mathbf{\Lambda}_i) = \max_{j\in\mathcal{N}(i)} \left\{W_{ij}(\mathbf{h}_i, \mathbf{\Lambda}_i)\right\}. \tag{3.87}$$

Note that the objective on the right hand side of (3.86) is a linear function of $\mathbf{\Lambda}_i$. Given $\mathbf{h}_i$, $W_{ij}(\mathbf{h}_i, \mathbf{\Lambda}_i)$ is the maximum of a set of linear functions of $\mathbf{\Lambda}_i$. As a consequence, $W_{ij}(\mathbf{h}_i, \mathbf{\Lambda}_i)$ is a convex function of $\mathbf{\Lambda}_i$ given $\mathbf{h}_i$. Moreover, note that $W_i(\mathbf{h}_i, \mathbf{\Lambda}_i)$ is the maximum of $W_{ij}(\mathbf{h}_i, \mathbf{\Lambda}_i)$ for all $j \in \mathcal{N}(i)$, then given $\mathbf{h}_i$ it is a convex function of $\mathbf{\Lambda}_i$ as well. Since convexity implies continuity, $W_i(\mathbf{h}_i, \mathbf{\Lambda}_i)$ is a continuous function of $\mathbf{\Lambda}_i$ for any given $\mathbf{h}_i$. This implies

$$\lim_{n\to\infty} W_i(\mathbf{h}_i, \mathbf{\Lambda}_i(n)) = W_i(\mathbf{h}_i, \mathbf{\Lambda}_i). \tag{3.88}$$

Recall that $q_i(\mathbf{h}_i, \mathbf{\Lambda}_i)$ equals to 1 if $W_i(\mathbf{h}_i, \mathbf{\Lambda}_i) > 0$ and 0 otherwise. Therefore, $q_i(\mathbf{h}_i, \mathbf{\Lambda}_i(n))$ converges pointwise to $q_i(\mathbf{h}_i, \mathbf{\Lambda}_i)$ almost everywhere. Furthermore, note that $q_i(\mathbf{h}_i, \mathbf{\Lambda}_i(n))$ is upper bounded by 1. Using dominated convergence theorem [44, Chapter 5.9], (3.85) follows. The argument for the continuity of the expectation $\mathbb{E}_{\mathbf{h}_i}[q_{ij} C_{ij}(h_{ij} p_{ij})|\mathbf{\Lambda}_i]$ is analogous.

Chapter 4

Optimal wireless communications with imperfect CSI

In Chapter 2 and Chapter 3, we developed distributed algorithms for optimal random access channels and networks, respectively. In both cases, terminals are assumed to have access to perfect local CSI. In practice, however, perfect CSI is rarely available due to estimation errors and, perhaps more fundamentally, to feedback delay. Algorithms to handle imperfect CSI in the transmission over wireless channels are the subject matter of this chapter. We focus on three types of channels: single user point-to-point block fading channels [15], multiuser downlink orthogonal frequency division multiplexing (OFDM) [38], and multiuser uplink random access (RA) [29]. In all three cases we develop algorithms adapting to imperfect CSI that maximize ergodic throughputs subject to average power constraints.

As in the case of perfect CSI, transmitters adapt their power and coding mode to channel observations in order to exploit favorable channel conditions. However, due to the inaccuracy of imperfect CSI, channel outages occur when the rate selected turns out too aggressive for the actual channel realization. From a practical perspective it is recognized that to mitigate the negative effect of outages caused by imperfect CSI a channel backoff function is needed to enforce the selection of more conservative coding modes; see e.g. [52]. Instead of selecting a code adapted to the channel estimate, we select a code adapted to a smaller channel realization. This reduces the transmission rate but also reduces the likelihood of a channel outage resulting on overall larger throughput. Ideally, power allocation and rate backoff should be jointly optimized but this results in a nonconvex optimization problem. Since we need to determine power allocation and backoff for each fading state and fading takes on a continuum of values it further follows that the problem is infinite dimensional. Infinite dimensionality compounded with lack of convexity results in computational intractability.

Computational intractability notwithstanding, the problem can be simplified through the imposition of additional restrictions to yield more tractable formulations that lead to the successful development of transmission strategies for various types of wireless channels. Most relevant to the work presented here are works on point-to-point channels, e.g., [24, 48, 49], broadcast channels [2, 5, 40–42, 45] and random access channels [12, 43, 52]. E.g., when power is fixed and only rate adaptation is considered the problem is reduced to the determination of the optimal backoff function; e.g. [41]. A second possibility is to fix a target outage probability and separate the optimization problem into the determination of a backoff function for target outage, followed by optimal power allocation over estimated channels [42]. A third possible restriction is to assume that the backoff function takes a certain parametric form and proceed to optimize the corresponding parameters, e.g. [52]. These different reformulations yield tractable problems but the resulting throughputs are not optimal for the original problem.

Rather than reformulating the original problem into a suboptimal tractable alternative, the contribution of this chapter is to develop algorithms that jointly find optimal power allocations and channel backoff functions. Key in achieving this goal is the recognition that the structure of the resulting optimization problem makes it part of a class of problems that despite their lack of convexity have null Lagrangian duality gap [34]. The Lagrangian dual problem of the joint power and backoff function optimization is convex, because dual problems are convex regardless of the convexity of the primal problem, and their dimensionality is given by the number of power constraints which is typically equal to the number of terminals. The combination of convexity and small finite dimensionality results in computational tractability that has to be contrasted with the computational intractability that follows from the infinite dimensionality and lack of convexity of the primal problem. Let us emphasize that lack of duality gap makes primal and dual problems equivalent.

We begin by studying optimal transmission over a single user point-to-point channel with imperfect CSI to illustrate the methodology we will later generalize to multiuser OFDM and RA channels (Section 4.1). In the case of point-to-point channels there is only one constraint and consequently the dual problem is one-dimensional. Lack of convexity is leveraged to show that the optimal power allocation and channel backoff functions are uniquely determined by the optimal dual variable. With the optimal multiplier available, determination of optimal power allocation and channel backoff decomposes into two-dimensional per-fading state optimization subproblems (Section 4.1.2). We further develop a stochastic subgradient descent algorithm in the dual domain that converges to the optimal Lagrange multiplier and yields the optimal power allocation and channel backoff function as a byproduct (Section 4.1.3). This algorithm operates based on instantaneous channel estimates and does not require access to the channel's probability distribution function (pdf).

We then consider optimal transmission over a downlink multiuser OFDM channel with imperfect CSI (Section 4.2). The objective is to maximize a convex utility of the ergodic rates of all users subject to an average sum power budget. In addition to power allocations and channel backoffs, the algorithm for OFDM needs to determine subcarrier assignments for each channel realization. Similar to the case of single-user channels, jointly optimal backoff and frequency and power allocations are uniquely determined by a finite number of Lagrange multipliers equal to the number of users served plus one. With the optimal multiplier available, the problem of determining optimal operating points decomposes into two-dimensional per-frequency, per-terminal, and per-fading state subproblems (Section 4.2.1). Stochastic subgradient descent algorithms to find optimal operating points are developed as well (Section 4.2.2).

We finally investigate uplink multiuser RA channels whereby users contend for communication with a common receiver (Section 4.3). In this case, terminals do not coordinate their transmission attempts and make transmission decisions based on estimates of their own channels only. If they decide to transmit, they choose a power and a rate for their communication attempt. The objective is to maximize proportional fair utility of ergodic rates subject to individual power constraints at each terminal. Decompositions and stochastic subgradient descent algorithms analogous to those derived for single user and OFDM channels are derived (sections 4.3.1 and 4.3.2).

Numerical results are presented in Section 4.4 and concluding remarks in Section 4.5.

4.1 Point-to-point channels

Consider a wireless channel with time slots indexed by t. The channel at time t is denoted as $h(t)$. The channel is assumed to be block fading. The pdf $m_h(h)$ of the channel h is unknown. We assume channels have continuous pdf. In each time slot the transmitter computes an estimate $\hat{h}(t)$ of the current gain $h(t)$ to adapt transmitted power and code selection to the channel state. The accuracy of estimates $\hat{h}(t)$ is characterized through the conditional probability distribution $m_{h|\hat{h}}(h|\hat{h})$ that determines the probability of the actual channel being h when the estimate is $\hat{h}$. The probability distribution $m_{h|\hat{h}}(h|\hat{h})$ depends on the channel estimation method and is assumed known, although we make no assumptions on its specific form – see Remark 7 below.

Based on the value of the channel estimate $\hat{h}$, the transmitter decides on a power allocation $P = P(\hat{h}) : \mathbf{R}_+ \to [0, p^{\text{inst}}]$, where $p^{\text{inst}} > 0$ is the maximum instantaneous power the transmitter can use. The communication rate through the channel is a function of the transmitted power $P(\hat{h})$ and the *actual* channel gain h that we generically denote as $C(P(\hat{h}), h)$. The function $C(P(\hat{h}), h)$ depends on how signals are coded and modulated at the physical layer. To achieve rate $C(P(\hat{h}), h)$ the transmitter has to select an appropriate code adapted to the received SNR $P(\hat{h})h/N_0$, e.g., the appropriate modulation and coding mode if AMC is used. This is not possible however, because the code selection depends on the unknown channel gain h. A feasible alternative is to adapt the code to the estimated received SNR $P(\hat{h})\hat{h}/N_0$ and attempt transmission at a rate $C(P(\hat{h}), \hat{h})$. Observe that the transmitted rate does not coincide with the channel throughput due to the possibility of lost packets. Indeed, a channel outage is assumed to occur when the transmitted rate $C(P(\hat{h}), \hat{h})$ exceeds the maximum rate $C(P(\hat{h}), h)$ the channel can afford, i.e. when $C(P(\hat{h}), \hat{h}) > C(P(\hat{h}), h)$ or simply when $\hat{h} > h$. The instantaneous rate achieved in the channel is therefore given by

$$R(h, \hat{h}) = C(P(\hat{h}), \hat{h})\mathbb{I}[\hat{h} \leq h], \tag{4.1}$$

which corrects for lost packets through the indicator $\mathbb{I}[\hat{h} \leq h]$.

Selecting a code to attempt transmission at rate $C(P(\hat{h}), \hat{h})$ would likely result in a substantial number of dropped packets. For the sake of argument suppose that the conditional distribution $m_{h|\hat{h}}(h|\hat{h})$ is symmetric around $\hat{h}$. In such case about half of the packets are lost as the outage probability would be $\mathrm{P}\left\{\hat{h} > h\right\} = 0.5$. To alleviate the negative effect of outages, a channel backoff function $B = B(\hat{h}) : \mathbf{R}_+ \to \mathbf{R}_+$ is used to determine a backed-off channel gain $B(\hat{h})$. The code is then adapted to the received SNR $P(\hat{h})B(\hat{h})/N_0$ – as opposed to $P(\hat{h})\hat{h}/N_0$ – and communication proceeds at a rate $C(P(\hat{h}), B(\hat{h}))$. With codes adapted to $P(\hat{h})B(\hat{h})/N_0$, an outage occurs if $B(\hat{h}) > h$. Thus, the instantaneous transmission rate can be written as

$$R(h, \hat{h}) = C(P(\hat{h}), B(\hat{h}))\mathbb{I}\left\{B(\hat{h}) \leq h\right\}. \tag{4.2}$$

The idea is that making $B(\hat{h}) < \hat{h}$ reduces the chance of an outage thereby increasing the effective rate $R(h, \hat{h})$ even if the attempted transmission is more conservative [cf. (4.1) and (4.2)]. However, as we shall show later, making $B(\hat{h}) > \hat{h}$ is optimal in some cases.

4.1.1 Ergodic rate optimization

Our goal is to find the optimal power allocation function P and channel backoff function B such that the expected transmission rate is maximized subject to an average power constraint P_0,

$$\begin{aligned} \mathsf{P_s} = &\max \mathbb{E}_{h,\hat{h}}\left[C(P(\hat{h}), B(\hat{h}))\mathbb{I}\left\{B(\hat{h}) \le h\right\}\right] \\ &\text{s.t. } \mathbb{E}_{\hat{h}}\left[P(\hat{h})\right] \le P_0. \end{aligned} \tag{4.3}$$

Solving (4.3) is challenging because: (I1) The objective includes an expectation over the random channel gain h, whose realizations are not available at the transmitter and whose pdf is unknown. (I2) Variables in this optimization problem are functions P and B defined on $\mathbf{R}_{+}$, implying the dimensionality of the problem is infinite. (I3) The objective and the constraint involve expectations over channel estimates $\hat{h}$, whose realizations are observed in each time slot but whose pdf is unknown. (I4) The channel capacity function $C(P(\hat{h}), B(\hat{h}))$ may be nonconvex or even discontinuous as in the case of AMC.

To overcome issue (I1), we rewrite the expectation in the objective of (4.3) as a conditional expectation over h with $\hat{h}$ given, followed by an expectation over $\hat{h}$, i.e.

$$\mathbb{E}_{h,\hat{h}}\left[C(P(\hat{h}), B(\hat{h}))\mathbb{I}\left\{B(\hat{h}) \le h\right\}\right] = \mathbb{E}_{\hat{h}}\left[C(P(\hat{h}), B(\hat{h}))\mathbb{E}_{h|\hat{h}}\left[\mathbb{I}\left\{B(\hat{h}) \le h\right\}\right]\right]. \tag{4.4}$$

Note that the inner expectation in (4.4) is just the probability $\Pr(B(\hat{h}) \le h|\hat{h})$ of the backed off channel being smaller than the actual channel h for a given estimate $\hat{h}$. This probability can be written in terms of the complementary cumulative distribution function (ccdf) $M_{h|\hat{h}}(\cdot)$ of h given $\hat{h}$ as

$$M_{h|\hat{h}}(B(\hat{h})) := \Pr(B(\hat{h}) \le h|\hat{h}) = \mathbb{E}_{h|\hat{h}}\left[\mathbb{I}\left\{B(\hat{h}) \le h\right\}\right]. \tag{4.5}$$

Since $m_{h|\hat{h}}(\cdot)$ is known – see Remark 7 – the ccdf $M_{h|\hat{h}}(B(\hat{h}))$ is available. This allows us to simplify (4.4) to

$$\mathbb{E}_{h,\hat{h}}\left[C(P(\hat{h}), B(\hat{h}))\mathbb{I}\left\{B(\hat{h}) \le h\right\}\right] = \mathbb{E}_{\hat{h}}\left[C(P(\hat{h}), B(\hat{h}))M_{h|\hat{h}}(B(\hat{h}))\right]. \tag{4.6}$$

Using (4.6) the objective in (4.3) can be written as a single expectation over $\hat{h}$ yielding the equivalent formulation

$$\begin{aligned} \mathsf{P_s} = &\max \mathbb{E}_{\hat{h}} \left[C(P(\hat{h}), B(\hat{h})) M_{h|\hat{h}}(B(\hat{h})) \right] \\ &\text{s.t. } \mathbb{E}_{\hat{h}} \left[P(\hat{h}) \right] \leq P_0. \end{aligned} \tag{4.7}$$

Problems (4.3) and (4.7) are equivalent. Our goal is to find the optimal power allocation P^* with values $P^*(\hat{h})$ and backoff function B^* with values $B^*(\hat{h})$ that jointly solve problem (4.7). Since actual channel gains h are not present in (4.7), issue (I1) has been resolved. Issues (I2)-(I4), however, still hold for problem (4.7). Sections 4.1.2 and 4.1.3 discuss a method to solve (4.7) that overcomes these issues. We pursue this after the following remark.

Remark 7. The probability distribution $m_{h|\hat{h}}(h|\hat{h})$ depends on the channel estimation method. A typical way of estimating the channel is to send a training signal that is known to both the transmitter and the receiver and get feedback from the receiver on the measured channel gain. Due to estimation error and/or feedback delays, estimated channels $\hat{h}$ are different from actual channels h and are modeled as

$$\hat{h} = h + e, \tag{4.8}$$

where e is a complex Gaussian random noise $\mathcal{CN}(0, \sigma_e^2)$. For the model in (4.8) it holds that the pdf of h given $\hat{h}$ is a noncentral chi-square given by [28]

$$m_{h|\hat{h}}(h|\hat{h}) = \frac{1}{\sigma_e^2} \exp\left(-\frac{h + \hat{h}}{\sigma_e^2} \right) I_0 \left(\frac{2\sqrt{h\hat{h}}}{\sigma_e^2} \right), \tag{4.9}$$

where $I_0(x) = \sum_{i=0}^{\infty} (x^2/4)^i/(i!)^2$ is the zeroth order modified Bessel function of the first kind. This particular form for the conditional pdf $m_{h|\hat{h}}(h|\hat{h})$ is used to provide numerical results in Section 4.4. The rest of the development in the chapter holds independently of the particular form of this pdf. Note that we assume the conditional pdf $m_{h|\hat{h}}(h|\hat{h})$ does not change over time.

4.1.2 Optimal power allocation and channel backoff functions

The optimization problem in (4.7) has only one constraint, implying that while the primal problem is infinite dimensional, the dual problem is one-dimensional. More importantly, it has been shown that problems like (4.7), where the non-convex functions appear inside expectations, have null duality gap as long as the pdf of the random variable with respect to which we take the expected value has no points of strictly positive probability (see Appendix 4.6.1). As a result, working in the dual domain is equivalent. To introduce the dual function associate Lagrange multiplier λ with the power constraint and define the Lagrangian as

$$\begin{aligned}\mathcal{L}(P,B,\lambda) &= \mathbb{E}_{\hat{h}}\left[C(P(\hat{h}),B(\hat{h}))M_{h|\hat{h}}(B(\hat{h}))\right] + \lambda\left[P_0 - \mathbb{E}_{\hat{h}}\left[P(\hat{h})\right]\right] \\ &= \mathbb{E}_{\hat{h}}\left[C(P(\hat{h}),B(\hat{h}))M_{h|\hat{h}}(B(\hat{h})) - \lambda P(\hat{h})\right] + \lambda P_0, \end{aligned} \tag{4.10}$$

where we rearranged terms to write the second equality. The dual function is then defined as the maximum of the Lagrangian over the sets of feasible functions P and B, i.e.,

$$g(\lambda) = \max_{P,B} \mathcal{L}(P,B,\lambda). \tag{4.11}$$

We now can write the dual problem as the minimum of $g(\lambda)$ over nonnegative λ, i.e.,

$$\mathsf{D_s} = \min_{\lambda \geq 0} g(\lambda) \tag{4.12}$$

Since the problem (4.7) and its dual (4.12) have been shown to have null gap we have that $\mathsf{P_s} = \mathsf{D_s}$. This property can be exploited to characterize the optimal power allocation and channel backoff functions as is done in the following theorem.

Theorem 4. *The optimal power allocation function P^* with values $P^*(\hat{h})$ and optimal backoff function B^* with values $B^*(\hat{h})$ that solve problem* (4.7) *are determined by the optimal dual variable λ^* of the dual problem* (4.12)*. In particular,*

$$\left\{P^*(\hat{h}), B^*(\hat{h})\right\} \in \underset{b\in[0,\infty),p\in[0,p^{\text{inst}}]}{\operatorname{argmax}} \left\{C(p,b)M_{h|\hat{h}}(b) - \lambda^* p\right\}. \tag{4.13}$$

Proof. According to the definition of the dual function [cf. (4.11)], $g(\lambda^*)$ is the maximum of the Lagrangian $\mathcal{L}(P, B, \lambda^*)$ across all functions P and B. Since optimal functions P^* and B^* are possible arguments of the Lagrangian in this maximization it follows that $\mathcal{L}(P^*, B^*, \lambda^*)$ must be bounded above by $g(\lambda^*)$, i.e.,

$$\mathsf{D_s} = g(\lambda^*) = \max_{P,B} \mathcal{L}(P, B, \lambda^*) \geq \mathcal{L}(P^*, B^*, \lambda^*). \tag{4.14}$$

As per its definition in (4.10) the Lagrangian $\mathcal{L}(P^*, B^*, \lambda^*)$ can be written as

$$\mathcal{L}(P^*, B^*, \lambda^*) = \mathbb{E}_{\hat{h}}\left[C(P^*(\hat{h}), B^*(\hat{h}))M_{h|\hat{h}}(B^*(\hat{h}))\right] + \lambda^*\left[P_0 - \mathbb{E}_{\hat{h}}\left[P^*(\hat{h})\right]\right].$$

Since B^* and P^* are feasible for the primal problem, the average power constraint must be satisfied, i.e., $P_0 - \mathbb{E}_{\hat{h}}\left[P^*(\hat{h})\right] \geq 0$. Since we also know that $\lambda^* \geq 0$ we conclude that $\lambda^*\left[P_0 - \mathbb{E}_{\hat{h}}\left[P^*(\hat{h})\right]\right] \geq 0$ and as a result

$$\mathcal{L}(P^*, B^*, \lambda^*) \geq \mathbb{E}_{\hat{h}}\left[C(P^*(\hat{h}), B^*(\hat{h}))M_{h|\hat{h}}(B^*(\hat{h}))\right] = \mathsf{P_s}. \tag{4.15}$$

Substituting (4.15) into (4.14) gives us

$$\mathsf{D_s} = g(\lambda^*) \geq \mathcal{L}(P^*, B^*, \lambda^*) \geq \mathsf{P_s}. \tag{4.16}$$

Since the duality gap is null, i.e. $\mathsf{D_s} = \mathsf{P_s}$, the inequalities in (4.16) must be satisfied with equalities, i.e.

$$\mathsf{D_s} = g(\lambda^*) = \mathcal{L}(P^*, B^*, \lambda^*) = \mathsf{P_s}. \tag{4.17}$$

The equality $g(\lambda^*) = \mathcal{L}(P^*, B^*, \lambda^*)$ in (4.17) implies that P^* and B^* are maximizers of the Lagrangian $\mathcal{L}(P, B, \lambda^*)$,

$$\{P^*, B^*\} \in \operatorname*{argmax}_{P,B} \mathcal{L}(P, B, \lambda^*). \tag{4.18}$$

Note that in (4.18) we used set inclusion instead of equality because the maximizer may not be unique. Using the definition of the Lagrangian [cf. (4.10)], we can rewrite (4.18) as

$$\{P^*, B^*\} \in \operatorname*{argmax}_{P,B} \mathbb{E}_{\hat{h}}\left[C(P(\hat{h}), B(\hat{h}))M_{h|\hat{h}}(B(\hat{h})) - \lambda^* P(\hat{h})\right], \tag{4.19}$$

where we ignored the term $\lambda^* P_0$ since it does not depend on $P(\hat{h})$ or $B(\hat{h})$. Due to the linearity of the expectation operator, the maximization in (4.19) can be carried out inside the expectation. This yields separate maximizations for each channel state estimate $\hat{h}$ as indicated in (4.13). $\square$

Provided that λ^* is available, Theorem 4 states that $P^*(\hat{h})$ and $B^*(\hat{h})$ can be obtained by solving the maximization in (4.13). Although the problem in (4.13) might be nonconvex, solving it is by no means a difficult task as it only involves two variables. This provides a great advantage because the problem dimensionality is reduced from infinity to 1. Also, we remark that Theorem 4 is true no matter what the capacity function is and how the underlying channel is distributed. Next, we shall develop online algorithms that find the optimal solutions for problem (4.7) using only instantaneous imperfect CSI.

4.1.3 Online learning algorithms

Unlike the nonconvex primal problem, the dual problem in (4.12) is always convex. This suggests that gradient descent algorithms in the dual domain are guaranteed to converge to the optimal multiplier λ^*. In particular, we use stochastic subgradient descent algorithms that iteratively compute primal and dual variables. Given dual variable $\lambda(t)$, the algorithm proceeds to a primal iteration in which it computes power allocation $p(t)$ and backoff function $b(t)$ as

$$\{p(t), b(t)\} \in \underset{p \in [0, p^{\text{inst}}], b \geq 0}{\operatorname{argmax}} \left\{ C(p, b) M_{h|\hat{h}(t)}(b) - \lambda(t) p \right\}. \tag{4.20}$$

Multipliers $\lambda(t+1)$ are then updated based on $\lambda(t)$ and $p(t)$ as

$$\lambda(t+1) = \left[\lambda(t) - \epsilon(t) \left[P_0 - p(t)\right]\right]^+, \tag{4.21}$$

where $[x]^+ = \max\{0, x\}$ denotes projection on the nonnegative reals and $\epsilon(t) > 0$ is a possibly time dependent step size. The difference $P_0 - p(t)$ in (4.21) is a stochastic subgradient of the dual function as it can be shown that the expected value of $P_0 - p(t)$ is a (deterministic) subgradient of the dual function [16,32]. This property implies that $P_0 - p(t)$ points to λ^* on an average sense

Algorithm 2: Optimal power control and channel backoff for point-to-point channels

Initialize Lagrangian multiplier $\lambda(0)$;

for $t = 0, 1, 2, \cdots$ **do**

  Compute primal variables as per (4.20):

  $$\{p(t), b(t)\} \in \underset{p \in [0, p^{\text{inst}}], b \geq 0}{\text{argmax}} \Big\{ C(p, b) M_{h|\hat{h}(t)}(b) - \lambda(t) p \Big\};$$

  Transmit with power $p(t)$ and rate $C(p(t), b(t))$;

  Update dual variables as per (4.21):

  $$\lambda(t+1) = [\lambda(t) - \epsilon(t) [P_0 - p(t)]]^+;$$

end

and can be exploited to prove convergence in the dual domain. The computations in (4.20) and (4.21) are summarized in Algorithm 2.

Particular convergence properties depend on whether constant or time varying step sizes are used. We first consider diminishing step sizes. If $\epsilon(t)$ is nonsummable but square summable, i.e., $\sum_{t=0}^{\infty} \epsilon(t) = \infty$ and $\sum_{t=0}^{\infty} \epsilon^2(t) < \infty$, then using standard stochastic approximation techniques it can be shown that $\lambda(t)$ converges to λ^* almost surely [20]. As a consequence of Theorem 4, this indicates that the primal variables almost surely converge to the optimal values as time grows, i.e., $p(t) = P^*(\hat{h}(t))$ and $b(t) = B^*(\hat{h}(t))$ almost surely as t goes to infinity.

