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A Sharper Ratio

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Abstract
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A SHARPER RATIO

Xingtan Zhang

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Dedication

To my parents, for all the love and encouragement.

To my wife, for all the love, support and dedication.

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ABSTRACT

A SHARPER RATIO

Xingtan Zhang

Kent Smetters

The Sharpe ratio is the dominant measure for ranking risky assets and funds. This paper derives a generalized ranking measure which, under a regularity condition, is valid in the presence of a much broader assumption (utility, probability) space yet still preserves wealth separation for the broad HARA utility class. Our ranking measure, therefore, can be used with "fat tails" as well as multi-asset class portfolio optimization. We also explore the foundations of asset ranking, including proving a key impossibility theorem: any ranking measure that is valid at non-Normal "higher moments" cannot generically be free from investor preferences. Finally, we derive a closed-form approximate measure (that can be used without numerical analysis), which nests some previous attempts to include higher moments. Despite the added convenience, we demonstrate that approximation measures are unreliable even with an infinite number of higher moments.

Contents

Chapter 1

Introduction

Sharpe (1966) demonstrated that picking a portfolio with the largest expected risk premium relative to its standard deviation is equivalent to picking the portfolio that maximizes the original investor's expected utility problem, assuming that portfolio returns are Normally distributed.[1] The Sharpe ratio, therefore, is a convenient "sufficient statistic" for the investor's problem since it does not rely on the investor's preferences or level of wealth.

The immense power of the Sharpe ratio stems from the fact that it allows the investment management process to be decoupled from the specific attributes of the heterogeneous investor base. Indeed, the multi-trillion dollar money management (mutual fund and hedge fund) industry relies heavily on this separation. While investors in a fund might differ in their risk aversion and level of wealth (including

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[1]Of course, the Sharpe ratio relies on pioneering mean-variance work by Markowitz (1952).

outside assets), an investment manager only needs to correctly estimate the first two moments of the Normal distribution that characterizes the fund's risk.[2] It is not surprising, therefore, that the Sharpe ratio is tightly integrated into the investment management practice and embedded into virtually all institutional investment analytics and trading platforms. Even consumer-facing investment websites like Google Finance reports the Sharpe ratio for most mutual funds along with just a few other basic statistics, including the fund's alpha, beta, expected return, R^2 tracking (if an indexed fund), and standard deviation.

Of course, it is well known that investment returns often exhibit "higher order" moments that might differ from Normality (Fama 1965; Brooks and Kat 2002; Agarwal and Naik 2004, and Malkiel and Saha 2005).[3] In practice, investment professionals, therefore, often look for investment opportunities that would have historically – that is, in a "back test" – produced unusually large Sharpe ratios under the belief that an extra large value provides some "buffer" in case the under-

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[2]The Sharpe ratio, however, only ranks the various *risky* portfolios in the presence of a risk-free numeraire instrument. It does not determine the optimal split between the risky asset and the risk-free asset, which must be determined in the second stage using consumer preferences. This two-step process is generally known as "two fund separation." Hence, there is still a role for "personalized financial advice" in the Sharpe world and the need to understanding an investors tolerance to risk.

[3]A related literature has examined how disaster risk can explain equilibrium pricing within the neoclassical growth model (Barro 2006, 2009; Gabaix 2008, 2012; Gourio 2012; and, Wachter 2012)

lying distribution is not Normal. Academic researchers, however, know that this convention is often a mistake. A large Sharpe ratio often does not provide much information outside of the *assumption (utility, probability) space* where the Sharpe ratio is valid. Indeed, it is easy to create portfolios with large Sharpe ratios that are actually first-order stochastically dominated by portfolios with smaller Sharpe ratios.[4] Multi-asset class portfolios with bonds, derivatives, and other securities often produce left skewed distributions even if the if the core equity risk is Normally distributed (Leland 1999; Spurgin 2001; and Ingersoll et al. 2007).[5]

The potential limits of the Sharpe ratio to correctly rank risky portfolios has led to an interest in producing risk measures that take into account non-Normal "higher-order" distribution moments. This literature dates back to at least the early work by Paul Samuelson (1970). A short list of other contributors include the seminal paper by Kraus and Litzenberger (1976), Scott and Horvath (1980), Owen and Rabinovitch (1983), Brandt et al (2005), Jurczendo and Maillet (2006), Zakamouline and Koekebakker (2008), Dvila (2010) and Pierro and Mosevich (2011). Much of this important work has extended Sharpe's mean-variance ranking measure to some additional moments of the risk distribution, typically under some restric-

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[4]In other words, the portfolio with the smaller Sharpe ratio should be preferred by all expected utility maximizers with positive marginal utility in wealth.

[5]Ingersoll et al. (2007) also examine the potential of manipulation of returns by managers according to a criterion that they outline. In contrast, our focus is on how to rank risky investments consistent with the investor's original problem.

tions on preferences or some other mathematical simplifications. A second area of research bypasses the investor's expected utility problem altogether and instead focuses on producing risk measures that satisfy certain mathematical properties such as "coherence."[6] Examples of coherent risk measures include "average VaR," "entropic VaR," and "super-hedging." While these measures satisfy certain axioms, a portfolio that best maximizes one or more of these measures does not necessarily maximize the investor's expected utility (i.e., the standard investor problem). A third line of work has evolved, often from practitioners, and has produced more heuristic measures including "value at risk (VaR),"[7] Omega, Sortino ratio, Treynor ratio, Jensen's alpha, upside potential ratio, Roy's safety-first criterion, and many more.[8] In practice, investment managers combine the Sharpe ratio with one or more

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[6]A "coherent" risk measure, for example, satisfies monotonicity, sub-additivity, homogeneity, and translational invariance (Artzner et al 1999). More recent work has emphasized risk measures that avoid "worst case" scenarios and are monotonic in first-order stochastic dominance. See, for example, the excellent work by Aumann and Serrano (2008); Foster and Hart (2009); and Hart (2011).

[7]Standard VaR is not coherent, whereas the variants on VaR noted in the previous sentence are coherent.

[8]Modigliani (1997) proposed a transformation of the Sharpe ratio, which became known as the "risk-adjusted performance measure." This measure attempts to characterize how well a risk rewards the investor for the amount of risk taken relative to a benchmark portfolio and the risk-free rate. This measure is not included in the list provided in the text because it provides a way of interpreting the unit-free Sharpe ratio rather than an alternative measure in the presence of non-Normally distributed risk.

of these other types of measures.

This paper makes three contributions. First, we derive a generalized ratio that correctly ranks risky returns under a broad assumption (utility, probability) space, including allowing for an unbounded number of higher moments.[9] By "correctly ranks," we mean it in the original tradition of Sharpe: the generalized ratio picks the portfolio that is preferred under the investor expected utility problem. Allowing for a broad *utility* space is critical for capturing realistic investor attitudes toward risk. Accounting for higher moments of the *risk distribution* is important for allowing for (i) "fat tails" distributions and/or (ii) multi-asset class optimization, where Normality is often violated. Like the original Sharpe ratio, our measure preserves wealth separation under the broad functional form of HARA utility, which includes many standard utility functions as special cases.[10] Unlike the original Sharpe ratio, our generalized ratio does not preserve separation from investor preferences. Indeed, we prove a related impossibility theorem: preference separation is generically impossible in the presence of non-Normal higher-order moments.[11]

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[9] Our core mathematical advancement is summarized in our Lemma 3.1.2, which demonstrates how to solve an infinite-order Maclaurin expansion for its correct root when no closed form solution exists. Previously, even numerical analysis required solving finite N order series, which produced N real and complex solutions.

[10] In other words, the generalized ratio can correctly rank without knowing the investor's wealth.

[11] Our language is a little loose at this point at this point. We introduce the concept of an *adjusted cumulant*, where Normality implies that the third and higher order adjusted cumulants are zero.

Second, we explore the theoretical foundations of the ranking measures in more detail. Besides the impossibility theorem just mentioned, we more closely examine the assumption space where the traditional Sharpe ratio is a valid ranking measure. Despite its extensive usage in industry, very little is actually known about the Sharpe ratio, that is, beyond the few cases where it is known to correctly rank risks (e.g., Normally distributed risk or quadratic utility). We show that the Sharpe ratio is valid under a larger assumption space than currently understood. We also explore why it is challenging to write down a *necessary* condition for the Sharpe ratio to be a valid ranking measure. In answering these questions, we are also able to generalize the seminal Kaus-Litzenberger (1976) "preference for skewness" result to an unlimited number of higher moments. This generalization is useful because plausible utility functions produce an infinite number of non-zero higher-order derivatives, and there does not exist any probability distribution that can be fully described by its first three order of cumulants.

Third, we derive a linear approximation of our generalized ratio, which allows for closed-form solutions. Our formula allows for an infinite number of higher moments, thereby nesting some previous attempts to generalize the Sharpe ratio to higher moments. Our simulation results, however, show that approximations can be very inaccurate, unstable and even divergent. Serious risk management, therefore, generally requires calculating the generalized ratio.

The paper is organized as follows. Section 2 provides an overview of the standard

investor problem. Section 3 derives the generalized ratio described earlier. Section 4 provides theoretical insights into the Sharpe ratio. Section 5 provides closed-form approximation formulas to the generalized rato. Section 6 provides numerical examples comparing the generalized ratio, approximation, and the Sharpe for a range of potential applications. Section 7 concludes. Proofs of lemmas and theorems are provided in the Appendices.

Chapter 2

The Investor Problem

The investor problem that we consider is fairly standard.

2.1   Investor Problem

The investor has preferences characterized by the utility function u and wants to allocate wealth w among the risk free asset paying a constant rate r and a risky asset paying a net return Y. More formally:

$$\max_{a} \mathbb{E} u\left(w(1+r) + a(Y-r)\right) \tag{2.1.1}$$

where the variable a is the amount of wealth invested in the risky asset. To reduce notation, we will often write $w_r \equiv w(1+r)$. Now suppose that u belongs to the function space $\mathcal{U}_s$ that denotes all the smooth utility functions defined on the real number line with positive odd-order derivatives and negative even-order derivatives.

$\mathcal{U}_s$, therefore, incorporates a broad set of utility classes including HARA.

Lemma 2.1.1. *For any given increasing and concave utility function u, maximization problem (2.1.1) has a unique solution a^*. Furthermore,*

if $\mathbb{E}Y > r$ then $a^* > 0$;

if $\mathbb{E}Y = r$ then $a^* = 0$;

if $\mathbb{E}Y < r$ then $a^* < 0$

In other words, the investor problem that we are considering for a given risky asset is standard: the best portfolio exists and risk taking follows standard behavior. If the expected return to the risky security exceeds the risk-free rate then at least some risky position will be held; if the two returns are equal then no risky asset is held; otherwise, a short position is taken.

Given utility function u and initial wealth w, for two different risky asset Y_1, Y_2 and risk free rate r, it is convenient to write $(Y_1, r) \geq_u^w (Y_2, r)$ if the following holds

$$\max_a \mathbb{E}u(w(1+r) + a(Y_1 - r)) \geq \max_a \mathbb{E}u(w(1+r) + a(Y_2 - r))$$

Notice that the optimal allocation in above two maximization problem are not necessary to be same. The interpretation of $(Y_1, r) \geq_u^w (Y_2, r)$ is that for investor with given preference and initial wealth and the risk free rate is r, then investing in Y_1 is better than Y_2. For simplicity we write $(Y_1, r) \geq_u (Y_2, r)$ if we have $(Y_1, r) \geq_u^w (Y_2, r)$ holds for all $w > 0$.

2.2 Ranking Definitions

Like the original Sharpe Ratio, we want to pairwise rank two risky investments with random returns denoted as Y_1 and Y_2. Of course, if we know the investor's preferences, the investor's wealth, and the exact functional form of the distribution of the underlying risky asset produce a return Y, we can then simply integrate the expectation operator in equation (2.1.1) to determine investor's indirect utility associated with each risk at the investor's optimum. However, in practice, we are typically missing some of this information, and so we would like to be able to rank among investments based on a subset of this information. Indeed, the real power of the Sharpe Ratio stems from its ability to correctly pairwise rank two investment risks simply by knowing something about the underlying risk distribution and the risk-free rate. Toward that end, the following definitions will be useful:

Definition 2.2.1. *[Ranking Measures] For any risky assets Y and risk free rate r, we say that:*

- A *distribution-only ranking measure* is a function q_D which only depends on Y, r:

- A *distribution-preference ranking measure* is a function q_{DU} which only depends on Y, r, u.

