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Moduli Problems in Derived Noncommutative Geometry

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Abstract
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The final chapter studies the moduli of Noncommutative Calabi-Yau Spaces (oriented 2D-topological field theories). The Cobordism Hypothesis and Deligne's Conjecture are used to outline an approach to proving the unobstructedness of this space, and constructing a Frobenius structure on it.

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ABSTRACT

MODULI PROBLEMS IN DERIVED NONCOMMUTATIVE GEOMETRY

Pranav Pandit

Tony Pantev, Advisor

We study moduli spaces of boundary conditions in 2D topological field theories. To a compactly generated linear ∞-category $\mathcal{X}$, we associate a moduli functor $\mathcal{M}_{\mathcal{X}}$ parametrizing compact objects in $\mathcal{X}$. The Barr-Beck-Lurie monadicity theorem allows us to establish the descent properties of $\mathcal{M}_{\mathcal{X}}$, and show that $\mathcal{M}_{\mathcal{X}}$ is a derived stack. The Artin-Lurie representability criterion makes manifest the relation between finiteness conditions on $\mathcal{X}$, and the geometricity of $\mathcal{M}_{\mathcal{X}}$. If $\mathcal{X}$ is fully dualizable (smooth and proper), then $\mathcal{M}_{\mathcal{X}}$ is geometric, recovering a result of Toën-Vaquie from a new perspective. Properness of $\mathcal{X}$ does not imply geometricity in general: perfect complexes with support is a counterexample. However, if $\mathcal{X}$ is proper and perfect (symmetric monoidal, with "compact = dualizable"), then $\mathcal{M}_{\mathcal{X}}$ is geometric.

The final chapter studies the moduli of Noncommutative Calabi-Yau Spaces (oriented 2D-topological field theories). The Cobordism Hypothesis and Deligne's Conjecture are used to outline an approach to proving the unobstructedness of this space, and constructing a Frobenius structure on it.

Contents

Chapter 1

Introduction

The subject of this thesis, derived noncommutative geometry, is the natural coming together of two fundamental paradigm shifts: one within mathematics, and the other in physics. The first is a movement away from mathematics based on sets, to a mathematics where the primitive entities are shapes. The second, is the radical idea in physics that the notion of space-time is not intrinsic to a physical theory. Sections §1.1 and §1.2 are devoted to a cursory overview of these two incipient revolutions in the way we perceive reality. In section §1.3, we sketch in quick, broad strokes the emerging contours of derived noncommutative geometry. Our purpose in including these sections is to place this thesis within the broader context into which it naturally fits, and to lend some perspective to the results proven here. Section §1.4 details, in a semi-informal tone, the main results of this article, and outlines the organizational structure of the document.

1.1 Brave New Mathematics

For several centuries, the scientific approach has been conflated with the reductionist paradigm. The tremendous advances in human knowledge during the aforementioned era, stand testimony, no doubt, to the power and efficacy of reductionism. Nevertheless, as with any approach to understanding and perceiving reality that has thought as its basis, it is inherently limited. What follows is a brief discussion of one particular limitation.

The dominance of the atomic, reductionist world-view is apparent in the very foundations of mathematics. Modern mathematics is based on sets. The notion of a set abstracts the essence out of the everyday experience of collections of objects. Central to this abstraction is a notion of “sameness” or “equality”: one posits that the only reasonable question that one can ask of two members of a set is whether they are equal. Furthermore a set is determined by its members, and the only question that one can ask of two sets is whether they are equal, and so on.

In as much as mathematics is an attempt to mirror the phenomena of Nature within the mental structures created by thought, this set-theoretic model is fundamentally flawed. The notion of equality does not accurately mirror relationships between physical entities. Rather than say that two such entities are equal, it is more natural and useful to specify a particular identification of the two. Often, there are several. The idea that there can be a multitude of ways of identifying an object with itself is the notion of symmetry, which has played such a vital role throughout the history of science.

If one adopts the viewpoint put forth in the previous paragraph, then it becomes logically incumbent upon one to apply the same reasoning to the "identifications" (which we will henceforth suggestively refer to as either "morphisms" or "paths") of objects: rather than ask whether two morphisms are equal, one seeks to describe the totality of all morphisms between the two given morphisms. This reasoning continues ad-infinitum, applying to the morphisms between morphisms, as so on.

This leads one to contemplate a mathematical notion, which we will call "shape", that captures the everyday experience of an aggregate of objects, together with the totality of all identifications between them, and the totality of all identifications between those identifications, ad infinitum. Fortunately, such structures have been studied in algebraic topology for several decades, under the name "homotopy types".

Within the world of set-theoretical mathematics, there are several models for the notion of a "homotopy type", such as topoligical spaces and simplicial sets. In each of these models, the notions described above have concrete avatars. For instance, the points of a topological space incarnate the objects of the "shape" it represents, the paths represent the identifications between objects, the homotopies of paths represent the identifications between identifications, and so forth. It is important to emphasize, however, that each of these models contains redundant information - for instance the real line and a point represent the same shape, while being distinct topological spaces.

The point of view that we adopt is that there exists a universe of discourse which has shapes as the primordial entities in place of sets, and which exists without reference to the set-theoretical world. Following Waldhausen (who in turn borrowed the term from Aldous Huxley's book "Brave New Worlds"), we will refer to this universe of discourse as "Brave New Mathematics".

The axiomatization of the shape theoretical world (Brave New Foundations) seems to be a thing of the future. However, thanks to the monumental efforts of Grothendieck [Gro83], Simpson [Sim10], Toën-Vezzosi [TV05, TV08], Lurie [Lur09a, Lur11b, Lur04] and several others, we do have a shape-theoretical universe of discourse. Their approach has been to construct models for Brave New Mathematics within the set-theoretical world.

Most of the objects of classical mathematics have brave new analogues. The table on the next page summarizes some of the important examples from algebra, topology and geometry. In many respects, these shape-theoretical analogues have many of the same formal properties as their classical counterparts. In fact, a cursory glance at the table will convince the reader that many of these structures have been studied under one guise or another in classical mathematics. For instance, the subject of stable homotopy theory is largely concerned with the analogues of abelian groups, while homological algebra is concerned with the brave new analogue of algebra over $\mathbb{Z}$. The notions of geometric ∞-stacks have also had classical precursors - namely, Artin stacks.

Classical Entity	Brave New Analogue
Sets	Spaces
Categories	∞-categories
Abelian groups	Spectra
Associative rings	A_∞-ring spectra
	$\mathbb{E}_n$-rings
Commutative rings	$\mathbb{E}_\infty$-rings
Topoi	∞-topoi
Algebraic Spaces	Geometric ∞-stacks
Manifolds	Derived Smooth Manifolds
Abelian Categories	Stable ∞-categories
k-linear Grothendieck categories	k-linear presentable ∞-categories

Throughout this paper, the objects that we study will live on the right hand side of this column. The reader who is willing to accept the existence of the brave new analogues, and take for granted that they have certain formal similarities to their classical counterparts, can read most of this paper without an intimiate knowlegde of inner workings of the shape theoretical world.

For a detailed discussion of the myriad ways in which the shape theoretical perspective clarifies our understanding of various questions in classical mathematics, we refer the reader to the beautiful survey articles [Toë03, Toë09, TV07b].

1.2 Quantum Geometry

The reductionist worldview has also been challenged by the revolutions within physics during the last century. Relativity has destroyed the illusion of space and time (resp. matter and energy) as separate entities, while the Quantum Revolution has shaken even the most fundamental assumptions about causality, and the dichotomy between the observer and the observed. It is perhaps accurate to say that most singular contribution of String Theory has been to question the very notion of space-time itself. It is to this radical idea that we devote this section.

After Feynman's fundamental insights, the art of doing physics (or, at least, QFT) can be described as follows. One starts out by choosing a space-time manifold, together with some additional geometric structure on it, such as a Riemannian metric or a complex structure. This geometric structure gives rise to a Lagrangian, which in turn is used to write down a "path-integral". The final output of the path-integral formalism are certain "correlation functions" which organize themselves naturally into a mathematical structure called a Quantum Field Theory. The precise meanings of the terms used above are not important for our purposes. The point we wish to make is that entire process of "doing physics" can be summarized schematically as follows:

$$\begin{pmatrix} \text{Geometric} \\ \text{Background} \\ \text{Data} \end{pmatrix} \xrightarrow{\text{Path integral}} \begin{pmatrix} \text{Quantum Field} \\ \text{Theory} \end{pmatrix}$$

As shown in the next figure, there are a variety of different types of geometric background data ("space-times") that give rise to Quantum Field Theories. This list is by no means exhaustive - the possible types of geometric background data that can be used to construct a field theory being limited only by our imagination. Other examples include representations of groups, and differential equations.

$$\begin{tikzcd}
\begin{pmatrix}\text{Symplectic}\\ \text{Spaces}\end{pmatrix} \arrow[r, leftrightarrow, dotted] \arrow[dr, bend right, "\text{Path Integral}"'] & \begin{pmatrix}\text{Algebraic}\\ \text{Spaces}\end{pmatrix} \arrow[r, leftrightarrow, dotted] \arrow[d, "\text{Path}"', "\text{Integral}"] & \begin{pmatrix}\text{Germs of}\\ \text{Singularities}\end{pmatrix} \arrow[dl, bend left, "\text{Path Integral}"] \\
& (\text{ Quantum Field Theory }) &
\end{tikzcd}$$

The remarkable discovery of the string theorists is that space-times of superficially disparate origins can give rise to equivalent Quantum Field Theories (QFTs). Thus, for instance, a symplectic space and an algebraic variety may give rise to the same physical theory. Since the correlation functions of the QFT are the only experimentally verifiable aspect of the model, the two geometric backgrounds are indistinguishable from a physical point of view. This suggests that the very notion of space-time, in the classical sense, is not intrinsic to the physical theory. Rather, it is an auxiliary construct that proves expedient for obtaining a better understanding of the theory, in much same way that choosing a basis for a vector space may prove useful for certain computations.

A natural question that arises in this context is the following: is there a "notion of geometry" (which one might refer to as "quantum geometry"), that is intrinsic to a Quantum Field Theory? A tautological solution to this problem is to take the very notion of a Quantum Field Theory as a proxy for the notion of a quantum geometry or quantum space-time. While conceptually sound, this approach has two major drawbacks. The first is that QFTs are very complicated, and not very well understood as mathematical objects. The second, more serious drawback is that the path integral is not, at the present time, a well-defined mathematical construct.

1.3 Derived Noncommutative Geometry

In this section, we will provide a brief synopsis of the noncommutative geometry program proposed in [KKP08], as understood by the present author.

In order to circumvent the difficulties of working with the notion of "quantum space" introduced in the previous section, one may choose to work with a toy model for physical theories. One possible toy model for the QFTs arising in physics is the notion of a 2-dimensional topological field theory (2D-TFT). After the seminal work of Atiyah [Ati88], it is understood that topological field theories should be seen as symmetric monoidal functors from a certain fixed symmetric monoidal category Cob_n to another symmetric monoidal category $\mathcal{C}$ (we will call such a thing a n-dimensional-TFT with values in $\mathcal{C}$). In light of the cobordism hypothesis, it is clear that it is better to work with the shape theoretical analogue of these TFTs - namely the extended topological field theories of [Lur09b]. We refer the reader to *loc. cit.* for precise definitions, and for a beautiful exposition of this circle of ideas. The 2D-TFTs of greatest relevance in physics are symmetric monoidal functors from a certain symmetric monoidal $(\infty, 2)$-category $\mathrm{Bord}_2^{\mathrm{fr}}$ (and its variants) to the symmetric monoidal $(\infty, 2)$-category of k-linear ∞-categories, for some commutative ring k. The cobordism hypothesis asserts that such a functor is uniquely determined by its value on a single object (a framed point), and thus by a single k-linear ∞-category.

In keeping with the philosophy of the previous section, we would like to take the point of view that a derived noncommutative space over k is a 2D-TFT valued in k-linear ∞-categories. Or equivalently, by the cobordism hypothesis, a derived noncommutative space over k is a k-linear ∞-category. Derived noncommutative spaces themselves organize themselves into a k-linear $(\infty, 2)$-category.

While the path integral is not a well defined mathematical operation, there are rigorous constructions associating derived noncommutative spaces to various types of geometric background data. For instance, to a commutative space (geometric ∞-stack), one can associate the ∞-category of quasi-coherent sheaves on it. This is discussed in some detail in Chapter 2. To a symplectic manifold, one can associate its derived Fukaya category. Another example of a different flavor is furnished by associating to a commutative space X the ∞-category of $\mathcal{D}_X$-modules. Thus, derived noncommutative geometry provides a unified framework in which to discuss superficially disparate notions of geometry. It provides a natural setting in which to understand the mysterious "string dualities", such as Mirror Symmetry and Langlands Duality, which we alluded to in the previous section.

It is a remarkable fact that much of the geometry of a "geometric background" (algebraic space, symplectic manifold,...) is encoded in the noncommutative shadow attached to it, and the associated 2D-TFT. For instance, the smoothness and properness of a scheme X is encoded in the (full) dualizability of $\mathrm{QC}(X)$ as an object of the symmetric monoidal $(\infty, 2)$-category of k-linear presentable ∞-categories. Conjecturally, the Hodge structure of X can be recovered from the TFT associated to $\mathrm{QC}(X)$. There are similar statements for other geometric backgrounds, such as symplectic manifolds. We will briefly touch upon these ideas in Chapter 5. For a detailed treatment, the reader is referred to [KKP08, Lur09b].

1.4 About this work

We have seen that to a commutative space (geometric ∞-stack), one can associate its category of quasi-coherent sheaves, which is a noncommutative space. There is a construction going in the other direction, which associates to a noncommutative space, the moduli of compact objects in it, which is a derived stack. On certain subcategories, this pair of functors restricts to an adjunction (see [TV07a]). The first question that we investigate is the following:

Question 1: What conditions on a linear ∞-category $\mathcal{X}$ ensure that the moduli of compact objects in $\mathcal{X}$ is a *geometric* ∞-stack?

The other question that finds mention in this paper is the following:

Question 2: Is there a geometric ∞-stack parametrizing smooth and proper noncommutative spaces? What additional geometric structures exist on this stack, if it exists?

Organization of this document:

Each chapter begins with a detailed description of its contents, and most sections begin with a brief synopsis of what follows. Therefore, we will be brief here.

In Chapter 2, we will introduce the primary characters in our story: geometric stacks and derived noncommutative spaces. The final section of this chapter defines the moduli of compact branes on a noncommutative space, and several related moduli functors. These will be main objects of study in Chapters 3 and 4.

Chapter 3 is devoted to a study of the descent properties of the moduli functors introduced in 2. Theorem 3.5.3 summarizes the results of this chapter. The main result from this chapter that will be used in the next chapter is the fact that the moduli functor parametrizing compact branes is a derived stack.

In Chapter 4, which is the heart of this dissertation, we investigate the geometricity of the moduli of compact branes on a noncommutative space. The main results of this chapter are Theorems 4.3.1 and 4.4.1. The first asserts that there is a geometric stack parametrizing compact branes on a smooth and proper noncommutative space. The second describes a certain algebraic structure on a proper noncommutative space that ensures that the moduli functor of compact branes is geometric. Proposition 4.5.2 give says that the moduli of perfect complexes with support along a subscheme is almost never geometric. In particular, properness does not imply geometricity.

The final chapter outlines ongoing work on the moduli of noncommutative spaces. We state a conjecture regarding the existence of a geometric stack parametrizing certain noncommutative spaces. Furthermore, we conjecture that the deformation theory of Calabi-Yau noncommutative spaces is unobstructed, and sketch a proof of this fact. Finally, we outline an approach to constructing a Frobenius structure on the moduli space of noncommutative Calabi-Yau spaces.

1.5 Background and Notation

Throughout this work, we will assume familiarity with the language of homotopical mathematics as developed by Lurie in [Lur09a, Lur11b, Lur04]. Specifically, we will assume that the reader has at least a fleeting acquaintance with the rudiments of topology, algebra and algebraic geometry in the ∞-categorical context. Having said that, we would like to emphasize that an intimate knowledge of the inner workings of the theory in *loc. cit.* is not needed in order to read this paper.

An attempt has been made to keep the statements of the results and the proofs devoid of references to a particular model for ∞-categories. Any equivalent (in a suitable sense) model will suffice. In particular, the reader who is more comfortable with the parlance of Toën/Toën-Vezzosi [Toë07, TV05, TV08], should encounter little difficulty in translating most results of this paper into that language. There is one caveat: for statements that involve functor categories, monads and the Barr-Beck-Lurie theorem, model categories must be replaced by a more flexible notion such as Segal Categories, as is done, for instance, in [TV02].

To a large extent, the notation used in this paper is consistent with the notation in [Lur09a, Lur11b, Lur04]. The following is a list of some frequently used notation.

Notation 1.5.1 (*Bibliographical Convention*)**.** We will use the letter

- "T" to refer to the book Higher Topos Theory [Lur09a].
- "A" to refer to the book, Higher Algebra [Lur11b].
- "G" to refer to the thesis Derived Algebraic Geometry [Lur04].

Thus, for example, T.3.2.5.1. refers to [Lur09a, Remark 3.2.5.1.], while A.6.3.6.10. refers to [Lur11b, Theorem 6.3.6.10].

Notation 1.5.2 (*Spaces*)**.** We will denote by $\mathcal{S}$ (resp. $\widehat{\mathcal{S}}$) the ∞-category of small (resp. large) spaces (T.1.2.1.6.), and by $\mathcal{S}_\infty := \mathrm{Stab}(\mathcal{S})$ the stable ∞-category of spectra. For an ∞-category $\mathcal{C}$, $\mathrm{Stab}(\mathcal{C})$ is its stabilization (A.1.4.) We will denote by $(-)^{\simeq}$ the functor that associates to an ∞-category the maximal subgroupoid.

Notation 1.5.3 (∞-*categories*)**.** Throughout, κ will denote an arbitrary regular cardinal, and ω is the smallest one. We will denote by

- Cat_∞ (resp. $\widehat{\mathrm{Cat}}_\infty$) the ∞-category of essentially small (resp. large) ∞-categories.

- $\mathrm{Cat}_\infty^{\mathrm{Ex}}$ (resp. $\mathrm{Cat}_\infty^{\vee}$) the subcategory of Cat_∞ consisting of small stable (resp. idempotent complete stable) ∞-categories and exact functors.

- $\mathcal{P}r^L$ (resp. $\mathcal{P}r^R$) the subcategory of $\widehat{\mathrm{Cat}}_\infty$ consisting of presentable ∞-categories and left adjoints (resp. accessible right adjoints). See T.5.5.

- $\mathcal{P}r^L_\kappa$ (resp. $\mathcal{P}r^R_\kappa$) the subcategory of $\mathcal{P}r^L$ (resp. $\mathcal{P}r^R$) consisting of κ-compactly generated ∞-categories and functors that preserve κ-compact objects (resp. are κ-accessible). See T.5.5.7.

- $\mathcal{P}r^L_{st}$ the subcategory of $\mathcal{P}r^L$ consisting of stable ∞-categories.

Notation 1.5.4 (*Functor categories*)**.** For $\mathcal{C}$, $\mathcal{D}$ in Cat_∞, $\mathrm{Fun}(\mathcal{C},\mathcal{D})$ denotes the ∞-category of functors $\mathcal{C} \to \mathcal{D}$. We will denote by

- $\mathrm{Fun}^{\mathrm{L}}(\mathcal{C},\mathcal{D})$ (resp. $\mathrm{Fun}^{\mathrm{R}}(\mathcal{C},\mathcal{D})$) the full subcategory of $\mathrm{Fun}(\mathcal{C},\mathcal{D})$ consisting of functors that preserve all small colimits (resp. are accessible, and preserve all small limits).

- $\mathrm{Fun}^{\mathrm{LAd}}(\mathcal{C},\mathcal{D})$ (resp. $\mathrm{Fun}^{\mathrm{RAd}}(\mathcal{C},\mathcal{D})$) the full subcategory of $\mathrm{Fun}(\mathcal{C},\mathcal{D})$ consisting of functors that have right adjoints (resp. have left adjoints).

- $\mathrm{Fun}^{\mathrm{L}}_{\kappa}(\mathcal{C},\mathcal{D})$ (resp. $\mathrm{Fun}^{\mathrm{R}}_{\kappa}(\mathcal{C},\mathcal{D})$) the full subcategory of $\mathrm{Fun}(\mathcal{C},\mathcal{D})$ consisting of functors that preserve all small colimits and κ-compact objects (resp. are κ-accessible, and preserve all small limits).

By the adjoint functor theorem (T.5.5.2.9.), if $\mathcal{C}$ and $\mathcal{D}$ are presentable, then we have natural equivalences $\mathrm{Fun}^{\mathrm{L}}(\mathcal{C},\mathcal{D}) \simeq \mathrm{Fun}^{\mathrm{LAd}}(\mathcal{C},\mathcal{D})$ and $\mathrm{Fun}^{\mathrm{R}}(\mathcal{C},\mathcal{D}) \simeq \mathrm{Fun}^{\mathrm{RAd}}(\mathcal{C},\mathcal{D})$.

Notation 1.5.5. *Small objects: compact objects and dualizable objects.* We will denote by $\mathcal{X}^{\kappa}$ the ∞-category of κ-compact objects in an ∞-category $\mathcal{X}$. $(-)^{\kappa}$ defines a functor $\mathcal{P}r^L_{\kappa} \to \mathrm{Cat}_{\infty}$. If $\mathcal{X}$ is the underlying category of a symmetric monoidal ∞-category $\mathcal{X}^{\otimes}$, we will denote by $\mathcal{X}^{\mathrm{fd}}$ the full subcategory of dualizable objects. More generally if $\mathcal{X}$ is the underlying category of a symmetric monoidal (∞,n)-category, $\mathcal{X}^{\mathrm{fd}}$ denotes the full (∞,n)-subcategory of fully dualizable objects.

Notation 1.5.6 (*Categorical hom and tensor*)**.** The categories $\mathcal{P}r^L$ and $\mathcal{P}r^L_{\kappa}$ are symmetric monoidal, and the inclusion functor $\mathcal{P}r^L_{\kappa} \subseteq \mathcal{P}r^L$ is symmetric monoidal (A.6.3.) with unit $\mathcal{S}$. We will denote by $\otimes$ the tensor product on $\mathcal{P}r^L$. This is not to be confused with the Cartesian monoidal structure "$\times$"on $\widehat{\mathrm{Cat}}_{\infty}$. The functors $\mathrm{Fun}^{\mathrm{L}}(-,-)$ (resp. $\mathrm{Fun}^{\mathrm{L}}_{\kappa}(-,-)$) defines an internal hom on $\mathcal{P}r^L$ (resp. $\mathcal{P}r^L_{\kappa}$).

Notation 1.5.7 (*Algebras and modules*)**.** Let $\mathcal{O}^{\otimes}$ be an ∞-operad, $\mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be an $\mathcal{O}$-monoidal category and let $\mathcal{M}$ be an ∞-category tensored over $\mathcal{C}$ (A.4.2.1.9.). We will denote by $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ the ∞-category of $\mathcal{O}$-algebra objects in $\mathcal{C}$. For A in $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$, we will write $\mathrm{Mod}^{\mathcal{O}}_{A}(\mathcal{M})$ for the ∞-category of A-modules in $\mathcal{M}$. When

$\mathcal{O}$ is the commutative operad $\mathrm{CAlg}(\mathcal{C}) := \mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$, and $\mathrm{Mod}_A(\mathcal{M}) := \mathrm{Mod}_A^{\mathcal{O}}(\mathcal{M})$. When $\mathcal{O}$ is the associative operad (A.4.1.1.), $\mathrm{Alg}(\mathcal{C}) := \mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$. We will use the abbreviation $\mathrm{Mod}_A := \mathrm{Mod}_A(\mathcal{S}_\infty)$ for A in $\mathrm{CAlg}(\mathcal{S}_\infty)$.

Notation 1.5.8 (*1-Categories*)**.** We will denote by Cat the ∞-category of 1-categories. In the quasicategorical model, this is the simplicial nerve of the Dwyer-Kan localization of the 1-category of categories along the subcategory of weak equivalences. We will denote by

- $\mathrm{N}(-)$ the natural inclusion $\mathrm{Cat} \to \mathrm{Cat}_\infty$. In the quasicategorical model, this is the nerve functor.
- $h : \mathrm{Cat}_\infty \to \mathrm{Cat}$ the left adjoint to $\mathrm{N}(-)$. We will refer to $h\mathcal{C}$ as the homotopy category of $\mathcal{C}$.

Notation 1.5.9 (*Ground ring*)**.** Throughout, we will fix a connective $\mathbb{E}_\infty$-ring k. We will assume that k is a Derived G-ring in Chapter 4.

Notation 1.5.10 (*Algebraic geometry*)**.** . We will denote by $\mathrm{CAlg}_{\mathrm{k}}$ the ∞-category of *connective* $\mathbb{E}_\infty$-algebras in $\mathrm{Mod}_{\mathrm{k}}$. We will denote by $\mathrm{Aff}_{\mathrm{k}}$ the category of derived affine schemes. By definition $\mathrm{Aff}_{\mathrm{k}} := \mathrm{CAlg}_{\mathrm{k}}^{op}$. We will denote by $\mathrm{Spec} : \mathrm{CAlg}_{\mathrm{k}}^{op} \to \mathrm{Aff}_{\mathrm{k}}$ and $\mathcal{O} : \mathrm{Aff}_{\mathrm{k}} \to \mathrm{CAlg}$ the tautological equivalences. We will denote by $\mathcal{S}t_{\mathrm{k}}$ the ∞-topos of derived stacks over k. For $\mathcal{F}$ in $\mathcal{S}t_{\mathrm{k}}$ we will write $\mathcal{S}t_{\mathcal{F}}$ the ∞-topos $(\mathcal{S}t_{\mathrm{k}})_{\mathcal{F}/}$.

Notation 1.5.11 (*Diagrams and limits*)**.** For K in Cat_∞, $K^\triangleleft$ (resp. $K^\triangleright$) will denote the category obtained from K by adjoining an initial (resp. final) object $\{\infty\}$. For x in K we will usually denote by ψ_x the unique morphism $\infty \to x$ (resp. $x \to \infty$). Our terminology regarding limits follows T.4.

Notation 1.5.12 (*Moduli Functors*)**.** Let $\mathcal{X} \in \mathcal{P}r^L_{\omega,\mathrm{k}}$. For the definition of the moduli functors $\mathfrak{M}_\mathcal{X}$, $\mathcal{M}^\dagger_\mathcal{X}$, $\mathcal{M}^\natural_\mathcal{X}$, $\mathcal{M}^\flat_\mathcal{X}$, $\mathcal{M}_\mathcal{X}$ and $\mathcal{M}^\vee_\mathcal{X}$ we refer the reader to §2.4 and Notation 2.4.1.

Chapter 2

Brane Moduli

The purpose of this chapter is twofold. Firstly, we will elucidate the structure of commutative and noncommutative spaces to a point where we will be able to define precisely the primary objects of study in this thesis: moduli of objects ("branes") in linear ∞-categories. Secondly, we will collect together several definitions and propositions that will play an important role in the sequel. The reader would do well to just skim over this chapter on the first reading, or to skip it altogether, referring back when necessary.

We begin, in §2.1 with a rapid overview of derived stacks. The notion of geometric stacks is recalled. These are derived stacks that are, in a certain precise sense, sufficiently close to derived affine schemes so as to make them amenable to study via the techniques of algebraic geometry. One of the main goals of this thesis is to determine whether certain stacks are geometric.

In §2.2 we will recall some of the main features of the theory of presentable ∞-categories and compactly generated ∞-categories. We will then recall how the notion of dualizability arising from the symmetric monoidal structure on presentable categories gives rise to finiteness conditions of geometric content on categories. The symmetric monoidal structure will also be used to define the notion of an ∞-category that is linear over a ground ring k. We will then go on to introduce the notion of perfect symmetric monoidal ∞-categories, which will play an important role in this paper. These are presentable symmetric monoidal ∞-categories where the compact objects coincide with the dualizable ones. Some basic properties of perfect categories are noted in this section; however, the primary reason for their importance in the context of this paper will only become apparent in §2.3. The section closes with the definition of quasi-coherent sheaves on derived stacks, and a "derived" version of faithfully flat cohomological descent.

In §2.3 we collect together various useful facts about limits of ∞-categories that will be used frequently later in the paper. Special attention is paid to the behavior of compact objects in this context.

In §2.4 we define the central objects of study in this thesis - moduli of objects in k-linear ∞-categories.

2.1 Commutative Spaces

The following schema for defining the notion $\mathcal{G}$ of a geometric space is ubiquitous in mathematics. One starts with some (ordinary) category $\mathcal{C}$ of "affine models". One then defines $\mathcal{G}$-spaces by "gluing" together objects of $\mathcal{C}$ in a prescribed way. One way of making precise this notion of gluing, is to say that $\mathcal{G}$-spaces are certain sheaves on $\mathcal{C}$ with respect to a Grothendieck topology τ. Algebraic geometry is characterized by the assumption that $\mathcal{C}$ is the opposite of a category of "algebraic objects", such as, for instance, the category of commutative rings in a symmetric monoidal category. This schema carries over verbatim to the shape theoretical context, giving rise to derived algebraic geometry.

Our reference for what follows is [Lur04]. The main purpose of what follows is to fix terminology. We take $\mathrm{Aff}_\mathrm{k} := \mathrm{CAlg}_\mathrm{k}^\mathrm{op}$ as our ∞-category of affine builing blocks, objects of which will be called derived affine schemes. Let τ be a topology on Aff_k in the sense of [Lur04]. Recall that a simplicial object $U_\bullet : \mathrm{N}(\Delta^\mathrm{op}) \to \mathrm{Aff}_\mathrm{k}$ is a τ-hypercover if for all n the natural morphism $U_{n+1} \to (\mathrm{cosk}_n U_\bullet)_{n+1}$ is in τ. By a 1-coskeletal hypercover, we mean one of the form $\mathrm{cosk}_0(f)$ for some f in τ. We will refer to $\mathrm{cosk}_0(f)$ as the Čech nerve of f.

Definition 2.1.1. Let $\mathcal{F} \in \mathcal{P}(\mathrm{Aff}_\mathrm{k}) := \mathrm{Fun}(\mathrm{Aff}_\mathrm{k}^\mathrm{op}, \mathcal{S})$. We say that $\mathcal{F}$ is a sheaf for the τ-topology if it preserves products and carries the Čech nerve of any morphism τ-cover $U \to X$ to a limit diagram. We will say that $\mathcal{F}$ is a sheaf for the τ-hypertopology if it preserves products and carries any τ-hypercover to a limit diagram.

Recall from [Lur04] the definitions of the various topologies (flat, étale, etc) on $\mathrm{Aff}_{\mathrm{k}}$.

Definition 2.1.2. The ∞-topos $\mathcal{S}t_{\mathrm{k}}$ of derived ∞-stacks over k (or simply, derived stacks), is the full subcategory of the category $\mathcal{P}(\mathrm{Aff}_{\mathrm{k}})$ consisting of functors that are sheaves for the étale topology.

Definition 2.1.3.

1. We adopt definition G.5.1.3. as our definition of a relative n-stack, or n-representable morphism. We will say that $\mathcal{F} \in \mathcal{P}(\mathrm{Aff}_{\mathrm{k}})$ is a derived algebraic n-stack if $\mathcal{F}$ is a derived n-stack in the sense of [Lur04]. That is, $\mathcal{F}$ is a derived algebraic n-stack if $\mathcal{F} \to \mathrm{Spec(k)}$ is a relative n-stack. The terms geometric n-stack and algebraic n-stack will be used interchangeably.

2. We will say that $\mathcal{F}$ is a locally geometric ∞-stack if it can be written as a filtered colimit of a diagram $\{\mathcal{F}_\alpha\}$ of stacks such that

   (a) Each $\mathcal{F}_\alpha$ is a derived algebraic n-stack for some n.

   (b) Every morphism in the diagram is a monomorphism.

   We will say $\mathcal{F}$ is locally of finite presentation if each $\mathcal{F}_\alpha$ can be chosen locally of finite presentation.

2.2 Noncommutative Spaces

Presentable Categories. Throughtout this paper, we will work with large ∞-categories. It will usually be necessary to know that these large ∞-categories are "controlled", in a certain precise sense, by a "small amount of data". The theory of presentable ∞-categories developed by Lurie in [Lur09a] offers a framework where one can make precise statements of this type. We refer the reader to *loc. cit.* for a detailed discussion of presentable categories. In the paragraphs that follow, we will collect together some of the main definitions and theorems about presentable categories that will be used frequently in the rest of the paper.

Definition 2.2.1. Let κ be a regular cardinal, and let $\mathcal{X}$ be a large ∞-category. We will say that

(1) $\mathcal{X}$ is *κ-accessible* if there exists a small ∞-category $\mathcal{X}^0$ and an equivalence $\mathrm{Ind}_\kappa(\mathcal{X}^0) \simeq \mathcal{X}$. We will say $\mathcal{X}$ is accessible if it is κ-accessible for some regular cardinal κ. $\mathcal{X}$ is *presentable* if it is accessible and admits all small colimits.

(2) $\mathcal{X}$ is *κ-compactly generated* if it is presentable and κ-accessible. $\mathcal{X}$ is compactly generated if it is ω-compactly generated.

(3) A functor $f : \mathcal{X} \to \mathcal{Y}$ between presentable ∞-categories is *κ-good* if it preserves κ-compact objects, and *κ-accessible* if it preserves κ-filtered colimits.

(4) An object X in $\mathcal{X}$ is a *κ-compact generator* for $\mathcal{X}$ if $\mathcal{X}$ is presentable, X is κ-compact, and for every object Y in $\mathcal{X}$, we have that $\mathcal{X}(X, Y) \simeq \{*\}$ implies

that Y is a final object of $\mathcal{X}$. A compact generator is an ω-compact generator.

We will make frequent use of the following theorem of Schwede-Schipley. The reader is referred to [Lur11b] for a treatment in the language of the current paper.

Theorem 2.2.2. *Let $\mathcal{X}$ be a ω-compactly generated A-linear ∞-category. Then we have an equivalence* $\mathrm{Fun}(\mathcal{X}^{\omega}, \mathrm{Mod}_A) \simeq \mathcal{X}$. *If $\mathcal{X}$ admits a compact generator X, then we have an equivalence $\mathcal{X} \simeq \mathrm{LMod}_{\mathcal{E}}$, where $\mathcal{E}$ is the associative algebra* $\mathbb{M}\mathrm{or}_A(X, X)$ *in* Mod_A.

For the definition of linear ∞-categories, the reader is referred to the subsection on quasi-coherent sheaves in this section.