In addition to diminishing step size, constant step size can be used for the algorithm. However, with a constant step size the dual iterates $\lambda(t)$ no longer converge to the optimal value almost surely. Instead, they stay within a small distance of λ^* with probability close to 1 as t goes to infinity and convergence can be established in a time average sense only [32]. Specifying Theorem 1 of [32] to the stochastic subgradient descent algorithm in (4.20)-(4.21) yields the following property.

Property 1. *If constant step sizes $\epsilon(t) = \epsilon > 0$ for all t are used in Algorithm 2, the average power* *constraint is almost surely satisfied*

$$\lim_{t\to\infty} \frac{1}{t}\sum_{u=1}^{t} p(u) \le P_0 \quad \textit{a.s.}, \tag{4.22}$$

and the ergodic limit of $C(p(t), b(t))M_{h(t)|\hat{h}(t)}(b(t))$ almost surely converges to a value within $\kappa\epsilon/2$ of optimal,

$$\mathsf{P_s} - \lim_{t\to\infty} \frac{1}{t}\sum_{u=1}^{t} C(p(u), b(u))M_{h(u)|\hat{h}(u)}(b(u)) \le \kappa\epsilon/2 \quad \textit{a.s.}, \tag{4.23}$$

where $\kappa \ge \mathbb{E}\left[\left(P_0 - p(t)\right)^2 \,\middle|\, \lambda(t)\right]$ is a constant bounding the second moment $\mathbb{E}\left[\left(P_0 - p(t)\right)^2 \,\middle|\, \lambda(t)\right]$ of the stochastic subgradient $P_0 - p(t)$.

Since $p(t)$ can only take values in $[0, p^{\text{inst}}]$, the constant κ in (4.23) is upper bounded by $\max\{P_0^2, (P_{\mathsf{max}} - P_0)^2\}$. It follows that the time average of $C(p(t), b(t))M_{h(t)|\hat{h}(t)}(b(t))$ can be made arbitrarily close to optimal by reducing the step size ϵ. Notice however that the rate $C(p(t), b(t))M_{h(t)|\hat{h}(t)}(b(t))$ is an average across possible channel realizations $h(t)$ for given estimate $\hat{h}(t)$, which is in general different from the instantaneous transmission rate $C(p(t), b(t))$ $\mathbb{I}\{b(t) \le h(t)\}$ achieved by the algorithm. Despite this disparity in instantaneous values, their ergodic limits are almost surely equal. To see this just note that according to its definition in (4.5) it holds $M_{h(t)|\hat{h}(t)}(b(t)) = \mathbb{E}_{h(t)|\hat{h}(t)}\Big[\mathbb{I}\{b(t) \le h(t)\}\Big]$. With estimates $\hat{h}(\mathbb{N})$ given for all times t, the stochastic process $h(\mathbb{N})$ is ergodic. It then follows that we must have

$$\lim_{t\to\infty} \frac{1}{t}\sum_{u=1}^{t} C(p(u), b(u))\mathbb{I}\{b(u) \le h(u)\} = \lim_{t\to\infty} \frac{1}{t}\sum_{u=1}^{t} C(p(u), b(u))\mathbb{E}_{h(u)|\hat{h}(u)}\Big[\mathbb{I}\{b(u) \le h(u)\}\Big] \quad \text{a.s.,} \tag{4.24}$$

because with $\hat{h}(\mathbb{N})$ given the term $C(p(u), b(u))$ is just a constant. Substituting the equality $M_{h(t)|\hat{h}(t)}(b(t)) = \mathbb{E}_{h(t)|\hat{h}(t)}\Big[\mathbb{I}\{b(t) \le h(t)\}\Big]$ into (4.24) and the resulting expression into (4.23) gives

$$\mathsf{P_s} - \lim_{t\to\infty} \frac{1}{t}\sum_{u=1}^{t} C(p(u), b(u))\mathbb{I}\{b(u) \le h(u)\} \le \kappa\epsilon/2 \quad \text{a.s..} \tag{4.25}$$

Eq. (4.25) shows that although the algorithm with constant step sizes does not find $P^*(\hat{h}(t))$ and $B^*(\hat{h}(t))$ it generates sequences $p(t)$ and $b(t)$ whose time averages are almost surely near optimal. The near optimality gap can be made arbitrarily small by reducing the step size ϵ as we have already noted. The advantage of using a constant step size is that if the channel distributions change slowly the algorithm can adapt to that change.

4.2 Orthogonal frequency division multiplexing

Consider now an OFDM channel where a common access point (AP) spends an average power budget P_0 to communicate with N terminals $\{T_n\}_{n=1}^N$ using a set of orthogonal frequencies $\mathcal{F}$. As in the point-to-point channel case of Section 4.1, time is slotted and indexed by t. The time-varying channel gain between the AP and terminal T_n for all frequencies $f \in \mathcal{F}$ is modeled as block fading and denoted by $h_n^f(t)$. In each time slot the AP observes channel gain estimates for all terminals and frequencies which we denote as a vector $\hat{\mathbf{h}}(t) := \{\hat{h}_n^f(t) : n \in \mathcal{N}, f \in \mathcal{F}\}$. Based on $\hat{\mathbf{h}}(t)$, the AP decides on frequency allocation $q_n^f(t) := Q_n^f(\hat{\mathbf{h}}(t)) \in \{0,1\}$ and power allocation $p_n^f(t) := P_n^f(\hat{\mathbf{h}}(t)) \in [0, p^{\text{inst}}]$. If $q_n^f(t) = 1$, it transmits to T_n using frequency f. Since a given frequency cannot be used by more than one terminal in the same time slot, we require $\sum_{n=1}^N Q_n^f(\hat{\mathbf{h}}) \leq 1$ for all $f \in \mathcal{F}$. Define the vector $\mathbf{Q}^f(\hat{\mathbf{h}}) := [Q_1^f(\hat{\mathbf{h}}), \cdots, Q_n^f(\hat{\mathbf{h}})]^T$ grouping the schedules of all terminals for given frequency and channel realization. We can then express the frequency exclusion constraint as

$$\mathbf{Q}^f(\hat{\mathbf{h}}) \in \mathcal{Q} := \left\{ \mathbf{q} = [q_1, \cdots, q_N]^T : q_n \in \{0,1\}, \mathbf{q}^T \mathbf{1} \leq 1 \right\}, \tag{4.26}$$

which simply states that at most one component of $\mathbf{Q}^f(\hat{\mathbf{h}})$ can be 1.

If frequency f is scheduled for communication to T_n the AP determines power allocations $P_n^f(\hat{\mathbf{h}}(t))$ for the communication to terminal T_n in frequency f for joint channel estimates $\hat{\mathbf{h}}(t)$ as well as a channel backoff value $B_n^f(\hat{\mathbf{h}}(t))$ that we also let be a function of all channel estimates $\hat{\mathbf{h}}(t)$. The intent of the backoff function $B_n^f(\hat{\mathbf{h}}(t))$ is to reduce the likelihood of channel outages as in the point-to-point channel case discussed in Section 4.1. Therefore, channel coding is selected according to the value of the product $P_n^f(\hat{\mathbf{h}}(t))B_n^f(\hat{\mathbf{h}}(t))$ and communication is attempted at a rate $C\left(P_n^f(\hat{\mathbf{h}}(t)), B_n^f(\hat{\mathbf{h}}(t))\right)$. The instantaneous throughput for the link to terminal T_n has to discount for channel outages and to account for all frequencies $f \in \mathcal{F}$ scheduled for this transmission yielding the instantaneous rate

$$R_n(\mathbf{h}, \hat{\mathbf{h}}) = \sum_{f\in\mathcal{F}} Q_n^f(\hat{\mathbf{h}}(t))C\left(P_n^f(\hat{\mathbf{h}}(t)), B_n^f(\hat{\mathbf{h}}(t))\right)\mathbb{I}\left\{B_n^f\left(\hat{\mathbf{h}}(t)\right) \le h_n^f(t)\right\}. \qquad (4.27)$$

The term $Q_n^f(\hat{\mathbf{h}}(t))$ in (4.27) is a binary indicator of wether T_n is scheduled in frequency f for channel realization $\hat{\mathbf{h}}(t)$, the factor $C\bigg(P_n^f(\hat{\mathbf{h}}(t)), B_n^f(\hat{\mathbf{h}}(t))\bigg)$ is the attempted transmission rate in such case, and the indicator $\mathbb{I}\bigg\{B_n^f(\hat{\mathbf{h}}(t)) \le \hat{h}_n^f(t)\bigg\}$ accounts for dropped packets.

Since we are interested in ergodic rates, we define the average rate $r_n := \mathbb{E}_{\mathbf{h},\hat{\mathbf{h}}}\left[R_n(\mathbf{h}, \hat{\mathbf{h}})\right]$. Upon defining $M_{h_n^f|\hat{h}_n^f}(\cdot)$ as the ccdf of h_n^f given $\hat{h}_n^f$, we can express the ergodic rate from the AP to T_n as [cf. (4.6)]

$$\begin{aligned} r_n =\ & \mathbb{E}_{\hat{\mathbf{h}}}\left[\sum_{f\in\mathcal{F}} Q_n^f(\hat{\mathbf{h}}(t))C\left(P_n^f(\hat{\mathbf{h}}(t)), B_n^f(\hat{\mathbf{h}}(t))\right)M_{h_n^f|\hat{h}_n^f}\left(B_n^f(\hat{\mathbf{h}}(t))\right)\right] \\ :=\ & \mathbb{E}_{\hat{\mathbf{h}}}\left[\sum_{f\in\mathcal{F}} Q_n^f(\hat{\mathbf{h}}(t))R_n^f\left(P_n^f(\hat{\mathbf{h}}(t)), B_n^f(\hat{\mathbf{h}}(t)); \hat{h}_n^f(t)\right)\right], \end{aligned} \qquad (4.28)$$

where in the second equality we defined $R_n^f\Big(P_n^f(\hat{\mathbf{h}}),B_n^f(\hat{\mathbf{h}});\hat{h}_n^f\Big):=C\Big(P_n^f(\hat{\mathbf{h}}),B_n^f(\hat{\mathbf{h}})\Big)M_{h_n^f|\hat{h}_n^f}\Big(B_n^f(\hat{\mathbf{h}})\Big)$ as the expected throughput of terminal n on frequency f. The expected throughput is the rate at which the AP expects to convey information to terminal T_n on frequency f when the channel estimate is $\hat{\mathbf{h}}$. By expected throughput here we refer to the conditional expectation with respect to $\mathbf{h}$ given $\hat{\mathbf{h}}$.

To evaluate the performance of the system, introduce utility functions $U_n(r_n)$ to measure the value of ergodic rate r_n for terminal n. The AP's goal is to find optimal subcarrier assignment, power allocation and channel backoff functions such that the sum utility $\sum_{n=1}^N U_n(r_n)$ is maximized. Recalling the expression for r_n in (4.28) and introducing an average sum power constraint, the optimal operating point is obtained as the solution of the optimization problem

$$\begin{aligned}
\mathsf{P_b} = \max \sum_{n=1}^{N} U_n(r_n) \\
\text{s.t. } r_n \le \mathbb{E}_{\hat{\mathbf{h}}}\left[\sum_{f\in\mathcal{F}} Q_n^f(\hat{\mathbf{h}}) R_n^f\left(P_n^f(\hat{\mathbf{h}}), B_n^f(\hat{\mathbf{h}}); \hat{h}_n^f\right)\right], \qquad P_0 \ge \mathbb{E}_{\hat{\mathbf{h}}}\left[\sum_{n=1}^{N}\sum_{f\in\mathcal{F}} Q_n^f(\hat{\mathbf{h}}) P_n^f(\hat{\mathbf{h}})\right], \\
\mathbf{Q}^f(\hat{\mathbf{h}}) \in \mathcal{Q}, \quad P_n^f(\hat{\mathbf{h}}) \in [0, p^{\text{inst}}], \quad B_n^f(\hat{\mathbf{h}}) \ge 0, \quad r_n \in [0, r_{\mathsf{max}}],
\end{aligned} \tag{4.29}$$

where r_{max} is a given upper bound on the rates r_n of each user. The relaxation of the rate equality constraint in (4.28) to the corresponding inequality constraint in (4.29) is without loss of optimality. The average sum power constraint is enforced by the inequality $P_0 \ge \mathbb{E}_{\hat{\mathbf{h}}}\left[\sum_{n=1}^{N}\sum_{f\in\mathcal{F}} Q_n^f(\hat{\mathbf{h}}) P_n^f(\hat{\mathbf{h}})\right]$ in (4.29). The factor $Q_n^f(\hat{\mathbf{h}})P_n^f(\hat{\mathbf{h}})$ is the power used for communication to T_n on frequency f for channel estimate $\hat{\mathbf{h}}$. This term is not null only if $Q_n^f(\hat{\mathbf{h}}) = 1$ which means that terminal T_n is scheduled on frequency f. Individual power consumptions $Q_n^f(\hat{\mathbf{h}})P_n^f(\hat{\mathbf{h}})$ are summed for all terminals $n = 1, \ldots, n$ and all frequencies $f \in \mathcal{F}$ to determine the total power consumption for gain estimate $\hat{\mathbf{h}}$. These sums of instantaneous power consumptions are averaged over the distribution of $\hat{\mathbf{h}}$ to determine the average power expenditure that cannot exceed the budget P_0.

Solving problem (4.29) bears the same challenges as solving (4.7). The problem is infinite dimensional due to the optimization variables being functions of the channel estimates $\hat{\mathbf{h}}$ and the expectations are with respect to the random variable $\hat{\mathbf{h}}$ whose pdf is unknown. The problem is also not convex because the function $R_n^f\left(P_n^f(\hat{\mathbf{h}}), B_n^f(\hat{\mathbf{h}}); \hat{h}_n^f\right)$ is not concave on the variables $P_n^f(\hat{\mathbf{h}})$ and $B_n^f(\hat{\mathbf{h}})$. In this case we also have the requirement of variables $Q_n^f(\hat{\mathbf{h}})$ being binary as represented by the nonconvex set constraint $\mathbf{Q}^f(\hat{\mathbf{h}}) \in \mathcal{Q}$. As we show in the next section, and as in the case of point-to-point channels, these issues are resolved by working on the dual domain.

4.2.1 Optimal solution

The optimization problem in (4.29) also has the structure of the problems shown to have null duality gap in [34, 36]. Therefore, we can work on its dual problem which is finite dimensional and convex without loss of optimality. To do so, introduce multipliers λ_n associated with the ergodic rate constraint of user T_n and μ associated with the average power constraint [cf. (4.29)]. Further define the vector $\mathbf{\Lambda} := \{\lambda_1, \cdots, \lambda_N, \mu\}$ grouping all dual variables and vectors $\mathbf{P}(\hat{\mathbf{h}}) := \left\{Q_n^f(\hat{\mathbf{h}}), P_n^f(\hat{\mathbf{h}}), B_n^f(\hat{\mathbf{h}}) : n \in \{1, \cdots, N\}, f \in \mathcal{F}\right\}$ and $\mathbf{r} := \{r_1, \cdots, r_N\}$ respectively grouping resource allocation and ergodic rates. Further let $\mathbf{P}$ stand for the resource allocation function with values $\mathbf{P}(\hat{\mathbf{h}})$. The Lagrangian of the optimization problem in (4.29) can then be written as

$$\mathcal{L}(\mathbf{P}, \mathbf{r}, \mathbf{\Lambda}) = \sum_{n=1}^{N} U_n(r_n) + \sum_{n=1}^{N} \lambda_n \left[\mathbb{E}_{\hat{\mathbf{h}}}\left[\sum_{f\in\mathcal{F}} Q_n^f(\hat{\mathbf{h}}) R_n^f\left(P_n^f(\hat{\mathbf{h}}), B_n^f(\hat{\mathbf{h}}); \hat{h}_n^f\right)\right] - r_n\right]$$
$$+ \mu\left[\mathbb{E}_{\hat{\mathbf{h}}}\left[P_0 - \sum_{n=1}^{N}\sum_{f\in\mathcal{F}} Q_n^f(\hat{\mathbf{h}}) P_n^f(\hat{\mathbf{h}})\right]\right]. \tag{4.30}$$

The dual function and the dual problem are then given by

$$\mathsf{D_b} = \min_{\lambda_n \geq 0, \mu \geq 0} g(\mathbf{\Lambda}) = \min_{\lambda_n \geq 0, \mu \geq 0} \max_{\mathbf{P}, \mathbf{r}} \mathcal{L}(\mathbf{P}, \mathbf{r}, \mathbf{\Lambda}). \tag{4.31}$$

Note that the Lagrangian in (4.30) exhibits a separable structure because all summands involve a single primal variable. To explain this observation consider all summands of (4.30) that involve transmission rate r_n associated with terminal n and define the Lagrangian component associated with $\mathbf{r}$ as

$$\mathcal{L}^{(1)}(\mathbf{r}, \mathbf{\Lambda}) := \sum_{n=1}^{N} \left[U_n(r_n) - \lambda_n r_n\right]. \tag{4.32}$$

Define also the per channel Lagrangian components $\mathcal{L}^{(2)}\left(\mathbf{P}(\hat{\mathbf{h}}), \hat{\mathbf{h}}, \mathbf{\Lambda}\right)$ grouping all summands of (4.30) that involve resource allocation $\mathbf{P}(\hat{\mathbf{h}})$ and a given channel estimate $\hat{\mathbf{h}}$, i.e.,

$$\mathcal{L}^{(2)}\left(\mathbf{P}(\hat{\mathbf{h}}), \hat{\mathbf{h}}, \mathbf{\Lambda}\right) := \sum_{n=1}^{N}\sum_{f\in\mathcal{F}} Q_n^f(\hat{\mathbf{h}})\left[\lambda_n R_n^f\left(P_n^f(\hat{\mathbf{h}}), B_n^f(\hat{\mathbf{h}}); \hat{h}_n^f\right) - \mu P_n^f(\hat{\mathbf{h}})\right]. \tag{4.33}$$

It is easy to see by reordering summands in (4.30) that we can rewrite the Lagrangian as a sum of the component $\mathcal{L}^{(1)}(\mathbf{r}, \mathbf{\Lambda})$ and an expectation of the per channel components $\mathcal{L}^{(2)}\left(\mathbf{P}(\hat{\mathbf{h}}), \hat{\mathbf{h}}, \mathbf{\Lambda}\right)$,

$$\mathcal{L}(\mathbf{P}, \mathbf{r}, \mathbf{\Lambda}) = \mathcal{L}^{(1)}(\mathbf{r}, \mathbf{\Lambda}) + \mathbb{E}_{\hat{\mathbf{h}}}\left[\mathcal{L}^{(2)}\left(\mathbf{P}(\hat{\mathbf{h}}), \hat{\mathbf{h}}, \mathbf{\Lambda}\right)\right] + \mu P_0. \tag{4.34}$$

By leveraging the null duality gap, i.e., the equivalence $\mathsf{P_b} = \mathsf{D_b}$, and the Lagrangian separability in (4.34) we can characterize the optimal solution of the primal problem using the optimal solution of the dual problem, as shown in the following theorem.

Theorem 5. *The optimal subcarrier assignment function Q_n^{f*} with values $Q_n^{f*}(\hat{\mathbf{h}})$, channel backoff function B_n^{f*} with values $B_n^{f*}(\hat{\mathbf{h}})$ and power allocation function P_n^{f*} with values $P_n^{f*}(\hat{\mathbf{h}})$ for solving problem* (4.29) *are determined by the optimal variables λ_n^* and μ^* of the dual problem* (4.31). *In particular, for a given frequency $f \in \mathcal{F}$ and channel estimate $\hat{\mathbf{h}}$ values $P_n^{f*}(\hat{\mathbf{h}})$ and $B_n^{f*}(\hat{\mathbf{h}})$ of the optimal power and backoff functions are given by*

$$\left\{P_n^{f*}(\hat{\mathbf{h}}), B_n^{f*}(\hat{\mathbf{h}})\right\} \in \underset{p\in[0,p^{\text{inst}}],b\geq 0}{\operatorname{argmax}} \lambda_n^* R_n^f\left(p, b; \hat{h}_n^f\right) - \mu^* p. \tag{4.35}$$

To determine optimal frequency allocation values $Q_n^{f*}(\hat{\mathbf{h}})$ compute discriminants $\lambda_n^* R_n^f\left(P_n^{f*}(\hat{\mathbf{h}}), B_n^{f*}(\hat{\mathbf{h}}); \hat{h}_n^f\right) - \mu^* P_n^{f*}(\hat{\mathbf{h}})$ for all n and f. Determine the index of the terminal with maximum discriminant,

$$n^f = \underset{n}{\operatorname{argmax}}\, \lambda_n^* R_n^f\left(P_n^{f*}(\hat{\mathbf{h}}), B_n^{f*}(\hat{\mathbf{h}}); \hat{h}_n^f\right) - \mu^* P_n^{f*}(\hat{\mathbf{h}}), \tag{4.36}$$

and set $Q_n^{f*}(\hat{\mathbf{h}}) = 0$ for all $n \neq n^f$. For $n = n^f$ set $Q_n^{f*}(\hat{\mathbf{h}}) = 1$ if $\lambda_n^* R_n^f\left(P_n^{f*}(\hat{\mathbf{h}}), B_n^{f*}(\hat{\mathbf{h}}); \hat{h}_n^f\right) - \mu^* P_n^{f*}(\hat{\mathbf{h}}) > 0$.

Proof. As we have shown in the proof of Theorem 4, the fact that $\mathsf{P_b} = \mathsf{D_b}$ implies that optimal functions $\mathbf{P}^*$ and variables $\mathbf{r}^*$ are maximizers of the Lagrangian $\mathcal{L}(\mathbf{P}, \mathbf{r}, \mathbf{\Lambda}^*)$, i.e.

$$\{\mathbf{P}^*, \mathbf{r}^*\} \in \underset{\mathbf{P},\mathbf{r}}{\operatorname{argmax}}\, \mathcal{L}(\mathbf{P}, \mathbf{r}, \mathbf{\Lambda}^*). \tag{4.37}$$

Since $\mathbf{P}$ and $\mathbf{r}$ appear in different summands in $\mathcal{L}(\mathbf{P}, \mathbf{r}, \mathbf{\Lambda}^*)$ [cf. (4.32)-(4.34)], we can separate the maximizations for $\mathbf{P}(\hat{\mathbf{h}})$ and $\mathbf{r}$ and write $\mathbf{P}^*$ as

$$\mathbf{P}^* \in \underset{\mathbf{P}}{\operatorname{argmax}} \sum_{n=1}^{N} \sum_{f\in\mathcal{F}} \mathbb{E}_{\hat{\mathbf{h}}}\left[Q_n^f(\hat{\mathbf{h}})\left[\lambda_n^* R_n^f\left(P_n^f(\hat{\mathbf{h}}), B_n^f(\hat{\mathbf{h}}); \hat{h}_n^f\right) - \mu^* P_n^f(\hat{\mathbf{h}})\right]\right]. \tag{4.38}$$

Due to the linearity of expectation, the maximization in (4.38) can be carried out inside the expectation. Using the definition of $\mathbf{P}^*(\hat{\mathbf{h}})$ we have the following relationship

$$\left\{Q_n^{f*}(\hat{\mathbf{h}}), P_n^{f*}(\hat{\mathbf{h}}), B_n^{f*}(\hat{\mathbf{h}})\right\} \in \operatorname*{argmax}_{\mathbf{P}} \sum_{f\in\mathcal{F}} Q_n^f(\hat{\mathbf{h}}) \left[\lambda_n^* R_n^f \left(P_n^f(\hat{\mathbf{h}}), B_n^f(\hat{\mathbf{h}}); \hat{h}_n^f\right) - \mu^* P_n^f(\hat{\mathbf{h}})\right]. \quad (4.39)$$

Since for a fixed $f \in \mathcal{F}$ at most one $Q_n^{f*}(\hat{\mathbf{h}})$ can be 1 [cf. (4.26)], the computation of $Q_n^{f*}(\hat{\mathbf{h}})$, $P_n^{f*}(\hat{\mathbf{h}})$, and $B_n^{f*}(\hat{\mathbf{h}})$ as per (4.39) can be further separated into the computations in (4.35) and (4.36). Indeed, if $Q_n^{f*}(\hat{\mathbf{h}}) = 1$ the best possible values for $P_n^{f*}(\hat{\mathbf{h}})$ and $B_n^{f*}(\hat{\mathbf{h}})$ are the ones that maximize the factor $\lambda_n^* R_n^f \left(P_n^f(\hat{\mathbf{h}}), B_n^f(\hat{\mathbf{h}}); \hat{h}_n^f\right) - \mu^* P_n^f(\hat{\mathbf{h}})$. If $Q_n^{f*}(\hat{\mathbf{h}}) = 0$ any value of $P_n^{f*}(\hat{\mathbf{h}})$ and $B_n^{f*}(\hat{\mathbf{h}})$ is optimal, in particular the one that maximizes this factor. Therefore

$$\left\{P_n^{f*}(\hat{\mathbf{h}}), B_n^{f*}(\hat{\mathbf{h}})\right\} \in \operatorname*{argmax}_{P_n^f(\hat{\mathbf{h}})\in[0,p^{\text{inst}}], B_n^f(\hat{\mathbf{h}})\geq 0} \lambda_n^* R_n^f \left(P_n^f(\hat{\mathbf{h}}), B_n^f(\hat{\mathbf{h}}); \hat{h}_n^f\right) - \mu^* P_n^f(\hat{\mathbf{h}}). \quad (4.40)$$

Upon the change of variables $p = P_n^f(\hat{\mathbf{h}})$ and $b = B_n^f(\hat{\mathbf{h}})$ (4.35) follows.