- A *distribution-preference-wealth ranking measure* is a function q_{DUW} which only depends on Y, r, u, w.

Definition 2.2.2. *[Valid Ranking Measure, Assumption Space] Suppose* $\mathcal{U}$ *is a set of utility functions and* $\mathcal{Y}$ *is a set of random variables, we call q to be a* valid *ranking measure with respect to* $\mathcal{U} \times \mathcal{Y}$ *if,* $\forall u \in \mathcal{U}, Y_1, Y_2 \in \mathcal{Y}$,

$$q(Y_1, r) \geq q(Y_2, r) \iff (Y_1, r) \geq_u (Y_2, r)$$

We call $\mathcal{U} \times \mathcal{Y}$ *the* assumption space *of the ranking measure.*

In words, a valid *distribution-only* ranking measure does not require knowing the investor's preferences or level of wealth in order to properly rank risky gambles. As we show below, the Sharpe ratio is such an example. A *distribution-preference* ranking measure then also requires knowing the investor's preferences. We show below that our generalized ranking measure for HARA utility is one such example. We also prove that all ranking measures that are "valid" at arbitrary "higher moments" across a wide range of utility functions must at least be a distribution-preference ranking measure. The *distribution-preference-wealth* ranking measure requires also knowing the investor's wealth. The last ranking measure is the least interesting of the three: relative to the original investor problem, the only advantage of the distribution-preference-wealth measure is that it allows for the ranking to be performed on the distribution's moments (cumulants), which are often estimated empirically in practice.

In theory, we could define our assumption space to be a bundle, where the random variables set could be different for different utility function. To be specific, assumption space could be $\mathcal{A}$, which element are of form (u, Y), now we call q to

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be a *valid* ranking measure with respect to $\mathcal{A}$, if for any $(u, Y_1), (u, Y_2) \in \mathcal{A}$ we have $q(Y_1, r) \geq q(Y_2, r) \iff (Y_1, r) \geq_u (Y_2, r)$. In this article, we restrict all the utility-risk pair such that the investor's problem makes sense. For example, our assumption space does not include *log* utility and *normal* distribution because the maximization problem is not meaningful in this context.

When there is no confusion, we call our ranking measure q to be a valid ranking measure with respect to the utility assumption space only if the random variables could be anything from the given probability space; we call it valid ranking measure with respect to the random variables space if the utility function could be any increasing concave function. We can also define q_n to be a valid ranking measure *sequence* if it point-wise converges to a valid ranking measure.

Remark 2.2.3. *It is straightforward to show that a sufficient condition for function q to be valid ranking measure is for $\max_a \mathbb{E} u \left(w(1 + r) + a(Y - r) \right)$ to be increasing in q. Moreover, two ranking measures are equivalent if one measure is a strictly increasing transformation of the other.*

Example 2.2.4. *Let $\mathcal{U}_e = u(\cdot) : u(w) = -exp(-\gamma w), \gamma > 0$ that is CARA class utility function. It is easy to show that $\gamma\{u[\max_a \mathbb{E} u \left(w(1 + r) + a(Y - r) \right)] - w(1 + r)\}$ only depends on Y and r for $\forall u \in \mathcal{U}_e$. If we let*

$$q(Y, r) := \gamma\{u[\max_a \mathbb{E} u \left(w(1 + r) + a(Y - r) \right)] - w(1 + r)\}$$

then it is a valid ranking measure with respect to $\mathcal{U}_e$.

2.3   The Sharpe Ratio

Suppose that the underlying risky return Y is draw from a Normal distribution $N\left(\mathbb{E}Y, \sqrt{Var(Y)}\right)$, and let

$$q_S(Y, r) = \left(\frac{\mathbb{E}Y - r}{\sqrt{Var(Y)}}\right)^2.$$

Sharpe (1996) showed that the investor's indirect utility,

$$\max_a \mathbb{E}u\left(w(1+r) + a(Y-r)\right)$$

is an increasing function of $q_S(Y, r)$. In addition, if we restrict random variables space such that $\mathbb{E}Y \geq r$ then the maximized expected utility is an increasing function of Sharpe ratio $\frac{\mathbb{E}Y - r}{\sqrt{Var(Y)}}$. Hence, q_S is a distribution-only based ranking measure when the underlying return distribution is Normal. In our language, we could say q is a valid ranking measure with respect to $\mathcal{U}_q$, which contains all the quadratic function; q is also a valid ranking measure with respect to $\mathcal{Y}_n$, which contains all the normal distribution. In words, q_S correctly ranks two investment risks without knowing the investor's preferences or wealth; the only information required are the parameters (expected returns and standard deviation) of the underlying risk distribution and, of course, the risk-free rate. As discussed later, the Sharpe ratio squared is valid for some non-Normal distributions when utility takes the (implausible) quadratic form.

However, the Normal shock assumption is only a sufficient condition for q_S to be a distribution-only based ranking measure. The Sharpe ratio ranking measure

q_S is actually valid over a wider class of return distributions.

Theorem 2.3.1. *Let χ_α denote a parametrized distribution family, where α is the vector of parameters. If for every element $Y_\alpha \in \chi_\alpha$, the random variable $\frac{Y_\alpha - \mathbb{E}Y_\alpha}{\sqrt{Var(Y_\alpha)}}$ is independent of α and symmetric, then $\max_a \mathbb{E}u\left(w(1+r) + a(Y-r)\right)$ is an increasing function of q_S.*

Normally distributed risk is a special case of this result.

Example 2.3.1. *If Y_α is Normally distributed, then $\frac{Y_\alpha - \mathbb{E}Y_\alpha}{\sqrt{Var(Y_\alpha)}}$ is $N(0,1)$, which is independent of α and symmetric.*

So is the symmetric bivariate distribution.

Example 2.3.2. *Consider the random variable X where $X = \alpha_1 + \sqrt{\alpha_2}$ w.p. 1/2, $X = \alpha_1 + \sqrt{\alpha_2}$ w.p. 1/2. Then $\frac{X_\alpha - \mathbb{E}X_\alpha}{\sqrt{Var(X_\alpha)}}$ is a bivariate random variable. Therefore, Sharpe Ratio is correct.*

Indeed, we can construct many probability spaces where the Sharpe ratio properly ranks risky returns.

Example 2.3.3. *Suppose T is a t distribution with degree of freedom 4, construct distribution family by letting $\chi_{\alpha_1, \alpha_2} = \{X : X = \alpha_1 + \alpha_2 * T\}$ i.e. $\chi_{\alpha_1, \alpha_2}$ is the set of all random variables that can be written as linear function of T. Then for this family of distribution, we have $\forall Y_\alpha \in \chi_{\alpha_1, \alpha_2}$, the random variable $\frac{Y_\alpha - \mathbb{E}Y_\alpha}{\sqrt{Var(Y_\alpha)}} = T$ is independent of parameter and symmetric. From Theorem 2.3.1, we know that Sharpe ratio could properly rank risky returns inside $\chi_{\alpha_1, \alpha_2}$.*

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While these results demonstrate that Sharpe is potentially more robust than commonly understood, when the ratio is no longer valid at ranking, the ratio can "break, not bend." Consider the following example that comes from Hodge's (1998).

Example 2.3.4. *Consider two risky assets described by their risk net returns Y_1 and Y_2.*

Probability	0.01	0.04	0.25	0.40	0.25	0.04	0.01
Excess Return Y_1	-25%	-15%	-5%	5%	15%	25%	35%
Excess Return Y_2	-25%	-15%	-5%	5%	15%	25%	45%

Clearly the first asset with return Y_1 is first-order stochastically dominated by the second asset with return Y_2. However, the Sharpe ratio for the first asset is 0.500 whereas the Sharpe ratio for the second asset is only 0.493. Indeed, as noted in Section 1, one can produce large Sharpe ratios simply by introducing option and other derivative contracts into the portfolio. We return to this topic in Section 6.

Chapter 3

Generalized Ratio Ranking

Measure

We now derive our generalized ratio for ranking risks that is applicable to a larger

assumption space.

3.1 The Regularity Condition

Using Taylor's theorem, we can rewrite the first-order condition of the investor's

problem (2.1.1) as follows:

$$0 = \mathbb{E}u'\left(w_r + a(Y-r)\right)(Y-r)$$

$$= \mathbb{E}\left(\sum_{n=0}^{\infty} u^{(n+1)}(w_r)\frac{a^n(Y-r)^n}{n!}\right)(Y-r)$$

$$= \sum_{n=0}^{\infty} \frac{u^{(n+1)}(w_r)\mathbb{E}(Y-r)^{n+1}}{n!}a^n$$

$$= \sum_{n=1}^{\infty} \frac{u^{(n)}(w_r)\mathbb{E}(Y-r)^n}{(n-1)!}a^{n-1} \tag{3.1.1}$$

Definition 3.1.1. *[The* n-*th* t-*moment] We will call* $t_n^Y \equiv \mathbb{E}(Y-r)^n$ *the* n-th *translated moment (or* n-th t-moment *for shorthand) for the risky investment with return* Y.

A closed-form solution of equation 3.1.1 is typically not available. (Section 5, however, provides some closed-form solutions for approximate measures.) Moreover, in practice, we can't let computers (or our pencils) run indefinitely, and so we must truncate the expansion to a finite (but potentially large) N number of terms. However, such a truncation will typically produce *many* roots, even though the original infinite expansion in equation (3.1.1) has a single root by Lemma (2.1.1).

Fortunately, the following lemma provides the central mechanism for selecting the correct root in the N-term expansion. The general nature of this lemma suggests that it could have fairly broad application outside of the current study.

Lemma 3.1.2. *Suppose real function* $f(x) = 0$ *has unique real solution* x_0. *Denote*

Maclaurin expansion of f to be $\sum_{n=0}^{\infty} c_n x^n$. Consider

$$f_N(x) = \sum_{n=0}^{N} c_n x^n.$$

$f_N = 0$ has N solutions on complex plane S_N. Denote the convergent radius for the series as λ. If $\lambda > |x_0|$, then: (i) the smallest absolute real root in S_N converges to x_0 as $N \to \infty$ and (ii) there is a finite value of N such that there is only one real root.

Remark 3.1.3. *In general, the smallest absolute root does not necessarily converge monotonically (even in absolute value) as N grows. Hence, it is technically not possible to impose a "stopping rule" on the value of N to be used for calculating the root. However, in practice, our simulations suggest that N does indeed converge after a reasonable value, especially after the only one real root emerges.*

Definition 3.1.4. *[Regularity Condition] We will say that the utility-risk pair (u, Y) satisfies the* Regularity Condition *if the corresponding series of 3.1.1 satisfies the requirement $\lambda > |x_0|$ in Lemma 3.1.2.*

Denote assumption space $\mathcal{A}_{\mathcal{RC}}$ to be all the utility-risk pair such that regularity condition holds. In all the following result, we always assume our assumption to be a subset of $\mathcal{A}_{RC}$

Corollary 3.1.5. *The regularity condition trivially holds if the convergence radius is infinite (i.e., $\lambda = \infty$).*

3.2 HARA Utility

Consider the HARA utility function $u \in \mathcal{U}_H$ class where $u(w) = \frac{\rho}{1-\rho}\left(\frac{\lambda w}{\rho} + \phi\right)^{1-\rho}$.

