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Abstract
A model space for the Virasoro group is constructed and some remarks on its properties are given. Presented by Philip C. Nelson at Texas A&M Superstring Workshop, March 12-17, 1990.

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Remarks on Virasoro Model Space*

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ABSTRACT

A model space for the Virasoro group is constructed and some remarks on its properties are given.

The structures of rational conformal field theories (CFT) have been fairly well studied[1], but beyond that not many things are known yet. In search of a framework that can be used to investigate even the non-rational CFT's, one may be tempted to study the promising relationship between the CFT and the representation theory of Virasoro algebra more seriously. Here we will briefly describe some steps taken in this direction taken in [2]; see also [3].

Since Kirillov-Kostant's method of orbits using geometric quantization has been successful to produce the representations of some noncompact groups, it is natural to attempt this method for the Virasoro group and we can further investigate how the representations form the Hilbert space of any CFT. But the actual construction is still elusive.

One of the main difficulties is that we do not know how to quantize the individual coadjoint orbits of Virasoro. In particular, the geometrical meaning of the appearance of degenerate Verma modules for the discrete minimal CFT's is

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a complete mystery. We may speculate that all such difficulties are due to the fact that a proper phase space is not correctly identified and individual coadjoint orbits may not be proper spaces. From the Kac determinant formula we already know that once the central charge c is given, a collection of highest weights are automatically specified. Thus somehow we could expect that not just a single Verma module but a set of them will appear at the same time if we have a nice representation theory. Besides, this is a desirable structure of CFT. But each coadjoint orbit can at most provide one Verma module, thus we should enlarge the phase space to accommodate a sufficient number of coadjoint orbits, presumably infinitely many for irrational CFT's. Furthermore, for $c < 1$ the unitary Hilbert spaces for CFT are not the Verma modules but their quotients so that again a single coadjoint orbit seems not to be good enough.

As proposed by Alekseev and Shatashvili one may try to use a somewhat larger space in which all the possible coadjoint orbits are contained[4]. In fact, for some compact groups such a larger space has been known for a long time. For example, consider SU(2); then the coadjoint orbits are $S^2 \simeq SU(2)/U(1)$, but there is a larger space $A \equiv SL(2, \mathbf{C})/N_+$, the principal affine space of $SL(2, \mathbf{C})$, where N_+ is the maximal unipotent subgroup. The following is satisfied for the complexified maximal torus T_c of $SL(2, \mathbf{C})$:

$$A \equiv SL(2, \mathbf{C})/N_+ \simeq SU(2)/U(1) \times T_c.$$

One remarkable property of such a space A is that the space of holomorphic functions on A which are square integrable with proper weights provides the unitary representation containing every irreducible one exactly once. Such a construction is due to Segal and Bargmann.

Gelfand generalized the above construction for arbitrary SU(N) and conjectured that the unitary representation which appears in this way also contains every irreducible representation exactly once[5]. Gelfand also coined the term "model space" for a space that has a property that the derived unitary representation is a direct product of irreducible ones with multiplicity exactly one.

The generalization for the Virasoro group at first sight is not obvious because the model space $A = G_c/N_+$ requires the existence of the complexification G_c of the given group G; as is well known Virasoro has no complexification. But, carefully observing the relation between the coadjoint orbit and the model space for a compact group, we can construct a space with all the required properties anyway.

Note that $\dim_{\mathbf{c}} G_c/N_+ = \frac{1}{2}(\dim G + \mathrm{rank} G)$ and $\dim_{\mathbf{c}} G/T = \frac{1}{2}(\dim G - \mathrm{rank} G)$.

Thus

$$\dim_{\mathbf{c}} A - \dim_{\mathbf{c}} G/T = \mathrm{rank} G.$$

If there exists any model space $\mathcal{A}$ of the Virasoro group, presumably it would satisfy

$$\text{“}\dim_{\mathbf{R}} \mathcal{A} - \dim_{\mathbf{R}} \mathit{Diff}\ S^1/S^1 = 2\ \mathrm{rank} \mathit{Diff}\ S^1 S^1 = 2.''$$

This relation is formal because both terms on the LHS are infinite. Since we have an analytic realization of *Diff* S^1/S^1 by Kirillov and Yur'ev[6] as

$$\mathit{Diff}\ S^1/S^1 = \{f : f(0) = 0, \quad f'(0) = 1\}, \tag{1}$$

we can enlarge by complex dimension 1 to

$$\mathcal{A} = \{f : f(0) = 0\}. \tag{2}$$

In both cases f is a holomorphic function on the unit disk $D = \{: z :< 1\}$, continuous and univalent up to the boundary.

This space $\mathcal{A}$ turns out to be isomorphic to *Diff* $S^1 \times \mathbf{R}_+ = \{(\gamma, s)\}$, and it has a natural complex structure. The symplectic form on *Diff* $S^1 \times \mathbf{R}_+$ can be induced by a map from *Diff* $S^1 \times \mathbf{R}_+$ into the cotangent space $T^*\mathit{Diff}\ S^1$ and is invariant under the action of *Diff* S^1, which acts naturally on *Diff* $S^1 \times \mathbf{R}_+$. With respect to the complex structure just mentioned this symplectic form is a nondegenerate pseudo-Kähler form, i.e. the corresponding hermitian form is not positive definite. For the technical details we refer [2].

