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Theoretical Studies of Cosmic Acceleration

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Abstract
In this thesis we describe theoretical approaches to the problem of cosmic acceleration in the early and late universe. The first approach we consider relies upon the modification of Einstein gravity by the inclusion of mass terms as well as couplings to higher-derivative scalar fields possessing generalized internal shift symmetries - the Galileons. The second half of the thesis is concerned with the quantum-mechanical consistency of a theory of the early universe known as the pseudo-conformal mechanism which, in contrast to inflation, relies not on the effects of gravity but on conformal field theory (CFT) dynamics.

It is possible to couple Dirac-Born-Infeld (DBI) scalars possessing generalized Galilean internal shift symmetries (Galileons) to nonlinear massive gravity in four dimensions, in such a manner that the interactions maintain the Galilean symmetry. Such a construction is of interest because it is not possible to couple such fields to massless General Relativity in the same way. Using tetrad techniques we show that this massive gravity-Galileon theory possesses a primary constraint necessary to ensure propagation with the correct number of degrees of freedom.

We study the background cosmology of this theory around cosmologically relevant spacetimes and find that, as in pure massive gravity, spatially flat solutions do not exist. Spatially open solutions do exist - consisting of a branch of self-accelerating solutions that are identical to those of pure massive gravity, and a new second branch of solutions which do not appear without the inclusion of Galileons. We study the propagating degrees of freedom of the massive gravity-Galileon theory around the self-accelerating solutions and identify the conditions necessary for the theory to remain free of ghost-like instabilities. We show that on the self-accelerating branch the kinetic terms for the vector and scalar modes of the massive graviton vanish, as in the case of pure massive gravity.

We conclude our exploration of massive gravity by considering the possibility of variable-mass massive gravity, where the fixed graviton mass is replaced by the expectation value of a rolling scalar field. We ask whether self-inflation can be driven by the self-accelerated branch of this theory, and we find that, while such solutions can exist for a short period, they cannot be sustained for a cosmologically useful time. Furthermore, we demonstrate that there generally exist future curvature singularities of the ``big brake" form in cosmological solutions to these theories.

In the second half of the thesis we construct the gravitational dual of the pseudo-conformal universe, a proposed alternative to inflation in which a CFT in nearly flat space develops a time dependent vacuum expectation value. Constructing this dual amounts to finding five-dimensional domain-wall spacetimes with anti-de Sitter asymptotics, for which the wall has the symmetries of four-dimensional de Sitter space. This holographically realizes the characteristic symmetry breaking pattern O(2,4) to O(1,4) of the pseudo-conformal universe. We present an explicit example with a massless scalar field, using holographic renormalization to obtain general expressions for the renormalized scalar and stress-tensor one-point functions. We discuss the relationship between these solutions and those of four-dimensional holographic CFTs with boundaries, which break O(2,4) to O(2,3).

Finally, we undertake a systematic study of one and two point functions of CFTs on spaces of maximal symmetry with and without boundaries and investigate their spectral representations. Integral transforms are found, relating the spectral decomposition to renormalized position space correlators. Several applications are presented, including the holographic boundary CFTs as well as spacelike boundary CFTs, which provide realizations of the pseudo-conformal universe.

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In this thesis we describe theoretical approaches to the problem of cosmic acceleration in the early and late universe. The first approach we consider relies upon the modification of Eintein gravity by the inclusion of mass terms as well as couplings to higher-derivative scalar fields possessing generalized internal shift symmetries — *the Galileons*. The second half of the thesis is concerned with the quantum-mechanical consistency of a theory of the early universe known as the *pseudo-conformal mechanism* which, in contrast to inflation, relies not on the effects of gravity but on conformal field theory (CFT) dynamics.

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Contents

1 **Introduction** 1

2 **Massive Gravity Coupled to Galileons is Ghost Free** 7

2.1 The model . . . 9

2.2 Hamiltonian analysis . . . 12

3 **Cosmological perturbations of massive gravity coupled to DBI Galileons** 16

3.1 Background Cosmology and Self Accelerating Solutions . . . 17

3.2 Perturbations . . . 20

4 **Cosmologies of extended massive gravity** 27

5 **Holography of the Pseudo-Conformal Universe** 35

5.1 Probe Scalar Limit . . . 38

5.2 Including Back-Reaction . . . 43

5.3 Appendices . . . 61

6 **Holographic CFTs on maximally symmetric spaces: correlators, integral transforms and applications** 64

6.1 CFT correlators on maximally symmetric spaces and their UV singularities 66

6.2 Holographic Boundary CFT . . . 81

6.3 Holographic pseudo-conformal CFT . . . 91

6.4 Appendices . . . 96

7 **Conclusion** 111

Bibliography **114**

LIST OF PAPERS COMPLETED DURING PHD

1. J. Khoury, B. A. Ovrut and J. Stokes, "The Worldvolume Action of Kink Solitons in AdS Spacetime," JHEP **1208**, 015 (2012) [arXiv:1203.4562 [hep-th]].

2. B. A. Ovrut and J. Stokes, "Heterotic Kink Solitons and their Worldvolume Action," JHEP **1209**, 065 (2012) [arXiv:1205.4236 [hep-th]].

3. K. Hinterbichler, J. Stokes and M. Trodden, "Cosmologies of extended massive gravity," Phys. Lett. B **725**, 1 (2013) [arXiv:1301.4993 [astro-ph.CO]].

4. M. Andrews, G. Goon, K. Hinterbichler, J. Stokes and M. Trodden, "Massive Gravity Coupled to Galileons is Ghost-Free," Phys. Rev. Lett. **111**, no. 6, 061107 (2013) [arXiv:1303.1177 [hep-th]].

5. M. Andrews, K. Hinterbichler, J. Stokes and M. Trodden, "Cosmological perturbations of massive gravity coupled to DBI Galileons," Class. Quant. Grav. **30**, 184006 (2013) [arXiv:1306.5743 [hep-th]].

6. K. Hinterbichler, J. Stokes and M. Trodden, "Holography for a Non-Inflationary Early Universe," JHEP **1501**, 090 (2015) [arXiv:1408.1955 [hep-th]].

7. A. Polyakov, Z. H. Saleem and J. Stokes, "The curious case of large-N expansions on a (pseudo)sphere," Nucl. Phys. B **893**, 54 (2015) [arXiv:1412.1437 [hep-th]].

8. K. Hinterbichler, J. Stokes and M. Trodden, "Holographic CFTs on maximally symmetric spaces: correlators, integral transforms and applications," Phys. Rev. D **92**, no. 6, 065025 (2015) [arXiv:1505.05513 [hep-th]].

9. S. Gubser, Z. H. Saleem, S. S. Schoenholz, B. Stoica and J. Stokes, "Nonlinear Sigma Models with Compact Hyperbolic Target Spaces," arXiv:1510.02129 [hep-th].

List of Figures

Figure 1 : Penrose diagrams showing the Poincaré coordinates and de Sitter slice coordinates. The left-hand figure shows the global AdS cylinder; the de Sitter slice coordinate region is bounded from above by the lightcone which emanates downward from $t = 0$, and is bounded from below by the slanted ellipse, which also marks the lower boundary of the Poincare patch (the upper slanted ellipse shown in outline form is the upper boundary of the Poincaré patch). The right-hand figure shows a two dimensional slice down the axis of the AdS cylinder; thin lines are Poincaré lines of constant z and t, thick lines de Sitter slice lines of constant ρ and η. 41

Figure 2 : A typical solution. Here $b = 1$. On the horizontal axis is the ρ coordinate on a logarithmic scale, and on the vertical axis is the scale factor $A(\rho)$, normalized by ρ so that it can be seen that $A \to \rho$ as $\rho \to \infty$, corresponding to asymptotically AdS boundary conditions. The singularity occurs at the value $\rho_* \approx 0.7$, for which the scale factor goes to $-\infty$. 45

Figure 3 : Spectral representations $\langle \mathcal{O}_2(n)\mathcal{O}_2(n')\rangle_{\mathbb{H}_3}$ (left) and $f(\vec{n} \cdot \vec{n}')$ (right) obtained by numerical interpolation. 81

Chapter 1

Introduction

Modern cosmology poses a host of physics questions that demand answers from fundamental theory. For instance: what is the resolution of the cosmic singularity? What explains the anomalous acceleration seen in the present-day universe and what is the origin of primordial fluctuations in the very early universe, which seeded the cosmic structure seen today? This thesis explores possible solutions to these problems and others by drawing upon novel field theory techniques such as infra-red modifications of gravity and dualities originating in string theory.

One of the most perplexing puzzles in the late universe is the problem of the cosmological constant. In classical general relativity the cosmological constant is free parameter Λ which appears in the action for the gravitational field,

$$S = \frac{1}{2}M_{\mathrm{P}}^2 \int d^4x\sqrt{-g}(R[g_{\mu\nu}] - 2\Lambda) + S_{\mathrm{matter}}[g_{\mu\nu}, \Psi] \tag{1.0.1}$$

where the first term is the Einstein-Hilbert action, the second is the cosmological constant and the final term represents the mixing of matter fields Ψ with the gravitational field $g_{\mu\nu}$. Solving the Einstein field equations in a cosmogical Friedman-Roberston-Walker spacetime we find that the cosmological constant contributes a constant term in the Friedman equation, while all other energy components are suppressed by factors of the scale factor,

$$\left(\frac{\dot{a}}{a}\right)^2 = \Lambda + \frac{1}{3M_{\mathrm{P}}^2}\left[-\frac{k}{a^2} + \cdots\right]. \tag{1.0.2}$$

where the dots represent contributions from matter fields which are suppressed by higher powers of a. It follows that at the classical level, the cosmological constant is an adjustable parameter which can be simply fixed from experimental observations of the expansion rate of the universe. The story changes signficantly when quantum mechanics is taken into account. Vacuum fluctuations of the graviton and other fields are expected to drive the cosmological constant to the short-distance cut-off of the theory. On the other hand, the observed cosmic acceleration suggests that the cosmological constant is determined by the infra-red (long-distance) cut-off. This strongly suggests that the the solution of the cosmological constant problem lies in the infrared interactions of gravity. Recently it has become clear that a consistent infra-red modification of gravity based on mass terms exists and can be applied to the late universe. We will explore this theory and some of its extensions.

The late universe is not the only place where the techniques of quantum field theory can address theoretical issues. In the very early universe we are faced with a number of questions in inflation and some of its alternatives. The prevailing hypothesis for the origin of nearly scale-invariant temperature fluctuations observed in the cosmic microwave background is that the early universe underwent a period of exponentially accelerated expansion called inflation (see [1] for a review) which stretched microscopic quantum fluctuations to macroscopic scales. In addition to explaining temperature anisotropies, inflation solves the horizon problem (why the universe appears uniform across apparently causually disconnected regions) and the flatness problem (why the universe is spatially flat when curvature is expected to dominate at late times). It is important to understand if these successes are unique to inflation, and if there exist experimentally distinguishable alternatives.

Over the past decade, attempts to find a consistent infrared modification of gravity have led to two seemingly distinct discoveries. The first is a consistent, ghost free nonlinear realization of massive gravity, known as dRGT massive gravity [2, 3]. The second is a class of intriguing scalar field theories — the Galileons [4] — with novel classical and quantum properties that can be traced to their nonlinear derivative interactions. There is at least one connection between these ideas, in that the Galileon interactions govern the longitudinal degree of freedom of a ghost-free massive graviton in the decoupling limit [2]. Furthermore, if one is to consider covariantizing Galileons, while preserving second order equations of motion and the special symmetries of those theories, then it is natural to couple not to General Relativity (GR) as in [5, 6] (which breaks the Galileon symmetry), but rather to massive gravity itself [7]. Thus one is led to a theory of massive gravity Galileons, which we will study extensively in this thesis.

One of the general challenges faced in modifying gravity by mass terms is to ensure that the resulting theory propagates the correct number of degrees of freedom to describe a massive spin-2 particle. It has long been known that the Fierz-Pauli action [8] provides a consistent description of the linear fluctuations of a massive graviton in flat spacetime. Nonlinear theories of massive gravity, however, tend to suffer from an instability known as the Boulware-Deser (BD) ghost [9]. The dRGT theory is a 3-parameter family of potentials whose special structure ensures that there is a dynamical constraint which removes the ghost degree of freedom. This has been demonstrated by explicitly counting degrees of freedom in the Hamiltonian formalism [10, 11], and through other methods [12–15]. It is therefore essential to ensure that extensions of the dRGT massive gravity theory are theoretically well-grounded. In the first chapter we bring the techniques of field theory to bear on this issue by proving that massive gravity coupled to Galileons propagates the correct number of degrees of freedom. This is a crucial requirement of the theory that opens the door to the model building in the subsequent chapters.

An appealing feature of pure dRGT massive gravity is that it admits self-accelerating solutions [16–21], in which the de Sitter Hubble factor is of order the mass of the graviton. Since having a light graviton is technically natural [22, 23], such a solution is of great interest in the late-time universe to account for cosmic acceleration. We derive the background cosmological equations for massive gravity coupled to Galileons, and find that the presence of the scalar leads to a more complicated constraint than in pure dRGT. We discuss the possible solutions in the case of zero and negative spatial curvature. We find that, as in pure dRGT theory, this constraint forbids flat FRW solutions. For an open FRW ansatz, however, solutions can exist and they come in two branches. The first branch consists of self-accelerating solutions that are identical to the self-accelerating solutions of pure dRGT theory. The second branch consists of novel solutions which are not found in pure massive gravity.

Although the dRGT theory possesses a self-accelerating solution with negatively curved spatial slices [21], the study of fluctuations on top of this background has shown that the kinetic terms for the vector and scalar perturbations vanish [24].

It has been shown that the vanishing of these terms can be remedied by departing from isotropic and homogeneous cosmologies [25, 26] or by introducing new degrees of freedom. There are many ways to achieve the latter option, and several possibilities have been explored in the so-called quasi-dilaton [27–30] and mass-varying extensions of dRGT [20, 30, 31]. Motivated by these examples, we perform a study of cosmological perturbations around the self-accelerating branch for the massive gravity Galileon theory and show that the kinetic terms for the scalar and vector modes vanish, similar to the case of the pure dRGT theory.

It is tempting to theorize that massive gravity might play a role the early universe too. A natural question is whether massive gravity might provide an alternative to inflation by driving accelerated expansion in the early universe. To use the self-accelerating solution of massive gravity for inflation (i.e. "self-inflation"), the graviton mass would have to be of order the Hubble scale during inflation. Yet, we know that the current graviton mass cannot be much larger than the Hubble scale today [32].

Thus, for self-inflation to be possible, the graviton mass must change in time. One idea of how to realize this is to promote the graviton mass to a scalar field, Φ, which has its own dynamics and can roll [20, 31]. The expectation value (VEV) of Φ then sets the mass of the graviton. We can imagine that at early times Φ has a large VEV, so that the graviton is very massive, and the universe self-inflates with a large Hubble constant. Then, at late times, Φ rolls to a smaller VEV, self-inflation ends and the graviton mass attains a small value consistent with present day measurements.

We will see that, in practice, such an inflation-like implementation of massive gravity is difficult to achieve in this model. Pure dRGT theory has a constraint, stemming from the Bianchi identity, which forbids standard FRW evolution in the flat slicing [20] (the self-accelerating solutions are found in other slicings). There appears an analogous constraint in the variable mass theory, and this constraint, while it no longer forbids flat FRW solutions, implies that self-inflation cannot be sustained for a cosmologically relevant length of time. In addition, we show that non-inflationary cosmological solutions to this theory may exhibit future curvature singularities of the "big brake" type.

In the remainder of the thesis we seek to overcome one of the challenges which is faced by alternatives to inflation which do not rely on modified gravity. Typically such scenarios involve a slowly contracting 'pre-big bang' phase in which the universe crunches and subsequently expands. An interesting implementation of this idea postulates that the early universe is described by a CFT whose conformal algebra is spontaneously broken by operator vacuum expectation values to a de Sitter subalgebra — the so-called pseudo-conformal universe [33–36]. Despite the theoretically appealing features of this model, it is challenging to find ultra-violet-complete examples of CFTs which possess the required set of vacuum states to realize the pseudo-conformal mechanism. In chapter 5 we address this issue by providing an explicit embedding of the pseudo-conformal mechanism in the so-called AdS/CFT correspondence [64]. Our construction involves computing the unambiguous parts of the exact one-point functions of the scalar operator and stress tensor, in the presence of a general source and boundary metric, using the techniques of holographic renormalization [106].

We will see that the holographic perspective of the pseudo-conformal universe leads us naturally to the mathematical problem of calculating CFT correlation functions in curved spacetimes. Such correlation functions are crucial when confronting these theories with observational cosmology. Correlation functions in general quantum field theories can present both short and long distance singularities. The short-distance singularities are regularization dependent and parametrize un-calculable high energy effects which are renormalized into the undetermined local couplings of the effective action[1]. Theories without a mass gap exhibit long-range correlations, which can lead to infra-red singularities in Fourier space. These are calculable universal features, not dependent on regularization ambiguities or absorbable into local couplings. The last chapter is devoted to exploring the two-point functions of CFTs in curved spaces such as those arising in the pseudo-conformal universe. In particular, we explain how short-distance singularities can be absorbed into local counter-terms. The initial motivation for this work was to holographically compute two-point functions in the pseudo-conformal universe (which would correspond to the power spectra of interest in cosmology), but the results apply more widely to other conformal field theories with boundaries.

[1] There are some exceptions, e.g. [117].

Chapter 2

Massive Gravity Coupled to Galileons is Ghost Free

In this chapter, we describe the explicit counting of degrees of freedom which shows that massive gravity coupled to DBI Galileons is not afflicted by the BD ghost. Most of this chapter is taken from [37], which was written in collaboration with Garret Goon, Kurt Hinterbichler, Mark Trodden and Melinda Andrews.

A general Lagrangian for nonlinear massive gravity can be formulated by introducing a non-dynamical reference metric $\bar{g}_{\mu\nu}$ (e.g. the Minkowski one, $\bar{g}_{\mu\nu} = \eta_{\mu\nu}$) and constructing a potential of the form $V(g^{\mu\sigma}\bar{g}_{\sigma\nu})$. The potential explicitly breaks diffeomorphism invariance and it is expected that the theory generally propagates 12 phase-space degrees of freedom, rather than the 10 necessary to describe a massive graviton. The extra degree of freedom is the BD ghost.

The diffeomorphism invariance broken by the mass term can be restored through the Stückelberg method [22], which involves introducing four auxiliary scalars $\phi^{\mathcal{A}}(x)$ through the replacement $\bar{g}_{\mu\nu} \to \partial_\mu\phi^{\mathcal{A}}\partial_\nu\phi^{\mathcal{B}}\eta_{\mathcal{A}\mathcal{B}}$. The Stückelberg fields are pure gauge and the original theory is recovered by choosing unitary gauge $\phi^{\mathcal{A}} = \delta^{\mathcal{A}}_{\mu} x^\mu$. In this formulation, the scalars can be regarded as the embedding mapping of a sigma model $\Sigma \to \mathcal{M}$, where both Σ and $\mathcal{M}$ are four dimensional Minkowski space. There is an internal Poincaré symmetry corresponding to the isometries of the target space. From this point of view, the dynamical metric $g_{\mu\nu}(x)$ is a worldvolume metric living on Σ.

The target space may be higher dimensional, and need not be flat – we may generalize the sigma model to map to an arbitrary target space of dimension $D \geq 4$ with coordinates $\phi^{\mathcal{A}}$

(so that now $\mathcal{A}, \mathcal{B}, \cdots$ run over D values) and a fixed target space metric $G_{\mathcal{AB}}(\phi)$,

$$\bar{g}_{\mu\nu} = \partial_\mu \phi^{\mathcal{A}} \partial_\nu \phi^{\mathcal{B}} G_{\mathcal{AB}}(\phi). \tag{2.0.1}$$

There are now $(\dim \mathcal{M}) - 4$ embedding fields which cannot be gauged away and these become physical Dirac-Born-Infeld (DBI) scalars coupled to the physical metric through the dRGT potential [7]. Apart from the dRGT terms, curvature invariants constructed solely from $\bar{g}_{\mu\nu}$ and extrinsic curvatures of the embedding can be included in the action. The leading term in the derivative expansion is the DBI action $\sim \int \mathrm{d}^4x \sqrt{-\bar{g}}$, and higher Lovelock invariants give Galileons [38, 39]. The theory will possess a Galileon-like internal symmetry for every isometry of $G_{\mathcal{AB}}$, and the resulting Galileons will be the generalized curved space Galileons discussed in [40–43].

Apart from generalizing dRGT, the construction of [7] is of interest because it provides a method of coupling the Galileons to (massive) gravity while preserving the Galileon invariance. When coupling to ordinary massless gravity, non-minimal couplings must be added to ensure second-order equations of motion, and the Galileon symmetry is broken [5, 6]. In the present construction, there is no such problem, suggesting that the Galileons more naturally couple to a massive graviton.

In [7], it was shown that the theory is ghost-free, for a flat target space metric, in the decoupling limit, and for a certain simplifying choice of parameters. In this chapter, using methods similar to those of [15], we demonstrate that the full theory, for any target space metric $G_{\mathcal{AB}}$, to all orders beyond the decoupling limit, and for all choices of parameters, has the primary constraint necessary to eliminate the Boulware-Deser ghost.

2.1 The model

The dynamical variables are the physical metric $g_{\mu\nu}$ and the D scalars $\phi^{\mathcal{A}}$, which appear through the induced metric (2.0.1). The action is

$$S = S_{\text{EH}}[g] + S_{\text{mix}}[g, \bar{g}] + S_{\text{Galileon}}[\bar{g}] \ . \tag{2.1.1}$$

Here $S_{\text{EH}}[g]$ is the Einstein-Hilbert action for $g_{\mu\nu}$, with a possible cosmological constant Λ,

$$S_{\text{EH}}[g] = \frac{M_P^2}{2} \int \mathrm{d}^4x \sqrt{-g} \left(R[g] - 2\Lambda\right) . \tag{2.1.2}$$

The action mixing the two metrics is

$$S_{\text{mix}}[g, \bar{g}] = -\frac{M_P^2 m^2}{8} \sum_{n=1}^{3} \beta_n S_n \left(\sqrt{g^{-1}\bar{g}}\right) \ , \tag{2.1.3}$$

where $\sqrt{g^{-1}\bar{g}}$ is the matrix square root of the matrix $g^{\mu\sigma}\bar{g}_{\sigma\nu}$, and $S_n(M)$ of a matrix M are the symmetric polynomials[2] $S_n(M) = M^{[\mu_1}{}_{\mu_1} \cdots M^{\mu_n]}{}_{\mu_n}$. The β_n are three free parameters (one combination of which is redundant with the mass m). $S_{\text{Galileon}}[\bar{g}]$ stands for any Lagrangian constructed from diffeomorphism invariants of $\bar{g}$ (and extrinsic curvatures of the embedding) whose equations of motion remain second order in time derivatives. The possible terms in $S_{\text{Galileon}}[\bar{g}]$ are the Lovelock invariants and their boundary terms (see [38] and Sec. IV.B of [39] for a discussion). The structure of the dRGT-DBI coupled system (3.0.2) is nearly identical to that of ghost-free bi-gravity [45], the difference being that one of the two metrics is induced from a target space, and so it fundamentally depends on the embedding scalars.

Following [15], we will find it convenient to write the theory in vierbein form[3]. We write

[2] Our anti-symmetrization weight is $[\mu_1 \ldots \mu_n] = \frac{1}{n!}(\mu_1 \cdots \mu_n + \cdots)$. See appendix A of [15] for more details on the symmetric polynomials.

[3] See also [46, 47] for covariant methods of degree of freedom counting in the vierbein formulation of massive gravity.

the physical metric and induced metric in terms of vierbeins $E^A = E_\mu{}^A \mathrm{d}x^\mu$, $\bar{E}^A = \bar{E}_\mu{}^A \mathrm{d}x^\mu$,

$$g_{\mu\nu} = E_\mu{}^A E_\nu{}^B \eta_{AB}, \quad \bar{g}_{\mu\nu} = \bar{E}_\mu{}^A \bar{E}_\nu{}^B \eta_{AB}, \tag{2.1.4}$$

where η_{AB} is the 4 dimensional Minkowski metric. For the induced metric $\bar{g}_{\mu\nu}$, we choose a frame in which the vierbein is in an upper triangular form

$$\bar{E}_\mu{}^B = \begin{pmatrix} \bar{N} & \bar{N}^i \bar{e}_i{}^a \\ 0 & \bar{e}_i{}^a \end{pmatrix} . \tag{2.1.5}$$

Here $\bar{N}$ and $\bar{N}^i$ are ADM lapse and shift variables, and $\bar{e}_i{}^a$ is an upper triangular spatial dreibein for the spatial part of the induced metric and $\bar{e}^i{}_a$ its inverse transpose (in what follows $i, j, \ldots$ are spatial coordinate indices raised and lowered with the spatial metric $\bar{g}_{ij}$, and $a, b, \ldots$ are spatial Lorentz indices raised and lowered with δ_{ab}). These are obtained in terms of $\phi^{\mathcal{A}}$ by solving

$$\begin{aligned} \bar{g}_{00} &= \dot{\phi}^{\mathcal{A}} \dot{\phi}^{\mathcal{B}} G_{\mathcal{AB}}(\phi) = -\bar{N}^2 + \bar{N}^i \bar{N}_i \\ \bar{g}_{0i} &= \dot{\phi}^{\mathcal{A}} \partial_i \phi^{\mathcal{B}} G_{\mathcal{AB}}(\phi) = \bar{N}_i \\ \bar{g}_{ij} &= \partial_i \phi^{\mathcal{A}} \partial_j \phi^{\mathcal{B}} G_{\mathcal{AB}}(\phi) = \bar{e}_i{}^a \bar{e}_j{}^b \delta_{ab} . \end{aligned} \tag{2.1.6}$$

The upper triangular vierbein (2.1.5) has 10 components, and is merely a repackaging of the 10 components of $\bar{g}_{\mu\nu}$(which in turn depend on the $\phi^{\mathcal{A}}$).

For the physical metric $g_{\mu\nu}$, we parameterize its 16 component vierbein as a local Lorentz transformation (LLT) M, which has 6 components, times a vierbein $\hat{E}$ which is constrained in some way so that it has only 10 components,

$$E_\mu{}^A = M^A{}_B \hat{E}_\mu{}^B \tag{2.1.7}$$

The freedom to choose the constraints for $\hat{E}$ allows us to make different aspects of the theory manifest. The mixing term (3.0.4), in terms of vierbeins, takes the form

$$
\begin{aligned}
S_{\text{mix}} &\equiv -\frac{M_P^2 m^2}{8}\sum_{n=1}^{3}\frac{\beta_n}{n!(4-n)!}S_{\text{mix}}^{(n)},\\
S_{\text{mix}}^{(1)} &= \int \epsilon_{ABCD}\bar{E}^A\wedge E^B\wedge E^C\wedge E^D,\\
S_{\text{mix}}^{(2)} &= \int \epsilon_{ABCD}\bar{E}^A\wedge \bar{E}^B\wedge E^C\wedge E^D,\\
S_{\text{mix}}^{(3)} &= \int \epsilon_{ABCD}\bar{E}^A\wedge \bar{E}^B\wedge \bar{E}^C\wedge E^D.
\end{aligned}
\tag{2.1.8}
$$

The dynamical vierbein has 16 components, 6 more than the metric. If we choose the 6 constraints which $\hat{E}$ must satisfy to be the symmetry condition,

$$
\hat{E}_{\mu[A}\bar{E}^{\mu}{}_{B]} = 0,
\tag{2.1.9}
$$

then we can show using the arguments in [15] (see also [48] for subtleties) that the extra 6 fields in M are auxiliary fields which are eliminated by their own equations of motion, setting $M = 1$, and the resulting theory is dynamically equivalent to the metric formulation (3.0.2).

Instead, we take $\hat{E}$ to be of upper triangular form

$$
\hat{E}_{\mu}{}^{A} = \begin{pmatrix} N & N^i e_i{}^a \\ 0 & e_i{}^a \end{pmatrix}.
\tag{2.1.10}
$$

Here the spatial dreibein $e_i{}^a$ is arbitrary, containing 9 components. The LLT M in (2.1.7) depends now on 3 boost parameters p^a and can be written as

$$
M(p)^A{}_B = \begin{pmatrix} \gamma & p^b \\ p_a & \delta^a_b + \frac{1}{1+\gamma}p^a p_b \end{pmatrix},
\tag{2.1.11}
$$

where $\gamma \equiv \sqrt{1+p^a p_a}$. Using this decomposition, the 16 component vierbein $E_\mu{}^A$ is parameterized in terms of the 3 components of p^a, one N, 3 components of N^i and the 9 components of $e_i{}^b$.

2.2 Hamiltonian analysis

We start the Hamiltonian analysis by Legendre transforming with respect to the spatial vierbein $e_i{}^a$, defining canonical momenta $\pi^i{}_a = \frac{\partial \mathcal{L}}{\partial \dot{e}_i{}^a}$. Since S_{mix} contains no time derivatives of the physical metric, and S_{Galileon} has no dependence on the physical metric at all, the expressions for the canonical momenta are the same as their GR counterparts. In particular, there will be three primary constraints

$$\mathcal{P}_{ab} = e_{i[a}\pi^i{}_{b]} = 0\,. \tag{2.2.1}$$

In GR, these are first class constraints which generate local rotations. In our case, the local Lorentz symmetry is broken, and these constraints will generate secondary constraints and form second class pairs.

The Einstein-Hilbert part of the action takes the form[4]

$$S_{\text{EH}} = \int \mathrm{d}^4x\, \pi^i{}_a \dot{e}_i{}^a - \frac{1}{2}\lambda^{ab}\mathcal{P}_{ab} - N\mathcal{C}\,(e,\pi) - N^j\mathcal{C}_j\,(e,\pi)\,. \tag{2.2.2}$$

The anti-symmetric λ^{ab} holds the three Lagrange multipliers for the three primary constraints (2.2.1). The N and N^i appear as Lagrange multipliers enforcing respectively the Hamiltonian and momentum constraints of GR: $\mathcal{C} = 0,\ \mathcal{C}_i = 0$.

We now look at the mixing terms (2.1.8). The contributions to $\mathcal{L}_{\text{mix}}$ are of the form $\sim \epsilon^{\mu\nu\rho\sigma}\epsilon_{ABCD}E_\mu{}^A E_\nu{}^B \bar{E}_\rho{}^C \bar{E}_\sigma{}^D$, containing various numbers of copies of E and $\bar{E}$. From (2.1.5), (2.1.10) and (2.1.11), we see that the $\mu = 0$ components of $E_\mu{}^A$ and $\bar{E}_\mu{}^A$ are strictly linear in their respective lapses and shifts and the $\mu = i$ components are independent of the lapse and shift. Therefore, due to the anti-symmetry of the epsilons, the entire mixing

[4]See [49] or Appendix B of [15] for details of the Hamiltonian formulation of GR in vierbein form.

term is linear in the lapses and shifts, so we may write

$$\begin{aligned}\mathcal{L}_{\rm mix} = &-N\mathcal{C}_{\rm mix}(e,\bar{e},p) - N^i\mathcal{C}_{{\rm mix},i}(e,\bar{e},p) - \bar{N}\bar{\mathcal{C}}_{\rm mix}(e,\bar{e},p) \\ &- \bar{N}^i\bar{\mathcal{C}}_{{\rm mix},i}(e,\bar{e},p) - \mathcal{H}_{\rm mix}(e,\bar{e},p)\ .\end{aligned} \tag{2.2.3}$$

The lapse and shift remain as Lagrange multipliers, and the p^a appear algebraically. We now solve the constraint enforced by N^i for the p^a: $\mathcal{C}_i + \mathcal{C}_{{\rm mix},i} = 0 \Rightarrow p^a = p^a(e,\bar{e},\pi)$. Plugging back into the action we obtain

$$\begin{aligned}S = \int {\rm d}^4x\, &\pi^i{}_a\dot{e}_i{}^a - \frac{1}{2}\lambda^{ab}\mathcal{P}_{ab} - N\left[\mathcal{C}\left(e,\pi\right) + \mathcal{C}_{\rm mix}(e,\bar{e},\pi)\right] \\ &- \bar{N}\bar{\mathcal{C}}_{\rm mix}(e,\bar{e},\pi) - \bar{N}^i\bar{\mathcal{C}}_{{\rm mix},i}(e,\bar{e},\pi) \\ &- \mathcal{H}_{\rm mix}(e,\bar{e},\pi) + \mathcal{L}_{\rm Galileon}(\bar{e},\bar{N},\bar{N}^i)\ .\end{aligned} \tag{2.2.4}$$

It remains to Legendre transform with respect to the scalars $\phi^{\mathcal{A}}$, which appear through the dependence of $\bar{N}$, $\bar{N}^i$ and $\bar{e}_i{}^a$, as determined by (2.1.6). In order to avoid dealing with the complications of diffeomorphism invariance, we first fix unitary gauge, setting the first four fields equal to the space-time coordinates: $\phi^\mu = x^\mu$ (this can be done consistently in the action, since the missing equations of motion are implied by the remaining equations). The Lagrangian (2.2.4) then depends on the remaining $(\dim\mathcal{M}) - 4$ scalars and their time derivatives, and has no further gauge symmetry. Crucially, we see from (2.1.6) that while $\bar{N}$ and $\bar{N}_i$ depend on time derivatives of the scalars, the $\bar{e}_i{}^a$'s do not, and this in turn implies that the momenta conjugate to the scalars are independent of the dynamical lapse N. Thus, when the scalar velocities are eliminated in terms of the momenta, the action will remain linear in N. (If this were not the case, the lapse would no longer be a Lagrange multiplier, but would instead become an auxiliary field which does not impose a constraint on the remaining variables.) The phase space is spanned by the 9 independent components of $e_i{}^a$, the physical scalars, and the canonical momenta. Since the interaction terms break the local rotation invariance of GR (i.e. a local spatial rotation of the Lorentz index on the vielbein), the 3 primary constraints (2.2.1) associated with the local rotations will generate secondary constraints and form 3 second class pairs, thus removing 3 degrees of freedom. The constraint enforced by N is precisely the special primary constraint needed to remove the Boulware-Deser sixth degree of freedom, leaving 5 degrees of freedom for the massive graviton. Analogously to what happens in massive gravity, we expect this special primary constraint to generate a secondary constraint to eliminate the ghost's conjugate momentum [11], as was recently confirmed in a special case of the theory [50]. (If there were no secondary constraint, the theory would propagate a fractional number of degrees of freedom, a strange situation for a Lorentz-invariant theory.)