Denote $\mathcal{U}_H^\rho \subset \mathcal{U}_H$ be the subset of HARA utility function where ρ is given. Then:

$$u^{(n)}(w) = \frac{\rho}{1-\rho}(1-\rho)(-\rho)\cdots(2-n-\rho)(\frac{\lambda}{\rho})^n\left(\frac{\lambda w}{\rho} + \phi\right)^{1-n-\rho}$$

From equation (3.1.1), we need to solve

$$\sum_{n=1}^{N} u^{(n)}(w_r)\frac{t_n^Y}{(n-1)!}a^{n-1} = 0 \tag{3.2.1}$$

as $N \to \infty$. With some simple substitution, this equation becomes:

$$\sum_{n=1}^{N} \frac{\rho}{1-\rho}(1-\rho)(-\rho)\cdots(2-n-\rho)(\frac{\lambda}{\rho})^n\left(\frac{\lambda w_r}{\rho} + \phi\right)^{1-n-\rho}\frac{t_n^Y}{(n-1)!}a^{n-1} = 0$$

or

$$\lambda\left(\frac{\lambda w_r}{\rho} + \phi\right)^{-\rho}\sum_{n=1}^{N}(\rho)\cdots(\rho+n-2)\frac{t_n^Y}{(n-1)!}\left(-\frac{\lambda}{\rho}\frac{a}{\frac{\lambda w_r}{\rho} + \phi}\right)^{n-1} = 0$$

Let

$$b_n = \begin{cases} 1, & n = 1 \\ \\ (\rho)\cdots(\rho+n-2), & n \geq 2 \end{cases} \tag{3.2.2}$$

Also, let $z = -\frac{\lambda}{\rho}\frac{a}{\frac{\lambda w_r}{\rho} + \phi}$. With these change of variables, equation (3.2.1) can be rewritten as:

$$-\sum_{n=1}^{N} \frac{b_n t_n^Y}{(n-1)!}z^{n-1} = 0. \tag{3.2.3}$$

Definition 3.2.1. *[Generalized Ranking Measure with HARA Utility[1]] Let denote $z_{N,Y}$ as the smallest absolute real root z that solves equation (3.2.3). The (N-th order) HARA ranking measure is defined as:*

$$q_H^N(t_n^Y, b_n) = -\sum_{n=1}^{N} \frac{b_n t_n^Y}{n!} z_{N,Y}^n. \tag{3.2.4}$$

where b_n is shown in equation (3.2.2) and t_n^Y is the nth t-moment of the risky investment with return Y.

Notice that the root $z_{N,Y}$ is only a function of preferences b_n and the *t-moments* t_n^Y of the underlying risk distribution. In particular, $z_{N,Y}$ does not depend on the investor's wealth.[2]

Theorem 3.2.1. *$q_H^N(t_n^Y, b_n)$ is a valid ranking measure sequence w.r.t. assumption space where utility belongs to $\mathcal{U}_H^\rho$ and random variables belongs to anything that satisfy the regularity condition.*

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[1]For brevity, we don't consider the case of non-HARA utility in this section since it is not generically wealth independent. However, see Section 5, where we derive approximation formulas starting with the most generic case.

[2]Zakamouline and Koekebakker (2008) also note that HARA should be wealth independent, but they don't solve for a ranking measure. Instead, they solve a three-moment distribution with skewness using approximation methods. See Section (5).

3.3   An Impossibility Theorem

Like the Sharpe ratio, the generalized ratio can rank risk without know the wealth of the underlying investors. Relative to the Sharpe ratio, the disadvantage of the HARA ranking measure is that it requires knowledge of the investor's preferences for ranking. This outcome, however, is not simply a feature of our particular construction of a ranking measure.

Theorem 3.3.1. *There does not exist a distribution-only ranking measure for HARA utility if portfolio risk Y can be any random variable.*

Indeed, we can conclude that there is no generic distribution-only ranking measure when Y can take on any distribution, leading to the following impossibility theorem.

Corollary 3.3.1. *There does not exist a generic distribution-only ranking measure if portfolio risk Y can be any random variable.*

In particular, if we want to accommodate non-Normally distributed risk, using a distribution-only ranking measure, like Sharpe, is typically not valid across a wide range of investor preferences. In contrast, the distribution-only Sharpe ratio ranking measure is valid across a range of preferences (that is, preferences that are consistent with Normal returns) if the random variable is restricted to be Normally distributed.

3.4 Two Special Cases

However, in two special cases of HARA utility, we can simplify things a bit more.

3.4.1 CARA Utility

For the case of constant absolute risk aversion (CARA), the value of $\phi = 1$, $\rho \to \infty$, and so $b_n = 1$. Hence, the corresponding value of $z_{N,Y}$ is only a function of the *t-moments* of the underlying risk.

Corollary 3.4.1. *For CARA utility, if the regularity condition holds, a distribution-only ranking measure exists and takes the form* $q_{CARA}(t_n^Y) = -\sum_{n=1}^{N} \frac{t_n^Y}{n!} z_{N,Y}^n$.

In other words, we can construct a valid ranking measure that requires only a characterization of the shock distribution, like the Sharpe ratio. Intuitively, the absence of the income effect with CARA utility means that risk aversion over final wealth risk drops out on the assumption space. Unlike the Sharpe ratio, however, this measure is valid for non-Normally distributed risk if preferences are restricted to the CARA form. Of course, while CARA utility is commonly used for theoretical analysis, its application to actual investor problems is limited.

3.4.2 CRRA Utility

Now consider the case of constant relative risk aversion (CRRA) utility where $\phi = 0$, $\rho > 0$, and $\lambda = \rho$. The CRRA form is commonly used to model investor preferences

since the level of risk aversion scales with investor wealth. The CRRA ranking function, however, is still a *distribution-preference* ranking measure, as in the more general HARA case. However, we obtain a nice portfolio choice simplification for CRRA.

Remark 3.4.2. *For CRRA utility, the quantity $-z(1+r)$ is equal to $\frac{a}{w}$, the percentage of wealth w that is invested into the risky asset.*

In other words, with CRRA utility, we can solve "two fund separation" investor problem simultaneously: (1) picking the best risky investment Y from the assumption space $\mathcal{A}_{\mathcal{H}}$ with the CRRA restrictions ($\phi = 0$, $\rho > 0$, $\lambda = \rho$) and (2) picking the share of wealth to be placed into this risky investment (versus bonds).[3] However, the quantity $-z(1+r)$ itself is not a valid ranking measure since the generalized ranking measure under CRRA is not a monotone transformation of $-z(1+r)$.[4]

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[3]Recall that both the Sharpe ratio and the generalized ratio herein ranks only risky investments, given the numeraire risk-free security. Neither ratio typically determines the allocation between the best ranked risky security and and the *risk-free* security. That allocation typically requires returning to the original investor problem (2.1.1), which includes the investor's investor preferences and wealth. However, with CRRA, the value of z also indicates the share of the investor's wealth that should be allocated into the best ranked risky investment relative to the risk-free security, even with non-Normally distributed risk.

[4]For example, consider two risky asset payoffs, Y and tY, where t is a positive constant. The generalized ranking measure produces identical values since the investor should be indifferent between the two risky assets. However, the percent invested into each asset will differ.

3.5 Extension to Multiple Asset Classes

The generalized ratio is more powerful than just being able to handle "fat tail" risks. The generalized ratio allows one to consider *composite risks* consistent with multi-asset class portfolios that are typically not Normally distributed, especially with derivatives, thinly traded securities, corporate bonds, and other security classes. The generalized ratio, therefore, can be used as the core foundation for multi-asset class portfolio optimization due to its ability to correctly pairwise rank different composite assets. The only additional practical step for computational purposes is to combine the ranking function q_H with a globally stable optimization routine that searches over the space of potential composite permutations of the security space. We provide some examples in Section 6.

Chapter 4

Foundations of Ranking

The Sharpe ratio appears to be a puzzling ranking measure because it appears to hold under seemingly unrelated restrictions on the assumption space. For example, it is well know that the Sharpe ratio correctly ranks Normally distributed risks for most types of preferences consistent with a Normally distributed shock (that is, where the standard Inada condition is not imposed). It is also well known that the Sharpe ratio correctly ranks *non-Normally* distributed risks provided that preferences are quadratic. Furthermore, as we showed in Section 2, the Sharpe ratio is a valid ranking measure in some cases where the underlying probability distribution is not Normal and preferences are not quadratic. This section uses perturbation analysis to explore the foundations of the Sharpe ratio in more detail. In the process, we extend the classic Klaus-Litzenberger (1976) result, demonstrating that investors prefer skewness in their returns, to infinite (adjusted) cumulants.

4.1 Adjusted Cumulants

4.1.1 Definitions

Definition 4.1.1. *[Infinitely Divisible] For given probability space, we say that random variable Y has an infinitely divisible distribution, if for each positive integer T, there is a sequence of i.i.d. random variables $X_{T,1}$, $X_{T,2}$,...,$X_{T,T}$ such that*

$$Y \overset{d}{=} X_{T,1} + X_{T,2} + \cdots + X_{T,T}$$

where the symbol $\overset{d}{=}$ denotes equality in distribution. Loosely, we say Y has the infinitely divisibility property. We call $X_{T,1}$ the T-th component of Y.

We can think of a single unit of time as being divided into T equal length subintervals, Δt, i.e., $\Delta t = \frac{1}{T}$. Each variable $X_{T,i}$ then represents the return in the i-th subinterval. For notational simplicity, since the $X_{T,1}, X_{T,2}, \cdots, X_{T,T}$ subintervals of risk Y are i.i.d., we drop the subscripts and simply express each subinterval as X.

Definition 4.1.2. *[Adjusted Cumulant] Suppose Y has an infinitely divisible distribution and let*

$$\epsilon_n = \frac{X - \frac{\mu}{T}}{\sigma\sqrt{\frac{1}{T}}} = \frac{X - \mu\Delta t}{\sigma\sqrt{\Delta t}},$$

where μ, σ are mean and standard deviation of Y. We define Y's kth's adjusted cumulant as

$$\nu_k = \nu_k(Y) = \lim_{T\to\infty} \frac{\mathbb{E}\epsilon_T^k}{\left(\frac{1}{T}\right)^{-\frac{k-2}{2}}} = \lim_{\Delta t\to 0} \frac{\mathbb{E}\epsilon^k}{(\Delta t)^{-\frac{k-2}{2}}}, \forall k \geq 2.$$

Lemma 4.1.3. *The adjusted cumulant ν_n is related to a distribution's (more tra-ditionally defined) cumulant as follows:*

$$\kappa_n = \nu_n \sigma^n, \forall n \geq 2.$$

where κ_n is the distribution's n-th cumulant.

Remark 4.1.4. *The adjusted cumulant concept is easier to interpret than the more traditional cumulant of a distribution. In particular, ν_3 corresponds to a random variable Y's skewness while ν_4 is its excess kurtosis. Moreover, if Y is Normally distributed then $\nu_n = 0$, $\forall n \geq 3$.*

4.1.2 Calculating Adjusted Cumulants

In general, for random variable Y, we denote μ_1 the mean and for $k \geq 2$,

$$\mu_k = \mathbb{E}(Y - \mathbb{E}Y)^k \text{ and } \xi_k = \frac{\mu_k}{\mu_2^{\frac{k}{2}}}$$

So ξ_3 represents the skewness and ξ_4 the kurtosis. We can calculate adjusted cumulants using two methods: by induction or from a distribution's moment generating function. Each has its relative advantages. We start with the induction approach.

Theorem 4.1.1. *For any integer $n \geq 4$,*

$$\nu_n = \xi_n - \sum_{\substack{n = i_1 + i_2 + \cdots + i_k \\ i_1 \geq i_2 \cdots \geq i_k \geq 2}} \frac{\binom{n}{i_1}\binom{n-i_1}{i_2}\cdots\binom{n-i_1-\cdots-i_{k-1}}{i_k}}{k!}\nu_{i_1}\cdots\nu_{i_k}.$$

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The key advantage of the inductive approach is that adjusted cumulants can be easily calculated using actual data, which is quite useful for many practical applications where the function form of the risk distribution is not known.