Dualizability. For the definition of symmetric monoidal ∞-categories, and for the various facts about dualizability that we will need, we refer the reader to [Lur11b, Lur09b, BZFN10, TV08]. Here we recall some of the main definitions and facts that we will need.

The proof of the following lemma is straightforward.

Lemma 2.2.3. *If $\mathcal{X} \in \mathcal{P}r^L_{\omega,\mathrm{k}}$, then $\mathcal{X}$ is dualizable as an object of the symmetric monoidal $(\infty, 1)$-category $\mathcal{P}r^L_{\mathrm{k}}$. A dual of $\mathcal{X}$ is $\mathcal{X}^{\mathrm{op}}$.*

Definition 2.2.4. We will say that $\mathcal{X} \in \mathcal{P}r^L_{\omega,\mathrm{k}}$ is smooth and proper if it is dualizable as an object of the symmetric monoidal $(\infty, 1)$-category $(\mathcal{P}r^L_{\omega,\mathrm{k}})^{\otimes}$.

For a discussion of the properties of smooth and proper categories, we refer the reader to [TV07a]. The lemma that follows can be found in [BZFN10]. It will play a crucial role in Chapters 3 and 4.

Lemma 2.2.5. *Let $\mathcal{C}^{\otimes}$ be a symmetric monoidal ∞-category, and let X be a dualizable object in the underlying category $\mathcal{C}$. Then the functor $X \otimes (-) : \mathcal{C} \to \mathcal{C}$ commutes with all limits.*

Quasi-coherent sheaves. To a derived stack X, or more generally any functor $X \in \mathrm{Fun}(\mathrm{Aff}_{\mathrm{k}}, \mathcal{S})$, one can associate a k-linear presentable ∞-category $\mathrm{QC}(X)$, which can be thought of as a noncommutative shadow of the commutative space X. This subsection is devoted to giving a definition of this category and taking note of some of its basic properties.

There is a functor $\mathcal{M} : \mathrm{CAlg}_{\mathrm{k}} \to \widehat{\mathrm{Cat}}_{\infty}$ whose action on objects and 1-morphisms can be described as follows. To an object A in $\mathrm{CAlg}_{\mathrm{k}}$, $\mathcal{M}$ assigns the ∞-category of A-modules, Mod_A. The action of $\mathcal{M}$ on 1-morphisms $f : A \to B$ in $\mathrm{CAlg}_{\mathrm{k}}$ is given by base change (left Kan extension). In symbols:

$$\mathcal{M}(A) := \mathrm{Mod}_A$$

$$\mathcal{M}(f) := B \otimes_A (-)$$

It is important to note that the formulas given above do not, by themselves, guarantee the existence of an ∞-functor $\mathcal{M}$ with the prescribed action on objects and 1-morphisms: in order to specify $\mathcal{M}$ it is also necessary to specify various "higher order" coherences. The existence of the ∞-functor $\mathcal{M}$ can be established in several ways. In the language of [Lur11b, §6.6.3., §6.3.5.9.], $\mathcal{M}$ is the composite:

$$\mathrm{CAlg_k} \longrightarrow \mathrm{Alg}(\mathrm{Mod_k}) \longrightarrow \widehat{\mathrm{Cat}}_\infty^{\mathrm{alg}} \xrightarrow{\widehat{\Theta}} \widehat{\mathrm{Cat}}_\infty^{\mathrm{Mod}} \longrightarrow \widehat{\mathrm{Cat}}_\infty$$

Recall that, roughly speaking, the ∞-category $\widehat{\mathrm{Cat}}_\infty^{\mathrm{alg}}$ consists of pairs $(\mathcal{C}^\otimes, A)$, where where $\mathcal{C}^\otimes$ is a (not necessarily small) symmetric monoidal ∞-category and A is an object in $\mathrm{Alg}(\mathcal{C})$. Similarly, the ∞-category $\widehat{\mathrm{Cat}}_\infty^{\mathrm{Mod}}$ consists of pairs $(\mathcal{C}^\otimes, \mathcal{N})$, where $\mathcal{C}^\otimes$ is a symmetric monoidal ∞-category, and $\mathcal{N}$ is a (not necessarily small) ∞-category tensored over $\mathcal{C}^\otimes$. In the diagram above, the first arrow is the forgetful functor from $\mathbb{E}_\infty$-algebras to $\mathbb{E}_1$-algebras, and the last arrow is the forgetful functor that sends $(\mathcal{C}^\otimes, \mathcal{N})$ to the underlying ∞-category $\mathcal{N}$. The functor $\mathrm{Alg}(\mathrm{Mod_k}) \to \widehat{\mathrm{Cat}}_\infty^{\mathrm{alg}}$ is the inclusion of the subcategory consisting of pairs $(\mathcal{C}^\otimes, A)$, where $\mathcal{C}^\otimes \simeq \mathrm{Mod}_\mathrm{k}^\otimes$, and the morphisms are equivalent to the identity on $\mathcal{C}^\otimes$. Roughly speaking, $\widehat{\Theta}$ associates to $(\mathcal{C}^\otimes, A)$ the pair $(\mathcal{C}^\otimes, \mathrm{RMod}_A(\mathcal{C}))$.

Remark 2.2.6. The ∞-category $\mathcal{M}(A) = \mathrm{Mod}_A$ is presentable. This follows, for instance, from A.4.2.3.7., and the fact the ∞-category of spectra is presentable. Furthermore, the category Mod_A is ω-compactly generated.

Remark 2.2.7. For any $f : A \to B$ in $\mathrm{CAlg_k}$, the functor $\mathcal{M}(f)$ has a right adjoint, namely, the forgetful functor $\mathrm{Mod}_B \to \mathrm{Mod}_A$. More is true: the right adjoint $\mathcal{M}(B) \to \mathcal{M}(A)$ preserves all colimits. In particular, it is ω-accessible. It follows that $\mathcal{M}(f) : \mathcal{M}(A) \to \mathcal{M}(B)$ preserves ω-compact objects.

Remark 2.2.8. The categories Mod_A have a symmetric monoidal structure induced by the symmetric monoidal structure on $\mathcal{S}_\infty$. Thus, $\mathcal{M}(A)$ can be viewed as commutative algebra object in $\mathcal{P}r^L$. In particular, it $\mathcal{M}(A)$ is a module over itself; i.e., it can viewed as an object in $\mathrm{Mod}_{\mathcal{M}(A)}(\mathcal{P}r^L) =: \mathcal{P}r^L_A$.

For A in $\mathrm{CAlg}_{\mathrm{k}}$, functor $\mathcal{M}(\theta_A) : \mathcal{M}(\mathrm{k}) \to \mathcal{M}(A)$ induced by the structure morphism $\theta_A : \mathrm{k} \to A$ is symmetric monoidal (A.4.4.3.1), i.e., it is a morphism in $\mathrm{CAlg}(\mathcal{P}r^L)$. Restricting the action of $\mathcal{M}(A)$ along $\mathcal{M}(\theta_A)$, we get an induced $\mathcal{M}(\mathrm{k})$-module structure on $\mathcal{M}(A)$. Morphisms in $\mathrm{CAlg}_{\mathrm{k}}$ commute with the structure maps $\theta_{(-)}$ by definition; this immediately implies that the functors $\mathcal{M}(f)$ are $\mathcal{M}(\mathrm{k})$-linear.

Recall that $\mathcal{P}r^L_\omega$ (resp. $\mathcal{P}r^L_{\omega,\mathrm{k}}$) denotes the subcategory of $\mathcal{P}r^L$ (resp. $\mathcal{P}r^L_{\mathrm{k}}$) consisting of all compactly generated ∞-categories (resp. all $\mathrm{Mod}_{\mathrm{k}}$-linear compactly generated ∞-categories), and morphisms that preserve small colimits and ω-compact objects. The preceding three remarks are summarized by the following proposition:

Lemma 2.2.9. *There exists a functor $\mathfrak{M}_{\mathbf{1}}^{\otimes} : \mathrm{CAlg}_{\mathrm{k}} \to \mathrm{CAlg}(\mathcal{P}r^L_{\omega,\mathrm{k}})$ such that the diagram below is (homotopy) commutative:*

$$\begin{tikzcd}
 & \mathrm{CAlg}(\mathcal{P}r^L_{\omega,\mathrm{k}}) \arrow[d] \\
 & \mathcal{P}r^L_{\omega,\mathrm{k}} \arrow[d] \\
\mathrm{CAlg}_{\mathrm{k}} \arrow[uur, dotted, "\mathfrak{M}_{\mathbf{1}}^{\otimes}"] \arrow[r, "\mathcal{M}"'] & \widehat{\mathrm{Cat}}_\infty
\end{tikzcd}$$

Notation 2.2.10. Let $\mathfrak{M}_{\mathbf{1}}$ denote the functor obtained by composing $\mathfrak{M}_{\mathbf{1}}^{\otimes}$ with the forgetful functor $\mathrm{CAlg}(\mathcal{P}r^L_{\omega,\mathrm{k}}) \to \mathcal{P}r^L_{\omega,\mathrm{k}}$. Let $\mathrm{QC}^{\mathrm{aff}}$ denote the composite

$$\mathrm{Aff}_{\mathrm{k}}^{op} \xrightarrow{\mathcal{O}} \mathrm{CAlg}_{\mathrm{k}} \xrightarrow{\mathfrak{M}_1} \mathcal{P}r^L_{\omega,\mathrm{k}} \longrightarrow \mathcal{P}r^L_{\mathrm{k}}$$

For a derived affine scheme X in $\mathrm{Aff}_{\mathrm{k}}$, $\mathrm{QC}^{\mathrm{aff}}(X)$ is the ∞-category of quasicoherent sheaves on X. We would like to extend this functor to arbitrary derived stacks.

Let $j : \mathrm{Aff}_{\mathrm{k}} \to \mathcal{P}(\mathrm{Aff}_{\mathrm{k}})$ denote the Yoneda embedding. By the universal property of categories of presheaves, left Kan extension defines an equivelance $\mathrm{Fun}(\mathrm{Aff}_{\mathrm{k}}, \mathcal{C}) \simeq \mathrm{Fun}^{\mathrm{L}}(\mathcal{P}(\mathrm{Aff}_{\mathrm{k}}), \mathcal{C})$, for any $\mathcal{C}$ that admits all small colimits. Take $\mathcal{C} = (\mathcal{P}r^L_{\mathrm{k}})^{op}$, and let $\widetilde{\mathrm{QC}}$ denote that image of $\mathcal{M}$ under the induced equivalence $\mathrm{Fun}(\mathrm{Aff}_{\mathrm{k}}^{op}, \mathcal{P}r^L_{\mathrm{k}}) \simeq \mathrm{Fun}^{\mathrm{R}}(\mathcal{P}(\mathrm{Aff}_{\mathrm{k}})^{op}, \mathcal{P}r^L_{\mathrm{k}})$.

Notation 2.2.11. Let $a : \mathcal{P}(\mathrm{Aff}_{\mathrm{k}}) \to \mathcal{S}t_{\mathrm{k}}$ be the localization functor, with right adjoint i, and let QC denote the composite $\widetilde{\mathrm{QC}} \circ i^{op}$. We will often implicitly identify $\mathcal{S}t_{\mathrm{k}}$ with the essential image of the fully faithful functor i.

Definition 2.2.12. For a derived stack X over k, the ∞-category $\mathrm{QC}(X)$ is called the ∞-*category of quasicoherent sheaves* on X.

Remark 2.2.13. The ∞-category $\mathrm{QC}(X)$, is stable. This follows, for instance, from the fact that $\mathrm{Mod}_{\mathrm{k}}$-linear ∞-categories are stable [].

Remark 2.2.14. Let X be a discrete scheme. Then the relationship between the ∞-category QC and the abelian category $\mathrm{Qcoh}(X)$ of quasicoherent sheaves on X

is as follows: there is a t-structure on QC such that $\mathrm{QC}^{\heartsuit} \simeq \mathrm{Qcoh}(X)$, and we have an equivalence $h\mathrm{QC} \simeq D(\mathrm{Qcoh}(X))$.

Remark 2.2.15. The étale topology is subcanonical, so for A in $\mathrm{CAlg}_{\mathrm{k}}$, $\mathrm{Spec}(A)$ is a derived stack. Furthermore, we have $\mathcal{M}(A) = \mathrm{QC}(\mathrm{Spec}(A))$.

Remark 2.2.16. Let $\mathcal{F} \in \mathcal{P}(\mathrm{Aff}_{\mathrm{k}})$, and $\Phi : (j/\mathcal{F}) \to \mathcal{P}(\mathrm{Aff}_{\mathrm{k}})$ be the functor that carries $\mathrm{Spec}(A) \to \mathcal{F}$ to $\mathrm{Spec}(A)$. Then we have a natural equivalence $\mathcal{F} \simeq \mathrm{colim}\, \Phi$. Take $\mathcal{F} = i(X)$ for some derived stack X. Using the preceding remark and the fact that $\widetilde{\mathrm{QC}}$ preserves limits, we have

$$\mathrm{QC}(X) \simeq \lim(\widetilde{\mathrm{QC}} \circ \Phi^{op}) \simeq \lim_{\mathrm{Spec}(A) \to X} \mathrm{QC}(\mathrm{Spec}(A))$$

The diagram $\Phi : (j/\mathcal{F}) \to \mathcal{P}(\mathrm{Aff}_{\mathrm{k}})$ is large, and consequently the description of QC in 2.2.16 is not very useful in practice. In the category $\mathcal{S}t_{\mathrm{k}}$, one often has small diagrams taking values in (the essential image of) $\mathrm{Aff}_{\mathrm{k}}$ whose colimit is a given derived stack X. For example, if $U_{\bullet} \to X$ is an étale (or flat) hypercover then we have $\mathrm{colim}_n\, U_n \simeq X$. However, since i^{op} does not preserve limits, one has to work much harder to show that $\mathrm{QC}(X) \simeq \lim_n \mathrm{QC}(U_n)$. The following proposition is the homotopical/derived analogue of flat descent for quasicoherent sheaves on ordinary schemes:

Proposition 2.2.17. *The functor* $\mathrm{QC} : \mathcal{S}t_{\mathrm{k}}^{op} \to \mathcal{P}r_{\mathrm{k}}^{L}$ *is a sheaf for the flat hypertopology.*

Proof. This is known to the experts; see e.g. [Lur04, Example 4.2.5] and [TV08, Theorem 1.3.7.2]. It also follows from Theorem 3.2.1; indeed, it is the special case of that theorem when $\mathcal{X} \simeq \hat{\mathrm{k}}$. □

Dualizability vs. Compactness: Perfect Symmetric Monoidal Categories. In this subsection, we will introduce the notion of perfect symmetric monoidal ∞-categories, and observe that the class of perfect categories is stable under the tensor product on presentable ∞-categories. Some examples of perfect categories will be given. The relevance of perfect categories to this paper lies in the fact that limit diagrams taking values in perfect categories are particularly well behaved, in a sense that will be made precise in §2.3.

Definition 2.2.18. Let $\mathcal{X}$ be in $\mathcal{P}r^L_{\omega,\mathrm{k}}$. A *perfect* symmetric monoidal structure on $\mathcal{X}$ is a symmetric monoidal structure $\otimes$ that distributes over colimits, and is such that $\mathcal{X}^\omega \simeq \mathcal{X}^{\mathrm{fd}}$. Denote by $\mathcal{P}r^L_{\mathrm{perf}}$ the full subcategory of $\mathrm{CAlg}(\mathcal{P}r^L_{\omega,\mathrm{k}})$ consisting of perfect symmetric monoidal ∞-categories.

Remark 2.2.19. By the Eckmann-Hilton argument all symmetric monoidal structures compatible with a perfect symmetric monoidal structure are perfect.

For the purposes of this paper, the most important example of a perfect symmetric monoidal category is the category of modules over an $\mathbb{E}_\infty$-ring.

Proposition 2.2.20 ([TV08, BZFN10, Lur11b])**.** *Let $A \in \mathrm{CAlg}(\mathcal{S}_\infty)$. Then the symmetric monoidal ∞-category Mod_A is perfect. In particular, the functor $\mathfrak{M}_{\mathbf{1}}^{\otimes}$ de-*

fined in Lemma 2.2.9 factors through the ∞-category of perfect symmetric monoidal ∞-categories:

$$\begin{tikzcd}
& \mathcal{P}r^L_{\mathrm{perf}} \arrow[d] \\
& \mathcal{P}r^L_{\omega,\mathrm{k}} \arrow[d] \\
\mathrm{CAlg}_{\mathrm{k}} \arrow[ruu, dotted, "\mathfrak{M}^{\otimes}_{\mathbf{1}}"] \arrow[r, "\mathcal{M}"'] & \widehat{\mathrm{Cat}}_{\infty}
\end{tikzcd}$$

The category of commutative algebra objects in a symmetric monoidal category $\mathcal{X}$ inherits a symmetric monoidal structure from the underlying category. In particular, presentable k-linear symmetric monoidal ∞-categories inherit a symmetric monoidal structure from $\mathcal{P}r^L_{\mathrm{k}}$. The next lemma says that perfect symmetric monoidal categories are closed under the tensor product:

Lemma 2.2.21. *The category $\mathcal{P}r^L_{\mathrm{perf}}$ admits a symmetric monoidal structure, and the inclusion $\mathcal{P}r^L_{\mathrm{perf}} \subseteq \mathrm{CAlg}(\mathcal{P}r^L_{\omega,\mathrm{k}})$ is symmetric monoidal.*

Proof. Let $\mathcal{X}$ and $\mathcal{Y}$ be perfect symmetric monoidal categories. We will show that the induced symmetric monoidal structure on $\mathcal{X} \otimes \mathcal{Y}$ is perfect. Since $\mathcal{X}$ and $\mathcal{Y}$ are perfect and the unit is always dualizable, we have $\mathbf{1}_{\mathcal{X}} \in \mathcal{X}^{\omega}$ and $\mathbf{1}_{\mathcal{Y}} \in \mathcal{Y}^{\omega}$. Consequently $\mathbf{1}_{\mathcal{X}\otimes\mathcal{Y}} = \mathbf{1}_{\mathcal{X}} \otimes \mathbf{1}_{\mathcal{Y}} \in \mathcal{X}^{\omega} \otimes \mathcal{Y}^{\omega} \subseteq (\mathcal{X} \otimes \mathcal{Y})^{\omega}$. Recall that if $\mathbf{1}_{\mathcal{C}}$ is a compact object in a symmetric monoial ∞-category $\mathcal{C}$ then every dualizable object is compact. This follows easily from the equivalence $\mathcal{C}(X \otimes Y, Z) \simeq \mathcal{C}(X, Y^\vee \otimes Z)$ for any X, Y, Z. See for e.g., [TV08, Prop 1.2.3.7]. Therefore we have $(\mathcal{X} \otimes \mathcal{Y})^{\mathrm{fd}} \subseteq (\mathcal{X} \otimes \mathcal{Y})^{\omega}$.

To prove the converse, note that by our hypotheses, we have $\mathcal{X}^\omega \subseteq \mathcal{X}^{\mathrm{fd}}$ and $\mathcal{Y}^\omega \subseteq \mathcal{Y}^{\mathrm{fd}}$, and consequently, since dualizable objects are stable by tensor product, we have $\mathcal{X}^\omega \otimes \mathcal{Y}^\omega \subseteq (\mathcal{X} \otimes \mathcal{Y})^{\mathrm{fd}}$. Since $(\mathcal{X} \otimes \mathcal{Y})^\omega$ is the idempotent completion of $\mathcal{X}^\omega \otimes \mathcal{Y}^\omega$, and $(\mathcal{X} \otimes \mathcal{Y})^{\mathrm{fd}}$ is stable under retracts, it follows that $(\mathcal{X} \otimes \mathcal{Y})^\omega \subseteq (\mathcal{X} \otimes \mathcal{Y})^{\mathrm{fd}}$.

To complete the proof, it remains only to observe that the unit $\mathrm{Mod}_{\mathrm{k}}$ for the monoidal structure on $\mathrm{CAlg}(\mathcal{P}r^L_{\omega,\mathrm{k}})$ is perfect by Lemma 2.2.20. $\square$

The inclusion $\mathcal{P}r^L_{\omega,\mathrm{k}} \subseteq \mathcal{P}r^L_{\mathrm{k}}$ does not reflect limits in general: the limit in $\mathcal{P}r^L_{\mathrm{k}}$ (or equivalently in $\widehat{\mathrm{Cat}}_\infty$) of a diagram of compactly generated categories need not be compactly generated. Consequently, $\mathrm{QC}(X)$ need not be compactly generated. Following [BZFN10], we make the following definition:

Definition 2.2.22. A derived stack X is *perfect* if it has affine diagonal and $\mathrm{QC}(X)$ is an ω-compactly generated ∞-category. Let $\mathcal{S}t^{\mathrm{perf}}_{\mathrm{k}}$ denote the full subcategory of $\mathcal{S}t_{\mathrm{k}}$ consisting of perfect stacks.

Let X and Y be commutative spaces (schemes, or more generally, derived stacks). Then any quasi-coherent sheaf $\mathcal{F}$ on $X \times Y$ gives rise to a functor $FM(\mathcal{F}) : \mathrm{QC}(X) \to \mathrm{QC}(Y)$ defined by $FM(\mathcal{F})(\mathcal{E}) := p_{Y*}(\mathcal{F} \otimes p_X^*\mathcal{E})$, where p_X and p_Y are the projections from the product to the individual factors. In particular, any correspondence $j : Z \to X \times Y$ gives rise to a functor $FM(j_*\mathcal{O}_Z)$. Correspondences may be thought of as morphisms in a suitable category of commutative motives. Thus one may think of FM as a construction that assigns to a morphism of commutative motives X and Y, a morphism between the noncommutative shadows $\mathrm{QC}(X)$ and $\mathrm{QC}(Y)$. Perhaps one of the main reasons for the utility of the notion of perfect stacks is that for perfect stacks, every map between the noncommutative shadows arises in this way. More precisely, we have the following theorem:

Theorem 2.2.23 ([Toë07],[BZFN10])**.** *The cartesian symmetric monoidal structure on $\mathcal{S}t_{\mathrm{k}}$ restricts to a symmetric monoidal structure on $\mathcal{S}t_{\mathrm{k}}^{\mathrm{perf}}$. Furthermore, the restriction of* QC *to $\mathcal{S}t_{\mathrm{k}}^{\mathrm{perf}}$ is symmetric monoidal. In other words, if X and Y are perfect stacks over* k*, then $X \times_{\mathrm{k}} Y$ is perfect and we have a natural equivalence:*

$$\mathrm{QC}(X) \otimes_{\mathrm{Mod}_{\mathrm{k}}} \mathrm{QC}(Y) \simeq \mathrm{QC}(X \times_{\mathrm{k}} Y)$$

Furthermore, we have a natural equivalence

$$\mathrm{QC}(X \times_{\mathrm{k}} Y) \simeq \mathrm{Fun}^{\mathrm{L}}_{\mathrm{k}}(\mathrm{QC}(X), \mathrm{QC}(Y))$$

2.3 Gluing Noncommutative Spaces

The operation of passing to compact objects is not, in general, compatible with taking limits in $\mathcal{P}r^{L}$. This section is devoted to careful study of this phenomenon. There is no single "main proposition" in this section - our purpose is simply to collect together several results about limits of ∞-categories that will be used in the sequel. The reader would do well to skip this section on the first reading.

We begin with some simple observations about the relationship between linear structures and limits. In the sequel, we will prove several results about diagrams taking values in $\mathcal{P}r^L_\omega$. By virtue of the next two lemmas, each of these results remains true if we replace $\mathcal{P}r^L_\omega$ by $\mathcal{P}r^L_{\omega,\mathrm{k}}$, and $\mathcal{P}r^L$ by $\mathcal{P}r^L_{\mathrm{k}}$.

Lemma 2.3.1. *The forgetful functor $\mathcal{P}r^L_{\omega,\mathrm{k}} := \mathrm{Mod}_{\mathrm{Mod}_k}(\mathcal{P}r^L_\omega) \to \mathcal{P}r^L_\omega$ preserves and reflects all small limits.*

Proof. This follows from the general statement that the forgetful functor from a module category to the underlying category preserves and reflects all small limits A.4.2.3.3. □

Lemma 2.3.2. *The forgetful functor $i^L_{\mathrm{k}} : \mathcal{P}r^L_{\mathrm{k}} \to \widehat{\mathrm{Cat}}_\infty$ preserves and reflects all limits.*

Proof. We have $i^L_{\mathrm{k}} = i^L \circ \pi^L_{\mathrm{k}}$, where $i^L : \mathcal{P}r^L \to \widehat{\mathrm{Cat}}_\infty$ is the natural inclusion, and $\pi^L_{\mathrm{k}} : \mathcal{P}r^L_{\mathrm{k}} := \mathrm{Mod}_{\mathrm{Mod}_{\mathrm{k}}}(\mathcal{P}r^L) \to \mathcal{P}r^L$ is the forgetful functor. The functor π^L_{k} preserves and reflects all limits by A.4.2.3.1. According to T.5.5.3.13., the categories $\mathcal{P}r^L$ and $\widehat{\mathrm{Cat}}_\infty$ admits all small limits and i^L preserves all small limits. The fact that i^L is conservative, together with the following lemma, implies that i^L reflects all small limits. □

Lemma 2.3.3. *Let $f : \mathcal{C} \to \mathcal{D}$ be a functor between ∞-categories, and let K be a simplicial set. Assume that $\mathcal{C}$ admits limits of diagrams of shape K, that f preserves these limits, and that f is conservative. Then f reflects limits of shape K.*

Proof. Let $\phi : K^{\triangleleft} \to \mathcal{C}$ be a diagram, and suppose that $f \circ \phi : K^{\triangleleft} \to \mathcal{D}$ is a limit diagram. Since $\mathcal{C}$ admits limits of diagrams of shape K, there exists a limit diagram $\psi : K^{\triangleleft} \to \mathcal{C}$ with $\psi_{|K} \simeq \phi_{|K}$. By the definition of limits, there is a morphism $\alpha : \phi \to \psi$ in $\mathrm{Fun}_K(K^{\triangleleft}, \mathcal{C})$. We will complete the proof by showing that α is an equivalence. Since f is conservative, it will suffice to show that $f(\alpha)$ is an equivalence.

Since f preserves K-limit diagrams, $f \circ \psi$ is also a limit diagram. Furthermore, we have $(f \circ \psi)_{|K} = (f \circ \phi)_{|K}$. Since the restriction $\mathrm{Fun}_K(K^{\triangleleft}, \mathcal{D}) \to \mathrm{Fun}_K(K, \mathcal{D}) \simeq \{f \circ \phi\}$ is a trivial Kan fibration, we have a natural equivalence $\beta : f \circ \psi \to f \circ \phi$ in $\mathrm{Fun}_K(K^{\triangleleft}, \mathcal{D})$. Using the fact that $\mathrm{Fun}_K(K^{\triangleleft}, \mathcal{D}) \to \mathrm{Fun}_K(K, \mathcal{D})$ is a trivial fibration again, we conclude that $\beta \circ f(\alpha)$ is an equivalence. By the two out of three property, $f(\alpha)$ is an equivalence. $\square$

Let $\nu : K \to \mathcal{P}r^L_{\omega,\mathrm{k}}$ be a diagram. We have induced diagrams, $\nu' : K \to \widehat{\mathrm{Cat}}_\infty$ and $(-)^\omega \circ \nu : K \to \mathrm{Cat}_\infty$. Understanding the relatioships between the limits of these three diagrams will play a central role in this paper. Lemma 2.3.5 says that there is essentially no difference between computing limits in $\mathcal{P}r^L_\omega$ and Cat_∞. We begin with a simple observation that we will need in the proof of that lemma.

Lemma 2.3.4. *The functor $(-)^\omega : \mathcal{P}r^L_\omega \to \mathrm{Cat}_\infty$ is conservative.*

Proof. Let $\mathrm{Cat}_\infty^{\mathrm{REx}(\omega)^\vee} \subseteq \mathrm{Cat}_\infty$ denote the subcategory consisting of essentially small idempotent complete ∞-categories that admit ω-small colimits, and functors that preserve ω-small colimits. Since the subcategory of ω-compact objects in a compactly generated ∞-category is stable under ω-small colimits, and the morphisms in $\mathcal{P}r^L_\omega$ preserve all colimits, the functor $(-)^\omega$ factors through $\mathrm{Cat}_\infty^{\mathrm{REx}(\omega)^\vee}$:

$$\begin{tikzcd}
\mathcal{P}r^L_\omega \arrow[r, dotted] \arrow[dr, "(-)^\omega"'] & \mathrm{Cat}_\infty^{\mathrm{REx}(\omega)^\vee} \arrow[d] \\
& \mathrm{Cat}_\infty
\end{tikzcd}$$

The right vertical map is manifestly conservative. The dotted arrow is an equivalence by virtue of T.5.5.7.9 and T.5.5.7.10. The lemma follows. $\square$

Lemma 2.3.5. *The categories $\mathcal{P}r^L_\omega$ and Cat_∞ admit all small limits, and the functor $(-)^\omega : \mathcal{P}r^L_\omega \to \mathrm{Cat}_\infty$ preserves and reflects and limits.*

Proof. The fact that $\mathcal{P}r^L_\omega$ admits all limits is a consequence of A.6.3.7.9., and A.6.3.7.10., which state that $\mathcal{P}r^L_\omega$ is in fact presentable. The fact that Cat_∞ admits all limits is proven in T.3.3.3.

Let K be a simplicial set, let $\nu : K^\triangleleft \to \mathcal{P}r^L_\omega$ be a diagram, and let $\mathcal{V} \to K^\triangleleft$ be a coCartesian fibration classified by ν. Then $\mathcal{V}$ is characterized by: $\mathrm{Fun}^{\mathrm{L}}_\omega(\mathcal{W}, \mathcal{V}_\infty) \simeq \lim_{x \in K} \mathrm{Fun}^{\mathrm{L}}_\omega(\mathcal{W}, \mathcal{V}_x)$ for all $\mathcal{W}$ in $\mathcal{P}r^L$. Here, the limit on the right is computed in Cat_∞ (note that the functor categories are essentially small). Taking $\mathcal{W} = \mathcal{S}$, the ∞-category freely generated under colimits by a single object, we find that $\mathcal{V}^\omega_\infty \simeq \lim \mathcal{V}^\omega_x$. This proves that $(-)^\omega$ preserves small limits. The fact that it also reflects limits follows from the fact that $(-)^\omega$ is conservative (Lemma 2.3.4), and Lemma 2.3.3. $\square$

The relationship between the limit of a diagram $\nu : K \to \mathcal{P}r^L_\omega$, and the limit of the induced diagram $\nu' : K \to \widehat{\mathrm{Cat}}_\infty$ is more subtle. The definition that follows isolates a key property of diagrams with values in $\widehat{\mathrm{Cat}}_\infty$ that facilitates the comparison of the two limits.

Definition 2.3.6. Let K be a simplicial set, and let $\nu : K^{\triangleleft} \to \widehat{\mathrm{Cat}}_\infty$ be a limit diagram classifying a coCartesian fibration $\nu^{\flat} : \mathcal{V} \to K^{\triangleleft}$. Let $\psi_x : \mathcal{V}_\infty \to \mathcal{V}_k$ be the natural functor. We will say that ν has Property ¶ is the following condition is satisfied: An object X in $\mathcal{V}_\infty$ is compact if and only if $\psi_x(X)$ is compact for all x in K.

The two propositions that follow describe restrictions on the codomain of a diagram that ensure that it have Property ¶. One of the main reasons why perfect categories play an important role in this paper is that diagrams taking values in perfect categories have Property ¶.

Lemma 2.3.7. *Let K be a simplicial set and let $\mu : K^{\triangleleft} \to \mathcal{P}r^L_{\mathrm{perf}}$ be a limit diagram taking values in perfect categories. Let $\pi : \mathcal{P}r^L_{\mathrm{perf}} \to \mathcal{P}r^L_{\omega,\mathrm{k}}$ be the forgetful functor and let $\nu = \pi \circ \mu$. Then ν has Property ¶.*

Proof. This follows from the definition of a perfect category, and the following propostion. □

Proposition 2.3.8 (Lurie)**.** *Let $\nu : K^{\triangleleft} \to \mathrm{CAlg}(\mathcal{P}r^L)$ be a diagram of symmetric monoidal categories, and let $\mathcal{V} \to K^{\triangleleft}$ be a coCartesian fibration classified by ν. Then $X \in \mathcal{V}^{\mathrm{fd}}_\infty$ if and only if $\psi_x(X) \in \mathcal{V}^{\mathrm{fd}}_\infty$ for all x in K.*

Proof. This is A.2.4.5.11. Here is a rough outline. To a symmetric monoidal ∞-category $\mathcal{C}$ one associates the ∞-groupoid $\mathrm{DDat}(\mathcal{C})$ of duality data in $\mathcal{C}$. Roughly speaking, $\mathrm{DDat}(\mathcal{C})$ classifies triplies $(X, X^\vee, \mathrm{ev}_X, \mathrm{coev}_X)$, where $\mathrm{ev}_X : X \otimes X^\vee \to \mathbf{1}$ and $\mathrm{coev}_X : \mathbf{1} \to X^\vee \otimes X$ are morphisms exhibiting $X^\vee$ as dual to X. The essential thing to check is that the map $\mathrm{DDat}(\mathcal{C}) \to \mathcal{C}^{\mathrm{fd}}$ that carries $(X, X^\vee, \mathrm{ev}_X, \mathrm{coev}_X)$ to X is a trivial fibration. This is A.4.2.5.10. The result then follows from the observation that the functor $\mathcal{C} \to \mathrm{DDat}(\mathcal{C})$ commutes with limits. $\square$

In contrast with the previous two results, the following lemma describes a restriction on the domain of a diagram that ensures that Property ¶ holds. This observation will play an important role in proving that compact objects descend along étale morphisms.

Lemma 2.3.9. *Let K be a finite simplicial set. Then every limit diagram $\nu : K^\triangleleft \to \mathcal{P}r^L_\omega$ has Property ¶.*

Proof. Let $\mathcal{V} \to K^\triangleleft$ be a coCartesian fibration classified by ν. Let $X \in \mathcal{V}_\infty$The only thing that needs proof is that if $\psi_x(X)$ is compact for all x in K then X is compact in $\mathcal{V}_\infty$.

The category $\mathrm{Sect}(K, \mathcal{V})$ of coCartesian sections of $\mathcal{V}_{|K}$ are a full subcategory of the functor category $\mathrm{Fun}(K, \mathcal{V})$. Mapping spaces in functor categories are computed by ends. In particular, (by virtue of the equivalence $\mathrm{Sect}(K, \mathcal{V}) \simeq \mathcal{V}_\infty$) we have for X, A in $\mathcal{V}_\infty$, $\mathcal{V}_\infty(X, A) \simeq \mathrm{End}_{x \in K}(\chi, \alpha)$, where χ, α are coCartesian sections of $\mathcal{V}_{|K}$

with $\chi_\infty = X$ and $\alpha_\infty = A$. Since K is a finite simplicial set, this end is a finite limit.