We have implicitly assumed that S_{Galileon} can be written in such a way that the $(\dim \mathcal{M}) - 4$ unitary gauge scalar fields appear with at most first time derivatives, so that we may define canonical momenta in the usual way. This is not immediately obvious, because the higher order Galileons in S_{Galileon} possess higher derivative interactions. However, the higher derivative interactions within S_{Galileon} are special in that they generate equations of motion which are no higher than second order in time. This means it should be possible, after integrations by parts, to express these Lagrangians in terms of first time derivates only (though we shouldn't expect to be able to do the same with both the spatial and time derivates simultaneously). For example, take the case of a flat 5D target space, so that there is a single physical scalar ϕ. The unitary gauge induced metric is $\bar{g}_{\mu\nu} = \eta_{\mu\nu} + \partial_\mu\phi\,\partial_\nu\phi$. The first higher-derivative Galileon is the cubic, coming from the extrinsic curvature term

$$S_K \sim \int \mathrm{d}^4x\sqrt{-\bar{g}}\bar{K} \sim \int \mathrm{d}^4x \frac{\partial_\mu\partial_\nu\phi\,\partial^\mu\phi\,\partial^\nu\phi}{1+(\partial\phi)^2} \ . \tag{2.2.5}$$

Looking at the structure of the possible higher-order time derivatives, the only offending term is

$$\frac{\ddot{\phi}\dot{\phi}^2}{1+(\partial\phi)^2} \subset \mathcal{L}_K \ . \tag{2.2.6}$$

Expanding the denominator in powers of $(\partial\phi)^2$ we see that every term in this expansion is of the form $\ddot{\phi}\dot{\phi}^n(\vec{\nabla}\phi)^{2m}$ for some integer m and n. Integrating by parts, we can express each one in terms of first time derivatives only: $\ddot{\phi}\dot{\phi}^n(\vec{\nabla}\phi)^{2m} \sim \frac{\mathrm{d}}{\mathrm{d}t}(\dot{\phi}^{n+1})(\vec{\nabla}\phi)^{2m} \sim \dot{\phi}^{n+1}\frac{\mathrm{d}}{\mathrm{d}t}(\vec{\nabla}\phi)^{2m}$. The same can be done with the higher Galileons and with a curved target space (see for example the Hamiltonian analysis of [51, 52] in the non-relativistic case).

Chapter 3

Cosmological perturbations of massive gravity coupled to DBI Galileons

Having established the theoretical self-consistency of the massive gravity Galileon model, we now undertake the analysis of the theory's background cosmological spacetime solutions and their fluctuations. Much of the material is taken from [53, 54] which were written in collaboration with Melinda Andrews, Kurt Hinterbichler and Mark Trodden.

As was explained in the previous chapter, the construction of massive gravity coupled to Galileons is carried out using an extension of the probe brane approach [38–42, 55] for constructing general Galileon models and the bi-metric approach for constructing the dRGT nonlinear massive gravity theory [44, 45]. We introduce a physical metric $g_{\mu\nu}$ and a second, induced metric $\bar{g}_{\mu\nu}$ which is the pull-back through a dynamical embedding $\phi^A(x)$ into a five-dimensional Minkowski space with metric $\eta_{AB} = \mathrm{diag}(-1, 1, 1, 1, 1)$,

$$\bar{g}_{\mu\nu} = \eta_{AB}\partial_\mu\phi^A\partial_\nu\phi^B \ . \tag{3.0.1}$$

The action contains three kinds of terms:

$$S = S_{\mathrm{EH}}[g] + S_{\mathrm{mix}}[g, \bar{g}] + S_{\mathrm{Galileon}}[\bar{g}] \ . \tag{3.0.2}$$

The first part $S_{\mathrm{EH}}[g]$ is the Einstein-Hilbert action for $g_{\mu\nu}$

$$S_{\mathrm{EH}}[g] = \frac{M_{\mathrm{P}}^2}{2}\int \mathrm{d}^4x\,\sqrt{-g}\,R[g] \ . \tag{3.0.3}$$

The second part is the action mixing the two metrics,

$$S_{\rm mix}[g,\bar{g}] = M_{\rm P}^2 m^2 \int {\rm d}^4x\sqrt{-g}(\mathcal{L}_2+\alpha_3\mathcal{L}_3+\alpha_4\mathcal{L}_4) \ , \tag{3.0.4}$$

where

$$\begin{aligned}
\mathcal{L}_2 &= \frac{1}{2}\left([\mathcal{K}]^2-[\mathcal{K}^2]\right) \ , \\
\mathcal{L}_3 &= \frac{1}{6}\left([\mathcal{K}]^3-3[\mathcal{K}][\mathcal{K}^2]+2[\mathcal{K}^3]\right) , \\
\mathcal{L}_4 &= \frac{1}{24}\left([\mathcal{K}]^4-6[\mathcal{K}]^2[\mathcal{K}^2]+3[\mathcal{K}^2]^2+8[\mathcal{K}][\mathcal{K}^3]-6[\mathcal{K}^4]\right) \ ,
\end{aligned}$$

and where the brackets are traces of powers of the matrix $\mathcal{K}^\mu{}_\nu = \delta^\mu{}_\nu - \sqrt{g^{\mu\sigma}\bar{g}_{\sigma\nu}}$. The final part is the DBI Galileon action $S_{\rm Galileon}[\bar{g}]$ consisting of the Lovelock invariants constructed from $\bar{g}$, and their boundary terms (see [38, 40, 56] and Sec. IV.B of [39]; we use normalizations consistent with [40, 56]),

$$S_{\rm Galileon} = \Lambda^4 \int {\rm d}^4x\sqrt{-\bar{g}}\left\{-a_2+\frac{a_3}{\Lambda}[\bar{K}]-\frac{a_4}{\Lambda^2}\left([\bar{K}]^2-[\bar{K}^2]\right)+\frac{a_5}{\Lambda^3}\left([\bar{K}]^3-3[\bar{K}][\bar{K}^2]+2[\bar{K}^3]\right)\right\} , \tag{3.0.5}$$

where $\bar{K}_{\mu\nu}$ is the extrinsic curvature of the brane embedding $\phi^A(x)$ into the flat five-dimensional Minkowski space and indices are raised with $\bar{g}^{\mu\nu}$ (since the bulk is flat, we may use Gauss-Codazzi to eliminate all intrinsic curvatures in favor of extrinsic curvatures).

Note that we have set the cosmological constant and a possible tadpole term in $S_{\rm Galileon}$ to zero. This ensures the existence of a flat space solution with constant π. Restoring these terms does not change our essential conclusion.

3.1 Background Cosmology and Self Accelerating Solutions

For our purposes, we take a Friedmann-Robertson-Walker (FRW) ansatz for the physical metric

$${\rm d}s^2 = -N^2(t){\rm d}t^2 + a^2(t)\Omega_{ij}{\rm d}x^i{\rm d}x^j \ , \quad \Omega_{ij} = \delta_{ij} + \frac{\kappa}{1-\kappa r^2}x^i x^j \ , \tag{3.1.1}$$

where $\kappa < 0$ is the spatial curvature. (As shown in [53], this model does not admit non-trivial cosmological solutions for a flat FRW ansatz, just as pure dRGT massive gravity does not [20], and there are no solutions for $\kappa > 0$ since the fiducial Minkowski metric cannot be foliated by closed slices.) The embedding (the Stückelbergs) are chosen so that the fiducial metric has the symmetries of an open FRW spacetime [21],

$$\phi^0 = f(t)\sqrt{1-\kappa\vec{x}^2}, \quad \phi^i = \sqrt{-\kappa}f(t)x^i, \quad \phi^5 \equiv \pi(t) \ . \tag{3.1.2}$$

where $f(t)$ plays the role of a Stückelberg field which restores time reparametrization invariance. The induced metric then takes the form

$$\bar{g}_{\mu\nu}dx^\mu dx^\nu = \left(-\dot{f}(t)^2 + \dot{\pi}(t)^2\right)dt^2 - \kappa f(t)^2\Omega_{ij}(\vec{x})dx^i dx^j. \tag{3.1.3}$$

Note that we can obtain the extended massive gravity mass terms from the dRGT mass terms by replacing $\bar{g}_{\mu\nu}$ with $\bar{g}_{\mu\nu} + \partial_\mu\pi\partial_\nu\pi$.

This ansatz leads to the mini-superspace action

$$S_R = 3M_{\rm P}^2\int {\rm d}t\,\left[-\frac{\dot{a}^2 a}{N} + \kappa N a\right] \ , \tag{3.1.4}$$

$$S_{\rm mass} = 3M_{\rm P}^2\int {\rm d}t\, m^2\left[NF(a,f) - G(a,f)\sqrt{\dot{f}^2 - \dot{\pi}^2}\right] \ , \tag{3.1.5}$$

$$S_\pi = -\Lambda^4\int {\rm d}t\,(\sqrt{-\kappa}f)^3\sqrt{\dot{f}^2-\dot{\pi}^2} \ , \tag{3.1.6}$$

where

$$F(a,f) = a(a-\sqrt{-\kappa}f)(2a-\sqrt{-\kappa}f) + \frac{\alpha_3}{3}(a-\sqrt{-\kappa}f)^2(4a-\sqrt{-\kappa}f) + \frac{\alpha_4}{3}(a-\sqrt{-\kappa}f)^3 \ , \tag{3.1.7}$$

$$G(a,f) = a^2(a-\sqrt{-\kappa}f) + \alpha_3 a(a-\sqrt{-\kappa}f)^2 + \frac{\alpha_4}{3}(a-\sqrt{-\kappa}f)^3 \ .$$

Note that the cubic and higher Galileons in (3.0.5) do not contribute to the mini-superspace action.

Varying with respect to N, we obtain the Friedmann equation,

$$\frac{H^2}{N^2} + \frac{\kappa}{a^2} + m^2 \frac{F(a,f)}{a^3} = 0 \ . \tag{3.1.8}$$

The equations obtained by varying the action with respect to f and π, respectively, are

$$\begin{aligned} \frac{\delta S}{\delta f} = 3M_{\rm P}^2 m^2 \left[N\frac{\partial F}{\partial f} - \frac{\partial G}{\partial f}\sqrt{\dot{f}^2 - \dot{\pi}^2} \right] - 3\Lambda^4 (\sqrt{-\kappa})^3 f^2 \sqrt{\dot{f}^2 - \dot{\pi}^2} \\ + \frac{d}{dt}\left[\left(3M_{\rm P}^2 m^2 G + \Lambda^4 (\sqrt{-\kappa} f)^3\right) \frac{\dot{f}}{\sqrt{\dot{f}^2 - \dot{\pi}^2}} \right] = 0 \ , \end{aligned} \tag{3.1.9}$$

$$\frac{\delta S}{\delta \pi} = -\frac{d}{dt}\left[\left(3M_{\rm P}^2 m^2 G + \Lambda^4 (\sqrt{-\kappa} f)^3\right) \frac{\dot{\pi}}{\sqrt{\dot{f}^2 - \dot{\pi}^2}} \right] = 0 \ . \tag{3.1.10}$$

The acceleration equation obtained by varying with respect to a is redundant, due to the time reparametrization invariance of the action.

In contrast to GR, these equations enforce a constraint: the combination $\dot{f}\frac{\delta S}{\delta f} + \dot{\pi}\frac{\delta S}{\delta \pi}$ becomes

$$\frac{\partial G(a,f)}{\partial a}\left(\dot{a}\sqrt{\dot{f}^2 - \dot{\pi}^2} - \sqrt{-\kappa} N \dot{f} \right) = 0 \ , \tag{3.1.11}$$

the analogue of which for pure massive gravity is responsible for the well-known absence of flat FRW solutions in that theory.

There exist two branches of solutions depending on whether the constraint equation is solved by setting $\frac{\partial G}{\partial a} = 0$ or instead by setting $\dot{a}\sqrt{\dot{f}^2 - \dot{\pi}^2} - \sqrt{-\kappa} N\dot{f} = 0$. In this work we shall focus on the former choice, since this corresponds to de Sitter space – the self-accelerating branch of the theory [53], in which the metric takes the same form as the self-accelerating solution of the original massive gravity theory. Defining $X \equiv \frac{\sqrt{-\kappa} f}{a}$, we find an algebraic equation for f that can be written in the form $J_\phi = 0$, where

$$J_\phi \equiv 3 - 2X + \alpha_3(1-X)(3-X) + \alpha_4(1-X)^2 \ . \tag{3.1.12}$$

The solutions read

$$f(t) = \frac{1}{\sqrt{-\kappa}} X_\pm a(t) \,, \qquad X_\pm \equiv \frac{1 + 2\,\alpha_3 + \alpha_4 \pm \sqrt{1 + \alpha_3 + \alpha_3^2 - \alpha_4}}{\alpha_3 + \alpha_4} \ . \tag{3.1.13}$$

These are the same self-accelerated solutions that were found in pure massive gravity [21]. The solution for the extra Galileon field π can then be determined by solving (3.1.10).

However, for the theory at hand, there exists a new possibility. This second branch consists of solutions for which

$$\dot{a}\sqrt{\dot{f}^2 - \dot{\pi}^2} = \sqrt{-\kappa}\dot{f}. \tag{3.1.14}$$

In the case of the pure dRGT theory where $\pi = 0$, this branch gives only solutions for which $a = \sqrt{-\kappa}t$, which is just Minkowski space in Milne coordinates. Here we have the possibility of non-trivial solutions on this branch. Solving for $\dot{\pi}$ gives $\dot{\pi} = \pm\dot{f}\sqrt{1 + \frac{\kappa}{\dot{a}^2}}$, and substituting this into the π equation of motion (3.1.10) we see that $\dot{f}$ cancels and we are left with an algebraic equation in f which can in be solved. (For example, in the case where $\alpha_3 = \alpha_4 = 0$, we have

$$f = \frac{\Lambda^4 + 3m^2 M_{\rm P}^2 a^3 + \mathcal{C}\sqrt{\frac{-\kappa}{\kappa + \dot{a}^2}}}{3m^2 M_{\rm P}^2 a^2 \sqrt{-\kappa}}, \tag{3.1.15}$$

where $\mathcal{C}$ is the integration constant from integrating (3.1.10).) Using this to eliminate f from the Friedmann equation (3.1.8) yields an equation of motion for the scale factor and which can have solutions with non-trivial evolution.

3.2 Perturbations

We now turn to the behavior of perturbations around this background cosmological solution. To obtain the quadratic action for perturbations, we work in unitary gauge and expand the metric and π fields to second order in fluctuations around the background. We write the metric as $g_{\mu\nu} = g^{(0)}_{\mu\nu} + \delta g_{\mu\nu}$, with

$$\delta g_{\mu\nu} = \begin{pmatrix} -2N^2\Phi & NaB_i \\ NaB_j & a^2 h_{ij} \end{pmatrix} . \tag{3.2.1}$$

Here, Φ, B_i and h_{ij} are the small perturbations, N and a are the background lapse and scale factor, and we henceforth raise and lower latin indices with respect to Ω_{ij}.

The vector perturbation B_i can be decomposed into transverse and longitudinal components via

$$B_i = B_i^T + \partial_i B \ , \quad D^i B_i^T = 0, \tag{3.2.2}$$

where D_i denotes the covariant derivative with respect to Ω_{ij}. The tensor perturbations h_{ij} decompose into a transverse traceless component h_{ij}^{TT}, a transverse vector E_i^T, a longitudinal component E, and a trace Ψ as follows:

$$h_{ij} = 2\Psi\Omega_{ij} + \left(D_i D_j - \frac{1}{3}\Omega_{ij}\triangle\right) E + \frac{1}{2}(D_i E_j^T + D_j E_i^T) + h_{ij}^{TT} \ , \tag{3.2.3}$$

where $\triangle \equiv D^i D_i$, and the transverse traceless conditions read

$$D^i h_{ij}^{TT} = h_i^{TT\, i} = 0, \quad D^i E_i^T = 0 \ . \tag{3.2.4}$$

We denote the remaining dynamical scalar field – the Galileon perturbation – by τ, via

$$\phi^5 = \pi + \tau \ . \tag{3.2.5}$$

3.2.1 Preliminaries

Before writing the full quadratic actions for the various perturbations, we first write some intermediate expressions obtained from the expansions of the mass terms (3.0.4). This will serve to highlight the manner in which the kinetic terms vanish, and illustrate the similarities with pure dRGT.

For convenience, we introduce the quantities

$$s = \sqrt{1 - (\dot{\pi}/\dot{f})^2} \ , \quad r = \frac{\dot{f}a}{N\sqrt{-\kappa f}} \ , \tag{3.2.6}$$

and we will continue to use J_ϕ to denote the quantity (3.1.12) which vanishes on the equations of motion.

Expanding the mass term to linear order in the fluctuations yields

$$S_{\rm mix} = S^{(0)}_{\rm mix} + \int dx^4 N a^3 \sqrt{\Omega} \left[-\left(\Phi + \frac{1}{2} h \right) \rho_g + \frac{1}{2} M_{\rm P}^2 m_g^2 (1 - rs) X h J_\phi + M_{\rm P}^2 m_g^2 (r\dot{\pi}/\dot{f}^2 s) Y \dot{\tau} \right] \ , \tag{3.2.7}$$

where we have defined

$$\rho_g \equiv -M_{Pl}^2 m_g^2 (1-X) \left[3(2-X) + \alpha_3 (1-X)(4-X) + \alpha_4 (1-X)^2 \right], \tag{3.2.8}$$

$$Y \equiv X(1-X) \left[3 + 3\alpha_3 (1-X) + \alpha_4 (1-X)^2 \right] \ . \tag{3.2.9}$$

When the background equation of motion for the Stückelberg fields are satisfied, the terms linear in the metric match the corresponding terms of pure massive gravity. This suggests that we follow the massive gravity analysis of [24] and define

$$\tilde{S}_{\rm mix}[g_{\mu\nu}, \tau] \equiv S_{\rm mix}[g_{\mu\nu}, \tau] + \int {\rm d}^4 x \sqrt{-g} \rho_g \equiv M_{\rm P}^2 m_g^2 \int d^4 x N a^3 \sqrt{\Omega} \tilde{\mathcal{L}}_{\rm mix} \ . \tag{3.2.10}$$

Expanding to second order in perturbations we have,

$$\tilde{\mathcal{L}}^{(0)}_{\rm mix} = - rsY, \tag{3.2.11}$$

$$\tilde{\mathcal{L}}^{(1)}_{\rm mix} = 3(1-rs)XJ_\phi\Psi + (r\dot{\pi}/\dot{f}^2 s)Y\dot{\tau}, \tag{3.2.12}$$

$$\begin{aligned}\tilde{\mathcal{L}}^{(2)}_{\rm mix} =& \frac{1}{2}\frac{r}{s}\frac{1}{\dot{f}^2 s^2}Y\dot{\tau}^2 + \frac{1}{2}\left(6\Phi\Psi + \frac{B^T_i B^{Ti}}{1+rs}\right)XJ_\phi + 3\frac{r}{s}\frac{\dot{\pi}}{\dot{f}^2}XJ_\phi\dot{\tau}\Psi \\ &+ \frac{\dot{\pi}}{\sqrt{-\kappa}\dot{f}f}\left(\frac{r}{1+rs}\right)XJ_\phi\tau\triangle B - \frac{1}{2\kappa f^2}\left[\left(\frac{1-r^2}{1+rs}\right)XJ_\phi + \frac{r}{s}Y\right]\tau\triangle\tau \\ &+ \frac{1}{8}(1-rs)\left(12\Psi^2 - 2h^{TT}_{ij}h^{TTij} + E^T_j\triangle E^{Tj}\right)XJ_\phi \\ &+ \frac{1}{8}m_g^{-2}M^2_{GW}\left(24\Psi^2 - h^{TT}_{ij}h^{TTij} + \frac{1}{2}E^T_j\triangle E^{Tj}\right),\end{aligned} \tag{3.2.13}$$

where we have defined a quantity which will turn out to be the graviton mass term:

$$m_g^{-2}M^2_{GW} \equiv XJ_\phi + (1-rs)X^2\left[1+\alpha_3(2-X)+\alpha_4(1-X)\right] . \tag{3.2.14}$$

Here we have not imposed any equations of motion on the background. We note that all of the terms in (3.2.13) which depend upon Φ or B_i are proportional to J_ϕ, and therefore vanish on the de Sitter self-accelerating branch, on which $J_\phi = 0$. As we will see, this implies the vanishing of the graviton scalar and vector kinetic terms on this background.

3.2.2 Tensor perturbations

We now write the full second order action obtained from expanding (3.0.2) and decomposing the perturbations according to (3.2.1) and (3.2.2), (3.2.3).

The tensor perturbations take the same form as in pure massive gravity, but with a different time-dependent mass term,

$$S^{(2)}_{\rm tensor} = \frac{M_{\rm P}^2}{8}\int {\rm d}^4x\sqrt{\Omega}\,Na^3\left[\frac{1}{N^2}\dot{h}^{TTij}\dot{h}^{TT}_{ij} + \frac{1}{a^2}h^{TTij}(\triangle - 2\kappa)h^{TT}_{ij} - M^2_{GW}h^{TTij}h^{TT}_{ij}\right], \tag{3.2.15}$$

where M^2_{GW}, in terms of the definitions (3.2.6), (3.1) made above, takes the following value on the de Sitter self-accelerating branch,

$$M_{GW}^2 = \pm(rs-1)m_g^2 X_\pm^2 \sqrt{1+\alpha_3+\alpha_3^2-\alpha_4}\ . \tag{3.2.16}$$

As in pure massive gravity, the tensor perturbation maintains the correct sign for both the kinetic and gradient terms. However, the new mass term implies a more complicated region of parameter space in which the tensors are non-tachyonic, $M_{GW}^2 > 0$ (the sign of the mass term is given by the sign of $\pm(rs-1)$). Note, however, that even if this term is negative, so that we have a tachyonic instability, then barring any fine tuning such instabilities are of order the Hubble scale if we have chosen $m \sim H$ to ensure late time acceleration of the correct magnitude. This agrees qualitatively with the result found in pure massive gravity.

3.2.3 Vector perturbations

Since the vector field B_i^T obtained from δg_{0i} does not appear in the Lagrangian with any time derivatives, it can be eliminated as an auxiliary field. Leaving the background fields arbitrary for the moment, we find the solution

$$B_i^T = \frac{a(1+rs)\,(-\Delta-2\kappa)}{2\left[(1+rs)(-\Delta-2\kappa)+2a^2 J_\phi m^2 X\right] N}\dot{E}_i^T . \tag{3.2.17}$$

This matches the result of pure dRGT theory $B_i^T = \frac{a}{2N}\dot{E}_i^T$ when the Stückelberg equation of motion for the de Sitter self-accelerating branch is imposed, $J_\phi = 0$. It is instructive, however, to leave the backgrounds arbitrary so that we can explicitly see the kinetic term vanish. Substituting the general expression for the non-dynamical vector we obtain

$$S^{(2)}_{\rm vector} = \frac{M_{\rm P}^2}{8}\int d^4x\sqrt{\Omega}\,a^3 N\left\{\mathcal{T}_V(\dot{E}_i^T)^2 - \left[\frac{1}{2}M_{GW}^2(-\Delta-2\kappa)+J_\phi k^2 m^2(1-rs)\right](E_i^T)^2\right\}\ , \tag{3.2.18}$$

where

$$\mathcal{T}_V = \frac{a^2 J_\phi m^2 X\,(-\Delta-2\kappa)}{\left[(1+rs)(-\Delta-2\kappa)+2a^2 J_\phi m^2 X\right] N^2}\ . \tag{3.2.19}$$

The vanishing of the vector kinetic terms is now obvious on the de Sitter self-accelerating branch where $J_\phi = 0$. The vector Lagrangian has the same form as pure dRGT theory, only with a different time-dependent mass,

$$S^{(2)}_{\text{vector}} = -\frac{M_{\rm P}^2}{16}\int d^4x\sqrt{\Omega}\, a^3 N M_{GW}^2(-\Delta - 2\kappa)(E_i^T)^2. \tag{3.2.20}$$

3.2.4 Scalar perturbations

The analysis of the scalar perturbations simplifies considerably on the de Sitter self-accelerating branch since all the terms mixing scalar degrees of freedom from the graviton with the fluctuation of the Galileon vanish when $J_\phi = 0$, as can be seen from the expression (3.2.13). The scalars Φ and B coming from perturbations of δg_{00} and δg_{0i} appear without time derivatives and we may eliminate them as auxiliary fields. We obtain (this time imposing the self-accelerating background equation of motion $J_\phi = 0$)

$$\Phi = \frac{\kappa\Delta}{6a^2H^2}E - \frac{\Delta}{6HN}\dot{E} - \frac{\kappa}{a^2H^2}\Psi + \frac{1}{HN}\dot{\Psi} \tag{3.2.21}$$

$$B = \frac{\Delta}{6aH}E + \frac{a}{2N}\dot{E} - \frac{1}{aH}\Psi \tag{3.2.22}$$

which are the same as in pure dRGT theory. The calculation of the graviton scalar quadratic action mirrors the dRGT case and we find that the kinetic terms vanish and the action once again has the same form as pure dRGT, only with a modified time-dependent mass,

$$S^{(2)}_{\text{scalar}} = \frac{M_{\rm P}^2}{2}\int d^4x\sqrt{\Omega}\, a^3 N\left(6M_{GW}^2\Psi^2 + \frac{1}{6}M_{GW}^2\Delta(-\Delta - 3\kappa)E^2\right). \tag{3.2.23}$$

We now turn to the expansion of the Galileon action (3.0.5), using (3.2.5). We start by expanding the lowest Galileon, the DBI term (the one proportional to a_2 in (3.0.5)) to quadratic order in τ. We obtain $S_{\rm DBI} = -a_2\Lambda^4 \int d^4x N a^3 \sqrt{\Omega}\mathcal{L}_{\rm DBI}$, where

$$\mathcal{L}^{(0)}_{\rm DBI} = -rsX^4\ , \tag{3.2.24}$$

$$\mathcal{L}^{(1)}_{\rm DBI} = \left(\frac{r}{s}\frac{\dot\pi}{\dot f^2}\dot\tau\right) X^4\ , \tag{3.2.25}$$

$$\mathcal{L}^{(2)}_{\rm DBI} = \frac{1}{2}\frac{r}{s}\left(\frac{1}{\dot f^2 s^2}\dot\tau^2 + \frac{1}{\kappa f^2}\tau\Delta\tau\right) X^4\ . \tag{3.2.26}$$

From $\mathcal{L}^{(2)}_{\rm DBI}$ we see that the effect of including the DBI Lagrangian is to shift $Y \to Y - (a_2\Lambda^4/m^2M_{\rm P}^2)X^4$ in the quadratic action (3.2.13). Note that on the de Sitter self-accelerating branch, where $J_\phi = 0$, this is equivalent to shifting the brane tension by

$$\Lambda^4 \to \tilde\Lambda^4 = \Lambda^4 - \frac{m^2M_{\rm P}^2}{a_2}\frac{Y_\pm}{X_\pm^4}\ . \tag{3.2.27}$$

We therefore see that on the self-accelerating de Sitter branch, the Galileon has the correct-sign kinetic term provided

$$\frac{m^2M_{\rm P}^2}{a_2\Lambda^4}\frac{Y_\pm}{X_\pm^4} < 1\ . \tag{3.2.28}$$

It is clear that this constraint can always be satisfied by choosing $a_2\Lambda^4$ appropriately large. Note that the background Stückelberg and Galileon fields do not lead to any simplification for the DBI quadratic action.

The higher Galileon terms in (3.0.5) can be similarly expanded to quadratic order. After imposing the background equation for the Stueckelberg/Galileon and its time derivatives, we obtain $S_{\rm Galileon} = \Lambda^4 \int d^4x N a^3\sqrt{\Omega}\mathcal{L}_{\rm Galileon}$, where

$$\begin{aligned}\mathcal{L}^{(2)}_{\rm Galileon} = &-\frac{r}{\dot f^2 s^3}\left[-\frac{1}{2}a_2 + 3\frac{a_3}{\Lambda}\left(\frac{\dot\pi}{sf\dot f}\right) - 9\frac{a_4}{\Lambda^2}\left(\frac{\dot\pi}{sf\dot f}\right)^2 + 12\frac{a_5}{\Lambda^3}\left(\frac{\dot\pi}{sf\dot f}\right)^3\right]X^4\dot\tau^2 \quad (3.2.29)\\ &-\frac{r}{s}\frac{1}{\kappa f^2}\left[\frac{1}{2}a_2 + \frac{a_3}{\Lambda}\frac{\dot\pi}{sf\dot f} - \frac{a_4}{\Lambda^2}\frac{3\dot\pi^4 + 11\dot f^2\dot\pi^2 - 2\dot f^4}{s^2f^2\dot f^4} + 6\frac{a_5}{\Lambda^3}\frac{3\dot\pi^4 + 2\dot f^2\dot\pi^2 + 2\dot f^4}{s^3f^3\dot f^5}\dot\pi\right]X^4\tau\triangle\tau\ .\end{aligned}$$

The conditions for stability can now be read off by requiring that these kinetic terms have the correct sign.

Chapter 4

Cosmologies of extended massive gravity

In this chapter we explore the cosmology of variable mass massive gravity and its viability to produce self-inflation. Most of the content can be found in [53] which was coauthored with Kurt Hinterbichler and Mark Trodden.