Remark 4.1.5. *By Remark 4.1.3, the standard cumulant of a distribution can, therefore, also be computed inductively using Theorem 4.1.1.*

Another way to calculated adjusted cumulants is by exploiting the fact that an infinitely divisible distribution corresponds to a Levy process. Suppose X_t is Levy Process, $Y = X_1$ then Y is infinitely divisible distribution with $X = X_{\frac{1}{n}}$. By Levy-Khinchine representation, we have

$$\mathbb{E}e^{i\theta X_t} = exp\left(bit\theta - \frac{1}{2}\sigma_0^2 t\theta^2 + t\int_{\mathbb{R}\backslash\{0\}}(e^{i\theta x} - 1 - i\theta x I_{|x|<1})W(dx)\right)$$

where $b \in \mathbb{R}$, and I is the indicator function. The Levy measure W must be such that

$$\int_{\mathbb{R}\backslash\{0\}} min\{x^2, 1\}W(dx) < \infty$$

Denote

$$\phi(\theta, t) \equiv bit\theta - \frac{1}{2}\sigma_0^2 t\theta^2 + t\int_{\mathbb{R}\backslash\{0\}}(e^{i\theta x} - 1 - i\theta x I_{|x|<1})W(dx)$$

and

$$\psi(\theta, t) \equiv bt\theta + \frac{1}{2}\sigma_0^2 t\theta^2 + t\int_{\mathbb{R}\backslash\{0\}}(e^{\theta x} - 1 - \theta x I_{|x|<1})W(dx)$$

i.e $\phi(\theta, t) = \psi(i\theta, t)$ and $e^{\psi(\theta,t)}$ is the moments generating function of X_t.

Theorem 4.1.2. *Suppose $e^{\psi(\theta,t)}$ is the moment generating function of the Levy Process X_t, let $Y = X_1$, and σ equal the standard deviation of Y. Then:*

$$\nu_k(Y) = \frac{\frac{\partial^k \psi(\theta,1)}{\partial \theta^k}\big|_{\theta=0}}{\sigma^k}, \forall k \geq 2.$$

4.2 Ranking Measure with Short Trading Times

For the risky return Y with an infinitely divisible distribution, consider the Δt period investor problem.

$$\max_a \mathbb{E}u(w(1 + r\Delta t) + a(X - r\Delta t)) \tag{4.2.1}$$

$$= \max_a \sum_{n=0}^{+\infty} u^{(n)}(w(1 + r\Delta t))\frac{a^n}{n!}\mathbb{E}(X - r\Delta t)^n$$

$$= \max_a \sum_{n=0}^{+\infty} u^{(n)}(w(1 + r\Delta t))\frac{a^n}{n!}\mathbb{E}(\mu\Delta t + \sigma\sqrt{\Delta t}\epsilon - r\Delta t)^n$$

By definition of adjusted cumulants, the leading term of $\mathbb{E}(\mu\Delta t + \sigma\sqrt{\Delta t}\epsilon - r\Delta t)^n$ is $\sigma^n(\Delta t)^{\frac{n}{2}}\mathbb{E}\epsilon^n$, which is of order $\sigma^n\nu_n\Delta t$ for $n \geq 2$, and it is $(\mu - r)\Delta t$ when $n = 1$. Denote $\nu_1 = \frac{\mu-r}{\sigma}$. Then $\mathbb{E}(\mu\Delta t + \sigma\sqrt{\Delta t}\epsilon - r\Delta t)^n \sim \sigma^n\nu_n\Delta t$ for any $n \geq 1$. Denote $w_r = w(1 + r\Delta t)$. So

$$\max_a \sum_{n=0}^{+\infty} u^{(n)}(w(1 + r\Delta t))\frac{a^n}{n!}\mathbb{E}(\mu\Delta t + \sigma\sqrt{\Delta t}\epsilon - r\Delta t)^n$$

$$= \max_a \left(u(w_r) + \sum_{n=1}^{+\infty} u^{(n)}(w_r)\frac{\sigma^n\nu_n\Delta t}{n!}a^n + o(\Delta t) \right)$$

$$= \max_a \left(u(w_r) + \left(\sum_{n=1}^{+\infty} u^{(n)}(w_r)\frac{\nu_n}{n!}(\sigma a)^n \right)\Delta t + o(\Delta t) \right).$$

Theorem 4.2.1. *As $\Delta t \to 0$, the maximization problem (4.2.1) is :*

$$\sum_{n=1}^{+\infty} u^{(n)}(w_r) \frac{\nu_n}{(n-1)!} (\sigma a)^{n-1} = 0, \tag{4.2.2}$$

assuming that this series converges.

Now, consider the *finite* series:

$$\sum_{n=1}^{N} u^{(n)}(w_r) \frac{\nu_n}{(n-1)!} (\sigma a)^{n-1} = 0 \tag{4.2.3}$$

and let a_N^* equal the smallest absolute real root that solves equation (4.2.3). By

Lemma 3.1.2, this value will converge to to root of series (4.2.2) for a large enough

value of N, if the series regularity condition holds. Inserting a_N^* into the investor

problem:

$$\max_a \mathbb{E}u(w(1 + r\Delta t) + a(X - r\Delta t))$$

$$= \mathbb{E}u(w(1 + r\Delta t) + a_N^*(X - r\Delta t))$$

$$= u(w_r) + \left(\sum_{n=1}^{+\infty} u^{(n)}(w_r) \frac{\nu_n}{n!} (\sigma a_N^*)^n \right) \Delta t + o(\Delta t)$$

$$\approx u \left(w_r + \frac{\left(\sum_{n=1}^{+\infty} u^{(n)}(w_r) \frac{\nu_n}{n!} (\sigma a_N^*)^n \right) \Delta t}{u'(w_r)} \right)$$

$$= u \left(w_r + \left(\sum_{n=1}^{+\infty} \frac{u^{(n)}(w_r)}{u'(w_r)} \frac{\nu_n}{n!} (\sigma a_N^*)^n \right) \Delta t \right)$$

Definition 4.2.1. *[Adjusted Cumulant Ranking Measure] The* N-th order adjusted

cumulant ranking *measure is*

$$\sum_{n=1}^{N} \frac{u^{(n)}(w_r)}{u'(w_r)} \frac{\nu_n}{n!} (\sigma a_N^*)^n \tag{4.2.4}$$

Remark 4.2.2. *If N approaches to infinity, we call it* adjusted cumulant ranking measure.

Definition 4.2.3. *[Scalable] We say that utility function u is* scalable *with respect to random variable space χ, a subset of all infinite divisible distributions, if the following equivalence holds*

$$(Y, r) \geq_u (Y', r) \Leftrightarrow (X, r\Delta t) \geq_u (X', r\Delta t)$$

where $Y, Y' \in \chi$ and X, X' are any component of Y and Y' of length Δt, respectively. In other words, if u is scalable, an investor prefers Y over Y' if and only if he prefers X over X' in the Δt time period.

Lemma 4.2.4. *CARA utility and Quadratic utility are both scalable with respect to all infinite divisible distributions. HARA utility is scalable with respect to all Poisson distributions. Any utility function is scalable with respect to a Normal risk distribution.*

Theorem 4.2.2. *If u is scalable with respect to χ, then the adjusted cumulant ranking measure is a valid ranking measure with respect to $u \times \chi$.*

Proof. Since u is scalable with respect to χ, we have $(Y, r) \geq_u (Y', r) \Leftrightarrow (X, r\Delta t) \geq_u (X', r\Delta t)$. By above derivation, we have $(X, r\Delta t) \geq_u (X', r\Delta t) \Leftrightarrow (X, r\Delta t) \geq_{ACRM} (X', r\Delta t)$, where $\geq_{ACRM}$ means adjusted cumulant ranking measure score function. Then one can show (in the appendix) that $(X, r\Delta t) \geq_{ACRM} (X', r\Delta t) \Leftrightarrow$

$(Y, r) \geq_{ACRM} (Y', r)$. So we establish that $(Y, r) \geq_{ACRM} (Y', r) \Leftrightarrow (Y, r) \geq_u$

(Y', r). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Example 4.2.5. *Suppose utility is CARA and the underlying portfolio risk distributions are infinitely divisible and they satisfy the regularity condition, then adjusted cumulant ranking measure is a valid ranking measure here*

$$\sum_{n=1}^{+\infty} \frac{\nu_n^{Y_1}}{n!} (\sigma_1 a_{N,Y_1})^n > \sum_{n=1}^{+\infty} \frac{\nu_n^{Y_2}}{n!} (\sigma_2 a_{N,Y_2})^n \iff (Y_1, r) \geq_u (Y_2, r)$$

where σ_i is the standard deviation of Y_i, $i = 1, 2$.

Note that this result is also consistent with Davila(2010) 's optimal allocation with higher cumulant. In words, with CARA utility, ranking the Δt component problem with the adjusted cumulant Measure correctly ranks the original investor problem where the investment problem is made over the discrete time length T. Intuitively, the absence of wealth effects with CARA utility means that there is no need for rebalancing. (A formal proof is provided in the Appendix.) The Sharpe Ratio is well defined for a utility function like CARA since the standard Inada condition $(u'(w \to 0) = \infty)$ does not hold. For function forms like CRRA where the Inada condition holds, the demand for a Normally distributed risky asset would be zero due to unlimited liability.

4.3 Understanding the Sharpe Ratio

Suppose that for all $n \geq 3$, $\nu_n = 0$ or $u^{(n)} = 0$. The first order condition (4.2.2) is then

$$\sum_{n=1}^{2} u^{(n)}(w_r)\frac{\nu_n}{(n-1)!}(\sigma a)^{n-1} = 0 \tag{4.3.1}$$

and the 2nd order adjusted cumulant ranking measure implies:

$$\sigma a^* = -\frac{u'(w_r)\nu_1}{u''(w_r)\nu_2} = -\frac{u'(w_r)}{u''(w_r)}\frac{\mu - r}{\sigma}$$

The investors indirect utility is then given by:

$$\max_{a} \mathbb{E}u(w_r + a(X - r))$$

$$= u(w_r) + \left(\sum_{n=1}^{2} u^{(n)}(w_r)\frac{\nu_n}{n!}(\sigma a^*)^n\right)\Delta t + o(\Delta t)$$

$$= u(w_r) + \left(\sum_{n=1}^{2} u^{(n)}(w_r)\frac{\nu_n}{n!}\left(-\frac{u'(w_r)}{u''(w_r)}\frac{\mu - r}{\sigma}\right)^n\right)\Delta t + o(\Delta t)$$

$$= u(w_r) + u'(w_r)\left(-\frac{u'(w_r)}{u''(w_r)}(\frac{\mu - r}{\sigma})^2 + \frac{1}{2}\frac{u'(w_r)}{u''(w_r)}(\frac{\mu - r}{\sigma})^2\right)\Delta t + o(\Delta t)$$

$$= u(w_r) - u'(w_r)\left(\frac{1}{2}\frac{u'(w_r)}{u''(w_r)}(\frac{\mu - r}{\sigma})^2\right)\Delta t + o(\Delta t)$$

$$\approx u\left(w_r - \frac{1}{2}\frac{u'(w_r)}{u''(w_r)}(\frac{\mu - r}{\sigma})^2\Delta t\right)$$

$$= u\left(w_r - \frac{1}{2}\frac{u'(w_r)}{u''(w_r)}\left(\frac{\mu\Delta t - r\Delta t}{\sigma\sqrt{\Delta t}}\right)^2\right)$$

Notice that the Sharpe Ratio of X is exactly $\frac{\mu\Delta t - r\Delta t}{\sigma\sqrt{\Delta t}}$. Hence, the Sharpe ratio is a valid ranking measure.[1]

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[1]Technically, when using perturbation analysis, the ranking measure is the square of the Sharpe Ratio, which implies the Sharpe Ratio when the expected equity premium, $\mu - r$, is positive. For

Remark 4.3.1. *If risk Y is Normally distributed, then $\nu_n = 0$, $n \geq 3$, and so the Sharpe ratio is a valid ranking measure for almost any utility function u compatible with Normally distributed risk, including CARA. Alternatively, if u takes the quadratic form then $u^{(n)} = 0$, $n \geq 3$, and so the Sharpe ratio is a valid ranking across a wide range of risk distributions. However, it is possible that equation (4.3.1) emerges if, for example, $\nu_n = 0$ for odd values of n and $u^{(n)} = 0$ for even values, or some other combination. Hence, it is not generally true that Normally distributed risk or quadratic utility is required for the Sharpe ratio to be a valid ranking measure.*

In fact, in turns out that equation (4.3.1) is not even a *necessary* condition for the Sharpe ratio to be valid. In fact, many of the examples we provided in Section (2) were Sharpe is a correct ranking measure do *not* produce the condition shown in equation (4.3.1) emerges. In other words, an even more general sufficient condition for Sharpe exists, as provided in the following Theorem.