Now assume that $\chi_x \in \mathcal{V}_x^\omega$ for all x in K, and let Λ be an ω-filtered category, and let $A_\bullet : \Lambda \to \mathcal{V}_\infty$ be a diagram. Let $\alpha_\bullet : \Lambda \to \mathrm{Sect}(K, \mathcal{V})$ be the induced diagram. We have the commutative diagram:

$$\begin{tikzcd}
\operatorname{colim} \mathcal{V}_\infty(X, A_\lambda) \arrow[r] \arrow[d, "\wr"] & \mathcal{V}_\infty(X, \operatorname{colim} A_\lambda) \arrow[d, "\wr"] \\
\operatorname{colim} \mathrm{End}_{x\in K}\mathcal{V}_x(\chi_x, \alpha_{\lambda,x}) \arrow[r] \arrow[d, "\wr"] & \mathrm{End}_{x\in K}\mathcal{V}_x(\chi_x, \operatorname{colim} \alpha_{\lambda,x}) \arrow[d, "\wr"] \\
\mathrm{End}_{x\in K}\operatorname{colim} \mathcal{V}_x(\chi_x, \alpha_{\lambda,x}) \arrow[r, "\sim"] & \mathrm{End}_{x\in K}\mathcal{V}_x(\chi_x, \operatorname{colim} \alpha_{\lambda,x})
\end{tikzcd}$$

The upper vertical maps are equivalences by the paragraph above. The bottom vertical maps are equivalences because ω-filtered colimits commute with ω-small limits in $\mathcal{S}$ (T.5.3.3.3). Finally, the bottom horizontal map is an equivalence by our assumption that $\chi_x \in \mathcal{V}_x^\omega$. It follows that the top horizontal morphism is an equivalence, proving that $X \in \mathcal{V}_\infty^\omega$. $\square$

We now turn our attention to establishing a relationship between the limit of a diagram $\nu : K \to \mathcal{P}r_\omega^L$ and the limit of the induced diagram $\nu' : K \to \widehat{\mathrm{Cat}}_\infty$, obtained by composing ν with the inclusion $\mathcal{P}r_\omega^L \subseteq \widehat{\mathrm{Cat}}_\infty$. Property ¶ will play a central role in this discussion.

Lemma 2.3.10. *Let K be a simplicial set, and let $\nu : K^\triangleleft \to \mathcal{P}r_\omega^L$ be a diagram. Let $i : \mathcal{P}r_\omega^L \to \widehat{\mathrm{Cat}}_\infty$ be the natural inclusion. Assume that $i \circ \nu$ is a limit diagram which has Property ¶. Then ν is a limit diagram.*

Proof. Since the inclusion $\mathcal{P}r^L \subseteq \widehat{\mathrm{Cat}}_\infty$ preserves and reflects limits, we may view ν as a limit diagram in $\mathcal{P}r^L_\omega$. The limit $\mathcal{V}_\infty$ of $\nu_{|K}$ in $\mathcal{P}r^L$ is charaterized upto equivalence by $\mathrm{Fun}^{\mathrm{L}}(\mathcal{W}, \mathcal{V}_\infty) \simeq \lim_{x\in K}\mathrm{Fun}^{\mathrm{L}}(\mathcal{W}, \mathcal{V}_x)$, for any $\mathcal{W}$ in $\mathcal{P}r^L$. Our hypothesis that ν takes values in $\mathcal{P}r^L_\omega$ implies that each of the functors ψ_x preserves ω-compact objects, and therefore this equivalence restricts to a fully faithful functor $\mathrm{Fun}^{\mathrm{L}}_\omega(\mathcal{W}, \mathcal{V}_\infty) \to \lim_{x\in K}\mathrm{Fun}^{\mathrm{L}}_\omega(\mathcal{W}, \mathcal{V}_x)$, where $\mathrm{Fun}^{\mathrm{L}}_\omega(-,-) \subseteq \mathrm{Fun}^{\mathrm{L}}(-,-)$ denotes the full subcategory of functors that preserve ω-compact objects. We will show that this functor is essentially surjective.

Now let $\mathcal{W} \in \mathcal{P}r^L_\omega$, and let $w^\flat : \mathcal{W}^\sharp \to K^\triangleleft$ be a cocartesian fibration classified by the constant functor $K^\triangleleft \to \mathcal{P}r^L_\omega$ that sends every object to $\mathcal{W}$, and let $\sigma_x : \mathcal{W}^\sharp_\infty \to \mathcal{W}^\sharp_x$ denote the functor (equivalence) induced by the unique morphism $\infty \to x$. Let $X \in \lim_{x\in K}\mathrm{Fun}^{\mathrm{L}}_\omega(\mathcal{W}, \mathcal{V}_x)$, and let $\chi : \mathcal{W}^\sharp_{|K} \to \mathcal{V}_{|K}$ be the corresponding cocartesian section. Note that $\chi_x : \mathcal{W}^\sharp_x \to \mathcal{V}_x$ preserves ω-compact objects for all x in K. The equivalence $\mathrm{Fun}^{\mathrm{L}}(\mathcal{W}, \mathcal{V}_\infty) \simeq \lim_{x\in K}\mathrm{Fun}^{\mathrm{L}}(\mathcal{W}, \mathcal{V}_x)$ implies that χ extends to a map $\chi : \mathcal{W}^\sharp \to \mathcal{V}$ defined by a cocartesian section such that $\chi_\infty \in \mathrm{Fun}^{\mathrm{L}}(\mathcal{W}_\infty, \mathcal{V}_\infty)$. We have natural equivalences $\chi_x \circ \sigma_x \simeq \psi_x \circ \chi_x$, since χ is cocartesian.

Let $X \in \mathcal{W}^\sharp_\infty$ be a compact object. For every x in K, we have an equivalence $\chi_x(\sigma_x(X)) \simeq \psi_x(\chi_\infty(X))$ in $\mathcal{V}_x$. Since σ_x is an equivalence and σ_x preserves compact objects, we conclude that $\psi_x(\chi_\infty(X))$ is compact. Property ¶ now implies that $\chi_\infty(X)$ is compact. Thus $\chi_\infty \in \mathrm{Fun}^{\mathrm{L}}_\omega(\mathcal{W}^\sharp_\infty, \mathcal{V}_\infty)$, so χ defines an element X' in $\mathrm{Fun}^{\mathrm{L}}_\omega(\mathcal{W}, \mathcal{V}_\infty)$ that maps to $\mathcal{X}$. This proves essential surjectivity of the natural functor mapping $\mathrm{Fun}^{\mathrm{L}}_{\omega}(\mathcal{W}, \mathcal{V}_{\infty})$ to $\lim_{x \in K} \mathrm{Fun}^{\mathrm{L}}_{\omega}(\mathcal{W}, \mathcal{V}_x)$.

So we have $\mathrm{Fun}^{\mathrm{L}}_{\omega}(\mathcal{W}, \mathcal{V}_{\infty}) \simeq \lim_{x \in K} \mathrm{Fun}^{\mathrm{L}}_{\omega}(\mathcal{W}, \mathcal{V}_x)$. This equivalence characterizes $\mathcal{V}_{\infty}$ as a limit of $\nu_{|K}$ in $\mathcal{P}r^{L}_{\omega}$, so $\phi : K^{\triangleleft} \to \mathcal{P}r^{L}_{\omega}$ is a limit diagram.

□

Lemma 2.3.11. *Let K be a simplicial set, and let $\nu : K^{\triangleleft} \to \mathcal{P}r^{L}_{\omega}$ be a diagram, and let $i : \mathcal{P}r^{L}_{\omega} \to \widehat{\mathrm{Cat}}_{\infty}$ be the natural inclusion. Let $\nu' : K^{\triangleleft} \to \widehat{\mathrm{Cat}}_{\infty}$ be a limit diagram with $\nu'_{|K} \simeq i \circ \nu_{|K}$, and let $\mathcal{V}' \to K^{\triangleleft}$ be a coCartesian fibration classified by ν'. Assume that ν' has Property ¶, and that $\mathcal{V}'_{\infty}$ is compactly generated, i.e., $\mathcal{V}'_{\infty} \in \mathcal{P}r^{L}_{\omega}$. Then the following are equivalent:*

(1) $i \circ \nu$ is a limit diagram and has Property ¶.

(2) ν is a limit diagram.

(3) The induced diagram $(-)^{\omega} \circ \nu : K^{\triangleleft} \to \mathrm{Cat}_{\infty}$ is a limit diagram.

Proof. We have already proven that (1) ⇒ (2) and (2) ⇒ (3) (Lemmas 2.3.10 and 2.3.5 respectively). These implications do not require that additional hypothesis that $\mathcal{V}'_{\infty}$ is compactly generated.

We will now prove that (3) ⇒ (1). Assume that $(-)^{\omega} \circ \nu$ is a limit diagram. Let $\phi : \mathcal{V}_{\infty} \to \mathcal{V}'_{\infty}$ be the morphism induced by the universal property of $\mathcal{V}'$. We will complete the proof by showing that ϕ is an equivalence. By virtue of the fact that $\mathcal{P}r^{L} \subseteq \widehat{\mathrm{Cat}}_{\infty}$ reflects limits, we have that $\phi \in \mathrm{Fun}^{\mathrm{L}}(\mathcal{V}_{\infty}, \mathcal{V}'_{\infty})$. Since $\mathcal{V}'_{\infty}$ is compactly generated by hypothesis, it will suffice to show that ϕ induces an equivalence $\phi^\omega : \mathcal{V}^\omega_\infty \to (\mathcal{V}'_\infty)^\omega$ on the subcategories of compact objects. We now proceed to identify the compact objects on both sides.

The coCartesian fibration $\mathcal{V}^\flat \to K^\triangleleft$ classifying $(-)^\omega \circ \nu$ can be described as follows: it is the full subcategory $\mathcal{V}^\flat \subseteq \mathcal{V}$ whose objects are determined by the condition $\mathcal{V}^\flat_x = \mathcal{V}^\omega_x$ for all x in $K^\triangleleft$. Since $\mathcal{V}^\flat$ classifies a limit diagram, it follows that $\mathcal{V}^\omega_\infty$ can be identified with the full subcategory of cocartesian sections χ of $\mathcal{V}_{|K}$ that are levelwise compact (those for which χ_x is compact in $\mathcal{V}_x$ for all x in K).

One the other hand, since $\mathcal{V}'$ is a classifies $\widehat{\mathrm{Cat}}_\infty$-limit diagram and has property ¶, $(\mathcal{V}'_\infty)^\omega$ can be identified with the cocartesian sections of $\mathcal{V}'_{|K}$ that are levelwise compact. Since $\mathcal{V}'_{|K} \simeq \mathcal{V}_{|K}$ by hypothesis, this show that $\mathcal{V}^\omega_\infty \simeq (\mathcal{V}')^\omega_\infty$. This completes the proof. □

We conclude this section with two lemmas that will be used in the sequel

Lemma 2.3.12. *The functor $\pi : \mathcal{P}r^L_{\omega,\mathrm{k}} \to \mathcal{P}r^L_{\mathrm{k}}$ preserves reflects ω-small products.*

Proof. This follows from the following fact: If $\{\mathcal{C}_\alpha\}$ is a finite family of ∞-categories with product $\mathcal{C}$, then an object X in $\mathcal{C}$ is ω-compact as soon as its image in each $\mathcal{C}_\alpha$ is ω-compact (T.5.3.4.10). □

Lemma 2.3.13. *The functors $(-)^\simeq : \widehat{\mathrm{Cat}}_\infty \to \widehat{\mathcal{S}}$ and $(-)^\simeq : \mathrm{Cat}_\infty \to \mathcal{S}$, which carry an ∞-category to the maximal ∞-groupoid that it contains, preserve all limits.*

Proof. The functor $(-)^\simeq$ is a right adjoint, and therefore preserves all limits: the natural inclusion $\pi_{\leq\infty}$ of spaces into ∞-categories is left adjoint to $(-)^\simeq$. □

2.4 Moduli of Compact Branes

The purpose of this section is to give precise definitions of the moduli functors that will be the central object of study in the next two chapters of this thesis.

To a quasi-compact quasi-separated scheme X over a field k, one can associate two moduli functors $\mathcal{M}_X^{\mathrm{QC}} : \mathrm{CAlg}_{\mathrm{k}} \to \widehat{\mathrm{Cat}}_\infty$ and $\mathcal{M}_X^{\mathrm{perf}} : \mathrm{CAlg}_{\mathrm{k}} \to \mathrm{Cat}_\infty$ parametrizing quasi-coherent sheaves on X and perfect complexes on X respectively. These functors can be described by the formulas

$$\mathcal{M}_X^{\mathrm{QC}}(A) := \mathrm{QC}(X \times_{\mathrm{k}} \mathrm{Spec}(A))$$

$$\mathcal{M}_X^{\mathrm{perf}}(A) := \mathrm{Perf}(X \times_{\mathrm{k}} \mathrm{Spec}(A))$$

The tensor product theorem, Proposition 2.2.23, empowers us with the following enlightening description of these functors

$$\mathcal{M}_X^{\mathrm{QC}}(A) := \mathrm{QC}(X) \otimes_{\mathrm{Mod}_{\mathrm{k}}} \mathrm{Mod}_A$$

$$\mathcal{M}_X^{\mathrm{perf}}(A) := (\mathrm{QC}(X) \otimes_{\mathrm{Mod}_{\mathrm{k}}} \mathrm{Mod}_A)^\omega$$

This last description of the functors makes manifest the fact that the functors $\mathcal{M}_X^{\mathrm{QC}}$ and $\mathcal{M}_X^{\mathrm{perf}}$ are in fact invariants of the noncommutative shadow $\mathrm{QC}(X)$ of X. It suggests that for any $\mathrm{k} \in \mathrm{CAlg}_{\mathrm{k}}$ and any $\mathcal{X} \in \mathcal{P}r_{\mathrm{k}}^L$ (i.e., for any noncommutative space) one should introduce moduli functors $\mathcal{M}_{\mathcal{X}}^{\dagger} : \mathrm{CAlg}_{\mathrm{k}} \to \mathcal{P}r_{\mathrm{k}}^L$, and $\mathcal{M}_{\mathcal{X}}^{\flat}$:

$\mathrm{CAlg}_{\mathrm{k}} \to \mathrm{Cat}_\infty$ defined by the formulae

$$\mathcal{M}^{\dagger}_{\mathcal{X}}(A) := \mathcal{X} \otimes_{\mathrm{Mod}_{\mathrm{k}}} \mathrm{Mod}_A$$

$$\mathcal{M}^{\flat}_{\mathcal{X}}(A) := (\mathcal{X} \otimes_{\mathrm{k}} \mathrm{Mod}_A)^{\omega}$$

If $\mathcal{X}$ is, furthermore, compactly generated, then $\mathcal{X} \otimes_{\mathrm{Mod}_{\mathrm{k}}} \mathrm{Mod}_A$ is compactly generated. We have already seen in the section on quasi-coherent sheaves, that for any $f : A \to B$ in $\mathrm{CAlg}_{\mathrm{k}}$, the induced functor $B \otimes_A : \mathrm{Mod}_A \to \mathrm{Mod}_B$ preserves compact objects. Thus, the functor $\mathcal{M}^{\dagger}_{\mathcal{X}}$ admits a natural lift to a functor $\mathfrak{M}_{\mathcal{X}} : \mathrm{CAlg}_{\mathrm{k}} \to \mathcal{P}r^L_{\omega,\mathrm{k}}$. The association $\mathcal{X} \mapsto \mathfrak{M}_{\mathcal{X}}$ defines a functor $\mathcal{P}r^L_{\omega,\mathrm{k}} \to \mathrm{Fun}(\mathrm{CAlg}_{\mathrm{k}}, \mathcal{P}r^L_{\omega,\mathrm{k}})$. Indeed, this is the functor that is adjoint to the natural functor

$$\mathcal{P}r^L_{\omega,\mathrm{k}} \times \mathrm{CAlg}_{\mathrm{k}} \xrightarrow{\mathrm{id} \times \mathfrak{M}_{\mathbf{1}}} \mathcal{P}r^L_{\omega,\mathrm{k}} \times \mathcal{P}r^L_{\omega,\mathrm{k}} \xrightarrow{\otimes} \mathcal{P}r^L_{\omega,\mathrm{k}}$$

If $\mathcal{X}$ is the underlying category of a symmetric monoidal category $\mathcal{X}$, there is yet another analogue of the moduli of perfect complexes on a scheme, namely the functor $\mathcal{M}^{\vee}_{\mathcal{X}} : \mathrm{CAlg}_{\mathrm{k}} \to \mathrm{Cat}_\infty$ defined by

$$\mathcal{M}^{\vee}_{\mathcal{X}}(A) := (\mathcal{X} \otimes_{\mathrm{Mod}_{\mathrm{k}}} \mathrm{Mod}_A)^{\mathrm{fd}}$$

Recall that $(\mathcal{X} \otimes_{\mathrm{Mod}_{\mathrm{k}}} \mathrm{Mod}_A)^{\mathrm{fd}}$ denotes the subcategory of fully dualizable objects in the symmetric monoidal ∞-category $(\mathcal{X} \otimes_{\mathrm{Mod}_{\mathrm{k}}} \mathrm{Mod}_A)^{\mathrm{fd}}$.

Notation 2.4.1. We will need several variants of the functors that we have just introduced. These functors are defined by the requirement that the following diagram be commutative

$$
\begin{tikzcd}
 & \widehat{\mathrm{Cat}}_\infty \\
 & \mathcal{P}r^L_{\mathrm{k}} \arrow[u] \\
\mathrm{CAlg}_{\mathrm{k}} \arrow[uur, "\mathcal{M}^\natural_{\mathcal{X}}"] \arrow[ur, "\mathcal{M}^\dagger_{\mathcal{X}}"] \arrow[r, "\mathfrak{M}_{\mathcal{X}}"] \arrow[dr, "\mathcal{M}^\flat_{\mathcal{X}}"] \arrow[ddr, "\mathcal{M}_{\mathcal{X}}"] & \mathcal{P}r^L_{\omega,\mathrm{k}} \arrow[u] \arrow[d, "(-)^\omega"] \\
 & \mathrm{Cat}_\infty \arrow[d, "(-)^\simeq"] \\
 & \mathcal{S}
\end{tikzcd}
$$

Chapter 3

Brane Descent

In this chapter, we will study the descent properties of the moduli functors introduced in §2.4. The material is organized as follows. We begin, in §3.1 by recalling the Barr-Beck-Lurie monandicity theorem and its relation to descent. In §3.2 we apply the monadicity theorem to show that the moduli functor $\mathcal{M}^{\natural}_{\mathcal{X}}$ parametrizing all objects in a compactly generated category is a sheaf for the flat topology (Proposition 3.2.1). In §3.3, we use the results of the previous section to deduce that dualizable objects in compactly generated symmetric monoidal ∞-categories descend along flat maps (Proposition 3.3.1). The next section, §3.4, is devoted to the study of the descent properties of compact objects. Since the notion of compactness is not local for the flat topology, one cannot immediately deduce the descent properties of the functor $\mathcal{M}_{\mathcal{X}}$, parametrizing compact objects in $\mathcal{X}$, from the corresponding properties for $\mathcal{M}^{\natural}_{\mathcal{X}}$. Nevertheless, it turns out that for an arbitrary compactly generated category $\mathcal{X}$, the functor $\mathcal{M}_{\mathcal{X}}$ is a sheaf for the étale topology. Furthermore, if one imposes some finiteness conditions on $\mathcal{X}$, then this functor is in fact a sheaf for the flat topology. The main result of this section, Proposition 3.4.1, is an essential ingredient in Chapter 4, where we investigate the geometricity of the moduli of compact objects in a compactly generated category $\mathcal{X}$. However, the reader who is willing to accept this theorem on faith, can read that chapter independently of this one. Finally, in the last section, §3.5, we point out that all of the sheaves that we have consider are in fact hypersheaves - that is, they satisfy descent with respect to arbitrary flat/étale hypercovers. Theorem 3.5.3 summarizes all the results of this chapter. We will not need the results of this section in the rest of the paper.

3.1 The Barr-Beck-Lurie Theorem

The involution on the $(\infty, 2)$-category of ∞-categories that takes an ∞-category to its opposite, interchanges left adjoints with right adjoints, and monads with comonads. Consequently, every theorem about monads has a dual comonadic analogue. In particular, we have the following comonadic analogue of the Barr-Beck-Lurie theorem.

Theorem 3.1.1 (Lurie [Lur11b, Theorem 6.2.2.5])**.** *Let $f : \mathcal{C} \to \mathcal{D}$ be an ∞-functor that admits a right adjoint $g : \mathcal{D} \to \mathcal{C}$. Then the following are equivalent:*

1. *f exhibits $\mathcal{C}$ as comonadic over $\mathcal{D}$.*

2. *f satisfies the following two conditions:*

   (a) *f is conservative, i.e., it reflects equivalences.*

   (b) *Let U be a cosimplicial object in $\mathcal{C}$, which is f-split. Then U has a limit in $\mathcal{C}$, and this limit is preserved by f.*

In practice, we will use the following consequence of the comonadic Barr-Beck-Lurie theorem, which is the dual version of A.6.2.4.3. Recall the notion of a right adjointable diagram (A.6.3.2.13).

Proposition 3.1.2. *Let $\mathcal{C}^\bullet : \mathrm{N}(\Delta_+) \to \widehat{\mathrm{Cat}}_\infty$ be a coaugmented cosimplicial ∞-category, and set $\mathcal{C} := \mathcal{C}^{-1}$. Let $f : \mathcal{C} \to \mathcal{C}^0$ be the evident functor. Assume that:*

1. *The ∞-category $\mathcal{C}$ admits totalizations of f-split cosimplicial objects, and those totalizations are preserved by f.*

2. (Beck-Chevalley conditions) *For a morphism $\alpha : [m] \to [n]$ in Δ_+, let $\tilde{\alpha}$ be the morphism defined by $\tilde{\alpha}(0) = 0$ and $\tilde{\alpha}(i) = \alpha(i-1)$ for $1 \leq i \leq m$. Then for every α, the diagram*

$$\begin{tikzcd} \mathcal{C}^m \arrow[r, "d^0"] \arrow[d, "\alpha"] & \mathcal{C}^{m+1} \arrow[d, "\tilde{\alpha}"] \\ \mathcal{C}^n \arrow[r, "d^0"] & \mathcal{C}^{n+1} \end{tikzcd}$$

   is right adjointable.

Then the canonical map $\theta : \mathcal{C} \to \lim_\Delta \mathcal{C}^\bullet$ admits a fully faithful right adjoint. If f is conservative, then θ is an equivalence.

3.2 Flat Descent for Branes

In this section we will use comonadic yoga outlined in the previous section to show that families of branes descend along faithfully flat and quasi-compact morphisms. More precisely, we will prove the following theorem:

Proposition 3.2.1. *Let $\mathcal{X}$ be a presentable* k*-linear ∞-category. Assume that $\mathcal{X}$ is dualizable as an object of the symmetric monoidal ∞-category $\mathcal{P}r_{\mathrm{k}}^{L,\otimes}$ (this holds, in particular, when $\mathcal{X}$ is ω-compactly generated). Then the $\widehat{\mathrm{Cat}}_\infty$-valued presheaf $\mathcal{M}^\sharp_{\mathcal{X}}$ on $\mathrm{Aff}_{\mathrm{k}}$ defined in Notation 2.4.1 is a sheaf for the flat topology.*

The proof of the proposition will occupy the rest of this section. The essential point is to verify that $\mathcal{M}^\natural_{\mathcal{X}}$ carries the Čech nerve of a faithfully flat morphism $f : A \to A^0$ to a limit diagram in $\widehat{\mathrm{Cat}}_\infty$. For this, we will appeal to Proposition 3.1.2. We begin by collecting together some preliminary results that will allow us the verify the hypotheses of that Proposition.

The lemma that follows facilitates the verification of the "Beck-Chevalley conditions" of Proposition 3.1.2:

Lemma 3.2.2. *(Base change). For any $\mathcal{X}$ in $\mathcal{P}r^L_{\omega,\mathrm{k}}$, the functor $\mathcal{M}^\natural_{\mathcal{X}} : \mathrm{CAlg}_{\mathrm{k}} \to \widehat{\mathrm{Cat}}_\infty$ of Notation 2.4.1 carries cocartesian squares to right adjointable squares.*

Proof. The proof is essentially the same as [TV08, Prop 1.1.0.8]. Let

$$\begin{tikzcd} A \arrow[r, "p"] \arrow[d, "f"] & A' \arrow[d, "f'"] \\ B \arrow[r, "p'"'] & B' \end{tikzcd}$$

be a cocartesian square in $\mathrm{CAlg}_{\mathrm{k}}$, and let

$$\begin{tikzcd} \mathrm{Mod}_A(\mathcal{X}) \arrow[r, "p^*"] \arrow[d, "f^*"] & \mathrm{Mod}_{A'}(\mathcal{X}) \arrow[l, bend right, dotted, "p_*"'] \arrow[d, "f'^*"] \\ \mathrm{Mod}_B(\mathcal{X}) \arrow[r, "p'^*"'] & \mathrm{Mod}_{B'}(\mathcal{X}) \arrow[l, bend right, dotted, "p'_*"'] \end{tikzcd}$$

be the diagram in $\widehat{\mathrm{Cat}}_\infty$ by induced by $\mathcal{M}^\natural_{\mathcal{X}}$. Here, for a morphism $p : A \to A'$ in $\mathrm{CAlg}_{\mathrm{k}}$, $p^* := \mathcal{M}^\natural_{\mathcal{X}}(p) = \mathfrak{M}_X(p) = A \otimes_{A'} (-)$, and $p_* : \mathrm{Mod}_{A'}(\mathcal{X}) \to \mathrm{Mod}_A(\mathcal{X})$ is the forgetful functor, which is right adjoint to p^* (see Remark **??**).

Let $M \in \mathrm{Mod}_{A'}(\mathcal{X})$. To prove the lemma, we must show that the natural morphism $\nu_M : f^* p_* M \to p'_* f'^* M$ is an equivalence. This follows from the following peculiarity of *commutative* algebras: pushouts coincide with tensor products in $\mathrm{CAlg}_{\mathrm{k}}$, i.e., we have $B' \simeq A' \coprod_A B \simeq A' \otimes_A B$. Consequently, we have a chain of equivalences:

$$M \otimes_{A'} B' \xrightarrow{\sim} M \otimes_{A'} (A' \otimes_A B) \xrightarrow{\sim} M \otimes_A B$$

which, as the reader will readily check, is a homotopy inverse of ν_M. $\square$

Let $A^\bullet : \mathrm{N}(\Delta_+) \to \mathrm{CAlg}_{\mathrm{k}}$ be the Čech nerve of a faithfully flat morphism $f : A \to A^0$, and let $\mathcal{X} \in \mathcal{P}r^L_{\omega,\mathrm{k}}$. In order to deduce from the base change lemma that the cosimplicial ∞-category $\mathcal{M}^\natural_{\mathcal{X}}(A^\bullet)$ satisfies the Beck-Chevalley conditions, we need following simple observation:

Lemma 3.2.3. *Let $f : A \to A^0$ be a morphism in* $\mathrm{CAlg}_{\mathrm{k}}$*, and let $A^\bullet : \mathrm{N}(\Delta_+) \to$* $\mathrm{CAlg}_{\mathrm{k}}$ *be the Čech nerve $\check{C}(f) := cosk_0(f)$. For a morphism α in Δ_+, let $\tilde{\alpha}$ be the* *morphism defined in Proposition 3.1.2. Then for each $\alpha : [m] \to [n]$ in Δ_+, the following diagram is cocartesian:*

$$\begin{tikzcd} A^m \arrow[r, "d^0"] \arrow[d, "A(\alpha)"'] & A^{m+1} \arrow[d, "A(\tilde{\alpha})"] \\ A^n \arrow[r, "d^0"] & A^{n+1} \end{tikzcd} \tag{$\dagger$}$$

Proof. Since $A^\bullet$ is the 0-coskeleton of f, we have $A^n \simeq \otimes_A^{n+1} A^0 \simeq \coprod_A^{n+1} A^0$, and d^0 is the inclusion of the summand $\coprod_A^{n+1} A^0 \to A^0 \coprod_A (\coprod_A^{n+1} A^0)$. It follows immediately that the square ($\dagger$) is cocartesian for any $\alpha : [m] \to [n]$. □

Lemma 3.2.5 almost says that if $A^\bullet : \mathrm{N}(\Delta_+) \to \mathrm{CAlg}_{\mathrm{k}}$ is the Čech nerve of a flat morphism $f : A \to A^0$ in $\mathrm{CAlg}_{\mathrm{k}}$, then $\mathcal{M}_{\mathcal{X}}^{\natural}(A^\bullet)$ satisfies condition 1. of Proposition 3.1.2. In preparation for the proof of Lemma 3.2.5, we prove the following special case of that lemma:

Lemma 3.2.4. *Let $f : A \to B$ be a morphism in $\mathrm{CAlg}_{\mathrm{k}}$, and let $f^* : \mathrm{Mod}_A \to \mathrm{Mod}_B$ be the functor defined by $f^*(M) := B \otimes_A M$. Assume that f is flat. Then f^* preserves totalizations of f^*-split cosimplicial objects.*

Proof. Let $M^\bullet \in \mathrm{Fun}(\mathrm{N}(\Delta), \mathrm{Mod}_A)$ be an f^*-split cosimplicial module. We wish to show that the natural map

$$B \otimes_A |M^\bullet| \longrightarrow |B \otimes_A M^\bullet| \tag{$*$}$$

is an equivalence. Since the forgetful functor, $\mathrm{Mod}_B \to \mathcal{S}_\infty$ is conservative, Whitehead's theorem implies that it will suffice to show that the induced morphism on π_n

is an isomorphism for all $n \in \mathbb{Z}$. We will use the Bousfield-Kan spectral sequence to compute the homotopy groups, and show that we have an isomorphism on the E_2 page.

There is a Bousfield-Kan spectral sequence with $E_2^{p,q} = \pi^{-p}\pi_q M^\bullet$, and $E_\infty^{p+q} = \pi_{p+q}|M^\bullet|$. Here $\pi^{-p}\pi_q M^\bullet$ is the $(-p)$th cohomotopy group of the cosimplicial abelian group $\pi_q M^\bullet$. Since B is a flat A-module, $\pi_0 B$ is a flat $\pi_0 A$-module. So we have an induced spectral sequence with $E_2^{p,q} = \pi_0 B \otimes_{\pi_0 A} \pi^{-p}\pi_q M^\bullet$ and $E_\infty^{p+q} = \pi_0 B \otimes_{\pi_0 A} \pi_{p+q}|M^\bullet|$. Finally, since B is flat over A, we have $\pi_{p+q}(B \otimes_A |M^\bullet|) \simeq \pi_0 B \otimes_{\pi_0 A} \pi_{p+q}|M^\bullet|$. So, in summary, we have a spectral sequence

$$E_2^{p,q} = \pi_0 B \otimes_{\pi_0 A} \pi^{-p}\pi_q M^\bullet \Longrightarrow \pi_{p+q}(B \otimes_A |M^\bullet|)$$

Similarly, we have a Bousfeld-Kan spectral sequence for the right hand side of $(*)$, with $E_2^{p,q} = \pi^{-p}\pi_q(B \otimes_A M^\bullet)$ and $E_\infty^{p+q} = \pi_{p+q}|B \otimes_A M^\bullet|$. Using the flatness of B over A again, we have $\pi^{-p}\pi_q(B \otimes_A M^\bullet) \simeq \pi^{-p}(\pi_0 B \otimes_{\pi_0 A} \pi_q M^\bullet) \simeq \pi_0 B \otimes_{\pi_o A} \pi^{-p}\pi_q M^\bullet$. So the spectral sequence becomes

$$E_2^{p,q} = \pi_0 B \otimes_{\pi_0 A} \pi^{-p}\pi_q M^\bullet \Longrightarrow \pi_{p+q}|B \otimes_A M^\bullet|$$

Thus, the E_2 pages of the spectral sequences for the left and right hand sides of $(*)$ coincide. To complete the proof, it will suffice to show that these spectral sequences degenerate. Let $N \to N^\bullet$ be a split coaugmented cosimplicial B-module. Then $\pi_q N \to \pi_q N^\bullet$ is a split coaugmented cosimplical abelian group for all q, and so we have $\pi^{-p}\pi_q N^\bullet = 0$ for $p \neq 0$, and $\pi^0 \pi_q N^\bullet = \pi_q N$. Applying this to $N^\bullet := B \otimes_A M^\bullet$, and $N = |B \otimes_A M^\bullet|$, we see that both the spectral sequences above degenerate at the E_2 page. $\square$

Lemma 3.2.5. *Let $f : A \to B$ be a flat morphism in $\mathrm{CAlg}_{\mathrm{k}}$, and let $\mathcal{X}$ be an object of $\mathcal{P}r^L_{\omega,\mathrm{k}}$. Then the category $\mathcal{M}^\natural_{\mathcal{X}}(A)$ admits all small limits, and the functor $\mathcal{M}^\natural_{\mathcal{X}}(f) : \mathcal{M}^\natural_{\mathcal{X}}(A) \to \mathcal{M}^\natural_{\mathcal{X}}(B)$ preserves totalizations of $\mathcal{M}^\natural_{\mathcal{X}}(f)$-split cosimplicial objects.*

Proof. The first statement is clear: the ∞-category $\mathcal{M}^\natural_{\mathcal{X}}(A) \simeq \mathcal{X} \otimes_{\mathrm{Mod}_{\mathrm{k}}} \mathrm{Mod}_A$ is presentable (2.2.6), and in particular admits all small limits and colimits.

Since $\mathcal{X}$ is a compactly generated k-linear ∞-category, Proposition **??** says that the restricted Yoneda embedding gives an equivalence $\mathcal{X} \simeq \mathrm{Fun}_{\mathrm{k}}(\mathcal{X}^\omega, \mathrm{Mod}_{\mathrm{k}})$. Furthermore, for any A in $\mathrm{CAlg}_{\mathrm{k}}$, we have $\mathcal{M}^\natural_{\mathcal{X}}(A) \simeq \mathrm{Mod}_A(\mathrm{Fun}_{\mathrm{k}}(\mathcal{X}^\omega, \mathrm{Mod}_{\mathrm{k}})) \simeq \mathrm{Fun}_{\mathrm{k}}(\mathcal{X}^\omega, \mathrm{Mod}_A)$.