To decide on indicator variables $Q_n^f(\hat{\mathbf{h}})$ substitute the optimal power and backoff values in (4.40) into the sum maximization in (4.39) to obtain

$$\left\{Q_n^{f*}(\hat{\mathbf{h}})\right\} \in \operatorname*{argmax}_{\mathbf{Q}^f(\hat{\mathbf{h}})\in\mathcal{Q}} Q_n^f(\hat{\mathbf{h}}) \left[\lambda_n^* R_n^f \left(P_n^{f*}(\hat{\mathbf{h}}), B_n^{f*}(\hat{\mathbf{h}}); \hat{h}_n^f\right) - \mu^* P_n^{f*}(\hat{\mathbf{h}})\right]. \quad (4.41)$$

If all discriminants $\lambda_n^* R_n^f \left(P_n^{f*}(\hat{\mathbf{h}}), B_n^{f*}(\hat{\mathbf{h}}); \hat{h}_n^f\right) - \mu^* P_n^{f*}(\hat{\mathbf{h}})$ are negative, the maximum in (4.41) is attained by making $Q_n^f(\hat{\mathbf{h}}) = 0$ for all n implying that frequency f is not used by any terminal during the time slot. Otherwise, the largest objective in (4.41) is obtained by making $Q_n^{f*}(\hat{\mathbf{h}})$ for the terminal with the largest discriminant. These computations coincides with (4.36). □

With optimal multipliers given, optimal power allocation and channel backoff can be computed using (4.35). Optimal frequency allocations are determined by comparing the discriminants in (4.36) and assigning frequency f to the terminal with the largest discriminant if this discriminant is positive. Notice that the maximization required in (4.35) is of a nonconvex objective, but this involves just two variables and is analogous to the maximand in (4.20) for the case of point-to-point channels. We can interpret (4.20) as establishing a decomposition on per terminal, per frequency and per fading state subproblems. Further observe that Theorem 5 indicates that the optimal solution is opportunistic. Frequency f is used only when at least one terminal observes a good channel on this frequency so as to have a positive discriminant $\lambda_n^* R_n^f \left(P_n^{f*}(\hat{\mathbf{h}}), B_n^{f*}(\hat{\mathbf{h}}); \hat{h}_n^f \right) - \mu^* P_n^{f*}(\hat{\mathbf{h}})$ [cf. (4.36)].

4.2.2 Online learning algorithms

Similar to the case of point-to-point channels we can solve the optimization problem in (4.29) using stochastic subgradient descent in the dual domain. To determine stochastic subgradients of the dual function start with given dual variable $\mathbf{\Lambda}(t)$ and channel realization $\hat{\mathbf{h}}(t)$ and proceed to determine the Lagrangian maximizers

$$\mathbf{r}(t) \in \underset{\mathbf{r}}{\operatorname{argmax}} \, \mathcal{L}^{(1)}\left(\mathbf{r}, \mathbf{\Lambda}(t)\right), \tag{4.42}$$

$$\mathbf{p}(t) \in \underset{\mathbf{p}}{\operatorname{argmax}} \, \mathcal{L}^{(2)}\left(\mathbf{p}, \hat{\mathbf{h}}(t), \mathbf{\Lambda}(t)\right), \tag{4.43}$$

Notice that in (4.43) we determine a power allocation that corresponds to the current channel estimate $\hat{\mathbf{h}}(t)$. Using the definition of the Lagrangian components $\mathcal{L}^{(1)}\left(\mathbf{r}, \mathbf{\Lambda}\right)$, $\mathcal{L}^{(2)}\left(\mathbf{P}(\hat{\mathbf{h}}), \hat{\mathbf{h}}, \mathbf{\Lambda}\right)$ [cf. (4.32) and (4.33)] and attempted transmission rate $R_n^f\left(P_n^f(\hat{\mathbf{h}}), B_n^f(\hat{\mathbf{h}}); \hat{h}_n^f\right)$, the primal iterates in (4.42) and (4.43) can be computed as

$$r_n(t) = \underset{r_n \in [0, r_{\max}]}{\operatorname{argmax}} \; U_n(r_n) - \lambda_n(t) r_n, \tag{4.44}$$

$$\{q_n^f(t), p_n^f(t), b_n^f(t)\} = \underset{\mathbf{a} \in \mathcal{Q},\, p \in [0, p^{\text{inst}}],\, b > 0}{\operatorname{argmax}} \sum_{n,f} q_n^f \left[\lambda_n(t) R_n^f(b, p; \hat{h}_n^f(t)) - \mu(t) p \right]. \tag{4.45}$$

Following the logic used for deriving (4.35)-(4.36) the maximization in (4.45) can be further simplified to the computation of power allocation and backoff function

$$\{p_n^f(t), b_n^f(t)\} = \underset{p \in [0, p^{\text{inst}}], b \geq 0}{\operatorname{argmax}} \; \lambda_n(t) R_n^f(b, p; \hat{h}_n^f(t)) - \mu(t) p, \tag{4.46}$$

followed by the determination of terminal indices

$$n^f(t) = \underset{n}{\operatorname{argmax}} \, \lambda_n(t) R_n^f(p_n^f(t), b_n^f(t); \hat{h}_n^f(t)) - \mu(t) p_n^f(t). \tag{4.47}$$

We then set $q_n^f(t) = 0$ for all $n \neq n^f(t)$ and $q_n^f(t) = 1$ for $n = n^f(t)$ if $\lambda_n(t) R_n^f(p_n^f(t), b_n^f(t); \hat{h}_n^f(t)) - \mu(t) p_n^f(t) > 0$.

A subgradient of the dual function $g(\mathbf{\Lambda}(t))$ can be obtained by evaluating the instantaneous constraint slacks associated with the Lagrangian maximizers. Denoting as $s_{\lambda_n}(t)$ the subgradient components along the λ_n direction and $s_\mu(t)$ the component along the μ direction we have

$$\begin{aligned} s_{\lambda_n}(t) &= \sum_{f \in \mathcal{F}} q_n^f(t) R_n^f(p_n^f(t), b_n^f(t); \hat{h}_n^f(t)) - r_n(t), \\ s_\mu(t) &= P_0 - \sum_{n=1}^{N} \sum_{f \in \mathcal{F}} q_n^f(t) p_n^f(t). \end{aligned} \tag{4.48}$$

The algorithm is completed with an update of the dual variables along the stochastic subgradient direction moderated by a possibly time varying step size $\epsilon(t)$,

$$\lambda_n(t+1) = \left[\lambda_n(t) - \epsilon(t) \left[\sum_{f \in \mathcal{F}} q_n^f(t) R_n^f(p_n^f(t), b_n^f(t); \hat{h}_n^f(t)) - r_n(t) \right] \right]^+, \tag{4.49}$$

$$\mu(t+1) = \left[\mu(t) - \epsilon(t) \left[P_0 - \sum_{n=1}^{N} \sum_{f \in \mathcal{F}} q_n^f(t) p_n^f(t) \right] \right]^+, \tag{4.50}$$

As in the case of point-to-point channels, particular convergence properties depend on whether constant or time varying step sizes are used. With diminishing step size, $\lambda_n(t)$ and $\mu(t)$ converge to optimal dual variables λ_n^* and μ^* almost surely. With constant step size convergence is established in an ergodic sense by applying Theorem 1 of [32] to the optimization problem in (4.29) and the stochastic dual descent algorithm in (4.44) and (4.46)-(4.50). The resulting property is specified in the following.

Property 2. *If constant step sizes $\epsilon(t) = \epsilon > 0$ for all t are used in Algorithm 3, rate and power constraints are almost surely satisfied on average*

$$\lim_{t \to \infty} \frac{1}{t} \sum_{u=1}^{t} r_n(u) \leq \lim_{t \to \infty} \frac{1}{t} \sum_{u=1}^{t} \left[\sum_{f \in \mathcal{F}} q_n^f(u) R_n^f \left(p_n^f(u), b_n^f(u); \hat{h}_n^f(u) \right) \right] \quad a.s., \tag{4.51}$$

$$\lim_{t \to \infty} \frac{1}{t} \sum_{u=1}^{t} \left[\sum_{n=1}^{N} \sum_{f \in \mathcal{F}} q_n^f(u) p_n^f(u) \right] \leq P_0 \quad a.s. \tag{4.52}$$

The sum utility of the ergodic limit of $r_n^f(t)$ almost surely converges to a value within $\kappa\epsilon/2$ of optimal,

$$\mathsf{P_b} - \sum_{n=1}^{N} U_n \left(\lim_{t\to\infty} \frac{1}{t} \sum_{u=1}^{t} \sum_{f\in\mathcal{F}} r_n^f(u) \right) \leq \kappa\epsilon/2 \quad \textit{a.s.}, \tag{4.53}$$

where $\kappa \geq \mathbb{E}\left[\sum_n s_{\lambda_n}^2(t) + s_\mu^2(t) \,\middle|\, \lambda(t) \right]$ is a constant bounding the second moment of the norm of the stochastic subgradient with components given as in (4.48).

Property 2 establishes optimality of the sequences of primal variables generated by (4.44) and (4.46)-(4.50). In particular, ergodic limits of these sequences almost surely satisfy problem constraints and are within a small factor of optimal. As in the case of point to point channels the rate $q_n^f(u)C\Big(p_n^f(u), b_n^f(u)\Big)M_{h_n^f(u)|\hat{h}_n^f(u)}\Big(b_n^f(u)\Big)$ in the ergodic limit in (4.51) is different from the instantaneous transmission rate $q_n^f(u)C\Big(p_n^f(u), b_n^f(u)\Big)\mathbb{I}\Big\{b_n^f(u) \leq h_n^f(u)\Big\}$ achieved by the algorithm. However, their ergodic limits are equivalent since the stochastic process $\mathbf{h}(\mathbb{N})$ is ergodic given estimates $\hat{\mathbf{h}}(\mathbb{N})$, i.e.,

$$\begin{aligned} &\lim_{t\to\infty} \frac{1}{t} \sum_{u=1}^{t} q_n^f(u)C\left(p_n^f(u), b_n^f(u)\right) M_{h_n^f(u)|\hat{h}_n^f(u)}\left(b_n^f(u)\right) \\ &= \lim_{t\to\infty} \frac{1}{t} \sum_{u=1}^{t} q_n^f(u)C\left(p_n^f(u), b_n^f(u)\right) \mathbb{I}\left\{b_n^f(u) \leq h_n^f(u)\right\} \quad \text{a.s..} \end{aligned} \tag{4.54}$$

Substituting (4.54) into (4.51) and the resulting expression into (4.53) we have

$$\mathsf{P_b} - \sum_{n=1}^{N} U_n \left(\lim_{t\to\infty} \frac{1}{t} \sum_{u=1}^{t} \sum_{f\in\mathcal{F}} q_n^f(u)C\left(p_n^f(u), b_n^f(u)\right) \mathbb{I}\left\{b_n^f(u) \leq h_n^f(u)\right\} \right) \leq \kappa\epsilon/2 \quad \text{a.s..} \tag{4.55}$$

The inequality in (4.55) establishes that the utility of the ergodic limits of the transmission rates achieved by the algorithm is within $\kappa\epsilon/2$ of the optimal value $\mathsf{P_b}$. Since κ is a constant, the optimality gap can be made arbitrarily small by reducing the step size ϵ.

The procedure is summarized in Algorithm 3. Multipliers $\lambda_n(0)$ and $\mu(0)$ are initialized at time slot 0. Primal and dual variables are computed iteratively in subsequent time slots. In particular, for each time slot t the algorithm first computes variable $r_n(t)$ for all users as per (4.44) which decides the number of packets to be accepted into T_n's queue awaiting for transmission (line 3). The algorithm then iterates over frequencies and calculates power allocation

$p_n^f(t)$, channel backoff $b_n^f(t)$ for all n by solving the two-variable maximization as per (4.46) (line 7). The subcarrier assignments $q_n^f(t)$ are then determined by setting $q_n^f(t) = 1$ for n such that $\lambda_n(t)R_n^f(t) - \mu(t)p_n^f(t)$ the largest positive discriminant among all n as per (4.47) while setting $q_n^f(t) = 0$ for all the rest users (line 11). Note that for a given frequency there is at most one $q_n^f(t) = 1$. If $q_n^f(t) = 1$, the AP transmit to T_n over frequency f using power $p_n^f(t)$ and rate $C(p_n^f(t), b_n^f(t))$. The algorithm then proceeds to update multipliers for the next time slot based on multipliers and primal variables of the current time slot according to (4.49) and (4.50) (lines 15-16).

4.3 Random access

Consider now a multiple access channel in which N terminals contend for communication to a common AP using random access. The channel between terminals and the AP is modeled as block fading and denoted as $h_n(t)$. Assume each terminal only observes an imperfect version of its local channel $\hat{h}_n(t)$. Based on its local channel, terminals decide channel access $q_n(t) := Q_n(\hat{h}_n) \in \{0, 1\}$, power allocations $p_n(t) := P_n(\hat{h}_n) \in [0, p^{\text{inst}}]$ and channel backoffs $b_n(t) := B_n(\hat{h}_n) \geq 0$. We remark that Q_n, P_n and B_n are functions of local channels only as opposed to functions of all channel realizations as in the case of OFDM considered in Section 4.2. Since terminals contend for channel access, a transmission from terminal T_n in time slot t is successful if and only if $q_n(t) = 1$ and $q_m(t) = 0$ for all $m \neq n$. If the transmission of T_n is successful, its transmission rate is determined by $C(p_n(t), b_n(t))$. As as consequence, the instantaneous transmission rate for T_n in time slot t is

$$r_n(t) = C\left(p_n(t), b_n(t)\right) \mathbb{I}\left\{b_n(t) \leq h_n(t)\right\} q_n(t) \prod_{m=1, m\neq n}^{N} \left[1 - q_m(t)\right]. \tag{4.56}$$

Algorithm 3: Optimal subcarrier assignment, power control and channel backoff for OFDM

Initialize Lagrangian multipliers $\lambda_n(0), \mu(0)$;

for $t = 0, 1, 2, \cdots$ **do**

  Compute primal variables $r_n(t) = \underset{r_n \in [0, r_{\max}]}{\operatorname{argmax}} \; U_n(r_n) - \lambda_n(t) r_n$ as per (4.44) for all n

  Accept $r_n(t)$ packets to T_n's queue for transmission

  for *each frequency* $f \in \mathcal{F}$ **do**

    For all n, compute primal variables $p_n^f(t), b_n^f(t)$ as per (4.46):

    $$\{p_n^f(t), b_n^f(t)\} = \underset{p \in [0, p^{\text{inst}}], b \geq 0}{\operatorname{argmax}} \; \lambda_n(t) R_n^f(p, b; \hat{h}_n^f(t)) - \mu(t) p;$$

    Find terminal with maximum discriminant as per (4.47):

    $$n^f(t) = \underset{n}{\operatorname{argmax}} \, \lambda_n(t) R_n^f(p_n^f(t), b_n^f(t); \hat{h}_n^f(t)) - \mu(t) p_n^f(t);$$

    if $n = n^f(t)$ and $\lambda_n(t) R_n^f(p_n^f(t), b_n^f(t); \hat{h}_n^f(t)) - \mu(t) p_n^f(t) > 0$ **then**

      Set $q_n^f(t) = 1$, transmit to T_n on frequency f with power $p_n^f(t)$ and rate $C(p_n^f(t), b_n^f(t))$;

    end

  end

  Update dual variables as per (4.49) and (4.50):

  $$\lambda_n(t+1) = \left[\lambda_n(t) - \epsilon(t)\left[\sum_{f \in \mathcal{F}} q_n^f(t) R_n^f(p_n^f(t), b_n^f(t); \hat{h}_n^f(t)) - r_n(t)\right]\right]^+;$$

  $$\mu(t+1) = \left[\mu(t) - \epsilon(t)\left[P_0 - \sum_{n=1}^{N} \sum_{f \in \mathcal{F}} q_n^f(t) p_n^f(t)\right]\right]^+;$$

end

The ergodic rate is given by a time average of the instantaneous rates in (4.56) which due to ergodicity can be equivalently written as

$$\begin{aligned} r_n &= \mathbb{E}_{\hat{h}} \left[C\left(P_n\left(\hat{h}_n\right), B_n\left(\hat{h}_n\right)\right) M_{h_n|\hat{h}_n(t)} \left(B_n(\hat{h}_n)\right) Q_n(\hat{h}_n) \prod_{m=1, m\neq n}^{N} \left[1 - Q_m\left(\hat{h}_m\right)\right]\right] \\ &= \mathbb{E}_{\hat{h}} \left[R_n\left(P_n\left(\hat{h}_n\right), B_n\left(\hat{h}_n\right); \hat{h}_n\right) Q_n(\hat{h}_n) \prod_{m=1, m\neq n}^{N} \left[1 - Q_m\left(\hat{h}_m\right)\right]\right], \end{aligned} \tag{4.57}$$

where in the second equality we defined the average attempted transmission rate for terminal n as $R_n\left(P_n\left(\hat{h}_n\right), B_n\left(\hat{h}_n\right); \hat{h}_n\right) = C\left(P_n\left(\hat{h}_n\right), B_n\left(\hat{h}_n\right)\right) M_{h_n|\hat{h}_n}\left(B_n(\hat{h}_n)\right)$. An important observation here is that since terminals are required to make channel access and power control decisions independently of each other, $Q_n(\hat{h}_n), P_n(\hat{h}_n), B_n(\hat{h}_n)$ are independent of $Q_m(\hat{h}_m)$, $P_m(\hat{h}_m)$, $B_m(\hat{h}_m)$ for all $n \neq m$. This allows us to rewrite r_n as

$$r_n = \mathbb{E}_{\hat{h}_n} \left[R_n\left(P_n\left(\hat{h}_n\right), B_n\left(\hat{h}_n\right); \hat{h}_n\right) Q_n(\hat{h}_n)\right] \prod_{m=1, m\neq n}^{N} \left[1 - \mathbb{E}_{\hat{h}_m}\left[Q_m\left(\hat{h}_m\right)\right]\right]. \tag{4.58}$$

The objective is to maximize the proportional fair utility of the ergodic rates r_n, i.e.,

$$U(\mathbf{r}) = \sum_{n=1}^{N} \log(r_n), \tag{4.59}$$

where $\mathbf{r} := \{r_n : n \in \{1, \cdots, N\}\}$. In a network where channel pdfs vary among users, maximizing sum log utility $U(\mathbf{r})$ yields solutions that are fair since it prevents users from having very low transmission rates. The optimal random access with imperfect CSI is formulated as the following optimization problem

$$\begin{aligned} \mathsf{P_r} \; &= \max \; U(\mathbf{r}) \\ \text{s.t.} \;\; & r_n = \mathbb{E}_{\hat{h}_n} \left[R_n\left(P_n\left(\hat{h}_n\right), B_n\left(\hat{h}_n\right); \hat{h}_n\right) Q_n(\hat{h}_n)\right] \prod_{m=1, m\neq n}^{N} \left[1 - \mathbb{E}_{\hat{h}_m}\left[Q_m(\hat{h}_m)\right]\right], \\ & \mathbb{E}_{\hat{h}_n}\left[Q_n(\hat{h}_n) P_n(\hat{h}_n)\right] \leq P_{0n}, \\ & Q_n(\hat{h}_n) \in \{0,1\}, P_n(\hat{h}_n) \in [0, p^{\text{inst}}], B_n(\hat{h}_n) \geq 0, \end{aligned} \tag{4.60}$$

where the second inequality indicates each terminal has an average power budget of P_{0n}. Since we require Q_n, P_n and B_n to be functions of local channel estimates only, we need a distributed solution for problem (4.60). However, its formulation is not amenable for distributed implementations because the rate constraint involves actions of all terminals. Thus, we need to separate problem (4.60) into per terminal subproblems. To do so, we substitute r_n into $U(\mathbf{r})$ and express the logarithm of a product as a sum of logarithms so as to write

$$U(\mathbf{r}) = \sum_{n=1}^{N} \left[\log \mathbb{E}_{\hat{h}_n} \left[R_n \left(P_n \left(\hat{h}_n \right), B_n \left(\hat{h}_n \right); \hat{h}_n \right) Q_n(\hat{h}_n) \right] + \sum_{m=1, m \neq n}^{N} \log \left[1 - \mathbb{E}_{\hat{h}_m} \left[Q_m(\hat{h}_m) \right] \right] \right] \tag{4.61}$$

$$= \sum_{n=1}^{N} \left[\log \mathbb{E}_{\hat{h}_n} \left[R_n \left(P_n \left(\hat{h}_n \right), B_n \left(\hat{h}_n \right); \hat{h}_n \right) Q_n(\hat{h}_n) \right] + (N-1) \log \left[1 - \mathbb{E}_{\hat{h}_n} \left[Q_n(\hat{h}_n) \right] \right] \right]. \tag{4.62}$$

where in (4.62) we grouped terms related to T_n. To maximize $U(\mathbf{r})$ for the whole system it suffices to separately maximize corresponding summand for each terminal n. Upon introducing auxiliary variables $x_n = \mathbb{E}_{\hat{h}_n} \left[Q_n(\hat{h}_n) R_n \left(P_n \left(\hat{h}_n \right), B_n \left(\hat{h}_n \right); \hat{h}_n \right) \right]$ and $y_n = \mathbb{E}_{\hat{h}_n} \left[Q_n(\hat{h}_n) \right]$, it follows that (4.60) is equivalent to the following per terminal subproblems

$$\begin{aligned}
\mathsf{P}_{\mathsf{r,n}} \quad = \quad & \max \sum_{f \in \mathcal{F}} \log x_n + (N-1) \log(1 - y_n) \\
& \text{s.t.} \;\; x_n \leq \mathbb{E}_{\hat{h}_n} \left[Q_n(\hat{h}_n) R_n \left(P_n \left(\hat{h}_n \right), B_n \left(\hat{h}_n \right); \hat{h}_n \right) \right], \qquad y_n \geq \mathbb{E}_{\hat{h}_n} \left[Q_n(\hat{h}_n) \right], \\
& P_{0n} \geq \mathbb{E}_{\hat{h}_n} \left[Q_n(\hat{h}_n) P_n(\hat{h}_n) \right], \\
& x_n \geq 0, \; 0 \leq y_n \leq 1, \; Q_n(\hat{h}_n) \in \{0, 1\}, \; P_n(\hat{h}_n) \in [0, p^{\text{inst}}], \; B_n(\hat{h}_n) \geq 0.
\end{aligned} \tag{4.63}$$

In particular, we have $\mathsf{P}_{\mathsf{r}} = \sum_{n=1}^{N} \mathsf{P}_{\mathsf{r,n}}$. Therefore, to solve problem (4.60) we only need to solve problem (4.63) for all terminals in a distributed manner.

4.3.1 Optimal solution

Similar to the case of point-to-point channel, problem (4.63) has null duality gap which allows us to work on its dual domain without loss of optimality. To define the problem's Lagrangian, associate multipliers α_n with the constraint involving x_n in (4.63), β_n with the constraint involving y_n, and ν_n with the average power constraint. Further define $\mathbf{\Lambda}_n := \{\alpha_n, \beta_n, \nu_n\}$, $\mathbf{P}_n(\hat{h}_n) :=$

$\Big\{Q_n(\hat{h}_n), P_n(\hat{h}_n), B_n(\hat{h}_n)\Big\}$, $\mathbf{x}_n := \{x_n, y_n\}$ grouping all multipliers, resource allocation variables and auxiliary variables, respectively. The Lagrangian is then given by

$$\mathcal{L}_n(\mathbf{P}_n, \mathbf{x}_n, \mathbf{\Lambda}_n) = \log x_n + (N-1)\log(1-y_n) + \alpha_n \Big[\mathbb{E}_{\hat{h}_n}\Big[Q_n(\hat{h}_n) R_n\Big(P_n\Big(\hat{h}_n\Big), B_n\Big(\hat{h}_n\Big); \hat{h}_n\Big)\Big] - x_n\Big]$$
$$+ \beta_n \Big[y_n - \mathbb{E}_{\hat{h}_n}\Big[Q_n(\hat{h}_n)\Big]\Big] + \nu_n \Big[P_{0n} - \mathbb{E}_{\hat{h}_n}\Big[Q_n(\hat{h}_n) P_n(\hat{h}_n)\Big]\Big]. \tag{4.64}$$

The dual problem can then be written as

$$\mathsf{D}_{\mathsf{r,n}} = \min_{\alpha_n \geq 0, \beta_n \geq 0, \nu_n \geq 0} g_n(\mathbf{\Lambda}_n) = \min_{\alpha_n \geq 0, \beta_n \geq 0, \nu_n \geq 0} \max_{\mathbf{P}_n, \mathbf{x}_n} \mathcal{L}_n(\mathbf{P}_n, \mathbf{x}_n, \mathbf{\Lambda}_n). \tag{4.65}$$

Observe that primal variables $\mathbf{P}_n$ and $\mathbf{x}_n$ appear in different summands in (4.64). This allows us to regroup terms involving $\mathbf{P}_n$ and $\mathbf{x}_n$ and decompose the Lagrangian. To do so, define $\mathcal{L}_n^{(1)}(\mathbf{x}_n, \mathbf{\Lambda}_n)$ as the per terminal local Lagrangian component involving auxiliary variable $\mathbf{x}_n$

$$\mathcal{L}_n^{(1)}(\mathbf{x}_n, \mathbf{\Lambda}_n) = [\log x_n - \alpha_n x_n] + [(N-1)\log(1-y_n) + \beta_n y_n], \tag{4.66}$$

and $\mathcal{L}_n^{(2)}(\mathbf{P}_n(\hat{h}_n), \hat{h}_n, \mathbf{\Lambda}_n)$ as the per terminal per fading state Lagrangian component involving resource allocation variable $\mathbf{P}_n(\hat{h}_n)$

$$\mathcal{L}_n^{(2)}(\mathbf{P}_n(\hat{h}_n), \hat{h}_n, \mathbf{\Lambda}_n) = Q_n(\hat{h}_n) \Big[\alpha_n R_n\Big(P_n\Big(\hat{h}_n\Big), B_n\Big(\hat{h}_n\Big); \hat{h}_n\Big) - \beta_n - \nu_n P_n(\hat{h}_n)\Big]. \tag{4.67}$$

As a result, the Lagrangian in (4.64) can be rewritten as

$$\mathcal{L}_n(\mathbf{P}_n, \mathbf{x}_n, \mathbf{\Lambda}_n) = \mathcal{L}_n^{(1)}(\mathbf{x}_n, \mathbf{\Lambda}_n) + \mathbb{E}_{\hat{h}_n}\Big[\mathcal{L}_n^{(2)}(\mathbf{P}_n(\hat{h}_n), \hat{h}_n, \mathbf{\Lambda}_n)\Big] + \nu_n P_{0n}, \tag{4.68}$$

By leveraging the property of null duality gap, i.e., $\mathsf{P}_{\mathsf{r,n}} = \mathsf{D}_{\mathsf{r,n}}$, we can characterize the optimal solution of the primal problem using the optimal solution of the dual problem, as shown in the following theorem.