The variable mass massive gravity is the dRGT theory in which the graviton mass squared is promoted to a scalar field Φ,

$$S = S_{\rm EH} + S_{\rm mass} + S_{\Phi} \ , \tag{4.0.1}$$

where

$$\begin{aligned} S_{\rm EH} &= \frac{1}{2}M_{\rm P}^2 \int {\rm d}^4x \, \sqrt{-g} \, R \ , && (4.0.2) \\ S_{\rm mass} &= M_{\rm P}^2 \int {\rm d}^4x \, \sqrt{-g} \, \Phi \left(\mathcal{L}_2 + \alpha_3 \mathcal{L}_3 + \alpha_4 \mathcal{L}_4 \right) \ , && (4.0.3) \\ S_{\Phi} &= - \int {\rm d}^4x \, \sqrt{-g} \, \left[\frac{1}{2} g(\Phi) (\partial\Phi)^2 + V(\Phi) \right] \ . && (4.0.4) \end{aligned}$$

Here α_3, α_4 are the two free parameters of dRGT theory. We have allowed for an arbitrary kinetic function $g(\Phi)$ and potential $V(\Phi)$, so that there is no loss of generality in the scalar sector. The mass term consists of the ghost-free combinations [3],

$$\begin{aligned} \mathcal{L}_2 &= \frac{1}{2} \left([\mathcal{K}]^2 - [\mathcal{K}^2] \right) \ , \\ \mathcal{L}_3 &= \frac{1}{6} \left([\mathcal{K}]^3 - 3[\mathcal{K}][\mathcal{K}^2] + 2[\mathcal{K}^3] \right) , \\ \mathcal{L}_4 &= \frac{1}{24} \left([\mathcal{K}]^4 - 6[\mathcal{K}]^2[\mathcal{K}^2] + 3[\mathcal{K}^2]^2 + 8[\mathcal{K}][\mathcal{K}^3] - 6[\mathcal{K}^4] \right) \ , \end{aligned} \tag{4.0.5}$$

where $K^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} - \sqrt{g^{\mu\sigma}\eta_{\sigma\nu}}$, $\eta_{\mu\nu}$ is the non-dynamical fiducial metric which we have taken to be Minkowski, and the square brackets are traces. To work in the gauge invariant formalism, we introduce four Stückelberg fields ϕ^a through the replacement $\eta_{\mu\nu} \to \partial_\mu \phi^a \partial_\mu \phi^b \eta_{ab}$.

Variable mass massive gravity was first considered in [20], and further studied in [31, 57–59] (see also [27] for a more symmetric scalar extension of dRGT). dRGT gravity has been demonstrated to be ghost-free through a variety of different approaches [10, 11, 13–15], and the introduction of the scalar field does not introduce any new Boulware-Deser like ghost degrees of freedom into the system [31].

For cosmological applications we take a Friedmann, Robertson-Walker (FRW) ansatz for the metric, so that

$$\mathrm{d}s^2 = -N^2(t)\mathrm{d}t^2 + a^2(t)\Omega_{ij}\mathrm{d}x^i\mathrm{d}x^j, \tag{4.0.6}$$

where

$$\Omega_{ij} = \delta_{ij} + \frac{\kappa}{1-\kappa r^2}x^i x^j \tag{4.0.7}$$

is the line element for a maximally symmetric 3-space of curvature κ and $r^2 = x^2 + y^2 + z^2$. We also take the assumptions of homogeneity and isotropy for the scalar field,

$$\Phi = \Phi(t). \tag{4.0.8}$$

Consider first the case of flat Euclidean sections ($\kappa = 0$). We work in the gauge invariant formulation, and the Stueckelberg degrees of freedom take the ansatz [20, 21].

$$\phi^i = x^i, \quad \phi^0 = f(t), \tag{4.0.9}$$

where $f(t)$, like $a(t)$, is a monotonically increasing function of t.

Inserting (4.0.6) and (4.0.9) into the action, we obtain the mini-superspace action

$$S_R = 3M_{\mathrm{P}}^2 \int \mathrm{d}t \left[-\frac{\dot{a}^2 a}{N} \right], \tag{4.0.10}$$

$$S_{\mathrm{mass}} = 3M_{\mathrm{P}}^2 \int \mathrm{d}t\, \Phi \left[NF(a) - \dot{f} G(a) \right], \tag{4.0.11}$$

$$S_\phi = \int \mathrm{d}t\, a^3 \left[\frac{1}{2} N^{-1} g(\Phi) \dot{\Phi}^2 - NV(\Phi) \right]. \tag{4.0.12}$$

where

$$F(a) = a(a-1)(2a-1) + \frac{\alpha_3}{3}(a-1)^2(4a-1) + \frac{\alpha_4}{3}(a-1)^3, \tag{4.0.13}$$

$$G(a) = a^2(a-1) + \alpha_3 a(a-1)^2 + \frac{\alpha_4}{3}(a-1)^3. \tag{4.0.14}$$

This mini-superspace action is invariant under time reparametrizations, under which f transforms like a scalar.

There are four equations of motion, obtained by varying with respect to F, N, Φ and a. As in GR, the Noether identity for time reparametrization invariance tells us that the acceleration equation obtained by varying with respect to a is a consequence of the other equations, so we may ignore it. After deriving the equations, we will fix the gauge $N = 1$ (this cannot be done directly in the action without losing equations).

Varying with respect to f we obtain the constraint pointed out in [57],

$$\Phi = \frac{\mathcal{C}}{G(a)}, \tag{4.0.15}$$

where $\mathcal{C}$ is an arbitrary integration constant. (Note that the analogous equation in the fixed mass theory implies that $a =$ constant, so there are no evolving flat FRW solutions in that case [20].) Varying with respect to N, we obtain the Friedmann equation,

$$3M_{\mathrm{P}}^2\left[H^2+\frac{\Phi F(a)}{a^3}\right]=\frac{1}{2}g(\Phi)\dot{\Phi}^2+V, \tag{4.0.16}$$

and varying with respect to Φ we obtain the scalar field equation

$$\begin{aligned} g(\Phi)\left[\ddot{\Phi}+3H\dot{\Phi}\right]+\frac{1}{2}g'(\Phi)\dot{\Phi}^2+V'(\Phi) \\ =3M_{\mathrm{P}}^2\left[\frac{F(a)}{a^3}-\dot{f}\frac{G(a)}{a^3}\right]. \end{aligned} \tag{4.0.17}$$

Rather than solving the coupled second-order Einstein-scalar equations of motion, one can instead reduce the system to a single first-order Friedmann equation. The relation (4.0.15) can be used to eliminate Φ and its first derivative from (4.0.16), which then becomes a first-order differential equation in a which determines the scale factor,

$$H^2=\frac{V\left(\frac{\mathcal{C}}{G(a)}\right)-3M_P^2\mathcal{C}\frac{F(a)}{a^3G(a)}}{3M_P^2-\frac{1}{2}\mathcal{C}^2g\left(\frac{\mathcal{C}}{G(a)}\right)\frac{G'(a)^2a^2}{G(a)^4}}. \tag{4.0.18}$$

Once we have solved for the scale factor, the scalar Φ is determined from (4.0.15) and the Stueckelberg field f is determined by solving (4.0.17)[5].

4.0.5 Singularities

One feature of this model that has not been noticed previously is that it allows for the possibility of curvature singularities at finite values of a. These can happen when the evolution attempts to pass through values of a for which the denominator of the right hand side of (4.0.18) goes to zero.

When this happens a is finite, but the Hubble parameter, and hence $\dot{a}$, is blowing up. The scalar curvature is also blowing up, so this is a genuine curvature singularity; a "big brake"

[5] Note that in general the Stueckelberg field cannot be chosen arbitrarily as in [57] but is non-trivially constrained by the choice of mass term, or in this case, kinetic function $g(\Phi)$.

where the universe comes to some finite scale factor and gets stuck [60, 61]. Similar types of singularities also occur in DGP [62].

For example, Taylor expanding the denominator of (4.0.18) for large a we obtain a critical value of the scale factor at which the Hubble parameter diverges,

$$a_{\rm cr} = \sqrt{3}\left(\frac{g(0)}{2M_{\rm P}^2}\right)^{1/6}\left(\frac{\mathcal{C}}{3+3\alpha_3+\alpha_4}\right)^{1/3}. \tag{4.0.19}$$

4.0.6 Self-Inflation

We now consider the possibility that the graviton has a large mass in the early universe, through some natural displacement (and resulting VEV) of the scalar field from its true minimum near zero. We seek dynamics such that the scalar field slowly rolls down its potential, during which time the graviton remains massive, resulting in a large self-acceleration of the universe. This self-acceleration comes from the second term on the left-hand side of the Friedmann equation (4.0.16). For this to be true self-acceleration this term should be much larger than both the scalar kinetic energy and potential on the right hand side, so that the acceleration is primarily driven by the modification to gravity and not by the scalar field. After many e-folds, Φ should roll towards zero, self-inflation should end, and the graviton mass should become small at late times.

Thus, assume we have an inflationary solution $a \sim e^{Ht}$, with $H \sim$ constant. The scale factor is growing exponentially, so we Taylor expand the entire right hand side of (4.0.18) for large a, as

$$H^2 = \frac{V(0)}{3M_P^2} + \mathcal{C}\left[\frac{\frac{V'(0)}{M_P^2} - (6+4\alpha_3+\alpha_4)}{3+3\alpha_3+\alpha_4}\right]\frac{1}{a^3} + \mathcal{O}\left(\frac{1}{a^4}\right). \tag{4.0.20}$$

We see that the dependence on all of the massive gravity modifications redshifts away exponentially, at least as fast as a^{-3}, and we are left with inflation driven only by the value of the potential at $\Phi = 0$. (In particular, contributions sensitive to the function g only start to enter at $\mathcal{O}(1/a^7)$.)

Said another way, we only have self-inflation if the quantity $\frac{\phi F(a)}{a^3}$ in (4.0.16) is approximately constant when $a \sim e^{Ht}$. But the constraint equation (4.0.15) makes this impossible: we see from (4.0.15) that $\Phi \sim \frac{1}{a^3}$, since $G(a) \sim a^3$ for large a. Since $F(a) \sim a^3$, the quantity $\frac{\Phi F(a)}{a^3}$ behaves like $\sim \Phi \sim a^{-3}$, so it goes to zero exponentially fast and we cannot sustain self-inflation.

Having encountered an obstacle to the possibility of self-inflation in the flat slicing, we now investigate the possibility in the open slicing ($\kappa < 0$). Following [21] we take the Stueckelberg ansatz to be

$$\phi^0 = f(t)\sqrt{1-\kappa r^2}, \quad \phi^i = \sqrt{-\kappa} f(t) x^i. \tag{4.0.21}$$

The mini-superspace action then becomes

$$S_R = 3M_{\mathrm{P}}^2 \int \mathrm{d}t \left[-\frac{\dot{a}^2 a}{N} + \kappa N a \right], \tag{4.0.22}$$

$$S_{\mathrm{mass}} = 3M_{\mathrm{P}}^2 \int \mathrm{d}t \, \Phi \left[N F(a,f) - \dot{f} G(a,f) \right], \tag{4.0.23}$$

$$S_\phi = \int \mathrm{d}t \, a^3 \left[\frac{1}{2} N^{-1} g(\Phi) \dot{\Phi}^2 - N V(\Phi) \right], \tag{4.0.24}$$

where

$$\begin{aligned} F(a,f) &= a(a-\sqrt{-\kappa}f)(2a-\sqrt{-\kappa}f) \\ &\quad + \frac{\alpha_3}{3}(a-\sqrt{-\kappa}f)^2(4a-\sqrt{-\kappa}f) + \frac{\alpha_4}{3}(a-\sqrt{-\kappa}f)^3, \end{aligned} \tag{4.0.25}$$

$$\begin{aligned} G(a,f) &= a^2(a-\sqrt{-\kappa}f) + \alpha_3 a (a-\sqrt{-\kappa}f)^2 \\ &\quad + \frac{\alpha_4}{3}(a-\sqrt{-\kappa}f)^3. \end{aligned} \tag{4.0.26}$$

Again, we have time reparametrization invariance so we can ignore the acceleration equation, and we will fix the gauge $N = 1$ after deriving the equations of motion. The constraint equation arising from varying with respect to f is

$$(\dot{a} - \sqrt{-\kappa})\Phi \frac{\partial G(a,f)}{\partial a} + G(a,f)\dot{\Phi} = 0 \; . \tag{4.0.27}$$

The Friedmann equation obtained by varying with respect to N is

$$3M_{\rm P}^2 \left[H^2 + \frac{\kappa}{a^2} + \frac{\Phi F(a,f)}{a^3} \right] = \frac{1}{2} g(\Phi)\dot{\phi}^2 + V(\Phi), \tag{4.0.28}$$

and the equation of motion for Φ is

$$\begin{aligned} g(\Phi)\left[\ddot{\Phi} + 3H\dot{\Phi}\right] &+ \frac{1}{2} g'(\Phi)\dot{\Phi}^2 + V'(\Phi) \\ &= 3M_{\rm P}^2 \left[\frac{F(a,f)}{a^3} - \dot{f}\frac{G(a,f)}{a^3} \right] . \end{aligned} \tag{4.0.29}$$

In order to obtain inflation driven by the graviton mass, the term $\Phi F(a,f)/a^3$ in the Friedmann equation (4.0.28) must be approximately constant when $a \sim e^{Ht}$. Rearranging the constraint equation (4.0.27) to isolate Φ gives

$$\frac{\dot{\Phi}}{\Phi} = -\left(H - \frac{\sqrt{-\kappa}}{a} \right) \frac{a \frac{\partial G(a,f)}{\partial a}}{G(a,f)} . \tag{4.0.30}$$

Now there are three possibilities: the first is that $a(t) \sim e^{Ht}$ grows faster than $f(t)$. In this case, the right-hand side of (4.0.30) approaches a constant at late times, $\dot{\Phi}/\Phi \sim -3H + \mathcal{O}(1/a)$, which tells us that Φ decreases exponentially at late times, $\Phi(t) \sim e^{-3Ht}$. This, in turn, implies that the self-acceleration quantity $\Phi F(a,f)/a^3$ in (4.0.28) decreases exponentially like Φ, since $F(a,f)/a^3$ approaches a constant. So once again, we cannot sustain inflation in this case. The second possibility is that $f(t)$ grows faster than $a(t) \sim e^{Ht}$. In this case, the right-hand side of (4.0.30) goes to zero at late times, so Φ becomes constant. The self-acceleration quantity $\Phi F(a,f)/a^3$ in (4.0.28) now grows without bound as $\sim f^3(-\kappa)^{3/2}/a^3$ at late times, so again we do not achieve sustained inflation. Finally, there is the possibility that $f(t) \sim e^{-Ht}$, growing at the same rate as $a(t)$. This case follows the same pattern as the first possibility – the right-hand side of (4.0.30) approaches a constant at late times, and the self-acceleration quantity decreases exponentially.

In summary, flat and open FRW solutions exist in mass-varying massive gravity, but the constraint equation (the one which forbids flat FRW solutions in pure dRGT theory) does not allow for long-lasting self-inflation. Finally, homogeneous and isotropic closed FRW solutions are not possible for the same reason they are not in dRGT – the fiducial Minkowski metric cannot be foliated by closed slices.

Chapter 5

Holography of the Pseudo-Conformal Universe

This chapter describes the construction of the pseudo-conformal universe in AdS/CFT framework as well as the holographic calculation of renormalized one-point functions. Most of its content can be found in [63] which was coathored with Kurt Hinterbichler and Mark Trodden.

In the most common examples of the AdS/CFT correspondence, the boundary field theories are SU(N) gauge theories and the bulk gravitational theories are string theories which reduce to Einstein gravity in an appropriate large-N limit [64]. Indeed, as previously anticipated [65], 4D gauge theories in flat space should admit an exact description in terms of string theories on 5D backgrounds with a curved fifth dimension ρ

$$ds^2 = d\rho^2 + a(\rho)^2 \eta_{\mu\nu} dx^\mu dx^\nu \ , \tag{5.0.1}$$

where $\eta_{\mu\nu}$ is the Poincaré invariant metric of the 4D gauge theory and ρ is related to the Liouville (longitudinal) mode which arises in the quantization of the non-critical string. The gauge theory must be located at a position ρ_* such that the warp factor either vanishes or becomes infinite. If the gauge theory is a conformal field theory (CFT) then conformal invariance fixes $a(\rho) = e^{\rho/R}$, corresponding to AdS_5 with radius R, and it is convenient to choose $\rho_* = \infty$ which corresponds to placing the CFT at the boundary of AdS. The AdS/CFT correspondence can thus be viewed as a particular example of a more general gauge-string duality. Wick rotating this picture, $a(\rho)$ is essentially a scale factor, and the conformal limit is de Sitter space [66, 67]. This observation has been used to provide an alternative way to look at the behavior of inflationary spacetimes in the early universe, in which a Euclidean CFT at the asymptotic future is dual to an inflationary spacetime in the bulk [68–71].

This raises the question of whether it is possible to holographically realize other, non-inflationary proposals for the physics of the early universe. The pseudo-conformal universe [33–36] is an early universe scenario whose characteristic feature is spontaneous symmetry breaking of the conformal group in 4D to a subalgebra which is isomorphic to the algebra of de Sitter isometries. The idea is to postulate that the early universe is described by a CFT containing a set of scalar primary operators $\mathcal{O}_I$ with conformal dimensions Δ_I. The CFT is assumed to exist in a state which spontaneously breaks the so$(2,4)$ conformal algebra to an so$(1,4)$ de Sitter subalgebra, which happens when the operators develop time-dependent vacuum expectation values of the form

$$\langle \mathcal{O}_I \rangle \propto \frac{1}{(-t)^{\Delta_I}}. \tag{5.0.2}$$

The time t runs from $-\infty$ to 0, and the scenario breaks down at late times near $t=0$, where a re-heating transition to a traditional radiation dominated expansion must take place. This symmetry breaking pattern ensures that spectator fields of vanishing conformal dimension will automatically acquire scale-invariant spectra [35], which under favorable conditions can be transferred to become the scale invariant curvature perturbations we see today [72].

During the pseudo-conformal phase, spacetime is approximately flat, in contrast to the exponential expansion of inflation. As a consequence, gravity wave production in the pseudo-conformal scenario is exponentially suppressed. Thus, an observation of a large primordial tensor to scalar ratio (for example the interpretation of the result reported by BICEP2 [73]) would provide a crucial test of the pseudo-conformal proposal.

In this chapter, we will be interested in constructing a bulk dual to a CFT in the pseudo-conformal phase. In the case of dS/CFT and duals to inflation, the physics of interest is in the bulk and the dual is the boundary CFT. Here, by contrast, the physics of interest is the non-gravitational CFT on the boundary, and the dual is the bulk gravitational theory.

If a CFT possesses a gravitational dual, then the conformal vacuum corresponds to empty AdS, and other states of the CFT Hilbert space correspond to activating configurations of fields in the bulk. These field configurations can break the $\mathrm{so}(2,4)$ isometry group of AdS_5 to a subgroup, which in turn breaks the isometry group of the spacetime through gravitational backreaction. At large values of the radial coordinate ρ, however, the warp factor should approach $\sim e^{\rho/R}$, corresponding to restoration of the full conformal group at high energies. In general the broken/unbroken isometries of the asymptotically anti-de Sitter background map to the corresponding broken/unbroken conformal generators of the field theory. It follows that to implement the pseudo-conformal mechanism in AdS/CFT, we need a spacetime with the isometries of $\mathrm{so}(1,4)$, which are enhanced to $\mathrm{so}(2,4)$ at the boundary. Geometrically, this corresponds to a domain-wall spacetime asymptoting to anti-de Sitter space, where the domain wall is foliated by four-dimensional de Sitter slices. Since the isometry group of dS_4 is $\mathrm{so}(1,4)$, this realizes the required breaking pattern. In the limit in which backreaction is ignored, and near the boundary, this should revert to AdS_5 in the de Sitter foliation.

In Section 5.1 we will first consider the simpler case of a probe scalar, ignoring backreaction, in which the background profile for the scalar should be preserved by a $\mathrm{so}(1,4)$ subgroup of $\mathrm{so}(2,4)$. We find exact solutions of the wave equation for any value of the mass of the scalar. We then generalize in Section 5.2 to the fully interacting Einstein-scalar theory and obtain the background equations of motion for domain-wall spacetimes which have the appropriate symmetries, providing an explicit solution for the case of a massless scalar. We determine the VEV of the operators dual to the metric and scalar field and show that they have the correct form (5.0.2) to break the conformal group of the boundary field theory to a de Sitter subalgebra. We review the appropriate holographic renormalization machinery and we calculate the exact renormalized one-point functions for arbitrary scalar source configurations and boundary metric. We describe the analytic continuations which relate the our setup to the interface CFTs in Appendix 5.3.2.

5.1 Probe Scalar Limit

As a warmup for the full back-reacted problem, we first consider a probe scalar on AdS_5. The coordinates adapted for working with a dual Minkowski CFT are those of the Poincaré patch, in which AdS_5 is foliated by Minkowski slices parametrized by $x^\mu = (t, x^i)$,

$$ds^2 = \frac{1}{z^2}\left(dz^2 + \eta_{\mu\nu}dx^\mu dx^\nu\right) \ . \tag{5.1.1}$$

Here the radial coordinate is z, which ranges over $(0, \infty)$, with 0 the boundary and ∞ deep in the bulk. We have set the AdS radius to unity. The Killing vectors for AdS_5 in these coordinates are

$$P_\mu = -\partial_\mu, \tag{5.1.2}$$

$$L_{\mu\nu} = x_\mu\partial_\nu - x_\nu\partial_\mu, \tag{5.1.3}$$

$$K_\mu = x^2\partial_\mu - 2x_\mu x^\nu\partial_\nu + z^2\partial_\mu - 2x_\mu z\partial_z, \tag{5.1.4}$$

$$D = -x^\mu\partial_\mu - z\partial_z \ . \tag{5.1.5}$$

The first two sets of Killing vectors are the generators of the 4D Poincaré subalgebra $\mathrm{iso}(1,3)$ preserved by constant z slices. Going to the boundary at $z = 0$, the Killing vectors become the generators of the 4D conformal group $\mathrm{so}(2,4)$, which is the statement that the isometries of anti-de Sitter act on the boundary as conformal transformations.

A configuration of a bulk scalar ϕ of mass m will have an expansion near the boundary of the form

$$\phi(x,z) = z^{\Delta_-}\left[\phi_0(x) + \mathcal{O}(z^2)\right] + z^{\Delta_+}\left[\psi_0(x) + \mathcal{O}(z^2)\right], \tag{5.1.6}$$

where

$$\Delta_\pm = 2 \pm \sqrt{4 + m^2}. \tag{5.1.7}$$

(There can be additional logarithmic terms if $\sqrt{4+m^2}$ is an integer.) The coefficient $\phi_0(x)$ is a source term in the lagrangian of the CFT which sources an operator $\mathcal{O}$ of dimension Δ_+, and the coefficient $\psi_0(x)$ determines the vacuum expectation value (VEV) of this operator [74],

$$\langle \mathcal{O} \rangle = (2\Delta_+ - 4)\psi_0. \tag{5.1.8}$$

We are interested in the case in which there is a VEV of the form (5.0.2) in the absence of sources, so that the symmetries are spontaneously broken by the VEV rather than explicitly broken by a source. Thus we seek configurations for which $\psi_0 \propto 1/(-t)^{\Delta_+}$ and $\phi_0 = 0$.

The field profile we want must preserve the D, P_i, L_{ij} and K_i conformal generators, which form the unbroken de Sitter so$(1,4)$ subalgebra of interest [35]. We seek the most general bulk scalar field configuration which preserves these symmetries. Preservation of the spatial momentum and rotations, P_i and L_{ij}, implies that the scalar depends only on t and z,

$$\phi = \phi(z,t)\ . \tag{5.1.9}$$

Demanding invariance under $D = -z\partial_z - x^\mu\partial_\mu$, gives

$$z\partial_z\phi + t\partial_t\phi = 0\ , \tag{5.1.10}$$

which implies that the field is a function only of the ratio z/t,

$$\phi = \phi(z/t)\ . \tag{5.1.11}$$

This is now automatically invariant under the spatial special conformal generators, K_i,

$$K_i\phi = -2x^i(z\partial_z + t\partial_t)\phi = 0\ . \tag{5.1.12}$$

Thus, a profile $\phi(z/t)$ is the most general one which preserves the required so$(1,4)$ symmetries.

It will be useful to work in coordinates adapted to the unbroken so$(1,4)$ symmetries. Define a radial coordinate $\rho \in (0,\infty)$ and a time coordinate $\eta \in (-\infty, 0)$ by the relations

$$\begin{aligned} t &= \eta \coth \rho, \quad z = (-\eta)\,\mathrm{csch}\,\rho, \\ \rho &= \cosh^{-1}\left[\frac{(-t)}{z}\right], \qquad \eta = -\sqrt{t^2 - z^2}. \end{aligned} \tag{5.1.13}$$

In these coordinates, the metric becomes

$$ds^2 = d\rho^2 + \sinh^2\rho \left[\frac{-d\eta^2 + d\vec{x}^2}{\eta^2}\right] \ , \tag{5.1.14}$$

which we recognize as the foliation of AdS_5 by inflationary patch dS_4 slices. These coordinates cover the region $t < 0$, $(-t) > z$, which is the bulk causal diamond associated with the time interval $t = (-\infty, 0)$. This is the region we expect to be dual to the pseudo-conformal scenario[6]. The boundary is approached as $\rho \to \infty$, and the line $(-t) = z$ is approached as $\rho \to 0$. Near the boundary, the coordinate η corresponds with t, and ρ approaches $e^\rho \simeq 2\frac{(-t)}{z}$. The region and coordinates are illustrated in Figure 5.1.

We now consider the scalar wave equation in these coordinates. The equation of motion for a minimally coupled real scalar field of mass m on curved space is

$$(\Box^{(5)} - m^2)\phi = 0 \ . \tag{5.1.15}$$

In the de Sitter adapted coordinates, a scalar configuration with the desired profile (5.1.11) is one which depends only on the ρ coordinate. The wave equation then reduces to

$$\phi''(\rho) + 4\coth\rho\, \phi'(\rho) - m^2\phi(\rho) = 0 \ . \tag{5.1.16}$$

[6]Note that this region does not satisfy the criterion proposed in [75], which suggests that a full duality may require non-local operators in an essential way. See also [76–79].

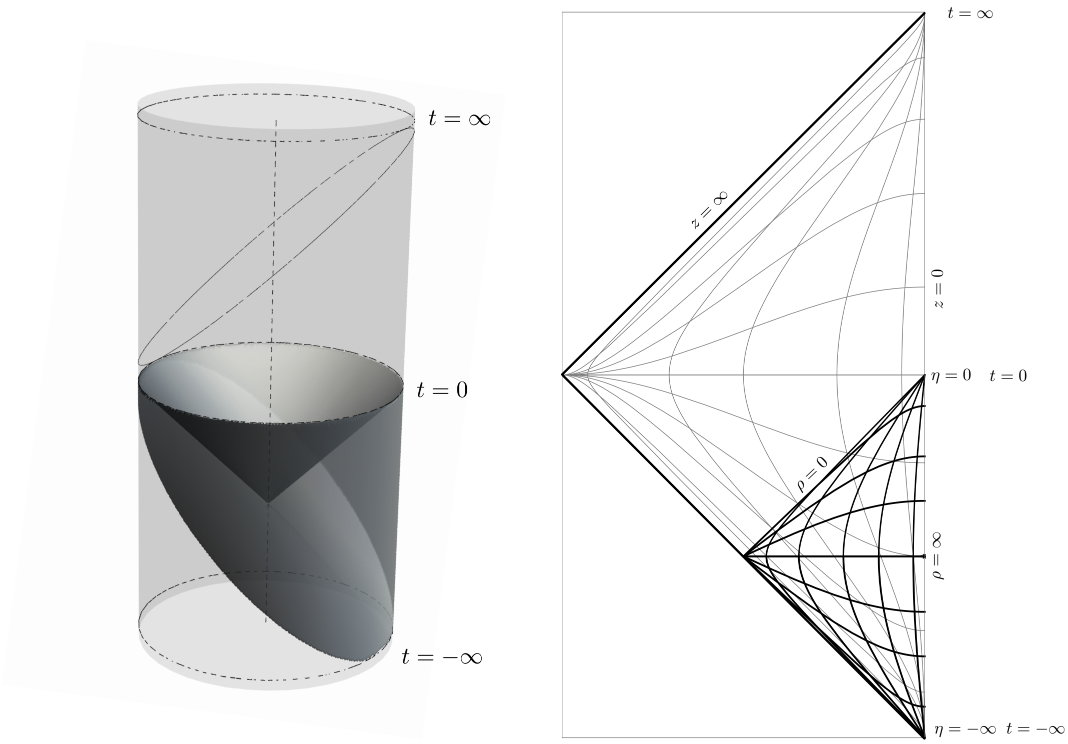

Figure 1: Penrose diagrams showing the Poincaré coordinates and de Sitter slice coordinates. The left-hand figure shows the global AdS cylinder; the de Sitter slice coordinate region is bounded from above by the lightcone which emanates downward from $t = 0$, and is bounded from below by the slanted ellipse, which also marks the lower boundary of the Poincare patch (the upper slanted ellipse shown in outline form is the upper boundary of the Poincaré patch). The right-hand figure shows a two dimensional slice down the axis of the AdS cylinder; thin lines are Poincaré lines of constant z and t, thick lines de Sitter slice lines of constant ρ and η.

The general solution is

$$\begin{aligned}
\phi(\rho) &= C_+ \frac{e^{-\sqrt{4+m^2}\rho}\left(\sqrt{4+m^2}+\coth\rho\right)\operatorname{csch}^2\rho}{\sqrt{4+m^2}} \\
&\qquad + C_- \frac{e^{\sqrt{4+m^2}\rho}\left(\sqrt{4+m^2}-\coth\rho\right)\operatorname{csch}^2\rho}{3+m^2}, \quad m^2 \neq -3, -4\ , \\
\phi(\rho) &= C_+ \operatorname{csch}^3\rho + C_- \operatorname{csch}^3\rho(\sinh 2\rho - 2\rho)\ , \quad m^2 = -3\,, \\
\phi(\rho) &= C_+ \operatorname{csch}^2\rho(\rho\coth\rho - 1) + C_- \operatorname{csch}^2\rho\coth\rho\ , \quad m^2 = -4\,,
\end{aligned} \tag{5.1.17}$$

where $C_\pm$ are the two integration constants of the second order wave equation. The mass must satisfy the Breitenlohner-Freedman stability bound $m^2 \geq -4$ [80]. The mass range $m^2 > 0$ corresponds to irrelevant operators $\Delta_+ > 4$, $-4 \leq m^2 < 0$ corresponds to relevant operators $\Delta_+ < 4$, and $m^2 = 0$ to marginal operators $\Delta_+ = 4$.

At large ρ (the boundary) the solutions (5.1.17) behave as

$$\phi(\rho) \simeq C_+ e^{-\Delta_+ \rho}\left[1 + \mathcal{O}(e^{-2\rho})\right] + C_- e^{-\Delta_- \rho}\left[1 + \mathcal{O}(e^{-2\rho})\right] \ , \tag{5.1.18}$$

where $\Delta_\pm = 2 \pm \sqrt{4+m^2}$ and we have absorbed unimportant constants into C_+, C_-. (There are also terms proportional to ρ and mixing between the two coefficients in the cases $m^2 = -3, -4$.) Changing back to the Poincaré coordinates z, t and using the asymptotic relation $e^\rho \sim \frac{(-t)}{z}$, we have

$$\phi(\rho) \simeq C_+ z^{\Delta_+}\left[\frac{1}{(-t)^{\Delta_+}} + \mathcal{O}(z^2)\right] + C_- z^{\Delta_-}\left[\frac{1}{(-t)^{\Delta_-}} + \mathcal{O}(z^2)\right] \ . \tag{5.1.19}$$

(We have again absorbed unimportant constants into $C_\pm$, and there are also terms logarithmic in z in the cases $m^2 = -3, -4$). Comparing with (5.1.6) we see that this configuration has $\phi_0 = \frac{C_-}{(-t)^{\Delta_-}}$, $\psi_0 = \frac{C_+}{(-t)^{\Delta_+}}$. Since we are interested in spontaneous breaking for which there is no source, $\phi_0 = 0$, we must fix $C_- = 0$, after which we obtain a spontaneously generated VEV proportional to ψ_0,

$$\langle \mathcal{O} \rangle \propto \frac{C_+}{(-t)^{\Delta_+}}. \tag{5.1.20}$$

We thus have configurations which correctly reproduce the time-dependent vacuum expectation value required to break $\mathrm{so}(2,4) \to \mathrm{so}(1,4)$, where Δ_+ is the scaling dimension of the dual operator, defined over the causal diamond of the region $t \in (-\infty, 0)$ of the Poincaré patch. This is precisely what is required to realize the pseudo-conformal mechanism.

A feature generic to these solutions is that they diverge as $\sim 1/\rho^3$ as $\rho \to 0$, which means the scalar is blowing up and back-reaction is becoming important as we approach the line

$(-t) = z$ in the Poincaré patch. This is dual to the approach to $t \to 0$ in the boundary, where the pseudo-conformal phase ends and additional physics must kick in to reheat to a traditional radiation dominated universe. We now turn to the fully back-reacted case with dynamical bulk gravity.