Let $X \in \mathcal{X}^\omega$ be an object classified by a morphism of small k-linear ∞-categories $\psi_X : \mathfrak{B}\mathrm{k} \to \mathcal{X}^\omega$. Using the natural identifications $\mathrm{Fun}_{\mathrm{k}}(\mathfrak{B}\mathrm{k}, \mathrm{Mod}_A) \simeq \mathrm{Mod}_{\mathrm{k} \otimes A} \simeq \mathrm{Mod}_A$, we see that pullback along ψ_X defines a functor $\psi^*_{X,A} : \mathrm{Fun}_{\mathrm{k}}(\mathcal{X}^\omega, \mathrm{Mod}_A) \to \mathrm{Mod}_A$.

Let $f : A \to B$ be a morphism in $\mathrm{CAlg}_{\mathrm{k}}$. Under the identification $\mathcal{M}^\natural_{\mathcal{X}}(A) \simeq \mathrm{Fun}_{\mathrm{k}}(\mathcal{X}^\omega, \mathrm{Mod}_A)$, the functor $\mathcal{M}^\natural_{\mathcal{X}}(f)$ corresponds to the functor $f^* \circ (-)$, where $f^* := \mathcal{M}^\natural_{\mathbf{1}}(f) = B \otimes_A (-)$. Furthermore, for every X in $\mathcal{X}^\omega$, we have a homotopy commutative diagram in $\widehat{\mathrm{Cat}}_\infty$

$$\begin{tikzcd}
\mathrm{Fun}_{\mathrm{k}}(\mathcal{X}^\omega, \mathrm{Mod}_A) \arrow[r, "f^*\circ(-)"] \arrow[d, "\psi^*_{X,A}"'] & \mathrm{Fun}_{\mathrm{k}}(\mathcal{X}^\omega, \mathrm{Mod}_B) \arrow[d, "\psi^*_{X,B}"] \\
\mathrm{Mod}_A \arrow[r, "f^*"'] & \mathrm{Mod}_B
\end{tikzcd}$$

Now suppose that $f : A \to B$ is a flat morphism. Let $M^\bullet$ be a cosimplicial object in $\mathrm{Fun}_{\mathrm{k}}(\mathcal{X}^\omega, \mathrm{Mod}_A)$, for which the induced cosimplicial object $f^*M^\bullet$ is split. To complete the proof of the lemma, it will suffice to show that the natural morphism $\nu_{M^\bullet} : f^*(\lim M^\bullet) \to \lim f^*(M^\bullet)$ is an equivalence.

Since the family of functors $\{\psi^*_{X,B}\}_{X\in\mathcal{X}^\omega}$ is jointly conservative, it is enough to show that for each X in $\mathcal{X}^\omega$, the morphism $\psi^*_{X,B}(\nu_{M^\bullet})$ is an equivalence. The commutativity of the diagram above, together with the fact that the functors $\psi^*_{X,-}$ commute with all limits, implies that this equivalent to showing that the natural morphism $\nu_{\psi^*_{X,A}(M^\bullet)} : f^*(\lim \psi^*_{X,A}) \to \lim f^*(\psi^*_{X,A}(M^\bullet))$ is an equivalence for every object X in $\mathcal{X}^\omega$.

Note that the cosimplicial B-module $f^*(\psi^*_{X,A}(M^\bullet))$ is split, being the image under $\psi^*_{X,B}$ of the split cosimplicial object $f^*(M^\bullet)$. Applying Lemma 3.2.4 to the A-module $\psi^*_{X,A}(M^\bullet)$, we see that the morphism $\nu_{\psi^*_{X,A}(M^\bullet)}$ is an equivalence for every X is $\mathcal{X}^\omega$. $\square$

Lemma 3.2.6. *Let $f : A \to B$ be a faithfully flat morphism in* $\mathrm{CAlg}_{\mathrm{k}}$*, and let $\mathcal{X}$ be* *an object in $\mathcal{P}r^L_{\omega,\mathrm{k}}$. Then the functor $\mathcal{M}^\natural_X(f) : \mathcal{M}^\natural_{\mathcal{X}}(A) \to \mathcal{M}^\natural_{\mathcal{X}}(A)$ is conservative.*

Proof. We will retain the notation from Lemma 3.2.5. Since the family $\{\psi^*_{X,B}\}_{X\in\mathcal{X}^\omega}$ is jointly conservative, $\mathcal{M}^\natural_{\mathcal{X}}(f)$ is conservative if and only if $\{\psi^*_{X,B}\circ\mathcal{M}^\natural_{\mathcal{X}}(f)\}_{X\in\mathcal{X}^\omega} = \{f^*\circ\psi^*_{X,A}\}_{X\in\mathcal{X}^\omega}$ is a jointly conservative family. Using the fact that $\{\psi^*_{X,A}\}_{X\in\mathcal{X}^\omega}$ is jointly conservative, we see that this is equivalent to asking that $f^* : \mathrm{Mod}_A \to \mathrm{Mod}_B$ is conservative. Since Mod_B is stable, this is equivalent to asking that f^* reflects zero objects. But this is what is means for a flat morphism to be *faithfully* flat. $\square$

Lemma 3.2.7. *Let $\mathcal{X}$ be an object in $\mathcal{P}r^L_{\omega,\mathrm{k}}$. The functor $\mathcal{M}^\natural_{\mathcal{X}}$ preserves finite products.*

Proof. This is formal. Let A_i, $i = 1, 2$, be commutative k-algebras, and let $A := A_1 \times A_2$. Consider the adjunction

$$\begin{tikzcd} \mathcal{M}^\natural_{\mathcal{X}}(A) \arrow[r, bend left] & \mathcal{M}^\natural_{\mathcal{X}}(A_1) \times \mathcal{M}^\natural_{\mathcal{X}}(A_2) \arrow[l] \end{tikzcd}$$

The left adjoint, which is the natural morphism $\mathcal{M}^\natural_{\mathcal{X}}(A) \to \lim \mathcal{M}^\natural_{\mathcal{X}}(A_i)$, carries $M \in \mathrm{Mod}_A(\mathcal{X})$ to $(M \otimes_A A_1, M \otimes_A A_2)$. The right adjoint carries (M_1, M_2) to $p_{1*}M_1 \times p_{2*}M_2$, where $p_i : A \to A_i$ is the natural projection, and $p_{i*} : \mathrm{Mod}_{A_i}(\mathcal{X}) \to \mathrm{Mod}_A(\mathcal{X})$ is the forgetful functor. We will show that the unit and counit of this adjunction are equivalences.

The ∞-category $\mathrm{Mod}_A(\mathcal{X})$ is stable, and therefore we have natural equivalences $M \oplus N \simeq M \times N$ for M, N in $\mathrm{Mod}_A(\mathcal{X})$. Using this, together with the projection formula, we have, for M in $\mathrm{Mod}_A(\mathcal{X})$:

$$\begin{aligned} p_{1*}(M \otimes_A A_1) \times p_{2*}(M \otimes_A A_2) &\simeq M \otimes_A p_{1*}A_1 \times M \otimes_A p_{2*}A_2 \\ &\simeq (M \otimes_A p_{1*}A_1) \oplus (M \otimes_A p_{2*}A_2) \\ &\simeq M \otimes_A (p_{1*}A_1 \oplus p_{2*}A_2) \\ &\simeq M \otimes_A A \\ &\simeq M \end{aligned}$$

One checks that the composite morphism $p_{1*}(M \otimes_A A_1) \times p_{2*}(M \otimes_A A_2) \to M$ is inverse to the unit of the adjunction, proving that the unit is an equivalence.
For M_i in $\mathrm{Mod}_{A_i}(\mathcal{X})$, we have natural equivalences $p_{i*}M_i \otimes_A A_i \simeq M_i$ and $p_{i*}M_i \otimes_A A_j \simeq 0$ for $i \neq j$. From this, one immediately deduces that the natural maps $(p_{1*}M_1 \times p_{2*}M_2) \otimes_A A_i \simeq (p_{1*}M_1 \otimes_A A_i) \oplus (p_{2*}M_2 \otimes_A A_i) \to M_i$ are equivalences. This shows that the counit is an equivalence. $\square$

We are now in a position to prove the main proposition of this section.

Proof of Proposition 3.2.1. We will first consider the case where $\mathcal{X}$ is compactly generated. Let $\mathcal{X}$ be in $\mathcal{P}r^L_{\omega,\mathrm{k}}$. We must show that $\mathcal{M}^\sharp_\mathcal{X}$ preserves finite products and carries the Čech nerve of and flat morphism to a limit diagram. By virtue of Lemma 3.2.7, only the second statement remains to be proved.

Let $U^\bullet : \mathrm{N}(\Delta_+^{op}) \to \mathrm{Aff}_\mathrm{k}$ be the Čech nerve of a flat morphism, and let $A^\bullet := \mathcal{O}(U^\bullet) : \mathrm{N}(\Delta_+) \to \mathrm{CAlg}_\mathrm{k}$. Put $A := A^{-1}$. Lemma 3.2.5 says that the associated diagram $\mathcal{M}^\natural_\mathcal{X}(A^\bullet) : \mathrm{N}(\Delta_+^{op}) \to \widehat{\mathrm{Cat}}_\infty$, satisifies condition 1. of the corollary of the

Barr-Beck-Lurie theorem, Proposition 3.1.2. The base change lemma for branes (Lemma 3.2.2), together with Lemma 3.2.3, implies that $\mathcal{M}^{\natural}_{\mathcal{X}}(A^{\bullet})$ satisfies condition 2. of Proposition 3.1.2. Finally, Lemma 3.2.6 tells us that the natural map $\mathcal{M}^{\natural}_{\mathcal{X}}(A) \to \mathcal{M}^{\natural}_{\mathcal{X}}(A^0)$ is conservative. Thus, by Proposition 3.1.2, the natural map $\mathcal{M}^{\natural}_{\mathcal{X}}(A) \to \lim \mathcal{M}^{\natural}{}_{\mathcal{X}}(A^n)$ is an equivalence. This proves the proposition when $\mathcal{X}$ is compactly generated.

For the general case, let $\mathcal{X}$ be a dualizable object in $\mathcal{P}r^{L,\otimes}_{\mathrm{k}}$, and let $A^{\bullet}$ be as above. By the compactly generated case, we know that $\mathcal{M}^{\natural}_{\mathbf{1}}(A^{\bullet})$ is a limit diagram in $\widehat{\mathrm{Cat}}_{\infty}$, and therefore $\mathrm{Mod}_{A^{\bullet}}$ is a limit diagram in $\mathcal{P}r^{L}_{\mathrm{k}}$ by Lemma 2.3.2. Since $\mathcal{X}$ is a dualizable object of $\mathcal{P}r^{L}_{\mathrm{k}}$, the functor $\mathcal{X} \otimes_{\mathrm{Mod}^{\otimes}_{\mathrm{k}}} (-) : \mathcal{P}r^{L}_{\mathrm{k}} \to \mathcal{P}r^{L}_{\mathrm{k}}$ commutes with all limits. Consequently, $\mathrm{Mod}_{A^{\bullet}}(\mathcal{X})$ is a limit diagram in $\mathcal{P}r^{L}_{\mathrm{k}}$, and hence, by Lemma 2.3.2, $\mathcal{M}^{\natural}_{\mathcal{X}}(A^{\bullet})$ is a limit diagram in $\widehat{\mathrm{Cat}}_{\infty}$. This proves that $\mathcal{M}^{\natural}_{\mathcal{X}}$ carries the Čech nerve of any faithfully flat morphism to a limit diagram in $\widehat{\mathrm{Cat}}_{\infty}$.

The proof of the fact that $\mathcal{M}^{\natural}_{\mathcal{X}}$ preserves products is identical - in the previous paragraph, one only need replace the simplicial set $\mathrm{N}(\Delta_{+})$ by the simplicial set K that indexes product diagrams. □

3.3 Flat Descent for Dualizable Branes

Recall that in §2.4 we associated with a k-linear presentable symmetric monoidal ∞-category $\mathcal{X}^{\otimes}$, a moduli functor $\mathcal{M}^{\vee}_{\mathcal{X}} : \mathrm{CAlg}_{\mathrm{k}} \to \widehat{\mathrm{Cat}}_{\infty}$, which carries a commutative k-algebra A to the ∞-category of dualizable objects in $\mathrm{Mod}_{A}(\mathcal{X})$. The purpose of this section is to state and prove the following proposition.

Proposition 3.3.1. *Let $\mathcal{X}^{\otimes}$ be a $\mathrm{Mod}_{\mathrm{k}}$-linear symmetric monoidal presentable ∞-category. Assume that the underlying category $\mathcal{X}$ is dualizable as an object of $\mathcal{P}r^L_{\mathrm{k}}$ (this holds, in particular, if $\mathcal{X}$ is in $\mathcal{P}r^L_{\omega,\mathrm{k}}$). Then the functor $\mathcal{M}^{\vee}_{\mathcal{X}} : \mathrm{CAlg}_{\mathrm{k}} \to \widehat{\mathrm{Cat}}_{\infty}$ defined in §2.4 is a sheaf for the flat hypertopology.*

The proposition follows almost immediately from the results of the previous section. The only additional ingredient that we will need is the following analogue of Lemma 2.3.5:

Lemma 3.3.2. *The functor $(-)^{\mathrm{fd}} : \mathrm{CAlg}(\mathcal{P}r^L_{\mathrm{k}}) \to \widehat{\mathrm{Cat}}_{\infty}$, which carries a symmetric monoidal category to its subcategory of dualizable objects, preserves all limits.*

Proof. Let $\nu^{\flat} : K^{\triangleleft} \to \mathrm{CAlg}(\mathcal{P}r^L_{\mathrm{k}})$ be a diagram. According to A.3.2.2.5, $\nu^{\flat}$ is a limit diagram if and only if the induced diagram $\nu' : K^{\triangleleft} \to \mathcal{P}r^L_{\mathrm{k}}$, obtained by composing $\nu^{\flat}$ with the functor that carries a symmetric monoidal category $\mathcal{X}^{\otimes}$ to the underlying category $\mathcal{X} : \mathcal{X}^{\otimes}_{\langle 1 \rangle}$, is a limit diagram. We have already seen that the forgetful functor $\psi : \mathcal{P}r^L_{\mathrm{k}} \to \widehat{\mathrm{Cat}}_{\infty}$ preserves and reflects all limits (Lemma 2.3.2). So $\nu^{\flat}$ is a limit diagram if and only if the induced diagram $\nu := \psi \circ \nu' : K^{\triangleleft} \to \widehat{\mathrm{Cat}}_{\infty}$ is a limit diagram.

Assume that $\nu^{\flat}$ is a limit diagram, and let $\mathcal{V} \to K^{\triangleleft}$ (resp. $\mathcal{V}^{\mathrm{fd}} \to K^{\triangleleft}$) be a coCartesian fibration classifiying ν (resp. $(-)^{\mathrm{fd}} \circ \nu$). Note that $\mathcal{V}^{\mathrm{fd}}$ can be identified with a full subcategory of $\mathcal{V}$. Theorem 5.17 in [Lur11a] characterizes $\widehat{\mathrm{Cat}}_{\infty}$-valued limit diagrams as those $\widehat{\mathrm{Cat}}_\infty$-valued diagrams that classify coCartesian fibrations that have certain properties. Since $\nu^\flat$ is a limit diagram, $\mathcal{V}$ has all of these properties. One checks that Proposition 2.3.8 implies that the full subcategory $\mathcal{V}^{\mathrm{fd}}$ inherits all these properties. Applying [Lur11a, Theorem 5.17] again, we see that $(-)^{\mathrm{fd}} \circ \nu$ is a limit diagram. This completes the proof. □

We now turn to the proof of the descent property for dualizable branes:

Proof of Proposition 3.3.1. Let $\mathcal{X}^\otimes$ be a symmetric monoidal ∞-category satisfying the hypotheses of the proposition. Let $A^\bullet : \mathrm{N}(\Delta_+) \to \mathrm{CAlg}_{\mathrm{k}}$ be the Čech nerve of a flat morphism $A^{-1} \to A^0$. By Proposition 3.2.1, the induced coaugmented cosimplicial object $\mathcal{M}^\natural_{\mathcal{X}}(A^\bullet)$ in $\widehat{\mathrm{Cat}}_\infty$ is a limit diagram. By virtue of Lemma 2.3.2, the coaugmented cosimplicial object $\mathrm{Mod}_{A^\bullet}(\mathcal{X})$ is a limit diagram in $\mathcal{P}r^L_{\mathrm{k}}$. Applying Lemma 3.3.2, we have that $\mathcal{M}^\vee_{\mathcal{X}}(A^\bullet)$ is a limit diagram in $\widehat{\mathrm{Cat}}_\infty$. Similarly, if $A \simeq A_1 \times A_2$, then $\mathrm{Mod}_A(\mathcal{X}) \simeq \mathrm{Mod}_{A_1}(\mathcal{X}) \times \mathrm{Mod}_{A_2}(\mathcal{X})$ in $\mathcal{P}r^L_{\mathrm{k}}$ by Proposition 3.2.1, and by Lemma 3.3.2, we have that $\mathcal{M}^\vee_{\mathcal{X}}(A) \simeq \mathcal{M}^\vee_{\mathcal{X}}(A_1) \times \mathcal{M}^\vee_{\mathcal{X}}(A_2)$ in $\widehat{\mathrm{Cat}}_\infty$. □

3.4 Étale Descent for Compact Branes

While families of branes descend along arbitrary faithfully flat maps, descent may destroy the property of being compact. This is due to the fact that an object X in the limit $\mathcal{C}$ of a diagram $\{\mathcal{C}_\alpha\}$ of ∞-categories need not be compact even if it has compact image in each $\mathcal{C}_\alpha$. In other words, the inclusion $\mathcal{P}r^L_{\omega,\mathrm{k}} \subseteq \widehat{\mathrm{Cat}}_\infty$ does not reflect limits. One solution to this problem is to pass to coarser topology τ, for which the notion of being compact is τ-local:

Proposition 3.4.1. *Let $\mathcal{X}$ be a presentable* $\mathrm{Mod}_{\mathrm{k}}$*-linear ∞-category. Let $\mathcal{M}^{\flat}_{\mathcal{X}}$ be the functor defined in Notation 2.4.1.*

(1) Assume that $\mathcal{X}$ admits a compact generator. Then $\mathcal{M}^{\flat}_{\mathcal{X}}$ is a sheaf for the étale topology.

(2) Assume that $\mathcal{X}$ is smooth and proper. Then $\mathcal{M}^{\flat}_{\mathcal{X}}$ is a sheaf for the flat topology.

(3) Assume that there is a perfect symmetric monoidal ∞-category $\mathcal{X}^{\otimes}$ whose underlying category is $\mathcal{X}$. Then $\mathcal{M}^{\flat}_{\mathcal{X}}$ is a sheaf for the flat topology.

The proposition will be the outcome of the next several lemmas. We will deduce descent for compact branes from the descent property of big branes (Proposition 3.2.1) by appealing to Lemma 2.3.11. This in turn is facilitated by the fact that the étale topology is generated by the Nisnevich topology and the finite étale topology.

Let $\{A \to A_\alpha\}_{\alpha\in\Lambda}$ be a collection of étale morphisms in $\mathrm{CAlg}_{\mathrm{k}}$, and let $X_\alpha := \mathrm{Spec}(\pi_0 A_\alpha)$, and $X := \mathrm{Spec}(A)$. Recall from G.4.2. that the family $\{A \to A_\alpha\}_{\alpha\in\Lambda}$ is a covering family for the Nisnevich topology τ_{Nis} on $\mathrm{CAlg}_{\mathrm{k}}$ if and only if the following condition is satisfied: there exists a finite subset $\{A_1, A_2, ..., A_n\} \subseteq \{A_\alpha\}_{\alpha\in\Lambda}$ and a sequence of compact open subsets $\emptyset = U_0 \subseteq U_1 \subseteq ... \subseteq U_n = X$ such that $X_i \times_X (U_i - U_{i-1})$ contains an open subscheme which maps isomorphically onto

$U_i - U_{i-1}$. Let us say that a cartesian square in $\mathrm{Aff}_{\mathrm{k}}$

$$\begin{tikzcd} U' \arrow[r] \arrow[d] & X' \arrow[d, "\pi"] \\ U \arrow[r] & X \end{tikzcd}$$

is a distinguished Nisnevich square if U is open in X and π is an isomorphism over $X - U$. The next Proposition, which is a version of the Morel-Voevodsky descent theorem, will allow us to deduce Nisnevich descent for $\mathcal{M}^{\flat}_{\mathcal{X}}$ from the corresponding descent property for $\mathcal{M}^{\natural}_{\mathcal{X}}$. This is Propostion 4.4.2 in [Lur04].

Proposition 3.4.2. *Let $\mathcal{C}$ be an ∞-category and let $\mathcal{F} : \mathrm{Aff}^{\mathrm{op}}_{\mathrm{k}} \to \mathcal{C}$ be a functor. Then $\mathcal{F}$ is a sheaf for the Nisnevich topology if and only if it carries distinguished Nisnevich squares to homotopy pullback squares.*

With this proposition at our disposal, Nisnevich descent for compact branes follows immediately from results that we have already proven.

Proposition 3.4.3. *Let $\mathcal{X}$ be a compactly generated* k*-linear ∞-category. Then the functor $\mathcal{M}^{\flat}_{\mathcal{X}}$ (see Notation 2.4.1) is a sheaf for the Nisnevich topology.*

Proof. Since the flat topology is finer than the Nisnevich topology, the presheaf $\mathcal{M}^{\natural}_{\mathcal{X}}$ is a Nisnevich sheaf by Proposition 3.2.1. Let K be the simplicial set whose nondegenerate simplices are pictured below:

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

Since K is finite, Lemma 2.3.9 implies that any diagram $\nu : K \to \widehat{\mathrm{Cat}}_\infty$ has Property ¶ (see Definition 2.3.6). In particular, for any distinguished Nisnevich square $\mu : K \to \mathrm{CAlg}_{\mathrm{k}}$, the induced diagram $\mathcal{M}^{\natural}_{\mathcal{X}} \circ \mu : K \to \widehat{\mathrm{Cat}}_\infty$ has Property ¶, and is also a limit diagram by virtue of Proposition 3.4.2, since $\mathcal{M}^{\natural}_{\mathcal{X}}$ is a Nisnevich sheaf. Applying Lemma 2.3.11, we conclude that $\mathcal{M}^{\flat}_{\mathcal{X}} \circ \mu = (-)^{\omega} \circ \mathcal{M}^{\natural}_{\mathcal{X}} \circ \mu : K \to \mathrm{Cat}_\infty$ is a limit diagram. Since ν was an arbitrary distinguished Nisnevich square, applying Proposition 3.4.2 again, we conclude that $\mathcal{M}^{\flat}_{\mathcal{X}}$ is a sheaf for the Nisnevich topology. □

To verify the étale descent property for compact branes, it will suffice to check that $\mathcal{M}^{\flat}_{\mathcal{X}}$ is also a sheaf for the finite etale topology. Recall that a morphism $A \to B$ in $\mathrm{CAlg}_{\mathrm{k}}$ is finite if the $\pi_0 B$ is a finite $\pi_0 A$-module. The covering families for the finite étale topology are the collections $\{A \to A_\alpha\}_{\alpha \in \Lambda}$ of morphisms in $\mathrm{CAlg}_{\mathrm{k}}$ for which there exists a finite subset $\Lambda_0 \subseteq \Lambda$ such that $\prod_{\alpha \in \Lambda_0} A_\alpha$ is faithfully flat, étale and finite over A. The key property of finite étale maps that allows us to deduce descent for compact branes is the following:

Lemma 3.4.4. *Let $\mathcal{X}$ be in $\mathcal{P}r^{L}_{\omega,\mathrm{k}}$, and assume that $\mathcal{X}$ admits a compact generator. Let $f : A \to B$ be an finite étale morphism in $\mathrm{CAlg}_{\mathrm{k}}$. Then the forgetful functor $f_* : \mathrm{Mod}_B(\mathcal{X}) \to \mathrm{Mod}_A(\mathcal{X})$ preserves ω-compact objects.*

Proof. Let X be a compact generator for $\mathcal{X}$, and let $\mathcal{E} := \mathbb{M}\mathrm{or}_{\mathcal{X}}(X, X)$ be its endormorphism object, which is an $\mathbb{E}_1$-algebra in $\mathrm{Mod}_{\mathrm{k}}$. By Theorem **??**, we have an equivalence $\mathcal{X} \simeq \mathrm{RMod}_{\mathcal{E}} \simeq \mathrm{LMod}_{\mathcal{E}^{\mathrm{op}}}$. We will write f^* for $(-) \otimes_A B$, the left adjoint to f_*.

By virtue of being a right adjoint, f_* preserves arbitrary limits, and hence, in particular, finite limits. Since the categories involved are stable, it also preserves finite colimits. The category $\mathrm{LMod}^{\omega}_{\mathcal{E}^{\mathrm{op}} \otimes A}$ is stable under finite colimits and retracts, and the category $\mathrm{LMod}^{\omega}_{\mathcal{E}^{\mathrm{op}} \otimes B}$ is generated by $\mathcal{E} \otimes_{\mathrm{k}} B$ under finite colimits and retracts. Consequently, it will suffice to show that $\mathcal{E} \otimes_{\mathrm{k}} B \in \mathrm{LMod}^{\omega}_{\mathcal{E}^{\mathrm{op}} \otimes A}$.

Following [TV08], let us say that an A-module M is *strong* if the natural morphism $\pi_* A \otimes_{\pi_0 A} \pi_0 M \to \pi_* M$ is an equivalence. Since $A \to B$ is finite étale, we have that $\pi_0 A \to \pi_0 B$ is finite étale, and B is a strong A module. In particular, we have that $\pi_0 B$ is a flat $\pi_0 A$-module of finite presentation, and is therefore a projective $\pi_0 A$-module. Since B is also a strong A module, it follows from [TV08, Lemma 2.2.2.2.] that B is a projective A-module of finite presentation. Since the tensor produc distributes over colimits, it follows that $\mathcal{E} \otimes_{\mathrm{k}} B$ is finitely presented over $\mathcal{E} \otimes_{\mathrm{k}} A$. Since the compact modules are precisely the retracts of modules of finite presentation, the result follows. $\square$

In order to verify that $\mathcal{M}^{\flat}$ satisfies the Beck-Chevalley conditions of Proposition 3.1.2. we will need an analogue of the base change lemma 3.2.2. The previous lemma allows us to deduce such a base change result from Lemma 3.2.2.

Lemma 3.4.5. *Let $\mathcal{X}$ be an object of $\mathcal{P}r^{L}_{\omega,\mathrm{k}}$ which admits a compact generator. Let $f : A^{-1} \to A^0$ be a finite étale morphism in $\mathrm{CAlg}_{\mathrm{k}}$ and let $A^{\bullet} : \mathrm{N}(\Delta_+) \to \mathrm{CAlg}_{\mathrm{k}}$*

be its Čech nerve. Then for each morphism $\alpha : [m] \to [n]$ in Δ_+, the functor $\mathcal{M}^\flat$ carries the commutative diagram

$$\begin{tikzcd} A^m \arrow[r, "d^0"] \arrow[d, "\alpha"'] & A^{m+1} \arrow[d, "\tilde{\alpha}"] \\ A^n \arrow[r, "d^0"'] & A^{n+1} \end{tikzcd}$$

to a right adjointable square in Cat_∞.

Proof. Let $f : [m] \to [n]$ be an arbitrary morphism in Δ_+. To simplify notation, let $A := A^m$, $A' := A^{m+1}$, $B := A^n$, $B' := A^{n+1}$, $p := d^0 : A \to A'$ and $p' := d^0 : B \to B'$. By Lemma 3.2.3 and Lemma 3.2.2, we have a right adjointable square

$$\begin{tikzcd} \mathrm{Mod}_A(\mathcal{X}) \arrow[r, "p^*"] \arrow[d, "f^*"'] & \mathrm{Mod}_{A'}(\mathcal{X}) \arrow[d, "f'^*"] \arrow[l, dotted, bend right, "p_*"'] \\ \mathrm{Mod}_B(\mathcal{X}) \arrow[r, "p'^*"'] & \mathrm{Mod}_{B'}(\mathcal{X}) \arrow[l, dotted, bend right, "p'_*"'] \end{tikzcd}$$

Since $A^\bullet$ is a Čech nerve, the map $p : A \to A'$ is the inclusion of the summand $\otimes_{A^{-1}}^{n+1} A^0 \to A^0 \otimes_{A^{-1}} (\otimes_{A^{-1}}^{n+1} A^0)$. Finite étale maps are stable under base change and composition, and $A^{-1} \to A^0$ is finite étale by hypothesis. It follows that p is finite étale. Similarly, p' is finite étale. By Lemma 3.4.4, the right adjointable square above restricts to a right adjointable square

$$\begin{tikzcd}
\mathrm{Mod}_A(\mathcal{X})^\omega \arrow[r, "p^*"] \arrow[d, "f^*"] & \mathrm{Mod}_{A'}(\mathcal{X})^\omega \arrow[d, "f'^*"] \arrow[l, bend right, dotted, "p_*"'] \\
\mathrm{Mod}_B(\mathcal{X})^\omega \arrow[r, "p'^*"'] & \mathrm{Mod}_{B'}(\mathcal{X})^\omega \arrow[l, bend right, dotted, "p'_*"']
\end{tikzcd}$$

Since $\mathcal{M}^\flat_{\mathcal{X}}(A) = \mathrm{Mod}_{\mathcal{X}}(A)^\omega$ by definition, and $\mathcal{M}^\flat_{\mathcal{X}}(f) = f^*$ for any morphism $f : A \to B$ in rings, this completes the proof. □

The proof that $\mathcal{M}^\flat_{\mathcal{X}}$ satisfies the remaining conditions of Proposition 3.1.2 is almost identical to the corresponding proof for $\mathcal{M}^\natural_{\mathcal{X}}$. The following lemma says that condition 1. of that proposition is satisfied.

Lemma 3.4.6. *Let* $f : A \to B$ *be a faithfully flat morphism in* $\mathrm{CAlg}_{\mathrm{k}}$*, and let* $\mathcal{X}$ *be an object of* $\mathcal{P}r^L_{\omega,\mathrm{k}}$ *that admits a compact generator. Then the category* $\mathcal{M}^\flat_{\mathcal{X}}(A)$ *admits limits of* $\mathcal{M}^\flat_{\mathcal{X}}(f)$*-split cosimplicial objects, and the functor* $\mathcal{M}^\flat_{\mathcal{X}}(f) : \mathcal{M}^\flat_{\mathcal{X}}(A) \to \mathcal{M}^\flat_{\mathcal{X}}(B)$ *preserves these limits.*

Proof. The proof is essentially identical to that of Lemma 3.2.5. The only additional thing that needs proof is the fact that $\mathcal{M}^\flat_{\mathcal{X}}(A) := \mathrm{Mod}_A(\mathcal{X})^\omega$ admits totalizations of $\mathcal{M}^\flat_{\mathcal{X}}(f)$-split cosimplicial objects.

As in the proof of Lemma 3.4.4, we fix a compact generator X in $\mathcal{X}$, and identify $\mathcal{M}^\flat_{\mathcal{X}}(A)$ with $\mathrm{LMod}^\omega_{\mathcal{E}^{\mathrm{op}} \otimes A}$. We have $\mathcal{M}^\flat_{\mathcal{X}}(A) \simeq \mathcal{M}^\natural_{\mathcal{X}}(A)^\omega \simeq \mathrm{Mod}_A \mathcal{X}^\omega \simeq (\mathrm{LMod}_A(\mathrm{LMod}_{\mathcal{E}^{\mathrm{op}}}))^\omega \simeq \mathrm{LMod}^\omega_{A \otimes \mathcal{E}^{op}}$. The functor $\mathcal{M}^\flat_{\mathcal{X}}(f)$ can be identified with the functor $B \otimes_A (-)$.

Let $M^\bullet : \mathrm{N}(\Delta) \to \mathrm{LMod}^\omega_{A\otimes\mathcal{E}^{op}}$ be a $\mathcal{M}^\flat_\mathcal{X}(f)$-split cosimplicial object. The proof of Lemma 3.2.5 shows that $\pi_0 B \otimes_{\pi_0 A} \pi^{-p}\pi_q M$ vanishes for $p \neq 0$. Since $A \to B$ is faithfully flat, the same is true of $\pi_0 A \to \pi_0 B$. Consequently, it follows from the previous statement that $\pi^{-p}\pi_q M$ vanishes for $p \neq 0$. Since the standard t-structure on $\mathrm{LMod}^\omega_{A\otimes\mathcal{E}}$ is left and right complete by A.7.1.1.13, Corollary A.1.2.4.10 (applied to $(\mathrm{LMod}^\omega_{A\otimes\mathcal{E}})^{\mathrm{op}}$) now implies that $M^\bullet$ admits a totalization in $\mathrm{LMod}^\omega_{A\otimes\mathcal{E}}$. □

Lemma 3.4.7. *Let $f : A \to B$ be a faithfully flat morphism in CAlg_k, and let $\mathcal{X}$ be an object in $\mathcal{P}r^L_{\omega,\mathrm{k}}$. Then the functor $\mathcal{M}^\flat_X(f) : \mathcal{M}^\flat_\mathcal{X}(A) \to \mathcal{M}^\flat_\mathcal{X}(A)$ is conservative.*

Proof. The functor $\mathcal{M}^\flat_X(f)$ is the restriction of the functor $\mathcal{M}^\natural_X(f)$ (see Notation 2.4.1) to the subcategory of compact objects in $\mathcal{M}^\flat_X(A)$. Therefore, the lemma is an immediate consequence of Lemma 3.2.6. □

Lemma 3.4.8. *Let $\mathcal{X}$ be an object of $\mathcal{P}r^L_{\omega,\mathrm{k}}$. Then functor $\mathcal{M}^\flat_\mathcal{X} : \mathrm{CAlg}_\mathrm{k} \to \widehat{\mathrm{Cat}}_\infty$ preserves products.*

Proof. Let A_i, $i = 1, 2$, be objects of CAlg_k, and let $A := A_1 \times A_2$. By Lemma 3.2.7, the natural diagram

$$\begin{tikzcd} & \mathrm{Mod}_A(\mathcal{X}) \arrow[dl] \arrow[dr] & \\ \mathrm{Mod}_{A_1}(\mathcal{X}) & & \mathrm{Mod}_{A_2}(\mathcal{X}) \end{tikzcd}$$

is a limit diagram in $\widehat{\mathrm{Cat}}_\infty$. Since every finite diagram has Property ¶ (Lemma 2.3.9), it follows from Lemma 2.3.10, that it is also a limit diagram in $\mathcal{P}r^L_{\omega,\mathrm{k}}$. Lemma

2.3.5 then asserts that the induced diagram

$$\begin{tikzcd} & \mathrm{Mod}_A(\mathcal{X})^\omega \arrow[dl] \arrow[dr] & \\ \mathrm{Mod}_{A_1}(\mathcal{X})^\omega & & \mathrm{Mod}_{A_2}(\mathcal{X})^\omega \end{tikzcd}$$

is a limit diagram in Cat_∞, which is exactly what we set out to prove. □

We will have now assembled all of the essential facts that are necessary to prove the main proposition of this section.