Theorem 4.3.1. *For given utility function u, suppose χ_u is not empty. Then the Sharpe ratio is a valid ranking measure on set χ_u if the higher-order adjusted cumulants of all risks in set χ_u are equal to each other (i.e. $\nu_k(Y) = \nu_k(Y'), \forall Y, Y' \in \chi_u, k \geq 3$), with the odd-numbered cumulants equal to zero (i.e., $\nu_k(Y) = \nu_k(Y') = 0, \forall Y, Y' \in \chi_u, k = 3, 5, 7, ...$).*

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brevity, we won't continue to make this distinction in the discussion below under the assumption that, in equilibrium, risky (maybe composite) securities must pay a risk premium.

Example 4.3.2. *Suppose T is a given symmetric infinitely divisible distribution and construct a new distribution family of the form $\chi_{\alpha_1,\alpha_2} = \{X : X = \alpha_1 + \alpha_2 * T\}$. Then, the adjusted cumulants are $\nu_{2k}(X) = \nu_{2k}(T)$ and $\nu_{2k+1}(X) = \nu_{2k+1}(T) = 0$, $\forall k \geq 1$. Therefore, the Sharpe ratio (squared) is a valid ranking measure on the set χ_{α_1,α_2}.*

While Theorem 4.3.1 provides a broad sufficiency condition for the Sharpe ratio to be a valid ranking measure, producing a generic *necessary* condition turns out to be highly intractable. Mechanically, a necessary condition could be produced implicitly from the investor's first-order conditions, but it provides no real economic insight.

4.4   Generalization of the Skewness Preference

Using a three-moment distribution, Kraus and Litzenberger (1976) well-cited paper demonstrates that investors with cubic utility prefer skewness in their returns.[2] More recently, Peirro and Mosevich (2011) nicely demonstrate that, in the special case of CARA utility, investors dislike kurtosis as well.

Of course, most interesting utility functions have infinite non-zero higher-order terms. Moreover, the case of Normally distributed risk ($\nu_k = 0, k > 2$) turns out to be extremely special. In particular, there are no distributions than can be sufficiently characterized by just adding a few additional higher-order cumulants in

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[2]See also Kane (1982).

order to expand on mean-variance analysis.

Lemma 4.4.1. *There does not exist a random probability distribution for which*
$\nu_m = \nu_{m+1} = ... = 0$ *for some $m > 3$, with the lower-order adjusted cumulants (orders 3 to $m - 1$) being nonzero.*

Hence, it is interesting to consider high-order terms as well. The following result, however, generalizes the previous results.

Theorem 4.4.1. *If $\mu > r$, the adjusted cumulant risk measure (equation 4.2.4) is increasing with respect to odd adjusted cumulants $\nu_3, \nu_5, \cdots$ and decreasing with respect to even adjusted cumulants $\nu_4, \nu_6, \cdots$. If $\mu < r$, the adjusted cumulant risk measure is decreasing with respect to odd adjusted cumulant $\nu_3, \nu_5, \cdots$ and increasing with respect to even adjusted cumulant $\nu_4, \nu_6, \cdots$.*

The previous results, therefore, are special cases.

Corollary 4.4.2. *Suppose $\mu > r$, the investor prefers high skewness and low kurtosis.*

Chapter 5

An Approximation Formula

The papers by Levy and Markowitz (1979) and Kroll, Levy and Markowitz (1984) were influential in suggesting that the investor's expected utility problem could be well approximated by the first two moments, the mean and variance. Of course, these influential papers were written before concerns arose about "fat tail" events and derivatives were being commonly used by risk managers. Later analysis then added some additional moments using linear approximation. This section adds an infinite number of additional moments. The simulation results in following section, however, demonstrates that, even with a large number of additional terms, these sorts of approximations often fail to correctly rank portfolios, are unstable, and can even diverge.

Although it is impossible to get a closed-form formula for equation 4.2.3, we can

solve for linearized closed-form solution. We need to solve

$$\sum_{n=1}^{N} \frac{u^{(n)}(w_r)}{u'(w_r)} \frac{\nu_n}{(n-1)!}(\sigma a)^{n-1} = 0.$$

or

$$\frac{\mu - r}{\sigma} + \frac{u''(w_r)}{u'(w_r)}(\sigma a) + \sum_{n=3}^{N} \frac{u^{(n)}(w_r)}{u'(w_r)} \frac{\nu_n}{(n-1)!}(\sigma a)^{n-1} = 0.$$

The solution σa^* only depends on the coefficients of the polynomial. Denote

$$\sigma a^* = -\frac{u'(w_r)}{u''(w_r)} \frac{\mu - r}{\sigma} + g\left(\frac{u^{(n)}(w_r)}{u'(w_r)}\nu_n; 1 \leq n \leq N\right)$$

The easiest approximation of g as $\nu_n \to 0$ for all $3 \leq n \leq N$ is a linear function of $\{\nu_n; 3 \leq n \leq N\}$. Suppose

$$g\left(\frac{u^{(n)}(w_r)}{u'(w_r)}\nu_n; 1 \leq n \leq N\right) \approx \sum_{n=3}^{N} c_n \frac{u^{(n)}(w_r)}{u'(w_r)}\nu_n$$

i.e.

$$\sigma a^* \approx -\frac{u'(w_r)}{u''(w_r)} \frac{\mu - r}{\sigma} + \sum_{n=3}^{N} c_n \frac{u^{(n)}(w_r)}{u'(w_r)}\nu_n$$

Define $p_n = \frac{u^{(n)}(w_r)}{u'(w_r)}$. Then:

$$\sigma a^* \approx -\frac{1}{p_2}\nu_1 + \sum_{n=3}^{N} \frac{(-\frac{1}{p_2})^n \nu_1^{n-1}}{(n-1)!} p_n \nu_n. \tag{5.0.1}$$

Theorem 5.0.2. *If equation 5.0.1 held with equality, then the investor's expected utility is increasing in the value of*

$$-\frac{\nu_1^2}{2p_2} + \sum_{k=3}^{N} \frac{p_k \nu_k}{k!}\left(-\frac{\nu_1}{p_2}\right)^k \tag{5.0.2}$$

which we will call the "approximate ranking measure."

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Remark 5.0.3. *For HARA utility, the approximate ranking measure becomes:*

$$-\frac{1}{p_2}\left(\frac{\nu_1^2}{2} + \sum_{k=3}^{N} \frac{b_k \nu_1^k \nu_k (-1)^{k-1}}{k!}\right) \tag{5.0.3}$$

- When $N = 2$, we have $-\frac{1}{p_2}\left(\frac{SR^2}{2}\right)$, where SR denotes the Sharpe Ratio, corresponding to the mean-variance framework by Levy and Markowitz (1979) and Kroll, Levy and Markowitz (1984).

- When $N = 3$, we have $\frac{-1}{p_2}\left(\frac{SR^2}{2} + \frac{b_3}{6}SR^3(\nu_3(\Delta t)^{-/2})\right)$. Notice that $\nu_3(\Delta t)^{-/2}$ is the skewness of X. Hence, this formula matches the extension of the mean-variance framework by Zakamouline and Koekebakker (2008) to include skewness.

- When $N = 4$, we have $\frac{-1}{p_2}\left(\frac{SR^2}{2} + \frac{b_3}{6}SR^3 * Skew - \frac{b_4}{24}SR^4 * (Kurt - 3)\right)$, where $Skew$ corresponds to the skewness.

- For $N > 4$, additional terms can be easily computed using equation 5.0.3.

Chapter 6

Applications

6.1 Hodge's (1998) Paradox

Hodge(1998) provided the following example, which he notes produces a paradox.

Probability	0.01	0.04	0.25	0.40	0.25	0.04	0.01
Excess Return of Asset A	-25%	-15%	-5%	5%	15%	25%	35%
Excess Return of Asset B	-25%	-15%	-5%	5%	15%	25%	45%

Clearly asset B first order stochastic dominates asset A. However asset A has a Sharpe ratio of 0.500, whereas asset B has a Sharpe ratio of 0.493. We use our generalized ratio to re-evaluate this paradox. Even with the distribution-only CARA version of our ranking function, the generalized ratio is able to correctly rank Asset B greater than Asset at a value of $N \geq 5$ or more adjusted cumulants (Table (6.1)). Our approximation formula also seems reasonable (Table (6.2)).

Table 6.1: Hodge's Paradox – The Generalized Ratio

N	3	4	5	10
Asset A	NaN	0.1150	0.1172	0.1166
Asset B	NaN	0.1140	0.1190	0.1173

Explanation: Ranking measures for the distribution-only CARA version of the generalized ratio for Hodge's example, where N is the largest adjusted cumulant used in the shown calculation.

Table 6.2: Hodge's Paradox – Approximation Formula

N	3	4	5	10
Asset A	0.1237	0.1227	0.1227	0.1227
Asset B	0.1263	0.1228	0.1239	0.1236

Explanation: Ranking measures for the distribution-only CARA version of the approximate ratio for Hodge's example, where N is the largest adjusted cumulant used in the shown calculation.

6.2 Single Fund Asset Allocation and Ranking

Allocation

We use the generalized ratio to calculate the optimal asset allocation into the S&P500, calculated based on monthly returns from January 1950 to June 2012, versus a risk-free investment paying an annual interest rate $r_f = 5\%$. As noted in Section (3.4.2), in the CRRA case we solve the "two fund separation" problem simultaneously as part of the generating the ranking index. Table 6.3 reports the associated values for the CRRA distribution-preference ranking measure.

Because the values shown in Table 6.3 are produced directly from the CRRA ranking measure, we can "double check" the accuracy of our generalized ranking calculation by also performing simulations on the orginal investor problem (2.1.1). Recall that the generalized ranking measure is calculated based only on the translated moments of the underlying data. For the original investor problem, however, we need to know the actual distribution in order to integrate the expectation operator. Since, we don't have that information, we assume that the "true" distribution is simply given by the histogram of our data and we then sample that data 100,000 times. Of course, this assumption could, in general, produce considerable error because it effectively eliminates the tails of the distribution, which could be especially problematic with non-Normal risk. In the case of broad S&P500 index, however, this effect appears to be small. Table 6.4 shows the results from simulation analysis on the original investor problem. Notice that the results are very close to

Table 6.3: Portfolio Allocation into the S&P500 using the CRRA Generalized Ratio

N	2	3	4	5	6	20
CRRA(1)	147.66%	145.65%	143.35%	143.15%	143.06%	143.03%
CRRA(2)	73.83%	73.07%	72.47%	72.44%	72.43%	72.43%
CRRA(3)	49.22%	48.77%	48.47%	48.46%	48.45%	48.45%
CRRA(4)	36.92%	36.6%	36.41%	36.4%	36.4%	36.4%
CRRA(5)	29.53%	29.29%	29.15%	29.15%	29.15%	29.15%

Explanation: CRRA(X) shows the optimal allocation into the S&P500, as a percentage of wealth, where X is the coefficient of risk aversion

and N is the largest adjusted cumulant used in the shown calculation.

Table 6.4: S&P500 – Simulation Results using the Investor Problem (2.1.1)

optimal allocation

CRRA(1)	143.03%
CRRA(2)	72.41%
CRRA(3)	48.45%
CRRA(4)	36.4%
CRRA(5)	29.15%

Explanation: CRRA(X) shows the distribution-preference CRRA ranking, where X is the coefficient of risk aversion. CRRA results expressed as a percentage of wealth to be invested in the fund.

calculations produced by the Generalized Ratio with $N = 20$.

Table 6.5 also shows the results for the approximate risk measure. Notice that the measure does fairly well at low values of risk aversion, but performs poorly at higher values.