Theorem 6. *The optimal subcarrier assignment function Q_n^* with values $Q_n^*(\hat{h}_n)$, channel backoff function B_n^* with values $B_n^*(\hat{h}_n)$ and power allocation function P_n^* with values $P_n^*(\hat{h}_n)$ for solving problem* (4.63) *are uniquely determined by the optimal variables α_n^*, β_n^* and ν_n^* of the dual problem* (4.65)*. In* *particular, for given terminal n we have*

$$\left\{P_n^*(\hat{h}_n), B_n^*(\hat{h}_n)\right\} \in \underset{p\in[0,p^{\text{inst}}],b\geq 0}{\operatorname{argmax}} \; \alpha_n^* R_n\left(p, b; \hat{h}_n\right) - \beta_n^* - \nu_n^* p, \tag{4.69}$$

$$Q_n^*(\hat{h}_n) = \mathbb{I}\left\{\alpha_n^* R_n\left(P_n^*(\hat{h}_n), B_n^*(\hat{h}_n); \hat{h}_n\right) - \beta_n^* - \nu_n^* P_n^*(\hat{h}_n)) > 0\right\}. \tag{4.70}$$

Proof. As we did in the proof of Theorem 4, by exploiting the null duality gap we can show that the optimal functions $\mathbf{P}_n^*$ and variables $\mathbf{x}_n$ are maximizers of the Lagrangian $\mathcal{L}_n(\mathbf{P}_n, \mathbf{x}_n, \mathbf{\Lambda}_n^*)$, i.e.

$$\{\mathbf{P}_n^*, \mathbf{x}_n^*\} \in \underset{\mathbf{P}_n,\mathbf{x}_n}{\operatorname{argmax}} \, \mathcal{L}_n(\mathbf{P}_n, \mathbf{x}_n, \mathbf{\Lambda}_n^*). \tag{4.71}$$

Note that since $\mathbf{P}_n^*$ and $\mathbf{x}_n^*$ appear in different summands in $\mathcal{L}_n(\mathbf{P}_n, \mathbf{x}_n, \mathbf{\Lambda}_n^*)$ [cf. (4.66)-(4.68)], we can write $\mathbf{P}_n^*$ as the maximizer of the corresponding summand, i.e.,

$$\mathbf{P}_n^* \in \underset{\mathbf{P}_n}{\operatorname{argmax}} \, \mathbb{E}_{\hat{h}_n}\left[Q_n(\hat{h}_n)\left[\alpha_n^* R_n\left(P_n\left(\hat{h}_n\right), B_n\left(\hat{h}_n\right); \hat{h}_n\right) - \beta_n^* - \nu_n^* P_n(\hat{h}_n)\right]\right]. \tag{4.72}$$

Due to the linearity of expectation, the maximization in (4.72) can be carried out inside the expectation. We therefore have

$$\left\{Q_n^*(\hat{h}_n), P_n^*(\hat{h}_n), B_n^*(\hat{h}_n)\right\} \in \underset{a\in\{0,1\},p\in[0,p^{\text{inst}}],b>0}{\operatorname{argmax}} \; a\left[\alpha_n^* R_n\left(p, b; \hat{h}_n\right) - \beta_n^* - \nu_n^* p\right]. \tag{4.73}$$

where we have used the definition of the aggregate variable $\mathbf{P}_n(\hat{h}_n) := \left\{Q_n(\hat{h}_n), P_n(\hat{h}_n), B_n(\hat{h}_n)\right\}$.

Since the variable a in (4.73) can only take values in $\{0, 1\}$, the objective in (4.72) can only be 0 or $\alpha_n^* R_n\left(p, b; \hat{h}_n\right) - \beta_n^* - \nu_n^* p$. Thus, to solve (4.73) we just need to find the optimal $P_n^*(\hat{h}_n), B_n^*(\hat{h}_n)$ when $a = 1$ and set $Q_n^*(\hat{h}_n) = 1$ if the resulting objective is strictly positive. This procedure is what (4.69) and (4.70) state. $\square$

Given the optimal dual variable $\mathbf{\Lambda}_n^*$, optimal functions for power allocation $P_n^*(\hat{h}_n)$, channel backoff $B_n^*(\hat{h}_n)$, and channel access $Q_n^*(\hat{h}_n)$ can be determined in a distributed manner through (4.69) and (4.70) using local information only. This satisfies the design requirement that terminals have to operate independently of each other. It is also worth remarking that the resulting transmission policy is opportunistic with respect to channel estimates $\hat{h}_n$ because terminal n transmits only when $\alpha_n^* R_n\left(P_n^*(\hat{h}_n), B_n^*(\hat{h}_n); \hat{h}_n\right) - \beta_n^* - \nu_n^* P_n^*(\hat{h}_n) > 0$. For this inequality to be true we need to have a sufficiently large rate $R_n\left(P_n^*(\hat{h}_n), B_n^*(\hat{h}_n); \hat{h}_n\right)$, which in turn requires large channel estimates $\hat{h}_n$. In fact it is not difficult to see that (4.70) implies a threshold policy in which terminals transmit if and only if the channel $\hat{h}_n$ exceeds a threshold that can be computed in terms of the optimal multiplier values. This is consistent with similar observations in the case of perfect CSI [16, 17].

The computation of the optimal power allocation and optimal channel backoff in random access [cf. (4.69)] is similar to that of OFDM [cf. (4.35)] in the sense that they both solve a two-variable nonconvex optimization problem. However, determination of their corresponding optimal subcarrier assignments is different. For a given frequency, the optimal subcarrier assignment $Q_n^{f*}(\hat{\mathbf{h}})$ in OFDM is determined jointly for all n and at most one $Q_n^{f*}(\hat{\mathbf{h}})$ can be 1. The optimal $Q_n^*(\hat{h}_n)$ in the case of random access is computed locally and there might be more than one $Q_n^*(\hat{h}_n)$ set to 1 in for different n. This is because in the case of random access all terminals act independently of each other and there is no coordination among them while in the case of OFDM the AP plays the role of a central decision maker.

4.3.2 Online learning algorithms

To solve problem (4.63) without knowledge of the channel pdf we implement the stochastic subgradient descent algorithm in the dual domain as we did in the case of point-to-point and OFDM channels. To find stochastic subgradients we compute maximizers of the local Lagrangian components $\mathcal{L}^{(1)}(\mathbf{x}_n, \mathbf{\Lambda}_n)$ and $\mathcal{L}^{(2)}(\mathbf{p}_n, \hat{h}_n, \mathbf{\Lambda}_n)$ for given channel estimate $\hat{h}_n(t)$ and Lagrangian multiplier $\mathbf{\Lambda}_n(t)$, i.e.,

$$\mathbf{x}_n(t) = \underset{\mathbf{x}_n}{\operatorname{argmax}}\, \mathcal{L}_n^{(1)}\left(\mathbf{x}_n, \mathbf{\Lambda}_n(t)\right), \tag{4.74}$$

$$\mathbf{p}_n(t) = \underset{\mathbf{p}_n}{\operatorname{argmax}}\, \mathcal{L}_n^{(2)}\left(\mathbf{p}_n, \hat{h}_n(t), \mathbf{\Lambda}_n(t)\right), \tag{4.75}$$

Recall that x_n and y_n appear in different summands in $\mathcal{L}_n^{(1)}(\mathbf{x}_n, \mathbf{\Lambda}_n(t))$ [cf. (4.66)]. As a result, we can separate the maximizations for $x_n(t)$ and $y_n(t)$, i.e.,

$$x_n(t) = \operatorname*{argmax}_{x \geq 0} \log x - \alpha_n(t)x = \frac{1}{\alpha_n(t)}, \tag{4.76}$$

$$y_n(t) = \operatorname*{argmax}_{0 \leq y \leq 1} (N-1)\log(1-y) + \beta_n(t)y = \left[1 - \frac{N-1}{\beta_n(t)}\right]^+. \tag{4.77}$$

Furthermore, using the definition of $\mathbf{p}_n(t)$ and $\mathcal{L}_n^{(2)}\left(\mathbf{p}_n, \hat{h}_n(t), \mathbf{\Lambda}_n(t)\right)$, the resource allocation computation in (4.75) can be rewritten as

$$\{q_n(t), p_n(t), b_n(t)\} = \operatorname*{argmax}_{a \in \{0,1\}, p \in [0, p^{\text{inst}}], b \geq 0} a\left[\alpha_n(t) R_n(p_n(t), b_n(t); \hat{h}_n(t)) - \beta_n(t) - \nu_n(t)p\right]. \tag{4.78}$$

Optimizations for $x_n(t)$ and $y_n(t)$ are relatively easy because their objectives are both convex functions with a single variable [cf. (4.76) and (4.77)]. Determination of $q_n(t), b_n(t)$ and $p_n(t)$ as per (4.78) is more complicated but it can be simplified since $q_n(t)$ can only take values in $\{0, 1\}$. Using this fact as we did in the proof of Theorem 6 we conclude that (4.78) is equivalent to

$$\{b_n(t), p_n(t)\} = \operatorname*{argmax}_{p \in [0, p^{\text{inst}}], b \geq 0} \alpha_n(t) R_n(p_n(t), b_n(t); \hat{h}_n(t)) - \beta_n(t) - \nu_n(t)p, \tag{4.79}$$

$$q_n(t) = \mathbb{I}\left\{\left(\alpha_n(t) R_n(p_n(t), b_n(t); \hat{h}_n(t)) - \beta_n(t) - \nu_n(t) p_n(t)\right)\right\}. \tag{4.80}$$

The stochastic subgradients of the dual function are obtained by evaluating the instantaneous constraint violations using $\mathbf{p}_n(t)$ and $\mathbf{x}_n(t)$. The dual variables are then updated using the stochastic subgradient as

$$\alpha_n(t+1) = \left[\alpha_n(t) - \epsilon(t)\left[q_n(t) R_n(p_n(t), b_n(t); \hat{h}_n(t)) - x_n(t)\right]\right]^+, \tag{4.81}$$

$$\beta_n(t+1) = \left[\beta_n(t) - \epsilon(t)\left[y_n(t) - q_n(t)\right]\right]^+, \tag{4.82}$$

$$\nu_n(t+1) = \left[\nu_n(t) - \epsilon(t)\left[P_n - q_n(t) p_n(t)\right]\right]^+. \tag{4.83}$$

As in algorithms point-to-point channel and OFDM channel, the use of diminishing step sizes results in almost sure convergence whereas use of constant step sizes results in an ergodic mode of convergence that we summarize in the following property.

Property 3. *If constant step size $\epsilon(t) = \epsilon > 0$ is used in Algorithm 4, it follows from [16, Theorem 1] that primal variables generated by the algorithm are almost surely feasible and almost surely near optimal in an ergodic sense for problem* (4.63)*. In particular, the average power constraint in* (4.63) *is almost surely satisfied, i.e.,*

$$\lim_{t\to\infty} \frac{1}{t} \sum_{u=1}^{t} q_n(u) p_n(u) \le P_{0,n} \quad \textit{a.s.}, \tag{4.84}$$

and the utility of the ergodic limit of the transmission rates almost surely converges to a value within $\kappa\epsilon$ of optimal,

$$\mathsf{P}_\mathsf{r} - \sum_{n=1}^{N} \log\left(\lim_{t\to\infty} \frac{1}{t} \sum_{u=1}^{t} R_n\left(p_n(u), b_n(u); \hat{h}_n(u)\right) q_n(u) \prod_{m=1, m\neq n}^{N} [1 - q_m(u)]\right) \le \kappa\epsilon \quad \textit{a.s.}, \tag{4.85}$$

where κ is a constant upper bounding the second moment of the norm of the stochastic subgradient.

Note that the term $R_n\left(p_n(u), b_n(u); \hat{h}_n(u)\right) q_n(u) \prod_{m=1, m\neq n}^{N} [1 - q_m(u)]$ in (4.85) is different from the instantaneous transmission rate $r_n(u)$ in (4.56) achieved by the policy. To establish optimality results for the ergodic limits of the instantaneous transmission rate, we write the following relationship by using the definition of $R_n\left(p_n(u), b_n(u); \hat{h}_n(u)\right)$ and the ergodicity property of $h_n(\mathbb{N})$ given $\hat{h}_n(\mathbb{N})$

$$\begin{aligned}&\lim_{t\to\infty} \frac{1}{t} \sum_{u=1}^{t} R_n\left(p_n(u), b_n(u); \hat{h}_n(u)\right) q_n(u) \prod_{m=1, m\neq n}^{N} [1 - q_m(u)] \\ &= \lim_{t\to\infty} \frac{1}{t} \sum_{u=1}^{t} C\left(p_n(u), b_n(u)\right) \mathbb{I}\left\{b_n(u) < \hat{h}_n(u)\right\} q_n(u) \prod_{m=1, m\neq n}^{N} [1 - q_m(u)] = \lim_{t\to\infty} \frac{1}{t} \sum_{u=1}^{t} r_n(u) \quad \text{a.s.}.\end{aligned} \tag{4.86}$$

Substituting (4.86) into (4.85) we can show the sum logarithm of the ergodic limits of the instantaneous transmission rate $r_n(t)$ is within $\kappa\epsilon$ of the optimal value P_r, i.e.,

$$\mathsf{P}_\mathsf{r} - \sum_{n=1}^{N} \log\left(\lim_{t\to\infty} \frac{1}{t} \sum_{u=1}^{t} r_n(u)\right) \le \kappa\epsilon \quad \text{a.s.}, \tag{4.87}$$

The procedure is summarized in Algorithm 4. The algorithm initializes multipliers $\alpha_n(0), \beta_n(0)$ and $\nu_n(0)$ at time 0. In each time slot t, it iteratively computes primal variables $x_n(t)$, $y_n(t)$,

$p_n(t)$, $b_n(t)$, $q_n(t)$ by (4.76) - (4.80) (lines 4-7). If $\alpha_n(t)R_n(p_n(t), b_n(t); \hat{h}_n(t)) - \beta_n(t) - \nu_n(t)p_n(t)$ is greater than 0, $q_n(t)$ is set to 1 and terminal n transmits on frequency f using power $p_n(t)$ and rate $C(p_n(t), b_n(t))$. Dual variables $\alpha_n(t), \beta_n(t), \nu_n(t)$ are then computed according to (4.81) - (4.83) (lines 12-14). Note that while the algorithm for OFDM (see Algorithm 3) is applied to the system with n users the presented algorithm for RA is for each individual terminal. In RA channels, each terminal distributedly operates based on Algorithm 4. Since each terminal makes channel access decisions based only on its local imperfect CSI and channels for different terminals are assumed to be independent, terminals' actions are independent of each other.

4.4 Numerical results

The performance of the proposed algorithms is further evaluated through numerical tests. We consider point-to-point channels in Section 4.4.1, OFDM channels in Section 4.4.2, and random access channels in Section 4.4.3.

4.4.1 Point-to-point channel

We assume the real channel coefficient h follows a complex Gaussian distribution $\mathcal{CN}(0, 2)$ and the channel estimation error is modeled by (4.8). The average power budget is $P_0 = 1$ and the channel capacity function takes the form of $\log(1 + P(\hat{h})B(\hat{h})/N_0)$. Without loss of generality, we assume N_0 is normalized to 1.

In the first set of tests, two channel estimation error variances $\sigma_e^2 = 0.1$ and $\sigma_e^2 = 0.7$, corresponding to small and large channel errors, are simulated. We apply diminishing step size $\epsilon(t) = 1/\sqrt{t}$ to obtain the optimal dual variable λ^* for both cases and then find the optimal power allocation function $P^*(\hat{h})$ and the optimal channel backoff function $B^*(\hat{h})$ according to Theorem 4, as shown in Fig. 4.1. For comparison purposes, $P^*(\hat{h})$ and $B^*(\hat{h})$ for $\sigma_e^2 = 0$ are also depicted. For both small and large channel errors, the optimal power allocation functions are given by

Algorithm 4: Optimal channel access, power control and channel backoff for T_n in random access

Initialize Lagrangian multipliers $\alpha_n(0), \beta_n(0)$ and $\nu_n(0)$;

for $t = 0, 1, 2, \cdots$ **do**

  Compute primal variables as per (4.76) - (4.80):

  $$x_n(t) = \operatorname*{argmax}_{x \geq 0} \log x - \alpha_n(t) x = \frac{1}{\alpha_n(t)};$$

  $$y_n(t) = \operatorname*{argmax}_{0 \leq y \leq 1} (N-1)\log(1-y) + \beta_n(t) y = \left[1 - \frac{N-1}{\beta_n(t)}\right]^+;$$

  $$\{b_n(t), p_n(t)\} = \operatorname*{argmax}_{p \in [0, p^{\text{inst}}], b \geq 0} \alpha_n(t) R_n(p, b; \hat{h}_n(t)) - \beta_n(t) - \nu_n(t) p;$$

  $$a_n(t) = \mathbb{I}\left\{\Big(\alpha_n(t) R_n(p_n(t), b_n(t); \hat{h}_n(t)) - \beta_n(t) - \nu_n(t) p_n(t)\Big)\right\};$$

  if $q_n(t) = 1$ **then**

    Transmit using power $p_n(t)$ and rate $C(p_n(t), b_n(t))$;

  end

  Update dual variables as per (4.81) - (4.83):

  $$\alpha_n(t+1) = \left[\alpha_n(t) - \epsilon(t)\left[q_n(t) R_n(p_n(t), b_n(t); \hat{h}_n(t)) - x_n(t)\right]\right]^+;$$

  $$\beta_n(t+1) = \left[\beta_n(t) - \epsilon(t)\left[y_n(t) - q_n(t)\right]\right]^+;$$

  $$\nu_n(t+1) = \left[\nu_n(t) - \epsilon(t)\left[P_n - q_n(t) p_n(t)\right]\right]^+;$$

end

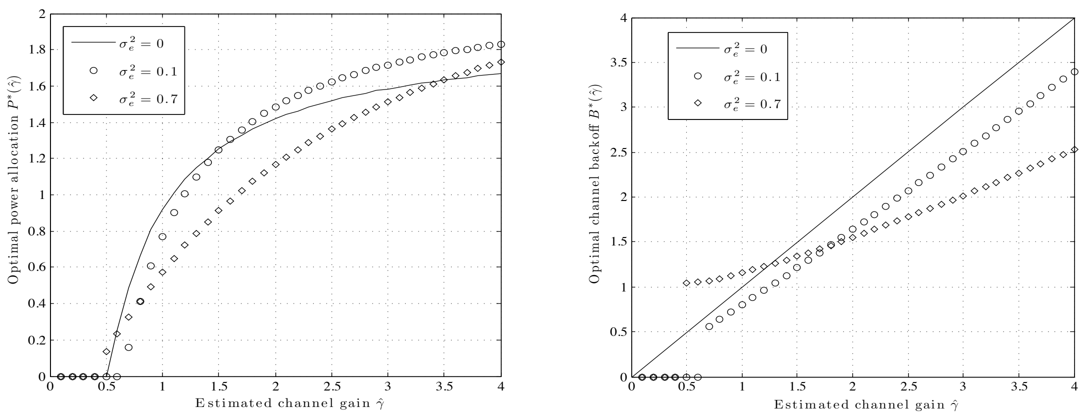


Figure 4.1: Optimal power allocation function $P^*(\hat{h})$ (left) and channel backoff function $B^*(\hat{h})$ (right) for single user point-to-point channel. Curves shown for channel state information (CSI) variance $\sigma_e^2 = 0.1$, $\sigma_e^2 = 0.1$, and $\sigma_e^2 = 0$, corresponding to perfect CSI. As CSI variance increases power allocation is more conservative for small channel values. When the CSI variance is large, the backoff function selects codes of a higher rate than what is dictated by the channel estimate. Channel coefficient follows a complex Gaussian distribution $\mathcal{CN}(0, 2)$, average power budget $P_0 = 1$, and channel conditional pdf $m_{h|\hat{h}}$ as in (4.9).

water-filling as in the case of perfect CSI. However, as the error in channel estimates increases, power is allocated more conservatively when channel gain estimates are small. The difference between the channel backoff functions for small and large channel errors are more significant. When $\sigma_e^2 = 0.1$, the channel backoff is almost linear and $B^*(\hat{h}) < \hat{h}$ for all $\hat{h}$, i.e. making $B^*(\hat{h})$ smaller is always beneficial. When $\sigma_e^2 = 0.7$ the channel backoff function is farther away from linear. It is interesting to note that $B^*(\hat{h}) > \hat{h}$ for small channel estimates $0.5 \leq \hat{h} \leq 1.2$. In that sense the use of the term backoff is a misnomer as it is actually beneficial to select a transmission mode more aggressive than what the channel estimate indicates. The intuition here is that when σ_e^2 is comparable to $\hat{h}$, it is likely that h is greater than $\hat{h}$ because we must have $h \geq 0$. Therefore, making $B(\hat{h})$ a little bigger than h is not likely to result in an outage.

In our next simulation, we test the algorithm with constant step size $\epsilon = 0.01$ and assuming channel error $\sigma_e^2 = 0.1$. Other parameters remain the same as before. We define the average transmission rate $\bar{r}(t) = (1/t)\sum_{u=1}^{t} C(p(u), b(u)) \cdot \mathbb{I}\{b(u) \leq h(u)\}$ and average power consumption $\bar{p}(t) = (1/t)\sum_{u=1}^{t} p(u)$. We compare average rates achieved by: 1) with both power allocation and channel backoff; 2) with channel backoff only (i.e., $p(t) = P_0$); 3) with power allocation only (i.e., $b(t) = \hat{h}(t)$). Fig. 4.2 (left) shows average rates achieved by these algorithms. There is a considerable improvement in average transmission rate when power allocation and channel backoff are jointly optimized. Furthermore, Fig. 4.2 (right) shows that the average power constraint is always satisfied, coinciding with the almost sure feasibility result in (4.22).

4.4.2 Downlink OFDM channel

To test Algorithm 3 for downlink OFDM channels, we assume that the number of users is $N = 8$ and that there are $|\mathcal{F}| = 4$ frequency tones available. As in the case of single user point-to-point channel, we model the complex channel coefficients h_n^f as random variables with complex Gaussian distributions $\mathcal{CN}(0, 2)$ and the channel estimation error as having a complex Gaussian distribution $\mathcal{CN}(0, \sigma_e^2)$ with $\sigma_e^2 = 0.1$ modeled by (4.8). The total average power budget

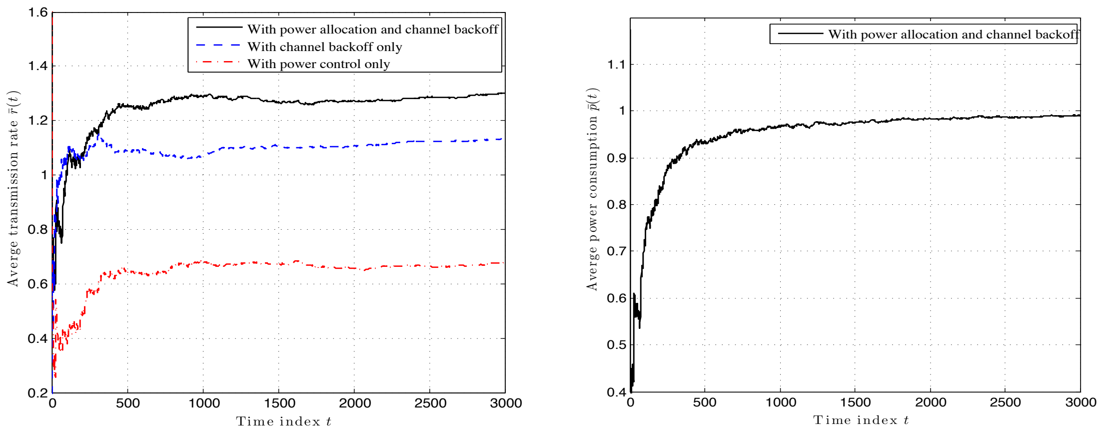


Figure 4.2: Convergence of average transmission rate (left) and average power consumption (right) for Algorithm 2. Average transmission rate as a function of time is shown for Algorithm 2 and cases in which only the backoff function is optimized – meaning $p(t) = P_0$ – or only the power allocation function is optimized – implying $b(t) = \hat{h}(t)$. Joint optimization yields substantial increase of average communication rate. Average power budget $P_0 = 1$, constant step size $\epsilon = 0.01$, and channel estimation error $\sigma_e^2 = 0.1$.