Proof of Proposition 3.4.1. We will first prove (1). Let $\mathcal{X}$ be an object of $\mathcal{P}r^L_{\omega,\mathrm{k}}$, and assume that $\mathcal{X}$ admits a compact generator. The étale topology on CAlg_k is generated by the Nisnevich topology and the finite étale topology (see [Ryd10]). We have already proven that $\mathcal{M}^\flat_\mathcal{X}$ is a Nisnevich sheaf (Proposition 3.4.3). To complete the proof, we must show that $\mathcal{M}^\flat_\mathcal{X}$ is also a sheaf for the finite étale topology, i.e., we must show that $\mathcal{M}^\flat_\mathcal{X}$ preserves products and carries the Čech nerve of any surjective finite étale morphism to a limit diagram in Cat_∞.

The first statement is Lemma 3.4.8. To prove the second statement, let $A^\bullet : \mathrm{N}(\Delta_+) \to \mathrm{CAlg}_\mathrm{k}$ be the Čech nerve of a faithfully flat finite étale morphism $A^{-1} \to A^0$. By Lemmas 3.4.6, 3.4.5 and 3.4.7, the induced coaugmented cosimplicial ∞-category $\mathcal{M}^\flat_\mathcal{X}(A^\bullet)$ satisfies all the hypotheses of Proposition 3.1.2, and consequently, is a limit diagram in Cat_∞. This completes the proof of (1).

We will now prove (2). Let $\mathcal{X}$ be a smooth and proper object in $\mathcal{P}r^L_{\omega,\mathrm{k}}$. We will show that $\mathcal{M}^\flat_\mathcal{X}$ carries the Čech nerve of any flat morphism to a limit diagram, and preserves produces. We will deduce these statements from the corresponding statements for $\mathcal{M}^{\natural}_{\mathcal{X}}$, the moduli of "big" branes.

Let $A^{\bullet} : \mathrm{N}(\Delta_+) \to \mathrm{CAlg}_{\mathrm{k}}$ be the Čech nerve of a flat morphism. Let $i : \mathcal{P}r^L_{\omega} \to \widehat{\mathrm{Cat}}_{\infty}$ denote the natural inclusion. By Proposition 3.2.1, the diagram $\mathcal{M}^{\natural}_{\mathbf{1}}(A^{\bullet}) := i \circ \mathfrak{M}_{\mathbf{1}}(A^{\bullet}) : \mathrm{N}(\Delta_+) \to \widehat{\mathrm{Cat}}_{\infty}$ is a limit diagram. Lemmas 2.2.20 and 2.3.7 imply that $\mathfrak{M}_{\mathbf{1}}(A^{\bullet})$ has Property ¶. Consequently, by Lemma 2.3.10, the diagram $\mathfrak{M}_{\mathbf{1}}(A^{\bullet}) : \mathrm{N}(\Delta_+) \to \mathcal{P}r^L_{\omega,\mathrm{k}}$ is a limit diagram. Since $\mathcal{X}$ is dualizable in $\mathcal{P}r^L_{\omega,\mathrm{k}}$, the functor $\mathcal{X} \otimes_{\mathrm{Mod}_{\mathrm{k}}} (-) : \mathcal{P}r^L_{\omega,\mathrm{k}} \to \mathcal{P}r^L_{\omega,\mathrm{k}}$ commutes with limits. It follows that $\mathfrak{M}_{\mathcal{X}}(A^{\bullet}) := \mathcal{X} \otimes_{\mathrm{Mod}_{\mathrm{k}}} \mathfrak{M}_{\mathbf{1}}(A^{\bullet})$ is a limit diagram. Lemma 2.3.5 now implies that $\mathcal{M}^{\flat}_{\mathcal{X}}(A^{\bullet}) := \mathfrak{M}_{\mathcal{X}}(A^{\bullet})^{\omega}$ is a limit diagram, which is what we set out to prove. The proof that $\mathcal{M}^{\flat}_{\mathcal{X}}$ preserves products similar - in the proof above, one only need replace the simplicial set $\mathrm{N}(\Delta_+)$ by the simplicial set that indexes product diagrams. This completes the proof of (2).

We turn now to (3). Assume that $\mathcal{X}$ is the underlying category of a perfect symmetric monoidal category $\mathcal{X}^{\otimes}$. Then, by definition, we have an equivalence $\mathcal{M}^{\flat}_{\mathcal{X}} \simeq \mathcal{M}^{\vee}_{\mathcal{X}}$. The result is therefore a consequence of the flat descent property of dualizable branes (Proposition 3.3.1). □

3.5 Hyperdescent

All of the sheaves that we have considered so far in this chapter are hypercomplete in the sense of [Lur09a]. That is, all of these sheaves satisfy descent with respect to hypercovers. It is possible to deduce this from the results of this chapter, and certain simplicial techniques from the theory of cohomological descent.

Definition 3.5.1. Let $\mathcal{D}$ be an ∞-category that admits all colimits and limits, and let $\mathcal{F}$ be a $\mathcal{D}$-valued presheaf on $\mathrm{Aff_k}$. Let $U^\bullet : \mathrm{N}(\Delta_+^{op}) \to \mathrm{Aff_k}$ be an augmented simplicial derived affine scheme. Put $U := U^{-1}$. We will say that $U^\bullet$ is of $\mathcal{F}$-*cohomological descent* if the natural map $\mathcal{F}(U) \to \lim \mathcal{F}(U^\bullet)$ is an equivalence. We will say that $U^\bullet$ is *universally* of $\mathcal{F}$-cohomological descent, if any base change of $U^\bullet$ is of $\mathcal{F}$-cohomological descent.

For a map $f : U^0 \to U$ in $\mathrm{Aff_k}$, the Čech nerve of f, denoted $\check{C}(f)$, is the 0-coskeleton of f computed in $(\mathrm{Aff_k})_{/U}$. Let $\mathbb{P}_{\mathcal{F}}$ denote the class of morphisms in $\mathrm{Aff_k}$ whose Čech nerve is *universally* of $\mathcal{F}$-cohomological descent.

Theorem 3.5.2. *$\mathbb{P}_{\mathcal{F}}$-hypercovers are universally of $\mathcal{F}$-cohomological descent.*

The proof of this theorem is essentially contained in the theory of cohomological descent developed in [Del74, SGA72]. The techniques needed to prove this theorem are of a simplicial nature, and do not depend on the specific choice of an ∞-site $\mathrm{Aff_k}$ - any ∞-site with certain formal properties (that are almost always satisfied) will suffice. Since we do not need the results of this section in the rest of this paper, we will relegate the proof of this theorem to a forthcoming paper. We close this chapter by observing that in light of Theorem 3.5.2, the results of this chapter can be summarized as follows:

Theorem 3.5.3. *Let $\mathcal{X}$ be a dualizable object of $\mathcal{P}r^L_{\mathrm{k}}$. Let the notation be as in Notation 2.4.1. Then the following is true*

(1) The functor $\mathcal{M}^{\natural}_{\mathcal{X}}$ is a sheaf for the flat hypertopology

(2) Assume that $\mathcal{X}$ admits a compact generator. Then $\mathcal{M}_{\mathcal{X}}$ and $\mathcal{M}^{\flat}_{\mathcal{X}}$ are sheaves for the étale hypertopology.

(3) Assume that $\mathcal{X}$ is smooth and proper. Then $\mathcal{M}^{\flat}_{\mathcal{X}}$ and $\mathcal{M}_{\mathcal{X}}$ are sheaves for the flat hypertopology.

(4) Assume that $\mathcal{X}$ admits a symmetric monoidal structure. Then $\mathcal{M}^{\vee}_{\mathcal{X}}$ is a sheaf for the flat hypertopology.

(5) Assume that $\mathcal{X}$ admits a perfect symmetric monoidal strucuture. Then $\mathcal{M}^{\flat}_{\mathcal{X}}$ and $\mathcal{M}_{\mathcal{X}}$ are sheaves for the flat hypertopology.

Proof. In light of the fact that flat maps and étale maps are stable under base change, this theorem is an immediate consequence of Theorem 3.5.2 and Propositions 3.2.1, 3.4.1 and 3.3.1.

□

Chapter 4

Geometricity

In the previous chapter, we saw that the moduli functors defined in §2.4 are, in fact, derived ∞-stacks. That was a significant first step in the direction of understanding whether these moduli functors are represented by geometric objects. In this chapter, we will continue and conclude our analysis of the geometricity of the moduli of objects in linear ∞-categories.

Our main tool in this investigation is the Artin-Lurie representability criterion, Theorem 4.1.4. Section 4.1 is devoted to collecting together the various definitions needed to formulate this theorem, and to recalling the theorem itself.

In §4.2, we study the deformation theory of objects in linear ∞-categories. The main result of this section, Proposition 4.2.4, describes conditions under which the moduli functor $\mathcal{M}_{\mathcal{X}}$ (see Notation 2.4.1) admits a cotangent complex. The existence of the cotangent complex is one of the conditions in the Artin-Lurie criterion, and often the hardest to verify in practice. So this result represents a major step in the direction of understanding the geometricity of $\mathcal{M}_{\mathcal{X}}$. Along the way, we will establish conditions on a category $\mathcal{X}$ that guarantee the geometricity of the stack of maps (resp. equivalences) between any two compact objects in $\mathcal{X}$. In the sections that follow, we will see that these conditions are always satisfied when $\mathcal{X}$ is smooth and proper, or proper and perfect symmetric monoidal.

The main result of §4.3, Theorem 4.3.1, states that if $\mathcal{X}$ is smooth and proper, then $\mathcal{M}_{\mathcal{X}}$ is a locally geometric ∞-stack. This is a result of Toën-Vacquié ([TV07a]). The proof given here is quite different in flavor from the one given in loc. cit. We find that the Artin-Lurie criterion makes manifest the role of dualizability (recall that $\mathcal{X} \in \mathcal{P}r^L_{\omega,\mathrm{k}}$ is smooth and proper iff it is dualizable) in establishing the geometricity of $\mathcal{M}_{\mathcal{X}}$. The essential point here is that if $\mathcal{X}$ is dualizable, then $\mathcal{X} \otimes (-)$ commutes with limits.

In §4.4, we drop the hypothesis of smoothness. Proposition 4.5.2 of §4.5 describes a large class of proper noncommutative spaces $\mathcal{X}$ for which $\mathcal{M}_{\mathcal{X}}$ is not geometric. However the main result of §4.4, Theorem 4.4.1, shows that in the presence of additional algebraic structure - namely, a perfect symmetric monoidal structure - the functor $\mathcal{M}_{\mathcal{X}}$ is representable by a locally geometric ∞-stack. In particular, when X is a proper scheme over k, or more generally a perfect stack, the moduli of perfect complexes on X is locally geometric. This generalizes a result of Lieblich [Lie06], and provides a new "homotopical" perspective on his work.

For the rest of this chapter, we assume that k is a derived G-ring in the sense of [Lur04]. This holds, in particular, when k- is a field of characteristic 0, which is the main case of interest to us.

4.1 The Artin-Lurie Criterion

In order to state the Artin-Lurie representability criterion, we need some basic definitions. Our reference for a detailed discussion of the contents of this section, and for any terms that are not defined here, is [Lur04].

Definition 4.1.1. Let $\mathcal{C}$ be an ∞-category, let $\mathcal{F} : \mathrm{CAlg}_{\mathrm{k}} \to \mathcal{C}$ be a functor. We will say that

(1) $\mathcal{F}$ is ω-accessible if it preserves ω-filtered colimits.

(2) $\mathcal{F}$ is *nilcomplete* if it carries Postnikov towers in $\mathrm{CAlg}_{\mathrm{k}}$ to limit diagrams in $\mathcal{C}$.

(3) $\mathcal{F}$ is *infinitesimally cohesive* if it carries small extensions to pullback squares.

(4) $\mathcal{F}$ is *formally effective* if for any complete discrete local Noetherian k-algebra A with maximal ideal $\mathfrak{m}$, the natural morphism $\mathcal{F}(A) \to \lim \mathcal{F}(A/\mathfrak{m}^n)$ is an equivalence.

We turn now to the definition of the cotangent complex, which is the brave new analogue of the cotangent space. Any geometric ∞-stack admits a cotangent complex, and the existence of one is perhaps the most non-trivial of the conditions in the Artin-Lurie recognition principle.

Definition 4.1.2. Let $f : \mathcal{F} \to \mathcal{F}'$ be a morphism of derived stacks over k. For $A \in \mathrm{CAlg}_{\mathrm{k}}$ and $M \in \mathrm{Mod}_A$, we will denote by $A\langle M\rangle$ the square-zero extension of A by M (see [Lur11b]).

1. Let $A \in \mathrm{CAlg}_{\mathrm{k}}$, and let $x \in \mathcal{F}(A)$. Then the functor

   $$\mathbb{D}\mathrm{er}_{\mathcal{F}}(x, -) : \mathrm{Mod}_A \to \mathcal{S}$$

   is defined to be the homotopy fiber at x of the natural morphism of functors

   $$\mathcal{F}(A\langle M\rangle) \to \mathcal{F}(A)$$

   For any $M \in \mathrm{Mod}_A$, the morphism $A\langle M\rangle \to A$ has a canonical section. Therefore $\mathbb{D}\mathrm{er}_{\mathcal{F}}(x, M)$ is a pointed space. One defines the functor

   $$\mathbb{D}\mathrm{er}_{\mathcal{F}/\mathcal{F}'}(x, -) : \mathrm{Mod}_A \to \mathcal{S}$$

   to be homotopy fiber of the natural morphism $df : \mathbb{D}\mathrm{er}_{\mathcal{F}}(x, -) \to \mathbb{D}\mathrm{er}_{\mathcal{F}'}(x, -)$.

2. Let $A \in \mathrm{CAlg}_{\mathrm{k}}$, and let $x \in \mathcal{F}(A)$. We will say that $f : \mathcal{F} \to \mathcal{F}'$ has a relative cotangent complex at x if there is an integer n for which the functor $\mathbb{D}\mathrm{er}_{\mathcal{F}/\mathcal{F}'}(x, -)$ is corepresented by a $(-n)$-connective A-module $\mathbb{L}_{\mathcal{F}/\mathcal{F}',x}$. The module $\mathbb{L}_{\mathcal{F}/\mathcal{F}',x}$ will be called the relative cotangent complex of $\mathcal{F}$ over $\mathcal{F}'$.

3. We will say that $f : \mathcal{F} \to \mathcal{F}'$ has a relative cotangent complex if it satisfies the following two conditions

   (a) For any $A \in \mathrm{CAlg}_{\mathrm{k}}$ and any $x \in \mathcal{F}(A)$, the morphism f has a relative cotangent complex $\mathbb{L}_{\mathcal{F}/\mathcal{F}',x}$ at x.

(b) Given any commutative diagram in $\mathcal{S}t_{\mathcal{F}'}$:

$$\begin{tikzcd} \mathrm{Spec}(A) \arrow[rr, "u"] \arrow[dr, "x"'] & & \mathrm{Spec}(A') \arrow[dl, "x'"] \\ & \mathcal{F} & \end{tikzcd}$$

The natural morphism $u^*\mathbb{L}_{\mathcal{F}/\mathcal{F}',x'} \to \mathbb{L}_{\mathcal{F}/\mathcal{F}',x}$ is an equivalence in Mod_A.

4. Let $A \in \mathrm{CAlg}_{\mathrm{k}}$, and $x \in \mathcal{F}(A)$. We will say that $\mathcal{F}$ has an absolute cotangent complex at x, if the structure morphism $\mathcal{F} \to \mathrm{Spec}(\mathrm{k})$ has a relative cotangent complex at x. The absolute cotangent complex at x will be denoted $\mathbb{L}_{\mathcal{F},x}$, and referred to simply as the cotangent complex at x. Similarly, $\mathcal{F}$ has a cotangent complex if the structure morphism $\mathcal{F} \to \mathrm{Spec}(\mathrm{k})$ has a relative cotangent complex.

Notation 4.1.3. When $\mathcal{F}'$ is a derived affine scheme $\mathrm{Spec}(A)$, we will simply write $\mathbb{L}_{\mathcal{F}/A,x}$ for $\mathbb{L}_{\mathcal{F}/\mathrm{Spec}(A),x}$.

Having collected together all of the necessary terminology, we are ready to state the main theorem of this section, which is a shape theoretical analogue of Artin's representability criterion. This deep theorem is due to Jacob Lurie.

Theorem 4.1.4 ([Lur04])**.** *Let* k *be a derived G-ring. Then a functor* $\mathcal{F} : \mathrm{CAlg}_{\mathrm{k}} \to \mathcal{S}$ *is a derived n-stack locally of finite presentation over* k *if and and only if the following conditions are satisfied:*

(1) The functor $\mathcal{F}$ *is* ω*-accessible.*

(2) The functor $\mathcal{F}$ is a sheaf for the étale topology.

(3) The functor $\mathcal{F}$ is formally effective

(4) The functor $\mathcal{F}$ has a cotangent complex.

(5) The functor $\mathcal{F}$ is infinitesimally cohesive.

(6) The functor $\mathcal{F}$ is nilcomplete.

(7) The restriction of $\mathcal{F}$ to discrete commutative rings factors through $\tau_{\leq n}\mathcal{S}$.

The Artin-Lurie criterion will play a central role in the rest of this chapter.

4.2 Infinitesimal theory: Brane Jets

In this section, we will investigate the deformation theory of objects in a k-linear category. Before we can state the main proposition, we need some definitions.

Definition 4.2.1. Let A be an object of $\mathrm{CAlg}_{\mathrm{k}}$.

1. Let $M \in \mathrm{Mod}_A$. For a, $b \in \mathbb{Z} \cup \{-\infty, \infty\}$, we will say that M has Tor amplitude contained in $[a, b]$ if for any discrete A-module N, we have that $\pi_k(M \otimes_A N) = 0$ for $k \notin [a, b]$. We will say that M has Tor amplitude $\leq n$ if it has Tor amplitude contained in $[-\infty, n]$.

2. Let $\mathcal{X}$ be an object of $\mathcal{P}r^L_A$, and let $X \in \mathcal{X}$. We will say that X is of amplitude $\leq n$ if the Mod_A-valued mapping object $\mathbb{M}\mathrm{or}_A(X, X)$ has Tor amplitude $\leq n$.

Note that M is flat iff it has Tor amplitude ≤ 0. For a discussion of the important properties of Tor amplitude, we refer the reader to [TV07a, § 2.4] and [Lur04].

Definition 4.2.2. Let A be in $\mathrm{CAlg}_{\mathrm{k}}$. An object $\mathcal{X}$ in $\mathcal{P}r^L_A$ is *locally compact* over A if for all X, Y, in $\mathcal{X}^\omega$, the mapping object $\mathbb{M}\mathrm{or}_{\mathcal{X}}(X, Y)$ is a compact object in Mod_A. We will say that $\mathcal{X}$ is locally compact if it is locally compact over k.

Remark 4.2.3. One only expects this to be a well behave notion when we restrict ourselves to categories that are ω-compactly generated.

Proposition 4.2.4. *Let $\mathcal{X}$ be a locally compact object of $\mathcal{P}r^L_{\mathrm{k}}$. In addition, assume that the functor $\mathcal{M}_{\mathcal{X}}$ defined in Notation 2.4.1 is*

(1) ω-accessible

(2) Nilcomplete

(3) Infinitesimally cohesive

(4) Formally effective

(5) A sheaf for the étale topology

Then the functor $\mathcal{M}_{\mathcal{X}} : \mathrm{CAlg}_{\mathrm{k}} \to \mathcal{S}$ has a cotangent complex.

The proof of this proposition, which will occupy the rest of this section, will be based on the following proposition and the Artin-Lurie representability criterion:

Proposition 4.2.5. *Let $\mathcal{F} : \mathrm{CAlg_k} \to \mathcal{S}$ be a derived stack. Assume that $\mathcal{F}$ is infinitesimally cohesive, and that the diagonal $\Delta : \mathcal{F} \to \mathcal{F} \times \mathcal{F}$ is n-representable for some n. The $\mathcal{F}$ has a cotangent complex.*

Proof. This is Proposition 1.4.2.7. in [TV08]. $\square$

Notation 4.2.6. Let $A \in \mathrm{CAlg_k}$ and let X, Y be objects in $\mathcal{M}^\flat_\mathcal{X}(A) := \mathrm{Mod}_A(\mathcal{X})^\omega$. We define a functor $\Omega_{X,Y}\mathcal{M}_\mathcal{X} : \mathrm{CAlg_k} \to \mathcal{S}$ by the requirement that the following square be cartesian:

$$\begin{tikzcd}
\Omega_{X,Y}\mathcal{M}_\mathcal{X} \arrow[r] \arrow[d] & \mathcal{M}_\mathcal{X} \arrow[d, "\Delta"] \\
\mathrm{Spec}(A) \arrow[r, "\{X,Y\}"] & \mathcal{M}_\mathcal{X} \times \mathcal{M}_\mathcal{X}
\end{tikzcd}$$

Our strategy for the proof of Proposition 4.2.4 is the following. We will use our hypothesis on $\mathcal{X}$, and on the associated functor $\mathcal{M}^\flat_\mathcal{X}$, to show that for any A in $\mathrm{CAlg_k}$, and any X, Y in $\mathcal{M}_\mathcal{X}(A)$, the functor $\Omega_{X,Y}\mathcal{M}_\mathcal{X}$ is an algebraic n-stack, for some n depending on A, X and Y. The proof of the algebraicity of $\Omega_{X,Y}\mathcal{M}_\mathcal{X}$ will itself be based on the Artin-Lurie criterion. We will then appeal to Propostion 4.2.5 to conclude that $\mathcal{M}_\mathcal{X}$ has a cotangent complex.

While we are proving the algebraicity of $\Omega_{X,Y}\mathcal{M}_\mathcal{X}$, we will also prove the algebraicity of a larger "linear" object $\overrightarrow{\Omega}_{X,Y}\mathcal{M}^\flat_\mathcal{X}$, which we now introduce. Intuitively, the functor $\overrightarrow{\Omega}_{X,Y}\mathcal{M}^\flat_\mathcal{X} : \mathrm{CAlg_k} \to \mathcal{S}$ is a natural "$(\infty, 2)$ categorical analogue" of $\Omega_{X,Y}\mathcal{M}_\mathcal{X}$, in that it can be thought as being defined by the requirment that the following is a "lax $(\infty, 2)$-categorical pullback square"

$$\begin{tikzcd} \overrightarrow{\Omega}_{X,Y}\mathcal{M}_{\mathcal{X}} \arrow[r] \arrow[d] & \mathrm{Spec}(A) \arrow[d, "Y"] \\ \mathrm{Spec}(A) \arrow[r, "X"] & \mathcal{M}^{\flat}_{\mathcal{X}} \end{tikzcd}$$

Rather than attempting to make precise the notion of a lax 2-pullback, we will simply give an explicit definition of $\overrightarrow{\Omega}_{X,Y}\mathcal{M}^{\flat}_{\mathcal{X}}$. Let $A \in \mathrm{CAlg}_{\mathrm{k}}$, let $X, Y \in \mathcal{M}^{\flat}_{\mathcal{X}}(A)$, and let B be a commutative A-algebra. We have the following description of the restriction of the functors $\overrightarrow{\Omega}_{X,Y}\mathcal{M}_{\mathcal{X}}$ and $\Omega_{X,Y}\mathcal{M}_{\mathcal{X}}$ to $\mathrm{CAlg}_{A/}$.

$$\overrightarrow{\Omega}_{X,Y}\mathcal{M}_{\mathcal{X}}(B) := \mathrm{Map}_{\mathrm{Mod}_B(\mathcal{X})}(X \otimes_A B, Y \otimes_A B)$$

$$\Omega_{X,Y}\mathcal{M}_{\mathcal{X}}(B) = \mathrm{Iso}_{\mathrm{Mod}_B(\mathcal{X})}(X \otimes_A B, Y \otimes_A B)$$

Here $\mathrm{Iso}(-,-) \subseteq \mathrm{Map}(-,-)$ is the full subcategory whose objects are morphisms that become invertible in $h\mathrm{Mod}_B(\mathcal{X})$. An immediate consequence of this explicit description of the functors, is the following lemma, which describes the relation between the two stacks that we have just introduced, in the special case where $\mathcal{X}$ admits a compact generator:

Lemma 4.2.7. *Let $\mathcal{X}$ be an object of $\mathcal{P}r^L_{\mathrm{k}}$, and assume that $\mathcal{X}$ has a compact generator. Let $\overrightarrow{\Omega}_{X,Y}\mathcal{M}^{\flat}_{\mathcal{X}}$ and $\Omega_{X,Y}\mathcal{M}_{\mathcal{X}}$ be as in Notation 4.2.6. Then the natural morphism $i : \Omega_{X,Y}\mathcal{M}_{\mathcal{X}} \to \overrightarrow{\Omega}_{X,Y}\mathcal{M}^{\flat}_{\mathcal{X}}$ is a Zariski open immersion. In particular, if $\overrightarrow{\Omega}_{X,Y}\mathcal{M}^{\flat}_{\mathcal{X}}$ is an algebraic n-stack, then the same is true of $\Omega_{X,Y}\mathcal{M}_{\mathcal{X}}$*

Proof. The second statement is an immediate consequence of the first statement and the fact that n-representable morphisms are stable under composition (G.5.1.4.).

To prove the first statement, suppose that we are given a morphism $\mathrm{Spec}(B) \to \overrightarrow{\Omega}_{X,Y}\mathcal{M}^{\flat}_{\mathcal{X}}$, classified by $\Phi \in \mathrm{Map}_{\mathrm{Mod}_B(\mathcal{X})}(X \otimes_A B, Y \otimes_A B)$. By Theorem **??**, we may identify $\mathrm{Mod}_B(\mathcal{X})$ with $\mathrm{LMod}_{\mathcal{E}^{\mathrm{op}} \otimes B}$. Let $U_n \subseteq \mathrm{Spec}(\pi_0 B)$ be the complement of the support of $\pi_n \mathrm{cone}(\Phi)$. By the upper semicontinuity of cohomology groups, U_n is open. Since X and Y are perfect A-modules, so is $\mathrm{cone}(\Phi)$. Therefore for all but finitely many n, we have $U_n = \mathrm{Spec}(\pi_0 B)$. It follows that $U := \bigcap U_n$ is Zariski open in $\mathrm{Spec}(\pi_0 B)$. Let $f_1, ..., f_n$ be elements in B that are lifts of elements in $\pi_0 B$ that cut out the complement of U. Let $V := \mathrm{Spec}(B[f_i^{-1}])$. Let $j : V \to \mathrm{Spec}(B)$ denote the natural inclusion. Note that $\Phi_{|V}$ is an equivalence, since its cone is contractible by construction our of V.

We will now prove that the map i is a Zariski open immersion. Let $\mathcal{F}$ be the fiber of i over $\mathrm{Spec}(B)$, so that square below is cartesian.

$$\begin{tikzcd}
V \arrow[r, "\pi"] & \mathcal{F} \arrow[r] \arrow[d, "i'"] & \Omega_{X,Y}\mathcal{M}_{\mathcal{X}} \arrow[d, "i"] \\
& \mathrm{Spec}(B) \arrow[r, "\{\Phi\}"] & \overrightarrow{\Omega}_{X,Y}\mathcal{M}^{\flat}_{\mathcal{X}}
\end{tikzcd}$$

For any C in $\mathrm{CAlg}_{\mathrm{k}}$, $\mathcal{F}(C)$ is a classifying space for triples $\{(f, \Phi', \alpha)\}$, where $f : \mathrm{Spec}(C) \to \mathrm{Spec}(B)$, $\Phi' \in \mathrm{Iso}_{\mathrm{Mod}_C(\mathcal{X})}(X \otimes_A C, Y \otimes_A C)$, and $\alpha : \Phi \otimes_B C \to \Phi'$ is an equivalence. The map π is sends a morphism $f : \mathrm{Spec}(C) \to V$ to the triple

$\{(j \circ f, \Phi \otimes_B C, \mathrm{id})\}$.

To prove that i is a Zariski open immersion, it will suffice to show that i' is a monomorphism, that π is surjective, and that $i' \circ \pi$ is a Zariski open immersion of affine schemes.

Recall that a morphism $f : S \to T$ in an ∞-category is a monomorphism if the induced map $S \to S \times_T S$ is an equivalence. Observe that i (and therefore i') is a monomorphism. Indeed, for any C in $\mathrm{CAlg}_{\mathrm{k}}$, $i(C)$ is the natural map $\mathrm{Iso}_{\mathrm{Mod}_C(\mathcal{X})}(X \otimes_A C, Y \otimes_A C) \to \mathrm{Map}_{\mathrm{Mod}_C(\mathcal{X})}(X \otimes_A C, Y \otimes_A C)$. Since the equivalences constitute a union of connected components in the space of maps, this is a monomorphism.

For any $(f, \Phi', \alpha) \in \mathcal{F}(C)$, we have that $\mathrm{cone}(\Phi')$ is a contractible, since Φ' is an equivalence. The equivalence α induces an equivalence $\mathrm{cone}(\Phi \otimes_B C) \simeq \mathrm{cone}(\Phi')$. Since V is the complement of the support of $\mathrm{cone}(\Phi)$, f must factor through V. This proves the surjectivity of π.

Finally, the composite $i' \circ \pi$ is just the natural inclusion $j : \mathrm{Spec}(B[f_i^{-1}]) \to \mathrm{Spec}(B)$, which is obviously a Zariski open immersion. $\square$

Apart from the existence of the cotangent complex, the fact that $\Omega_{X,Y}\mathcal{M}_{\mathcal{X}}$ satisfies the conditions of Theorem 4.1.4 follows formally from our hypotheses. The only observation one needs is the following evident lemma:

Lemma 4.2.8. *Let $\mathcal{C}$ be an ∞-category, and let*

$$\begin{tikzcd} \mathcal{F} \arrow[r] \arrow[d] & \mathcal{F}_1 \arrow[d] \\ \mathcal{F}_2 \arrow[r] & \mathcal{F}_0 \end{tikzcd}$$

be a carestian square in $\mathcal{P}(\mathcal{C})$. Let K be a simplicial set.

1. *Let $\nu : K^{\triangleleft} \to \mathcal{C}$ be a diagram, and suppose that $\mathcal{F}_i \circ \nu$ is a limit diagram for $i = 0, 1, 2$. Then $\mathcal{F} \circ \nu$ is a limit diagram.*

2. *Assume K is ω-filtered. Let $\nu : K^{\triangleright} \to \mathcal{C}$ be a diagram, and suppose that $\mathcal{F}_i \circ \nu$ is a colimit diagram for $i = 0, 1, 2$. Then $\mathcal{F} \circ \nu$ is a colimit diagram.*

Proof. The first statement follows from that fact that products (in fact, arbitrary limits) commute with all limits. The second statement is a consequence of the fact that products (in fact, all finite limits) commute with ω-filtered colimits in $\mathcal{S}$. $\square$

As an immediate consequence we have the following:

Lemma 4.2.9. *Let $\mathcal{X}$ be a presentable* k*-linear ∞-category satisfying conditions (1)-(5) of Proposition 4.2.4, and let $\Omega_{X,Y}\mathcal{M}_{\mathcal{X}}$ be as in Notation 4.2.6. Then $\Omega_{X,Y}\mathcal{M}_{\mathcal{X}}$ is ω-accessible, nilcomplete, infinitesimally cohesive, formally effective, and a sheaf for the étale topology.*

Proof. This follows immediately from our hypotheses, the definitions of the various terms, and Lemma 4.2.8. $\square$

Our next goal is to prove the analogue of the previous lemma for $\overrightarrow{\Omega}_{X,Y}\mathcal{M}^{\flat}_{\mathcal{X}}$. If we had enough $(\infty, 2)$-categorical machinery at our disposal, the proof would be very similar to that for $\Omega_{X,Y}\mathcal{M}_{\mathcal{X}}$. We will take a more hands-on approach.

Lemma 4.2.10. *Let $\nu : K^{\triangleleft} \to \mathrm{Cat}_{\infty}$ be a limit diagram classifying a coCartesian fibration $\nu^{\flat} : \mathcal{V} \to K^{\triangleleft}$. Let X, $X' \in \mathcal{V}_{\infty}$, and let χ, $\chi' \in \mathrm{Fun}_{K^{\triangleleft}}(K, \mathcal{V})$ be the corresponding coCartesian sections. Then we have a natural equivalence $\mathcal{V}(x, x') \simeq \lim_{x\in K}\mathcal{V}_x(\chi_x, \chi'_x)$ in $\mathcal{S}$.*

Proof. According to [Lur11a, Lemma 5.17], there are coCartesian sections $\overline{\chi}$, $\overline{\chi}'$ in $\mathrm{Fun}_{\mathrm{k}^{\triangleleft}}(K^{\triangleleft}, \mathcal{V})$ such that $\overline{\chi}_{|K} = \chi$, $\overline{\chi}'_{|K}$ and $\overline{\chi}_{\infty} = X$ and $\overline{\chi}'_{\infty} = X'$. Furthermore, the aforementioned lemma tells us that $\overline{\chi}'$ is a $\nu^{\flat}$-limit diagram. This implies that $\mathcal{V}(X', X) \simeq \lim_{x\in K}\mathcal{V}(X, \chi'_x)$. Furthermore, the lemma tells us that for each $x \in K$, $\overline{\chi}$ carries the morphism $\infty \to x$ to a $\nu^{\flat}$-cartesian morphism. Consequently, we have $\mathcal{V}(X, \chi'_x) \simeq \mathcal{V}_x(\chi_x, \chi'_x)$. Putting this together with the previous equivalence, and using the fact that $\mathcal{V}_{\infty} \subseteq \mathcal{V}$ is a full subcategory, we have $\mathcal{V}(X', X) \simeq \lim_{x\in K}\mathcal{V}_x(\chi_x, \chi'_x)$. □

Lemma 4.2.11. *Let $\mathcal{X}$ be an object of $\mathcal{P}r^{L}_{\mathrm{k}}$, let $\overrightarrow{\Omega}_{X,Y}\mathcal{M}^{\flat}_{\mathcal{X}}$ be as in Notation 4.2.6 and let $\mu : K^{\triangleleft} \to \mathrm{CAlg}_{\mathrm{k}}$ be a diagram. Assume that the induced diagram $\mathcal{M}^{\flat}_{\mathcal{X}} \circ \mu : K^{\triangleleft} \to \mathrm{Cat}_{\infty}$ is a limit diagram. The the diagram $\overrightarrow{\Omega}_{X,Y}\mathcal{M}^{\flat}_{\mathcal{X}} \circ \mu : K^{\triangleleft} \to \mathcal{S}$ is a limit diagram.*

Proof. Apply Lemma 4.2.10 with $\nu = \mathcal{M}^{\flat}_{\mathcal{X}} \circ \mu$, $X = X$, and $X' = Y$. □

Finally, we need to check that $\overrightarrow{\Omega}_{X,Y}\mathcal{M}^{\flat}_{\mathcal{X}}$ preserves filtered colimits.