5.2 Including Back-Reaction

We now consider a scalar field minimally coupled to Einstein gravity

$$S = \int d^5x\sqrt{-G}\left[\frac{1}{2}R[G] + 6 - \frac{1}{2}G^{MN}\partial_M\phi\partial_N\phi - V(\phi)\right] - \int d^4x\sqrt{-g}K \ . \tag{5.2.1}$$

G_{MN} is the bulk metric, and we have a Gibbons-Hawking term on the boundary [81, 82] which depends on the boundary metric and extrinsic curvature. We have set the radius of AdS_5 to unity and absorbed the overall factors of the Planck mass into the definition of the action. We seek a metric which respects the unbroken dS_4 isometries and approaches AdS_5 at the boundary. The appropriate ansatz is thus a domain wall spacetime in which the wall is foliated by dS_4 slices,

$$ds^2 = d\rho^2 + e^{2A(\rho)}\left[\frac{-d\eta^2 + d\vec{x}^2}{\eta^2}\right] \ , \quad \phi = \phi(\rho) \ . \tag{5.2.2}$$

We are interested in asymptotically AdS_5 solutions, for which the scale factor $A(\rho) \to \rho + const.$ as $\rho \to \infty$. We demand that the scalar potential has an extremum at $\phi = 0$ with the value $V(0) = 0$ into which the scalar flows as $\rho \to \infty$,

$$V = \frac{1}{2}m^2\phi^2 + \mathcal{O}(\phi^3) \ . \tag{5.2.3}$$

The independent equations of motion for the background fields are[7]

[7]We note that these are the same background equations for a spherically symmetric Euclidean domain wall,

$$ds^2 = d\rho^2 + a(\rho)^2(d\theta^2 + \sin^2\theta d\Omega_3^2) \ . \tag{5.2.4}$$

$$\phi''(\rho) + 4A'(\rho)\phi'(\rho) = \frac{\partial V}{\partial \phi}, \tag{5.2.6}$$

$$6A'(\rho)^2 - 6e^{-2A(\rho)} - 6 = \frac{1}{2}\phi'(\rho)^2 - V \ . \tag{5.2.7}$$

When $\phi = 0$, we find that the equations are solved by $A(\rho) = \log(\sinh\rho)$, which is empty AdS_5 corresponding to the conformal vacuum of the dual field theory.

A simple solution of the second-order system can be found by choosing a vanishing potential for the scalar, $V(\phi) = 0$. This choice corresponds to a particular truncation of type IIB supergravity, as we review in Appendix 5.3.1. The scalar is the string theory dilaton, and the dual operator is closely related to the $\mathcal{N} = 4$ SYM Lagrangian [91], which is marginal, $\Delta = 4$.

Now that we have chosen $V = 0$, we may integrate (5.2.6) once to obtain

$$\phi'(\rho) = ce^{-4A(\rho)} \ , \tag{5.2.8}$$

with integration constant c. Substituting this into (5.2.7) we obtain

$$A'(\rho)^2 = 1 + e^{-2A(\rho)} + be^{-8A(\rho)} \ , \tag{5.2.9}$$

where $b \equiv c^2/12 \geq 0$. Defining a new coordinate,

$$u = e^{-2A(\rho)} \ , \tag{5.2.10}$$

This can be seen by analytically continuing the angular coordinates as $\theta = it + \pi/2$. The S^4 metric continues to the global dS_4 metric and we obtain a domain wall foliated by global dS_4 slices

$$ds^2 \to d\rho^2 + a(\rho)^2(-dt^2 + \cosh^2 t\, d\Omega_3^2) \ . \tag{5.2.5}$$

Under the analytic continuation $\rho = it$, $\theta = i\lambda$, $\hat{a}(t) = -ia(it)$, the spherically symmetric Euclidean domain wall maps to a FLRW spacetime with hyperbolic spatial sections $ds^2 = -dt^2 + \hat{a}(t)^2(d\lambda^2 + \sinh^2\lambda d\Omega_3^2)$. This analytic continuation was used in [83–85] to holographically study crunching AdS cosmologies. See also [86–90] for related work on time-dependent solutions of IIB supergravity and their gauge theory duals.

we obtain, with the use of (5.2.9), the metric (5.2.2) in the form

$$ds^2 = \frac{du^2}{4u^2(1+u+bu^4)} + \frac{1}{u}\left[\frac{-d\eta^2 + d\vec{x}^2}{\eta^2}\right] . \tag{5.2.11}$$

This metric is AdS_5 for $b = 0$ and for $b \neq 0$ it approaches AdS_5 near the boundary at $u = 0$. We have arrived at a general solution for the metric with a single integration constant. This is one of the three integration constants expected in the general solution of the original system (5.2.6), (5.2.7). Of the other two, one is expressed as an arbitrary shift on the scalar (since the scalar only appears with derivatives) and the other can be absorbed into the scale of the de Sitter slices (which can be fixed to unity by demanding $A(\rho) \to \rho - \ln 2$ at infinity). The equation for the scalar (5.2.8) expressed in terms of the variable u is

$$\frac{d\phi}{du} = -\frac{c}{2}\frac{u}{\sqrt{1+u+bu^4}} . \tag{5.2.12}$$

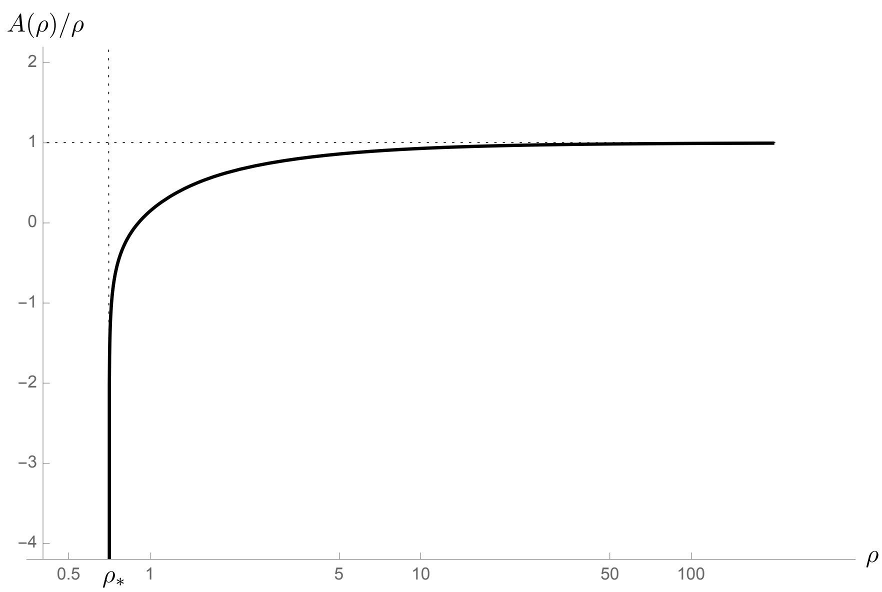

Figure 2: A typical solution. Here $b = 1$. On the horizontal axis is the ρ coordinate on a logarithmic scale, and on the vertical axis is the scale factor $A(\rho)$, normalized by ρ so that it can be seen that $A \to \rho$ as $\rho \to \infty$, corresponding to asymptotically AdS boundary conditions. The singularity occurs at the value $\rho_* \approx 0.7$, for which the scale factor goes to $-\infty$.

A typical solution is shown in Figure 5.2. In the interior, the background develops a curvature singularity at a value finite value $\rho = \rho_*$, determined by b, for which $A \to -\infty$. As $b \to 0$, $\rho_* \to 0$ in agreement with the probe calculation. From the definition (5.2.10), we see that the u coordinate tends to ∞ as we approach the singularity, so the u coordinate covers the region from the boundary to the singularity as we range over $u = (0, \infty)$. This singularity is a naked singularity[8]. The gauge-theory interpretation of this bulk singularity is that the theory is flowing from a conformal fixed point in the UV to a gapped theory at low energies [93, 94]. Since the singularity occurs at a fixed value ρ_*, we expect the gauge theory in flat space to possess a time-dependent IR cut-off. Transforming to the Poincaré patch we obtain $z_* \sim t \operatorname{sech} \rho_*$. Since $1/z$ parametrizes the RG scale, we find expect that the effective cut-off goes like $\Lambda_{\rm IR} \sim \frac{\cosh \rho_*}{t}$. Note that the geometry of our solution is analogous to the Janus solution of [95] in the sense that their symmetry breaking patterns are related by Wick rotation. Unlike our solution, however, the Janus solution is everywhere regular and corresponds to turning on the source of the dual operator rather than the VEV.

5.2.1 Fefferman-Graham expansion and one-point functions

To determine the VEVs of the stress tensor and the operator $\mathcal{O}$ dual to the scalar field, we must first write the metric and scalar in Fefferman-Graham coordinates [96], which is always possible in asymptotically AdS spacetime. We will calculate the holographic one point functions of the scalar plus gravity system for a scalar with no potential, including all the appropriate holographic renormalizations as discussed in [105–108]. Some steps of this calculation, such as the computation of the logarithmic term in the regulated on-shell action, have been carried out elsewhere [109–111].

At the boundary our spacetime approaches AdS, so in a neighborhood of the boundary it

[8] It maps under the domain-wall/cosmology correspondence [92] to a big bang in a spatially open universe.

is possible to transform to Fefferman-Graham coordinates [96],

$$ds^2 = \frac{1}{z^2}\left(dz^2 + \tilde{g}_{\mu\nu}(x,z)dx^\mu dx^\nu\right) \ , \qquad \phi(x,z) = \tilde{\varphi}(x,z),$$

$$\tilde{g}_{\mu\nu}(x,z) = g_{\mu\nu}(x) + z^2 g_{(2)\mu\nu}(x) + z^4(t_{\mu\nu}(x) + g_{(4)\mu\nu}(x)\log z) + \cdots \ , \tag{5.2.13}$$

$$\tilde{\varphi}(x,z) = \varphi(x) + z^2\varphi_{(2)}(x) + z^4(\psi(x) + \varphi_{(4)}(x)\log z) + \cdots \ . \tag{5.2.14}$$

The expansion for the scalar, which includes the logarithmic term, is the appropriate one for a massless field in the bulk. The leading term $g_{\mu\nu}(x)$ is the metric on which the boundary field theory lives (which for our purposes we will eventually take to be $\eta_{\mu\nu}$), and $\varphi(x)$ is the source of the dimension 4 scalar operator $\mathcal{O}$ in the boundary theory (which for our purposes we will eventually take to vanish). The sub-leading terms are determined by solving the equations of motion.

The purpose of the following is to show that the one-point functions of the scalar operator $\langle\mathcal{O}\rangle$ and the stress tensor $\langle T_{\mu\nu}\rangle$ are determined (up to numerical constants) by the coefficients of z^4 in the Fefferman-Graham expansions of the bulk fields (ψ and $t_{\mu\nu}$), and to quantify the ambiguities in these one-point functions due to different renormalization schemes.

In AdS/CFT, the generating function of boundary field theory correlators as a function of $g_{\mu\nu}(x), \varphi(x)$ is given by the bulk action evaluated on (5.2.14) [112, 113]. Breaking the bulk metric into sideways ADM [114] coordinates with respect to z, with lapse N, shift N^μ and spatial metric $\gamma_{\mu\nu}$,

$$G_{\text{bulk}} = \left(\begin{array}{c|c} N^2 + N^\mu N_\mu & N_\mu \\ \hline N_\mu & \gamma_{\mu\nu} \end{array}\right), \tag{5.2.15}$$

the bulk action (5.2.1) including the boundary term becomes

$$S = \frac{1}{2}\int_0^\infty dz \int d^4x\sqrt{-\gamma}N\left(R[\gamma] + K^2 - K_{\mu\nu}^2 + 12 - (\mathcal{L}_n\phi)^2 - (\nabla\phi)^2\right) \ , \tag{5.2.16}$$

where the extrinsic curvature and Lie derivative in the normal direction is

$$K_{\mu\nu} = \frac{1}{2}\mathcal{L}_n g_{\mu\nu} = \frac{1}{2N}\left(\gamma'_{\mu\nu} - \nabla_\mu N_\nu - \nabla_\nu N_\mu\right), \quad \mathcal{L}_n\phi = \frac{1}{N}\left(\phi' - N^\mu\partial_\mu\phi\right) \ , \tag{5.2.17}$$

primes denote derivatives with respect to z, and index movements and covariant derivatives are with respect to the spatial metric $\gamma_{\mu\nu}$.

In the Fefferman-Graham coordinates (5.2.14), we have

$$N = \frac{1}{z}, \qquad N^\mu = 0, \qquad \gamma_{\mu\nu} = \frac{1}{z^2}\tilde{g}_{\mu\nu} \ . \tag{5.2.18}$$

To obtain the equations of motion in Fefferman-Graham gauge we first vary with respect to N, N^μ, $\gamma_{\mu\nu}$ and ϕ, then impose (5.2.18), yielding

$$\tilde{g}^{\rho\sigma}\tilde{\nabla}_\rho\tilde{g}'_{\mu\sigma} - \tilde{\nabla}_\mu \operatorname{Tr}(\tilde{g}^{-1}\tilde{g}') = 2\phi'\partial_\mu\phi, \tag{5.2.19}$$

$$\operatorname{Tr}(\tilde{g}^{-1}\tilde{g}'') - \frac{1}{2}\operatorname{Tr}(\tilde{g}^{-1}\tilde{g}'\tilde{g}^{-1}\tilde{g}') - \frac{1}{z}\operatorname{Tr}(\tilde{g}^{-1}\tilde{g}') = -2\phi'^2, \tag{5.2.20}$$

$$\tilde{g}''_{\mu\nu} - (\tilde{g}'\tilde{g}^{-1}\tilde{g}')_{\mu\nu} + \frac{1}{2}\operatorname{Tr}(\tilde{g}^{-1}\tilde{g}')\tilde{g}'_{\mu\nu} - 2R_{\mu\nu}(\tilde{g}) - \frac{1}{z}\operatorname{Tr}(\tilde{g}^{-1}\tilde{g}')\tilde{g}_{\mu\nu} - \frac{3}{z}\tilde{g}'_{\mu\nu} = -2\partial_\mu\phi\partial_\nu\phi, \tag{5.2.21}$$

$$\phi'' + \frac{1}{2}\operatorname{Tr}(\tilde{g}^{-1}\tilde{g}')\phi' - \frac{3}{z}\phi' + \tilde{\Box}\phi = 0 \ . \tag{5.2.22}$$

Here (5.2.19) is the N^μ equation, (5.2.20) is a linear combination of the N equation and the trace of the $\gamma_{\mu\nu}$ equation (with the coefficient chosen so as to cancel the scalar curvature term $R[\tilde{g}]$), (5.2.21) is a linear combination of the $\gamma_{\mu\nu}$ equation, $\gamma_{\mu\nu}$ times the N equation, and $\gamma_{\mu\nu}$ times the trace of the $\gamma_{\mu\nu}$ equation (with the coefficients chosen to cancel the $\operatorname{Tr}(\tilde{g}'')$ term and the $R[\tilde{g}]\tilde{g}_{\mu\nu}$ term), and (5.2.22) is the ϕ equation.

We solve these equations of motion order by order in z by inserting the Fefferman-Graham expansion (5.2.14) into (5.2.19), (5.2.20), (5.2.21), (5.2.22) and collecting like powers of z. The important parts of the expansion are the following: the lowest two orders in the expansion of (5.2.19),

$$\nabla^\nu g^{(2)}_{\mu\nu} - \nabla_\mu g^{(2)} - 2\varphi^{(2)}\nabla_\mu\varphi = 0 \ , \tag{5.2.23}$$

$$\nabla^\nu t_{\mu\nu} - \nabla_\mu t + \frac{3}{4}g^{(2)\rho\sigma}\nabla_\mu g^{(2)}_{\rho\sigma} + \frac{1}{4}g^{(2)\rho}_{\mu}\nabla_\rho g^{(2)} - \frac{1}{2}g^{(2)\rho\sigma}\nabla_\rho g^{(2)}_{\mu\sigma} - \frac{1}{2}g^{(2)}_{\mu\rho}\nabla_\sigma g^{(2)\rho\sigma}$$
$$+\frac{1}{4}\left(\nabla^\nu g^{(4)}_{\mu\nu} - \nabla_\mu g^{(4)}\right) - \frac{1}{2}\varphi^{(4)}\nabla_\mu\varphi - 2\psi\nabla_\mu\varphi + \log z\left(\nabla^\nu g^{(4)}_{\mu\nu} - \nabla_\mu g^{(4)} - 2\varphi^{(4)}\nabla_\mu\varphi\right) = 0 \ ,$$

the lowest order in the expansion of (5.2.20),

$$t - \frac{1}{4}g^{(2)}_{\mu\nu}g^{(2)\mu\nu} + \frac{3}{4}g^{(4)} + (\varphi^{(2)})^2 + g^{(4)}\log z = 0 \ , \tag{5.2.24}$$

the lowest two orders in the expansion of (5.2.21),

$$2g^{(2)}_{\mu\nu} + g^{(2)}g_{\mu\nu} + R_{\mu\nu} - \nabla_\mu\varphi\nabla_\nu\varphi = 0 \ , \tag{5.2.25}$$

$$g^{(4)}_{\mu\nu} + \frac{1}{2}g_{\mu\nu}g^{(2)}_{\rho\sigma}g^{(2)\rho\sigma} - g^{(2)}_{\mu\rho}g^{(2)\rho}_{\nu} - \frac{1}{2}\nabla_{(\mu}\nabla^\rho g^{(2)}_{\nu)\rho} + \frac{1}{4}\nabla^2 g^{(2)}_{\mu\nu} + \frac{1}{4}\nabla_\mu\nabla_\nu g^{(2)} - \frac{1}{2}g^{(2)}_{\rho(\mu}R^\rho{}_{\nu)}$$
$$+\frac{1}{2}g^{(2)\rho\sigma}R_{\mu\rho\nu\sigma} - g_{\mu\nu}t - \frac{1}{4}g_{\mu\nu}g^{(4)} + \nabla_{(\mu}\varphi\nabla_{\nu)}\varphi^{(2)} - g_{\mu\nu}g^{(4)}\log z = 0 \ , \tag{5.2.26}$$

and the lowest two orders in the expansion of (5.2.22),

$$\varphi^{(2)} - \frac{1}{4}\Box\varphi = 0 \ , \tag{5.2.27}$$

$$\varphi^{(4)} + \frac{1}{2}g^{(2)}\varphi^{(2)} + \frac{1}{4}\Box\varphi^{(2)} + \frac{1}{8}\nabla_\mu g^{(2)}\nabla^\mu\varphi - \frac{1}{4}\nabla_\mu g^{(2)\mu\nu}\nabla_\nu\varphi - \frac{1}{4}g^{(2)\mu\nu}\nabla_\mu\varphi\nabla_\nu\varphi = 0 \ .$$

In these expressions, all index raising/lowering, covariant derivatives, curvatures, etc. are with respect to $g_{\mu\nu}$.

Equation (5.2.25) determines $g^{(2)}_{\mu\nu}$; by taking a trace and reinserting the result in to the equation we find

$$g^{(2)}_{\mu\nu} = -\frac{1}{2}\left(R_{\mu\nu} - \frac{1}{6}Rg_{\mu\nu} - \nabla_\mu\varphi\nabla_\nu\varphi + \frac{1}{6}g_{\mu\nu}(\nabla\varphi)^2\right) \ . \tag{5.2.28}$$

Equation (5.2.27) determines $\varphi^{(2)}$,

$$\varphi^{(2)} = \frac{1}{4}\Box\varphi \ , \tag{5.2.29}$$

and (5.2.23) is automatically satisfied by (5.2.28), (5.2.29), as can be checked using the Bianchi identity.

Equation (5.2.28) determines $\varphi^{(4)}$,

$$\begin{aligned}
\varphi^{(4)} \quad &= \frac{1}{2}g^{(2)}\varphi^{(2)} - \frac{1}{4}\Box\varphi^{(2)} - \frac{1}{8}\nabla_\mu g^{(2)}\nabla^\mu\varphi + \frac{1}{4}\nabla_\mu g^{(2)\mu\nu}\nabla_\nu\varphi + \frac{1}{4}g^{(2)\mu\nu}\nabla_\mu\varphi\nabla_\nu\varphi \\
&= -\frac{1}{16}\Box^2\varphi + \frac{1}{6}\nabla^\mu\varphi\nabla^\nu\varphi\nabla_\mu\nabla_\nu\varphi + \frac{1}{12}(\nabla\varphi)^2\Box\varphi - \frac{1}{8}R^{\mu\nu}\nabla_\mu\varphi\nabla_\nu\varphi + \frac{1}{24}R(\nabla\varphi)^2 - \frac{1}{48}\nabla^\mu R\nabla_\mu\varphi \ .
\end{aligned}$$

Moving to (5.2.24), the logarithm term must vanish separately, which tells us that $g^{(4)}_{\mu\nu}$ is traceless,

$$g^{(4)} = 0 \ , \tag{5.2.30}$$

and the remainder of the equation then determines the trace of $t_{\mu\nu}$,

$$t = \frac{1}{4}g^{(2)}_{\mu\nu}g^{(2)\mu\nu} - (\varphi^{(2)})^2 = \frac{1}{16}R^2_{\mu\nu} - \frac{1}{72}R^2 + \frac{1}{36}R(\nabla\varphi)^2 - \frac{1}{8}R^{\mu\nu}\nabla_\mu\varphi\nabla_\nu\varphi - \frac{1}{16}(\Box\varphi)^2 + \frac{7}{144}(\nabla\varphi)^4 \ .$$

The logarithm term of (5.2.24) must also vanish separately, which tells us the divergence of $g^{(4)}_{\mu\nu}$,

$$\nabla^\nu g^{(4)}_{\mu\nu} = 2\varphi^{(4)}\nabla_\mu\varphi \ , \tag{5.2.31}$$

and the rest of (5.2.24) then determines the divergence of $t_{\mu\nu}$,

$$\nabla^\nu t_{\mu\nu} = 2\psi\nabla_\mu\varphi - \varphi^{(2)}\nabla_\mu\varphi^{(2)} - \frac{1}{4}g^{(2)\rho\sigma}\nabla_\mu g^{(2)}_{\rho\sigma} - \frac{1}{4}g^{(2)\rho}_{\mu}\nabla_\rho g^{(2)} + \frac{1}{2}g^{(2)\rho\sigma}\nabla_\rho g^{(2)}_{\mu\sigma} + \frac{1}{2}g^{(2)}_{\mu\rho}\nabla_\sigma g^{(2)\rho\sigma} \ .$$

Now, equation (5.2.26) determines $g^{(4)}_{\mu\nu}$,

$$\begin{aligned} g^{(4)}_{\mu\nu} = \quad & -\frac{1}{4}g_{\mu\nu}g^{(2)}_{\rho\sigma}g^{(2)\rho\sigma} + g^{(2)}_{\mu\rho}g^{(2)\rho}_{\nu} + \frac{1}{2}\nabla_{(\mu}\nabla^{\rho}g^{(2)}_{\nu)\rho} - \frac{1}{4}\nabla^2 g^{(2)}_{\mu\nu} - \frac{1}{4}\nabla_{\mu}\nabla_{\nu}g^{(2)} + \frac{1}{2}g^{(2)}_{\rho(\mu}R^{\rho}{}_{\nu)} \\ & -\frac{1}{2}g^{(2)\rho\sigma}R_{\mu\rho\nu\sigma} - g_{\mu\nu}(\varphi^{(2)})^2 - \nabla_{(\mu}\varphi\nabla_{\nu)}\varphi^{(2)} \ , \end{aligned}$$

which is consistent with (5.2.30) and (5.2.31).

Note that $g^{(4)}_{\mu\nu}$ and $\varphi^{(4)}$ are functional derivatives

$$g^{(4)}_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_W}{\delta g^{\mu\nu}}, \qquad 2\varphi^{(4)} = \frac{1}{\sqrt{-g}}\frac{\delta S_W}{\delta\varphi} \ , \tag{5.2.32}$$

of an action

$$S_W = \frac{1}{8}\int d^4x\,\sqrt{-g}\left[-\frac{1}{4}C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} - \frac{1}{2}(\Box\varphi)^2 - \frac{1}{3}(\nabla\varphi)^4 + R^{\mu\nu}\nabla_{\mu}\varphi\nabla_{\nu}\varphi - \frac{1}{3}R(\nabla\varphi)^2\right] \ , \tag{5.2.33}$$

with $C_{\mu\nu\rho\sigma}$ the Weyl tensor. The expression (5.2.31) is nothing but the Ward identity for diffeomorphism invariance of this action. This action is also Weyl invariant, with φ transforming with Weyl weight 0, and (5.2.30) is the corresponding Ward identity.

We can now evaluate the on shell action by plugging the solution into (5.2.16). The action found in this way is divergent due to the infinite volume of AdS_5, and so we define a regulated on-shell action by placing the boundary at $z = \epsilon$,

$$S_{\epsilon}[g,\varphi] \quad = \quad S_{\epsilon,\mathrm{bulk}}[g,\varphi] + S_{\epsilon,\mathrm{boundary}}[g,\varphi] \ , \tag{5.2.34}$$

where

$$S_{\epsilon,\mathrm{bulk}}[g,\varphi] \quad = \quad -4\int d^4x\int_{\epsilon}^{\infty} dz\, N\sqrt{-\gamma}, \tag{5.2.35}$$

$$S_{\epsilon,\mathrm{boundary}}[g,\varphi] \quad = \quad -\int d^4x\left[\sqrt{-\gamma}K\right]_{z=\epsilon} \ . \tag{5.2.36}$$

We have simplified the bulk part by using the trace of the bulk Einstein equations, $R[G] - (\partial_A \phi)^2 = -20$. Expanding for small ϵ, the resulting expression contains local divergent terms and non-local finite terms,

$$S_\epsilon[g,\varphi] = \int d^4x\sqrt{-g}\left[\frac{1}{\epsilon^4}a_4 + \frac{1}{\epsilon^2}a_2 + \log\epsilon\, a_0 + \text{finite} + \cdots\right] , \tag{5.2.37}$$

where the a's are the following local quantities constructed from $g_{\mu\nu}$ and φ,

$$a_4 = 3, \tag{5.2.38}$$

$$a_2 = 0, \tag{5.2.39}$$

$$\begin{aligned} a_0 &= -g^{(2)}_{\mu\nu}g^{(2)\mu\nu} + \frac{1}{2}\left(g^{(2)}\right)^2 + 2t \\ &= -\frac{1}{8}\left(R_{\mu\nu}R^{\mu\nu} - \frac{1}{3}R^2\right) - \frac{1}{8}(\Box\varphi)^2 - \frac{1}{12}(\nabla\varphi)^4 + \frac{1}{4}R^{\mu\nu}\nabla_\mu\varphi\nabla_\nu\varphi - \frac{1}{12}R(\nabla\varphi)^2. \end{aligned} \tag{5.2.40}$$

Note that a_0 is proportional, up to a total derivative, to the Lagrangian of (5.2.33).

Counterterms must be chosen to cancel the infinite terms, and are ambiguous up to local finite terms,

$$S_{c.t.}[g,\varphi] = \int d^4x\sqrt{-g}\left[-\frac{1}{\epsilon^4}a_4 - \frac{1}{\epsilon^2}a_2 - \log\epsilon\, a_0 + \text{local finite}\right]. \tag{5.2.41}$$

The generating function is the regulated generating functional minus the counter terms and is finite,

$$S[g,\varphi] = \lim_{\epsilon\to 0}\left(S_\epsilon[g,\varphi] + S_{c.t.}[g,\varphi]\right) . \tag{5.2.42}$$

The stress tensor and scalar VEVs can then be calculated by functionally differentiating, and the result is ambiguous up to functional derivatives of local finite terms.

Calculating directly in this way would require finding the complicated non-local finite part of $S[g,\varphi]$. Instead we can proceed indirectly. Start by defining $\langle T_{\mu\nu}\rangle^\epsilon$ as the stress tensor obtained by functionally differentiating the regulated action, before subtracting counterterms. It is a function of the full boundary metric $\tilde{g}_{\mu\nu}(x,\epsilon)$ and boundary scalar $\tilde{\varphi}(x,\epsilon)$. Since the bulk variation is always just the equation of motion, in varying (5.2.16) the only contribution on shell is a boundary term at the cutoff $z=\epsilon$,

$$\delta S_\epsilon = -\frac{1}{2}\int d^4x\sqrt{-\gamma}\left(K_{\mu\nu}-K\gamma_{\mu\nu}\right)\delta\gamma^{\mu\nu}\Big|_{z=\epsilon} . \tag{5.2.43}$$

Thus we have,

$$\begin{aligned}\langle T_{\mu\nu}\rangle^\epsilon &= -\frac{2}{\sqrt{-\tilde{g}}}\frac{\delta S_\epsilon[\tilde{g},\tilde{\varphi}]}{\delta\tilde{g}^{\mu\nu}} = -\frac{1}{\epsilon^2}\frac{2}{\sqrt{-\gamma}}\frac{\delta S_\epsilon[\gamma,\phi]}{\delta\gamma^{\mu\nu}} = \frac{1}{\epsilon^2}\left(K_{\mu\nu}-K\gamma_{\mu\nu}\right)\Big|_\epsilon \\ &= \frac{1}{2\epsilon^3}\left(\tilde{g}'_{\mu\nu}-\tilde{g}_{\mu\nu}\,\mathrm{Tr}(\tilde{g}^{-1}\tilde{g}')+\frac{6}{z}\tilde{g}_{\mu\nu}\right)\Big|_\epsilon .\end{aligned} \tag{5.2.44}$$

The variation of the regulated action (5.2.16) with respect to the scalar gives the regulated scalar one-point function,

$$\delta S = -\frac{1}{2}\int d^4x\,\sqrt{-\gamma}\frac{1}{N}\left(\phi'-N^\mu\partial_\mu\phi\right)\delta\phi' = \int d^4x\,\frac{1}{\epsilon^3}\sqrt{-\tilde{g}}\,\phi'\delta\phi = \int d^4x\,\frac{1}{\epsilon^3}\sqrt{-\tilde{g}}\,\tilde{\varphi}'\delta\tilde{\varphi}, \tag{5.2.45}$$

$$\langle\mathcal{O}\rangle^\epsilon = \frac{1}{\sqrt{-\tilde{g}}}\frac{\delta S}{\delta\tilde{\varphi}} = \frac{1}{\epsilon^3}\tilde{\varphi}' . \tag{5.2.46}$$

Evaluating these to finite order using the Fefferman-Graham expansion solutions, we have

$$\begin{aligned}\langle T_{\mu\nu}\rangle^\epsilon &= \frac{3}{\epsilon^4}g_{\mu\nu}+\frac{1}{\epsilon^2}\left(4g^{(2)}_{\mu\nu}-g_{\mu\nu}g^{(2)}\right)+5\log\epsilon\, g^{(4)}_{\mu\nu}+5t_{\mu\nu}+\frac{1}{2}g^{(4)}_{\mu\nu} \\ &\quad -g^{(2)}_{\mu\nu}g^{(2)}+g_{\mu\nu}g^{(2)}_{\rho\sigma}g^{(2)\rho\sigma}-2g_{\mu\nu}t ,\end{aligned} \tag{5.2.47}$$

$$\langle\mathcal{O}\rangle^\epsilon = \frac{2}{\epsilon^2}\varphi^{(2)}+4\varphi^{(4)}\log z+4\psi+\varphi^{(4)} . \tag{5.2.48}$$

Next we define the counter-term stress tensor and scalar one-point function, which are obtained by differentiating the counterterm action with respect to $\tilde{g}^{\mu\nu}$,

$$T^{c.t.}_{\mu\nu} = -\frac{2}{\sqrt{-\tilde{g}}}\frac{\delta S_{c.t.}[\tilde{g},\tilde{\varphi}]}{\delta \tilde{g}^{\mu\nu}} \ , \tag{5.2.49}$$

$$\langle \mathcal{O} \rangle^{\epsilon}_{c.t} = \frac{1}{\sqrt{-\tilde{g}}}\frac{\delta S[\tilde{g},\tilde{\varphi}]_{c.t.}}{\delta \tilde{\varphi}}. \tag{5.2.50}$$