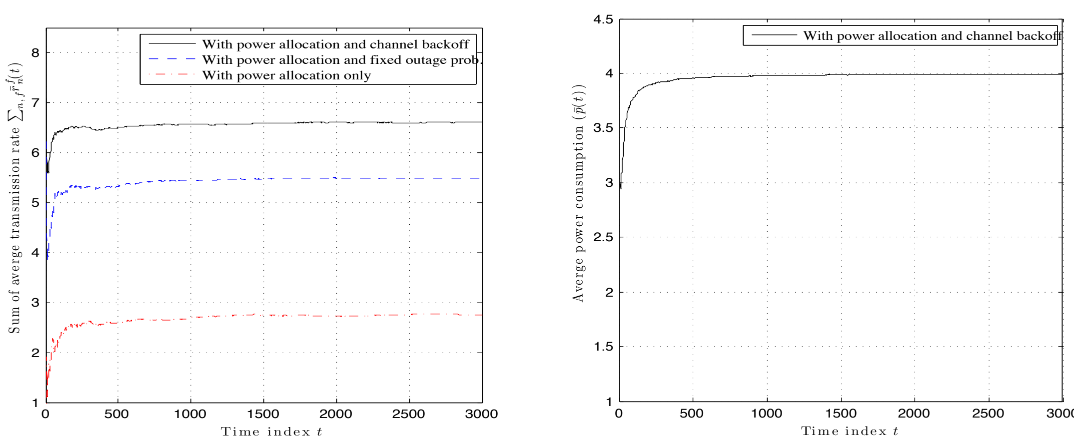


Figure 4.3: Rate (left) and power (right) convergence of Algorithm 3. Sum of average transmission rates is shown for Algorithm 3 and two suboptimal solutions. One case uses a backoff function with fixed outage probability 0.01 and the other case optimizes power allocation only – implying $b_n^f(t) = \hat{h}_n^f(t)$. Joint optimization yields substantial increase of average communication rate. Average power budget $P_0 = 4$, constant step size $\epsilon = 0.01$, and channel estimation error $\sigma_e^2 = 0.1$.

is $P_0 = 4$ and the channel capacity function takes the form of (1.1). Without loss of generality, we assume noise power is normalized to $N_0 = 1$. Sum utility $U_n(r_n) = r_n$ is used. We define the average utility as the sum of average transmission rates $\bar{U}(t) = \sum_{n=1}^{N} \bar{r}_n(t) = (1/t) \sum_{n=1}^{N} \sum_{f \in \mathcal{F}} \sum_{u=1}^{t} C(p_n^f(u), b_n^f(u)) \cdot \mathbb{I}\{b_n^f(u) \leq h_n^f(u)\}$ and the average total power consumption as the sum of average power allocated to each terminal $\bar{p}(t) = (1/t) \sum_{n=1}^{N} \sum_{f \in \mathcal{F}} \sum_{u=1}^{t} p_n^f(u)$. Average sum utility $\bar{U}(t)$ is shown in Fig. 4.3 (left) and average power $\bar{p}(t)$ is shown in Fig. 4.3 (right).

In addition to Algorithm 3, two alternative solutions are also implemented. For the first method, the value of the channel backoff function is chosen such that a fixed outage probability 0.01 is achieved [42], i.e., $b_n^f(t)$ is calculated such that $M_{h_n^f(t)|\hat{h}_n^f(t)}(b_n^f(t)) = 1 - 0.01 = 0.99$ for each observed $\hat{h}_n^f(t)$, and power is then allocated such that the average total power constraint is satisfied. For the second one, we do not perform any channel backoff, i.e. $b_n^f(t) = \hat{h}_n^f(t)$. We remark that both are suboptimal solutions since power allocation and channel backoff functions are not jointly optimized.

We run Algorithm 3 and these two suboptimal alternatives with constant step size $\epsilon(t) = 0.01$ and compare their performance in terms of average utility $\bar{U}(t)$. Fig. 4.3 (left) shows that the average utilities over 3000 time slots achieved by the proposed algorithm, the algorithm with fixed outage probability and the algorithm without channel backoff are 6.6, 5.5 and 2.8, respectively. By introducing channel backoff functions, there is a significant increase in average utility (6.6 vs. 2.8). This implies that channel backoff is indeed very important when dealing with imperfect CSI. Moreover, jointly optimizing power allocation and channel backoff results in 20% performance improvement (6.6 vs. 5.5). Fig. 4.3 (right) shows the total average power used by the proposed Algorithm 3. We see that the average power budget $P_0 = 4$ is satisfied.

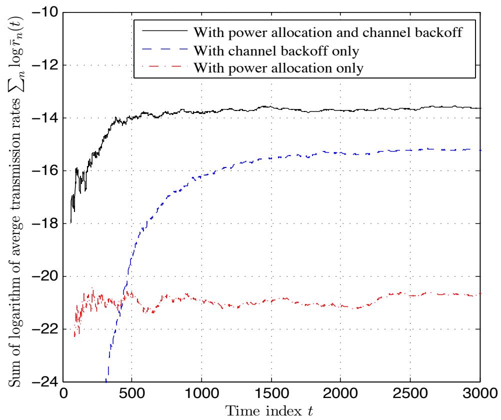


Figure 4.4: Rate (left) and power (right) convergence of Algorithm 4. Proportional fair utility of average transmission rates is shown for Algorithm 4 and two suboptimal solutions in which only the backoff function – meaning $p_n(t) = P_{0,n}$ – or only the power allocation function – implying $b_n(t) = \hat{h}_n(t)$ – are optimized. Joint optimization yields substantial increase of average communication rate. Average power budget $P_{0,n} = 1$ (right), constant step size $\epsilon = 0.01$, and channel estimation error $\sigma_e^2 = 0.1$.

4.4.3 Uplink RA channel

We run a set of simulations to test algorithms for the random access channel with imperfect CSI. Assume similar parameters as in the case of OFDM: $N = 8$, $N_0 = 1$, channel coefficient and channel estimation error modeled by complex Gaussian distributions $\mathcal{CN}(0, 2)$ and $\mathcal{CN}(0, 0.1)$, respectively. The power constraint for each terminal is set to $P_{0,n} = 1$.

The proposed Algorithm 4 is implemented in which channel access, power allocation and channel backoff functions are jointly optimized. Two other suboptimal solutions are also simulated: an alogrithm without power control – $p_n(t) = P_{0,n}$ is always constant – and an algorithm without channel backoff – $b_n(t) = \hat{h}_n(t)$ always equal to the real estimated channel gain. To compare their performance, define the average proportional fair utility as the sum of the logarithms of the average transmission rates, i.e., $\bar{U}(t) = \sum_{n=1}^{N} \log \bar{r}_n(t) = \sum_{n=1}^{N} \log(1/t) \sum_{u=1}^{t} r_n(u)$. Further define the average power consumption of each terminal as $\bar{p}_n(t) = (1/t) \sum_{u=1}^{t} p_n(u)$. Fig.

4.4 (left) compares the average proportional fair utility $\bar{U}(t)$ achieved by the three algorithms. The utility over 3000 time slots achieved by the proposed algorithm, the algorithm with fixed outage probability and the algorithm without channel backoff are -13.8, -15.3 and -20.7, respectively. Again, we observe that by jointly optimizing the channel access, power allocation and channel backoff the proposed algorithm achieves the highest utility. Moreover, Fig. 4.4 (right) shows that the average power budget for terminal 1 is satisfied. Note that the convergence rate of the algorithm for random access [cf. Fig. 4.4 left] is slower than the rate of OFDM [cf. Fig. 4.3 left]. This is because it takes longer to average out randomness in the case of RA since in OFDM the central decision maker has access to the channel of all users whereas in the case of RA each terminal only knows its own channel.

4.5 Summary

We considered optimal transmission over single user point-to-point channels, downlink OFDM channels and uplink RA channels with imperfect CSI in order to maximize expected transmission rates subject to average power constraints. For all cases we showed that the optimal solutions are determined by parameters in the form of the optimal multipliers of the Lagrange dual problem. We further developed stochastic subgradient descent algorithms on the dual domain that operate without knowledge of the channels' probability distributions. For vanishing step sizes these dual stochastic descent algorithms converge to the optimal multipliers. With constant step sizes optimal multipliers are not found but a policy that is optimal in an ergodic sense is determined. Numerical results showed significant performance gains of the proposed algorithms.

4.6 Appendices

4.6.1 Proof of null duality gap of problem (4.7)

To prove probem (4.7) has null duality gap, we introduce variable $c=\mathbb{E}_{\hat{h}}\Big[C(P(\hat{h}),B(\hat{h}))M_{h|\hat{h}}(B(\hat{h}))\Big]$ and rewrite problem (4.7) as

$$\begin{aligned}\mathsf{P_s} = \max \quad & c \\ \text{s.t. } & \mathbb{E}_{\hat{h}}\left[C(P(\hat{h}),B(\hat{h}))M_{h|\hat{h}}(B(\hat{h}))\right] \geq c, \\ & \mathbb{E}_{\hat{h}}\left[-P(\hat{h})\right] \geq -P_0,\end{aligned} \tag{4.88}$$

where we relaxed the first equality to inequality without loss of optimality. Further define $\mathbf{P}(\hat{h}) = [P(\hat{h}),B(\hat{h})]^T, \mathbf{f}_1(\hat{h},\mathbf{P}(\hat{h})) = [C(P(\hat{h}),B(\hat{h}))M_{h|\hat{h}}(B(\hat{h})),-P(\hat{h})]^T, \mathbf{x} = [c,-P_0]^T$ and $f_0(\mathbf{x}) = c$ and write (4.88) as

$$\begin{aligned}\mathsf{P_s} = \max\ & f_0(\mathbf{x}) \\ \text{s.t. } & \mathbb{E}_{\hat{h}}\left[\mathbf{f}_1(\hat{h},\mathbf{P}(\hat{h}))\right] - \mathbf{x} \geq 0.\end{aligned} \tag{4.89}$$

Note that problem (4.89) and (4.7) are equivalent. To establish zero duality gap, consider a perturbed version of (4.89)

$$\begin{aligned}\mathsf{P_s}(\boldsymbol{\delta}) = \max\ & f_0(\mathbf{x}) \\ \text{s.t. } & \mathbb{E}_{\hat{h}}\left[\mathbf{f}_1(\hat{h},\mathbf{P}(\hat{h}))\right] - \mathbf{x} \geq \boldsymbol{\delta},\end{aligned} \tag{4.90}$$

where we allow the constraint to be violated by $\boldsymbol{\delta}$. To prove that the duality gap for problem (4.89) is zero, it suffices to show that $\mathsf{P_s}(\boldsymbol{\delta})$ is a concave function of $\boldsymbol{\delta}$; see, e.g., [7, Sec. 6.2]. Let $\boldsymbol{\delta}$ and $\boldsymbol{\delta}'$ be a pair of perturbations, and $(\mathbf{P},\mathbf{x})$, $(\mathbf{P}',\mathbf{x}')$ be optimal solutions corresponding to the perturbations. Define $\boldsymbol{\delta}_\alpha = \alpha\boldsymbol{\delta} + (1-\alpha)\boldsymbol{\delta}'$ where $\alpha \in [0,1]$. We are interested in showing

$$\begin{aligned}\mathsf{P_s}(\boldsymbol{\delta}_\alpha) &= \mathsf{P_s}(\alpha\boldsymbol{\delta} + (1-\alpha)\boldsymbol{\delta}') \\ &\geq \alpha\mathsf{P_s}(\boldsymbol{\delta}) + (1-\alpha)\mathsf{P_s}(\boldsymbol{\delta}').\end{aligned} \tag{4.91}$$

To establish concavity of the perturbation function, we study properties of the expectation $\mathbb{E}_{\hat{h}}\left[\mathbf{f}_1(\hat{h}, \mathbf{P}(\hat{h}))\right]$. Define $\mathcal{Y}$ as a set that contains all possible values that $\mathbb{E}_{\hat{h}}\left[\mathbf{f}_1(\hat{h}, \mathbf{P}(\hat{h}))\right]$ can take, i.e., $\mathcal{Y} := \left\{\mathbf{y} : \exists \mathbf{P} \text{ for which } \mathbf{y} = \mathbb{E}_{\hat{h}}\left[\mathbf{f}_1(\hat{h}, \mathbf{P}(\hat{h}))\right]\right\}$. If channel pdf has no points of positive probability, then $\mathcal{Y}$ is convex [36, Theorem 3]. Therefore, there must exist $\mathbf{P}_\alpha(\hat{h})$ such that

$$\mathbb{E}_{\hat{h}}\left[\mathbf{f}_1(\hat{h}, \mathbf{P}_\alpha(\hat{h}))\right] = \alpha \mathbb{E}_{\hat{h}}\left[\mathbf{f}_1(\hat{h}, \mathbf{P}(\hat{h}))\right] + (1-\alpha)\mathbb{E}_{\hat{h}}\left[\mathbf{f}_1(\hat{h}, \mathbf{P}'(\hat{h}))\right]. \tag{4.92}$$

Since $\mathbf{P}(\hat{h})$ and $\mathbf{P}'(\hat{h})$ are feasible to problem (4.90), it follows that

$$\mathbb{E}_{\hat{h}}\left[\mathbf{f}_1(\hat{h}, \mathbf{P}(\hat{h}))\right] \geq \mathbf{x} + \boldsymbol{\delta}. \tag{4.93}$$

$$\mathbb{E}_{\hat{h}}\left[\mathbf{f}_1(\hat{h}, \mathbf{P}'(\hat{h}))\right] \geq \mathbf{x}' + \boldsymbol{\delta}'. \tag{4.94}$$

Substituting (4.93) and (4.94) into (4.92) yields

$$\begin{aligned}\mathbb{E}_{\hat{h}}\left[\mathbf{f}_1(\hat{h}, \mathbf{P}_\alpha(\hat{h}))\right] &\geq \alpha(\mathbf{x} + \boldsymbol{\delta}) + (1-\alpha)(\mathbf{x}' + \boldsymbol{\delta}') \\ &= \alpha\mathbf{x} + (1-\alpha)\mathbf{x}' + \boldsymbol{\delta}_\alpha. \end{aligned}\tag{4.95}$$

Define $\mathbf{x}_\alpha = \alpha\mathbf{x} + (1-\alpha)\mathbf{x}'$, then we have $\mathbb{E}_{\hat{h}}\left[\mathbf{f}_1(\hat{h}, \mathbf{P}_\alpha(\hat{h}))\right] \geq \mathbf{x}_\alpha + \boldsymbol{\delta}_\alpha$ implying that $\mathbf{P}_\alpha(\hat{h})$ and $\mathbf{x}_\alpha$ are feasible for problem (4.90) with perturbation $\boldsymbol{\delta}_\alpha$. In addition, since $f_0(\mathbf{x})$ is a linear function of $\mathbf{x}$ we have

$$f_0(\mathbf{x}_\alpha) = f_0(\alpha\mathbf{x} + (1-\alpha)\mathbf{x}') = \alpha f_0(\mathbf{x}) + (1-\alpha) f_0(\mathbf{x}'). \tag{4.96}$$

Since $(\mathbf{P}, \mathbf{x})$ is optimal for perturbation $\boldsymbol{\delta}$, we have $\mathsf{P}_\mathsf{s}(\boldsymbol{\delta}) = f_0(\mathbf{x})$, and likewise, $\mathsf{P}_\mathsf{s}(\boldsymbol{\delta}') = f_0(\mathbf{x}')$. Further note that the optimal solution $\mathsf{P}_\mathsf{s}(\boldsymbol{\delta}_\alpha)$ for perturbation $\boldsymbol{\delta}_\alpha$ must exceeds $f_0(\mathbf{x}_\alpha)$, we conclude that

$$\mathsf{P}_\mathsf{s}(\boldsymbol{\delta}_\alpha) \geq \alpha \mathsf{P}_\mathsf{s}(\boldsymbol{\delta}) + (1-\alpha)\mathsf{P}_\mathsf{s}(\boldsymbol{\delta}'). \tag{4.97}$$

(4.97) coincides with (4.91). This completes the proof since (4.97) holds for any $\boldsymbol{\delta}$ and $\boldsymbol{\delta}'$, and all $\alpha \in [0, 1]$.

Chapter 5

Cognitive access algorithms for wireless communications

Consider a multiple access fading channel in which terminals contend for communicating with a central access point. To exploit favorable channel conditions, terminals adapt their channel access as well as transmission power to the random states of the fading channel. The goal is to maximize the expected value of the sum transmission rates over all terminals subject to terminals' average power constraints. This problem has been studied extensively in the past and depending on the availability of the channel state information (CSI) the optimal solutions are different. When global CSI is available, i.e., each terminal knows the channel states of others, they can cooperate with each other to avoid collision. This is known as frequency-division multiple access (FDMA) in which the terminal with the largest channel state gets the opportunity to transmit, see e.g., [26, 35]. However, global CSI is usually not available in many practical scenarios and it is more reasonable to assume terminals only have access to local CSI. In this case, terminals make transmission decision and power allocation based on their local CSI without cooperating with each other. This is known as channel aware random access (RA), and it has been show the optimal solution is a threshold-based strategy, i.e., transmission is scheduled only when the local CSI exceeds certain threshold, see e.g., [16, 50].

FDMA and RA can be regarded as two special cases for multiple access channel where global CSI and local CSI are available for terminals. There are many other cases in between. For example, terminals may have some imperfect information about the global channel states. In other words, terminals are cognitive in the sense that each terminal has a different belief about the channel states. In this setting, it is natural to formulate the problem as a Bayesian game in which each terminal is a self-interested but rational player that maximizes the expected utility based on its belief. Bayesian game has been used to study random access channels in which each terminal knows the prior distributions of other channels [10, 47]. However, these algorithms cannot be generalized to the cognitive setting where beliefs change over terminals and time. This motivates us to develop cognitive access algorithms that determine an optimal allocation of resources taking into account the fact that different terminals have different beliefs on the channel state.

The rest of the chapter is organized as follows. In Section 5.1, we investigate multiple access channel where terminals have different beliefs about the channel states contending for communication with the central access point. We define the Bayesian game for this case and show that optimal solutions for FDMA and RA are Bayesian Nash Equilibrium (BNE) points of the game. Furthermore, we develop a cognitive access algorithm that finds solution for the Bayesian game approximately. These results are extended to the case of wireless networks in Section 5.2. Numerical results and summary are presented in Section 5.3 and 5.4, respectively.

5.1 Cognitive access algorithm for multiple access channel

Consider a multiple access channel as introduced in Section 1.1.1. Define the aggregate channel as $\mathbf{h}(t) := \{h_j(t)\}_{j=1}^n$ and the channel complement of terminal i as $\mathbf{h}_{-i}(t) := \{h_j(t)\}_{j=1, j\neq i}^n$. At time t, the state of the multiple access channel can be described by the aggregate channel $\mathbf{h}(t)$.

However, what terminals observe is not $\mathbf{h}(t)$ but an estimated version of $\mathbf{h}(t)$. In particular, from terminal i's perspective the channel state is $\tilde{\mathbf{h}}_i(t) := \{\tilde{h}_{ij}(t)\}_{j=1}^n$, where $\tilde{h}_{ij}(t)$ is terminal i's estimation of $h_j(t)$. We assume $\tilde{h}_{ij}$ has the same distribution as h_j and the accuracy of $\tilde{h}_{ij}$ is indicated by conditional probability distributions of h_j given $\tilde{h}_{ij}$, i.e., $f_{h_j|\tilde{h}_{ij}}(h)$ (see Remark 8). When the channel estimation is perfect, i.e., $\tilde{h}_{ij} = h_j$, it implies that $f_{h_j|\tilde{h}_{ij}}(h) = \delta(h - \tilde{h}_{ij})$. When $\tilde{h}_{ij}$ does not reveal any information about h_j, the conditional pdf $f_{h_j|\tilde{h}_{ij}}(h)$ is the same as the prior distribution of h_j, i.e. $f_{h_j|\tilde{h}_{ij}}(h) = f_{h_j}(h)$. The conditional pdf $f_{\mathbf{h}|\tilde{\mathbf{h}}_i}(\mathbf{h})$ is called terminal i's belief about the multiple access channel. We remark that for different terminals beliefs are different and the beliefs change over time. For future reference define the aggregate channel estimate as $\tilde{\mathbf{H}}(t) := \{\tilde{\mathbf{h}}_i(t)\}_{i=1}^n$ and the channel estimate complement of terminal i as $\tilde{\mathbf{h}}_{-i}(t) := \{\tilde{\mathbf{h}}_j(t)\}_{j=1, j\neq i}^n$.

We assume a backlogged system where all terminals have packets to transmit all the time. Upon observing its own channel estimate $\tilde{\mathbf{h}}_i(t)$ node i makes a decision on whether to transmit or not in the current time slot and if it chooses to do so it selects a transmit power for the communication attempt. Transmission decisions for node i are based on the attempt function $Q_i : \mathbb{R} \to \{0, 1\}$ and the power allocation function $P_i : \mathbb{R}^+ \to [0, p_i^{\text{inst}}]$, where $p_i^{\text{inst}} > 0$ is a limit on the instantaneous power transmitted by terminal i. Given the channel estimate $\tilde{\mathbf{h}}_i(t)$ terminal i makes a transmission attempt in time slot t if and only if $Q_i(\tilde{\mathbf{h}}_i(t)) = 1$ in which case it does so with power $P_i(\tilde{\mathbf{h}}_i(t))$. The pair of functions $\mathbf{P}_i := (Q_i, P_i)$ is termed the transmission strategy profile of terminal i. The joint strategy is defined as the grouping $\mathbf{P} := \{\mathbf{P}_j\}_{j=1}^n$ of all individual strategy profiles and the complementary strategy of terminal i as the grouping $\mathbf{P}_{-i} := \{\mathbf{P}_j\}_{j=1, j\neq i}^n$ of all strategies except the one of i. Observe that specifying the joint strategy $\mathbf{P}$ is equivalent to specifying the individual strategy $\mathbf{P}_i$ and the complementary strategy $\mathbf{P}_{-i}$.

A communication attempt with power $P_i(\tilde{\mathbf{h}}_i(t))$ when the channel is $h_i(t)$ proceeds at a rate $C[h_i(t), P_i(\tilde{\mathbf{h}}_i(t))]$, where $C : \mathbb{R}^+ \times [0, p_i^{\text{inst}}] \to \mathbb{R}^+$ is a function mapping channels and powers to transmission rates. Similar to the random access channel considered in Chapter 2, we assume a collision occurs if more than one terminal attempts transmission in the same time slot. Thus, user i is able to reach the AP at time t if and only if $Q_i(\tilde{\mathbf{h}}_i(t)) = 1$ and $Q_j(\tilde{\mathbf{h}}_j(t)) = 0$ for all $j \neq i$. Therefore, the instantaneous transmission rate for terminal i at time t is

$$r_i(h_i(t), \tilde{\mathbf{H}}(t), \mathbf{P}(\tilde{\mathbf{H}}(t))) \; = C\Big[h_i(t), P_i(\tilde{\mathbf{h}}_i(t))\Big] \times Q_i(\tilde{\mathbf{h}}_i(t)) \prod_{j=1, j\neq i}^{n} [1 - Q_j(\tilde{\mathbf{h}}_j(t))], \tag{5.1}$$

Furthermore, we define the instantaneous utility as the sum of instantaneous transmission rate, i.e.,

$$U(\mathbf{h}(t), \tilde{\mathbf{H}}(t), \mathbf{P}(\tilde{\mathbf{H}}(t))) = \sum_{i=1}^{n} r_i(h_i(t), \tilde{\mathbf{H}}(t), \mathbf{P}(\tilde{\mathbf{H}}(t))). \tag{5.2}$$

It then follows that the expected utility associated with policy $\mathbf{P}$ is given by

$$\bar{U}(\mathbf{P}) = \mathbb{E}_{\mathbf{h}, \tilde{\mathbf{H}}}\Bigg[U(\mathbf{h}, \tilde{\mathbf{H}}, \mathbf{P})\Bigg] = \mathbb{E}_{\mathbf{h}, \tilde{\mathbf{H}}}\Bigg[\sum_{i=1}^{n} r_i(h_i, \tilde{\mathbf{H}}, \mathbf{P})\Bigg]. \tag{5.3}$$

where we dropped the time index because the channel distribution is assumed stationary. Further note that each communication attempt, successful or not, incurs a power cost $P_i(\tilde{\mathbf{h}}_i)$. Therefore, the average power consumption of terminal i is the expectation $\mathbb{E}_{\tilde{\mathbf{h}}_i}\Big[Q_i(\tilde{\mathbf{h}}_i)P_i(\tilde{\mathbf{h}}_i)\Big]$ and in order to satisfy an average power budget p_i^{avg} we must have

$$\mathbb{E}_{\tilde{\mathbf{h}}_i}\Big[Q_i(\tilde{\mathbf{h}}_i)P_i(\tilde{\mathbf{h}}_i)\Big] \leq p_i^{\text{avg}}. \tag{5.4}$$

The goal of each terminal is to select the policy $\mathbf{P}_i$ that maximizes the average sum rate utility $\bar{U}(\mathbf{P})$ in (5.3) while satisfying the average power constraint in (5.4). However, terminal i lacks the information to do so. The rate $r_i(h_i, \tilde{\mathbf{H}}, \mathbf{P}(\tilde{\mathbf{H}}))$ attained by terminal i is dependent upon the scheduling $Q_i(\tilde{\mathbf{h}}_i)$ of all terminals [cf. (5.1)]. For terminal i to be able to solve this problem, it requires policies $\mathbf{P}_{-i}$ and channel estimations $\tilde{\mathbf{H}}_{-i}$ of all other terminals. Since neither $\mathbf{P}_{-i}$ nor $\tilde{\mathbf{H}}_{-i}$ is available to terminal i, it is impossible for terminal i to solve the problem without having access to other terminals' policies $\mathbf{P}_{-i}$ and channel estimations $\tilde{\mathbf{H}}_{-i}$. In this chapter, we aim at developing algorithms that allow terminals solve this problem approximately without cooperating with each other.

Remark 8. The probability distribution $f_{h_j|\tilde{h}_{ij}}(h)$ depends on the channel estimation method. A typical way is to assume that $\tilde{h}_{ij}$ is an outdated version of h_j modeled by an order-1 autoregressive (AR) process. For example, suppose h_j is complex Gaussian with pdf $\mathcal{CN}(0,2)$, then the estimation can be modeled by $h_j = \rho\tilde{h}_{ij} + e_j$ where ρ is the correlation coefficient between h_j and $\tilde{h}_{ij}$ and e_j is complex Gaussian random noise with pdf $\mathcal{CN}(0, 1-\rho^2)$. In this case, $f_{h_j|\tilde{h}_{ij}}(h)$ is a noncentral chi-square distribution [28]

$$f_{h_j|\tilde{h}_{ij}}(h) = \frac{1}{2(1-\rho^2)} \exp\left(-\frac{h + \rho^2\tilde{h}_{ij}}{2(1-\rho^2)}\right) I_0\left(\frac{\rho^2\sqrt{h\tilde{h}_{ij}}}{(1-\rho^2)}\right), \tag{5.5}$$

where $I_0(x) = \sum_{i=0}^{\infty}(x^2/4)^i/(i!)^2$ is the zeroth order modified Bessel function of the first kind. This particular form for the conditional pdf $f_{h_j|\tilde{h}_{ij}}(h)$ is used to provide numerical results in Section 5.3. The rest of the development in the chapter holds independently of the particular form of this pdf.