Lemma 4.2.12. *Let $\mathcal{X}$ be an object of $\mathcal{P}r^L_{\mathrm{k}}$, and let $\overrightarrow{\Omega}_{X,Y}\mathcal{M}^{\flat}_{\mathcal{X}}$ be as in Notation 4.2.6. The $\overrightarrow{\Omega}_{X,Y}\mathcal{M}^{\flat}_{\mathcal{X}} : \mathrm{CAlg}_{\mathrm{k}} \to \mathcal{S}$ is an ω-accessible functor.*

Proof. Let $\{A_\alpha\}$ be an ω-filtered diagram of commutative k-algebras with colimit A'. We have natural equivalences:

$$\begin{aligned}
\mathrm{colim}\ \overrightarrow{\Omega}_{X,Y}\mathcal{M}^{\flat}_{\mathcal{X}}(A_\alpha) &:= \mathrm{colim}\ \mathrm{Map}_{\mathrm{Mod}_{A_\alpha}(\mathcal{X})}(X \otimes_A A_\alpha, Y \otimes_A A_\alpha) \\
&\simeq \mathrm{colim}\ \mathrm{Map}_{\mathrm{Mod}_A(\mathcal{X})}(X, Y \otimes_A A_\alpha) \\
&\simeq \mathrm{Map}_{\mathrm{Mod}_A(\mathcal{X})}(X, \mathrm{colim}\ Y \otimes_A A_\alpha) \\
&\simeq \mathrm{Map}_{\mathrm{Mod}_A(\mathcal{X})}(X, Y \otimes_A \mathrm{colim}\ A_\alpha) \\
&\simeq \mathrm{Map}_{\mathrm{Mod}_{A'}(\mathcal{X})}(X \otimes_A A', Y \otimes_A A') \\
&= \overrightarrow{\Omega}_{X,Y}\mathcal{M}^{\flat}_{\mathcal{X}}(A')
\end{aligned}$$

We used the fact that X is (by definition, since $\mathcal{M}^{\flat}_{\mathcal{X}}(A) := \mathrm{Mod}_A(\mathcal{X})^{\omega}$) compact in going from the second line to the third line. The rest is formal algebraic manipulation. $\square$

We have thus proven that $\overrightarrow{\Omega}_{X,Y}\mathcal{M}^{\flat}_{\mathcal{X}}$ inherits the good properties of $\mathcal{M}^{\flat}_{\mathcal{X}}$:

Lemma 4.2.13. *Let $\mathcal{X}$ be an object of $\mathcal{P}r^L_{\mathrm{k}}$, and assume that $\mathcal{M}^{\flat}_{\mathcal{X}}$ satisfies hypotheses (1) -(5) of Proposition 4.2.4. Let $\overrightarrow{\Omega}_{X,Y}\mathcal{M}^{\flat}_{\mathcal{X}}$ be as in Notation 4.2.6. Then $\overrightarrow{\Omega}_{X,Y}\mathcal{M}^{\flat}_{\mathcal{X}}$ is ω-accessible, nilcomplete, infinitesimally cohesive, formally effective and a sheaf for the étale topology.*

Proof. This follows immediately from Lemmas 4.2.11 and 4.2.12. □

In order to apply the Artin-Lurie theorem to verify that, under the hypotheses of Proposition 4.2.4, the functors $\overrightarrow{\Omega}_{X,Y}\mathcal{M}^{\flat}_{\mathcal{X}}$ and $\Omega_{X,Y}\mathcal{M}_{\mathcal{X}}$ are algebraic stacks, it only remains to verify conditions (4) and (6) of that theorem. We now to turn to (4), which is the existence of the cotangent complex. We begin with two straightforward lemmas:

Lemma 4.2.14. *Let $\mathcal{X}$ be an object of $\mathcal{P}r^L_{\mathrm{k}}$, and let $X \in \mathrm{Mod}_A(\mathcal{X})^{\omega}$. Then the functor $\mathbb{M}\mathrm{or}(X,-) : \mathrm{Mod}_A(\mathcal{X}) \to \mathrm{Mod}_A$ commutes with all colimits. For any $Y \in \mathrm{Mod}_A(\mathcal{X})$, and $M \in \mathrm{Mod}_A$, we have $\mathbb{M}\mathrm{or}_A(X, Y \otimes_A M) \simeq \mathbb{M}\mathrm{or}_A(X,Y) \otimes_A M$, where $\mathbb{M}\mathrm{or}_A(-,-)$.*

Proof. The second statement follows from the first since Mod_A is generated under colimits by A, and $\otimes_A$ distributes over colimits. We will now prove the first statement. Note that since $\mathbb{M}\mathrm{or}(X,-)$ is a right adjoint by definition, it commutes with all limits, and in particular with finite limits. Since $\mathrm{Mod}_A(\mathcal{X})$ is stable (being k-linear), this implies the $\mathbb{M}\mathrm{or}(X,-)$ commutes with all finite limits. Given any filtered diagram $\{X_\alpha\}$ in $\mathrm{Mod}_A(\mathcal{X})$, we have a commutative diagram:

$$\begin{tikzcd}
\mathrm{Map}_A(A, \mathrm{colim}\ \mathbb{M}\mathrm{or}(X,X_\alpha)) \arrow[r] & \mathrm{Map}_A(A, \mathbb{M}\mathrm{or}(X, \mathrm{colim}\ X_\alpha)) \\
\mathrm{colim}\ \mathrm{Map}_A(A, \mathbb{M}\mathrm{or}(X,X_\alpha)) \arrow[r] \arrow[u, "\wr"] & \mathrm{Map}_A(A, \mathbb{M}\mathrm{or}(X, \mathrm{colim}\ X_\alpha)) \arrow[u, "\wr"'] \\
\mathrm{colim}\ \mathrm{Map}_A(X,X_\alpha) \arrow[r, "\sim"] \arrow[u, "\wr"] & \mathrm{Map}_A(X, \mathrm{colim}\ X_\alpha) \arrow[u, "\wr"']
\end{tikzcd}$$

The first row is equivalent to the second by compactness of A in Mod_A. The second and third rows are equivalent by adjunction. The bottom horizontal row is an equivalence by compactness of X in $\mathrm{Mod}_A(\mathcal{X})$. It follows that the top horizontal row is an equivalence. Since A is a generator for Mod_A, the functor $\mathrm{Map}_A(A, -)$ is conservative, since in the stable setting a map is an equivalence if and only if its fiber is contractible. It follows that the natural map colim $\mathbb{M}\mathrm{or}(X, X_\alpha) \to \mathbb{M}\mathrm{or}(X, \mathrm{colim}\ X_\alpha)$ is an equivalence. We have shown that $\mathbb{M}\mathrm{or}(X, -)$ commutes with finite colimits and ω-filtered colimits. Applying T.5.5.1.9., we conclude that $\mathbb{M}\mathrm{or}(X, -)$ commutes with all colimits. □

Lemma 4.2.15. *Let $\mathcal{X}$ be a presentable* k*-linear category that is locally compact over* k*. Then for all A in* CAlg_k*,* $\mathrm{Mod}_A(\mathcal{X})$ *is a locally compact category over A.*

Proof. We have a natural identification $\mathrm{Mod}_A(\mathcal{X}) \simeq \mathcal{X} \otimes_{\mathrm{Mod}_\mathrm{k}} \mathrm{Mod}_A$. Since $\mathcal{X} \otimes \mathrm{Mod}_A \simeq \mathrm{Ind}(\mathcal{X}^\omega \otimes_\mathrm{k} \mathrm{Mod}_A^\omega)$, and Mod_A^ω is generated under finite colimits by A, we see that every compact object in $\mathcal{X} \otimes_{\mathrm{Mod}_\mathrm{k}} \mathrm{Mod}_A$ is a retract of a finite colimit of objects of the form $X' \otimes_\mathrm{k} A$, with X' in $\mathcal{X}^\omega$. Furthermore, under the identification above, it is clear that for any two objects $X' \otimes_\mathrm{k} A$ and $Y' \otimes_\mathrm{k} A$, we have

$$\mathbb{M}\mathrm{or}_{\mathrm{Mod}_A(\mathcal{X})}(X' \otimes_\mathrm{k} A, Y' \otimes_\mathrm{k} A) \simeq \mathbb{M}\mathrm{or}_{\mathcal{X}}(X', Y') \otimes_\mathrm{k} A.$$

In particular, if X', $Y' \in \mathcal{X}^\omega$, then, since $\mathbb{M}\mathrm{or}_{\mathcal{X}}(X, Y)$ is compact in Mod_k by our hypothesis, it follows that $\mathbb{M}\mathrm{or}_{\mathrm{Mod}_A(\mathcal{X})}(X' \otimes_\mathrm{k} A, Y' \otimes_\mathrm{k} A)$ is compact in Mod_A.

Now let X, Y in $\mathrm{Mod}_A(\mathcal{X})^\omega$ be arbitrary. We will show that $\mathbb{M}\mathrm{or}_A(X,Y)$ is compact. Write X (resp. Y) as a retract of a finite colimit colim $X_\alpha \otimes_{\mathrm{k}} A$ (resp. colim $Y_\beta \otimes_{\mathrm{k}} A$), with X_α, Y_β in $\mathcal{X}^\omega$. Then $\mathbb{M}\mathrm{or}_A(X,Y)$ is a retract of $\mathbb{M}\mathrm{or}_A(\mathrm{colim}\ X_\alpha \otimes_{\mathrm{k}} A, \mathrm{colim}\ Y_\beta \otimes_{\mathrm{k}} A)$, so it will suffice to prove that the latter is compact in Mod_A, since compact objects are stable under retracts. $\mathbb{M}\mathrm{or}(-,-)$ commutes with finite colimits in the second variable by stability. Therefore we have

$$\mathbb{M}\mathrm{or}_A(\mathrm{colim}\ X_\alpha \otimes_{\mathrm{k}} A, \mathrm{colim}\ Y_\beta \otimes_{\mathrm{k}} A) \simeq \lim_\alpha \mathrm{colim}_\beta\ \mathbb{M}\mathrm{or}_A(X_\alpha \otimes_{\mathrm{k}} A, Y_\beta \otimes_{\mathrm{k}} A).$$

Since Mod_A is stable, the class of finite colimit diagrams coincides with class of finite limit diagrams, and so the right hand side can be written as a finite colimit, each of whose terms is compact by our remarks in the previous paragraph. It follows that $\mathbb{M}\mathrm{or}_A(\mathrm{colim}\ X_\alpha \otimes_{\mathrm{k}} A, \mathrm{colim}\ Y_\beta \otimes_{\mathrm{k}} A)$ is compact, completing the proof. □

Proposition 4.2.16. *Let $\mathcal{X}$ be a locally compact category over* k*. Let $A \in \mathrm{CAlg}_{\mathrm{k}}$, X, $Y \in \mathrm{Mod}_A(\mathcal{X})^\omega$, and let $\overrightarrow{\Omega}_{X,Y}\mathcal{M}^\flat_{\mathcal{X}}$ and $\Omega_{X,Y}\mathcal{M}_{\mathcal{X}}$ be as in Notation 4.2.6. Then the functors $\overrightarrow{\Omega}_{X,Y}\mathcal{M}^\flat_{\mathcal{X}}$ and $\Omega_{X,Y}\mathcal{M}_{\mathcal{X}}$ have cotangent complexes.*

Proof. For the duration of this proof we will use the following notation: $\mathcal{F} : \mathrm{CAlg}_{A/} \to \mathcal{S}$ (resp. $\mathcal{G} := \mathrm{CAlg}_{A/} \to \mathcal{S}$) denotes the restriction of $\overrightarrow{\Omega}_{X,Y}\mathcal{M}^\flat_{\mathcal{X}} : \mathrm{CAlg}_{\mathrm{k}} \to \mathcal{S}$ (resp. $\Omega_{X,Y}\mathcal{M}_{\mathcal{X}} : \mathrm{CAlg}_{\mathrm{k}} \to \mathcal{S}$) along the natural functor $\mathrm{CAlg}_{A/} \to \mathrm{CAlg}_{\mathrm{k}}$. Note that, by virtue of G.3.2.12, and the fact that the structure morphism $\mathrm{Spec}(A) \to \mathrm{Spec}(\mathrm{k})$ has a cotangent complex, it will suffice to prove the morphisms $\mathcal{F} \to \mathrm{Spec}(A)$ and $\mathcal{G} \to \mathrm{Spec}(A)$ have relative cotangent complexes.

We begin by considering $\mathcal{F}$. Let $B \in \mathrm{CAlg}_{A/}$, and $x : \mathrm{Spec}(B) \to \mathcal{F}$ be a morphism in $\mathcal{S}t_A$. We will now show that the functor $\mathbb{D}\mathrm{er}_{\mathcal{F}/A}(x, -) : \mathrm{Mod}_B \to \mathcal{S}$ is corepresentable. Let $M \in \mathrm{Mod}_B$ and let $B\langle M\rangle \in \mathrm{CAlg}_{B/}$ be the square zero extension of B by M, whose underlying B-module is $B \oplus M$. Consider the following diagram:

$$\begin{tikzcd}
\mathrm{Map}_{B\langle M\rangle}(X \otimes_A B\langle M\rangle, Y \otimes_A B\langle M\rangle) \arrow[r] & \mathrm{Map}_B(X \otimes_A B, Y \otimes_A B) \\
\mathrm{Map}_A(X, Y \otimes_A (B \oplus M)) \arrow[r] \arrow[u, "\wr"] & \mathrm{Map}_A(X, Y \otimes_A B) \arrow[u, "\wr"] \\
\mathrm{Map}_A(X, Y \otimes_A M) \times \mathrm{Map}_A(X, Y \otimes_A B) \arrow[r] \arrow[u, "\wr"] & \mathrm{Map}_A(X, Y \otimes_A B) \arrow[u, "\wr"]
\end{tikzcd}$$

The equivalence of the second row with the first is by adjunction. The second and third rows are equivalent by the stability of $\mathrm{Mod}_A(\mathcal{X})$, which allows us to identify sums with products. It is manifest in this diagram that $\mathbb{D}\mathrm{er}_{\mathcal{F}/A}(x, M)$, which by definition is the homotopy fiber at x of the top horizontal morphism, is equivalent to $\mathrm{Map}_A(X, Y \otimes_A M)$. We have a chain of equivalences

$$\begin{aligned}
\mathbb{D}\mathrm{er}_{\mathcal{F}/A}(x, M) &\simeq \mathrm{Map}_A(X, Y \otimes_A M) \\
&\simeq \Omega^\infty \mathbb{M}\mathrm{or}_A(X, Y \otimes_A M) \\
&\simeq \Omega^\infty (\mathbb{M}\mathrm{or}_A(X, Y) \otimes_A M) \\
&\simeq \Omega^\infty ((\mathbb{M}\mathrm{or}_A(X, Y) \otimes_A B) \otimes_B M) \\
&\simeq \Omega^\infty (\mathbb{M}\mathrm{or}_B((\mathbb{M}\mathrm{or}_A(X, Y) \otimes_A B)^\vee, M)) \\
&\simeq \mathrm{Map}_B((\mathbb{M}\mathrm{or}_A(X, Y) \otimes_A B)^\vee, M)
\end{aligned}$$

In going from the second line to the third, we have used Lemma 4.2.14, and in going from the fourth to the fifth, we have used the fact that $\mathbb{M}\mathrm{or}_A(X, Y)$ is compact by Lemma 4.2.15, and the fact that compact objects coincide with dualizable ones in Mod_B (Proposition 2.2.20). Thus, we see that the functor $\mathbb{D}\mathrm{er}_{\mathcal{F}/A}(x, -)$ is corepresented by the B-module $\mathbb{L}_{\mathcal{F}/A,x} := (\mathbb{M}\mathrm{or}_A(X, Y) \otimes_A B)^\vee$.

Now suppose that we are given a commutative diagram of functors $\mathrm{CAlg}_{A/} \to \mathcal{S}$:

$$\begin{tikzcd}
\mathrm{Spec}(C) \arrow[rr, "u"] \arrow[dr, "y"'] & & \mathrm{Spec}(B) \arrow[dl, "x"] \\
& \mathcal{F} &
\end{tikzcd}$$

We have an induced commutative diagram:

$$\begin{tikzcd}
u^*\mathbb{L}_{\mathcal{F}/A,x} \arrow[r] \arrow[d, "\wr"] & \mathbb{L}_{\mathcal{F}/A,y} \arrow[d, "\wr"] \\
(\mathbb{M}\mathrm{or}_A(X,Y)\otimes_A B)^\vee \otimes_B C \arrow[r] \arrow[d, "\wr"] & (\mathbb{M}\mathrm{or}_A(X,Y)\otimes_A C)^\vee \arrow[d, "\wr"] \\
\mathbb{M}\mathrm{or}_A(\mathbb{M}\mathrm{or}_A(X,Y),B)\otimes_B C \arrow[r] & \mathbb{M}\mathrm{or}_A(\mathbb{M}\mathrm{or}_A(X,Y),C)
\end{tikzcd}$$

The first and second rows are equivalent by the definitions of $\mathbb{L}_{\mathcal{F}/A,-}$ and the top horizontal morphism. The second and third rows are equivalent by adjunction. Finally, the bottom horizontal map is an equivalence by Lemma 4.2.14. Thus, the morphism $u^*\mathbb{L}_{\mathcal{F}/A,x} \to \mathbb{L}_{\mathcal{F}/A,y}$ is an equivalence. This proves the existence of a relative cotangent complex $\mathbb{L}_{\mathcal{F}/A} \in \mathrm{QC}(\mathcal{F})$ for the morphism $f : \mathcal{F} \to \mathrm{Spec}(A)$.

Now let $x : \mathrm{Spec}(B) \to \mathcal{G}$ be a point of $\mathcal{G}$, and let $i : \mathcal{G} \to \mathcal{F}$ be the natural map. Then we claim that we have equivalence of functor $\mathbb{D}\mathrm{er}_{\mathcal{G}/A}(x,-) \simeq \mathbb{D}\mathrm{er}_{\mathcal{F}/A}(i(x),-)$. To prove the claim, let $M \in \mathrm{Mod}_B$, and consider the commutative diagram:

$$\begin{tikzcd}
\mathbb{D}\mathrm{er}_{\mathcal{G}/A}(x,M) \arrow[r] \arrow[d] & \mathcal{G}(B\langle M\rangle) \arrow[r, "\pi'"] \arrow[d, "i_{B\langle M\rangle}"] & \mathcal{G}(B) \arrow[d, "i_B"] \\
\mathbb{D}\mathrm{er}_{\mathcal{F}/A}(i(x),M) \arrow[r] & \mathcal{F}(B\langle M\rangle) \arrow[r, "\pi"'] & \mathcal{F}(B)
\end{tikzcd}$$

For any $C \in \mathrm{CAlg}_{A/}$, $\mathcal{G}(C)$ is a union of connected components of $\mathcal{F}(C)$. Therefore, passing to the long exact sequences on homotopy groups associated with the fiber sequences above and applying the five lemma, we immediately see that $\pi_n\mathbb{D}\mathrm{er}_{\mathcal{G}/A}(x,M) \to \pi_n\mathbb{D}\mathrm{er}_{\mathcal{F}/A}(i(x),M)$ is an isomorphism for $n > 0$, and an injection for $n = 0$.

We will now prove that $\pi_0 \mathbb{D}\mathrm{er}_{\mathcal{G}/A}(x, M) \to \pi_0 \mathbb{D}\mathrm{er}_{\mathcal{F}/A}(i(x), M)$ is an isomporphism as well. In view of the fact that $\mathcal{G}(B)$ (resp. $\mathcal{G}(B\langle M\rangle)$) is a union of connected components of $\mathcal{F}(B)$ (resp. $\mathcal{F}(B\langle M\rangle)$), the only thing that needs proof is the following: if $\Phi \in \pi_0\mathcal{F}(B\langle M\rangle)$ is in the inverse image of $[i(x)] \in \pi_0\mathcal{F}(B)$, then Φ is in the image of $\mathcal{G}(B\langle M\rangle)$. Unravelling the definitions, one readily checks that this follows from the following more general fact: the functor $(-) \otimes_{B\langle M\rangle} B : \mathrm{Mod}_{B\langle M\rangle}(\mathcal{X}) \to \mathrm{Mod}_B(\mathcal{X})$ is conservative.

To prove the conservatism of this last functor, note that since the categories involved are stable, it will suffice to prove that the functor reflects zero objects. Let $X \in \mathrm{Mod}_{B\langle M\rangle}(\mathcal{X})$ and suppose that $X \otimes_{B\langle M\rangle} B \simeq 0$. The object of $\mathcal{X}$ underlying $X \otimes_{B\langle M\rangle} B$ is the cofiber of the action map $a : M \otimes X \to X$. Consequently, a is an equivalence in $\mathcal{X}$. On the other hand, the multiplication map $m : M \otimes M \to M$ is the zero map, by definition of a square zero extension. So in $h\mathcal{X}$, we have equalities $0 = [a \circ (m \otimes \mathrm{id})] = [a \circ (\mathrm{id} \otimes a)]$. The last map is an isomorphism with target X, so it follows that $X \simeq 0$. This completes the proof of the conservatism of $(-)\otimes_{B\langle M\rangle} B$, and therefore also the proof of the fact that the map $\pi_0\mathbb{D}\mathrm{er}_{\mathcal{G}/A}(x, M) \to \pi_0\mathbb{D}\mathrm{er}_{\mathcal{F}/A}(i(x), M)$ is an isomporphism.

To summarize: we have shown that the natural map from $\mathbb{D}\mathrm{er}_{\mathcal{G}/A}(x, -)$ to $\mathbb{D}\mathrm{er}_{\mathcal{F}/A}(i(x), -)$ is an equivalence of functors. It follows $\mathbb{D}\mathrm{er}_{\mathcal{G}/A}(x, -)$ is corepresented by $\mathbb{L}_{\mathcal{G}/A,x} := (\mathbb{M}\mathrm{or}_A(X, Y) \otimes_A B)^\vee = \mathbb{L}_{\mathcal{F}/A,i(x)}$. The fact that $\mathbb{L}_{\mathcal{G}/A,-}$ is compatible with base change follows from the corresponding statement for $\mathbb{L}_{\mathcal{F}/A,-}$.

This completes the proof of the fact $\mathcal{G} \to \operatorname{Spec}(A)$ has a relative cotangent complex. ∎

We now turn our attention to verifying that the functors $\overrightarrow{\Omega}_{X,Y}\mathcal{M}^{\flat}_{\mathcal{X}}$ and $\Omega_{X,Y}\mathcal{M}_{\mathcal{X}}$ satisfy condition (7) of the Artin-Lurie criterion (truncatedness).

Lemma 4.2.17. *Let $\mathcal{X}$ be in $\mathcal{P}r^L_{\mathrm{k}}$, and let X, Y be compact objects in $\operatorname{Mod}_A(\mathcal{X})$. Assume that $\mathbb{M}\mathrm{or}_A(X,Y)$ has Tor amplitude $\leq n$. Then the functors $\overrightarrow{\Omega}_{X,Y}\mathcal{M}^{\flat}_{\mathcal{X}}$ and $\Omega_{X,Y}\mathcal{M}_{\mathcal{X}}$ defined in Notation 4.2.6 are n-truncated. That is, the restriction of these functors to discrete commutative algebras factors through $\tau_{\leq n}\mathcal{S}$.*

Proof. Let B be a discrete object in $\mathrm{CAlg}_{A/}$. We have equivalences:

$$\begin{aligned}
\overrightarrow{\Omega}_{X,Y}\mathcal{M}^{\flat}_{\mathcal{X}}(B) &\simeq \operatorname{Map}_B(X \otimes_A B, Y \otimes_A B) \\
&\simeq \operatorname{Map}_A(X, Y \otimes_A B) \\
&\simeq \Omega^{\infty}\mathbb{M}\mathrm{or}_A(X, Y \otimes_A B) \\
&\simeq \Omega^{\infty}(\mathbb{M}\mathrm{or}_A(X,Y) \otimes_A B)
\end{aligned}$$

In passing from the third line to the fourth line we have made use of the fact that X is compact (Lemma 4.2.14). Since $\mathbb{M}\mathrm{or}_A(X,Y)$ is of Tor-amplitude $\leq n$, and B is discrete, we have that $\pi_k(\overrightarrow{\Omega}_{X,Y}\mathcal{M}^{\flat}_{\mathcal{X}}(B), x) = 0$ for $k > n$, and any choice of basepoint x. Since $\Omega_{X,Y}\mathcal{M}_{\mathcal{X}}(B)$ is the union of a set connected components of $\overrightarrow{\Omega}_{X,Y}\mathcal{M}^{\flat}_{\mathcal{X}}(B)$, it follows that $\pi_k(\Omega_{X,Y}\mathcal{M}_{\mathcal{X}}(B), x) = 0$ for $k > n$, and any choice of basepoint x. ∎

We are now ready to apply the Artin-Lurie criterion to prove the algebraicity of the stacks $\overrightarrow{\Omega}_{X,Y}\mathcal{M}^{\flat}_{\mathcal{X}}$ and $\Omega_{\mathcal{X},\mathcal{Y}}\mathcal{M}_{\mathcal{X}}$.

Proposition 4.2.18. *Let $\mathcal{X}$ be an object of $\mathcal{P}r^{L}_{\mathrm{k}}$ that is locally compact and satisfies conditions (1)-(5) of Proposition 4.2.4. Let $A \in \mathrm{CAlg}_{\mathrm{k}}$, and let X, $Y \in \mathrm{Mod}_A(\mathcal{X})^{\omega}$. Then there exists $n \geq 0$ such that the functors $\overrightarrow{\Omega}_{X,Y}\mathcal{M}^{\flat}_{\mathcal{X}}$ and $\Omega_{X,Y}\mathcal{M}_{\mathcal{X}}$ defined in Notation 4.2.6 are derived algebraic n-stacks, locally of finite presentation over* k.

Proof. Note that by our hypotheses on $\mathcal{X}$, and Lemma 4.2.15, $\mathbb{M}\mathrm{or}_A(X,Y)$ is a compact object of Mod_A. Consequently, by [TV07a, Proposition 2.22], there exists $n \geq 0$ such that $\mathbb{M}\mathrm{or}_A(X,Y)$ has Tor amplitude $\leq n$. Throughout this proof, n will refer to this number. We will prove that $\overrightarrow{\Omega}_{X,Y}\mathcal{M}^{\flat}_{\mathcal{X}}$ and $\Omega_{X,Y}\mathcal{M}_{\mathcal{X}}$ are derived algebraic n-stacks.

According to Lemma 4.2.13, $\overrightarrow{\Omega}_{X,Y}\mathcal{M}^{\flat}_{\mathcal{X}}$ satisfies conditions (1), (2), (3), (5) and (6) of Theorem 4.1.4. Proposition 4.2.16 guarantees the existence of a cotangent complex for $\overrightarrow{\Omega}_{X,Y}\mathcal{M}^{\flat}_{\mathcal{X}}$, which is condition (4). Lemma 4.2.17, it also satisfies condition (7) with n as above. It follows from Theorem 4.1.4 that $\overrightarrow{\Omega}_{X,Y}\mathcal{M}^{\flat}_{\mathcal{X}}$ is a derived algebraic n-stack.

Similarly, by Lemma 4.2.9, $\Omega_{X,Y}\mathcal{M}_{\mathcal{X}}$ satisfies conditions (1), (2), (3), (5) and (6) of the Artin-Lurie criterion. Proposition 4.2.16 tells us that it satisfies condition (4). Finally, Lemma 4.2.17, it also satisfies condition (7) with n as above. It follows from Theorem 4.1.4 that $\overrightarrow{\Omega}_{X,Y}\mathcal{M}^{\flat}_{\mathcal{X}}$ is a derived algebraic n-stack. □

Having established the algebraicity of the loop stack of $\mathcal{M}_{\mathcal{X}}$, we now return to our original purpose - to prove the existence of a cotangent complex for the functor $\mathcal{M}_{\mathcal{X}}$. Unfortunately, we cannot directly deduce Proposition 4.2.4 from Propositions 4.2.5 and 4.2.18. The reason is the following: although Proposition 4.2.18 does show that the fiber of the diagonal map $\Delta : \mathcal{M}_{\mathcal{X}} \to \mathcal{M}_{\mathcal{X}} \times \mathcal{M}_{\mathcal{X}}$ over any point $\mathrm{Spec}(A) \to \mathcal{M}_{\mathcal{X}} \times \mathcal{M}_{\mathcal{X}}$ is an algebraic n-stack for *some* n, the value of n depends on the point we choose. Thus, in general, there is no single number n for which the diagonal $\Delta : \mathcal{M}_{\mathcal{X}} \to \mathcal{M}_{\mathcal{X}} \times \mathcal{M}_{\mathcal{X}}$ is an n-representable morphism. This is not a major impediment, because $\mathcal{M}_{\mathcal{X}}$ is a filtered colimit of substacks $\mathcal{M}_{\mathcal{X}}^{n}$, each of which has a diagonal that is n-representable for some n. We now introduce the substacks $\mathcal{M}_{\mathcal{X}}^{n}$:

Lemma 4.2.19. *Let n be an integer, and let $\mathcal{X}$ be an object of $\mathcal{P}r_{\mathrm{k}}^{L}$. Then there is a monomorphism of functors $\mathcal{M}_{\mathcal{X}}^{\flat,n} \subseteq \mathcal{M}_{\mathcal{X}}^{\flat}$ such that for any A in* $\mathrm{CAlg}_{\mathrm{k}}$*, $\mathcal{M}_{\mathcal{X}}^{\flat,n}(A)$ is the full subcategory of $\mathcal{M}_{\mathcal{X}}^{\flat}(A)$ consisting of objects that are of amplitude $\leq n$.*

Proof. Let $f : A \to B$ be a morphism in $\mathrm{CAlg}_{\mathrm{k}}$. We must show that the natural functor $f^{*} : \mathcal{M}_{\mathcal{X}}^{\flat}(A) \to \mathcal{M}_{\mathcal{X}}^{\flat}(B)$ carries $\mathcal{M}_{\mathcal{X}}^{\flat,n}(A)$ into $\mathcal{M}_{\mathcal{X}}^{\flat,n}(B)$. Let $X \in \mathcal{M}_{\mathcal{X}}^{\flat}(A)$ be an object of amplitude $\leq n$. We will now show that $X \otimes_{A} B$ has amplitude $\leq n$. We have

$$\begin{aligned} \mathbb{M}\mathrm{or}_{B}(X \otimes_{A} B, X \otimes_{A} B) &\simeq \mathbb{M}\mathrm{or}_{A}(X, X \otimes_{A} B) \\ &\simeq \mathbb{M}\mathrm{or}_{A}(X, X) \otimes_{A} B \end{aligned}$$

The first equivalence is by adjunction, and the second is a consequence of Lemma 4.2.14, since X is compact by hypothesis. The A-module $\mathbb{M}\mathrm{or}_{A}(X, X)$ has amplitude

$\leq n$ by our assumptions on X. Tor amplitude is stable by base change ([TV07a, Proposition 2.22]). Therefore, it follows that $\mathbb{M}\mathrm{or}_A(X, X) \otimes_A B$ is of Tor-amplitude $\leq n$, proving that $X \otimes_A B$ is of amplitude $\leq n$. This completes the proof of the existence of the subfunctor $\mathcal{M}_{\mathcal{X}}^{\flat,n} \subseteq \mathcal{M}_{\mathcal{X}}^{\flat}$. □

Definition 4.2.20. Let $\mathcal{X}$ be an object of $\mathcal{P}r_{\mathrm{k}}^{L}$. For each integer n, we define the functor $\mathcal{M}_{\mathcal{X}}^{\flat,n} : \mathrm{CAlg}_{\mathrm{k}} \to \mathrm{Cat}_{\infty}$ as in Lemma 4.2.19. We define $\mathcal{M}_{\mathcal{X}}^{n} : \mathrm{CAlg}_{\mathrm{k}} \to \mathcal{S}$ to be the composite $\mathcal{M}_{\mathcal{X}}^{n} := (-)^{\simeq} \circ \mathcal{M}_{\mathcal{X}}^{\flat,n}$.

Lemma 4.2.21. *Let $\mathcal{X}$ be an object of $\mathcal{P}r_{\mathrm{k}}^{L}$, and assume that $\mathcal{X}$ is locally compact. Then $\mathcal{M}_{\mathcal{X}}^{\flat}$ is a filtered colimit of the functors $\mathcal{M}_{\mathcal{X}}^{\flat,n}$. Similarly, $\mathcal{M}_{\mathcal{X}}$ is the filtered colimit of the functors $\mathcal{M}_{\mathcal{X}}^{n}$.*

Proof. Let A in $\mathrm{CAlg}_{\mathrm{k}}$ and $\mathcal{X} \in \mathcal{M}_{\mathcal{X}}^{\flat}(A) = \mathrm{Mod}_A(\mathcal{X})^{\omega}$. The condition that $\mathcal{X}$ is locally compact implies, in particular, that $\mathbb{M}\mathrm{or}_A(X, X)$ is a compact object of Mod_A. Consequently, by [TV07a, Proposition 2.22], there is an integer n such that $\mathbb{M}\mathrm{or}_A(X, X)$ is of Tor amplitude $\leq n$. In other words, there is an integer n for which $X \in \mathcal{M}_{\mathcal{X}}^{\flat,n}(A)$. The rest is clear. □

We will not need the full strength of the following proposition until the next section. In this section, we will only need the fact that $\mathcal{M}_{\mathcal{X}}^{\flat,n}$ is infinitessimally cohesive, in order to apply Proposition 4.2.5.

Lemma 4.2.22. *Let $\mathcal{X}$ be an object of $\mathcal{P}r_{\mathrm{k}}^{L}$, and suppose that $\mathcal{M}_{\mathcal{X}}^{\flat}$ satisfies conditions (1)-(5) of Proposition 4.2.4. Then $\mathcal{M}_{\mathcal{X}}^{\flat,n}$ satisfies conditions (1)-(5) of Propo-*

sition 4.2.4, and so does $\mathcal{M}^n_{\mathcal{X}}$.