To calculate these, we must express the counterterm action in terms of $\tilde{g}_{\mu\nu}, \tilde{\varphi}$ rather than $g_{\mu\nu}, \varphi$ by inverting the Fefferman-Graham expansion up to the required order. We write

$$\begin{aligned} g_{\mu\nu} &= \tilde{g}_{\mu\nu} + z^2 \tilde{g}^{(2)}_{\mu\nu} + z^4 \tilde{g}^{(4)}_{\mu\nu} + \cdots \ , \\ \varphi &= \tilde{\varphi} + z^2 \tilde{\varphi}^{(2)} + z^4 \tilde{\varphi}^{(4)} + \cdots , \end{aligned} \tag{5.2.51}$$

then plug in the Fefferman-Graham expansion and equate powers of z to obtain

$$\begin{aligned} \tilde{g}^{(2)}_{\mu\nu} &= -g^{(2)}_{\mu\nu}\Big|_{\tilde{g},\tilde{\varphi}} , \\ \tilde{g}^{(4)}_{\mu\nu} &= -\frac{\delta g^{(2)}_{\mu\nu}}{\delta g_{\rho\sigma}} g^{(2)}_{\rho\sigma} - \frac{\delta g^{(2)}_{\mu\nu}}{\delta \varphi}\varphi^{(2)} - \Big(t_{\mu\nu} + g^{(4)}_{\mu\nu}\log z \Big)\Big|_{\tilde{g},\tilde{\varphi}} , \\ \tilde{\varphi}^{(2)} &= -\varphi^{(2)}\Big|_{\tilde{g},\tilde{\varphi}} , \\ \tilde{\varphi}^{(4)} &= -\frac{\delta \varphi^{(2)}}{\delta g_{\mu\nu}} g^{(2)}_{\mu\nu} - \frac{\delta \varphi^{(2)}}{\delta \varphi}\varphi^{(2)} - \Big(\psi + \varphi^{(4)}\log z \Big)\Big|_{\tilde{g},\tilde{\varphi}} . \end{aligned} \tag{5.2.52}$$

Expressing the counterterm action (5.2.41) in terms of $\tilde{g}_{\mu\nu}$ and $\tilde{\varphi}$, we find,

$$S_{c.t.}[\tilde{g},\tilde{\varphi}] = \int d^4x\sqrt{-\tilde{g}}\left[-\frac{3}{\epsilon^4} + \frac{1}{4\epsilon^2}\Big(-\tilde{R} + (\tilde{\nabla}\tilde{\varphi})^2 \Big) - \log\epsilon\, a_0[\tilde{g},\tilde{\varphi}] + \text{local finite} \right] . \tag{5.2.53}$$

Varying this, we obtain the counterterm stress tensor and VEV in terms of $\tilde{g}_{\mu\nu}$ and $\tilde{\varphi}$,

$$
\begin{aligned}
\langle T_{\mu\nu}\rangle^{c.t.}[\tilde{g},\tilde{\varphi}] &= -\frac{2}{\sqrt{-\tilde{g}}}\frac{\delta S_{c.t.}[\tilde{g},\tilde{\varphi}]}{\delta\tilde{g}^{\mu\nu}} \\
&= -\frac{3}{\epsilon^4}\tilde{g}_{\mu\nu} + \frac{1}{2\epsilon^2}\left(\tilde{R}_{\mu\nu} - \frac{1}{2}\tilde{R}\tilde{g}_{\mu\nu} - \tilde{\nabla}_\mu\tilde{\varphi}\tilde{\nabla}_\nu\tilde{\varphi} + \frac{1}{2}\tilde{g}_{\mu\nu}(\tilde{\nabla}\tilde{\varphi})^2\right) - 2\log\epsilon g^{(4)}(\tilde{g},\tilde{\varphi}) \\
&\quad +\delta(\text{local finite}) \ , \\
\langle\mathcal{O}\rangle^{\epsilon}_{c.t}[\tilde{g},\tilde{\varphi}] &= \frac{1}{\sqrt{-\tilde{g}}}\frac{\delta S_{c.t.}[\tilde{g},\tilde{\varphi}]}{\delta\tilde{\varphi}} \\
&= -\frac{1}{2\epsilon^2}\tilde{\Box}\tilde{\varphi} - 4\log\epsilon\varphi^{(4)}(\tilde{g},\tilde{\varphi}) + \delta(\text{local finite}) \ . \qquad (5.2.54)
\end{aligned}
$$

We now insert the Fefferman-Graham expansion into these in order to express them in terms of $g_{\mu\nu}$ and φ,

$$
\begin{aligned}
\langle T^{c.t.}_{\mu\nu}\rangle[g,\varphi] &= -\frac{3}{\epsilon^4}g_{\mu\nu} + \frac{1}{\epsilon^2}\left(2R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} - 2\nabla_\mu\varphi\nabla_\nu\varphi + g_{\mu\nu}(\nabla\varphi)^2\right) - 5\log\epsilon g^{(4)}_{\mu\nu} \\
&\quad -3t_{\mu\nu} + g^{(4)}_{\mu\nu} - \frac{1}{16}g_{\mu\nu}R_{\rho\sigma}R^{\rho\sigma} - \frac{1}{4}R_{\mu\rho}R_\nu^{\ \rho} + \frac{5}{24}R_{\mu\nu}R - \frac{1}{48}g_{\mu\nu}R^2 \\
&\quad +\frac{1}{2}R_{(\mu}^{\ \ \rho}\nabla_{\nu)}\varphi\nabla_\rho\varphi - \frac{5}{24}R_{\mu\nu}(\nabla\varphi)^2 + \frac{1}{24}g_{\mu\nu}R(\nabla\varphi)^2 - \frac{5}{24}R\nabla_\mu\varphi\nabla_\nu\varphi \\
&\quad -\frac{1}{24}\nabla_\mu\varphi\nabla_\nu\varphi(\nabla\varphi)^2 - \frac{1}{16}g_{\mu\nu}(\Box\varphi)^2 + \frac{1}{8}g_{\mu\nu}R^{\rho\sigma}\nabla_\rho\varphi\nabla_\sigma\varphi - \frac{1}{12}g_{\mu\nu}(\nabla\varphi)^4 \\
&\quad +\delta(\text{local finite}) \ , \\
\langle\mathcal{O}\rangle^{\epsilon}_{c.t}[g,\varphi] &= -\frac{1}{2\epsilon^2}\Box\varphi - 4\log\epsilon\varphi^{(4)} - \frac{1}{24}R\Box\varphi + \frac{1}{24}(\nabla\varphi)^2\Box\varphi + 2\varphi^{(4)} + \delta(\text{local finite}) \ .
\end{aligned}
$$

The renormalized stress tensor and one point function are now given by

$$
\langle T_{\mu\nu}\rangle = \lim_{\epsilon\to 0}\left(T^{\epsilon}_{\mu\nu}[g,\varphi] + T^{c.t.}_{\mu\nu}[g,\varphi]\right) \ , \qquad (5.2.55)
$$

$$
\langle\mathcal{O}\rangle = \lim_{\epsilon\to 0}\left(\langle\mathcal{O}\rangle^{\epsilon} + \langle\mathcal{O}\rangle^{\epsilon}_{c.t}[g,\varphi]\right) \ , \qquad (5.2.56)
$$

and should be finite. Because of the ambiguity of local finite contributions to the counterterms (5.2.41), the stress tensor is always ambiguous up to terms which are functional derivatives of local terms. Evaluating (5.2.55) and (5.2.56), the divergent pieces all cancel, as they should, yielding

$$\begin{aligned}
\langle T_{\mu\nu}\rangle &= 2t_{\mu\nu} + \frac{3}{2}g^{(4)}_{\mu\nu} + \frac{1}{16}g_{\mu\nu}R_{\rho\sigma}R^{\rho\sigma} - \frac{1}{4}R_{\mu\rho}R_{\nu}^{\ \rho} + \frac{1}{8}R_{\mu\nu}R - \frac{5}{144}g_{\mu\nu}R^2 \\
&\quad + \frac{1}{2}R_{(\mu}^{\ \ \rho}\nabla_{\nu)}\varphi\nabla_\rho\varphi - \frac{1}{8}R_{\mu\nu}(\nabla\varphi)^2 + \frac{5}{72}g_{\mu\nu}R(\nabla\varphi)^2 - \frac{1}{8}R\nabla_\mu\varphi\nabla_\nu\varphi \\
&\quad - \frac{1}{8}\nabla_\mu\varphi\nabla_\nu\varphi(\nabla\varphi)^2 + \frac{1}{16}g_{\mu\nu}(\Box\varphi)^2 - \frac{1}{8}g_{\mu\nu}R^{\rho\sigma}\nabla_\rho\varphi\nabla_\sigma\varphi + \frac{1}{36}g_{\mu\nu}(\nabla\varphi)^4 \\
&\quad \delta(\text{local finite})\ , \\
\langle\mathcal{O}\rangle &= 4\psi + 3\varphi^{(4)} - \frac{1}{24}R\Box\varphi + \frac{1}{24}(\nabla\varphi)^2\Box\varphi + \delta(\text{local finite})\ .
\end{aligned}$$

Note that the terms $\frac{3}{2}g^{(4)}_{\mu\nu}$, $3\varphi^{(4)}$ can be absorbed into δ(local finite) because they stem from the variation of the local action (5.2.33). The renormalized VEVs satisfy the Ward identity for diffeomorphisms,

$$\nabla^\nu\langle T_{\mu\nu}\rangle - \langle\mathcal{O}\rangle\nabla_\mu\varphi = 0\ . \tag{5.2.57}$$

Taking the trace of the stress tensor, we find the Weyl anomaly,

$$\begin{aligned}
\mathcal{A} &= g^{\mu\nu}\langle T_{\mu\nu}\rangle = -a_0 + \mathrm{Tr}\ \delta(\text{local finite}) \\
&= -\frac{1}{16}E_{(4)} + \frac{1}{16}C^2_{\mu\nu\rho\sigma} + \frac{1}{8}(\Box\varphi)^2 + \frac{1}{12}(\nabla\varphi)^4 - \frac{1}{4}R^{\mu\nu}\nabla_\mu\varphi\nabla_\nu\varphi + \frac{1}{12}R(\nabla\varphi)^2 + \mathrm{Tr}\ \delta(\text{local finite})\ ,
\end{aligned}$$

where $E_{(4)} \equiv R^2_{\mu\nu\rho\sigma} - 4R^2_{\mu\nu} + R^2$ is the four dimensional Euler density.

In the case of a flat background with no source (the case of interest in this chapter), $g_{\mu\nu} = \eta_{\mu\nu}$, $\varphi = 0$, all the curvature and φ terms vanish, and the only ambiguity is a term in the stress tensor proportional to $\eta_{\mu\nu}$, coming from a cosmological constant counterterm, so we have

$$\langle T_{\mu\nu}\rangle = 2t_{\mu\nu} + (const.)\eta_{\mu\nu}, \qquad \langle\mathcal{O}\rangle = 4\psi \qquad (\text{flat space, no source})\ . \tag{5.2.58}$$

The contribution of the conformal anomaly to the stress-tensor one-point function vanishes on flat space. The cosmological constant counterterm can be chosen so that the contribution proportional to $\eta_{\mu\nu}$ vanishes. Then we see that the one-point functions are indeed given by the order z^4 parts of the Fefferman-Graham expansion.

We have thus calculated the unambiguous parts of the exact one-point functions of the scalar operator and stress tensor, in the presence of a general source and boundary metric. The calculation shows that the one-point functions of the scalar operator $\langle\mathcal{O}\rangle$ and the stress tensor $\langle T_{\mu\nu}\rangle$, for zero source and flat background, are determined up to numerical constants by the coefficients of z^4 in the Fefferman-Graham expansions of the bulk fields,

$$\langle\mathcal{O}\rangle = 4\psi, \quad \langle T_{\mu\nu}\rangle = 2t_{\mu\nu}. \tag{5.2.59}$$

In order to find the VEVs corresponding to our background spacetime we must find the flat sliced Fefferman-Graham expansion of the metric (5.2.11). Following [97], we first define a new coordinate $y \in (1,\infty)$ through the definition $y^2 = 1 + u + bu^4$. For $b = 0$ the metric is AdS_5, which in the y coordinates reads

$$ds^2 = \frac{dy^2}{(y^2-1)^2} + \frac{1}{y^2-1}\left[\frac{-d\eta^2 + d\vec{x}^2}{\eta^2}\right] . \tag{5.2.60}$$

Our goal is to change to flat-sliced Fefferman-Graham coordinates, and in the $b = 0$ case the desired coordinates are the Poincaré coordinates

$$ds^2 = \frac{dz^2 - dt^2 + d\vec{x}^2}{z^2}, \tag{5.2.61}$$

and the coordinate transformation is (recalling that $(-t) > z$)

$$y = \frac{(-t)}{\sqrt{t^2 - z^2}}, \quad \eta = \sqrt{t^2 - z^2} . \tag{5.2.62}$$

For $b = c^2/12 \neq 0$ we can work perturbatively in c. First solve the equation $y^2 = 1+u+bu^4$

to linear order in b,

$$u = y^2 - 1 - b(y^2-1)^4 + \mathcal{O}(b^2) \ , \tag{5.2.63}$$

and use this to write the metric in terms of the y variable to linear order in b,

$$ds^2 = \frac{dy^2}{(y^2-1)^2}\left[1 - 6b(y^2-1)^3 + \mathcal{O}(b^2)\right] + \frac{1}{y^2-1}\left[1 + b(y^2-1)^3 + \mathcal{O}(b^2)\right] ds^2_{\mathrm{dS}_4} \ . \tag{5.2.64}$$

Now we need to find, to linear order in b, the coordinate transformation which takes us to flat sliced Fefferman-Graham coordinates. Writing the ansatz,

$$y = \frac{t}{\sqrt{t^2 - z^2}} + b f_1(x) + \mathcal{O}(b^2), \quad \eta = \sqrt{t^2 - z^2} + b z f_2(x) + \mathcal{O}(b^2) \ , \tag{5.2.65}$$

where $x \equiv (-t)/z$, the functions $f_1(x)$ and $f_2(x)$ can be fixed by demanding that the transformed metric has Fefferman-Graham form. In particular, we demand that the $\mathcal{O}(b)$ terms in g_{zz} and g_{tz} vanish. This gives two conditions which can be solved to give two coupled ordinary differential equations, the solutions of which are

$$f_1(x) = \frac{2 + 9x^2 - 6x^4 - 6x^2\left(x^2-1\right)^2 \log\left(1 - \frac{1}{x^2}\right)}{8x\left(x^2-1\right)^{7/2}}, \tag{5.2.66}$$

$$f_2(x) = \frac{1}{4}\sqrt{x^2-1}\left[\frac{17 - 42x^2 + 24x^4}{6\left(x^2-1\right)^3} + 4\log\left(1 - \frac{1}{x^2}\right) + \frac{3\log\left(1-\frac{1}{x^2}\right)}{x^2-1}\right] \ , \tag{5.2.67}$$

therefore,

$$ds^2 = \frac{dz^2\left[1 + \mathcal{O}(b^2)\right] + \left[-1 + b\beta_1(x) + \mathcal{O}(b)^2\right] dt^2 + \left[1 + b\beta_2(x) + \mathcal{O}(b)^2\right] d\vec{x}^2}{z^2} \ , \tag{5.2.68}$$

where

$$\beta_1(x) = \frac{1}{4} z^2 \left[\frac{-6t^4 + 9t^2 z^2 - 2z^4}{t^2 \left(t^2 - z^2\right)^2} - \frac{6 \log \left(1 - \frac{z^2}{t^2}\right)}{z^2} \right], \tag{5.2.69}$$

$$\beta_2(x) = \frac{6t^4 z^2 - 15t^2 z^4 + 11z^6 + 6\left(t^2 - z^2\right)^3 \log\left(1 - \frac{z^2}{t^2}\right)}{12\left(-t^2 + z^2\right)^3} . \tag{5.2.70}$$

We have chosen the integration constants in the solutions so that $\beta_1(x)$ is strictly real and $\beta_2(x)$ does not contain a constant term.

Thus, the Fefferman-Graham expansion of the metric (5.2.11) to order c^2 is

$$ds^2 = \frac{dz^2 + g_{tt}(z/t)dt^2 + g_{11}(z/t)d\vec{x}^2}{z^2} , \tag{5.2.71}$$

where the functions $g_{11}(z/t)$ and $g_{tt}(z/t)$ are

$$g_{tt}(z/t) = -1 + \frac{b}{8}\left(\frac{z}{t}\right)^8 + \mathcal{O}\left(\left(\frac{z}{t}\right)^{10}\right), \tag{5.2.72}$$

$$g_{11}(z/t) = 1 - \frac{b}{8}\left(\frac{z}{t}\right)^8 + \mathcal{O}\left(\left(\frac{z}{t}\right)^{10}\right) . \tag{5.2.73}$$

Comparing to (5.2.59), we see that the absence of a z^4 coefficient in the Fefferman-Graham expansion of the spatial metric reveals that all components of the one-point function $\langle T_{\mu\nu}\rangle$ vanish,

$$\langle T_{\mu\nu}\rangle = 0. \tag{5.2.74}$$

Integrating the scalar field equation (5.2.8) with the help of the chain rule

$$\phi = -\frac{c}{2}\int dx\, \frac{du}{dx}\frac{u}{\sqrt{1+u+bu^4}} , \tag{5.2.75}$$

and using the asymptotic expansion for u in terms of x we can obtain the z-dependence of ϕ,

$$\phi = \text{const.} - \frac{c}{4}\left(\frac{z}{t}\right)^4 + \mathcal{O}\left((z/t)^6\right). \tag{5.2.76}$$

According to (5.2.59), this fall-off behavior corresponds to the spontaneously generated VEV

$$\langle \mathcal{O} \rangle = -\frac{c}{(-t)^4}. \tag{5.2.77}$$

One can show by Taylor expanding that the next power in c is $\mathcal{O}(c^3(z/t)^{12})$ which is subdominant, as are all higher powers of c. Similarly, the c^4 corrections to the metric are at least $\mathcal{O}((z/t)^8)$, and so cannot contribute to the stress-tensor VEV.

The pseudo-conformal solutions studied in [34, 35] have the property that the energy density vanishes but the pressure does not, instead getting a profile $p \sim 1/t^4$ consistent with the symmetries. Once coupled to gravity, this makes for a very stiff equation of state which is essential for the scalar field component to dominate over other cosmological components such as matter, radiation, curvature, anisotropy, etc. This serves to empty out and smooth the universe and address the standard puzzles of big bang cosmology without the need for an exponentially expanding spacetime. Here instead we find a vanishing pressure. The difference is due to the fact that the AdS/CFT computation is computing the improved stress tensor of the CFT, which is traceless, and which one would use to couple the theory to gravity in a Weyl invariant manner. The pseudo-conformal solutions, on the other hand, are coupled minimally to gravity, and so the stress tensor which gets a $\sim 1/t^4$ profile is the minimal stress tensor which one uses to couple the theory minimally to gravity. A CFT on flat space with a VEV $\langle \mathcal{O} \rangle \propto 1/t^\Delta$ is equivalent via a Weyl transformation to a CFT on de Sitter with a constant VEV, so it is important that the pseudo-conformal scenario is minimally coupled to gravity rather than conformally coupled, otherwise it would be equivalent to inflation via a Weyl transformation.

5.3 Appendices

5.3.1 Supergravity Embedding

The choice of vanishing potential used in Section 5.2 can be uplifted to 10 dimensions as a particular truncation of ten-dimensional type IIB supergravity[9].

The ten-dimensional Einstein frame equations of motion for the metric, the dilaton and the self-dual five-form in IIB supergravity with all other fields vanishing are

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}(\partial\phi)^2 + \frac{1}{24}F_{\mu\rho\lambda\sigma\tau}F_\nu{}^{\rho\lambda\sigma\tau} - \frac{1}{240}g_{\mu\nu}F^2, \tag{5.3.1}$$

$$\Box\phi = 0, \tag{5.3.2}$$

$$dF_5 = 0, \tag{5.3.3}$$

$$*F_5 = F_5. \tag{5.3.4}$$

Consider a compactification of the form

$$ds^2 = d\rho^2 + e^{2A(\rho)}ds^2_{dS_4} + ds^2_{S^5}, \tag{5.3.5}$$

$$ds^2_{dS_4} = \frac{-d\eta^2 + d\vec{x}^2}{\eta^2}, \tag{5.3.6}$$

$$\phi = \phi(\rho), \tag{5.3.7}$$

$$F_5 = 4(*\omega_{S^5} + \omega_{S^5}) \ , \tag{5.3.8}$$

where ω_{S^5} is the volume form on the internal 5-sphere. The five-form is manifestly self-dual and satisfies $dF_5 = 0$. The scalar equation is solved by

$$\phi'(\rho) = ce^{-4A(\rho)} \ , \tag{5.3.9}$$

which is the same as the 5D equation (5.2.8). The five form flux acts as a source in the

[9] This simple choice of potential has been exploited to construct the AdS-sliced domain wall known as the Janus solution [95], originally formulated in ten-dimensional type IIB supergravity.

Einstein equation; we have $R_{\rho\rho} - \frac{1}{2}g_{\rho\rho}R = -10 - 6e^{-2A(\rho)} + 6A'(\rho)^2$, $F_{\rho....}F_{\rho}{}^{....} = -64(4!)$, and the self-duality constraint implies $F^2 = 0$, so the ρ-ρ component of the Einstein equation becomes, upon using (5.3.9),

$$A'(\rho)^2 = 1 + e^{-2A(\rho)} + be^{-8A(\rho)} \ , \tag{5.3.10}$$

where $b = c^2/12$. This is the same as the 5D equation (5.2.9). In the well-studied $\mathrm{AdS}_5 \times S^5$ compactification of IIB the boundary value of the dilaton is proportional to the Yang-Mills coupling parameter $e^{2\phi} \sim g^2_{\mathrm{YM}}$, whose inverse multiplies the $\mathcal{N} = 4$ SYM Lagrangian density. It is therefore natural to associate a spacetime varying dilaton with a running gauge coupling parameter.

5.3.2 Comparison with Holographic Defect CFTs

The pseudo-conformal solutions are essentially Wick rotated holographic interface CFTs, i.e. CFTs in which there is an interface that occurs around the spacelike surface $t = 0$. Consider a 4D CFT with a planar spatial boundary in Euclidean signature. Then the boundary breaks the conformal symmetry so$(1,5)$ to the subgroup so$(1,4)$ of conformal transformations which leave the boundary invariant. The unbroken symmetry group coincides with the isometry group of dS_4, so we expect correlation functions in this theory to be related in a trivial way to those of the pseudo-conformal universe. The one-point functions of scalar operators inserted away from the boundary are fixed by the residual so$(1,4)$ invariance

$$\langle \mathcal{O}_4(\vec{x}, y) \rangle \propto \frac{1}{y^{\Delta}} \ , \tag{5.3.11}$$

where y is the perpendicular distance to the the boundary and $\vec{x}$ denotes the remaining translationally invariant coordinates. We immediately see that the distance to the boundary corresponds to the cosmological time and that the temporal 'boundary' lies in the infinite future. Two-point functions between a boundary localized operator $\mathcal{O}_3(\vec{x}, 0)$ and an arbitrarily located operator $\mathcal{O}_4(\vec{x}', y)$ are likewise fully determined

$$\langle \mathcal{O}_3(\vec{x}, 0) \mathcal{O}_4(\vec{x}', y) \rangle \propto \frac{1}{y^{\Delta_4 - \Delta_3} [y^2 + (\vec{x} - \vec{x}')^2]^{\Delta_3}} \ . \tag{5.3.12}$$

However, correlation functions between two boundary-delocalized operators $\mathcal{O}_4(\vec{x}, y)$ and $\mathcal{O}_4'(\vec{x}', y')$ depend upon an unknown function of the conformal invariant

$$\langle \mathcal{O}_4(\vec{x}, y) \mathcal{O}_4'(\vec{x}', y') \rangle = \frac{1}{y^{\Delta_4} y'^{\Delta_4'}} f(\xi), \qquad \xi = \frac{(\vec{x} - \vec{x}')^2 + (y - y')^2}{4yy'} \ . \tag{5.3.13}$$

The form of the function $f(\xi)$ is constrained by the so-called boundary operator product expansion explained in [146].

Chapter 6

Holographic CFTs on maximally symmetric spaces: correlators, integral transforms and applications

In the previous chapter we saw that the AdS/CFT provides a tool to compute cosmologically relevant one-point functions in the pseudo-conformal universe using Einstein gravity. In this chapter, which is based on [116], we use a combination of quantum-field theoretic and holographic techniques to study the relationship between position and momentum space correlation functions in conformal field theories (CFTs) on maximally symmetric curved spaces, with and without boundaries. We focus primarily on one and two-point functions, paying special attention to the short-distance singularities and how they are to be renormalized into local counterterms.

Our general analysis encompasses the anti-de Sitter/boundary conformal field theory (AdS/BCFT) correspondence [118] (see [119, 120] for reviews). The AdS/BCFT correspondence can be regarded as a generalization of AdS/CFT [64] to situations in which the dual field theory itself has some boundary or defect [121]. In this case, the bulk theory possesses a boundary Q in addition to the usual asymptotic boundary M of AdS_{d+1}. The intersection $\partial M = Q \cap M$ of the new boundary Q with the CFT living on M represents the defect or boundary of the CFT. In this case the dual field theory is called a boundary conformal field theory. If the bulk boundary Q is chosen to preserve some subgroup of the $\mathrm{O}(2, d)$ isometries of the bulk AdS_{d+1}, then the dual field theory is invariant under the corresponding subgroup of the conformal group.

There exist a number of existing examples of this general setup. The metric for the Poincaré patch of AdS_{d+1} is

$$ds^2 = \frac{dz^2 - dt^2 + dx_1^2 + \cdots + dx_{d-2}^2 + dy^2}{z^2} \,, \tag{6.0.1}$$

where $z \in (0, \infty)$, and $(t, x_1, \ldots, x_{d-2}, y)$ label the coordinates of the d-dimensional dual field theory at $z = 0$. The Randall-Sundrum or hard-wall AdS/QCD models [122, 123] can be considered as an example, where the role of Q is played by the IR brane which lies at a fixed value $z = z_* > 0$ of the Poincaré radial coordinate, and the role of M is played by the $z = 0$ boundary; $M = \mathbb{R}^{1,d-1}$ at $z \to 0$. In this example Q does not intersect M. Poincaré symmetry $\mathrm{ISO}(1, d-1) \subset \mathrm{O}(2, d)$ is respected but the dilation and special conformal symmetry of $\mathrm{O}(2, d)$ is broken and this introduces a mass scale $1/z_*$ in the dual quantum field theory. The soft wall can be thought of as a generalization of the Randall-Sundrum I model [122], which contains a back-reacting scalar field in the bulk. The scalar field becomes singular in the interior of AdS and forms a naked singularity which plays the role of the IR boundary brane.

Locally localized gravity [124] is another example where Q is an AdS_d submanifold of AdS_{d+1} which intersects $M = \mathbb{R}^{1,d-1}$ along a flat, timelike surface $y = 0$. This holographically realizes a CFT on a half space $y \in [0, \infty)$ whose boundary at $y = 0$ breaks the conformal group $\mathrm{O}(2, d)$ but leaves unbroken an $\mathrm{O}(2, d-1)$ subgroup.

If we instead take a suitable de Sitter submanifold $Q = \mathrm{dS}_d$, then we find that Q intersects $M = \mathbb{R}^{1,d-1}$ on the flat spacelike surface $t = 0$ and the CFT is defined at times $t \in (-\infty, 0]$. This is our proposal for the holographic dual to a new kind of conformal field theory which possesses a spacelike boundary at future infinity. These new CFTs find application in the pseudo-conformal universe scenario for early universe cosmology. The pseudo-conformal universe [33–36] is an early universe scenario which serves as an alternative to inflation, in which the early universe is dominated by a CFT that spontaneously breaks the conformal group to a subgroup which is isomorphic to the group of de Sitter symmetries. Here, contrary to most applications of AdS/CFT or dS/CFT to cosmology, the theory of cosmological interest is the boundary CFT. Within this boundary CFT, there is a spacelike surface at

$t = 0$ which marks the point at which the pseudo-conformal phase ends and the universe must reheat and transition into a radiation dominated phase. This spacelike surface is the boundary of the CFT, which makes it a wick-rotated version of a BCFT. The boundary $t = 0$ now preserves a de Sitter subgroup $\mathrm{O}(1, d) \subset \mathrm{O}(2, d)$ and the most general vacuum expectation values for scalar operators of dimension Δ can evolve in time as $1/(-t)^{\Delta}$. Our proposal can be considered as the hard-wall version of [63, 125, 126].

The organization is as follows. In Section 6.1 we study two-point functions and their singularities in CFTs on flat space, the sphere, and hyperbolic space. In Section 6.2 we review the construction of holographic BCFTs, including their one- and two-point functions from the gravity dual. We additionally present a new derivation of the AdS/BCFT two-point function which exploits the AdS slicing of the bulk and provides an additional test of the of the construction laid out in section 6.1. In section 6.3 we provide additional calculations for the one-point and two-point function in the spacelike boundary (or pseudo-conformal) CFT.

6.1 CFT correlators on maximally symmetric spaces and their UV singularities

We would like to understand how two-point correlators, the treatment of their UV singularities, their interpretation as distributions, and their Fourier transforms, generalize to curved spaces. In particular, we consider maximally symmetric spaces, which have the same number of symmetries as flat $\mathbb{E}^d$ and are related to it by Weyl transformations. Physically, these spaces are solutions to the Einstein equations with a cosmological constant. In this section, we will consider the cases of Euclidean CFTs on spaces without boundaries, moving on to cases with boundaries in Section 6.2.

6.1.1 Flat space

We warm up by analyzing the singularity structure of the simplest possibility: a CFT on flat space without boundaries. We recall how local counter-terms must be introduced to remove the short-distance singularities of bare correlation functions. Our analysis differs from [127] in that we employ cut-off, rather than differential regularization, which we found easier to generalize to curved spaces. We will see in particular examples how the renormalized correlation functions thus defined are implicitly determined in terms of their Fourier transform.

Consider a CFT on flat $\mathbb{E}^d$ with $d \geq 3$. As is well known, the conformal symmetry fixes the form of the two-point correlator for scalar primary operators of dimension Δ to be $1/x^{2\Delta}$. Naively, the Fourier transform of the function $1/x^{2\Delta}$ is generally ill-defined both in the UV and IR. The naive definition of the Fourier transform of the two-point function is

$$\tilde{G}_{\Delta,d}(k) \stackrel{!}{=} \int d^d\vec{x}\, e^{-i\vec{k}\cdot\vec{x}}|\vec{x}|^{-2\Delta} \tag{6.1.1}$$

$$= V_{\mathbb{S}^{d-2}} \int_0^\pi d\theta (\sin\theta)^{d-2} \int_0^\infty dr\, r^{(d-2\Delta)-1} e^{-irk\cos\theta} \ , \tag{6.1.2}$$

with $V_{\mathbb{S}^{d-2}}$ the volume of the unit $d-2$ sphere. We see that this integral is only convergent if

$$\frac{d-2}{2} < \Delta < \frac{d}{2} \ , \tag{6.1.3}$$

where the upper and lower bounds are UV and IR constraints, respectively. There exist plenty of CFTs with operators violating this naive bound[10].

The above considerations underscore the well-known fact that correlation functions should be interpreted as distributions (i.e. generalized functions, see e.g. [108, 129, 130] in the AdS/CFT context). A distribution is a linear functional defined to act on some space of smooth test functions with nice prescribed behavior at infinity[11]. The action of the

[10] $\Delta \geq \frac{d-2}{2}$ is the unitarity bound for a scalar operator, saturated only for a free scalar, so the lower bound in (6.1.3) would be violated only for a free scalar [128].

[11] Usually the space of test functions is taken to be the Schwartz space, the space of smooth functions which fall at infinity, along with any of its multiple derivatives, faster than any inverse power of the coordinates. Unlike the space of functions with compact support, this has the advantage that the Fourier transform is always defined within the space of test functions. The corresponding distributions are known as tempered distributions.

correlation functional on a test function $f(x)$ is as follows

$$\frac{1}{x^{2\Delta}}[f] = \int d^d x \frac{1}{x^{2\Delta}} f(x) \ . \tag{6.1.4}$$

Due to the nice fall-off behavior of the test function, this interpretation of the two-point correlator is free from IR divergences. However it is still not defined because of possible UV divergences localized at $x = 0$. These are dealt with in the following way. We first define a regulated functional $1/x^{2\Delta}\big|_\epsilon$ which is UV finite for $\epsilon > 0$. There are many ways to do this. One way, which we illustrate below, is to cut off the integral within some ball around the origin of radius ϵ. Another is differential regularization [131] (reviewed in Appendix 6.4.4).