6.2.1   Fund-level Ranking

Sharpe ratios are routinely reported for publicly-traded mutual funds as well as private funds. The purpose of these ratios is to provide investors with guidance about how to rank funds. Mutual funds, hedge funds, and managers, therefore, could easily produce the generalized ratio measure for the CRRA case for a range of risk tolerances, say 2 (aggressive), 3 (moderate) and 5 (conservative), using a

Table 6.5: S&P500 – The Approximate Ratio

N	2	3	4	5	6	20
CRRA(1)	147.42%	143.64%	142.68%	142.47%	142.4%	142.39%
CRRA(2)	73.71%	68.03%	66.13%	65.59%	65.4%	65.34%
CRRA(3)	49.14%	41.57%	38.39%	37.31%	36.88%	36.71%
CRRA(4)	36.86%	27.39%	22.62%	20.74%	19.87%	19.47%
CRRA(5)	29.48%	18.13%	11.45%	8.43%	6.88%	6.07%

Explanation: CRRA(X) shows the optimal allocation into the S&P500, as a percentage of wealth, where X is the coefficient of risk aversion

and N is the largest adjusted cumulant used in the shown calculation.

Table 6.6: Fund Ranking: The S&P500 Index as an Example Fund

$N = 20$	optimal allocation	generalized ranking values
CRRA(1)	143.03%	0.00189
CRRA(2)	72.43%	0.00095
CRRA(3)	48.45%	0.00063
CRRA(4)	36.4%	0.00048
CRRA(5)	29.15%	0.00038

Explanation: Hypothetical rankings for CRRA(X), where X is the coefficient of risk aversion with $N = 20$ adjusted cumulants.

sufficiently large value of N.[1] Table 6.6 shows the values that could be reported for a hypothetical S&P500 indexed fund.

6.3 Multi Asset Class Allocation

Borrowing the example of Goetzmann et al (2002), and subsequently followed by Zakamouline and Koekebakker (2008), we now examine a portfolio with embedded options. The underlying stock follows geometric Brownian motion with today's price normalized to \$1. The price in 1 year is

$$S = \exp\left((\mu - \frac{1}{2}\sigma^2) + \sigma z \right),$$

_____

[1]As discussed in Section 3.4.2, the ranking measure is not monotone in the percent allocations invested into the fund (as shown in Table 6.4). Hence, the percent allocation is not a valid ranking measure

where z is the standard Normal random variable. For simplicity, we use parameters in Zakamouline and Koekebakker (2008): $\mu = 0.10, r_f = 0.05, \sigma = 0.15$. Also, suppose there is a European put option with strike \$0.88 and a European call option with strike \$1.12 that mature in 1 year. Using the Black-Scholes formula, we calculated the non-arbitrage price for the put and call option today are \$0.0079 and \$0.0345 respectively. Denote (a_1, a_2) as the allocation in put and call options, respectively. A positive value denotes buying the option while a negative value means selling (a writer). To compute the optimal multi-asset allocation over the puts and calls, we use a simple grid search with precision of 0.1 along the mesh, thereby essentially ensuring us that our results are not being driven by potential flaws in the global optimization routine.[2]

Our simulations find that $a_1 = -1.3739, a_2 = -0.5807$ maximizes the Sharpe ratio. In other words, the Sharpe ratio suggests a strong amount of put writing combined with a fair amount of call writing. In essence, the investor is short volatility, collecting the insurance premium as the reward. Our generalized ratio, however, suggests to short less put position and less call. In addition, we could match the result where allocation is calculated from the original investor problem.

---

[2]Of course, with more assets, a grid search would quickly run into the "curse of dimensionality" and be computationally impossible.

Table 6.7: (Put, Call) Position: The Generalized Ratio

N	2	3	4	5	6	7	15	20	simulation
CARA	(-1.374, -0.581)	(-1.129,0.902)	(-0.400,-0.284)	(-0.402,-0.229)	(-0.392, -0.241)	(-0.391,-0.240)	(-0.387,-0.237)	(-0.390, -0.239)	(-0.390,-0.239)
CRRA(3)	(-1.374,-0.581)	NaN	(-0.174,-0.270)	(-0.482,0.529)	(-0.138,-0.149)	(-0.120,0.007)	(-0.161,-0.075)	(-0.160,-0.077)	(-0.160,-0.076)
CRRA(4)	(-1.374,-0.581)	(-0.792,0.233)	(-0.226,-0.266)	(-0.317,0.065)	(-0.202,-0.158)	(-0.227,-0.096)	(-0.214,-0.120)	(-0.214,-0.120)	(-0.214,-0.120)
CRRA(5)	(-1.374,-0.581)	(-0.704,0.280)	(-0.259,-0.266)	(-0.310,-0.040)	(-0.240,-0.169)	(-0.254,-0.133)	(-0.247,-0.145)	(-0.247,-0.145)	(-0.247,-0.145)

Explanation: The optimal (put, call) allocation. For CRRA(X), X is the coefficient of risk aversion. For example, at CRRA $= 3$ and $N = 20$, premium collecting from the shorting put equals $0.0079·0.16 = 0.00126 while premiums collected on the call is $0.0345·0.077 = 0.00266, for a total of $0.00392. The last column, which labeled as simulation, is the optimal allocation we get if we directly used maximized expected utility using Monte Carlo simulation. Ingersoll et al (2007) develop a manager manipulation-proof measure around the $(0,0)$ position. In contrast, we focus on the allocation that maximizes the investor expected utility problem, which will typically produces some insurance demand with fairly priced options traded in discrete time.

Chapter 7

Conclusions

This paper derives a generalized ranking measure which, under a regularity condition, is valid in the presence of a much broader assumption (utility, probability) space than the Sharpe ratio and yet preserves wealth separation for the broad HARA utility class. Our ranking measure can be used with "fat tails" as well as multi-asset class portfolio optimization. We also explore the foundations of asset ranking, including proving a key impossibility theorem: any ranking measure that is valid at non-Normal "higher moments" cannot generically be free from investor preferences. Our simulation analysis demonstrates that the generalized ratio often produces very different optimal portfolios relative to Sharpe, especially for multi-asset portfolios where the assumption of Normality breaks down. The generalized ranking measure, therefore, can be used by investors and money managers, and could replace the numerous other measures that they are currently using for ranking portfolios.

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Appendix

Theorem 2.3.1

Denote that $Z_\alpha = \frac{Y_\alpha - \mathbb{E}Y_\alpha}{\sqrt{Var(Y_\alpha)}}$, by our assumption Z_α doesn't depend on α, thus we can ignore the subscript: $Z = Z_\alpha$. The FOC of the maximization problem is

$$\mathbb{E}u'(w(1+r) + a^*(Y_\alpha - r))(Y_\alpha - r) = 0$$

in terms of Z:

$$\mathbb{E}u'(w(1+r) + a^*(Z\sqrt{Var(Y_\alpha)} + \mathbb{E}Y_\alpha - r))(Z\sqrt{Var(Y_\alpha)} + \mathbb{E}Y_\alpha - r) = 0$$

i.e.

$$\mathbb{E}u'\left(w(1+r) + a^*\sqrt{Var(Y_\alpha)}(Z + \frac{\mathbb{E}Y_\alpha - r}{\sqrt{Var(Y_\alpha)}})\right)(Z + \frac{\mathbb{E}Y_\alpha - r}{\sqrt{Var(Y_\alpha)}}) = 0$$

Given information about u, w, r, the solution $a^*\sqrt{Var(Y_\alpha)}$ only depends on $\frac{\mathbb{E}Y_\alpha - r}{\sqrt{Var(Y_\alpha)}}$ and Z. Since we assume Z doesn't depend on parameters α, we can write $a^*\sqrt{Var(Y_\alpha)}$ as a function of $\frac{\mathbb{E}Y_\alpha - r}{\sqrt{Var(Y_\alpha)}}$. Let us assume

$$a^*\sqrt{Var(Y_\alpha)} = g\left(\frac{\mathbb{E}Y_\alpha - r}{\sqrt{Var(Y_\alpha)}}\right)$$

Then

$$\max_a \mathbb{E}u(w(1+r) + a(Y_\alpha - r)) = \mathbb{E}u(w(1+r) + a^*(Y_\alpha - r))$$

$$= \mathbb{E}u(w(1+r) + a^*(Z\sqrt{Var(Y_\alpha)} + \mathbb{E}Y_\alpha - r))$$

$$= \mathbb{E}u\left(w(1+r) + a^*\sqrt{Var(Y_\alpha)}(Z + \frac{\mathbb{E}Y_\alpha - r}{\sqrt{Var(Y_\alpha)}})\right)$$

$$= \mathbb{E}u\left(w(1+r) + g\left(\frac{\mathbb{E}Y_\alpha - r}{\sqrt{Var(Y_\alpha)}}\right)(Z + \frac{\mathbb{E}Y_\alpha - r}{\sqrt{Var(Y_\alpha)}})\right)$$

$$= f_{(u,w,r)}\left(\frac{\mathbb{E}Y_\alpha - r}{\sqrt{Var(Y_\alpha)}}\right)$$

In addition, since Z is symmetric, if Y_1 and Y_2 produce opposite value of Sharpe

ratio, i.e.

$$\frac{\mathbb{E}Y_1 - r}{\sqrt{Var(Y_{1)}}} = -\frac{\mathbb{E}Y_2 - r}{\sqrt{Var(Y_{2)}}}$$

The optimal allocation in these two cases would be opposite too, i.e.

$$a_1^*\sqrt{Var(Y_1)} = -a_2^*\sqrt{Var(Y_2)}$$

so $f_{(u,w,r)}$ is an even function. Let $\bar{f}(x) = f(x^2)$, clearly that $\bar{f}$ is an increasing function. Thus $\max_a \mathbb{E}u\left(w(1 + r_f) + a(Y - r_f)\right)$ is an increasing function of $\left(\frac{\mathbb{E}Y - r_f}{\sqrt{Var(Y)}}\right)^2$.

Proof of Lemma 3.1.2

On the complex plan, we can draw a small circle Γ around x_0 so that f have unique complex solution x_0 on Γ. Denote $\gamma = \partial\Gamma$ is the boundary of Γ. By Cauchy's

Theorem, we have

$$\frac{1}{2\pi i} \oint_\gamma \frac{f'}{f}\, dz = 1$$

We have

$$\frac{f'_N}{f_N} \to \frac{f'}{f}$$

and the fact that $\frac{1}{2\pi i} \oint_\gamma \frac{f'_N}{f_N}\, dz$ is always an integer, we conclude that for sufficient large N

$$\frac{1}{2\pi i} \oint_\gamma \frac{f'_N}{f_N}\, dz = 1$$

In other word, f_N has unique solution on Γ. In addition, f_Ns are polynomials with real coefficients, we conclude that this root is a real number as real polynomials has conjugate complex roots.

Now we show that the unique solution of f_N on Γ is the smallest absolute root of f_N on the complex plane. We show by contradiction, i.e. for any N, there is $n > N$ so that f_n has root that has smaller absolute value. First, they are uniform bounded, say $2|x_0|$. Then by bounded convergent subsequence, we know that we can find a convergent subsequence and it must converge to x_0. This is an contradiction because those points are outside Γ, meaning they have an positive distance from x_0, resulting in impossibility of converging to x_0.