Next we construct $\widehat{\mathcal{A}} = \widehat{\mathit{Diff}\ S^1} \times \mathbf{R}_+ \times \mathbf{R}$. As before we take s to be the coordinate for $\mathbf{R}_+$; we also take c to be the coordinate for the new $\mathbf{R}$. c is canonically conjugate to the central variable in $\widehat{\mathit{Diff}\ S^1}$. We can reduce $\widehat{\mathcal{A}}$ to $\mathcal{A}$ by a Hamiltonian reduction, in which we restrict c to one value and identify different values of the central variable in $\widehat{\mathit{Diff}\ S^1}$; then we find that $\mathcal{A}_c \equiv \widehat{\mathcal{A}}/\sim$ is the same as $\mathcal{A} = \mathit{Diff}\ S^1 \times \mathbf{R}_+$. Moreover, the left action of $\widehat{\mathit{Diff}\ S^1}$ on $\widehat{\mathcal{A}}$ descends to the usual left action of *Diff* S^1 on $\mathcal{A}$. We have not lost the central extension, however. Constructing the symplectic form and performing Hamiltonian reduction to fixed c yields a *family* of pseudo-Kähler symplectic forms on $\mathcal{A}$

$$\Omega_c = -i(\varphi_1\bar{\varphi}_1' - \bar{\varphi}_1\varphi_1') + 2i\sum_{n>0}\left(n + \frac{c}{24s}n^3\right)(\varphi_{n+1}\bar{\varphi}_{n+1}' - \bar{\varphi}_{n+1}\varphi_{n+1}'), \tag{3}$$

where $\varphi_1 = \frac{1}{2\sqrt{s}}\Delta - i\sqrt{s}v_0,\ \varphi_{k+1} = -i\sqrt{s}v_k, \overline{v}_k = v_{-k}$ and $(\gamma, s; v, \Delta)$ is a tangent vector to $\mathcal{A}$, $v = \sum v_k e^{in\theta},\ \theta \in S^1$. Similarly for the primes.

Ω_c and its corresponding hermitian metric are singular whenever $24s/c = -n^2$ for integer n. Hence we should really define $\overline{\mathcal{A}}_c$ as a singular symplectic variety. Also as before the hermitian metric is not positive definite. But the symplectic form is nondegenerate except at those discrete singular values so that the trouble for the nonexistence of nondegenerate symplectic form on some coadjoint orbits has been partly overcome.

Besides, far from being a pathology we expect the singularity of Ω to be the key to its correct quantization. As we cross the singularities the *signature* of the hermitian form changes. For indefinite hermitian form we know we should consider wavesections as Dolbeault cohomology classes[7], or equivalently introduce fermions. As noted by Alekseev and Shatashvili, such fermions are precisely what is needed to correct the signs in the character formula in [4]. Yet we do not know how to make this conjecture precise.

Once given a symplectic structure, we can try the usual geometric quantization, even though we have an indefinite hermitian form. With a certain integrality condition we can regard this symplectic 2-form as the curvature of a line bundle with connection (B, ∇). The wanted Hilbert space is supposed to show up as the space of sections of this line bundle and the quantization is basically the construction of a representation of Virasoro on this Hilbert space, corresponding to the Lie algebra of Poisson brackets acting on the symplectic manifold. If μ_a is the moment of some generator and X_{μ_a} its Hamiltonian vector field, then

$$\widehat{\mu}_a = -i\nabla_{X_{\mu_a}} + \mu_a \tag{4}$$

is the corresponding quantum operator on sections of B.

In our case *Diff* S^1 acts on $\mathcal{A}$ and (B, ∇) are defined in the usual way from the Kähler potential K of Ω and with the polarization defined by the *Diff* S^1-invariant complex structure on $\mathcal{A}$. Thus B is a holomorphic line bundle and $\nabla \equiv \mathrm{d} - i\partial K$ its Hermitian connection. Since $\Omega = i\partial\bar{\partial}K$ this is a suitable choice. We finally get an action on wavesections defined by eq.(4).

For model space $\mathcal{A}$ we want the space of holomorphic functions on $\mathcal{A}$ so that the bundle is holomorphically trivial. This amounts to finding a single global Kähler potential. We also want the group action (4) to be simply the left translation:

$$\widehat{\mu}_a = -iX_{\mu_a}. \tag{5}$$

In order for (4) to reduce to (5) we need to have that the Kähler potential K can be chosen everywhere to satisfy

$$\mu_a = -i\langle \partial K, X_{\mu_a} \rangle = -iX^{(1,0)}_{\mu_a} K. \tag{6}$$

Furthermore, eq.(6) amounts to requiring that the connection $-i\partial K$ be itself G-invariant. A little manipulation reduces this condition to

$$X^{(1,0)}_{\mu_a}\mu_b + X^{(0,1)}_{\mu_b}\mu_a = 0, \tag{7}$$

where $X^{(1,0)}_{\mu_a}$ is the holomorphic part of X_{μ_a}. The real part of (7) merely says $\{\mu_a,\mu_b\} = -\{\mu_b,\mu_a\}$, but the imaginary part is a new condition.