Because the divergence is associated with the singularity at $x = 0$, the divergent terms in (6.1.4) depend only on the value of the test function and its derivatives at $x = 0$. Because the divergences are localized, they can be cancelled by adding distributions which are delta functions and derivatives of delta functions, at the origin. We define the renormalized two-point correlator as a distribution of the following form

$$\langle \mathcal{O}_\Delta(x) \mathcal{O}_\Delta(0) \rangle^{\text{ren}}_{\mathbb{E}^d} = \lim_{\epsilon \to 0} \left[\frac{1}{x^{2\Delta}}\bigg|_\epsilon + c_1 \delta^d(x) + c_2 \Box \delta^d(x) + \cdots \right] \ , \tag{6.1.5}$$

where the coefficients $c_1, c_2, \ldots$ are chosen to depend on $1/\epsilon$ in such a way that the result is finite as $\epsilon \to 0$ when (6.1.5) is integrated against an arbitrary test function.

The infinite parts of the c's are fixed by requiring finiteness, but the finite parts are undetermined and represent ambiguities that are not calculable from the theory. Different regularization schemes will give different finite parts. If we let J be a source for the operator $\mathcal{O}$ and think in terms of the effective action $W[J]$ whose functional derivatives generate the correlators, these delta function ambiguities are precisely the local terms,

$$W[J] \supset \int d^d x \ c_1 J^2 + c_2 J \Box J + \cdots \ . \tag{6.1.6}$$

The local terms are ambiguous and contribute only to correlators at equal points, whereas the non-local terms are finite and unambiguous and contribute to the correlators at separate points.

For example, we can define the regulated functional by integrating only outside of a d-dimensional ball B_ϵ of radius ϵ,

$$\left.\frac{1}{x^{2\Delta}}\right|_\epsilon [f] = \int_{\mathbb{E}^d \backslash B_\epsilon} d^d x \frac{1}{x^{2\Delta}} f(x) \ , \tag{6.1.7}$$

in which case the coefficients $c_1, c_2, \ldots$ are either inverse powers or logarithms of ϵ,

$$\left.\frac{1}{x^{2\Delta}}\right|_\epsilon [f] = \int_\epsilon^\infty dr\, r^{d-1-2\Delta} \int d\Omega_{d-1} \left[f(0) + r\hat{x}^\mu \partial_\mu f(0) + \frac{r^2}{2} \hat{x}^\mu \hat{x}^\nu \partial_\mu \partial_\nu f(0) + \cdots \right] \ , \tag{6.1.8}$$

$$= V_{\mathbb{S}^{d-1}} \int_\epsilon^\infty dr\, r^{d-1-2\Delta} \left[f(0) + \frac{r^2}{2d} \Box f(0) + \cdots \right] \ . \tag{6.1.9}$$

The set of divergences ends with a logarithm if $\Delta = d/2 + k$ ($k = 0, 1, 2, \ldots$). Taking $d = 4$ and $\Delta = 2$, for instance, we find

$$\left.\frac{1}{x^4}\right|_\epsilon [f] = 2\pi^2 \int_\epsilon^\infty \frac{dr}{r} \left[f(0) + \mathcal{O}(r^2) \right] \ , \tag{6.1.10}$$

$$= -2\pi^2 \log(\mu\epsilon) f(0) + \text{finite} \ . \tag{6.1.11}$$

The mass scale μ is arbitrary and ambiguous, because it can be changed by the addition of a finite local piece. The renormalized two-point correlator is thus the following distribution

$$\langle \mathcal{O}_2(x) \mathcal{O}_2(0) \rangle^{\text{ren}}_{\mathbb{E}^4} = \lim_{\epsilon \to 0} \left[\left.\frac{1}{x^4}\right|_\epsilon + 2\pi^2 \log(\mu\epsilon) \delta^4(x) \right] \ . \tag{6.1.12}$$

This is finite and well-defined as a distribution, ambiguous only up to local delta contributions. Note that in cases in which there is a logarithmic divergence, such as this one, the coefficient of the logarithm is unambiguous and calculable, and is responsible for violation of scale invariance at coincident points,

$$\mu\frac{\partial}{\partial\mu}\langle\mathcal{O}_2(x)\mathcal{O}_2(0)\rangle^{\text{ren}}_{\mathbb{E}^4} = 2\pi^2\delta^4(x)\ . \tag{6.1.13}$$

Cases without logarithmic divergences, for example a $\Delta = 2$ operator in $d = 3$,

$$\langle\mathcal{O}_2(x)\mathcal{O}_2(0)\rangle^{\text{ren}}_{\mathbb{E}^3} = \lim_{\epsilon\to 0}\left[\left.\frac{1}{x^4}\right|_\epsilon - \frac{4\pi}{\epsilon}\delta^3(x)\right]\ , \tag{6.1.14}$$

do not exhibit scale-dependence at coincidence points. Another important case which we will return to later is a marginal operator for which $\Delta = d$, for example $\Delta = 3$ in $d = 3$,

$$\langle\mathcal{O}_3(x)\mathcal{O}_3(0)\rangle^{\text{ren}}_{\mathbb{E}^3} = \lim_{\epsilon\to 0}\left[\left.\frac{1}{x^6}\right|_\epsilon - \frac{2\pi}{3}\left(\frac{2}{\epsilon^3}\delta^3(x) + \frac{1}{\epsilon}\Box\delta^3(x)\right)\right]\ . \tag{6.1.15}$$

Now consider the Fourier transform (the appropriate integral transform in flat space). The ordinary Fourier transform of a test function f is another test function $\tilde{f}$. Given a distribution G, its Fourier transform is always defined and is the distribution $\tilde{G}$ which gives the same value acting on $\tilde{f}$ as G does acting on f. By this definition, the Fourier transform $\tilde{G}_{\Delta,d}(k)$ of the renormalized two-point distribution $\langle\mathcal{O}_\Delta(x)\mathcal{O}_\Delta(0)\rangle^{\text{ren}}_{\mathbb{E}^d}$ satisfies

$$\frac{1}{(2\pi)^d}\int d^dk\,\tilde{G}_{\Delta,d}(k)\tilde{f}(k) = \int d^dx\,\langle\mathcal{O}_\Delta(x)\mathcal{O}_\Delta(0)\rangle^{\text{ren}}_{\mathbb{E}^d}f(x)\ . \tag{6.1.16}$$

It can be shown that $\tilde{G}_{\Delta,d}(k)$ is given by

$$\tilde{G}_{\Delta,d}(k) = 2^{d-2\Delta}\pi^{d/2}\frac{\Gamma(d/2-\Delta)}{\Gamma(\Delta)}k^{2\Delta-d}\ , \tag{6.1.17}$$

when $\Delta \neq d/2 + k$, and contains terms logarithmic in k otherwise [131]. We are free to add to this arbitrary polynomials in k^2, since these are the Fourier transforms of the ambiguous local contact terms. Note that (6.1.17) is the expression which would be obtained by analytically continuing in Δ the Fourier transform from the region (6.1.3) in which it is defined without distributional considerations.

As a concrete example, consider a Gaussian test function of some width $a > 0$,

$$\tilde{f}(k) = e^{-ak^2} \iff f(x) = \frac{e^{-x^2/(4a)}}{(2\sqrt{\pi a})^3} \ . \tag{6.1.18}$$

In the example of $d = 3$, $\Delta = 2$ given above, the left-hand side of (6.1.16) trivially gives $-1/(4a^2)$ while the right-hand side evaluates to

$$\text{RHS} = \frac{1}{(2\sqrt{\pi a})^3} \lim_{\epsilon\to 0} \left[4\pi \int_\epsilon^\infty dr \frac{e^{-r^2/(4a)}}{r^2} - \frac{4\pi}{\epsilon} \right] = -\frac{1}{4a^2} \ . \tag{6.1.19}$$

In summary, we must regulate the UV singularities in position-space correlators, e.g. by imposing some short-distance cut-off around coincident points. After renormalization, the resulting correlators are finite, with ambiguous finite contact terms, and are related to their spectral decompositions by the integral transform (6.1.16). IR divergences, on the other hand, are calculable and unambiguous (and can be physically important, e.g. [133–135]) and are handled automatically by the distributional interpretation, requiring no special treatment.

6.1.2 Sphere

Next we consider a Euclidean CFT on the d-dimensional sphere $\mathbb{S}^d$, which is related by analytic continuation to a Lorentzian CFT on de Sitter space dS_d. This example will prove to be important for understanding the pseudo-conformal universe.

The two-point function for a CFT on $\mathbb{S}^d$ can be found by exploiting the fact that the round sphere is related to flat space by a Weyl transformation. The Euclidean space metric in spherical coordinates, and the standard round metric on the sphere are

$$ds^2_{\mathbb{E}^d} = dr^2 + r^2 d\Omega^2_{d-1}, \qquad ds^2_{\mathbb{S}^d} = d\theta^2 + \sin^2\theta d\Omega^2_{d-1} \ . \tag{6.1.20}$$

Consider the stereographic projection from $\mathbb{S}^d$ to $\mathbb{E}^d$, given by $r = \sin\theta/(1-\cos\theta) = \cot(\theta/2)$

and thus $dr = d\theta/(1-\cos\theta)$. Substituting, we find

$$ds^2_{\mathbb{S}^d} = (1-\cos\theta)^2 ds^2_{\mathbb{E}^d} \ . \tag{6.1.21}$$

Conformal field theory correlators transform under Weyl transformations (up to anomalies) as

$$\langle \mathcal{O}_{\Delta_1}(x_1)\cdots\mathcal{O}_{\Delta_n}(x_n)\rangle_{\Omega^2 g} = \Omega(x_1)^{-\Delta_1}\cdots\Omega(x_n)^{-\Delta_n}\langle \mathcal{O}_{\Delta_1}(x_1)\cdots\mathcal{O}_{\Delta_n}(x_n)\rangle_{g} \ . \tag{6.1.22}$$

Setting $\Omega = (1-\cos\theta)$ and using the known flat space form for the two-point function, we deduce the following bare two-point function on the sphere,

$$\langle \mathcal{O}_\Delta(\vec{n})\mathcal{O}_\Delta(\vec{n}')\rangle_{\mathbb{S}^d} = \frac{1}{2^\Delta(1-\cos\Theta)^\Delta} \ , \tag{6.1.23}$$

where

$$\cos\Theta = \cos\theta\cos\theta' + \cos\alpha\sin\theta\sin\theta' \ . \tag{6.1.24}$$

Θ is the geodesic distance between the two points in $\mathbb{S}^d$ and α is their angular separation in $\mathbb{S}^{d-1}$. It is noteworthy that the two-point function only depends on the geodesic distance between two points on the sphere, which follows from the symmetries of the problem. The normalization of $1/2^\Delta$ is such that the short-distance limit matches the normalization $1/x^{2\Delta}$ for flat space.

We now attempt to perform the analog of the Fourier transform, that is, expand the two-point distribution on the sphere into hyper-spherical harmonics as

$$\langle \mathcal{O}_\Delta(\vec{n})\mathcal{O}_\Delta(\vec{n}')\rangle^{\text{ren}}_{\mathbb{S}^d} = \sum_{l,\mathbf{m}} g_l \mathbb{Y}^*_{l\mathbf{m}}(\vec{n})\mathbb{Y}_{l\mathbf{m}}(\vec{n}') \ , \tag{6.1.25}$$

$$= \frac{1}{V_{\mathbb{S}^d}(d-1)}\sum_l (2l+d-1) g_l C_l^{(d-1)/2}(\vec{n}\cdot\vec{n}') \ , \tag{6.1.26}$$

where we have used the addition theorem on the d-dimensional sphere,

$$\sum_{\mathbf{m}} \mathbb{Y}^*_{l\mathbf{m}}(\vec{n})\mathbb{Y}_{l\mathbf{m}}(\vec{n}') = \frac{1}{V_{\mathbb{S}^d}(d-1)} \sum_l (2l+d-1) C_l^{(d-1)/2}(\vec{n}\cdot\vec{n}') \ , \tag{6.1.27}$$

and $C_l^\alpha(x)$ are the Gegenbauer polynomials defined by the generating function

$$\frac{1}{(1-2xt+t^2)^\alpha} = \sum_{l=0}^{\infty} C_l^\alpha(x) t^l \ . \tag{6.1.28}$$

The coefficients g_l are the analog of the Fourier transform. The inverse of this transform allows us to calculate the g_l's

$$g_l = \frac{1}{\mathbb{Y}_{l\mathbf{m}}(\vec{n}')} \int_{\mathbb{S}^d} d\vec{n}\, \mathbb{Y}_{l\mathbf{m}}(\vec{n}) \langle \mathcal{O}_\Delta(\vec{n})\mathcal{O}_\Delta(\vec{n}')\rangle^{\text{ren}}_{\mathbb{S}^d} \ , \tag{6.1.29}$$

$$= \frac{V_{\mathbb{S}^{d-1}}}{C_l^{(d-1)/2}(1)} \int_{-1}^{1} dx\, (1-x^2)^{(d-2)/2} \left[\frac{1}{2^\Delta(1-x)^\Delta} + \text{counter-terms}\right] C_l^{(d-1)/2}(x) \ , \tag{6.1.30}$$

where we have used rotational invariance to move $\vec{n}'$ to $\theta = 0$ and have also used that the expression is independent of $\mathbf{m}$ to set $\mathbf{m} = 0$, in which case the spherical harmonics become proportional to Gegenbauer polynomials.

As in the flat case, this integral transform is generally ill-defined unless counter-terms are included: the singularity of the integrand (6.1.30) at $x = 1$ leads to the non-physical bound

$$\Delta < \frac{d}{2} \ . \tag{6.1.31}$$

This is easy to understand because the sphere is locally flat, so we expect the same UV divergences as (6.1.3) on flat space. There is no lower bound, however, because the finite volume of the sphere naturally cuts off the IR divergence.

To study the UV singularity structure of the bare two-point correlator, we integrate it against a smooth test function on $\mathbb{S}^d \times \mathbb{S}^d$ of the form $f(\vec{n}\cdot\vec{n}')$ as follows,

$$\int_{\mathbb{S}^d} d\vec{n}\langle\mathcal{O}_\Delta(\vec{n})\mathcal{O}_\Delta(\vec{n}')\rangle_{\mathbb{S}^d} f(\vec{n}\cdot\vec{n}') = V_{\mathbb{S}^{d-1}} 2^{-\Delta}\int_{-1}^{1-\eta} dx(1-x^2)^{(d-2)/2}(1-x)^{-\Delta} f(x)\ ,$$
$$= V_{\mathbb{S}^{d-1}} 2^{-\Delta}\int_{-1}^{1-\eta} dx(1-x^2)^{(d-2)/2}(1-x)^{-\Delta}\left[f(1)+(x-1)f'(1)+\cdots\right] \tag{6.1.32}$$

where $0<\eta\ll 1$ is a UV regulator, cutting off the region $x=1$ in the integral where the two points come together. Expanding in powers of $\frac{1}{\eta}$, there will be divergent parts which must be cancelled off by local counterterms.

For example, consider the case $\Delta=2$ and $d=3$, which has the divergent part.

$$\int_{\mathbb{S}^3} d\vec{n}\,\langle\mathcal{O}_2(\vec{n})\mathcal{O}_2(\vec{n}')\rangle_{\mathbb{S}^3} f(\vec{n}\cdot\vec{n}') = \frac{2\pi\sqrt{2}}{\sqrt{\eta}} f(1) + \text{finite}\ . \tag{6.1.33}$$

As in flat space, the divergence is local, depending only on the value of the test function at the point $x=1$ where the two points come together. Subtracting off this divergence, the renormalized two-point correlator for an operator of this dimension should be defined as the distribution

$$\langle\mathcal{O}_2(\vec{n})\mathcal{O}_2(\vec{n}')\rangle^{\text{ren}}_{\mathbb{S}^3} = \lim_{\eta\to 0}\left[\frac{1}{2^2(1-\vec{n}\cdot\vec{n}')^2} - \frac{2\pi\sqrt{2}}{\sqrt{\eta}}\delta^3(\vec{n},\vec{n}')\right]\ , \tag{6.1.34}$$

where $\delta^d(\vec{n}_1,\vec{n}_2)$ is the covariant delta function on the sphere, defined such that

$$\int_{\mathbb{S}^d} d\vec{n}\,\delta^3(\vec{n},\vec{n}')f(\vec{n}\cdot\vec{n}') = f(1)\ . \tag{6.1.35}$$

In terms of the $x=\cos\Theta$ coordinate,

$$\delta^d(\vec{n},\vec{n}') = \frac{\delta(1-x)}{V_{\mathbb{S}^{d-1}}(1-x)^{(d-2)/2}}\ . \tag{6.1.36}$$

Expanding for small Θ we obtain

$$\delta^d(\vec{n},\vec{n}') \sim \frac{\delta(\Theta)}{V_{\mathbb{S}^{d-1}}\Theta^{d-1}} \sim \delta^d(x)\ , \tag{6.1.37}$$

where $\delta^d(x)$ is the delta function in flat space. Let us check that the short distance behavior of this correlator agrees with flat space. We have $\eta = 1-\cos\epsilon \sim \epsilon^2/2$ and thus we reproduce (6.1.14),

$$\langle \mathcal{O}_2(\vec{n})\mathcal{O}_2(\vec{n}')\rangle^{\text{ren}}_{\mathbb{S}^3} \sim \lim_{\epsilon\to 0}\left[\frac{1}{x^4} - \frac{4\pi}{\epsilon}\delta^3(x)\right]\ , \tag{6.1.38}$$

where x is now the physical distance between $\vec{n}$ and $\vec{n}'$ and ϵ is the physical cut-off distance. Now let us calculate the g_l's for our renormalized correlation function. Since we have a well defined distribution, the integral transform should exist and hence the g_l's will be finite. We get

$$g_l = \frac{1}{\mathbb{Y}_{l\mathbf{m}}(\vec{n}')}\int_{\mathbb{S}^3} d\vec{n}\,\mathbb{Y}_{l\mathbf{m}}(\vec{n})\langle \mathcal{O}_2(\vec{n})\mathcal{O}_2(\vec{n}')\rangle^{\text{ren}}_{\mathbb{S}^3}\ , \tag{6.1.39}$$

$$= \lim_{\eta\to 0}\frac{1}{\mathbb{Y}_{l\mathbf{m}}(\vec{n}')}\int_{\mathbb{S}^3} d\vec{n}\,\mathbb{Y}_{l\mathbf{m}}(\vec{n})\left[\frac{1}{2^2(1-\vec{n}\cdot\vec{n}')^2} - \frac{2\pi\sqrt{2}}{\sqrt{\eta}}\delta^3(\vec{n},\vec{n}')\right]\ , \tag{6.1.40}$$

$$= \lim_{\eta\to 0}\left[-\frac{2\pi\sqrt{2}}{\sqrt{\eta}} + \frac{V_{\mathbb{S}^2}}{U_l(1)}\int_{-1}^{1-\eta} dx(1-x^2)^{1/2}\frac{1}{2^2(1-x)^2}U_l(x)\right]\ . \tag{6.1.41}$$

If instead we define g_l by analytic continuation in Δ from the region in which (6.1.30) is defined, we obtain[12]

$$g_l = \frac{\pi^{d/2}}{2^{2\Delta-d}}\frac{\Gamma(d/2-\Delta)}{\Gamma(\Delta)}\frac{\Gamma(l+\Delta)}{\Gamma(d+l-\Delta)}\ , \tag{6.1.42}$$

where $\Delta \neq d/2+k$ $(k=0,1,2,\ldots)$. One can check by direct evaluation with $l=0,1,2,\ldots$ that (6.1.42) agrees with the formula (6.1.41) obtained by properly renormalizing, namely

$$g_l = -\pi^2(l+1)\ . \tag{6.1.43}$$

[12] See also [136], which is missing a factor of $2^{-\Delta}$.

We see that analytic continuation in Δ corresponds to minimal subtraction in the hard cut-off formalism, as was the case on flat space. The spectral decomposition (6.1.42) is thus related to the renormalized two-point function by the following integral transform,

$$\sum_{l,\mathbf{m}} g_l f_l^* \, \mathbb{Y}_{l\mathbf{m}}(\vec{n}')\mathbb{Y}^*_{l\mathbf{m}}(\vec{n}'') = \int_{\mathbb{S}^d} d\vec{n} \langle \mathcal{O}_\Delta(\vec{n})\mathcal{O}_\Delta(\vec{n}')\rangle^{\text{ren}}_{\mathbb{S}^d} f(\vec{n}\cdot\vec{n}'') \ , \tag{6.1.44}$$

or, written in terms of Gegenbauer polynomials,

$$\frac{1}{V_{\mathbb{S}^d}(d-1)} \sum_l g_l f_l^* (2l+d-1) C_l^{(d-1)/2}(\vec{n}'\cdot\vec{n}'') = \int_{\mathbb{S}^d} d\vec{n} \langle \mathcal{O}_\Delta(\vec{n})\mathcal{O}_\Delta(\vec{n}')\rangle^{\text{ren}}_{\mathbb{S}^d} f(\vec{n}\cdot\vec{n}'') \ . \tag{6.1.45}$$

The above formula is the analog of the flat-space Fourier transform (6.1.16).

To further illustrate, consider a gaussian test function on the sphere. We can make a gaussian on the sphere by stereographically mapping a gaussian on $\mathbb{E}^d$ to the sphere. Starting with the smooth test function e^{-r^2} on $\mathbb{E}^d$ (with r the polar radial coordinate) we obtain the following smooth test function on $\mathbb{S}^d$,

$$f(\vec{n}\cdot\vec{n}'') = \exp\left(-\frac{1+\vec{n}\cdot\vec{n}''}{1-\vec{n}\cdot\vec{n}''}\right) \ , \tag{6.1.46}$$

where we have made the following identifications,

$$r = \cot(\theta/2), \qquad \vec{n}\cdot\vec{n}'' = \cos\theta \ . \tag{6.1.47}$$

Computing the corresponding f_l's gives

$$f_l = \frac{1}{\mathbb{Y}_{l\mathbf{m}}(\vec{n}')} \int_{\mathbb{S}^3} d\vec{n} \, \mathbb{Y}_{l\mathbf{m}}(\vec{n}) f(\vec{n}\cdot\vec{n}') \ , \tag{6.1.48}$$

$$= \frac{4\pi}{U_l(1)} \int_{-1}^{1} dx (1-x^2)^{1/2} U_l(x) \exp\left(-\frac{1+x}{1-x}\right) \ . \tag{6.1.49}$$

It is convenient to choose $\vec{n}''\cdot\vec{n}' = -1$ so that the Gaussian is peaked when the arguments of the two-point distribution coincide. We then obtain for the right-hand side of (6.1.45),

$$\begin{aligned}\text{RHS} &= 4\pi \lim_{\eta\to 0}\int_{-1}^{1} dx(1-x^2)^{1/2}\left[\frac{1}{2^2(1-\vec{n}\cdot\vec{n}')^2} - \frac{2\pi\sqrt{2}}{\sqrt{\eta}}\frac{\delta(1-x)}{4\pi(1-x^2)^{1/2}}\right]\exp\left(-\frac{1-x}{1+x}\right) ,\\ &= 4\pi \lim_{\eta\to 0}\int_{-1}^{1-\eta} dx(1-x^2)^{1/2}\frac{\exp\left(-\frac{1-x}{1+x}\right)}{2^2(1-x)^2} - \frac{2\pi\sqrt{2}}{\sqrt{\eta}} ,\end{aligned} \tag{6.1.50}$$

while on the left-hand side we obtain an infinite sum over Chebyshev polynomials

$$\text{LHS} = -\frac{1}{2}\sum_{l=0}^{\infty} f_l^*(l+1)^2 U_l(-1) . \tag{6.1.51}$$

Numerically computing the f_l's it is easy to see that that LHS and RHS agree, as they should.

6.1.3 Hyperboloid

The next example we treat is the hyperboloid CFT (see e.g. [137]), where we will see that correlators continue from the sphere in a simple way by analytic continuation of the angular momentum to complex values, as in [138].

Analytically continuing the sphere $\mathbb{S}^d$ to negative curvature we obtain the d-dimensional hyperbolic space $\mathbb{H}_d$, which is the Euclidean continuation of anti de Sitter space AdS$_d$. The analysis for the hyperboloid CFT proceeds similarly to the sphere. The conformal map from $\mathbb{E}^d$ to $\mathbb{H}_d$ is given by $r = \coth(\rho/2) = \sinh\rho/(\cosh\rho - 1)$,

$$ds^2_{\mathbb{H}_d} = d\rho^2 + \sinh^2\rho d\Omega^2_{d-1} = (1-\cosh\rho)^2(dr^2 + r^2 d\Omega^2_{d-1}) , \tag{6.1.52}$$

and hence the conformal factor is $\Omega = (1-\cosh\rho)$. Using (6.1.22) and the known flat space form for the two-point function, it follows that the bare two-point function on the hyperboloid is given by

$$\langle\mathcal{O}_\Delta(n)\mathcal{O}_\Delta(n')\rangle_{\mathbb{H}_d} = \frac{1}{2^\Delta(\cosh\ell - 1)^\Delta} , \tag{6.1.53}$$

where

$$\cosh \ell = \cosh \rho \cosh \rho' - \cos \alpha \sinh \rho \sinh \rho' \ . \tag{6.1.54}$$

ℓ is the geodesic distance on $\mathbb{H}_d$, and α is the angular separation of the two points in $\mathbb{S}^{d-1}$. Expanding the two-point distribution into eigenfunctions $\psi_{p,l,\mathbf{m}}$ of the Laplacian on $\mathbb{H}_d$ (reviewed in appendix 6.4.2) gives

$$\langle \mathcal{O}_\Delta(n)\mathcal{O}_\Delta(n')\rangle^{\text{ren}}_{\mathbb{H}_d} = \int_0^\infty dp\, g(p) \sum_{l,\mathbf{m}} \psi_{p,l,\mathbf{m}}(n)\psi_{p,l,\mathbf{m}}(n')^* \ . \tag{6.1.55}$$

The right-hand side can be expressed in terms of the geodesic distance between n and n' with the help of the addition theorem [139]

$$\sum_{l,\mathbf{m}} \psi_{p,l\mathbf{m}}(n)\psi^*_{p,l\mathbf{m}}(n') = \frac{1}{2\pi}(2\pi \sinh \ell)^{(2-d)/2} \left| \frac{\Gamma((d-1)/2+ip)}{\Gamma(ip)} \right|^2 P^{(2-d)/2}_{-1/2+ip}(\cosh \ell) \ , \tag{6.1.56}$$

where $n \cdot n' = \cosh \ell$. We will focus on the case when d is odd for simplicity, since in this case the Legendre functions can be expressed in terms of Gegenbauer functions. The generalization to even d is straightforward. The addition theorem for d odd is

$$\sum_{l,\mathbf{m}} \psi_{p,l\mathbf{m}}(n)\psi^*_{p,l\mathbf{m}}(n') = \frac{2i}{V_{\mathbb{S}^d}(d-1)} p\, C^{(d-1)/2}_{-(d-1)/2+ip}(\cosh \ell) \ , \tag{6.1.57}$$

and thus

$$\langle \mathcal{O}_\Delta(n)\mathcal{O}_\Delta(n')\rangle^{\text{ren}}_{\mathbb{H}_d} = \frac{2i}{V_{\mathbb{S}^d}(d-1)} \int_0^\infty dp\, p\, g(p)\, C^{(d-1)/2}_{-(d-1)/2+ip}(\cosh \ell) \ . \tag{6.1.58}$$

Hence, we see that, at least in the case of odd dimension d, there is a simple relationship between the spectral decomposition of the two-point function on the sphere and the hyperboloid; namely, we simply take the corresponding expression on the sphere (6.1.26) and analytically continue the angular momentum quantum number to complex values, corresponding to the principal series of unitary irreducible representations of $\mathrm{SO}(1,d)$ [141]

$$l = -\frac{d-1}{2} + ip \ , \qquad p \geq 0 \ . \tag{6.1.59}$$

The spectral decomposition can be inverted to give

$$g(p) = \frac{1}{\psi_{p,l,\mathbf{m}}(n)} \int dn \, \psi_{p,l,\mathbf{m}}(n') \langle \mathcal{O}_\Delta(n) \mathcal{O}_\Delta(n') \rangle^{\mathrm{ren}}_{\mathbb{H}_d} \ , \tag{6.1.60}$$

$$= \frac{V_{\mathbb{S}^{d-1}}}{C^{(d-1)/2}_{-(d-1)/2+ip}(1)} \int_1^\infty dz (z^2-1)^{(d-2)/2} \left[\frac{1}{2^\Delta (z-1)^\Delta} + \text{counter-terms} \right] C^{(d-1)/2}_{-(d-1)/2+ip}(z) \ . \tag{6.1.61}$$

Here we have used that the expression is independent of l and $\mathbf{m}$ to set them both to zero. This allows us to make use of the following identity which expresses the wavefunctions in terms of Gegenbauer functions

$$C^{(d-1)/2}_{-(d-1)/2+ip}(\cosh \ell) = (\sinh \ell)^{(d-2)/2} P^{(2-d)/2}_{-1/2+ip}(\cosh \ell) \frac{2^{(d-2)/2} \Gamma(d/2) \Gamma((d-1)/2+ip)}{\Gamma(d-1) \Gamma(ip - d/2 + 3/2)} \ . \tag{6.1.62}$$

That is,

$$\psi_{p,0,0}(r,\Omega) \propto \frac{\Gamma((d-1)/2+ip)}{\Gamma(ip)} (\sinh r)^{(2-d)/2} P^{(2-d)/2}_{-1/2+ip}(\cosh r) \propto C^{(d-1)/2}_{-(d-1)/2+ip}(\cosh r) \ . \tag{6.1.63}$$

The generalization of the integral transformation (6.1.16) is now

$$\int_0^\infty dp \sum_{l,\mathbf{m}} g(p) f(p)^* \, \psi_{p,l,\mathbf{m}}(\vec{n}') \psi_{p,l,\mathbf{m}}(\vec{n}')^* = \int_{\mathbb{H}_d} dn \langle \mathcal{O}(n) \mathcal{O}(n') \rangle^{\mathrm{ren}}_{\mathbb{H}_d} f(n \cdot n') \ , \tag{6.1.64}$$

where

$$f(p) = \frac{V_{\mathbb{S}^{d-1}}}{C^{(d-1)/2}_{-(d-1)/2+ip}(1)} \int_1^\infty dz (z^2-1)^{(d-2)/2} f(z) C^{(d-1)/2}_{-(d-1)/2+ip}(z) \ . \tag{6.1.65}$$

Using the addition theorem this becomes simply

$$\frac{2i}{V_{\mathbb{S}^d}(d-1)} \int_0^\infty dp\, p\, C^{(d-1)/2}_{-(d-1)/2+ip}(1) g(p) f(p)^* = \int_{\mathbb{H}_d} dn \langle \mathcal{O}(n)\mathcal{O}(n')\rangle^{\text{ren}}_{\mathbb{H}_d} f(n\cdot n')\ . \tag{6.1.66}$$

Let us test this formula by focusing on the case of a $\Delta = 2$ scalar operator in $d = 3$ dimensions. Following the same steps as on the sphere we obtain the renormalized two-point correlator

$$\langle \mathcal{O}_2(n)\mathcal{O}_2(n')\rangle^{\text{ren}}_{\mathbb{H}_3} = \lim_{\eta\to 0}\left[\frac{1}{2^2(n\cdot n'-1)^2} - \frac{2\pi\sqrt{2}}{\sqrt{\eta}}\delta^3(n,n')\right]\ , \tag{6.1.67}$$

$$g(p) = \lim_{\eta\to 0}\left[-\frac{2\pi\sqrt{2}}{\sqrt{\eta}} + \frac{V_{\mathbb{S}^2}}{U_{-1+ip}(1)}\int_{1+\eta}^\infty dz (z^2-1)^{1/2}\frac{1}{2^2(z-1)^2}U_{-1+ip}(z)\right]\ . \tag{6.1.68}$$