Proof of theorem 3.3.1

Proof. We show by contradiction. Consider

$$
Y = \begin{cases} k\% & \text{w.p. } p \\ \\ -1\% & \text{w.p. } 1-p \end{cases}
$$

Suppose investor's utility function is $u(w) = \frac{w^{1-\gamma}}{1-\gamma}$. Without loss of generality, assume initial wealth $w_0 = 1$, then investor solves following problem

$$
\max_a \mathbb{E}u(1 + aY) = pu(1 + ak) + (1-p)u(1 - 0.01a)
$$

FOC gives

$$
pku'(1 + ak/100) = (1-p)u'(1 - 0.01a)
$$

So we have

$$
a^* = \frac{100(1 - (\frac{pk}{1-p})^{-1/\gamma})}{(1 + k * \frac{pk}{1-p}) - 1/\gamma)}
$$

The maximal value is then $\mathbb{E}u(1 + a^*Y) = pu(1 + a^*k) + (1-p)u(1 - 0.01a^*)$. Specifically, consider following example

$$
Y_1 = \begin{cases} 1.6\% & \text{w.p. } 0.77 \\ \\ -1\% & \text{w.p. } 0.23 \end{cases}
$$

$$
Y_2 = \begin{cases} 1.3\% & \text{w.p. } 0.81 \\ \\ -1\% & \text{w.p. } 0.19 \end{cases}
$$

Investors A and B are both CRRA with $\rho = 2$ and $\rho = 100$, respectively. Then

Investor	Y_1	Y_2	ranking
A	-0.8472	-0.8485	$Y_1 > Y_2$
B	-0.00722	-0.00717	$Y_1 < Y_2$

This implies there can not have distribution only ranking measure as if there is, we should expect investor A and B has same preference over Y_1 and Y_2. $\qquad\square$

Adjusted Moments

General

In general, for random variable Y, we denote μ_1 the mean and for $k \geq 2$,

$$\mu_k = \mathbb{E}(Y - \mathbb{E}Y)^k \text{ and } \xi_k = \frac{\mu_k}{\mu_2^{\frac{k}{2}}}$$

So ξ_3 represents the skewness and ξ_4 the kurtosis. We write S, K to represent skewness and kurtosis for simplicity. We also write $\mu_2 = \sigma^2$. We have

- $\nu_3 = \xi_3$

- $\nu_4 = \xi_4 - 3$

- $\nu_5 = \xi_5 - 5\xi_3$

- $\nu_6 = \xi_6 - \frac{15}{2}(\xi_4 - 3) - 10\xi_3^2 - 15 = \xi_6 - \frac{15}{2}\nu_4 - 10\nu_3^2 - 15$

- $\nu_7 = \xi_7 - \frac{21}{2}(\xi_5 - 5S) - \frac{35}{2}(K-3)S - \frac{175}{3}S = \xi_7 - \frac{21}{2}\nu_5 - \frac{35}{2}\nu_4\nu_3 - \frac{175}{3}\nu_3$

For integer n, there are numbers of ways to write it as sum of positive integers that greater than 1. For example, we can write

- $7=7$

- $7=5+2$

- $7=4+3$

- $7=3+2+2$

Those four ways to decomposing 7 matches the terms in ν_7 (noting that $\nu_2 = 1$). For a particular decomposition of n

$$n = i_1 + i_2 + \cdots + i_k$$

where $i_1 \geq i_2 \cdots \geq i_k \geq 2$. For $k \geq 2$, there is a corresponding term in ν_n that is $\nu_{i_1} \cdots \nu_{i_k}$ and the coefficient is

$$\frac{\binom{n}{i_1}\binom{n-i_1}{i_2}\cdots\binom{n-i_1-\cdots-i_{k-1}}{i_k}}{k!}$$

Thus, we conclude that

$$\nu_n = \xi_n - \sum_{\substack{n = i_1 + i_2 + \cdots + i_k \\ i_1 \geq i_2 \cdots \geq i_k \geq 2}} \frac{\binom{n}{i_1}\binom{n-i_1}{i_2}\cdots\binom{n-i_1-\cdots-i_{k-1}}{i_k}}{k!}\nu_{i_1}\cdots\nu_{i_k}$$

Indeed let $Y = X_1 + X_2 + \cdots + X_m$, where X_i are i.i.d. Let's denote X for simplicity. Expand

$$\mathbb{E}(Y - \mathbb{E}Y)^n$$

$$=\mathbb{E}(X_1 + X_2 + \cdots + X_m - \mathbb{E}(X_1 + X_2 + \cdots + X_m))^n$$

$$=\mathbb{E}((X_1 - \mathbb{E}X_1) + (X_2 - \mathbb{E}X_2) + \cdots + (X_m - \mathbb{E}X_m))^n$$

$$= \sum_{i_1+i_2+\cdots+i_m=n} \mathbb{E}(X_1 - \mathbb{E}X_1)^{i_1} \cdots (X_m - \mathbb{E}X_m)^{i_m}$$

$$=m\mathbb{E}(X - \mathbb{E}X)^n + \sum \mathbb{E}(X_1 - \mathbb{E}X_1)^{i_1} \cdots (X_m - \mathbb{E}X_m)^{i_m}$$

$$i_1 + i_2 + \cdots + i_m = n$$

$$2 \leq i_k < n, \ or \ i_k = 0$$

Infinitely divisible distribution

Suppose X_t is Levy Process, $Y = X_1$ then Y is infinitely divisible distribution with $X = X_{\frac{1}{n}}$. By Levy-Khinchine representation, we have

$$\mathbb{E}e^{i\theta X_t} = exp\left(bit\theta - \frac{1}{2}\sigma_0^2 t\theta^2 + t \int_{\mathbb{R}\backslash\{0\}} (e^{i\theta x} - 1 - i\theta x I_{|x|<1})W(dx) \right)$$

where $b \in \mathbb{R}$, and I is the indicator function. The Levy measure W must be such that

$$\int_{\mathbb{R}\backslash\{0\}} min\{x^2, 1\}W(dx) < \infty$$

Denote

$$\phi(\theta, t) = bit\theta - \frac{1}{2}\sigma_0^2 t\theta^2 + t \int_{\mathbb{R}\backslash\{0\}} (e^{i\theta x} - 1 - i\theta x I_{|x|<1})W(dx)$$

and

$$\psi(\theta, t) = bt\theta + \frac{1}{2}\sigma_0^2 t\theta^2 + t\int_{\mathbb{R}\backslash\{0\}} (e^{\theta x} - 1 - \theta x I_{|x|<1})W(dx)$$

i.e $\phi(\theta, t) = \psi(i\theta, t)$ and $e^{\psi(\theta,t)}$ is the moments generating function of X_t.

Suppose $e^{\psi(\theta,t)}$ is the moments generating function of Levy Process X_t, let $Y = X_1$, and σ is the standard deviation of Y, then we have

$$\nu_k(Y) = \frac{\frac{\partial^k \psi(\theta,1)}{\partial \theta^k}\big|_{\theta=0}}{\sigma^k}, \forall k \geq 2.$$

Relationship with cumulant

Since $e^{\psi(\theta,t)} = \mathbb{E}e^{\theta X_t}$, then the cumulant-generating function

$$g(\theta, t) = \log(\mathbb{E}e^{\theta X_t}) = \psi(\theta, t).$$

Denote k_n the n-th cumulant. Then

$$k_n = \frac{\partial g^n(\theta, 1)}{\partial \theta^n}\big|_{\theta=0} = \frac{\partial \psi^n(\theta, 1)}{\partial \theta^n}\big|_{\theta=0}.$$

So for $n \geq 2$,

$$k_n = \nu_n \sigma^n$$

Verification of Eq 3.2.3

For HARA utility $u(w) = \frac{\rho}{1-\rho}\left(\frac{\lambda w}{\rho} + \phi\right)^{1-\rho}$,

$$\max_a \mathbb{E}u\left(w(1+r_f) + a(Y - r_f)\right)$$

$$= u(w_r) + \lambda\left(\frac{\lambda w_r}{\rho} + \phi\right)^{-\rho}\sum_{n=1}^{\infty}(\rho)\cdots(\rho + n - 2)\frac{t_n^Y}{(n)!}\left(-\frac{\lambda}{\rho}\frac{a}{\frac{\lambda w_r}{\rho} + \phi}\right)^n$$

$$= u(w_r) - \lambda\left(\frac{\lambda w_r}{\rho} + \phi\right)^{-\rho}\sum_{n=1}^{\infty}\frac{b_n t_n^Y}{(n)!}z^n$$

So $q_H(t_n^Y, b_n) = -\sum_{n=1}^{N}\frac{b_n t_n^Y}{n!}z_{N,Y}^n$ is the ranking function.

Proof of Theorem 4.3.1

Proof. Suppose $\nu_n = c_n, \forall n \geq 3$. We need to solve

$$\sum_{n=1}^{2}u^{(n)}(w)\frac{\nu_n}{(n-1)!}(\sigma a)^{n-1} + \sum_{n=4,even}^{\infty}u^{(n)}(w)\frac{c_n}{(n-1)!}(\sigma a)^{n-1} = 0$$

Suppose we have $\sigma a = g(\nu_1)$. Then the let PM be the corresponding ranking furnction

$$PM = \sum_{n=1}^{2}u^{(n)}(w)\frac{\nu_n}{n!}(g(\nu_1))^n + \sum_{n=4,even}^{\infty}u^{(n)}(w)\frac{c_n}{(n-1)!}(g(\nu_1))^n$$

Then we have

$$\frac{\partial PM}{\partial \nu_1} = u'(w)g(\nu_1) + g'(\nu_1)\sum_{n=1}^{\infty}u^{(n)}(w)\frac{\nu_n}{(n-1)!}(\sigma a)^{n-1} = u'(w)g(\nu_1)$$

It it positive whenever ν_1 is poisitive and negative when ν_1 is negative. In addition, the measure is symmetric because $PM(\nu_1) = PM(-\nu_1)$. So Sharpe ratio is valid. $\qquad\square$

Theorem 4.4.1

We use chain rule:

$$\frac{\partial \sum_{n=1}^{N} \frac{u^{(n)}(w_r)}{u'(w_r)} \frac{\nu_n}{n!} (\sigma a^*)^n}{\partial \nu_i}$$

$$= \frac{\partial \sum_{n \neq i} \frac{u^{(n)}(w_r)}{u'(w_r)} \frac{\nu_n}{n!} (\sigma a^*)^n}{\partial \nu_i} + \frac{\partial \frac{u^{(i)}(w_r)}{u'(w_r)} \frac{\nu_i}{i!} (\sigma a^*)^i}{\partial \nu_i}$$

$$= \sum_{n \neq i} \frac{u^{(n)}(w_r)}{u'(w_r)} \frac{\nu_n}{n!} \frac{\partial (\sigma a^*)^n}{\partial \nu_i} + \frac{u^{(i)}(w_r)}{u'(w_r)} \frac{1}{i!} (\sigma a^*)^i + \frac{u^{(i)}(w_r)}{u'(w_r)} \frac{\nu_i}{i!} \frac{\partial (\sigma a^*)^i}{\partial \nu_i}$$

$$= \sum_{n=1}^{N} \frac{u^{(n)}(w_r)}{u'(w_r)} \frac{\nu_n}{n!} \frac{\partial (\sigma a^*)^n}{\partial \nu_i} + \frac{u^{(i)}(w_r)}{u'(w_r)} \frac{1}{i!} (\sigma a^*)^i$$

$$= \sum_{n=1}^{N} \frac{u^{(n)}(w_r)}{u'(w_r)} \frac{\nu_n}{n!} n (\sigma a^*)^{n-1} \frac{\partial (\sigma a^*)}{\partial \nu_i} + \frac{u^{(i)}(w_r)}{u'(w_r)} \frac{1}{i!} (\sigma a^*)^i$$

$$= \left(\sum_{n=1}^{N} \frac{u^{(n)}(w_r)}{u'(w_r)} \frac{\nu_n}{(n-1)!} (\sigma a^*)^{n-1} \right) \frac{\partial (\sigma a^*)}{\partial \nu_i} + \frac{u^{(i)}(w_r)}{u'(w_r)} \frac{1}{i!} (\sigma a^*)^i$$

$$= \frac{u^{(i)}(w_r)}{u'(w_r)} \frac{1}{i!} (\sigma a^*)^i$$

Use lemma 2.1 and the fact that u has positive odd derivatives and negative even derivatives, we can easily get the conclusion:

- If $\mu > r$, it is increasing with respect to odd adjusted cumulants $\nu_3, \nu_5, \cdots$ and decreasing with respect to even adjusted cumulants $\nu_4, \nu_6, \cdots$.

- If $\mu < r$, it is decreasing with respect to odd adjusted cumulant $\nu_3, \nu_5, \cdots$ and increasing with respect to even adjusted cumulant $\nu_4, \nu_6, \cdots$.