5.1.1 Multiple access channel without power control

Bayesian Nash Equilibrium

Let us first consider the case without power control, i.e. $P_i(\tilde{h}_i)$ is a constant for all terminals. In this case, we can ignore the power control functions $P_i(\tilde{\mathbf{h}}_i)$ and the instantaneous transmission rate for terminal i can be simplified to

$$\begin{aligned} r_i(h_i, \tilde{\mathbf{H}}, \mathbf{P}) &= C[h_i]Q_i(\tilde{\mathbf{h}}_i) \prod_{j=1, j\neq i}^{n} [1 - Q_j(\tilde{\mathbf{h}}_j)] \\ &= r_i(h_i, \tilde{\mathbf{h}}_i, \tilde{\mathbf{h}}_{-i}, Q_i, Q_{-i}), \end{aligned} \tag{5.6}$$

where the second equality holds true because $\tilde{\mathbf{H}} = \{\tilde{\mathbf{h}}_i, \tilde{\mathbf{h}}_{-i}\}$. From terminal i's perspective, we can rewrite the instantaneous sum rate utility as $U(\mathbf{h}, \tilde{\mathbf{h}}_i, \tilde{\mathbf{h}}_{-i}, Q_i, Q_{-i})$ and the expected sum rate utility as

$$\bar{U}(Q_i, Q_{-i}) = \mathbb{E}_{\mathbf{h}, \tilde{\mathbf{h}}_i, \tilde{\mathbf{h}}_{-i}}\left[U(\mathbf{h}, \tilde{\mathbf{h}}_i, \tilde{\mathbf{h}}_{-i}, Q_i, Q_{-i})\right]$$

$$= \mathbb{E}_{\tilde{\mathbf{h}}_i} \left[\mathbb{E}_{\mathbf{h},\tilde{\mathbf{h}}_{-i}|\tilde{\mathbf{h}}_i} \left[U(\mathbf{h}, \tilde{\mathbf{h}}_i, \tilde{\mathbf{h}}_{-i}, Q_i, Q_{-i}) \right] \right], \tag{5.7}$$

where the second equality follows from the fact that joint pdf equals to the product of prior pdf and conditional pdf, i.e., $f(\mathbf{h}, \tilde{\mathbf{h}}_i, \tilde{\mathbf{h}}_{-i}) = f(\tilde{\mathbf{h}}_i) \cdot f(\mathbf{h}, \tilde{\mathbf{h}}_{-i}|\tilde{\mathbf{h}}_i)$. The outer expectation in (5.7) is taken over terminal i's observations $\tilde{\mathbf{h}}_i$ while the inner expectation in (5.7) is with respect to terminal i's belief about the global channel states and beliefs of other terminals, i.e., $\mathbf{h}$ given $\tilde{\mathbf{h}}_i$ and $\tilde{\mathbf{h}}_{-i}$ given $\tilde{\mathbf{h}}_i$. Since we consider the case without power control, the average power constraint can be ignored [cf. (5.4)]. As a result, maximizing the expected sum rate utility $\bar{U}(Q_i, Q_{-i})$ is equivalent to maximizing the inner expectation $\mathbb{E}_{\mathbf{h},\tilde{\mathbf{h}}_{-i}|\tilde{\mathbf{h}}_i} \left[U(\mathbf{h}, \tilde{\mathbf{h}}_i, \tilde{\mathbf{h}}_{-i}, Q_i, Q_{-i}) \right]$ for every given $\tilde{\mathbf{h}}_i$. However, terminal i still lacks information to do so because the expectation depends on policies of other terminals. Since terminals are not allowed to cooperate with each other, each terminal has to make transmission decisions independently. This situation can be modeled as a game with incomplete information where each terminal makes decisions on its own belief about channels while receives a global utility.

Suppose terminal i's belief $\tilde{\mathbf{h}}_i$ and other terminals' policies Q_{-i} are given, the *best response* terminal i can take is to maximize the expected value of the sum rate utility based on its belief

$$Q_i^{\mathrm{BR}}(\tilde{\mathbf{h}}_i, Q_{-i}) := \underset{Q_i \in \{0,1\}}{\operatorname{argmax}} \, \mathbb{E}_{\mathbf{h},\tilde{\mathbf{h}}_{-i}|\tilde{\mathbf{h}}_i} \left[U(\mathbf{h}, \tilde{\mathbf{h}}_i, \tilde{\mathbf{h}}_{-i}, Q_i, Q_{-i}) \right]. \tag{5.8}$$

In this game, each terminal solves an optimization problem like (5.8). The equilibrium point of the game is defined by the following:

Definition 1. *For multiple access channel without power control, q_i^{BNE} is Bayesian Nash Equilibrium (BNE) if for all i the following holds true*

$$Q_i^{\mathrm{BNE}}(\tilde{\mathbf{h}}_i) = \underset{Q_i \in \{0,1\}}{\operatorname{argmax}} \, \mathbb{E}_{\mathbf{h},\tilde{\mathbf{h}}_{-i}|\tilde{\mathbf{h}}_i} \left[U(\mathbf{h}, \tilde{\mathbf{h}}_i, \tilde{\mathbf{h}}_{-i}, Q_i, Q_{-i}^{\mathrm{BNE}}) \right]. \tag{5.9}$$

At BNE, every terminal plays *best response* to the policies of other terminals. A natural question arises is how to obtain solutions for BNE. Let us first investigate two special cases when terminals have perfectly correlated beliefs ($f_{h_j|\tilde{h}_{ij}}(h) = \delta(h - \tilde{h}_{ij})$ for all j) and uncorrelated beliefs ($f_{h_j|\tilde{h}_{ij}}(h) = f_{h_j}(h)$ for all $j \neq i$).

When terminals' belief about others is perfect, this implies that each terminal has access to the global channel state, i.e., $\tilde{\mathbf{h}}_i = \mathbf{h}$ for all i. By using global CSI, terminals can cooperate with each other so that collision can be avoided. To do so, we introduce a constraint $\sum_{i=1}^n Q_i(\mathbf{h}) \leq 1$ which allows only at most one terminal to transmit in each time slot. Since collision is avoided, the instantaneous rate r_i can be rewritten as $Q_i(\mathbf{h})C(h_i)$. As a result, each terminal can solve the following global optimization problem locally

$$\begin{aligned} Q_i = &\operatorname{argmax} \mathbb{E}_{\mathbf{h}} \left[\sum_{i=1}^n Q_i(\mathbf{h}) C(h_i) \right] \\ &\text{s.t.} \sum_{i=1}^n Q_i(\mathbf{h}) \leq 1 \end{aligned} \tag{5.10}$$

Note that problem (5.10) is the optimal FDMA with single carrier and without average power constraint. To maximize the expectation in (5.10), it is equivalent to maximizing $\sum_{i=1}^n Q_i(\mathbf{h})C(h_i)$ for any given channel realization $\mathbf{h}$. Since at most one $Q_i(\mathbf{h})$ can be 1 for any given $\mathbf{h}$, the optimal solution is to set $Q_i(\mathbf{h}) = 1$ if the instantaneous transmission rate achieved by terminal i is greater than that of all other terminals, i.e., $C(h_i) > \max_{j\neq i} C(h_j)$. Therefore, the optimal solution $Q_i^{\text{FDMA}}(\mathbf{h})$ can be expressed as

$$Q_i^{\text{FDMA}}(\mathbf{h}) = H\left(C(h_i) > \max_{j\neq i} C(h_j) \right) \tag{5.11}$$

where $H(\cdot)$ is the Heaviside function. We show this solution has the following property.

Proposition 2. *For multiple access channel without power control, suppose terminals have perfect beliefs about the channel states, i.e.,* $f_{h_j|\tilde{h}_{ij}}(h) = \delta(h - \tilde{h}_{ij})$ *or* $\tilde{\mathbf{h}}_i = \mathbf{h}$, *then* Q_i^{FDMA} *obtained by* (5.11) *that solves problem* (5.10) *is BNE of the game as is defined by* (5.9).

Proof. When terminals have perfect beliefs about the channel states, we have $\tilde{\mathbf{h}}_i = \mathbf{h}$ for all i. As a result, the conditional pdfs $f(\mathbf{h}|\tilde{\mathbf{h}}_i)$ and $f(\tilde{\mathbf{h}}_j|\tilde{\mathbf{h}}_i)$ in (5.9) are delta functions and the BNE in this case is given by

$$Q_i^{\text{BNE}}(\mathbf{h}) = \underset{Q_i \in \{0,1\}}{\operatorname{argmax}} U(\mathbf{h}, Q_i, Q_{-i}^{\text{BNE}}). \tag{5.12}$$

We need to show that $Q_i^{\text{FDMA}}(\mathbf{h})$ obtained by (5.11) satisfies (5.12) for all i. Suppose for a given $\mathbf{h}$, h_i is the largest among all channels, i.e., $h_i > \max_{j \neq i} h_j$. According to (5.11), the optimal solution for FDMA is $Q_i^{\text{FDMA}}(\mathbf{h}) = 1$ and $Q_j^{\text{FDMA}}(\mathbf{h}) = 0$ for all $j \neq i$. It can be easily verified that $Q_i^{\text{FDMA}}(\mathbf{h})$ and $Q_j^{\text{FDMA}}(\mathbf{h})$ satisfies (5.12) for all i. □

We proceed to the case where terminals have uncorrelated beliefs. In this case, terminals have perfect knowledge about its local channel but only have prior knowledge about channels of others, i.e., $f_{h_i|\tilde{h}_{ii}}(h) = \delta(h - \tilde{h}_{ii})$ and $f_{h_j|\tilde{h}_{ij}}(h) = f_{h_j}(h)$. Since $\tilde{h}_{ij}$ does not reveal any information about h_j, we can write the policy of terminal i as a function of its local channel h_i only, i.e., $Q_i(h_i)$. As a result, the expected sum rate utility in (5.8) can be written as

$$\begin{aligned} \mathbb{E}_{\mathbf{h},\tilde{\mathbf{h}}_{-i}|\tilde{\mathbf{h}}_i}\left[U(\mathbf{h}, \tilde{\mathbf{h}}_i, \tilde{\mathbf{h}}_{-i}, Q_i, Q_{-i})\right] &= \mathbb{E}_{\mathbf{h}_{-i}}\left[U(\mathbf{h}, Q_i, Q_{-i})\right] \\ &= \mathbb{E}_{\mathbf{h}_{-i}}\left[\sum_{j=1}^{n} C(h_j) Q_j(h_j) \prod_{k=1, k\neq j}^{n} [1 - q_k(h_k)]\right]. \end{aligned} \tag{5.13}$$

Given h_i and Q_{-i}, the best response terminal i can take is to maximize the expected value of sum rate utility in (5.13)

$$Q_i^{\text{BR}}(h_i, Q_{-i}) := \underset{Q_i \in \{0,1\}}{\operatorname{argmax}} \mathbb{E}_{\mathbf{h}_{-i}}\left[\sum_{j=1}^{n} C(h_j) Q_j(h_j) \prod_{k=1, k\neq j}^{n} [1 - q_k(h_k)]\right]. \tag{5.14}$$

We show in this case the BNE is a threshold policy.

Proposition 3. *For multiple access channel without power control, suppose terminals have perfect knowledge about its local channel but only have prior knowledge about channels of others, i.e.,* $f_{h_i|\tilde{h}_{ii}}(h) = \delta(h - \tilde{h}_{ii})$ *and* $f_{h_j|\tilde{h}_{ij}}(h) = f_{h_j}(h)$*, then BNE of the game is a threshold policy.*

Proof. Since all terminals play best response at BNE, in order to show BNE is a threshold policy we need to show the best response defined by (5.14) is a threshold policy. To do so, we rewrite the expectation in (5.14) as

$$\begin{aligned}
&\mathbb{E}_{\mathbf{h}_{-i}}\left[\sum_{j=1}^{n} C(h_j)Q_j(h_j)\prod_{k=1,k\neq j}^{n}[1-q_k(h_k)]\right] \\
&= Q_i(h_i)C(h_i)\mathbb{E}_{\mathbf{h}_{-i}}\left[\prod_{j=1,j\neq i}^{n}[1-Q_j(h_j)]\right] \\
&\quad + [1-Q_i(h_i)]\mathbb{E}_{\mathbf{h}_{-i}}\left[\sum_{j=1,j\neq i}^{n}\left[C_j(h_j)Q_j(h_j)\prod_{k=1,k\neq j,k\neq i}^{n}[1-q_k(h_k)]\right]\right] \\
&:= Q_i(h_i)C(h_i)S_1(Q_{-i}) + [1-Q_i(h_i)]S_2(Q_{-i}),
\end{aligned} \tag{5.15}$$

where we defined expectations in the second equality as two functions of Q_{-i}, i.e., $S_1(Q_{-i})$ and $S_2(Q_{-i})$, respectively. Since $Q_i(h_i)$ can only take 0 or 1, and the expected sum rate utility equals to $C(h_i)S_1(Q_{-i})$ or $S_2(Q_{-i})$ when $Q_i(h_i)$ is 1 or 0, respectively. Therefore, to maximize $Q_i(h_i)C(h_i)S_1(Q_{-i}) + [1-Q_i(h_i)]S_2(Q_{-i})$ we just need to set $Q_i(h_i)$ to 1 if and only if $C(h_i)S_1(Q_{-i}) > S_2(Q_{-i})$, i.e.,

$$Q_i^{\text{BR}}(h_i, Q_{-i}) = H(C(h_i) > S_2(Q_{-i})/S_1(Q_{-i})). \tag{5.16}$$

Since channel capacity $C(h_i)$ is a nondecreasing function of h_i, there must exist a constant $h_{i,0} > 0$ such that $C(h_i) > S_2(Q_{-i})/S_1(Q_{-i})$ if and only if $h_i > h_{i,0}$. Therefore, the best response in (5.16) is equivalent to

$$Q_i^{\text{BR}}(h_i, Q_{-i}) = H(h_i > h_{i,0}), \tag{5.17}$$

where $h_{i,0}$ is a threshold. This completes the proof. □

When terminals have access to local CSI and operate independently without cooperating with each other, this is known as channel-aware random access, see e.g. [29,50]. For symmetric channel, the optimal solution is shown to be a threshold strategy and the optimal threshold h_0 is given by the solution to the following equation [50]

$$(n-1)\int_{h_0}^{\infty} C(h)f(h)dh = C(h_0)\int_0^{h_0} f(h)dh. \tag{5.18}$$

Therefore, the solution for optimal random access for symmetric channel is given by

$$Q_i^{\mathrm{RA}}(h_i) = H(h_i > h_0). \tag{5.19}$$

We show that $Q_i^{\mathrm{RA}}(h_i)$ obtained by (5.19) is the BNE of the game as is defined by (5.9).

Proposition 4. *For multiple access channel without power control, suppose channel is symmetric and terminals have perfect knowledge about its local channel but only have prior knowledge about channels of others, i.e.,* $f_{h_i|\tilde{h}_{ii}}(h) = \delta(h - \tilde{h}_{ii})$ *and* $f_{h_j|\tilde{h}_{ij}}(h) = f_{h_j}(h)$*, then the optimal random access policy* Q_i^{RA} *obtained by* (5.19) *is BNE of the game defined by* (5.9).

Proof. We have shown that the BNE is a threshold policy. For symmetric channel, the threshold for different terminals should be same which we assume to be $\hat{h}_0$. We need to show $\hat{h}_0$ is equal to h_0 given by the solution of (5.18). According to (5.16), we know that the following equality holds true when $h_i = \hat{h}_0$

$$C(\hat{h}_0) S_1(Q_{-i}^{\mathrm{BNE}}) = S_2(Q_{-i}^{\mathrm{BNE}}), \tag{5.20}$$

where $S_1(Q_{-i}^{\mathrm{BNE}})$ is given by

$$\begin{aligned} S_1(Q_{-i}^{\mathrm{BNE}}) &= \prod_{j=1, j\neq i}^{n} \mathbb{E}_{h_j}\left[1 - Q_j(h_j)\right] \\ &= \left(\int_0^{\hat{h}_0} f(h) dh \right)^{n-1}, \end{aligned} \tag{5.21}$$

where the first equality follows from the fact that h_i and h_j are independent for $i \neq j$ and for the same reason $S_2(Q_{-i}^{\mathrm{BNE}})$ can be expressed as

$$\begin{aligned} S_2(Q_{-i}^{\mathrm{BNE}}) &= \sum_{j=1, j\neq i}^{n} \mathbb{E}_{h_j}\left[C_j(h_j) Q_j(h_j)\right] \prod_{k=1, k\neq j, k\neq i}^{n} \mathbb{E}_{h_k}\left[1 - q_k(h_k)\right] \\ &= \sum_{j=1, j\neq i}^{n} \int_{\hat{h}_0}^{\infty} C(h) f(h) dh \left(\int_0^{\hat{h}_0} f(h) dh \right)^{n-2} \\ &= (n-1) \int_{\hat{h}_0}^{\infty} C(h) f(h) dh \left(\int_0^{\hat{h}_0} f(h) dh \right)^{n-2}. \end{aligned} \tag{5.22}$$

Substitute (5.21) and (5.22) into (5.20) we have

$$C(\hat{h}_0)\int_0^{\hat{h}_0} f(h)dh = (n-1)\int_{\hat{h}_0}^{\infty} C(h)f(h)dh, \tag{5.23}$$

which coincides with (5.18) implying $\hat{h}_0 = h_0$. This completes the proof. $\square$

In summary, when terminals have perfectly correlated and uncorrelated beliefs about other terminals it is possible to formulate the problem as FDMA or RA and find corresponding optimal solutions. Interestingly, optimal solutions for these two different problems both coincide with the BNE defined by (5.9). In other words, BNE can be used as a unified framework to model multiple access channels. Indeed, from an individual terminal's point of view the only difference between FDMA and RA is the knowledge about other channels which is captured by the belief in BNE. However, for intermediate cases where beliefs are neither perfectly correlated nor uncorrelated finding the BNE solution is not an easy task because the objective in (5.9) evolves policies of other terminals which is unknown to terminal i. Next, we develop algorithms that solve (5.9) approximately.

Cognitive access algorithm

The key in designing an algorithm for solving (5.9) is to model the actions of other terminals. Let $\tilde{Q}_{ij}(\cdot)$ be the modeling of terminal j's action from terminal i's perspective. The easiest way of modeling terminal j is to assume that $\tilde{h}_{ij}$ is the true channel gain and terminal j makes decision based on $\tilde{\mathbf{h}}_i$, i.e., $\tilde{Q}_{ij}(\tilde{\mathbf{h}}_i)$. The intuition behind this modeling is that each terminal finds a strategy that is optimal when their belief about other terminals are perfect as in the case of optimal FDMA (5.10). As a result, terminal i solves the following problem locally

$$\begin{aligned}\{Q_i^{\text{CA}}, \tilde{Q}_{-i}^{\text{CA}}\} = &\max \mathbb{E}_{\tilde{h}_{ii}}\left[Q_i(\tilde{\mathbf{h}}_i)C(\tilde{h}_{ii})\right] + \sum_{j=1,j\neq i}^{n} \mathbb{E}_{\tilde{h}_{ij}}\left[\tilde{Q}_{ij}(\tilde{\mathbf{h}}_i)C(\tilde{h}_{ij})\right] \\ &\text{s.t.} \quad Q_i(\tilde{\mathbf{h}}_i) + \sum_{j=1,j\neq i}^{n} \tilde{Q}_{ij}(\tilde{\mathbf{h}}_i) \leq 1.\end{aligned} \tag{5.24}$$

Note that problem (5.24) is the same as the one for FDMA (5.10) except that q_j is replaced by $\tilde{Q}_{ij}$. The optimal solution for (5.24) is given by

$$Q_i^{\mathrm{CA}}(\tilde{\mathbf{h}}_i) = H\left(C(\tilde{h}_{ii}) > \max_{j \neq i} C(\tilde{h}_{ij})\right), \tag{5.25}$$

In practice, terminal i makes transmission according to (5.25). When terminal i is scheduled for transmission, it assumes other terminals are silent, i.e., $\tilde{Q}_{ij} = 0$ for all $j \neq i$, and the attained transmission rate is $C(h_i)$. However, this may not be true since the modeled action $\tilde{Q}_{ij}$ may be different from the real action Q_j of terminal j which is obtained by terminal j solving another maximization problem like (5.24). Therefore, collision may happen when all terminals operate by following (5.25). However, we can show that the performance achieved by this algorithm is the same as FDMA when channel is perfect and is nearly good as RA when channel is uncorrelated. Let the expected utilities achieved by Q_i^{CA} and Q_i^{FDMA} are $\bar{U}^{\mathrm{CA}}$ and $\bar{U}^{\mathrm{FDMA}}$, respectively. We show that the performance of the proposed cognitive access algorithm for the following two cases.

Proposition 5. *If terminals have perfectly correlated belief, i.e., $f_{h_j|\tilde{h}_{ij}}(h) = \delta(h - \tilde{h}_{ij})$, the expected utility achieved by Q_i^{CA} is the same as that of FDMA, i.e., $\bar{U}^{\mathrm{CA}} = \bar{U}^{\mathrm{FDMA}}$.*

Proof. When terminals have perfectly correlated belief, $\tilde{h}_{ij} = h_j$. This implies that the problem (5.24) solved by the proposed algorithm is the same as the problem (5.10) solved by FDMA. Therefore, the performance achieved by both algorithms are identical, i.e., $\bar{U}^{\mathrm{CA}} = \bar{U}^{\mathrm{FDMA}}$. □

Proposition 6. *If terminals have perfect correlated belief about its local channel, i.e., $f_{h_i|\tilde{h}_{ii}}(h) = \delta(h - \tilde{h}_{ii})$, and uncorrelated belief about channels of other terminals, i.e., $f_{h_j|\tilde{h}_{ij}}(h) = f_{h_j}(h)$, the expected utility achieved by Q_i^{CA} is a fraction of that of FDMA, i.e., $\bar{U}^{\mathrm{CA}} = \beta\bar{U}^{\mathrm{FDMA}}$, where $\beta \in [0,1]$ is a constant. In particular, when channels are symmetric, $\beta = 1/e$ as the number of terminals goes to infinity.*

Proof. In both FDMA and the proposed algorithm, transmission decision for each terminal is made by comparing its local channel with maximum of the channels of others [cf. (5.11) and (5.25)]. Therefore, given local channel h_i terminal i transmits with certain probability under both policies. Let $\alpha_i^{\mathrm{FDMA}}(h_i)$ and $\alpha_i^{\mathrm{CA}}(h_i)$ be the probability that terminal i transmit in the proposed algorithm and FDMA, respectively. According to (5.11) and (5.25), $\alpha_i^{\mathrm{FDMA}}(h_i)$ and $\alpha_i^{\mathrm{CA}}(h_i)$ are given by

$$\alpha_i^{\mathrm{FDMA}}(h_i) = \Pr\left(C(h_i) > \max_{j\neq i} C(h_j)\right), \tag{5.26}$$

$$\alpha_i^{\mathrm{CA}}(h_i) = \Pr\left(C(h_i) > \max_{j\neq i} C(\tilde{h}_{ij})\right). \tag{5.27}$$

Since $\tilde{h}_{ij}$, h_j have the same distribution, we conclude that $\alpha_i^{\mathrm{CA}}(h_i) = \alpha_i^{\mathrm{FDMA}}(h_i) = \alpha_i(h_i)$. As a result, the expected utility achieved by Q_i^{FDMA} is

$$\bar{U}^{\mathrm{FDMA}} = \sum_i \mathbb{E}_{h_i}\left[C(h_i)\alpha_i(h_i)\right], \tag{5.28}$$

and the expected utility achieved by Q_i^{CA} is

$$\bar{U}^{\mathrm{CA}} = \sum_i \mathbb{E}_{h_i}[C(h_i)\alpha_i(h_i)] \prod_{j\neq i}\left[1 - \mathbb{E}_{h_j}[\alpha_j(h_j)]\right], \tag{5.29}$$

where the product $\prod_{j\neq i}\left[1 - \mathbb{E}_{h_j}[\alpha_j(h_j)]\right]$ in (5.29) represents the probability that all terminals other than terminal i are silient. Define $\beta_i := \prod_{j\neq i}\left[1 - \mathbb{E}_{h_j}[\alpha_j(h_j)]\right]$ and rewrite (5.29) as

$$\bar{U}^{\mathrm{CA}} = \sum_i \beta_i \mathbb{E}_{h_i}[C(h_i)\alpha_i(h_i)]. \tag{5.30}$$

Let $\beta_{\min} = \min_i \beta_i$ and $\beta_{\max} = \max_i \beta_i$, then there must exist $\beta \in [\beta_{\min}, \beta_{\max}]$ such that

$$\bar{U}^{\mathrm{CA}} = \beta \sum_i \mathbb{E}_{h_i}[C(h_i), P_i^{\mathrm{CA}}(h_i))\alpha_i(h_i)]. \tag{5.31}$$

Substitute (5.28) into (5.31) yields

$$\bar{U}^{\mathrm{CA}} = \beta \bar{U}^{\mathrm{FDMA}}. \tag{5.32}$$

When the channels are symmetric, it can be show that $\beta = (1 - 1/n)^{n-1}$. Since $\lim_{n\to\infty}(1 - 1/n)^{n-1} = 1/e$, $\bar{U}^{\mathrm{CA}}$ goes to $1/e \cdot \bar{U}^{\mathrm{FDMA}}$ as n goes to infinity. $\square$

5.1.2 Multiple access channel with power control

Bayesian Nash Equilibrium

We now proceed to the case with power control. Similar to what we did in the case without power control, we rewrite the expected sum rate utility as

$$\bar{U}(\mathbf{P}_i, \mathbf{P}_{-i}) = \mathbb{E}_{\tilde{\mathbf{h}}_i}\left[\mathbb{E}_{\mathbf{h},\tilde{\mathbf{h}}_{-i}|\tilde{\mathbf{h}}_i}\left[U(\mathbf{h}, \tilde{\mathbf{h}}_i, \tilde{\mathbf{h}}_{-i}, \mathbf{P}_i, \mathbf{P}_{-i})\right]\right], \tag{5.33}$$

Suppose other terminals' policies $\mathbf{P}_{-i}$ are given, the *best response* terminal i can take is to maximize the expected value of the sum rate utility based on its belief subject to the average power constraint

$$\begin{aligned} \mathbf{P}_i^{\text{BR}}(\mathbf{P}_{-i}) := \operatorname{argmax} \mathbb{E}_{\tilde{\mathbf{h}}_i}\left[\mathbb{E}_{\mathbf{h},\tilde{\mathbf{h}}_{-i}|\tilde{\mathbf{h}}_i}\left[U(\mathbf{h}, \tilde{\mathbf{h}}_i, \tilde{\mathbf{h}}_{-i}, \mathbf{P}_i, \mathbf{P}_{-i})\right]\right] \\ \text{s.t. } \mathbb{E}_{\tilde{\mathbf{h}}_i}\left[Q_i(\tilde{\mathbf{h}}_i)P_i(\tilde{\mathbf{h}}_i)\right] \le p_i^{\text{avg}}, \mathbf{P}_i \in \mathcal{P}_i \end{aligned} \tag{5.34}$$

where $\mathcal{P}_i = \{\mathbf{P}_i | Q_i(\tilde{\mathbf{h}}_i) \in \{0,1\}, P_i(\tilde{\mathbf{h}}_i) \in [0, p_i^{\text{inst}}]\}$ is the set of values that $\mathbf{P}_i$ can take. Comparing to the best response for the case without power control [cf. (5.8)], the objective in (5.34) is to maximize the double expectation and there exist additional average power constraints. In this game, each terminal solves an optimization problem like (5.34). The equilibrium point of the game is defined by the following:

Definition 2. $\mathbf{P}_i^{\text{BNE}}$ *is Bayesian Nash Equilibrium (BNE) if for all i the following holds true*

$$\begin{aligned} \mathbf{P}_i^{\text{BNE}} = \operatorname{argmax} \mathbb{E}_{\tilde{\mathbf{h}}_i}\left[\mathbb{E}_{\mathbf{h},\tilde{\mathbf{h}}_{-i}|\tilde{\mathbf{h}}_i}\left[U(\mathbf{h}, \tilde{\mathbf{h}}_i, \tilde{\mathbf{h}}_{-i}, \mathbf{P}_i, \mathbf{P}_{-i}^{\text{BNE}})\right]\right] \\ \text{s.t. } \mathbb{E}_{\tilde{\mathbf{h}}_i}\left[Q_i(\tilde{\mathbf{h}}_i)P_i(\tilde{\mathbf{h}}_i)\right] \le p_i^{\text{avg}}, \mathbf{P}_i \in \mathcal{P}_i \end{aligned} \tag{5.35}$$

At BNE, every terminal plays *best response* to the policies of other terminals. A natural question arises is how to obtain solutions for BNE. As we did for the case without power control, we start by investigating two special cases when terminals have perfectly correlated beliefs ($f_{h_j|\tilde{h}_{ij}}(h) = \delta(h - \tilde{h}_{ij})$ for all $j \neq i$) and uncorrelated beliefs ($f_{h_j|\tilde{h}_{ij}}(h) = f_{h_j}(h)$ for all $j \neq i$).