As before, consider a Gaussian test function to illustrate. We recall that ℓ is related to z by the relation $z = \cosh\ell$, so the natural analog of a Gaussian on the hyperboloid is the test function $f(z) = e^{-z}$. In order to evaluate the integral on the left-hand side of (6.1.66) we need the spectral representations of $\langle \mathcal{O}_2(n)\mathcal{O}_2(n')\rangle_{\mathbb{H}_3}$ and $f(\vec{n}\cdot\vec{n}')$ which are given by (6.1.68) and (6.1.65), respectively. The integral defining $f(p)$ was evaluated numerically for different values of p and numerically interpolated. The integral defining $g(p)$, while difficult to evaluate, can be guessed by analytical continuation from the sphere. Substituting $l = -(d-1)/2 + ip$, dropping a factor of i and multiplying by a measure factor of $\coth(\pi p)$ one finds agreement with the numerics. Substituting the approximate expression for $f(p)$ and the exact expression for $g(p)$ into the left-hand side of (6.1.66) and carrying out the final p-integral numerically leads to excellent agreement. Here we demonstrate the numerics for a $\Delta = 2$ operator in $d = 3$ dimensions,

$$\int_{\mathbb{H}_3} dn \langle \mathcal{O}_2(n)\mathcal{O}_2(n')\rangle^{\text{ren}}_{\mathbb{H}_3} f(n\cdot n') = \lim_{\eta\to 0}\left[4\pi\int_{1+\eta}^{\infty} dz (z^2-1)^{1/2}\frac{e^{-z}}{2^2(z-1)^2} - \frac{2\pi\sqrt{2}}{\sqrt{\eta}}e^{-1}\right] , \tag{6.1.69}$$

$$\simeq -5.118 . \tag{6.1.70}$$

$$\frac{i}{2\pi^2}\int_0^{\infty} dp\, p\, U_{-1+ip}(1) g(p) f(p)^* \simeq 5.118 . \tag{6.1.71}$$

Let us now consider a marginal operator $\Delta = d = 3$. We have

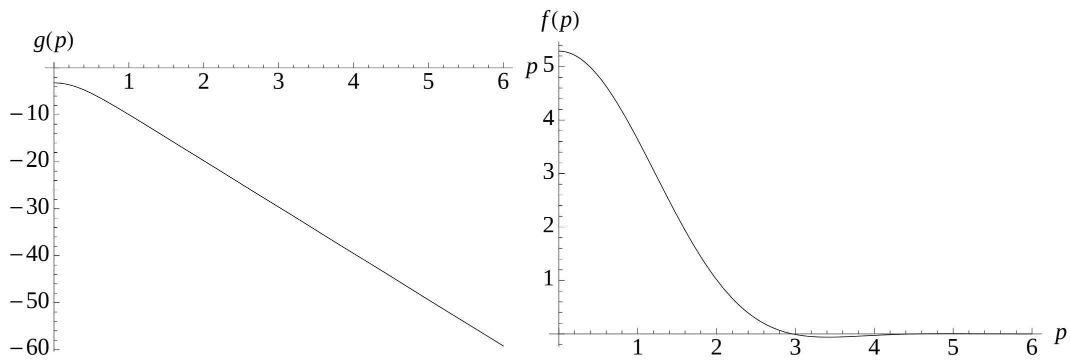

Figure 3: Spectral representations $\langle \mathcal{O}_2(n)\mathcal{O}_2(n')\rangle_{\mathbb{H}_3}$ (left) and $f(\vec{n}\cdot\vec{n}')$ (right) obtained by numerical interpolation.

$$\frac{i}{2\pi^2}\int_0^{\infty} dp\, p\, U_{-1+ip}(1) g(p) f(p)^* \simeq 1.1659 , \qquad g(p) = \frac{\pi^2}{12}p(1+p^2)\coth(\pi p) , \tag{6.1.72}$$

and

$$\int_{\mathbb{H}_3} dn \langle \mathcal{O}_3(n)\mathcal{O}_3(n')\rangle^{\text{ren}}_{\mathbb{H}_3} f(n\cdot n') = \lim_{\eta\to 0}\left[4\pi\int_{1+\eta}^{\infty} dz (z^2-1)^{1/2}\frac{e^{-z}}{2^3(z-1)^3} - \frac{\pi\sqrt{2}}{3\eta^{3/2}}e^{-1} + \frac{3\pi}{2\sqrt{2\eta}}e^{-1}\right] ,$$

$$\simeq 1.1659 . \tag{6.1.73}$$

6.2 Holographic Boundary CFT

In this section we will review the calculation of the one- and two-point functions for a boundary CFT from holography and then connect this with the integral transforms of the previous section. We begin with the AdS_{d+1} metric (6.0.1) and change to radial coordinates in the z and y variables: $(z, y) = (\eta\cos\phi, \eta\sin\phi)$. We will also use the coordinate ρ, defined by $(z, y) = (\eta\,\mathrm{sech}\,\rho, \eta\tanh\rho)$. The metric in these coordinates is

$$ds^2 = \frac{1}{\cos^2\phi}(d\phi^2 + ds^2_{AdS_d}) \tag{6.2.1}$$

$$= d\rho^2 + \cosh^2\rho\, ds^2_{AdS_d}\ . \tag{6.2.2}$$

This covers the full Poincaré patch of AdS_{d+1} if $-\infty < \rho < \infty$, or $-\pi/2 < \phi < \pi/2$ (the UV boundary is at $\rho = -\infty$, or $\phi = -\pi/2$). The claim of the AdS/BCFT correspondence is that we obtain the holographic dual to a half-space CFT by restricting $-\infty < \rho < \rho_*$ for some ρ_*. This effectively cuts the space off in the IR and introduces a second boundary Q at ρ_* in addition to the usual UV boundary at $\rho = -\infty$. The surface Q defined by $\rho = \rho_*$ is given in Poincaré coordinates by $z = \eta\,\mathrm{sech}\,\rho_*$ and $y = \eta\tanh\rho_*$. Hence, Q is defined by a curve in the (y, z) plane

$$y = z\sinh\rho_*\ . \tag{6.2.3}$$

Notice that if we choose $\rho_* = 0$ then Q is given simply by $y = 0$. For general ρ_* let us define $\tan\theta = \sinh\rho_*$ and consider the rotation

$$\begin{pmatrix} \tilde{y} \\ \tilde{z} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix}\ . \tag{6.2.4}$$

Now we see that Q lies at $\tilde{y} = 0$.

Assume that localized on Q there is a linear coupling

$$S_Q = \int_Q d^4x\sqrt{h}\, a\phi\ , \tag{6.2.5}$$

with $\sqrt{h}$ the induced volume form on Q and a a constant. It is natural to add such a term because, from a Witten diagram point of view, it can be seen to correspond to giving a vacuum expectation value to the dual operator [152]. An alternative possibility would be to add a boundary mass term $\int_Q d^dx\sqrt{h}\, b\, \phi^2$. Taking $b \to \infty$ realizes the Dirichlet boundary condition on Q. The variation of this term does not affect the bulk equations of motion but contributes to the boundary variation. The total boundary variation is

$$\delta S = \int_Q d^4x\sqrt{h}\delta\phi(n^\mu\partial_\mu\phi + a) \ , \tag{6.2.6}$$

where n^μ is the unit normal to Q. The first term is the boundary term coming from the variation of the bulk kinetic term after integration by parts, and the second term comes from varying (6.2.5). The variational principle requires (6.2.6) to vanish for arbitrary $\delta\phi$, which requires the boundary condition on Q to be of Neumann form

$$(n^\mu\partial_\mu\phi + a)|_Q = 0 \ . \tag{6.2.7}$$

In our case, we have $n_\mu dx^\mu = c\, d\tilde{y}$ where c is determined by the normalization condition $n_\mu n^\mu = 1$ or $g^{\tilde{y}\tilde{y}}c^2 = 1$. The metric written in terms of these variables is

$$ds^2 = \frac{d\tilde{z}^2 - dt^2 + dx_1^2 + dx_2^2 + d\tilde{y}^2}{z^2} = \frac{d\tilde{z}^2 - dt^2 + dx_1^2 + dx_2^2 + d\tilde{y}^2}{(\tilde{z}\cos\theta - \tilde{y}\sin\theta)^2} \ . \tag{6.2.8}$$

Hence, $g^{\tilde{y}\tilde{y}} = z^2$, $c = \pm 1/z$, $n^\mu\partial_\mu = c g^{\tilde{y}\tilde{y}}\partial_{\tilde{y}} = \pm z\partial_{\tilde{y}}$. Choosing the plus sign we obtain the boundary condition

$$\partial_{\tilde{y}}\phi|_{\tilde{y}=0} + \frac{a}{\tilde{z}\cos\theta} = 0 \ , \tag{6.2.9}$$

$$(\cos\theta\partial_y - \sin\theta\partial_z)\phi|_{y=z\tan\theta} + \frac{a}{z} = 0 \ . \tag{6.2.10}$$

6.2.1 One-point function

Let us consider the Fourier transform of the field configuration in the y-direction,

$$\phi(y,z) = z^{d/2} \int_{-\infty}^{\infty} dq\, f_q(z) c(q) e^{iqy} \ . \tag{6.2.11}$$

Substituting into the scalar equation in the flat slicing

$$z^2 \partial_z^2 \phi - (d-1) z \partial_z \phi + z^2 \eta^{\mu\nu} \partial_\mu \partial_\nu \phi = m^2 \phi \tag{6.2.12}$$

we obtain

$$z^{d/2+2} \int_{-\infty}^{\infty} dq\, f_q(z) c(q) e^{iqy} \left[\frac{f_q''(z)}{f_q(z)} + \frac{1}{z} \frac{f_q'(z)}{f_q(z)} - \frac{1}{z^2} (q^2 z^2 + m^2 + d^2/4) \right] = 0 \ , \tag{6.2.13}$$

which can be solved by choosing $f_q(z) = K_\nu(|q|z)$ or $I_\nu(|q|z)$ where

$$\nu = \sqrt{\left(\frac{d}{2}\right)^2 + m^2} \ . \tag{6.2.14}$$

To find $c(q)$ we need to substitute the ansatz into the inhomogeneous boundary condition. Choosing the solution which is regular in the interior we find that the boundary condition is satisfied if $c(q) \propto |q|^{d/2}/q$ [119]. Setting $d = 4$ and $m^2 = 0$, for example, we obtain

$$\phi(y,z) \propto \frac{(2y^3 + 3yz^2)}{(y^2 + z^2)^{3/2}} = 2 - \frac{3z^4}{4y^4} + \mathcal{O}(z^6/y^6) \ . \tag{6.2.15}$$

The vacuum expectation value can then be read off from coefficient of the normalizable term

$$\langle \mathcal{O}_4(y) \rangle \propto \frac{1}{y^4}, \tag{6.2.16}$$

which is of the form required by the unbroken subgroup of the conformal group, namely,

$$\langle \mathcal{O}_\Delta(y) \rangle = \frac{c}{y^\Delta}. \tag{6.2.17}$$

6.2.2 Two-point function

Unlike the holographic interface CFT [140, 152], the holographic BCFT two-point function is not of the form $1/x^{2\Delta}$ even when $a = 0$ [143]. If we insert a boundary at $z = z_*$ then we need to impose the Neumann boundary condition at $y = z \sinh \rho_*$. For convenience we will choose $\rho_* = 0$ so that the boundary condition is simply

$$\partial_y \phi|_{y=0} = 0 \ . \tag{6.2.18}$$

Substituting the ansatz $\phi = z^{d/2} f(z) h(y) e^{-i\vec{\omega}\cdot\vec{x}}$ we find that the boundary condition fixes $h(y) = e^{-iqy} + e^{iqy}$ and the general solution is thus of the form

$$\phi(\vec{x}, y, z) = z^{d/2} \int d^{d-1}\vec{\omega}\, dq\, (e^{-iqy} + e^{iqy}) e^{-i\vec{\omega}\cdot\vec{x}} k^{\nu} K_{\nu}(kz) \phi_{(0)}(\vec{\omega}, q) \ . \tag{6.2.19}$$

Since $(e^{-iqy} + e^{iqy}) K_{\nu}(kz)$ is an even function of q, the integral over q projects out the even part of $\phi_{(0)}(\vec{\omega}, q)$. Hence the only constraint on $\phi_{(0)}(\vec{\omega}, q)$ is that it is itself an even function of q,

$$\phi_{(0)}(\vec{\omega}, q) = \phi_{(0)}(\vec{\omega}, -q) \ . \tag{6.2.20}$$

Fourier transforming, we have

$$\phi_{(0)}(\vec{x}, y) = \int d^{d-1}\vec{\omega}\, dq\, e^{-i(qy + \vec{\omega}\cdot\vec{x})} \phi_{(0)}(\vec{\omega}, q) \ , \tag{6.2.21}$$

$$= \int d^{d-1}\vec{\omega}\, dq \cos(qy) e^{-i\vec{\omega}\cdot\vec{x}} \phi_{(0)}(\vec{\omega}, q) \ , \tag{6.2.22}$$

$$= \frac{1}{2} \int d^{d-1}\vec{\omega}\, dq (e^{iqy} + e^{-iqy}) e^{-i\vec{\omega}\cdot\vec{x}} \phi_{(0)}(\vec{\omega}, q) \ . \tag{6.2.23}$$

Inverting the Fourier transform we then find

$$\phi_{(0)}(\vec{\omega}, q) = \frac{1}{2(2\pi)^d} \int d^{d-1}\vec{x}\, dy (e^{iqy} + e^{-iqy}) e^{i\vec{\omega}\cdot\vec{x}} \phi_{(0)}(\vec{x}, y) \ , \tag{6.2.24}$$

which is automatically invariant under $q \to -q$ for any $\phi_{(0)}(\vec{x}, y)$. Substituting back we obtain

$$\begin{aligned}\phi(\vec{x}, y, z) = \frac{1}{2}\int d^{d-1}\vec{x}' dy' \big[& K_\Delta(\vec{x}, y; \vec{x}', y', z) + K_\Delta(\vec{x}, -y; \vec{x}', y', z) + \\ & + K_\Delta(\vec{x}, y; \vec{x}', -y', z) + K_\Delta(\vec{x}, -y; \vec{x}', -y', z)\big]\phi_{(0)}(\vec{x}', y') \ , \end{aligned} \tag{6.2.25}$$

$$= \int d^{d-1}\vec{x}' dy' \big[K_\Delta(\vec{x}, y; \vec{x}', y', z) + K_\Delta(\vec{x}, -y; \vec{x}', y', z)\big]\phi_{(0)}(\vec{x}', y') \ , \tag{6.2.26}$$

where

$$K_\Delta(\vec{x}, y; \vec{x}', y', z) = z^{d/2} \int \frac{d^{d-1}\vec{\omega}\, dq}{(2\pi)^d} e^{iq(y-y')+i\vec{\omega}(\vec{x}-\vec{x}')} k^\nu K_\nu(kz) \tag{6.2.27}$$

is the standard bulk-to-boundary propagator for an operator of dimension $\Delta = d/2 + \nu$. The two-point function is

$$\langle \mathcal{O}(X_1)\mathcal{O}(X_2)\rangle = \frac{1}{|X_1 - X_2|^{2\Delta}} + \frac{1}{|X_1 - X_2^*|^{2\Delta}} \ , \tag{6.2.28}$$

where $X \equiv (\vec{x}, y)$ and $X^* \equiv (\vec{x}, -y)$. Setting $\vec{x}_2 = \vec{0}$ without loss of generality we obtain

$$\langle \mathcal{O}(\vec{x}, y_1)\mathcal{O}(\vec{0}, y_2)\rangle = \frac{1}{(\vec{x}^2 + (y_1 - y_2)^2)^\Delta} + \frac{1}{(\vec{x}^2 + (y_1 + y_2)^2)^\Delta} \ , \tag{6.2.29}$$

$$= \frac{1}{(4y_1y_2)^\Delta}\left[\frac{1}{\xi^\Delta} + \frac{1}{(\xi+1)^\Delta}\right] \ , \tag{6.2.30}$$

which is of the correct form [145, 146] dictated by conformal invariance,

$$\langle \mathcal{O}_1(\vec{x}, y_1)\mathcal{O}_2(\vec{0}, y_2)\rangle = \frac{F(\xi)}{y_1^{\Delta_1} y_2^{\Delta_2}} \ , \qquad \xi = \frac{\vec{x}^2 + (y_1 - y_2)^2}{4y_1y_2} \ . \tag{6.2.31}$$

The function $F(\xi)$, which is not fixed by conformal invariance alone, is determined by the AdS/CFT calculation. If we take $y_2 \to 0$ then

$$\langle \mathcal{O}(\vec{x}, y)\mathcal{O}(\vec{0}, 0)\rangle = \frac{2}{(\vec{x}^2 + y^2)^\Delta} \ , \tag{6.2.32}$$

which is of the form fixed by $\mathrm{O}(1,4)$ invariance

$$\langle \mathcal{O}_1(\vec{x}, y_1)\mathcal{O}_2(\vec{0}, 0)\rangle \propto \frac{1}{y_1^{\Delta_1-\Delta_2}(\vec{x}^2+y_1^2)^{\Delta_2}} \ . \tag{6.2.33}$$

Repeating the calculation for the Dirichlet boundary condition we obtain

$$\langle \mathcal{O}(X_1)\mathcal{O}(X_2)\rangle = \frac{1}{|X_1-X_2|^{2\Delta}} - \frac{1}{|X_1-X_2^*|^{2\Delta}} \tag{6.2.34}$$

$$= \frac{1}{(4y_1y_2)^{\Delta}}\left[\frac{1}{\xi^{\Delta}} - \frac{1}{(\xi+1)^{\Delta}}\right] \ . \tag{6.2.35}$$

Note that, unlike in the case of Neumann boundary conditions, this two-point function vanishes in the limit $y_2 \to 0$.

6.2.3 *Two-point function in AdS slicing*

If we allow $\rho_* \neq 0$ then we encounter a difficulty because the boundary condition now mixes y derivatives with z derivatives on Q

$$\partial_y\phi|_{y=z\tan\theta} = \cot\theta\,\partial_z\phi|_{y=z\tan\theta} \ . \tag{6.2.36}$$

It is thus more natural to work in the slicing of AdS_{d+1} by AdS_d, where the boundary condition is replaced by $\partial_\rho\phi|_{\rho_*} = 0$. The metric in these coordinates is given by

$$ds^2 = d\rho^2 + \cosh^2\rho\, ds^2_{\mathbb{H}_d}. \tag{6.2.37}$$

We will mainly focus on the example of a marginal operator in three dimensions, but similar results hold for any d and Δ. As shown in appendix 6.4.2, the bulk-to-boundary propagator (assuming d odd) is given in these coordinates by

$$K(\rho, x; x') = \frac{2i}{V_{\mathbb{S}^d}(d-1)}\int_0^\infty dp\, f_p(\rho) p\, C^{(d-1)/2}_{-(d-1)/2+ip}(\cosh\ell) \ , \tag{6.2.38}$$

where $f_p(\rho)$ is some linear combination of

$$f(\rho) = (\operatorname{sech}\rho)^{d/2}\left\{P^{\nu}_{-1/2+ip}(\tanh\rho), \quad Q^{\nu}_{-1/2+ip}(\tanh\rho)\right\} \ , \tag{6.2.39}$$

to be fixed by boundary conditions. Given a marginal operator we should, according to the AdS/CFT correspondence, consider a $m^2 = 0$ scalar field in AdS$_4$. We obtain (assuming $d = 3$) the following asymptotics for the conical functions in (6.2.39) (the asymptotic boundary is at $\rho = -\infty$) ,

$$\begin{aligned}(\operatorname{sech}\rho)^{d/2}P^{\nu}_{-1/2+ip}(\tanh\rho) &= \sqrt{\frac{2}{\pi}}\cosh(p\pi) - 2\left(1+p^2\right)\sqrt{\frac{2}{\pi}}\cosh(p\pi)e^{2\rho} \\ &\quad + \frac{8}{3}p\left(1+p^2\right)\sqrt{\frac{2}{\pi}}\sinh(p\pi)e^{3\rho} + \mathcal{O}(e^{4\rho}) \ , \qquad (6.2.40) \\ (\operatorname{sech}\rho)^{d/2}Q^{\nu}_{-1/2+ip}(\tanh\rho) &= -i\sqrt{\frac{\pi}{2}}\sinh(p\pi) + i\left(1+p^2\right)\sqrt{2\pi}\sinh(p\pi)e^{2\rho} \\ &\quad - \frac{4}{3}ip\left(1+p^2\right)\sqrt{2\pi}\cosh(p\pi)e^{3\rho} + \mathcal{O}(e^{4\rho}) \ . \qquad (6.2.41)\end{aligned}$$

Regularity in the interior ($\rho_* = \infty$) demands that we drop the Legendre-P function and thus, according to the AdS/CFT dictionary, the bare two-point function is given by

$$\langle \mathcal{O}_3(n)\mathcal{O}_3(n')\rangle_{\mathbb{H}_3} \propto \int_0^\infty dp\, p^2(1+p^2)\coth(\pi p)U_{-1+ip}(n\cdot n') \ . \tag{6.2.42}$$

The right-hand side is clearly divergent, as is to be expected since we are dealing with the bare, rather than the renormalized correlator. We can gain considerable insight about this infinite expression with the help of the integral representation of the generalized Gegenbauer function [144]

$$C^{(d-1)/2}_{-(d-1)/2+ip}(z) = i(-1)^{(d-1)/2+1}2^{-(d-1)/2}\frac{\sinh(\pi p)}{\pi}\int_{-\infty}^{\infty} d\beta(\cosh\beta + z)^{-(d-1)/2}e^{-ip\beta} \ . \tag{6.2.43}$$

Rotating the contour to the imaginary axis by defining $\sigma = i\beta$, and using the Mellin transformation, we obtain the following generating function

$$p(\sigma) = (\cos\sigma + z)^{-(d-1)/2} = i(-2)^{(d-1)/2} \int_0^\infty dp \cosh(\sigma p) \, \frac{C^{(d-1)/2}_{-(d-1)/2+ip}(z)}{\sinh(\pi p)} \; . \qquad (6.2.44)$$

It is now possible to express the bare correlator as a linear combination of derivatives of $p(\sigma)$, thus extracting the finite part of the bare correlator

$$\langle \mathcal{O}_3(n)\mathcal{O}_3(n')\rangle_{\mathbb{H}_3} \propto \left(\frac{d^2p}{d\sigma^2} + \frac{d^4p}{d\sigma^4}\right)\bigg|_{\sigma=\pi} = \frac{6}{(n\cdot n' - 1)^3} \; . \qquad (6.2.45)$$

We thus see that the finite piece agrees with the expectations from conformal invariance.

For the BCFT we obtain a linear combination of Q and P Legendre functions determined by the boundary condition $f_p'(\rho_*) = 0$. In particular,

$$f_p(\rho) = (\operatorname{sech}\rho)^{d/2}\left[Q^\nu_{-1/2+ip}(\tanh\rho) + b_p P^\nu_{-1/2+ip}(\tanh\rho)\right] \; , \qquad (6.2.46)$$

where

$$b_p = -\frac{(1+2ip-2\nu)Q^\nu_{1/2+ip}(\tanh\rho_*) + (d-1-2ip)Q^\nu_{-1/2+ip}(\tanh\rho_*)\tanh\rho_*}{(1+2ip-2\nu)P^\nu_{1/2+ip}(\tanh\rho_*) + (d-1-2ip)P^\nu_{-1/2+ip}(\tanh\rho_*)\tanh\rho_*} \; . \quad (6.2.47)$$

Notice that for $\rho_* \to 0$ we obtain

$$b_p = -\frac{Q^\nu_{1/2+ip}(0)}{P^\nu_{1/2+ip}(0)} \; , \qquad (6.2.48)$$

while for $\rho_* \to \infty$ we have $b_p \to 0$ and we recover the formula for a pure CFT.

Choosing $d = 3$, $m^2 = 0$, $\rho_* = 0$ and using our Gaussian test function $f(z) = e^{-z}$, we obtain

$$\frac{i}{2\pi^2}\int_0^\infty dp\, p\, U_{-1+ip}(1) g(p) f(p)^* \simeq -1.19857 \; , \qquad g(p) = \frac{\pi\coth(\pi p) + 2ib_p}{2ib_p\coth(\pi p) + \pi} \; . \quad (6.2.49)$$

In order to evaluate the RHS of the distribution formula, we need to conformally map the BCFT two-point function (6.2.30) to the hyperboloid. This can be achieved by identifying $y > 0$ with the Poincaré radial coordinate of $\mathbb{H}_d$. It then follows that

$$\cosh \ell - 1 = \frac{(z - z')^2 + (\vec{x} - \vec{x}')^2}{2zz'} = 2\xi \ , \tag{6.2.50}$$

and thus

$$\langle \mathcal{O}(n)\mathcal{O}(n')\rangle_{\mathbb{H}_d} = \frac{1}{2^\Delta}\left[\frac{1}{(\cosh \ell - 1)^\Delta} + \frac{1}{(\cosh \ell + 1)^\Delta}\right] \ . \tag{6.2.51}$$

Subtracting divergences and smearing with the test function over the hyperboloid we obtain

$$\begin{aligned}\int_{\mathbb{H}_3} dn \langle \mathcal{O}_3(n)\mathcal{O}_3(n')\rangle^{\text{ren}}_{\mathbb{H}_3} f(n \cdot n') = \lim_{\eta\to 0}\Bigg[4\pi \int_{1+\eta}^\infty dz (z^2-1)^{1/2} e^{-z}\frac{1}{2^3}\left[\frac{1}{(z-1)^3}+\frac{1}{(z+1)^3}\right] + \\ - \frac{\pi\sqrt{2}}{3\eta^{3/2}}e^{-1} + \frac{3\pi}{2\sqrt{2\eta}}e^{-1}\Bigg] \simeq 1.19857 \ . \end{aligned}\tag{6.2.52}$$

In the case of a Dirichlet boundary condition at $\rho_* = 0$, the two-point function (6.2.35) expressed in terms of the geodesic distance is

$$\langle \mathcal{O}(n)\mathcal{O}(n')\rangle_{\mathbb{H}_d} = \frac{1}{2^\Delta}\left[\frac{1}{(\cosh \ell - 1)^\Delta} - \frac{1}{(\cosh \ell + 1)^\Delta}\right] \ , \tag{6.2.53}$$

and

$$b_p = -\frac{Q^\nu_{-1/2+ip}(0)}{P^\nu_{-1/2+ip}(0)} \ . \tag{6.2.54}$$

In this case

$$\begin{aligned}\int_{\mathbb{H}_3} dn \langle \mathcal{O}_3(n)\mathcal{O}_3(n')\rangle^{\text{ren}}_{\mathbb{H}_3} f(n \cdot n') &= \lim_{\eta\to 0}\Bigg[4\pi \int_{1+\eta}^\infty dz (z^2-1)^{1/2} e^{-z}\frac{1}{2^3}\left[\frac{1}{(z-1)^3}-\frac{1}{(z+1)^3}\right] + \\ &\quad - \frac{\pi\sqrt{2}}{3\eta^{3/2}}e^{-1} + \frac{3\pi}{2\sqrt{2\eta}}e^{-1}\Bigg] \\ &\simeq 1.1333 \ , \end{aligned}\tag{6.2.55}$$

$$\frac{i}{2\pi^2}\int_0^\infty dp\, p\, U_{-1+ip}(1) g(p) f(p)^* \simeq -1.1333 \ . \tag{6.2.56}$$

6.3 Holographic pseudo-conformal CFT

The pseudo-conformal CFT can be regarded as a CFT with a spacelike boundary at future infinity. We begin with the AdS_{d+1} metric (6.0.1) and perform the coordinate transformation $z = (-\eta)\operatorname{csch}\rho$ and $t = \eta \coth \rho$. Then

$$ds^2 = d\rho^2 + \sinh^2 \rho \frac{-d\eta^2 + d\vec{x}^2}{\eta^2} \tag{6.3.1}$$

$$= d\rho^2 + \sinh^2 \rho \, ds^2_{dS_d} \ , \tag{6.3.2}$$

where $\eta \in (-\infty, 0)$ and $\rho \in (0, \infty)$. Unlike the AdS_d slicing, this coordinate system only covers a subregion of the AdS_{d+1} Poincaré patch. The subregion already has a boundary given by the light-cone at $\rho = 0$. Rather than choosing Q to be this null boundary, however,we will instead fix Q at some $\rho_* > 0$.

We expect that the resulting VEV will be of the form $1/(-t)^\Delta$ with a ρ_*-dependent coefficient which vanishes as $\rho_* \to 0$. A general $\rho = \rho_*$ surface is given in Poincaré coordinates by a worldline

$$(-t) = z \cosh \rho_* \ , \tag{6.3.3}$$

which intersects the boundary on the spacelike surface $t = 0$. Let us define $\cosh \rho_* = \coth \phi$ so that the surface Q is defined by $t = -z \coth \phi$. Consider the Lorentz boost

$$\begin{pmatrix} \tilde{t} \\ \tilde{z} \end{pmatrix} = \begin{pmatrix} \cosh \phi & \sinh \phi \\ \sinh \phi & \cosh \phi \end{pmatrix} \begin{pmatrix} t \\ z \end{pmatrix} \ . \tag{6.3.4}$$

Now the surface is defined by $\tilde{z} = 0$, and the metric in these coordinates is

$$ds^2 = \frac{d\tilde{z}^2 - d\tilde{t}^2 + d\vec{x}^2}{z^2} \ . \tag{6.3.5}$$

The boundary condition on Q is now

$$\partial_{\tilde{z}}\phi|_{t=-z\coth\phi} + \frac{a}{z} = 0 \ , \tag{6.3.6}$$

$$(\sinh\phi\partial_t + \cosh\phi\partial_z)\phi|_{t=-z\coth\phi} + \frac{a}{z} = 0 \ . \tag{6.3.7}$$

6.3.1 One-point function

For $0 < a < \infty$ we choose the ansatz to be

$$\phi(t,z) = z^{d/2}\int_{-\infty}^{\infty} dq\, f_q(z)c(q)e^{iqt} \ . \tag{6.3.8}$$

Substituting into the scalar equation we obtain

$$z^{d/2+2}\int_{-\infty}^{\infty} dq\, f_q(z)c(q)e^{iqt}\left[\frac{f_q''(z)}{f_q(z)} + \frac{1}{z}\frac{f_q'(z)}{f_q(z)} - \frac{1}{z^2}(-q^2z^2 + m^2 + d^2/4)\right] = 0 \ . \tag{6.3.9}$$

Demanding that the terms in square brackets vanish we find that $f_q(z)$ should be a linear combination of Bessel functions, which have the following asymptotic behavior

$$\mathbb{Y}_\nu(|q|z) = z^\nu\left(-\frac{2^{-\nu}|q|^\nu\cos(\pi\nu)\Gamma(-\nu)}{\pi} + \mathcal{O}(z^2)\right) + z^{-\nu}\left(-\frac{2^\nu|q|^{-\nu}\Gamma(\nu)}{\pi} + \mathcal{O}(z^2)\right) \ , \tag{6.3.10}$$

$$J_\nu(|q|z) = z^\nu\left(\frac{2^{-\nu}|q|^\nu}{\Gamma(1+\nu)} + \mathcal{O}(z^2)\right) \ . \tag{6.3.11}$$

These asymptotic expansions suggest that in order to interpret the scalar field configuration as a spontaneously generated VEV, we should choose $f_q(z) = J_\nu(|q|z)$. Substituting this into the boundary condition we then obtain

$$\int_{-\infty}^{\infty} dq\,\left[iq\sinh\phi + \frac{1}{z}\left(\frac{d}{2} - 1 - q\frac{c'(q)}{c(q)} + iqz\coth\phi\right)\cosh\phi\right] J_\nu(|q|z)c(q)e^{-iqz\coth\phi} = -\frac{a}{z^{d/2+1}} \ , \tag{6.3.12}$$

where we have replaced $z\partial_z$ by $q\partial q$ and integrated by parts. Notice that for large arguments the Bessel function is oscillating rather than decaying exponentially

$$J_\nu(z) \sim \sqrt{\frac{2}{\pi z}} \cos\left(z - \nu\pi/2 - \pi/4\right) \ . \tag{6.3.13}$$

The function $c(q)$ cannot be a power law $|q|^\alpha$ because this would imply

$$\frac{1}{z^2}\int_{-\infty}^{\infty} d\tilde{q}\, \left[i\tilde{q}\sinh\phi + \left(\frac{d}{2} - 1 - \alpha + i\tilde{q}\coth\phi\right)\cosh\phi\right] J_\nu(\tilde{q})c(\tilde{q}/z)e^{-i\tilde{q}\coth\phi} = -\frac{a}{z^{d/2+1}} \ , \tag{6.3.14}$$

and then $\alpha = d/2 - 1$, which would cause the LHS to diverge. On the other hand, if we choose $c(q)$ to be a regulated delta function

$$c(q) = \frac{1}{2\alpha}e^{-\alpha|q|} \ , \tag{6.3.15}$$

then assuming $d = 4$ and $m^2 = 0$ we obtain

$$\phi(y, z) = z^2 \int_{-\infty}^{\infty} dq\, J_\nu(|q|z)\frac{1}{2\alpha}e^{-\alpha|q|}e^{iqt} \ , \tag{6.3.16}$$

$$= \left(-\frac{3}{4t^4} + \mathcal{O}(\alpha^2)\right) z^4 + \left(-\frac{5}{4t^6} + \mathcal{O}(\alpha^2)\right) z^6 + \mathcal{O}(z)^8 \ . \tag{6.3.17}$$

The correctly normalized scalar is

$$\phi(y, z) = \frac{a\left(2 - 2\sqrt{1-x^2} + x^2\left(-3 + 2\sqrt{1-x^2}\right)\right)\operatorname{csch}^4\phi\,\operatorname{sech}(2\phi)}{3\left(1-x^2\right)^{3/2}} \ , \tag{6.3.18}$$

where $x = z/(-t)$. By direct substitution it can be shown that this solves both the equation of motion and the boundary condition. For the same reason as in the timelike BCFT case, the dual operator acquires a VEV

$$\langle \mathcal{O}_4(t)\rangle \propto \frac{1}{t^4}. \tag{6.3.19}$$

6.3.2 Two-point function

In the Euclidean signature the space dS_d continues to $\mathbb{S}^d$ and the wave equation in AdS_{d+1} sliced by $\mathbb{S}^d$ is solved by

$$\phi(\rho,\hat{n}) = \sum_{l,\mathbf{m}} f_l(\rho)\mathbb{Y}_{l\mathbf{m}}(\hat{n})\phi_{(0)l\mathbf{m}} \tag{6.3.20}$$

$$= \frac{1}{V_{\mathbb{S}^d}(d-1)}\int d\Omega'_d \sum_l (2l+d-1) f_l(\rho) C_l^{(d-1)/2}(\hat{n}\cdot\hat{n}')\phi_{(0)}(\hat{n}') \ , \tag{6.3.21}$$

where

$$f_l(\rho) = (\sinh\rho)^{(1-d)/2}\left\{P^{(1-d)/2-l}_{-1/2+\nu}(\cosh\rho), \quad Q^{(1-d)/2-l}_{-1/2+\nu}(\cosh\rho)\right\} \ . \tag{6.3.22}$$

The asymptotics of the ring functions can be found from the relations (see sec. 3.13 of [159])

$$P^{\mu}_{-1/2+\nu}(\cosh\rho) = \frac{2^{2\mu}}{\Gamma(1-\mu)}(1-e^{-2\rho})^{-\mu}e^{-(\nu+1/2)\rho}F(1/2-\mu, 1/2+\nu-\mu; 1-2\mu; 1-e^{-2\rho}) \ , \tag{6.3.23}$$

$$Q^{\mu}_{-1/2+\nu}(\cosh\rho) = \frac{\pi^{1/2}e^{i\mu\pi}\Gamma(1/2+\nu+\mu)}{\Gamma(1+\nu)}(1-e^{-2\rho})^{\mu}e^{-(\nu+1/2)\rho}F(1/2+\mu, 1/2+\nu+\mu; 1+\nu; e^{-2\rho}) \ . \tag{6.3.24}$$

Setting $d=3$ and $m^2=0$, we obtain

$$(\sinh\rho)^{(1-d)/2}P^{(1-d)/2-l}_{-1/2+\nu}(\cosh\rho) = \frac{1}{\Gamma(3+l)} - \frac{2(1+l)^2e^{-2\rho}}{\Gamma(3+l)} + \frac{8e^{-3\rho}}{3\Gamma(l)} + \mathcal{O}(e^{-4\rho}) \ , \tag{6.3.25}$$

$$(\sinh\rho)^{(1-d)/2}Q^{(1-d)/2-l}_{-1/2+\nu}(\cosh\rho) = -\frac{8}{3}(-1)^l\Gamma(1-l)e^{-3\rho} + \mathcal{O}(e^{-5\rho}) \ . \tag{6.3.26}$$

For a massless scalar in AdS_{d+1} (assuming d odd) there are similar expressions in which 3 is replaced by d. It follows that the holographic two-point function on $\mathbb{S}^d$ is given by

$$\langle\mathcal{O}(\vec{n})\mathcal{O}(\vec{n}')\rangle \propto \sum_{l=0}^{\infty}(2l+d-1)\frac{\Gamma(d+l)}{\Gamma(l)}C_l^{(d-1)/2}(\vec{n}'\cdot\vec{n}') \ . \tag{6.3.27}$$

Comparing with (6.1.42) we obtain

$$\langle \mathcal{O}(\vec{n})\mathcal{O}(\vec{n}') \propto \frac{1}{(1-\cos\Theta)^d} \ , \tag{6.3.28}$$

which is the correct result for a marginal operator in d dimensions.