Proof. Assume that $\mathcal{X}$ satisfies the hypotheses of the lemma. Since $\mathcal{M}^\flat_{\mathcal{X}}$ is a sheaf for the étale topology by hypothesis, the fact that $\mathcal{M}^{\flat,n}_{\mathcal{X}}$ is a sheaf for the étale topology follows immediately from the fact (G.3.5.3) that the property of being of Tor amplitude $\leq n$ is local for the flat topology (and therefore also local for the étale topology). This proves that $\mathcal{M}^{\flat,n}_{\mathcal{X}}$ satisfies (5).

Let $\nu : \mathrm{N}(\mathbb{Z}_{\leq 0})^{\triangleleft} \to \mathrm{CAlg}_{\mathrm{k}}$ be a Postnikov tower. Let $\nu_{-\infty} = A$, so that $\nu_{-k} \simeq \tau_{\leq k}A$. By our assumption that $\mathcal{M}^\flat_{\mathcal{X}}$ is nilcomplete, the induced diagram $\mathcal{M}^\flat_{\mathcal{X}} \circ \nu : \mathrm{N}(\mathbb{Z}_{\leq 0})^{\triangleleft} \to \mathrm{Cat}_\infty$ is a limit diagram. According to [TV07a, Proposition 2.22(3)], an $M \in \mathrm{Mod}_A$ is perfect of amplitude $\leq n$ if and only if $M \otimes_A \pi_0 A \in \mathrm{Mod}_{\pi_0 A}$ is perfect of amplitude $\leq n$. It follows that $\mathcal{M}^\flat_{\mathcal{X}} \circ \nu$ has the following analogue of Property ¶ (see Definiton 2.3.6): an object $X \in \mathcal{M}^\flat_{\mathcal{X}}(A)$ is in $\mathcal{M}^{\flat,n}_{\mathcal{X}}(A)$ if and only if its image in $\mathcal{M}^\flat_X(\tau_{\leq k}A)$ lands in $\mathcal{M}^{\flat,n}_{\mathcal{X}}(\tau_{\leq k}A)$ for all k. Since $\mathcal{M}^\flat_{\mathcal{X}} \circ \nu$ is a limit diagram, it follows that $\mathcal{M}^{\flat,n}_{\mathcal{X}} \circ \nu$ is a limit diagram. Thus $\mathcal{M}^{\flat,n}_{\mathcal{X}}$ is nilcomplete (condition (2)).

Let A be a discrete Noetherian commutative k-algebra with maximal ideal $\mathfrak{m}$. Then $M \in \mathrm{Mod}_A$ has Tor amplitude $\leq n$ iff $M \otimes_A A/\mathfrak{m}$ has Tor amplitude $\leq n$. Therefore, by an argument similar to the one in the previous paragraph, one deduces the formal effectivity of $\mathcal{M}^{\flat,n}_{\mathcal{X}}$ from formal effectivity of $\mathcal{M}^\flat_{\mathcal{X}}$. The proof of the infinitesimal cohesiveness of $\mathcal{M}^{\flat,n}_{\mathcal{X}}$ is essentially identical. As in the proof of formal effectivity, the essential point is to note that the condition on Tor amplitude can be checked after base change to geometric points. We leave the details to the reader.

The statement about $\mathcal{M}^n$ follows immediately from the corresponding statement for $\mathcal{M}^{\flat,n}_{\mathcal{X}}$, and the fact that the functor $(-)^{\simeq}$ preserves all limits and filtered colimits.

$\square$

We can now apply Proposition 4.2.5 to deduce the existence of a cotangent complex for $\mathcal{M}^n_{\mathcal{X}}$:

Proposition 4.2.23. *Let $\mathcal{X}$ be an object of $\mathcal{P}r^L_{\mathrm{k}}$ satisfying all the hypotheses of Proposition 4.2.4. Then, for each integer n, the functor $\mathcal{M}^n_{\mathcal{X}}$ has a cotangent complex.*

Proof. By Lemma 4.2.22, the functor $\mathcal{M}^n_{\mathcal{X}}$ is infinitesimally cohesive, and by Proposition 4.2.18 its diagonal is a relative n-stack. Therefore by Proposition 4.2.5, $\mathcal{M}^n_{\mathcal{X}}$ admits a cotangent complex. $\square$

Proof of Proposition 4.2.4. The proposition follows from Proposition 4.2.23, and Lemma 4.2.21. $\square$

In practice, we will use Proposition 4.2.23. We decided not to state this Proposition at the beginning of the section, so as not to obscure the main ideas.

4.3 Dualizability implies Geometricity

In their seminal work, Toën and Vaquié prove the following beautiful theorem:

Theorem 4.3.1 (Toën-Vaquié, [TV07a]). *Let $\mathcal{X}$ be a smooth and proper* k*-linear ∞-category. That is, let $\mathcal{X}$ be a dualizable object of $\mathcal{P}r^L_{\omega,\mathrm{k}}$. Then the functor $\mathcal{M}_{\mathcal{X}}$: $\mathrm{CAlg}_{\mathrm{k}} \to \mathcal{S}$ defined in Notation 2.4.1 is a locally geometric derived ∞-stack, locally of finite presentation over* k*.*

This section is devoted to giving a new proof of this theorem. As we will see shortly, the proof given here makes manifest the role of dualizability. In order to exploit the dualizabilty of $\mathcal{X}$ as an object of $\mathcal{P}r^L_{\omega,\mathrm{k}}$, it will be necessary to work with the $\mathcal{P}r^L_{\omega,\mathrm{k}}$-valued functor $\mathfrak{M}_{\mathcal{X}}$, of which the space valued functor $\mathcal{M}_{\mathcal{X}}$ is a pale shadow. The importance and expediency of working with these more structured moduli functors is another point that we wish to bring out and emphasize. With this as our motivation, we will happy to take for granted the following theorem, which is also proven in [TV07a]. The methods of this paper could be used to give a direct proof of this result, using the Artin-Lurie theorem.

Proposition 4.3.2. *The functor $\mathcal{M}_{\mathbf{1}}$: $\mathrm{CAlg}_{\mathrm{k}} \to \mathcal{S}$ is a locally geometric derived ∞-stack, locally of finite presentation over* k*.*

We begin with some lemmas that will be used to prove that $\mathcal{M}_{\mathcal{X}}$ is infinitesimally cohesive, nilcomplete and formally effective (Propostion 4.3.6).

Lemma 4.3.3. *Let K be a simplicial set, and let $\nu : K^{\triangleleft} \to \mathcal{P}r^L_{\omega,\mathrm{k}}$ be a diagram. Assume that the induced diagram $(-)^{\simeq} \circ (-)^{\omega} \circ \nu : K^{\triangleleft} \to \mathcal{S}$ is a limit diagram, and that ν has Property ¶ (see Definition 2.3.6). Let η denote the unit of the natural* *adjunction*

$$\begin{tikzcd} \nu_\infty \arrow[r] & \lim \nu_{|K} \arrow[l, bend right] \end{tikzcd}$$

Then the following are equivalent:

(1) ν is a limit diagram

(2) For every compact object X in ν_∞, η_X is an equivalence.

Proof. The implication (1) $\Rightarrow$ (2) is obvious. We will now prove that (2) $\Rightarrow$ (1). Assume that (2) holds. By Lemma 2.3.5, it will suffice to show that the induced diagram $(-)^\omega \circ \nu$ is a limit diagram. To prove this, we must show that the natural map $\nu_\infty^\omega \to (\lim \nu_{|K})^\omega$ is an essential surjective and fully faithful.

We turn first to the essential surjectivity. Note that a functor $f : \mathcal{C} \to \mathcal{D}$ is essentially surjective iff the induced map of spaces $f^\simeq : \mathcal{C}^\simeq \to \mathcal{D}^\simeq$ is surjective on connected components. We have a commutative diagram

$$\begin{tikzcd} \nu_\infty^\simeq \arrow[r] \arrow[ddr] & ((\lim \nu_{|K})^\omega)^\simeq \arrow[d] \\ & (\lim(\nu_{|K})^\omega)^\simeq \arrow[d] \\ & \lim((\nu_{|K})^\omega)^\simeq \end{tikzcd}$$

The top vertical arrow is an equivalence by virtue of our assumption that ν has Property ¶. The bottom vertical map is an equivalence because $(-)^\simeq$ commutes with limits (Lemma 2.3.13). Finally, the diagonal map is an equivalence because of our assumtion that $(-)^{\simeq} \circ (-)^{\omega} \circ \nu$ is a limit diagram. It follows that the top horizontal morphism is an equivalence of spaces, and, in particular, is surjective on connected components. This proves the essential surjectivity.

We will now prove that the functor $\nu_{\infty}^{\omega} \to (\lim \nu_{|K})^{\omega}$ is fully faithful. Note that the fact that this functor is essentially surjective, together with the assumption (2), implies that the right adjoint $\lim \nu_{|K} \to \nu_{\infty}$ preserves compact objects, and consequently we have an induced adjunction

$$\begin{tikzcd} (\nu_{\infty})^{\omega} \arrow[r] & (\lim \nu_{|K})^{\omega} \arrow[l, bend right] \end{tikzcd}$$

The fact that $\nu_{\infty}^{\omega} \to (\lim \nu_{|K})^{\omega}$ is fully faithful now follows from the following more general fact, together with our assumption (2): a left adjoint $f : \mathcal{C} \to \mathcal{D}$ between arbitrary ∞-categories is fully faithful iff the unit of the adjunction is an equivalence. This in turn is manifest in the following commutative diagram:

$$\begin{tikzcd} \mathcal{C}(X, Y) \arrow[r, "\eta_Y \circ (-)"] \arrow[dr, "f"] & \mathcal{C}(X, gfY) \arrow[d, "\wr"] \\ & \mathcal{D}(fX, fY) \end{tikzcd}$$

Here g is the right adjoint to f, and η is the unit of the adjunction. The vertical equivalence is a consequence of adjointness. The diagonal map is an equivalence for all X, Y if and only if f is fully faithful, by definition. By the Yoneda lemma, the top horizontal map is an equivalence for all X, Y if and only if η_Y is an equivalence for all Y. Putting together the last three statements, one sees that η is an equivalence iff f is fully faithful. $\square$

Lemma 4.3.4. *Let $\mu : K^{\triangleleft} \to \mathrm{CAlg}_{\mathrm{k}}$ be a limit diagram, and suppose that $\mathcal{M}_{\mathbf{1}} \circ \mu : K^{\triangleleft} \to \mathcal{S}$ is a limit diagram. Then $\mathfrak{M}_{\mathbf{1}} \circ \mu : K^{\triangleleft} \to \mathcal{P}r^{L}_{\omega,\mathrm{k}}$ is a limit diagram.*

Proof. Let $A = \mu_{\infty}$, let $A_x = \mu_x$ for $x \in K$, and let $\nu = \mathfrak{M}_{\mathbf{1}} \circ \mu$. By definition, we have $\mathcal{M}_{\mathbf{1}} \circ \mu = (-)^{\simeq} \circ (-)^{\omega} \circ \nu$. Since $\mathcal{M}_{\mathbf{1}} \circ \mu$ is a limit diagram by hypothesis, Lemma 4.3.3 implies that, in order to prove the present lemma, it will suffice to show that the unit of the adjunction

$$\begin{tikzcd} \mathfrak{M}_{\mathbf{1}}(A) \arrow[r] & \lim_{x\in K}\mathfrak{M}_{\mathbf{1}}(A_x) \arrow[l, bend right] \end{tikzcd}$$

satisfies condition (2) of Lemma 4.3.3. For $M \in \mathfrak{M}_{\mathbf{1}}(A)^{\omega} = \mathrm{Mod}_A^{\omega}$, the unit of this adjunction is the natural morphism

$$M \otimes_A A \to \lim_{x\in K} M \otimes_A A_x$$

Since Mod_A is a perfect symmetric monoidal category by Lemma 2.2.20, the compact object M is dualizable. Consequently, by Lemma 2.2.5, the functor $M \otimes_A (-)$ commutes with all limits. It follows that the unit map above is an equivalence. $\square$

Proposition 4.3.5. *The functor $\mathfrak{M}_{\mathbf{1}} : \mathrm{CAlg}_{\mathrm{k}} \to \mathcal{P}r^{L}_{\omega,\mathrm{k}}$ is nilcomplete, infinitesimally cohesive and formally effective.*

Proof. Since $\mathcal{M}_{\mathbf{1}}$ is locally geometric and locally of finite presentation by Proposition 4.3.2, it is nilcomplete, infinitesimally cohesive and formally effective by virtue of the Artin-Lurie recognition principle (Theorem 4.1.4). The proposition now follows from Lemma 4.3.4 and the definitions. $\square$

Let $\mathcal{C}$ be an ∞-category. Let us say that a functor $\mathcal{F} : \mathrm{CAlg}_{\mathrm{k}} \to \mathcal{C}$ is good if it is infinitesimally cohesive, nilcomplete, a sheaf for the étale topology, formally effective and ω-accessible.

Proposition 4.3.6. *Let $\mathcal{X}$ be a smooth and proper* k*-linear ∞-category, i.e., a dualizable object of the symmetric monoidal ∞-category $(\mathcal{P}r^L_{\omega,\mathrm{k}})^{\otimes}$. Then the following is true*

1. *The functor $\mathfrak{M}_{\mathcal{X}} : \mathrm{CAlg}_{\mathrm{k}} \to \mathcal{P}r^L_{\omega,\mathrm{k}}$ is good.*

2. *The functor $\mathcal{M}^{\flat}_{\mathcal{X}} : \mathrm{CAlg}_{\mathrm{k}} \to \mathrm{Cat}_{\infty}$ is good.*

3. *For any $n \in \mathbb{Z}$, the functor $\mathcal{M}^{\flat,n}_{\mathcal{X}} : \mathrm{CAlg}_{\mathrm{k}} \to \mathrm{Cat}_{\infty}$ is good.*

4. *For any $n \in \mathbb{Z}$, the functor $\mathcal{M}^{n}_{\mathcal{X}} : \mathrm{CAlg}_{\mathrm{k}} \to \mathcal{S}$ is good.*

See Notation 2.4.1 and Definition 4.2.20 for the definitions of the various functors.

Proof. Note that all the conditions that go into the definition of a good functor, assert that the functor commutes with certain limits and filtered colimits. Since $(-)^{\omega} : \mathcal{P}r^L_{\omega,\mathrm{k}} \to \mathrm{Cat}_{\infty}$ commutes with all limits and filtered colimits (Lemma 2.3.5), we see that (1) $\Rightarrow$ (2). Lemma 4.2.22 says that (2) $\Rightarrow$ (3). Finally, (3) $\Rightarrow$ (4) by virtue of Lemma 2.3.13, which says that the functor $(-)^{\simeq}$ commutes with limits and filtered colimits. Thus, to prove the proposition, it will suffice to sow that (1) holds.

Since $\mathcal{X}$ is dualizable, the functor $\mathcal{X} \otimes_{\mathrm{Mod}_{\mathrm{k}}} (-) : \mathcal{P}r^L_{\omega,\mathrm{k}} \to \mathcal{P}r^L_{\omega,\mathrm{k}}$ commutes with all small limits (Lemma 2.2.5). Consequently, if $\mathfrak{M}_{\mathbf{1}}$ carries a diagram $K^{\triangleleft} \to \mathrm{CAlg}_{\mathrm{k}}$

to a limit diagram in $\mathcal{P}r^L_{\omega,\mathrm{k}}$, then so does $\mathfrak{M}_{\mathcal{X}} := \mathcal{X} \otimes_{\mathrm{Mod}_{\mathrm{k}}} \mathfrak{M}_{\mathbf{1}}$. In view of Proposition 4.3.5, this implies that $\mathfrak{M}_{\mathcal{X}}$ is infinitesimally cohesive, nilcomplete and formally effective.

Proposition 3.4.1 says that $\mathcal{M}^{\flat}_{\mathcal{X}}$ is a sheaf for the flat topology. By virtue of the fact that $(-)^{\omega} : \mathcal{P}r^L_{\omega,\mathrm{k}} \to \mathrm{Cat}_{\infty}$ reflects limits (Lemma 2.3.5), and the definition of the notion of sheaf, it follows that $\mathfrak{M}_{\mathcal{X}}$ is a sheaf for the flat topology, and consequently a sheaf for the étale topology. We would like to pause to point out that Proposition 3.4.1 was proven by first showing that $\mathfrak{M}_{\mathbf{1}}$ is a sheaf for the flat topology, and then using the fact that $\mathcal{X} \otimes_{\mathrm{Mod}_{\mathrm{k}}} (-)$ commutes with limits to deduce that $\mathfrak{M}_{\mathcal{X}}$ is a sheaf for the flat topology.

The ω-accessibility of $\mathfrak{M}_{\mathcal{X}}$ follows from the ω-accessibility of $\mathfrak{M}_{\mathbf{1}}$, and the fact that $\mathcal{X} \otimes_{\mathrm{Mod}_{\mathrm{k}}} (-)$ distributes over colimits for any $\mathcal{X} \in \mathcal{P}r^L_{\omega,\mathrm{k}}$. To complete the proof of the proposition, it remains only to show that $\mathfrak{M}_{\mathbf{1}}$ is ω-accessible. The argument for this is identical to [TV07a, Lemma 2.10]. $\square$

Lemma 4.3.7. *Let $\mathcal{X}$ be an object of $\mathcal{P}r^L_{\mathrm{k}}$, and let $\mathcal{M}^n_{\mathcal{X}} : \mathrm{CAlg}_{\mathrm{k}} \to \mathcal{S}$ be as in Definition 4.2.20. The functor $\mathcal{M}^n_{\mathcal{X}}$ is $(n+1)$-truncated for every $n \in \mathbb{N}$.*

Proof. Let A be a discrete commutative ring, and let $X \in \mathcal{M}^n_{\mathcal{X}}(A)$. By definition of $\mathcal{M}^n_{\mathcal{X}}$, $\mathbb{M}\mathrm{or}_A(X,X)$ is an A-module of Tor amplitude $\leq n$. Since $\mathrm{Map}_A(X,X) \simeq \Omega^{\infty}\mathbb{M}\mathrm{or}_A(X,X)$, and A is discrete, this immediately implies that $\mathrm{Map}_A(X,X)$ is n-truncated. The proposition now follows from the observation that $\pi_{k+1}(\mathcal{M}^n_{\mathcal{X}}, X) \simeq \pi_k(\mathrm{Map}_A(X,X), \mathrm{id}_X)$ for $k > 0$, since $\mathcal{M}^n_{\mathcal{X}}(A)$ is the underlying ∞-groupoid of a full subcategory of $\mathrm{Mod}_A(\mathcal{X})^\omega$. □

Proposition 4.3.8. *Let $\mathcal{X}$ be a dualizable object of $\mathcal{P}r^L_{\omega,\mathrm{k}}$, and let $n \in \mathbb{N}$. The functor $\mathcal{M}^n_{\mathcal{X}} : \mathrm{CAlg}_{\mathrm{k}} \to \mathcal{S}$ (see Definition 4.2.20) is a derived algebraic $(n+1)$-stack, locally of finite presentation over* k.

Proof. By virtue of Proposition 4.3.6 and Lemma 4.3.7, the functor $\mathcal{M}^n_{\mathcal{X}}$ satisfies conditions (1)-(3) and (5)-(7) of the Artin-Lurie recognition principle (Theorem 4.1.4). Since $\mathcal{X}$ is dualizable, it is, in particular, proper. Consequently, it is locally compact. Thus, $\mathcal{X}$ satisfies all the hypotheses of Proposition 4.2.23, and therefore $\mathcal{M}^n_{\mathcal{X}}$ admits a cotangent complex, which is condition (4) of Theorem 4.1.4. Thus, $\mathcal{M}^n_{\mathcal{X}}$ satisfies all the hypothesis of the Artin-Lurie theorem. The proposition follows.

□

We can now prove the main theorem of this section:

Proof of Theorem 4.3.1. The functor $\mathcal{M}_{\mathcal{X}}$ is the filtered colimit of the functors $\mathcal{M}^n_{\mathcal{X}}$ (Lemma 4.2.21), and furthermore each morphism $\mathcal{M}^n_{\mathcal{X}} \to \mathcal{M}_{\mathcal{X}}$ is a monomorphism (Lemma 4.2.19). The stack $\mathcal{M}^n_{\mathcal{X}}$ is a derived algebraic $(n+1)$-stack, locally of finite presentation, by Proposition 4.3.8. □

4.4 Proper Perfection implies Geometricity

The purpose of this section is to prove the following theorem.

Theorem 4.4.1. *Let $\mathcal{X}^{\otimes}$ be a perfect* k*-linear symmetric monoidal ∞-category in the sense of Definition 2.2.18. Assume that the underlying category $\mathcal{X}$ is compactly generated, and locally compact. Then the functor $\mathcal{M}_{\mathcal{X}} : \mathrm{CAlg}_{\mathrm{k}} \to \mathcal{S}$ (see Notation 2.4.1), is a locally geometric derived ∞-stack, locally of finite presentation over* k.

Example 4.4.2. If X is a proper perfect stack over k in the sense of Definition 2.2.22, then $\mathcal{X} := \mathrm{QC}(X)$ satisfies the hypotheses of the theorem. Thus, this theorem generalizes the main result of [Lie06], which asserts the existence of an Artin 1-stack parametizing certain "sufficiently rigid" perfect complexes on a proper scheme.

Remark 4.4.3. The role of local compactness in the proof is twofold. Firstly, it is needed in our proof of the existence of the cotangent complex (Proposition 4.2.4). The other aspect, which will play an important role in this section, is the following. If $\mathcal{X}$ is compactly generated, we have a natural equivalence $\mathrm{Fun}_{\mathrm{k}}(\mathcal{X}^{\omega}, \mathrm{Mod}_A) \simeq \mathrm{Mod}_A(\mathcal{X})$ for any $A \in \mathrm{CAlg}_{\mathrm{k}}$. We have the functor $p_A : \mathrm{Mod}_A(\mathcal{X}) \to \prod_{x \in \mathcal{X}^{\omega}} \mathrm{Mod}_A$, where $p_A := (\mathrm{ev}_x)_{x \in \mathcal{X}^{\omega}}$. The functor p_A is manifestly conservative. Secondly, our local compactness assumption implies that p_A preserves compact objects. This follows immediately from [TV07a, Lemma 2.8(1)] and T.5.3.4.10. The next several lemmas will show how these two properties of the functor p_A will allow us to deduce that $\mathcal{M}_{\mathcal{X}}$ satisfies several of the Artin-Lurie conditions, from that fact that $\mathcal{M}_{\mathbf{1}}$ satisfies the corresponding conditions.

Lemma 4.4.4. *Assume that the following diagram is a right adjointable square in*

$\widehat{\mathrm{Cat}}_\infty$, *that the maps p, p' are conservative, and that the pair (f', g') is an adjoint equivalence.*

$$\begin{tikzcd}
\mathcal{X} \arrow[r, "f"] \arrow[d, "p"'] & \mathcal{X}' \arrow[l, bend right, dotted, "g"'] \arrow[d, "p'"] \\
\mathcal{Y} \arrow[r, "f'"'] & \mathcal{Y}' \arrow[l, bend right, dotted, "g'"']
\end{tikzcd}$$

Then the pair (f, g) is an adjoint equivalence.

Proof. Let (η, ϵ) (resp. (η', ϵ')) be the pair (unit, counit) for the adjuntion (f, g) (resp. (f', g')). Let $\xi : p \circ g \to g' \circ p'$ be the natural morphism, which is an equivalence by virtue of the right adjointability of the diagram.

Assume that (f', g') defines an adjoint equivalence, i.e., that η' and ϵ' are equivalences. We will prove that the η and ϵ are equivalences. We have the following homotopy commutative diagram in $\mathrm{Fun}(\mathcal{X}, \mathcal{Y})$:

$$\begin{tikzcd}
p \arrow[r, "p\eta"] \arrow[d, "\eta' p"', "\wr"] & pgf \arrow[d, "\xi", "\wr"'] \\
g'f'p & g'p'f \arrow[l, "\sim"']
\end{tikzcd}$$

The bottom horizontal map is the equivalence that exists by the homotopy commutativity of the right adjointable square. Since the left and right vertical maps are equivalences by our hypotheses, it follows that $p \circ \eta$ is an equivalence. Since p is conservative, this implies that η is an equivalence. The proof that ϵ is an equivalence is essentially identical. $\square$

The following is a slightly refined version of the previous lemma, that will be useful. The proof ofis essentially the same - in fact, the previous lemma is a special case of what follows. We have decided to present them separately, so as not to obscure the simplicity of the proof.

Lemma 4.4.5. *Let the notation be as in Lemma 4.4.4, but do not assume that (f', g') is an adjoint equivalence. Suppose that there exist subcategories $\mathcal{X}_\sharp \subseteq \mathcal{X}$, $\mathcal{X}'_\sharp \subseteq \mathcal{X}'$, $\mathcal{Y}_\sharp \subseteq \mathcal{Y}$, $\mathcal{Y}'_\sharp \subseteq \mathcal{Y}'$, such that the following conditions are satisfied:*

(1) The functor p maps $\mathcal{X}_\sharp$ into $\mathcal{Y}_\sharp$, and the functor p' carries $\mathcal{X}'_\sharp$ into $\mathcal{Y}'_\sharp$.

(2) The adjoint pair (f', g') restricts to an adjoint equivalence between $\mathcal{Y}_\sharp$ and $\mathcal{Y}'_\sharp$.

Then for every X in $\mathcal{X}_\sharp$ the unit η_X is an equivalence, and for every X' in $\mathcal{X}'_\sharp$, the counit $\epsilon_{X'}$ is an equivalence.

Proof. The proof is identical to that of Lemma 4.4.4, except that one has to keep track of whether various objects live in the appropriate subcategories. We leave the details to the reader. $\square$

The next lemma explains the relevance of the previous two to the problem at hand.

Lemma 4.4.6. *Let $A_\bullet : K^{\triangleleft} \to \mathrm{CAlg}_{\mathrm{k}}$ be a limit diagram. Let $A := A_\infty$. We have a right adjointable square*

$$\begin{tikzcd}
\mathcal{M}^{\dagger}_{\mathcal{X}}(A) \arrow[r, "\phi_{\mathcal{X}}"'] \arrow[d, "p_A"'] & \lim_{x\in K}\mathcal{M}^{\dagger}_{\mathcal{X}}(A_x) \arrow[l, dotted, bend right, "\psi_{\mathcal{X}}"'] \arrow[d, "\lim p_{A_x}"] \\
\mathcal{M}^{\dagger}_{\mathbf{1}}(A) \arrow[r, "\phi_{\mathbf{1}}"'] & \lim_{x\in K}\mathcal{M}^{\dagger}_{\mathbf{1}}(A_x) \arrow[l, dotted, bend right, "\psi_{\mathbf{1}}"']
\end{tikzcd}$$

where p_A is the morphism defined in Remark 4.4.3.

Proof. The existence and commutativity of the diagram of left adjoints is clear. The essential point is to check the right adjointability of the diagram. The existence of the right adjoints is a formal consequence of the fact that the limits are computed in $\mathcal{P}r^L_{\mathrm{k}}$ (recall from 2.4.1 that $\mathcal{M}^{\dagger}_{\mathcal{X}}$ takes values in $\mathcal{P}r^L_{\mathrm{k}}$), since every morphism in $\mathcal{P}r^L_{\mathrm{k}}$ has a right adjoint (which is only a morphism in $\widehat{\mathrm{Cat}}_\infty$, a priori), by the adjoint functor theorem.

To check the adjointability of the diagram, we make use of the antiequivalence $\mathcal{P}r^L \simeq (\mathcal{P}r^R)^{\mathrm{op}}$, which allows us to identify limits in $\mathcal{P}r^L$ with colimits in $\mathcal{P}r^R$. Since limits in $\mathcal{P}r^L_{\mathrm{k}}$ can be computed in $\mathcal{P}r^L$, we have a natural equivalence

$$\lim^{\mathcal{P}r^L}_{x\in K}\mathcal{M}^{\dagger}_{\mathcal{X}}(A_x) \simeq \mathrm{colim}^{\mathcal{P}r^R}_{x\in K}\mathcal{M}^{\dagger}_{\mathcal{X}}(A_x)$$

Let $\alpha^x_{\mathcal{X}}$ denote the natural morphism $\mathcal{M}^{\dagger}_{\mathcal{X}}(A_x) \to \mathrm{colim}^{\mathcal{P}r^R}_{x\in K}\mathcal{M}^{\dagger}_{\mathcal{X}}(A_x)$, and let $\psi^x_{\mathcal{X}} := \psi_{\mathcal{X}} \circ \alpha^x_{\mathcal{X}}$. By the universal property of the colimit, it will suffice, in order to check the adjointability of the diagram, to verify the homotopy commutativity of the following diagram for every $x \in K$

$$\begin{tikzcd} \mathcal{M}^{\dagger}_{\mathcal{X}}(A) \arrow[d, "p_A"'] & \mathcal{M}^{\dagger}_{\mathcal{X}}(A_x) \arrow[l, "\alpha^x_{\mathcal{X}}"'] \arrow[d, "p_{A_x}"] \\ \mathcal{M}^{\dagger}_{\mathbf{1}}(A) & \mathcal{M}^{\dagger}_{\mathbf{1}}(A_x) \arrow[l, "\alpha^x_{\mathbf{1}}"'] \end{tikzcd}$$

By our construction of $\alpha^x_{\mathcal{X}}$, it is clear that $\alpha^x_{\mathcal{X}}$ is just the forgetful functor $\mathrm{Mod}_{A_x} \to \mathrm{Mod}_A$, left adjoint to the base change morphism $\mathrm{Mod}_A \to \mathrm{Mod}_{A_x}$ induced by the map $A = A_\infty \to A_x$. From this description of $\alpha^x_{\mathcal{X}}$, the commutativity of the above diagram is immediate. □

We will be interested in applying the previous lemma in the situation when the diagram $A_\bullet$ represents one of the following

1. The diagram $\nu : \mathrm{N}(\mathbb{Z}_{\leq 0})^{\triangleleft} \to \mathrm{CAlg}_{\mathrm{k}}$ witnessing A as the inverse limit of $A/\mathfrak{m}^N$, where$A$ is a discrete, complete, Noetherian local ring, with maximal ideal $\mathfrak{m}$.

2. Any diagram $\nu : K^{\triangleleft} \to \mathrm{CAlg}_{\mathrm{k}}$ realizing a small extension in $\mathrm{CAlg}_{\mathrm{k}}$.

3. Any Postnikov tower $\nu : \mathrm{N}(\mathbb{Z}_{\leq 0})^{\triangleleft} \to \mathrm{CAlg}_{\mathrm{k}}$.

Unfortunately, it is rarely, if ever, true that the bottom row of the equivalence in the previous lemma is an adjoint equivalence (even if we restrict ourselves to diagrams of the special types listed above). Therefore, we cannot apply Lemma

4.4.4 directly in order to deduce the good properties of $\mathcal{M}_{\mathcal{X}}$ from those of $\mathcal{M}_{\mathbf{1}}$. However, with some care, we will be able to apply Lemma 4.4.5. First, we need some preparatory lemmas.

Lemma 4.4.7. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism in $\mathcal{P}r^L_\omega$, let $g : \mathcal{Y} \to \mathcal{X}$ be a right adjoint of the functor f, and let $\mathcal{Y}^\sharp \subseteq \mathcal{Y}$ be a subcategory. Assume that f restricts to an equivalence $f' : \mathcal{X}^\omega \to \mathcal{Y}^\sharp$. Then the adjoint pair (f, g) restricts to an adjoint equivalence*

$$\begin{tikzcd} \mathcal{X}^\omega \arrow[r, "f'"'] & \mathcal{Y}^\sharp \arrow[l, bend right, "g'"'] \end{tikzcd}$$

In particular, g carries $\mathcal{Y}^\sharp$ into $\mathcal{X}^\omega$.

Proof. Let η (resp. ϵ) denote to unit (resp. counit) of the adjunction (f, g). Assume that $f' : \mathcal{X}^\omega \to \mathcal{Y}^\sharp$ is an equivalence. Then f' is fully faithful. For X, $X' \in \mathcal{X}^\omega$, consider the diagram

$$\begin{tikzcd} \mathcal{X}(X, X') \arrow[r, "\eta_{X'} \circ (-)"] \arrow[dr, "f"'] & \mathcal{X}(X, gfX') \arrow[d, "\wr"] \\ & \mathcal{Y}(fX, fX') \end{tikzcd}$$

Since f' is fully faithful, the diagonal map is an equivalence, and consequently, so is the upper horizontal map. Thus, $\mathcal{X}(X, \eta_{X'})$ is an equivalence for every ω-compact object X. Since $\mathcal{X}$ is compactly generated, this implies that $\eta_{X'}$ is an equivalence. This true for an arbitrary $X' \in \mathcal{X}^w$.

Let $Y \in \mathcal{Y}^\sharp$. Since f' is an equivalence, it is essentially surjective. Consequently, there exists $X \in \mathcal{X}^\omega$ and an equivalence $\alpha : fX \to Y$. By the previous paragraph, the unit $\eta_X : X \to gfX$ is an equivalence. Therefore, the composite $g\alpha \circ \eta_X : X \to gY$ is an equivalence. Since $X \in \mathcal{X}^\omega$ and $\mathcal{X}^\omega$ is replete, it follows that $gY \in \mathcal{X}^\omega$. This proves that the restriction of g to $\mathcal{Y}^\sharp$ factors through $\mathcal{X}^\omega$.