Proof of scaling property of adjusted cumulant measurement

By Levy-Khinchine representation, we have in general

$$\nu_n^X = \nu_n^Y k^{\frac{n-2}{2}}$$

Now look at the equations which solve a^*. a^{*Y} solves

$$\sum_{n=1}^{N} u^{(n)}(w) \frac{\nu_n^Y}{(n-1)!} (\sigma a)^{n-1} = 0 \tag{.0.1}$$

while a^{*X} solves

$$\sum_{n=1}^{N} u^{(n)}(w) \frac{\nu_n^X}{(n-1)!} (\sigma a)^{n-1} = 0 \tag{.0.2}$$

Since $\nu_n^X = \nu_n^Y k^{\frac{n-2}{2}}$, so (.0.2) is equivalent to

$$\sum_{n=1}^{N} u^{(n)}(w) \frac{\nu_n^Y k^{\frac{n-2}{2}}}{(n-1)!} (\sigma a)^{n-1} = 0$$

i.e.

$$k^{-\frac{1}{2}} \sum_{n=1}^{N} u^{(n)}(w) \frac{\nu_n^Y}{(n-1)!} (\sigma a \sqrt{k})^{n-1} = 0$$

Compared it to (.0.1), we have

$$a^{*Y} = a^{*X} \sqrt{k}$$

Therefore, the left hand side of theorem

$$\sum_{n=1}^{N} \frac{u^{(n)}(w)}{u'(w)} \frac{\nu_n^{X_{k,1}}}{n!} (\sigma a^{*X_{k,1}})^n$$

$$= \sum_{n=1}^{N} \frac{u^{(n)}(w)}{u'(w)} \frac{\nu_n^Y k^{\frac{n-2}{2}}}{n!} (\sigma a^{*Y} k^{-\frac{1}{2}})^n$$

$$= \frac{1}{k} \sum_{n=1}^{N} \frac{u^{(n)}(w)}{u'(w)} \frac{\nu_n^Y}{n!} (\sigma a^{*Y})^n$$

Or

$$PM(X_{k,1}, \frac{r_f}{k}) = \frac{1}{k} PM(Y, r_f)$$

where

$$PM(Y, r_f) = \sum_{n=1}^{N} \frac{u^{(n)}(w)}{u'(w)} \frac{\nu_n^Y}{n!} (\sigma a^{*Y})^n$$

Therefore, suppose we have infinitely divisible distributions Y and Y' in χ_m^N, we

must have the following equivalence:

$$PM(Y, r_f) \geq PM(Y', r_f) \Leftrightarrow PM(X_{k,1}, \frac{r_f}{k}) \geq PM(X'_{k,1}, \frac{r_f}{k}), \forall k$$

Example 4.2.5

Suppose X_t is Levy Process, $Y = X_1$ then Y is infinitely divisible distribution with

$X = X_{\frac{1}{n}}$. By Levy-Khinchine representation, we have

$$\mathbb{E}e^{i\theta X_t} = exp \left(bit\theta - \frac{1}{2}\sigma_0^2 t\theta^2 + t \int_{\mathbb{R}\backslash\{0\}} (e^{i\theta x} - 1 - i\theta x I_{|x|<1}) W(dx) \right)$$

where $b \in \mathbb{R}$, and I is the indicator function. The Levy measure W must be such

that

$$\int_{\mathbb{R}\backslash\{0\}} min\{x^2, 1\} W(dx) < \infty$$

Denote

$$\phi(\theta, t) = bit\theta - \frac{1}{2}\sigma_0^2 t\theta^2 + t \int_{\mathbb{R}\backslash\{0\}} (e^{i\theta x} - 1 - i\theta x I_{|x|<1})W(dx)$$

and

$$\psi(\theta, t) = bt\theta + \frac{1}{2}\sigma_0^2 t\theta^2 + t \int_{\mathbb{R}\backslash\{0\}} (e^{\theta x} - 1 - \theta x I_{|x|<1})W(dx)$$

i.e $\phi(\theta, t) = \psi(i\theta, t)$ and $e^{\psi(\theta,t)}$ is the moments generating function of X_t. From

Theorem 3.5 we have,

$$\nu_k = \lim_{n\to\infty} \mathbb{E}\epsilon^k n^{-k/2+1} = \frac{\mathbb{E}X_{\frac{1}{n}}^k}{\sigma^k}n = \frac{\frac{\partial^k \psi(\theta,1)}{\partial\theta^k}|_{\theta=0}}{\sigma^k}$$

Now suppose investor has CARA utility and as we mention before the risk

aversion doesn't matter because of $b_n = 1$, so for simplicity, we assume $u(w) =$

$-e^{-w}$.

$$a^* = arg\max_a \mathbb{E} - e^{-(w_r + a(Y-r))}$$

$$= arg\max_a \mathbb{E} - e^{-w_r + ar - aY}$$

$$= arg\max_a \mathbb{E} - e^{-w_r + ar}e^{-aY}$$

$$= arg\max_a -e^{-w_r + ar}\mathbb{E}e^{-aY}$$

$$= arg\max_a -e^{-w_r + ar}e^{\psi(-a,1)}$$

$$= arg\min_a e^{ar + \psi(-a,1)}$$

$$= arg\min_a ar + \psi(-a, 1)$$

i.e. a^* is the unique solution of

$$r = \psi'(-a, 1)$$

For adjusted cumulants performance measurement, we want to solve

$$\sum_{n=1}^{N} u^{(n)}(w_r) \frac{\nu_n}{(n-1)!} (\sigma a)^{n-1} = 0$$

Plug in $u(w) = -e^{-w}$ and $\nu_k = \frac{\frac{\partial^k \psi(\theta,1)}{\partial \theta^k}|_{\theta=0}}{\sigma^k}$ for $k \geq 2$ we get

$$\frac{\mu - r}{\sigma} + \sum_{n=2}^{N} \frac{\frac{\frac{\partial^n \psi(\theta,1)}{\partial \theta^n}|_{\theta=0}}{\sigma^n}}{(n-1)!} (-\sigma a)^{n-1} = 0$$

i.e.

$$\mu - r + \sum_{n=1}^{N-1} \frac{\frac{\partial^{n+1} \psi(\theta,1)}{\partial \theta^{n+1}}\Big|_{\theta=0}}{n!} (-a)^n = 0$$

Notice that $\mu = \frac{\partial \psi(\theta,1)}{\partial \theta}\Big|_{\theta=0}$, i.e.

$$r = \sum_{n=0}^{N-1} \frac{\frac{\partial^{n+1} \psi(\theta,1)}{\partial \theta^{n+1}}\Big|_{\theta=0}}{n!} (-a)^n$$

By Taylor expansion we knows that the right hand side approaches $\psi'(-a, 1)$. From Lemma 2.3, we have $a_N^* \to a^*$.

Approximation formula 5.0.1 and 5.0.2

We have

$$\sigma a^* \approx -\frac{u'(w_r)}{u''(w_r)} \frac{\mu - r}{\sigma} + \sum_{n=3}^{N} c_n \frac{u^{(n)}(w_r)}{u'(w_r)} \nu_n$$

Plug into the first order condition and only keeps terms of ν_n and ignore higher terms:

$$\frac{u''(w_r)}{u'(w_r)} \sum_{n=3}^{N} c_n \frac{u^{(n)}(w_r)}{u'(w_r)} \nu_n + \sum_{n=3}^{N} \frac{u^{(n)}(w_r)}{u'(w_r)} \frac{\nu_n}{(n-1)!} \left(-\frac{u'(w_r)}{u''(w_r)} \frac{\mu - r}{\sigma} \right)^{n-1} = 0.$$

Since $p_n = \frac{u^{(n)}(w_r)}{u'(w_r)}$, then

$$p_2 \sum_{n=3}^{N} c_n p_n \nu_n + \sum_{n=3}^{N} p_n \frac{\nu_n}{(n-1)!} \left(\left(-\frac{1}{p_2} \frac{\mu - r}{\sigma} \right)^{n-1} \right) = 0.$$

This implies that

$$c_n = \frac{\left(-\frac{1}{p2} \right)^n \left(\frac{\mu - r}{\sigma} \right)^{n-1}}{(n-1)!}$$

So

$$\sigma a^* = -\frac{1}{p_2} \frac{\mu - r}{\sigma} + \sum_{n=3}^{N} \frac{\left(-\frac{1}{p2} \right)^n \left(\frac{\mu - r}{\sigma} \right)^{n-1}}{(n-1)!} p_n \nu_n.$$

Now we plug in σa^* to the expected utility

$$\max_a \mathbb{E} u(w_r + a(X - r))$$

$$= u(w_r) + \left(\sum_{n=1}^{N} u^{(n)}(w_r) \frac{\nu_n}{n!} (\sigma a^*)^n \right) \Delta t + o(\Delta t)$$

$$= u(w_r) + u'(w_r) \left(\sum_{n=1}^{N} p_n \frac{\nu_n}{n!} \left(-\frac{1}{p_2} \frac{\mu - r}{\sigma} + \sum_{k=3}^{N} \frac{\left(-\frac{1}{p2} \right)^k \left(\frac{\mu - r}{\sigma} \right)^{k-1}}{(k-1)!} p_k \nu_k \right)^n \right) \Delta t + o(\Delta t)$$

$$\approx u \left(w_r + \sum_{n=1}^{N} p_n \frac{\nu_n}{n!} \left(-\frac{1}{p_2} \frac{\mu - r}{\sigma} + \sum_{k=3}^{N} \frac{\left(-\frac{1}{p2} \right)^k \left(\frac{\mu - r}{\sigma} \right)^{k-1}}{(k-1)!} p_k \nu_k \right)^n \Delta t \right)$$

then we have the approximation performance measure in Nth moments given by

$$-\frac{\left(\frac{\mu - r}{\sigma} \right)^2}{2p_2} \Delta t + \sum_{k=3}^{N} \frac{p_k \nu_k}{k!} \left(-\frac{\mu - r}{p_2 \sigma} \right)^k \Delta t$$

To show the scaling property, we define $b_n = \frac{p_n}{p_2^{n-1}}$ distribute the Δt, then we have

$$-\frac{(\frac{\mu-r}{\sigma})^2}{2p_2}\Delta t + \sum_{k=3}^{N}\frac{p_k\nu_k}{k!}(-\frac{\mu-r}{p_2\sigma})^k\Delta t$$

$$= -\frac{(\frac{\mu-r}{\sigma}\sqrt{\Delta t})^2}{2p_2} + \sum_{k=3}^{N}\frac{(-1)^k b_k\nu_k}{k!p_2}(\frac{\mu-r}{\sigma}\sqrt{\Delta t})^k(\Delta t)^{1-k/2}$$

$$= -\frac{(\frac{\mu-r}{\sigma}\sqrt{\Delta t})^2}{2p_2} + \frac{1}{p_2}\sum_{k=3}^{N}\frac{(-1)^k b_k}{k!}SR^k(\nu_k(\Delta t)^{1-k/2})$$

$$= -\frac{SR^2}{2p_2} + \frac{1}{p_2}\sum_{k=3}^{N}\frac{(-1)^k b_k}{k!}SR^k(\nu_k(\Delta t)^{1-k/2})$$

$$= \frac{-1}{p_2}\left(\frac{SR^2}{2} + \sum_{k=3}^{N}\frac{(-1)^{k-1}b_k}{k!}SR^k(\nu_k(\Delta t)^{1-k/2})\right)$$

$$\approx \frac{-1}{p_2}\left(\frac{SR^2}{2} + \sum_{k=3}^{N}\frac{(-1)^{k-1}b_k}{k!}SR^k\mathbb{E}\epsilon^k\right)$$

Regularity condition in numerical example

For CARA and discrete distribution, no need to worry as the convergence radius is always ∞. For HARA, the regularity condition is not always true.

[CARA with discrete distribution] Suppose utility is CARA and the underlying distribution is a discrete distribution then the Taylor expansion has infinite convergence radius, i.e. regularity condition always holds in this case. To prove, suppose the underlying distribution is characterized by

$$\{x_1, p_1; x_2, p_2, ; \cdots ; x_m, p_m\}$$

and $A = \max\{|x_i - r_f|\} = |x_j - r_f|$. Then one can show that the t-moments $t_n \approx A^n$ for sufficient large n for the reason below. Then using familiar convergence radius

formula, we obtain convergent radius

$$\limsup_{k} k/A = \infty$$

To calculate t_n, we have

$$p_j A^n \leq t_n = \sum_i p_i (x_i - r_f)^n \leq \sum_i p_i A^n = A^n$$