When terminals' belief about others is perfect, terminals can cooperate with each other so that collision can be avoided. As a result, each terminal can solve the following global optimization problem locally

$$\mathbf{P}_i^{\mathrm{FDMA}} = \operatorname{argmax} \sum_{i=1}^{n} \mathbb{E}_{\mathbf{h}} \left[Q_i(\mathbf{h}) C(h_i, P_i(\mathbf{h})) \right] \tag{5.36}$$

$$\text{s.t. } \mathbb{E}_{\mathbf{h}} \left[Q_i(\mathbf{h}) P_i(\mathbf{h}) \right] \leq p_i^{\mathrm{avg}}, \mathbf{P}_i \in \mathcal{P}_i \quad \forall i$$

$$\sum_{i=1}^{n} Q_i(\mathbf{h}) \leq 1$$

It can be shown that problems like (5.36) have null duality gap [34, 36], and the optimal solution is uniquely determined by the optimal solution for its dual problem. Let $\lambda_i^{\mathrm{FDMA}}$ be the optimal dual variable for the dual problem of (5.36), the optimal solution $\mathbf{P}_i^{\mathrm{FDMA}}$ for (5.36) is then given by

$$P_i^{\mathrm{FDMA}}(h_i) = \max_{P_i \in \mathcal{P}_i} C(h_i, P_i) - \lambda_i^{\mathrm{FDMA}} P_i, \tag{5.37}$$

$$Q_i^{\mathrm{FDMA}}(\mathbf{h}) = H\left(g_i^{\mathrm{FDMA}}(h_i) > \max\left\{ \max_{j \neq i} g_j^{\mathrm{FDMA}}(h_j), 0 \right\} \right) \tag{5.38}$$

where $g_i^{\mathrm{FDMA}}(h_i)$ is defined by

$$g_i^{\mathrm{FDMA}}(h_i) = C(h_i, P_i^{\mathrm{FDMA}}(h_i)) - \lambda_i^{\mathrm{FDMA}} P_i^{\mathrm{FDMA}}(h_i). \tag{5.39}$$

$g_i^{\mathrm{FDMA}}(h_i)$ can be regarded as a local utility function that only depends on terminal i's local CSI h_i. In each time slot, terminals compute their local utilities $g_i^{\mathrm{FDMA}}(h_i)$ and the one with the largest nonnegative utility gets the opportunity to transmit. We show this solution has the following property.

Proposition 7. *For multiple access channel with power control, suppose terminals have perfect beliefs about other terminals, i.e.,* $f_{h_j|\tilde{h}_{ij}}(h) = \delta(h - \tilde{h}_{ij})$ *or* $\tilde{\mathbf{h}}_i = \mathbf{h}$*, then* $\mathbf{P}_i^{\mathrm{FDMA}}$ *obtained by* (5.37) *and* (5.38) *that solves problem* (5.36) *is BNE of the game defined by* (5.35).

Proof. When terminal i has perfect beliefs about other terminals, we have $\tilde{\mathbf{h}}_i = \mathbf{h}$. As a result, we can write the maximization problem in (5.34) as

$$\mathbf{P}_i^{\mathrm{BR}}(\mathbf{P}_{-i}) = \operatorname{argmax} \mathbb{E}_{\mathbf{h}}\left[U(\mathbf{h}, \mathbf{P}_i, \mathbf{P}_{-i})\right] \qquad (5.40)$$
$$\text{s.t. } \mathbb{E}_{\mathbf{h}}\left[Q_i(\mathbf{h})P_i(\mathbf{h})\right] \leq p_i^{\mathrm{avg}}, \mathbf{P}_i \in \mathcal{P}_i.$$

To show $\mathbf{P}_i^{\mathrm{FDMA}}$ is BNE, we need to show that given $\mathbf{P}_{-i} = \mathbf{P}_{-i}^{\mathrm{FDMA}}$ the optimal solution for (5.40) is $\mathbf{P}_i^{\mathrm{BR}}(\mathbf{P}_{-i}^{\mathrm{FDMA}}) = \mathbf{P}_i^{\mathrm{FDMA}}$. We prove this by contradiction. Suppose given $\mathbf{P}_{-i} = \mathbf{P}_{-i}^{\mathrm{FDMA}}$ the optimal solution for (5.40) is $\mathbf{P}_i^{\mathrm{BR}}(\mathbf{P}_{-i}^{\mathrm{FDMA}}) \neq \mathbf{P}_i^{\mathrm{FDMA}}$. This implies

$$\mathbb{E}_{\mathbf{h}}\left[U(\mathbf{h}, \mathbf{P}_i^{\mathrm{BR}}(\mathbf{P}_{-i}^{\mathrm{FDMA}}), \mathbf{P}_{-i}^{\mathrm{FDMA}})\right] > \mathbb{E}_{\mathbf{h}}[U(\mathbf{h}, \mathbf{P}_i^{\mathrm{FDMA}}, \mathbf{P}_{-i}^{\mathrm{FDMA}})]. \qquad (5.41)$$

Moreover, the constraint in (5.40) implies that $\mathbf{P}_i^{\mathrm{BR}}(\mathbf{P}_{-i}^{\mathrm{FDMA}})$ is feasible for the FDMA problem. This contradicts with the fact that $\mathbf{P}_i^{\mathrm{FDMA}}$ is the global maximizer for the FDMA problem (5.36). Therefore, it must be $\mathbf{P}_i^{\mathrm{BR}}(\mathbf{P}_{-i}^{\mathrm{FDMA}}) = \mathbf{P}_i^{\mathrm{FDMA}}$. □

Next, we investigate the case where each terminal has perfect knowledge about its local channel but only has uncorrelated belief about channels of other terminals. In this case, terminals solves the following optimization problem

$$\mathbf{P}_i^{\mathrm{RA}} = \operatorname{argmax} \mathbb{E}_{\mathbf{h}}\left[U(\mathbf{h}, \mathbf{P})\right] \qquad (5.42)$$
$$\text{s.t. } \mathbb{E}_{h_i}\left[Q_i(h_i)P_i(h_i)\right] \leq p_i^{\mathrm{avg}}, \mathbf{P}_i \in \mathcal{P}_i \quad \forall i$$

This is known as the optimal random access channel as we studied in Chapter 2. We show that $\mathbf{P}_i^{\mathrm{RA}}$ has the following property:

Proposition 8. *For multiple access channel with power control, suppose terminals have perfect knowledge about its local channel but only have prior knowledge about channels of others, i.e.,* $f_{h_i|\tilde{h}_{ii}}(h) = \delta(h - \tilde{h}_{ii})$ *and* $f_{h_j|\tilde{h}_{ij}}(h) = f_{h_j}(h)$*, then* $\mathbf{P}_i^{RA}$ *that solves problem* (5.42) *is BNE of the game defined by* (5.35).

Proof. When terminal i has uncorrelated beliefs about other terminals, we have $f_{h_j|\tilde{h}_{ij}}(h) =$

$f_{h_j}(h)$. As a result, we can rewrite the maximization problem in (5.34) as

$$\mathbf{P}_i^{\text{BR}}(\mathbf{P}_{-i}) = \operatorname{argmax} \mathbb{E}_{\mathbf{h}} \left[U(\mathbf{h}, \mathbf{P}_i, \mathbf{P}_{-i}) \right] \tag{5.43}$$
$$\text{s.t. } \mathbb{E}_{h_i} \left[Q_i(h_i) P_i(h_i) \right] \leq p_i^{\text{avg}}, \mathbf{P}_i \in \mathcal{P}_i.$$

To show $\mathbf{P}_i^{\text{RA}}$ is BNE, we need to show that given $\mathbf{P}_{-i} = \mathbf{P}_{-i}^{\text{RA}}$ the solution for (5.43) is $\mathbf{P}_i^{\text{BR}}(\mathbf{P}_{-i}^{\text{RA}}) = \mathbf{P}_i^{\text{RA}}$. To see this is true, note that given $\mathbf{P}_{-i} = \mathbf{P}_{-i}^{\text{RA}}$ problem (5.42) and (5.43) are identical. In this case, optimal solution for (5.42) is the optimal solution for (5.43), i.e., $\mathbf{P}_i^{\text{BR}}(\mathbf{P}_{-i}^{\text{RA}}) = \mathbf{P}_i^{\text{RA}}$. □

Cognitive access algorithm

Similar to the case without power control, we develop cognitive access algorithms that solves problem (5.35) approximately. Let $\tilde{Q}_{ij}(\cdot)$ be terminal i's modeling of terminal j's action. From terminal i's perspective, it assumes that $\tilde{h}_{ij}$ is the true channel gain and terminal j makes decision based on it, i.e., $\tilde{Q}_{ij}(\tilde{h}_{ij})$. As a result, terminal i solves the following problem locally

$$\{\mathbf{P}_i^{\text{CA}}, \tilde{\mathbf{P}}_{-i}^{\text{CA}}\} = \max \mathbb{E}_{\tilde{\mathbf{h}}_i} \left[Q_i(\tilde{\mathbf{h}}_i) C(\tilde{h}_{ii}, P_i(\tilde{\mathbf{h}}_i)) \right] + \sum_{j=1, j\neq i}^{n} \mathbb{E}_{\tilde{\mathbf{h}}_i} \left[\tilde{Q}_{ij}(\tilde{\mathbf{h}}_i) C(\tilde{h}_{ij}, \tilde{p}_j(\tilde{\mathbf{h}}_i)) \right] \tag{5.44}$$
$$\begin{aligned} \text{s.t. } & \mathbb{E}_{\tilde{\mathbf{h}}_i} \left[Q_i(\tilde{\mathbf{h}}_i) P_i(\tilde{\mathbf{h}}_i) \right] \leq p_i^{\text{avg}}, \mathbf{P}_i \in \mathcal{P}_i \\ & \mathbb{E}_{\tilde{\mathbf{h}}_i} \left[\tilde{Q}_j(\tilde{\mathbf{h}}_i) \tilde{p}_j(\tilde{\mathbf{h}}_i) \right] \leq p_j^{\text{avg}}, \tilde{\mathbf{P}}_j \in \mathcal{P}_j \quad \forall j \neq i \\ & Q_i(\tilde{\mathbf{h}}_i) + \sum_{j=1, j\neq i}^{n} \tilde{Q}_j(\tilde{\mathbf{h}}_i) \leq 1. \end{aligned}$$

Note that problem (5.44) is the same as the one for FDMA (5.36) except that $\mathbf{P}_{-i}$ is replaced by $\tilde{\mathbf{P}}_{-i}$. Suppose the optimal dual variable for the dual problem of (5.44) is λ_i^{CA}, then the optimal solution for (5.44) is given by

$$P_i^{\text{CA}}(\tilde{h}_{ij}) = \max_{P_i \in \mathcal{P}_i} C(\tilde{h}_{ij}, P_i) - \lambda_i^{\text{CA}} P_i, \tag{5.45}$$
$$Q_i^{\text{CA}}(\tilde{\mathbf{h}}_i) = H\left(g_i^{\text{CA}}(\tilde{h}_{ii}) > \max\left\{ \max_{j\neq i} g_j^{\text{CA}}(\tilde{h}_{ij}), 0 \right\} \right), \tag{5.46}$$

where $g_i^{\mathrm{CA}}(h_i)$ is given by

$$g_i^{\mathrm{CA}}(\tilde{h}_{ij}) = C(\tilde{h}_{ij}, P_i^{\mathrm{CA}}(\tilde{h}_{ij})) - \lambda_i^{\mathrm{CA}} P_i^{\mathrm{CA}}(\tilde{h}_{ij}) \tag{5.47}$$

In practice, each terminal solves (5.44) offline locally to find the optimal multiplier λ_i^{CA}. Based on λ_i^{CA}, terminal i makes transmission and power allocation decisions according to (5.45)-(5.47). Before proceeding to show the performance of the proposed algorithm, we first prove the next properties.

Proposition 9. *Let λ_i^{CA} and $\lambda_i^{\mathrm{FDMA}}$ be optimal multipliers associated with terminal i's average power constraints in* (5.44) *and* (5.36)*, then $\lambda_i^{\mathrm{CA}} = \lambda_i^{\mathrm{FDMA}}$. Let $P_i^{\mathrm{CA}}(\tilde{h}_{ii})$ and $P_i^{\mathrm{FDMA}}(h_i)$ be the optimal power allocations for terminal i that solve problems* (5.44) *and* (5.36)*, then $P_i^{\mathrm{CA}}(\tilde{h}_{ii}) = P_i^{\mathrm{FDMA}}(h_i)$ if $\tilde{h}_{ii} = h_i$.*

Proof. Since $\tilde{h}_{ij}$ and h_j have the same pdf, the dual problems of (5.44) and (5.36) are the same. Therefore, the optimal dual variables for their dual problems are the same, i.e., $\lambda_i^{\mathrm{CA}} = \lambda_i^{\mathrm{FDMA}}$.

To see $P_i^{\mathrm{CA}}(\tilde{h}_{ii}) = P_i^{\mathrm{FDMA}}(h_i)$ for $\tilde{h}_{ii} = h_i$, observe that $P_i^{\mathrm{CA}}(\tilde{h}_{ii})$ and $P_i^{\mathrm{FDMA}}(h_i)$ are functions of λ_i^{CA} and $\lambda_i^{\mathrm{FDMA}}$, respectively [cf. (5.45) and (5.37)]. Since we have shown $\lambda_i^{\mathrm{CA}} = \lambda_i^{\mathrm{FDMA}}$, it follows that $P_i^{\mathrm{CA}}(\tilde{h}_{ii}) = P_i^{\mathrm{FDMA}}(h_i)$ if $\tilde{h}_{ii} = h_i$. This completes the proof. $\square$

Let the expected utilities achieved by $\mathbf{P}_i^{\mathrm{CA}}$ and $\mathbf{P}_i^{\mathrm{FDMA}}$ are $\bar{U}^{\mathrm{CA}}$ and $\bar{U}^{\mathrm{FDMA}}$, respectively. We show that the performance of the proposed algorithm for the following two cases.

Proposition 10. *If terminals have perfectly correlated belief, i.e., $f_{h_j|\tilde{h}_{ij}}(h) = \delta(h - \tilde{h}_{ij})$, the expected utility achieved by $\mathbf{P}_i^{\mathrm{CA}}$ is the same as that of FDMA, i.e., $\bar{U}^{\mathrm{CA}} = \bar{U}^{\mathrm{FDMA}}$.*

Proof. When terminals have perfectly correlated belief, $\tilde{h}_{ij} = h_j$. This implies that the problem (5.44) solved by the proposed algorithm is the same as the problem (5.36) solved by FDMA. Therefore, the performance achieved by both algorithms are identical, i.e., $\bar{U}^{\mathrm{CA}} = \bar{U}^{\mathrm{FDMA}}$. $\square$

Proposition 11. *If terminals have perfect correlated belief about its local channel, i.e., $f_{h_i|\tilde{h}_{ii}}(h) = \delta(h - \tilde{h}_{ii})$, and uncorrelated belief about channels of other terminals, i.e., $f_{h_j|\tilde{h}_{ij}}(h) = f_{h_j}(h)$, the expected* *utility achieved by $\mathbf{P}_i^{\mathrm{CA}}$ is a fraction of that of FDMA, i.e., $\bar{U}^{\mathrm{CA}} = \beta \bar{U}^{\mathrm{FDMA}}$, where $\beta \in [0, 1]$ is a constant. In particular, when channels are symmetric, $\beta = 1/e$ as the number of terminals goes to infinity.*

Proof. In both FDMA and the proposed algorithm, transmission decision for each terminal is made by comparing its local utility with maximum of the utilities of others [cf. (5.38) and (5.46)]. Therefore, given local channel h_i terminal i transmits with certain probability under both policies. Let $\alpha_i^{\mathrm{FDMA}}(h_i)$ and $\alpha_i^{\mathrm{CA}}(h_i)$ be the probability that terminal i transmit in the proposed algorithm and FDMA, respectively. According to (5.38) and (5.46), $\alpha_i^{\mathrm{FDMA}}(h_i)$ and $\alpha_i^{\mathrm{CA}}(h_i)$ are given by

$$\alpha_i^{\mathrm{FDMA}}(h_i) = \Pr\left(g_i^{\mathrm{FDMA}}(h_i) > \max\left\{ \max_{j \neq i} g_j^{\mathrm{FDMA}}(h_j), 0 \right\} \right) \tag{5.48}$$

$$\alpha_i^{\mathrm{CA}}(h_i) = \Pr\left(g_i^{\mathrm{CA}}(h_i) > \max\left\{ \max_{j \neq i} g_j^{\mathrm{CA}}(\tilde{h}_{ij}), 0 \right\} \right). \tag{5.49}$$

By definition, g_j^{FDMA} and g_j^{CA} are functions of the local channel h_i, corresponding optimal multipliers and optimal power allocations [cf. (5.39) and (5.47)]. Since $\lambda_i^{\mathrm{CA}} = \lambda_i^{\mathrm{FDMA}}$ and $P_i^{\mathrm{CA}}(h_i) = P_i^{\mathrm{FDMA}}(h_i)$ by Proposition 9, for the same channel we have $g_j^{\mathrm{CA}}(h_j) = g_j^{\mathrm{FDMA}}(h_j)$. Moreover, since $\tilde{h}_{ij}$, h_j have the same distribution, we conclude that $\alpha_i^{\mathrm{CA}}(h_i) = \alpha_i^{\mathrm{FDMA}}(h_i) = \alpha_i(h_i)$. As a result, the expected utility achieved by $\mathbf{P}_i^{\mathrm{FDMA}}$ is

$$\bar{U}^{\mathrm{FDMA}} = \sum_i \mathbb{E}_{h_i}\left[C(h_i), P_i^{\mathrm{FDMA}}(h_i))\alpha_i(h_i) \right], \tag{5.50}$$

and the expected utility achieved by $\mathbf{P}_i^{\mathrm{CA}}$ is

$$\bar{U}^{\mathrm{CA}} = \sum_i \mathbb{E}_{h_i}[C(h_i), P_i^{\mathrm{CA}}(h_i))\alpha_i(h_i)] \prod_{j \neq i} \left[1 - \mathbb{E}_{h_j}[\alpha_j(h_j)] \right], \tag{5.51}$$

where the product $\prod_{j \neq i} \left[1 - \mathbb{E}_{h_j}[\alpha_j(h_j)] \right]$ in (5.51) represents the probability that all terminals other than terminal i are silient. Define $\beta_i := \prod_{j \neq i} \left[1 - \mathbb{E}_{h_j}[\alpha_j(h_j)] \right]$ and rewrite (5.51) as

$$\bar{U}^{\mathrm{CA}} = \sum_i \beta_i \mathbb{E}_{h_i}[C(h_i), P_i^{\mathrm{CA}}(h_i))\alpha_i(h_i)]. \tag{5.52}$$

Let $\beta_{\min} = \min_i \beta_i$ and $\beta_{\max} = \max_i \beta_i$, then there must exist $\beta \in [\beta_{\min}, \beta_{\max}]$ such that

$$\bar{U}^{\mathrm{CA}} = \beta \sum_i \mathbb{E}_{h_i}[C(h_i), P_i^{\mathrm{CA}}(h_i))\alpha_i(h_i)]$$

$$= \beta \sum_i \mathbb{E}_{h_i}[C(h_i), P_i^{\mathrm{FDMA}}(h_i))\alpha_i(h_i)], \tag{5.53}$$

where the second equality follows from the fact that $P_i^{\mathrm{CA}}(h_i) = P_i^{\mathrm{FDMA}}(h_i)$. Substitute (5.50) into (5.53) yields

$$\bar{U}^{\mathrm{CA}} = \beta \bar{U}^{\mathrm{FDMA}}. \tag{5.54}$$

When the channels are symmetric, it can be show that $\beta = (1 - 1/n)^{n-1}$. Since $\lim_{n\to\infty}(1 - 1/n)^{n-1} = 1/e$, $\bar{U}^{\mathrm{CA}}$ goes to $1/e \cdot \bar{U}^{\mathrm{FDMA}}$ as n goes to infinity. □

5.2 Cognitive access algorithm for wireless networks

In the previous sections, we investigated multiple access channels in which terminals have beliefs about the global channel state and developed cognitive access algorithms that allow terminals to exploit this information to improve system performance. It is straightforward to extend the cognitive algorithms for multiple access channels to the case of wireless networks where terminals have beliefs about the global network CSI. In this section, we consider wireless networks from a game theoretical point of view and develop cognitive access algorithms similar to what we did in the case of multiple access channels.

Consider a random access wireless network as introduced in Section 1.1.2. Let $\mathbf{h}$ denote the actual channels of links in the network. Different from the assumptions made in Section 1.1.2 that each terminals only has access to its local CSI, we assume each terminal observes an estimated version of the global CSI denoted by $\tilde{\mathbf{h}}_i$. The accuracy of the estimation is reflected by the conditional pdf $f_{\mathbf{h}|\tilde{\mathbf{h}}_i}(\tilde{\mathbf{h}}_i)$. The channel access function Q_{ij} and power allocation function P_{ij} are functions of $\tilde{\mathbf{h}}_i$. Define channel complement of terminal i as $\tilde{\mathbf{h}}_{-i} := \{\tilde{\mathbf{h}}_j\}_{j=1, j\neq i}^n$. In this setting, the optimal operating point of the wireless network is given by the solution for the following optimization problem:

$$\max_{\{\mathbf{x}_i, \mathbf{P}_i\}\in\mathcal{B}_i} \sum_{i\in\mathcal{V}, k\in\mathcal{K}} U_i^k\left(a_i^k\right) - \sum_{i\in\mathcal{V}} V_i(p_i) \tag{5.55}$$

$$
\begin{aligned}
\text{s.t. } & a_i^k \leq \sum_{j \in \mathcal{N}(i)} \left(r_{ij}^k - r_{ji}^k \right) \quad \forall i, \quad \sum_{k \in \mathcal{K}} r_{ij}^k \leq c_{ij} \quad \forall (i,j), \\
& p_i \geq \mathbb{E}_{\tilde{\mathbf{h}}_i} \left[\sum_{j \in \mathcal{N}(i)} P_{ij}(\tilde{\mathbf{h}}_i) Q_{ij}(\tilde{\mathbf{h}}_i) \right] \quad \forall i, \\
& c_{ij} \leq \mathbb{E}_{\tilde{\mathbf{h}}_i} \left[\mathbb{E}_{\mathbf{h}, \tilde{\mathbf{h}}_{-i} | \tilde{\mathbf{h}}_i} \left[C_{ij} \left(h_{ij} P_{ij}(\tilde{\mathbf{h}}_i) \right) Q_{ij}(\tilde{\mathbf{h}}_i) \prod_{l \in \mathcal{M}_i(j)} \left[1 - Q_l(\tilde{\mathbf{h}}_l) \right] \right] \right] \quad \forall (i,j).
\end{aligned}
$$

Solving problem (5.55) is not an easy task because terminals are not allowed to cooperate with each other. Moreover, since $\tilde{\mathbf{h}}_i$ and $\tilde{\mathbf{h}}_j$ may be correlated the approximation method used to decompose the channel capacity constraint [cf. 3.1.2] cannot be applied here. On the other hand, we can model the problem as a game in which each terminal has its own belief about the global CSI and terminals' joint actions determine the system utility. The equilibrium point of the game is defined as follows.