We have a commutative diagram

$$\begin{tikzcd} fgfX \arrow[r, "fg\alpha"] \arrow[d, "\epsilon_{fX}"'] & fgY \arrow[d, "\epsilon_Y"] \\ fX \arrow[r, "\alpha"'] & X \end{tikzcd}$$

The horizontal morphisms are equivalences, since α is an equivalence. Therefore, in order to show that ϵ_Y is an equivalence, it will suffice to show that ϵ_{fX} is an equivalence. We have the triangular identity

$$\begin{tikzcd} fX \arrow[r, "f\eta_X"] \arrow[dr, "\mathrm{id}_{fX}"'] & fgfX \arrow[d, "\epsilon_{fX}"] \\ & fX \end{tikzcd}$$

We have already shown that η_X is an equivalence. It follows by the the "two-out-of-three" property of equivalences that ϵ_{fX} is an equivalence. By our remark above, ϵ_Y is an equivalence. Since Y was arbitrary, this completes the proof of the fact that the counit ϵ is an equivalence. Since the unit and counit of the adjoint pair (f', g') are equivalences, it follows that (f', g') is an adjoint equivalence. $\square$

Lemma 4.4.8. *Let* $A_\bullet : K^\triangleleft \to \mathrm{CAlg}_\mathrm{k}$ *be a limit diagram, and suppose that the induced diagram* $\mathfrak{M}_\mathbf{1}(A_\bullet) : K^\triangleleft \to \mathcal{P}r^L_{\omega,\mathrm{k}}$ *is a limit diagram. Put* $A := A_\infty$*. Then the adjoint pair*

$$\begin{tikzcd} \mathcal{M}^\dagger_\mathbf{1}(A) \arrow[r] & \lim \mathcal{M}^\dagger_\mathbf{1}(A_x) \arrow[l, bend right] \end{tikzcd}$$

restricts to an adjoint equivalence

$$\begin{tikzcd} \mathcal{M}^\dagger_\mathbf{1}(A)^{\mathrm{fd}} \arrow[r] & (\lim \mathcal{M}^\dagger_\mathbf{1}(A_x))^{\mathrm{fd}} \arrow[l, bend right] \end{tikzcd}$$

Proof. By virtue of Lemma 2.3.5 and our hypotheses, the natural morphism in Cat_∞, $\mathcal{M}^\flat_\mathcal{X}(A) \to \lim_{x\in K} \mathcal{M}^\flat_\mathcal{X}(A_x)$ is an equivalence. Since $\mathcal{X}$ is perfect, so is each of the categories $\mathrm{Mod}_A(\mathcal{X}) = \mathcal{X} \otimes \mathrm{Mod}_A$ (Proposition 2.2.21). Consequently, the previous statement is equivalent to the assertion that the natural morphism $\mathcal{M}^\dagger_\mathcal{X}(A)^{\mathrm{fd}} \to \lim_{x\in K} \mathcal{M}^\dagger_\mathcal{X}(A_x)^{\mathrm{fd}}$ is an equivalence. Proposition 2.3.8 tells us that we the map $(\lim_{x\in K} \mathcal{M}^\dagger_\mathcal{X}(A_x))^{\mathrm{fd}} \to \lim_{x\in K} \mathcal{M}^\dagger_\mathcal{X}(A_x)^{\mathrm{fd}}$ is an equivalence. In summary, we have shown that the natural map $\mathcal{M}^\dagger_\mathcal{X}(A)^{\mathrm{fd}} \to (\lim \mathcal{M}^\dagger_\mathcal{X}(A_x))^{\mathrm{fd}}$ is an equivalence. To complete the proof, we apply Lemma 4.4.7. $\square$

Lemma 4.4.9. *Let* $\mathcal{X}$ *be a locally compact, compactly generated, perfect symmetric monoidal* ∞*-category, and let* $A_\bullet : K^\triangleleft \to \mathrm{CAlg}_\mathrm{k}$ *be a diagram. Then the vertical arrows in the right adjointable square of Lemma 4.4.6 are conservative and preserve dualizable objects.*

Proof. The conservatism of p_A and p_{A_x} is Remark 4.4.3. The conservatism of $\lim p_{A_x}$ follows easily. To see this, note that it will suffice to show that this functor reflects zero objects, since the categories involved are stable. Let $\mathcal{V} \to K$ (resp. $\mathcal{V}' \to K$) be coCartesian fibrations representing $\lim \mathcal{M}^{\dagger}_{\mathcal{X}}(A_x)$ (resp. $\lim \mathcal{M}^{\dagger}_{\mathbf{1}}(A_x)$). Let $X \in \lim \mathcal{M}^{\dagger}_{\mathcal{X}}(A_x)$ be an element represented by a coCartesian section $\chi \in \mathrm{Fun}_K(K, \mathcal{V})$, and suppose that its image $\chi' \in \mathrm{Fun}_K(K, \mathcal{V}')$ is a zero object. Then $\chi'_x \simeq 0$ for all $x \in K$, whence, by the conservatism of p_{A_x}, we have that $\chi_x \simeq 0$ for all $x \in K$. It follows that $\chi \simeq 0$.

Since $\mathcal{X}$ is locally compact, Remark 4.4.3 says that p_A preserves compact objects. Since $\mathcal{M}^{\dagger}_{\mathcal{X}}(A)$ is perfect by our hypotheses and Proposition 2.2.21, this is equivalent to asserting that p_A preserves dualizable objects. The same proof shows that each functor p_{A_x} preserves dualizable objects. Since $(\lim \mathcal{M}^{\dagger}_{\mathcal{X}}(A_x))^{\mathrm{fd}} \simeq \lim \mathcal{M}^{\dagger}_{\mathcal{X}}(A_x)^{\mathrm{fd}}$ by Proposition 2.3.8, this show that $\lim p_{A_x}$ preserves dualizable objects, and completes the proof. $\square$

Lemma 4.4.10. *Let $\mathcal{X}$ be a compactly generated, locally compact, perfect symmetric monoidal ∞-category. Let $A_\bullet : K^{\triangleleft} \to \mathrm{CAlg}_{\mathrm{k}}$ be a diagram. Put $A := A_\infty$. Suppose that the induced diagram $\mathfrak{M}_{\mathbf{1}}(A_\bullet) : K^{\triangleleft} \to \mathcal{P}r^L_{\omega,\mathrm{k}}$ is a limit diagram. Let η (resp. ϵ) denote the unit (resp. counit) of the canonical adjunction*

$$\begin{tikzcd} \mathcal{M}^{\dagger}_{\mathcal{X}}(A) \arrow[r] & \lim \mathcal{M}^{\dagger}_{\mathcal{X}}(A_x) \arrow[l, bend right] \end{tikzcd}$$

Then for any $X \in \mathcal{M}^{\dagger}_{\mathcal{X}}(A)^{\mathrm{fd}}$, the unit η_X is an equivalence, and for any $X' \in (\lim \mathcal{M}^{\dagger}_{\mathcal{X}}(A_x))^{\mathrm{fd}}$, the counit $\epsilon_{X'}$ is an equivalence.

Proof. In view of Lemma 4.4.9 and Lemma 4.4.8, the right adjointable square of

Lemma 4.4.6 satisfies all the hypotheses of Lemma 4.4.5. The subcategories $\mathcal{X}_\sharp$, etc are the obvious ones - we leave the details to the reader. □

Lemma 4.4.11. *Let $\nu : K^{\triangleleft} \to \mathrm{CAlg}_{\mathrm{k}}$ be a limit diagram, and suppose that the induced diagram $\mathfrak{M}_{\mathbf{1}} \circ \nu : K^{\triangleleft} \to \mathcal{P}r^L_{\omega,\mathrm{k}}$ is a limit diagram. Let $\mathcal{X}$ be a perfect compactly generated symmetric monoidal ∞-category. Then the natural functor*

$$\phi^{\nu}_{\mathcal{X}} : \mathcal{M}^{\dagger}_{\mathcal{X}}(\nu_\infty) \to \lim\ \mathcal{M}^{\dagger}_{\mathcal{X}} \circ \nu_{|K}$$

admits a right adjoint, and any right adjoint preserves dualizable objects.

Proof. The existence of the right adjoint is immediate. Indeed, the limit is computed in $\mathcal{P}r^L_{\mathrm{k}}$, and therefore $\phi^{\nu}_{\mathcal{X}}$ is a morphism in $\mathcal{P}r^L_{\mathrm{k}}$. By the adjoint functor theorem every morphism in $\mathcal{P}r^L_{\mathrm{k}}$ has a right adjoint (the adjoint itself is only a morphism in $\widehat{\mathrm{Cat}}_\infty$). Let us denote the right adjoint by $\psi^{\nu}_{\mathcal{X}}$.

To begin, let us note that the following conditions are equivalent:

(1) $\mathfrak{M}_{\mathbf{1}} \circ \nu : K^{\triangleleft} \to \mathcal{P}r^L_{\omega,\mathrm{k}}$ is a limit diagram.

(2) $\mathcal{M}^{\flat}_{\mathbf{1}} \circ \nu : K^{\triangleleft} \to \mathrm{Cat}_\infty$ is a limit diagram.

(3) $(-)^{\mathrm{fd}} \circ \mathcal{M}^{\dagger}_{\mathbf{1}} \circ \nu : K^{\triangleleft} \to \mathrm{Cat}_\infty$ is a limit diagram.

The first two are equivalent by virtue of Lemma 2.3.5. The second and third are equivalent because the two functors are in fact equivalent, in view of Proposition 2.2.20, which asserts that $\mathcal{M}^{\dagger}_{\mathbf{1}}$ is a perfect category. Since (1) is true by our hypothesis, we conclude that the natural morphism $\mathcal{M}^{\dagger}_{\mathbf{1}}(\nu_\infty)^{\mathrm{fd}} \to \lim \mathcal{M}^{\dagger}_{\mathbf{1}}(\nu_{|K})^{\mathrm{fd}}$ is an equivalence. Proposition 2.3.8 says that natural morphism $(\lim \mathcal{M}_{\mathbf{1}}^{\dagger}(\nu_{|K}))^{\mathrm{fd}} \to \lim \mathcal{M}_{\mathbf{1}}^{\dagger}(\nu_{|K})^{\mathrm{fd}}$ is an equivalence. Furhtermore, since $\mathcal{M}_{\mathbf{1}}^{\dagger}(\nu_{\infty})$ is perfect, we have $\mathcal{M}_{\mathbf{1}}^{\dagger}(\nu_{\infty})^{\mathrm{fd}} \simeq \mathcal{M}_{\mathbf{1}}^{\dagger}(\nu_{\infty})^{\omega}$. Putting all this together, we see that $\phi_{\mathbf{1}}^{\nu}$ restricts to an equivalence

$$\phi_{\mathbf{1}}^{\nu,\flat} : \mathcal{M}_{\mathbf{1}}^{\dagger}(\nu_{\infty})^{\omega} \to (\lim \mathcal{M}_{\mathbf{1}}^{\dagger}(\nu_{|K}))^{\mathrm{fd}}$$

Applying Lemma 4.4.7 to the adjoint pair $(\phi_{\mathbf{1}}^{\nu}, \psi_{\mathbf{1}}^{\nu})$, we conclude that the restriction of $\psi_{\mathbf{1}}^{\nu}$ to $(\lim \mathcal{M}_{\mathbf{1}}^{\dagger}(\nu_{|K}))^{\mathrm{fd}}$ factors through $\mathcal{M}_{\mathbf{1}}^{\dagger}(\nu_{\infty})^{\omega}$, i.e., $\psi_{\mathbf{1}}^{\nu}$ carries dualizable objects to compact ones. Or, equivalently (since $\mathcal{M}_{\mathbf{1}}^{\dagger}$ is perfect), $\psi_{\mathbf{1}}^{\nu}$ preserves dualizable objects. Since $\mathcal{X}$ is compactly generated, it is dualizable as an object of $\mathcal{P}r_{\mathrm{k}}^{L}$, and so $\mathcal{X} \otimes (-) : \mathcal{P}r_{\mathrm{k}}^{L} \to \mathcal{P}r_{\mathrm{k}}^{L}$ commutes with limits. In particular , $\lim \mathcal{M}_{\mathcal{X}}^{\dagger}(\nu_{|K}) \simeq \mathcal{X} \otimes \lim \mathcal{M}_{\mathbf{1}}^{\dagger}(\nu_{|K})$, and hence $\psi_{\mathcal{X}}^{\nu} \simeq \mathcal{X} \otimes \psi_{\mathbf{1}}^{\nu}$. Since $\psi_{\mathbf{1}}^{nu}$ preserves dualizable objects, so does $\psi_{\mathcal{X}}^{\nu}$. $\square$

Lemma 4.4.12. *Let $\mathcal{X}$ be a compactly generated, locally compact, perfect symmetric monoidal ∞-category. Let $A_{\bullet} : K^{\triangleleft} \to \mathrm{CAlg}_{\mathrm{k}}$ be a diagram. Put $A := A_{\infty}$. Suppose that the induced diagram $\mathfrak{M}_{\mathbf{1}}(A_{\bullet}) : K^{\triangleleft} \to \mathcal{P}r_{\omega,\mathrm{k}}^{L}$ is a limit diagram. Then the cannonical adjunction*

$$\begin{tikzcd} \mathcal{M}_{\mathcal{X}}^{\dagger}(A) \arrow[r] & \lim \mathcal{M}_{\mathcal{X}}^{\dagger}(A_x) \arrow[l, bend right] \end{tikzcd}$$

restricts to an adjoint equivalence

$$\begin{tikzcd} (\mathcal{M}_{\mathcal{X}}^{\dagger}(A))^{\mathrm{fd}} \arrow[r] & (\lim \mathcal{M}_{\mathcal{X}}^{\dagger}(A_x))^{\mathrm{fd}} \arrow[l, bend right] \end{tikzcd}$$

Proof. Combine Lemmas 4.4.10 and 4.4.11. $\square$

Lemma 4.4.13. *Let the notation and hypotheses be exactly as in the previous lemma. Then the canonical adjunction*

$$\begin{tikzcd} \mathcal{M}^{\dagger}_{\mathcal{X}}(A) \arrow[r] & \lim \mathcal{M}^{\dagger}_{\mathcal{X}}(A_x) \arrow[l, bend right] \end{tikzcd}$$

induces an equivalence

$$(\mathcal{M}^{\dagger}_{\mathcal{X}}(A))^{\omega} \longrightarrow \lim \mathcal{M}^{\dagger}_{\mathcal{X}}(A_x)^{\omega}$$

Proof. According to 2.3.8 we have a natural equivalence

$$(\lim \mathcal{M}^{\dagger}_{\mathcal{X}}(A_x))^{\mathrm{fd}} \simeq \lim \mathcal{M}^{\dagger}_{\mathcal{X}}(A_x)^{\mathrm{fd}}$$

Since $\mathcal{X}$ is perfect, using Lemma 2.2.21 we have that

$$\mathcal{M}^{\dagger}_{\mathcal{X}}(A_x)^{\mathrm{fd}} \simeq \mathcal{M}^{\dagger}_{\mathcal{X}}(A_x)^{\omega}$$

for all $x \in K^{\triangleleft}$. In view of these facts, the lemma is an immediate consequence of Lemma 4.4.12. □

Proposition 4.4.14. *Let $\mathcal{X}$ be (the underlying category of) compactly generated, locally compact, perfect symmetric monoidal ∞-category. Then the functor $\mathcal{M}^{\flat}_{\mathcal{X}}$: $\mathrm{CAlg}_{\mathrm{k}} \to \mathrm{Cat}_{\infty}$ is infinitesimally cohesive, nilcomplete and formally effective.*

Proof. Since $\mathbf{1}$ is smooth and proper, it satisfies the hypotheses of Proposition 4.3.6. Combining this with Lemma 4.4.13, the result follows. □

We can now prove the main theorem of this section.

Proof of Proposition 4.4.1. Combining Proposition 4.4.14, Lemma 4.2.22 and Lemma 2.3.13, we see that $\mathcal{M}^n_{\mathcal{X}}$ is infinitesimally cohesive, nilcomplete and formally effective.

The proof of ω-accessibility in Lemma 4.3.6 only used the fact that $\mathcal{X} \otimes (-)$ preserves filtered colimits, which holds for an arbitrary $\mathcal{X} \in \mathcal{P}r^L_{\omega,\mathrm{k}}$. Thus $\mathcal{M}^n_{\mathcal{X}}$ is ω-accessible.

Theorem 3.3.1 implies that $\mathcal{M}^n_{\mathcal{X}}$ is a sheaf for the étale topology. Indeed, since $\mathcal{X}$ is perfect, we have natural equivalences $\mathcal{M}^\flat_{\mathcal{X}} \simeq \mathcal{M}^\vee_{\mathcal{X}}$. It follows that $\mathcal{M}^\flat_{\mathcal{X}}$ is a sheaf for the flat topology, and hence by Lemma 4.2.22 that $\mathcal{M}^{\flat,n}_{\mathcal{X}}$ is a sheaf for the flat topology. By Lemma 2.3.13, $\mathcal{M}^n_{\mathcal{X}}$ is a sheaf for the flat topology, and hence also for the étale topology.

In view of the previous two paragraphs, and our hypothesis that $\mathcal{X}$ is locally compact, $\mathcal{X}$ satisfies all the hypotheses of Proposition 4.2.23. Consequently,$\mathcal{M}^n_{\mathcal{X}}$ admits a cotangent complex.

Finally, Lemma 4.3.7 tells us that $\mathcal{M}^n_{\mathcal{X}}$ is $(n+1)$-truncated. Thus, $\mathcal{M}^n_{\mathcal{X}}$ satisfies all the hypotheses of Theorem 4.1.4, and therefore $\mathcal{M}^n_{\mathcal{X}}$ is a derived algebraic $(n+1)$-stack, locally of finite presentation over k. Since $\mathcal{M}_{\mathcal{X}}$ is the filtered colimit of the functor $\mathcal{M}^n_{\mathcal{X}}$, this $\mathcal{M}_{\mathcal{X}}$ is a locally geometric ∞-stack locally of finite presentation. $\square$

4.5 A Proper Counterexample

The result of the previous section might naturally lead one to wonder whether the functor $\mathcal{M}_{\mathcal{X}}$ is geometric for an arbitrary proper or locally compact category $\mathcal{X}$. This turns out not to be the case. In fact, there is a whole family of counterexamples, as the next proposition shows.

Notation 4.5.1. Let k be a field, and let X be a smooth and proper (ordinary, discrete) quasi-separated quasi-compact Noetherian scheme over k. Let $j : Z \hookrightarrow X$ be a closed immersion defined by an ideal sheaf $\mathcal{J} \subseteq \mathcal{O}_X$, and assume $Z \neq X$. Let $\mathrm{QC}_Z(X)$ be the full subcategory of quasi-coherent sheaves on X that are set-theoretically supported along Z.

Proposition 4.5.2. *Let $\mathcal{X} := \mathrm{QC}_Z(X)$ be as in Notation 4.5.1. Then $\mathcal{X}$ is a proper category (it is locally compact and admits a compact generator). However, the functors $\mathcal{M}^{\flat}_{\mathcal{X}}$ and $\mathcal{M}_{\mathcal{X}}$ are not formally effective. In particular, $\mathcal{M}_{\mathcal{X}}$ is not representable by a locally geoemtric ∞-stack.*

Proof. Since $\mathrm{QC}(X)$ is proper X is proper. In particular it is locally compact. It follows that the full subcategory $\mathcal{X} \subseteq \mathrm{QC}(X)$ is locally compact. By the main results of [BvdB03], Z admits a compact generator $\mathcal{E}$. By the standard devissage (see, for example, [Rou08]), every perfect complex with support along Z is generated under finite colimits by $j_*\mathcal{E}$. Thus, $j_*\mathcal{E}$ is a compact generator for $\mathcal{X} = \mathrm{QC}_Z(X)$. This proves that $\mathcal{X}$ is proper.

For a scheme Y and $A \in \mathrm{CAlg}_{\mathrm{k}}$, let Y_A denote the derived scheme $Y \times \mathrm{Spec}(A)$. By the results of [Pre11], for any other quasi-compact quasi-separated scheme X', and subscheme $Z' \hookrightarrow X'$, we have $\mathrm{Perf}_{Z \times Z'}(X \times X') \simeq \mathrm{Perf}_Z(X) \otimes_{\mathrm{k}} \mathrm{Perf}_{Z'}(X')$. This implies the following explicit description of the functor $\mathcal{M}^{\flat}_{\mathcal{X}}$: for any $A \in \mathrm{CAlg}_{\mathrm{k}}$, we have $\mathcal{M}^{\flat}_{\mathcal{X}}(A) = \mathrm{Perf}_{Z_A}(X_A)$.

Let $z \in Z$, and let $A := \widehat{\mathcal{O}_{X,z}}$ be the formal completetion of the structure sheaf at z. Let $\mathfrak{m}$ be the maximal ideal of A, and let $A_n := A/\mathfrak{m}^n$. Let $i_n : X \times \mathrm{Spec}(A_n) \to X \times X$ denote the natural inclusion, and let $\Delta : X \to X \times X$ be the diagonal. Since X is smooth, $\Delta_* \mathcal{O}_{\mathcal{X}}$ is a perfect complex. The family $\{i_n^* \Delta_* \mathcal{O}_X\}_{n \in \mathbb{N}}$ defines an object $\mathcal{F}$ in $\lim_n \mathrm{Perf}_{Z_{A_n}}(X_{A_n})$, which is clearly not $\mathcal{J}$-torsion. On the other hand, every sheaf in $\mathrm{Perf}_{Z_A}(X_A)$ is $\mathcal{J}$-torsion. Thus, $\mathcal{F}$ is not in the essential image of the map

$$\mathrm{Perf}_{Z_A}(X_A) \to \lim_n \mathrm{Perf}_{Z_{A_n}}(X_{A_n})$$

This proves that $\mathcal{M}^{\flat}_{\mathcal{X}}$ is not formally effective. Passing to the underlying groupoids, we see that the natural maps $\mathcal{M}_{\mathcal{X}}(A) \to \lim \mathcal{M}_{\mathcal{X}}(A_n)$ is not essentially surjective as well, showing that $\mathcal{M}_{\mathcal{X}}$ is not formally effective. By the Artin-Lurie critertion, this implies that $\mathcal{M}_{\mathcal{X}}$ is not representable by a geometric stack. $\square$

Chapter 5

Moduli of Noncommutative Spaces

The previous two chapters were devoted to the study of moduli of objects in k-linear ∞-categories. In view of the Cobordism Hypothesis, this can also be interpreted as the study of moduli spaces of boundary conditions in certain two dimensional topological field theories (2D-TFTs). In this chapter, we turn our attention to the moduli space of 2D-TFTs themselves. Once again, in light of the Cobordism Hypothesis, we may interpret this as a moduli space of certain k-linear ∞-categories.

We will provide almost no proofs in this chapter. The material discussed here is work in progress - detailed proofs of some of the results will appear in a forthcoming paper [Pan11]. Some other statements made here are of a speculative nature.

We assume k is a field of characteristic zero in this chapter. Some of the results hold without this assumption.

5.1 Geometricity

Compactly generated k-linear ∞-categories are themselves objects in a certain ∞-category: namely, the ∞-category $\mathcal{P}r^L_{\omega,\mathrm{k}}$. Thus, by analogy with the moduli functor $\mathcal{M}^\natural_{\mathcal{X}}$ which parametrizes objects in $\mathcal{X}$, we define moduli functors $\mathcal{M}^\natural_{\mathcal{P}r^L_{\omega,\mathrm{k}}} : \mathrm{CAlg}_\mathrm{k} \to \widehat{\mathrm{Cat}}_\infty$ and $\mathcal{M}^\natural_{\mathcal{P}r^L_\mathrm{k}} : \mathrm{CAlg}_\mathrm{k} \to \widehat{\mathrm{Cat}}_\infty$

$$\mathcal{M}^\natural_{\mathcal{P}r^L_{\omega,\mathrm{k}}}(A) := \mathrm{Mod}_{\mathrm{Mod}_A}(\mathcal{P}r^L_{\omega,\mathrm{k}})$$

$$\mathcal{M}^\natural_{\mathcal{P}r^L_\mathrm{k}}(A) := \mathrm{Mod}_{\mathrm{Mod}_A}(\mathcal{P}r^L_\mathrm{k})$$

Note that we are not using the notation from Notation 2.4.1 - the similarity is only suggestive. Since the categories Mod_A and $\mathcal{P}r^L_{\omega,\mathrm{k}}$ are symmetric monoidal, so is $\mathcal{M}^\natural_{\mathcal{P}r^L_{\omega,\mathrm{k}}}(A)$. We have two induced functors $\mathcal{M}^{\mathrm{fd}}_{\mathcal{P}r^L_{\omega,\mathrm{k}}} : \mathrm{CAlg}_\mathrm{k} \to \widehat{\mathrm{Cat}}_\infty$ and $\mathcal{M}_{\mathcal{P}r^L_{\omega,\mathrm{k}}} : \mathrm{CAlg}_\mathrm{k} \to \mathcal{S}$ given by

$$\mathcal{M}^{\mathrm{fd}}_{\mathcal{P}r^L_{\omega,\mathrm{k}}}(A) := \mathcal{M}^\natural_{\mathcal{P}r^L_{\omega,\mathrm{k}}}(A)^{\mathrm{fd}}$$

$$\mathcal{M}_{\mathcal{P}r^L_{\omega,\mathrm{k}}}(A) := \mathcal{M}_{\mathcal{P}r^L_{\omega,\mathrm{k}}}(A)^{\simeq}$$

The dualizable objects in the $(\infty,1)$-category $\mathrm{Mod}_{\mathrm{Mod}_A}(\mathcal{P}r^L_{\omega,\mathrm{k}})$ are precisely the categories that are smooth and proper over A, and compactly generated. This justifies the following definition

Definition 5.1.1. The functor $\mathcal{M}_{\mathcal{P}r^L_{\omega,\mathrm{k}}}$ is the moduli of smooth and proper (a.k.a.

saturated) noncommutative spaces. For the rest of this section, we will denote it by $\mathcal{M}$.

Conjecture 5.1.2. *The moduli of smooth and proper noncommutative spaces is a locally geometric ∞-stack, locally of finite presentation over* k*. At any* k*-valued point corresponding to a category $\mathcal{X}$, the tangent complex $\mathbb{T}_{\mathcal{M},\mathcal{X}}$ is computed by the shifted Hochschild cohomology spectrum:* $\mathrm{HH}^{\bullet}(\mathcal{X})[2]$.

This conjecture is by no means new. No claim of originality is being made. In fact, for some time it seems that this conjecture was commonly accepted as a theorem in many circles. Closer examination reveals that even the formal deformation theory is not completely understood. We refer the reader to [Toe10] for more details.

Here is a possible strategy for the proof of the conjecture, which is based on the techniques developed in the previous chapter. Observe that we have a natural consertive functor $\mathcal{M}^{\natural}_{\mathcal{P}r^L_{\omega,\mathrm{k}}}(A) = \mathcal{P}r^L_{\omega,A} \to \mathrm{Mod}_A$ given by

$$\mathcal{X} \mapsto \oplus_{x,y\in\mathcal{X}^{\omega}}\mathcal{X}(x,y)$$

Using Lemma 4.4.4 and the techniques of §4.4, it seems plausible that one may be able to deduce that the functor $\mathcal{M}$ satisfies the hypotheses of the Artin-Lurie criterion, from the fact that the functor $A \mapsto \mathrm{Mod}_A$ satisfies these conditions (Proposition 4.3.2).

It is worth noting that in order for any such approach to work, it is imperative that one work with lift of the moduli functor $\mathcal{M}^{\natural}_{\mathcal{P}r^L_{\omega,\mathrm{k}}}$ taking values in symmetric monoidal ∞-categories, and not merely its shadow $\mathcal{M}$, which takes values in spaces. Perhaps an even better approach would be the following. One first promotes the category $\mathcal{P}r^L_{\mathrm{k}}$ to a symmetric monoidal $(\infty, 2)$-category $\mathfrak{P}r^{L,\otimes}_{\mathrm{k}}$. We let $\mathfrak{P}r^{L,\otimes}_{\omega,\mathrm{k}}$ denote the full subcategory of compactly generated categories. One also promotes the moduli functor $\mathcal{M}^{\natural}_{\mathcal{P}r^L_{\omega,\mathrm{k}}}$ to a moduli functor $\mathfrak{M}$ taking values in symmetric monoidal $(\infty, 2)$-categories. This functor is defined by the formula

$$\mathfrak{M}(A) := \mathfrak{P}r^{L,\otimes}_{\omega,A}$$

It is a remarkable fact (see [Toe10]), that the ∞-groupoid of *fully* dualizable (see [Lur09b] for the definitions) objects in the symmetric monoidal $(\infty, 2)$-category $\mathfrak{P}r^{L,\otimes}_{\omega,A}$ coincides with the ∞-groupoid of dualizable objects in the symmetric monoidal $(\infty, 1)$-category $\mathcal{P}r^L_{\omega,A}$. Consequently, the space/∞-groupoid valued functor obtained from the functor $\mathfrak{M}$ by discarding noninvertible morphisms is precisely the functor $\mathcal{M}$. Based on our experiences in the previous chapters, it seems likely that working with the functor $\mathfrak{M}$, which is the most structured version of the moduli of noncommutative spaces that seems to be available, would be most expedient. In any case, we will use the functor $\mathfrak{M}$ in the next section.

5.2 Frobenius Manifolds: From TFTs to CohFTs

In this section, we will outline an approach to proving the unobstructedness of the moduli of Calabi-Yau noncommutative spaces, and constructing a Frobenius manifold structure on it. The approach proposed here generalizes, and provides a new perspective on the results of [BK98] and [Cos09].

Our main tool will be the Cobordism Hypothesis. This beautiful theorem was proven recently by Jacob Lurie. Our main reference for everything pertaining to the Cobordism Hypothesis and symmetric monoidal (∞, n)-categories is [Lur09b]. Another excellent reference for some of the material that follows is [Cos07].

Let $\mathrm{Cat}^{\mathrm{sm}}_{(\infty,n)}$ denote the ∞-category of rigid symmetric monoidal (∞, n)-categories. Denote by $\mathrm{Fun}^{\otimes}(-,-)$ the $\widehat{\mathrm{Cat}}_{\infty}$ valued internal hom on this category, and let $(-)^{\sim} : \mathrm{Cat}^{\mathrm{sm}}_{(\infty,n)} \to \mathrm{Cat}_{(\infty,0)}$ be the forgetful functor that takes an (∞, n)-category to the ∞-groupoid obtained by discarding all non-invertible morphisms. Recall that, roughly speaking the cobordism hypothesis asserts that this functor has a left adjoint $\mathrm{Bord}^{\mathrm{fr}}_n/(-)$. For an space/$\infty$-groupoid X, the category $\mathrm{Bord}^{\mathrm{fr}}_n/X$ is a certain (∞, n) category of "framed bordisms over X". The reader is referred to [Lur09b] for the details.

Now let $\mathcal{X}$ be an ∞-topos. We will need a version of the cobordism hypothesis that is "internal to $\mathcal{X}$". Let $\mathcal{X}^{\mathrm{sm}}_{(\infty,n)}$ denote the ∞-category of stacks in rigid symmetric monoidal ∞-categories on $\mathcal{X}$. That is $\mathcal{X}^{\mathrm{sm}}_{(\infty,n)} := \mathrm{Fun}^{\mathrm{R}}(\mathcal{X}, \mathrm{Cat}^{\mathrm{sm}}_{(\infty,n)})$. The category $\mathcal{X}^{\mathrm{sm}}_{(\infty,n)}$ is the ∞-category of rigid symmetric monoidal (∞, n)-categories internal to $\mathcal{X}$. One can also describe this category as, for instance, certain n-fold Complete Segal Space objects in $\mathcal{X}$. Similarly, the ∞-category of ∞-groupoid objects in $\mathcal{X}$ is the category $\mathcal{X}_{(\infty,0)} := \mathrm{Fun}^{\mathrm{R}}(\mathcal{X}, \mathrm{Cat}_{(\infty,0)}) \simeq \mathrm{Fun}^{\mathrm{R}}(\mathcal{X}, \mathcal{S}) \simeq \mathcal{X}$. This last chain of equivalences is just the statement that ∞-groupoid objects are effective in $\mathcal{X}$. The functor $(-)^{\sim} : \mathrm{Cat}^{\mathrm{sm}}_{(\infty,n)} \to \mathrm{Cat}_{(\infty,0)}$ induces a functor $\mathcal{X}^{\mathrm{sm}}_{(\infty,n)} \to \mathcal{X}$, which we will also denote by $(-)^{\sim}$. With this terminology, we can state the following theorem

Theorem 5.2.1. *The underlying groupoid functor* $(-)^{\sim} : \mathcal{X}^{\mathrm{sm}}_{(\infty,n)} \to \mathcal{X}_{(\infty,0)}$ *admits a left adjoint* $\mathrm{Bord}^{\mathrm{fr}}_n/(-)$.

This theorem is an immediate formal consequence of the cobordism hypothesis. The main point is that for an object X in an ∞-topos $\mathcal{X}$, and a manifold M, there exists an exponential X^M, and hence one had a notion of bordisms internal to $\mathcal{X}$.

Recall from [Lur09b] the notion of a Calabi-Yau object of dimension d in a symmetric monoidal $(\infty, 2)$-category $\mathcal{X}^{\otimes}$. We will say that an object $X \in \mathcal{X}$ admits a Calabi-Yau structure if it is in the essential image of the map $\mathcal{X}^{\mathrm{CY},d} \to \mathcal{X}$ for some d. Here $\mathcal{X}^{\mathrm{CY},d}$ is the category of Calabi-Yau objects of dimension d. Let $\mathcal{M}(A)^{\mathrm{CY}}$ denote the full subgroupoid of $\mathcal{M}(A)$ consisting of categories that admit a Calabi-Yau structure. We conjecture that the moduli functor $\mathcal{M}^{\mathrm{CY}}$ is representable by a locally geometric ∞-stack. We have the following generalization of the Bogomolov-Tian-Todorov Theorem.

Conjecture 5.2.2. *The deformation theory of* $\mathcal{M}^{\mathrm{CY}}$ *is unobstructed, and it admits a Frobenius structure.*

Idea of proof. Fix a point $\mathrm{Spec(k)} \to \mathcal{M}^{\mathrm{CY}}$. This corresponds to a smooth proper Calabi-Yau category $\mathcal{X}$. By Deligne's conjecture, the chain complex $\mathrm{HH}^{\bullet}(\mathcal{X})$ admits an action of the framed little discs operad. By the Cobordism Hypothesis, $\mathrm{HH}_\bullet(\mathcal{X})$ admits an action of the framed little discs operad. The circle action on $\mathrm{HH}_\bullet(\mathcal{X})$ induced by rotating the framings on the discs corresponds to Connes B-operator. By Hodge Theory (see [KKP08]), this circle action is homotopically trivial, since $\mathcal{X}$ is smooth and proper. The Calabi-Yau condition implies that Hochschild Homology can be identified with Hochschild cohomology, upto a shift. If we could also identify the circle actions on these two complexes, we would have that the circle action on $\mathrm{HH}^\bullet(\mathcal{X})$ is homotopically trivial. Using the triviality of the circle action and Conjecture 5.1.2. (which identifies the tangent complex as a shifted Hochschild cochain complex), one then shows that the $\mathbb{T}_{\mathcal{M},\mathcal{X}}[-1]$ is L_∞-formal. This implies the unobstructedness.

In order to construct the "Frobenius Structure", one promotes the moduli functor $\mathcal{M}^{\mathrm{CY}}$ to a functor $\mathfrak{M}^{\mathrm{CY}}$ taking values in rigid symmetric monoidal $(\infty, 2)$-categories. Then $\mathfrak{M}^{\mathrm{CY}}$, defines an object in $(\mathcal{S}t_{\mathrm{k}}^{\mathrm{sm}})_{(\infty,2)}$. It seems plausible that the triviality of the circle action together with Theorem 5.2.1 can be used to construct a Frobenius structure. We will present a more detailed discussion of this idea in [Pan11]. $\square$

The unobstructedness part of the previous conjecture also appears in [KKP08]. The author arrived at these ideas independently, and prior to the publication of [KKP08].

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