For $\mathcal{A}$ since it retracts to a circle epitomized by the phase of the first Taylor coefficient of f, so to study the triviality of B we can restrict our attention to the submanifold

$$\mathcal{A}_0 = \{f_u, u \in \mathbf{C}^\times\}, f_u(z) = uz.$$

Then $K = - : u :^2$, which is clearly global. Eqn. (7) is also satisfied by properly defining the moments.

But, further application of geometric quantization is not successful because of the indefinite hermitian structure and nonexistence of square integrability. We need a somewhat drastic new method.

The proposal by Alekseev, Faddeev and Shatashvili is to do functional integral quantization instead of canonically quantizing it[8]. They introduced the integral of the canonical one form for the symplectic structure as the required action for the functional integral and named it "geometric action". If necessary, the Cartan element can be used as the Hamiltonian.

In compact group cases the character formula is derived as a path integral for the geometric action of coadjoint orbit. For SU(2) the spin quantization is due to the quantizability of coadjoint orbit, or single-valuedness of the geometric action.

Surprisingly, the geometric action derived for loop groups (Virasoro group) turned out to be the WZW action (light-cone gauge fixed Liouville action) with an extra linear term[9]. This raises further question that this may be a right framework to investigate WZW CFT's from the representation theory of loop groups[10]. In the Virasoro group case it is less obvious how it can be related to the CFT's. But, H. Verlinde showed that without the linear term such a geometric action is related to the Virasoro conformal blocks[11]. With a linear term, the same Virasoro Ward identity can be satisfied only on a model space[3]. Thus in the Virasoro group case it seems to be more plausible to work on the model space rather than on each coadjoint orbit to investigate the possibility of CFT from this framework.

The geometric action corresponding to eq.(3) on $\mathcal{A}$ is

$$S[F, b_0] = \int d\theta dt \left[-b_0(t)\dot{F}F' + \frac{c}{48\pi}\frac{\dot{F}}{F'}\left(\frac{F'''}{F'} - 2\left(\frac{F''}{F'}\right)^2\right)\right], \tag{8}$$

where $b_0 = s/2\pi$ and $F \in \mathit{Diff}\ S^1$ are dynamical variables, while b_0 is just a constant in the geometric action of a coadjoint orbit. c is a classical central charge, which is supposed to get renormalized to be the actual Virasoro central charge. To perform the path integral quantization (8) needs to define some QFT, though not necessarily physical. Note that the parameter space has automatically a topology of a cylinder because $\theta \in S^1$ and $t \in \mathbf{R}$. $F(\theta + 2\pi, t) = F(\theta, t) + 2\pi$ and $F(\theta, t_i) = F(\theta, t_f)$. Thus in some sense this is equivalent to looking for the Lagrangian formulation of the chiral part of any CFT's, including the minimal models for which we still do not know any such formulation. But we know the Lagrangian formulation for the Ising model. Therefore it is worth while to check whether this defines any chiral part of the Ising model.

There are still a lot of unanswered questions. We do not know how to determine the renormalized Virasoro central charge. Once the renormalized c is given, we do not know what determines the highest weights h. We need some formula like the Kac determinant. Besides, without the linear term the geometrical action is almost Polyakov's 2d gravity action[12], except that the topology is a cylinder. Thus it will be extremely important to understand whether the linear term has any role in 2d gravity on the cylinder. Also if this is a right framework, one should be able to produce $c \geq 1$ CFT's, too. But, none of these are fully understood yet.

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References

[1] G. Moore and N. Seiberg, " Lectures on RCFT, " preprint RU-89-32 (≡YCTP-P13-89) and references therein.

[2] H.S. La, P. Nelson and A. Schwarz, "Virasoro Model Space," to appear in Comm. Math. Phys..

[3] H.S. La, Penn preprint UPR-0417T.

[4] A. Alekseev and S. Shatashvili, Commun. Math. Phys. **128** (1990) 197.

[5] I.M. Gelfand, Int. Cong. Math. 1970 **1** (1970) 95.

[6] A. Kirillov, Funct. Anal. Appl. **21** (1987) 122; A. Kirillov and D. Yur'ev, Funct, Anal. Appl. **21** (1987) 284.

[7] M. Narasimhan and K. Okamoto, Ann. Math. (2) **91** (1970) 486.
[8] A. Alekseev, L. Faddeev and S. Shatashvili, to appear in J. of Geom. Phys.
[9] A. Alekseev and S. Shatashvili, Nucl. Phys. **B323** (1989)719.
[10] A. Gerasimov, A. Marshakov, A. Morozov, M. Olshanetski and S. Shatashvili, ITEP preprint 64, 70, 72, 74 - 89.
[11] H. Verlinde, Princeton preprint PUPT-89/1140.
[12] A.M. Polyakov, Mod. Phys. Lett. **A2** (1987) 893.