Definition 3. $\mathbf{P}_i^{\text{BNE}}$ *is Bayesian Nash Equilibrium (BNE) if for all i the following holds true*

$$
\begin{aligned}
\{\mathbf{x}_i^{\text{BNE}}, \mathbf{P}_i^{\text{BNE}}\} = & \underset{\{\mathbf{x}_i, \mathbf{P}_i\} \in \mathcal{B}_i}{\operatorname{argmax}} \sum_{k \in \mathcal{K}} U_i^k \left(a_i^k \right) - V_i(p_i) \\
& \text{s.t. } a_i^k \leq \sum_{j \in \mathcal{N}(i)} \left(r_{ij}^k - r_{ji}^{k\text{BNE}} \right), \quad \sum_{k \in \mathcal{K}} r_{ij}^k \leq c_{ij} \quad \forall j \in \mathcal{N}(i), \\
& p_i \geq \mathbb{E}_{\tilde{\mathbf{h}}_i} \left[\sum_{j \in \mathcal{N}(i)} P_{ij}(\tilde{\mathbf{h}}_i) Q_{ij}(\tilde{\mathbf{h}}_i) \right], \\
& c_{ij} \leq \mathbb{E}_{\tilde{\mathbf{h}}_i} \left[\mathbb{E}_{\mathbf{h}, \tilde{\mathbf{h}}_{-i} | \tilde{\mathbf{h}}_i} \left[C_{ij} \left(h_{ij} P_{ij}(\tilde{\mathbf{h}}_i) \right) Q_{ij}(\tilde{\mathbf{h}}_i) \prod_{l \in \mathcal{M}_i(j)} \left[1 - Q_l^{\text{BNE}}(\tilde{\mathbf{h}}_l) \right] \right] \right], \forall j \in \mathcal{N}(i).
\end{aligned}
\tag{5.56}
$$

Comparing (5.56) and (5.55), note that only utilities and constraints associated with terminal i are present in (5.56). To find solutions to this game, we develop cognitive access algorithms similar to what we did in the previous sections. Recall that the key in designing cognitive access algorithms is to model the behaviors of other terminals and one of the easiest way is to assume $\tilde{\mathbf{h}}_i$ is the true network CSI and terminal i solves a local FDMA problem base on this belief. Let $\{\tilde{\mathbf{x}}_i^{\text{CA}}, \tilde{\mathbf{P}}_i^{\text{CA}}\}$ be the solution to the following problem

$$
\max_{\{\tilde{\mathbf{x}}_i, \tilde{\mathbf{P}}_i\} \in \mathcal{B}_i} \sum_{i \in \mathcal{V}, k \in \mathcal{K}} U_i^k \left(\tilde{a}_i^k \right) - \sum_{i \in \mathcal{V}} V_i(\tilde{p}_i) \tag{5.57}
$$

$$\text{s.t. } a_i^k \leq \sum_{j \in \mathcal{N}(i)} \left(\tilde{r}_{ij}^k - \tilde{r}_{ji}^k\right) \quad \forall i, \quad \sum_{k \in \mathcal{K}} \tilde{r}_{ij}^k \leq \tilde{c}_{ij} \quad \forall (i,j),$$

$$\tilde{p}_i \geq \mathbb{E}_{\tilde{\mathbf{h}}_i}\left[\sum_{j \in \mathcal{N}(i)} \tilde{P}_{ij}(\tilde{\mathbf{h}}_i)\tilde{Q}_{ij}(\tilde{\mathbf{h}}_i)\right], \quad \forall i$$

$$\tilde{c}_{ij} \leq \mathbb{E}_{\tilde{\mathbf{h}}_i}\left[C_{ij}\left(\tilde{h}_{ij}\tilde{P}_{ij}(\tilde{\mathbf{h}}_i)\right)\tilde{Q}_{ij}(\tilde{\mathbf{h}}_i)\right]. \quad \forall (i,j),$$

$$\tilde{\mathbf{Q}}\mathbf{A} \leq \mathbf{I}.$$

where in the last constraint $\mathbf{A}$ is a matrix with binary values and $\mathbf{I}$ is a vector with all elements equal to 1. When this constraint is satisfied, all the terminals coordinate with each other to prevent collisions from happening. However, since terminals beliefs are not always perfect collision may still occur if terminals operate according to $\{\tilde{P}_{ij}(\tilde{\mathbf{h}}_i), \tilde{Q}_{ij}(\tilde{\mathbf{h}}_i)\}$. As a consequence, the achieved channel capacity $\tilde{c}_{ij}$ for link (i,j) will be smaller than the solution $\tilde{c}_{ij}^{\text{CA}}$. In fact, the actual instantaneous link capacity achieved by terminals following $\{\tilde{P}_{ij}(\tilde{\mathbf{h}}_i), \tilde{Q}_{ij}(\tilde{\mathbf{h}}_i)\}$ is given by

$$\tilde{c}_{ij}(t) = C_{ij}\left(h_{ij}(t)\tilde{P}_{ij}(\tilde{\mathbf{h}}_i(t))\right)\tilde{Q}_{ij}(\tilde{\mathbf{h}}_i(t)) \prod_{l \in \mathcal{M}_i(j)}\left[1 - \tilde{Q}_l(\tilde{\mathbf{h}}_l(t))\right]. \tag{5.58}$$

Note that $\tilde{c}_{ij}$ is a function of the actual channel $\mathbf{h}$ and terminal i's belief $\tilde{\mathbf{h}}_i$. Given $\tilde{c}_{ij}$, we can find solutions for a_i^k, r_{ij}^k and c_{ij} by solving the following optimization problem

$$\max_{\mathbf{x}_i \in \mathcal{B}_i} \sum_{i \in \mathcal{V}, k \in \mathcal{K}} U_i^k\left(a_i^k\right) \tag{5.59}$$

$$\text{s.t. } a_i^k \leq \sum_{j \in \mathcal{N}(i)} \left(r_{ij}^k - r_{ji}^k\right) \quad \forall i, \quad \sum_{k \in \mathcal{K}} r_{ij}^k \leq c_{ij} \quad \forall (i,j),$$

$$c_{ij} \leq \mathbb{E}_{\mathbf{h}, \tilde{\mathbf{h}}_i}\left[\tilde{c}_{ij}\right] \quad \forall (i,j).$$

Note that the utility and constraints associated with $\tilde{p}_i$ and $\tilde{\mathbf{P}}_i$, i.e., the utility of average power consumption, the average power constraint and the channel capacity constraint, do not exist in the problem (5.59). This is because $\tilde{p}_i$ and $\tilde{\mathbf{P}}_i$ are obtained by solving problem (5.57). Since all the variables appear in summands of the constraints in (5.59), they can be decomposed in the dual domain. Given instantaneous values $\tilde{c}_{ij}(t)$, problem (5.59) can be solved using stochastic subgradient descent in the dual domain. In summary, the proposed cognitive access algorithm for wireless networks is consisted of three steps: 1) Each terminal obtains lower layer variables p_i^{CA} and $\mathbf{P}_i^{\mathrm{CA}}$ by solving a local optimization problem (5.57); 2) The instantaneous channel capacity $\tilde{c}_{ij}(t)$ is calculated by using (5.58); 3) All terminals jointly solve a global optimization problem (5.59) and obtain upper layer variables $a_i^{k\mathrm{CA}}$, r_{ij}^{CA} and c_{ij}^{CA}.

We now analyze the performance of the proposed cognitive access algorithm. Let the utilities achieved by $\mathbf{P}_i^{\mathrm{CA}}$ and $\mathbf{P}_i^{\mathrm{FDMA}}$ are $\bar{U}^{\mathrm{CA}}$ and $\bar{U}^{\mathrm{FDMA}}$, respectively. We show that the performance of the proposed algorithm for the following two cases.

Proposition 12. *If terminals have perfectly correlated belief, i.e.,* $f_{\mathbf{h}|\tilde{\mathbf{h}}_i}(\mathbf{h}) = \delta(\mathbf{h} - \tilde{\mathbf{h}}_i)$*, the utility achieved by* $\mathbf{P}_i^{\mathrm{CA}}$ *is the same as that of FDMA, i.e.,* $\bar{U}^{\mathrm{CA}} = \bar{U}^{\mathrm{FDMA}}$.

Proof. When terminals have perfectly correlated belief, this implies that the problem (5.57) solved by the proposed algorithm is the same as the problem solved by FDMA. Therefore, the performance achieved by both algorithms are identical, i.e., $\bar{U}^{\mathrm{CA}} = \bar{U}^{\mathrm{FDMA}}$. □

Proposition 13. *Consider sum rate utility, i.e.,* $U_i^k(a_i^k) = a_i^k$ *and* $V_i(p_i) = 0$*. If terminals have perfect correlated belief about its local channel, i.e.,* $f_{\mathbf{h}_i|\tilde{\mathbf{h}}_{ii}}(\mathbf{h}_i) = \delta(\mathbf{h}_i - \tilde{\mathbf{h}}_{ii})$*, and uncorrelated belief about channels of other terminals, i.e.,* $f_{\mathbf{h}_j|\tilde{\mathbf{h}}_{ij}}(\mathbf{h}_j) = f_{\mathbf{h}_j}(\mathbf{h}_j)$*, the expected utility achieved by* $\mathbf{P}_i^{\mathrm{CA}}$ *is a fraction of that of FDMA, i.e.,* $\bar{U}^{\mathrm{CA}} = \beta\bar{U}^{\mathrm{FDMA}}$*, where* $\beta \in [0, 1]$ *is a constant. In particular, when channels are symmetric,* $\beta = 1/e$ *as the number of terminals goes to infinity.*

Proof. In the first step of the proposed cognitive access algorithm, each terminal solves a local optimization problem like (5.57). This problem has the same structure as FDMA except for the variables are different. By following the same method used in proving Proposition (11), we can easily show that the expected value of the channel capacity achieved by the cognitive access algorithm is a fraction of the average channel capacity achieved by FDMA, i.e.,

$$\mathbb{E}[\tilde{c}_{ij}] = \beta_{ij} c_{ij}^{\mathrm{FDMA}}, \tag{5.60}$$

where β_{ij} is a constant between 0 and 1. Let $\beta = \min_{(i,j)} \beta_{ij}$, then we have

$$\mathbb{E}[\tilde{c}_{ij}] \geq \beta c_{ij}^{\mathrm{FDMA}}. \tag{5.61}$$

If we replace the capacity constraint $c_{ij} \leq \mathbb{E}_{\mathbf{h},\tilde{\mathbf{h}}_i}[\tilde{c}_{ij}]$ in (5.59) by $c_{ij} \leq \beta c_{ij}^{\mathrm{FDMA}}$, then the feasible set of the problem is reduced. Since all constraints in problem (5.59) are linear, it can be easily seen that $\beta a_i^{k\mathrm{FDMA}}$, $\beta r_{ij}^{\mathrm{FDMA}}$ and $\beta c_{ij}^{\mathrm{FDMA}}$ are feasible for problem (5.59). As a result, the achieved utility is given by $\bar{U} = \sum_{i,k} \beta a_i^{k\mathrm{FDMA}}$. Since $a_i^{k\mathrm{CA}}$ is the optimal solution, the utility achieved by $a_i^{k\mathrm{CA}}$ must be greater than that achieved by $\beta a_i^{k\mathrm{FDMA}}$, i.e.,

$$\bar{U}^{\mathrm{CA}} \geq \sum_{i,k} \beta a_i^{k\mathrm{FDMA}} = \beta \bar{U}^{\mathrm{FDMA}}. \tag{5.62}$$

This completes the proof. □

5.3 Numerical results

Numerical tests are conducted to evaluate performance of the proposed algorithm. We assume local channel h_i follows a complex Gaussian distribution $\mathcal{CN}(0,2)$ and the imperfect channel estimation $\tilde{h}_{ij}$ is modeled by (5.5). Assume capacity achieving codes are used for transmission and the capacity function takes the form of $C(h_i, P_i(h_i)) = \log(1 + h_i P_i(h_i)/N_0)$ where N_0 is normalized noise power. Without loss of generality, we assume $N_0 = 1$. The average power budget is 1 for all terminals, i.e. $p_i^{\mathrm{avg}} = 1$ for all i. We conducted simulations for different total number of terminals $n \in \{10, 20, 30, 40, 50\}$ and different correlation coefficient $\rho \in \{0, 0.1, 0.2, \cdots, 1\}$. In the simulation, stochastic subgradient descent algorithm [36] is used to iteratively compute the primal and dual variables. Optimal solutions for FDMA (when $\rho = 1$) and RA (when $\rho = 0$) are also computed.

Fig. 5.1 compares the expected sum rate achieved by optimal FDMA (when $\rho = 1$), optimal RA (when $\rho = 0$) and the proposed algorithm (when $\rho \in \{0, 0.1, 0.2, \cdots, 1\}$) for $n = 10$. When $\rho = 1$ the expected utility achieved by the proposed algorithm is 1.87 which is equal to that

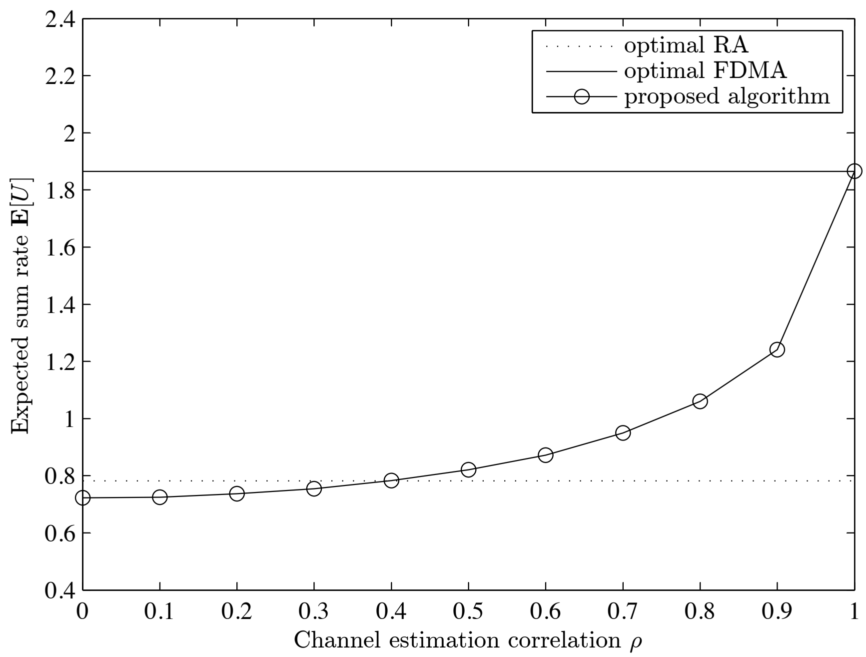


Figure 5.1: Comparison of the expected sum rate utility achieved by the optimal FDMA ($\rho = 1$), the optimal RA ($\rho = 0$) and the proposed algorithm ($\rho \in \{0, 0.1, 0.2, \cdots, 1\}$). The total number of terminals is $n = 10$.

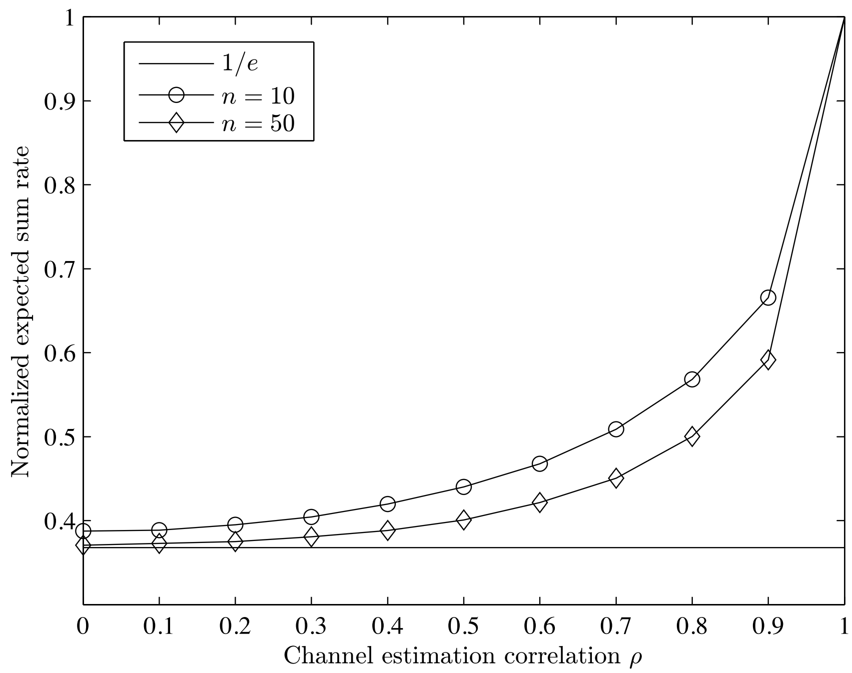


Figure 5.2: The expected sum rate utility achieved by the proposed algorithm normalized by that achieved by the optimal FDMA for $n = 10$ and $n = 50$. The horizontal line is $1/e \approx 0.368$.

achieved by the optimal FDMA. This corroborates the results in Proposition 10. As the correlation ρ decreases, the performance of the proposed algorithm degrades gracefully and achieves an expected utility of 0.72 when $\rho = 0$. This is very close to the expected utility achieved by the optimal RA (0.78).

In Proposition 11, it is shown for symmetric channel the expected utility achieved by the proposed algorithm for $\rho = 0$ is about $1/e$ of the utility achieved by optimal FDMA as n goes to infinity. To show this is true, we normalized the expected utility achieved by the proposed algorithm by the utility achieved by the optimal FDMA. Fig. 5.2 shows the normalized expected utility achieved by the proposed algorithm for $n = 10$ and $n = 50$. The horizontal line is $1/e \approx 0.368$. Indeed, for $n = 50$ the normalized utility converges to $1/e$ when $\rho = 0$. Moreover, notice that the normalized utility decreases as n increases. This is because when n increases the imperfect channel estimation is more likely to cause collisions.

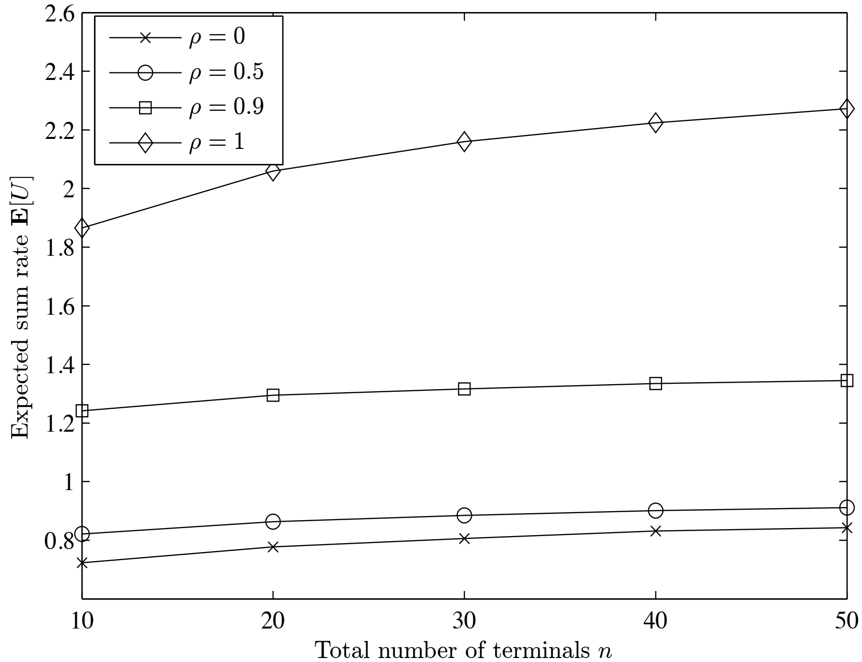


Figure 5.3: The expected sum rate utility achieved by the proposed algorithm for different ρ. For all cases, the expected utility increases as the total number of terminals grows.

Table 5.1: β for channels with different asymmetry levels.

a	0.500	1.581	5.000	15.811	50.000	158.114	500.000
$\bar{U}^{\mathrm{CA}}$	0.527	0.837	1.250	1.689	2.107	2.561	2.950
$\bar{U}^{\mathrm{FDMA}}$	1.358	2.119	3.245	4.303	5.423	6.614	7.801
β	0.388	0.395	0.385	0.393	0.389	0.387	0.378

For both FDMA and RA, it is well known that by adapting transmission power to random channel states multiuser diversity can be obtained, i.e., the expected sum rate utility increases as the number of terminals grows. This is also true for the proposed algorithm. Fig. 5.3 shows the expected utility achieved by the proposed algorithm for different ρ. As we can see, as n increases the system utility increases.

When terminals have uncorrelated beliefs about other channels, we proved that the expected utility achieved by the algorithm is a fraction of that of FDMA, i.e., $\bar{U}^{\mathrm{CA}} = \beta \bar{U}^{\mathrm{FDMA}}$, where β approaches $1/e$ when channels are symmetric (see Proposition 6 and 11). It is interesting to see how β changes when the channels are asymmetric. To do so, we conduct a set of simulations where channels have different levels of asymmetry. We assume the total number of terminals $n = 30$ and all the channels follow exponential distributions. The expected values of the channels are drawn from a uniform distribution $[0, 2a]$ where the parameter a controls the asymmetry of the channels. Table 5.1 shows the average utility achieved by the proposed algorithm and FDMA for different a. As we can see, the resultant β does not change much as a changes.

For the case of wireless network, we run simulations in a network with connectivity graph the same as Fig. 3.4. All the settings are the same as the simulation conducted in Chapter 3. We compared the average sum utility of the network achieved by the proposed cognitive access algorithm with the optimal FDMA and the RA proposed in Chapter 3. As we can see, when the correlation is perfect , the performance of the proposed algorithm is the same as the one achieved

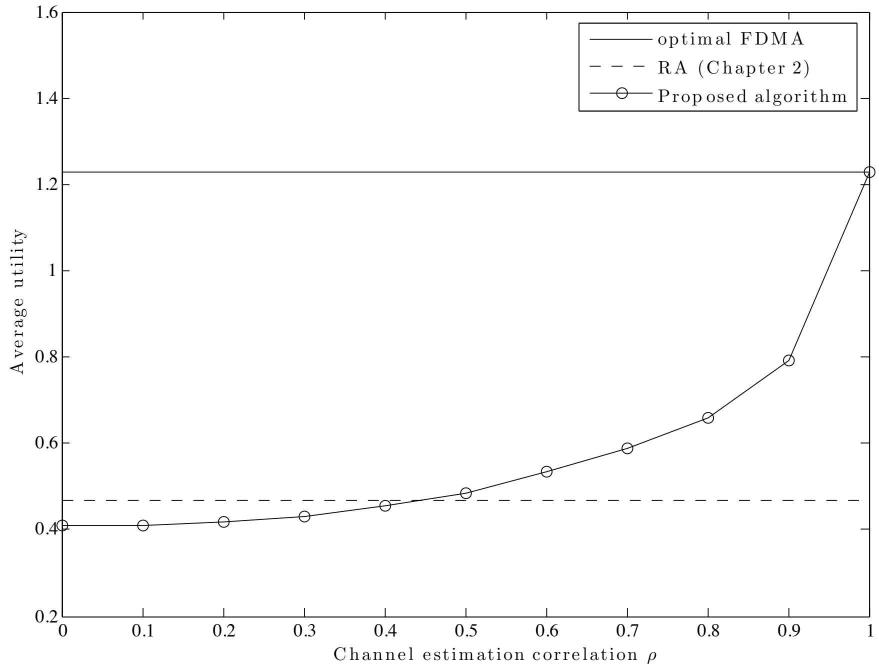


Figure 5.4: Comparison of the average sum rate utility of the network achieved by the optimal FDMA ($\rho = 1$), the RA ($\rho = 0$) and the proposed algorithm ($\rho \in \{0, 0.1, 0.2, \cdots, 1\}$). The network connectivity graph is the same as the one shown in Fig. 3.4.

by the optimal FDMA. As correlation decreases, the performance of the proposed algorithm degrades gracefully and achieves similar utility as the distributed algorithm proposed in Chapter 3 as ρ goes to zero.

5.4 Summary

We considered algorithms that adapts transmission policy to the random channel states in multiple access fading channels where each terminal has a different belief about the channel states. In this setting, we formulated the problem as a Bayesian game in which each terminal maximizes the expected utility based on its belief subject to an average power constraint. We showed that optimal solutions for both FDMA and RA are Bayesian Nash Equilibrium (BNE) points of the formulated game. Therefore, the proposed game theoretic formulation can be regarded as general framework for multiple access channels. Moreover, a cognitive algorithm is developed to solve the problem approximately. Numerical results show that the proposed algorithm achieves performance equal to as the optimal FDMA when the channel estimation is perfectly correlated and performance very close to the optimal RA when channel estimation is uncorrelated.

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