The infinite sums defining our holographic two-point functions do not converge. They can be regulated, however, by using the following generating function for Gegenbauer polynomials

$$p(t) = \frac{1}{(1-2tx+t^2)^\nu} = \sum_{l=0}^{\infty} C_l^\nu(x)t^l \ . \tag{6.3.29}$$

Let us check this explicitly for $d = 3$,

$$\langle \mathcal{O}(\vec{n})\mathcal{O}(\vec{n}')\rangle \propto \sum_{l=1}^{\infty} l(l+1)^2(l+2)U_l(\cos\Theta) \ , \tag{6.3.30}$$

$$= p''''(1) + 10p'''(1) + 24p''(1) + 12p'(1) \ . \tag{6.3.31}$$

$$= \frac{3}{(1-x)^3} \ . \tag{6.3.32}$$

In the general situation with a boundary in the bulk, we require a linear combination

$$f_l(\rho) = (\operatorname{sech}\rho)^{1-d/2}\left[P^{(1-d)/2-l}_{-1/2+\nu}(\cosh\rho) + b_l Q^{(1-d)/2-l}_{-1/2+\nu}(\cosh\rho)\right] \ , \tag{6.3.33}$$

where the coefficient b_l is determined by imposing either a Neumann ($f_l'(\rho_*) = 0$) or Dirichlet ($f_l(\rho_*) = 0$) condition at $\rho = \rho_*$. As we move $\rho_* \to \infty$ the boundary disappears, $b_l \to 0$ and we recover the pure CFT.

6.4 Appendices

6.4.1 Green's functions and propagators

Here we collect some results about scalar propagators and Green's functions in various maximally symmetric spaces.

6.4.2 Bulk-to-boundary propagator in anti-de Sitter slicing

The wave equation in the $\mathbb{H}_d$ slicing of $\mathbb{H}_{d+1}$ is

$$\partial_\rho^2\phi + d\tanh\rho\,\partial_\rho\phi + \operatorname{sech}^2\rho\,\nabla^2_{\mathbb{H}_d}\phi = m^2\phi\ . \tag{6.4.1}$$

Separating variables by writing $\phi = f(\rho)g(x)$ we obtain

$$\cosh^2\rho\left[\frac{f''(\rho)}{f(\rho)} + d\tanh\rho\frac{f'(\rho)}{f(\rho)} - m^2\right] + \frac{\nabla^2_{\mathbb{H}_d}g(x)}{g(x)} = 0\ , \tag{6.4.2}$$

and thus we need to solve the following eigenvalue problem

$$+l(l+d-1) = \frac{\nabla^2_{\mathbb{H}_d}g(x)}{g(x)}\ , \tag{6.4.3}$$

$$0 = \frac{f''(\rho)}{f(\rho)} + d\tanh\rho\frac{f'(\rho)}{f(\rho)} - m^2 + \lambda\operatorname{sech}^2\rho\ . \tag{6.4.4}$$

Consider the flat slicing of $\mathbb{H}_d$,

$$ds^2_{\mathbb{H}_d} = \frac{d\eta^2 + d\vec{x}^2}{\eta^2}\ . \tag{6.4.5}$$

There are two branches of solutions depending on whether $\lambda \equiv l(l+d-1)$ is above or below the Breitenlohner-Freedman bound [80, 147] for AdS_d

$$\lambda_{\mathrm{BF}} = -\left(\frac{d-1}{2}\right)^2\ . \tag{6.4.6}$$

We will focus on the range $\lambda < \lambda_{\mathrm{BF}}$ since this is required to obtain a complete set. Defining the complex angular momentum

$$l = -\frac{d-1}{2} + ip\ , \quad p > 0\ , \tag{6.4.7}$$

we have

$$\lambda = \lambda_{\mathrm{BF}} - p^2\ , \tag{6.4.8}$$

and the general solutions are given by the linear combinations

$$g(\vec{x},\eta) = \eta^{(d-1)/2}\left\{K_{ip}(k\eta), \quad I_{ip}(k\eta)\right\} e^{-i\vec{k}\cdot\vec{x}} \ , \tag{6.4.9}$$

$$f(\rho) = (\operatorname{sech}\rho)^{d/2}\left\{P^{\nu}_{-1/2+ip}(\tanh\rho), \quad Q^{\nu}_{-1/2+ip}(\tanh\rho)\right\} \ , \tag{6.4.10}$$

where $\nu = \sqrt{d^2/4+m^2}$. The modified Bessel functions of imaginary order (MacDonald functions) satisfy the Sturm-Liouville differential identity

$$\frac{d}{dx}\left[x\frac{d}{dx}K_{ip}(x)\right] - xK_{ip}(x) + \frac{p^2}{x}K_{ip}(x) = 0 \ , \tag{6.4.11}$$

where the weight function $w(x) = 1/x$ is positive for $x > 0$. This ensures that they obey the orthogonality relations [148]

$$\frac{2}{\pi^2}p\sinh(\pi p)\int_0^\infty dx\frac{K_{ip}(x)K_{ip'}(x)}{x} \ , = \delta(p-p') \tag{6.4.12}$$

$$\frac{2}{\pi^2 x}\int_0^\infty dp\, p\sinh(\pi p)K_{ip}(x)K_{ip}(y) = \delta(x-y) \ . \tag{6.4.13}$$

The normalized wavefunctions on $\mathbb{H}_d$ satisfy

$$\int d^dx\sqrt{g_{\mathbb{H}_d}}\psi_{k,p}(x)\psi_{k',p'}(x)^* = \delta^{d-1}(\vec{k}-\vec{k}')\delta(p-p') \ , \tag{6.4.14}$$

and are given by

$$\psi_{k,p}(\vec{x},\eta) = \eta^{(d-1)/2}\sqrt{\frac{p\sinh(\pi p)}{2^{d-2}\pi^{d+1}}}K_{ip}(k\eta)e^{-i\vec{k}\cdot\vec{x}} \ , \tag{6.4.15}$$

which agrees with [149] for $d = 2$. Notice that these wavefunctions vanish at $\eta = \infty$ because $K_{i\rho}(k\eta) \sim \sqrt{\frac{\pi}{2k\eta}}e^{-k\eta}$, while at small η they behave as $\psi_{k,p} \sim \eta^{(d-1)/2-ip}$, which vanishes as $\eta \to 0$. The $I_{i\rho}(k\eta)$ are not permissible wavefunctions because while they vanish at $\eta = 0$ they exponentially diverge as $\eta \to \infty$.

Now consider the spherical slicing of $\mathbb{H}_d$. The equation defining g is now

$$\partial_r^2 g + (d-1)\coth r\,\partial_r g(r) + \operatorname{csch}^2 r\nabla^2_{\mathbb{S}^{d-1}}g = \lambda g \ . \tag{6.4.16}$$

Separating variables as $g = Y(\hat{n})R(r)$ we then need to solve the following eigenvalue problem

$$\frac{\nabla^2_{\mathbb{S}^{d-1}} Y}{Y} = -l(l+d-2) \ , \tag{6.4.17}$$

$$0 = R''(r) + (d-1)\coth(r)R'(r) - \left[l(l+d-2)\,\mathrm{csch}^2\, r + \lambda\right] R(r) \ . \tag{6.4.18}$$

We find that Y is a hyper-spherical harmonic on $\mathbb{S}^{d-1}$ and the general solution for $R(r)$ (assuming $\lambda < \lambda_{\mathrm{BF}}$) is

$$R(r) = (\sinh r)^{(2-d)/2} \left\{ P^{(2-d)/2-l}_{-1/2+ip}(\cosh r), \quad Q^{(2-d)/2-l}_{-1/2+ip}(\cosh r) \right\} \ . \tag{6.4.19}$$

The associated Legendre functions $P^{\nu}_{-1/2+i\rho}(x)$, with complex degree $-1/2 + ip$, are called conical functions and satisfy the completeness relations

$$\int_0^\infty dp \left| \frac{\Gamma(ip - \nu + 1/2)}{\Gamma(ip)} \right|^2 P^{\nu}_{-1/2+ip}(x) P^{\nu}_{-1/2+ip}(y) = \delta(x-y) \ , \tag{6.4.20}$$

$$\left| \frac{\Gamma(ip - \nu + 1/2)}{\Gamma(ip)} \right|^2 \int_1^\infty dx\, P^{\nu}_{-1/2+ip}(x) P^{\nu}_{-1/2+ip'}(x) = \delta(p-p') \ . \tag{6.4.21}$$

We therefore find that the normalized wavefunctions in the $\mathbb{S}^{d-1}$ slicing of $\mathbb{H}_d$ are [150]

$$\psi_{p,l\mathbf{m}}(r,\Omega) = \frac{\Gamma((d-1)/2 + ip + l)}{\Gamma(ip)} (\sinh r)^{(2-d)/2} P^{(2-d)/2-l}_{-1/2+ip}(\cosh r) \mathbb{Y}_{l\mathbf{m}}(\Omega_{d-1}) \ . \tag{6.4.22}$$

We conclude that the general solution of the massive scalar wave equation in AdS_{d+1} is

$$\phi(\rho, x) = \sum_{l,\mathbf{m}} \int dp\, f_p(\rho) \psi_{p,l\mathbf{m}}(x) \phi_{(0)p,l\mathbf{m}} \ , \tag{6.4.23}$$

$$= \sum_{l,\mathbf{m}} \int dp\, f_p(\rho) \psi_{p,l\mathbf{m}}(r,\Omega) \left[\int dV'_{\mathbb{H}_d} \psi^*_{p,l\mathbf{m}}(x') \phi_{(0)}(x') \right] \ , \tag{6.4.24}$$

$$= \int dV'_{\mathbb{H}_d} \int dp\, f_p(\rho) \left[\sum_{l,\mathbf{m}} \psi_{p,l\mathbf{m}}(x) \psi^*_{p,l\mathbf{m}}(x') \right] \phi_{(0)}(x') \ , \tag{6.4.25}$$

$$= \int dV'_{\mathbb{H}_d} K(\rho, x; x') \phi_{(0)}(x') \ , \tag{6.4.26}$$

where we have used (6.1.56) and defined the bulk-to-boundary propagator

$$K(\rho, x; x') = \frac{1}{2\pi}(2\pi \sinh \ell)^{(2-d)/2} \int dp\, f_p(\rho) \left| \frac{\Gamma((d-1)/2 + ip)}{\Gamma(ip)} \right|^2 P^{(2-d)/2}_{-1/2+ip}(\cosh \ell)\ . \tag{6.4.27}$$

6.4.3 Scalar Green's functions

Here we review the scalar Green's functions on maximally symmetric spaces.

Sphere $\mathbb{S}^{d+1}$

We consider the scalar field action

$$S = \frac{1}{2} \int_{\mathbb{S}^{d+1}} d^{d+1}x \sqrt{g} \left[(\nabla\phi)^2 + m^2\phi^2 \right]\ , \tag{6.4.28}$$

The standard round metric on the sphere is

$$ds^2 = d\theta^2 + \sin^2\theta d\Omega_d^2\ , \tag{6.4.29}$$

where $\theta \in (0, \pi)$ and the wave equation for a scalar of mass m is

$$\left[\partial_\theta^2 + d \cot\theta \partial_\theta + \csc^2\theta \nabla^2_{\mathbb{S}^d} - m^2 \right] \phi = 0\ . \tag{6.4.30}$$

The Green's function when acted upon by the above differential operator gives a unit normalized delta function source. We can use rotational invariance to move the delta function source to $\theta = 0$ so that the Green's function only depends on the θ coordinate, and thus the $\nabla^2_{\mathbb{S}^d}$ term can be set to zero. Defining $z = \frac{1}{2}(1 + \cos\theta)$ we have $dz/d\theta = -[z(1-z)]^{-1/2}$ and thus the equation of motion away from coincident points ($z \neq 1$) becomes

$$z(1-z)G''(z) + (d+1)(1/2 - z)G'(z) - m^2 G(z) = 0\ . \tag{6.4.31}$$

Comparing with the hypergeometric equation

$$z(1-z)F''(z)+[c-(a+b+1)z]F'(z)-abF(z)=0 \tag{6.4.32}$$

we obtain $c=(d+1)/2$, $ab=m^2$ and $a+b=d$. The last two relations give $a(d-a)=m^2$. The hypergeometric function equation has three singular points at $z=0,1,\infty$. The linearly independent solutions around each of these points are

$$\begin{aligned}
z=0: &\quad F(a,b;c;z)\ , && z^{1-c}F(1+a-c,1+b-c;2-c;z)\ ,\\
z=1: &\quad F(a,b;1+a+b-c;1-z)\ , && (1-z)^{c-a-b}F(c-a,c-b;1+c-a-b;1-z)\ ,\\
z=\infty: &\quad z^{-a}F(a,1+a-c;1+a-b;z^{-1})\ , && z^{-b}F(b,1+b-c;1+b-a;z^{-1})\ .
\end{aligned} \tag{6.4.33}$$

For the sphere $z\in[0,1]$, and we expect a singularity at $\theta=0$ ($z=1$) but want to avoid a singularity at $\theta=\pi$ ($z=0$). Smoothness at $\theta=\pi$ implies that we discard the second solution around $z=0$ and the first solution around $z=1$. Moreover, we can discard the second solution around $z=1$ because it is singular at $\theta=\pi$. The solution is thus the original hypergeometric function,

$$\bar{G}(\theta)=F\left(\delta,d-\delta;\frac{d+1}{2};\frac{1}{2}(1+\cos\theta)\right)\ , \tag{6.4.34}$$

where bar indicates that we have dropped an overall normalization factor. The parameter δ (not to be confused with Δ) is chosen to be the larger root of the quadratic equation $\delta(d-\delta)=m^2$; namely,

$$\delta=\frac{d}{2}+\sqrt{\left(\frac{d}{2}\right)^2-m^2}\ . \tag{6.4.35}$$

This choice is without loss of generality because of the hypergeometric identity

$$F(a,b;c;z)=F(b,a;c;z)\ . \tag{6.4.36}$$

This Green functions behaves in the expected way for a conformally coupled scalar on $\mathbb{S}^{d+1}$ (which has[13] $m^2 = (d+1)(d-1)/4$ and $\delta = (d+1)/2$),

$$\bar{G}(\theta) = F\left(\frac{d+1}{2}, \frac{d-1}{2}; \frac{d+1}{2}; \frac{1}{2}(1+\cos\theta)\right) = \left(\frac{2}{1-\cos\theta}\right)^{(d-1)/2} . \tag{6.4.37}$$

For a massless scalar $m^2 = 0$, there is subtlety due to the shift symmetry of the action and the resultant divergence over the zero mode causes the propagator to be divergent in the massless limit. If we interpret the shift symmetry as a gauge symmetry, the two-point function turns out to be the coefficient of m^2 in the Taylor expansion of the normalized Green function $G(\theta)$ (see [153] for details). In the case of a massless scalar on $\mathbb{S}^4$ we obtain

$$\bar{G}(\theta) = 1 + \frac{1}{3}\left[\frac{1}{2}\frac{z}{1-z} - \log(1-z)\right] m^2 + \mathcal{O}(m^4) , \tag{6.4.38}$$

and the divergent normalization factor ($\sim 1/m^2$) selects the second term.

It is also interesting to express the Green's function in terms of the heat Kernel on $\mathbb{S}^{d+1}$,

$$K(\hat{n}, \hat{n}'; t) = \frac{1}{V_{\mathbb{S}^{d+1}} d} \sum_{l=0}^{\infty} (2l+d) C_\ell^{d/2}(\vec{n}\cdot\vec{n}') e^{-t\ell(\ell+d)} . \tag{6.4.39}$$

The Green's function is given by the Laplace transform of the heat kernel which provides a spectral decomposition analogous to (6.1.26),

$$\begin{aligned} G(\hat{n}, \hat{n}'; m^2) &= \int_0^\infty dt\, e^{-m^2 t} K(\vec{n}, \vec{n}'; t) && (6.4.40)\\ &= \frac{1}{V_{\mathbb{S}^{d+1}} d} \sum_{l=0}^{\infty} \frac{2l+d}{l(l+d)+m^2} C_\ell^{d/2}(\vec{n}\cdot\vec{n}') && (6.4.41)\\ &= \frac{1}{V_{\mathbb{S}^{d+1}} d} \frac{\pi}{\sin(\pi\nu)} C_\nu^{d/2}(-\vec{n}\cdot\vec{n}') , && (6.4.42) \end{aligned}$$

where ν satisfies $m^2 = -\nu(\nu+d)$. We can see that this agrees with the previous calculation by making use of the representation of the Gegenbauer function in terms of a hypergeometric

[13] Recall that a conformally coupled scalar has $m^2 = \frac{d-1}{4d} R$ and $R = (d+1)d$ is the scalar curvature for the unit $d+1$-sphere.

function

$$C_\nu^{d/z}(z) = C_\nu^{d/z}(1) F\left(-\nu, \nu + d; \frac{d+1}{2}; \frac{1-z}{2}\right) , \qquad C_\nu^{d/2}(1) = \frac{\Gamma(\nu + d)}{\Gamma(\nu+1)\Gamma(d)} . \quad (6.4.43)$$

$\mathbb{H}_{d+1}$

The wave equation on $\mathbb{H}_{d+1}$ can be obtained from that on $\mathbb{S}^{d+1}$ by analytically continuing the polar coordinate $\theta = ir$ and simultaneously flipping the sign of the curvature, which flips $m^2 \to -m^2$, yielding

$$\left[\partial_r^2 + d \coth r \, \partial_r + \operatorname{csch}^2 r \nabla^2_{\mathbb{S}^{d-1}} - m^2\right] \phi = 0 . \quad (6.4.44)$$

Here $r \in (0, \infty)$. Defining $z = \frac{1}{2}(1 + \cosh r)$, $z \in (0, \infty)$, we have $dz/dr = [z(z-1)]^{-1/2}$ and the equation of motion away from coincident points becomes

$$z(1-z)G''(z) + (d+1)(1/2 - z)G'(z) + m^2 G(z) = 0. \quad (6.4.45)$$

The unnormalized Green's function obtained by analytical continuation from the sphere is given by

$$\bar{G}_{\rm E}(\ell) = F\left(\Delta, d - \Delta; \frac{d+1}{2}; \frac{1}{2}(1 + \cosh \ell)\right) , \qquad \Delta = \frac{d}{2} + \sqrt{\left(\frac{d}{2}\right)^2 + m^2} , \quad (6.4.46)$$

where ℓ is the usual geodesic distance and Δ is the larger of the two roots $\Delta_\pm$ of the quadratic equation $\Delta(d - \Delta) = -m^2$. For large arguments the hypergeometric function has the following asymptotics (assuming $a - b$ is non-integer)

$$F(a, b; c; z) = z^{-a}\left[\lambda_1 + \mathcal{O}(z^{-1})\right] + z^{-b}\left[\lambda_2 + \mathcal{O}(z^{-1})\right] , \quad (6.4.47)$$

and thus the above Green's function behaves asymptotically for large ℓ as

$$G_{\mathrm{E}}(\ell) \sim Ae^{-\Delta_{+}\ell} + Be^{-\Delta_{-}\ell} \ . \tag{6.4.48}$$

The second term means that this Green's function is finite only if $m^2 \leq 0$. If we consider the first solution of the hypergeometric equation around $z = \infty$ we find

$$\bar{G}(\ell) = u^{\Delta} F\left(\Delta, \Delta + \frac{1-d}{2}; 2\Delta - d + 1; u\right) \ , \qquad u = \frac{2}{1+\cosh \ell} \ , \tag{6.4.49}$$

which behaves for large ℓ as $G(\ell) \sim e^{-\Delta \ell}$. The second solution around $z = \infty$ gives the same expression (6.4.49) with $\Delta = \Delta_{-}$. We can use the hypergeometric identity ([159] sec. 2.10, p. 109)

$$F(a, b; c, u) = (1-u)^{-a} F\left(a, c-b; c; \frac{u}{u-1}\right) \ , \tag{6.4.50}$$

to express (6.4.49) in the equivalent form

$$\bar{G}(\ell) = (2v^{-1})^{\Delta} F\left(\Delta, \Delta + \frac{1-d}{2}; 2\Delta - d + 1; -2v^{-1}\right) \ , \qquad v = \frac{1}{2}(n_{\mu} - n'_{\mu})(n_{\nu} - n'_{\nu})\eta^{\mu\nu} = \cosh \ell - 1 \ . \tag{6.4.51}$$

We can also use the hypergeometric identity from sec. 2.1.5 of [159] (p. 66)

$$F(a, b; 2b, u) = \left(1 - \frac{u}{2}\right)^{-a} F\left(\frac{a}{2}, \frac{a+1}{2}; b + \frac{1}{2}; \frac{u^2}{(2-u)^2}\right) \ , \tag{6.4.52}$$

which gives

$$\bar{G}(\ell) = (2\xi)^{\Delta} F\left(\frac{\Delta}{2}, \frac{\Delta+1}{2}; \Delta + 1 - \frac{d}{2}; \xi^2\right) \ , \qquad \xi = \operatorname{sech} \ell \ . \tag{6.4.53}$$

Remembering to remove the factor of 2^{Δ}, we fix notation be defining the 'Feynman' Green's function to be

$$\bar{G}_{\Delta}(\ell) = \xi^{\Delta} F\left(\frac{\Delta}{2}, \frac{\Delta+1}{2}; \Delta + 1 - \frac{d}{2}; \xi^2\right) . \tag{6.4.54}$$

If m^2 lies in the range

$$-\left(\frac{d}{2}\right)^2 < m^2 < -\left(\frac{d}{2}\right)^2 + 1 \ , \tag{6.4.55}$$

then the most general Green's function compatible with the AdS isometries is a linear combination

$$\bar{G}(\ell) = \alpha \bar{G}_{\Delta_+}(\ell) + \beta \bar{G}_{\Delta_-}(\ell) \ . \tag{6.4.56}$$

An important example is provided by a conformally coupled scalar on AdS_{d+1}, which has a mass given by

$$m^2 = -\left(\frac{d}{2}\right)^2 + \frac{1}{4} \ , \tag{6.4.57}$$

so both $\Delta_\pm$ branches are allowed. In fact, conformal covariance of $\bar{G}(\ell)$ actually requires that both Green's functions appear in the linear combination $\alpha = (d-1)/2$, $\beta = 1$. Let us see this explicitly for a conformally coupled scalar on AdS_4 which has $\Delta_+ = 2$ and $\Delta_- = 1$,

$$\bar{G}(\ell) = \alpha \frac{1}{\cosh^2 \ell - 1} + \beta \frac{\cosh \ell}{\cosh^2 \ell - 1} \ . \tag{6.4.58}$$

We notice that for $\alpha = \beta = 1$, $\bar{G}(\ell)$ is proportional to the Weyl transform of the CFT two-point function from flat space

$$\frac{1}{\cosh^2 \ell - 1} + \frac{\cosh \ell}{\cosh^2 \ell - 1} = \frac{1}{\cosh \ell - 1} \ . \tag{6.4.59}$$

This boundary condition can be interpreted [155] as allowing the scalar energy to pass through the AdS_4 boundary into a second copy of AdS_4. Another interesting interpretation of this boundary condition is that it is precisely the one for which $\bar{G}(\ell)$ is proportional to the analytic continuation from the sphere $\bar{G}_{\mathrm{E}}(\ell)$. Using hypergeometric identifies it can be shown that [156]

$$\bar{G}_\Delta(\ell) = A(d,\Delta) \left[\bar{G}_{\mathrm{E}}(\ell) + \tilde{G}_{\mathrm{E}}(\ell)\right] - B(d,\Delta) \left[\bar{G}_{\mathrm{E}}(\ell) - \tilde{G}_{\mathrm{E}}(\ell)\right] \ , \tag{6.4.60}$$

where $\tilde{G}_{\mathrm{E}}(\ell)$ is related to $\bar{G}_{\mathrm{E}}(\ell)$ by taking $\cosh\ell \to -\cosh\ell$ and

$$A(d,\Delta) = \frac{(-1)^{\Delta/2}\Gamma(\Delta - d/2 + 1)\Gamma\left(\frac{d+1-\Delta}{2}\right)}{2\Gamma\left(\Delta/2 - d/2 + 1\right)\Gamma\left(\frac{d+1}{2}\right)} \ , \tag{6.4.61}$$

$$B(d,\Delta) = \frac{(-1)^{(\Delta+1)/2}\Gamma(\Delta - d/2 + 1)\Gamma\left(\frac{d-\Delta}{2}\right)}{2\Gamma\left(\Delta/2 - d/2 + 1/2\right)\Gamma\left(\frac{d+1}{2}\right)} \ . \tag{6.4.62}$$

Demanding that the coefficient of $\tilde{G}_{\mathrm{E}}(\ell)$ vanishes leads to the boundary condition

$$\alpha = -\frac{A(d,\Delta_-) + B(d,\Delta_-)}{A(d,\Delta_+) + B(d,\Delta_+)}\beta \ . \tag{6.4.63}$$

Setting $\Delta_\pm = (d \pm 1)/2$ for a conformally coupled scalar we obtain $\alpha/\beta = (d-1)/2$.

Finally, let us note that there is a subtlety with using the Δ_- branch for a conformally coupled scalar in (6.4.49) or (6.4.51). This is because $\Delta_- = (d-1)/2$ so the hypergeometric function becomes $F(\Delta, 0, 0, u) = 1$. Instead one should first represent the hypergeometric function using (6.4.52) and then take the limit $b \to 0$.

Let us derive these results from the sum over Brownian motions on the hyperboloid. The heat kernel on $\mathbb{H}_{d+1}$ is given by [150]

$$K(x,y;t) = \frac{1}{2\pi}(2\pi\sinh\ell)^{(1-d)/2}\int_0^\infty dp\left|\frac{\Gamma(d/2+ip)}{\Gamma(ip)}\right|^2 P^{(1-d)/2}_{-1/2+ip}(\cosh\ell)e^{-t\left[p^2+d^2/4\right]} \ . \tag{6.4.64}$$

For $d=1$ we have

$$K(\ell;t) = \frac{1}{2\pi}\int_0^\infty dp\, p\tanh\pi p P_{-1/2+ip}(\cosh\ell)e^{-t\left[p^2+1/4\right]} \ , \tag{6.4.65}$$

and the associated Green's function is a ring function

$$G(\ell;m^2) = \int_0^\infty dt\, e^{-m^2 t} K(\ell;t) \tag{6.4.66}$$
$$= \frac{1}{2\pi}\int_0^\infty dp\, \frac{p\tanh\pi}{p^2+\nu^2} P_{-1/2+ip}(\cosh\ell) \tag{6.4.67}$$
$$= \frac{1}{2\pi} Q_{-1/2+\nu}(\cosh\ell)\ , \tag{6.4.68}$$

where now $\nu = \sqrt{1/4+m^2}$. The generalization to arbitrary d (ignoring normalization) is given by

$$\bar{G}(\ell;m^2) = (\sinh\ell)^{(1-d)/2} Q^{(1-d)/2}_{-1/2+\nu}(\cosh\ell)\ , \tag{6.4.69}$$

where $\nu = \sqrt{(d/2)^2+m^2}$. Let us now express this in terms of the hypergeometric function using sec. 3.2 of [159] (p. 122), namely,

$$Q^\mu_\nu(z) = e^{\mu i\pi} 2^{-\nu-1}\pi^{1/2}\frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu+3/2)} z^{-\nu-\mu-1}(z^2-1)^{\mu/2} F\left(\frac{\nu+\mu+2}{2}, \frac{\nu+\mu+1}{2}; \nu+\frac{3}{2}; z^{-2}\right)\ , \tag{6.4.70}$$
$$= e^{\mu i\pi} 2^{-\nu-1}\pi^{1/2}\frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu+3/2)} z^{-\nu-\mu-1}(z^2-1)^{\mu/2}(1-z^{-2})^{-\mu} F\left(\frac{-\nu+\mu+1}{2}, \frac{-\nu+\mu+2}{2}; \nu+\frac{3}{2}; z^{-2}\right)\ , \tag{6.4.71}$$

where in the second line we have used the Euler transformation

$$F(a,b;c;z) = (1-z)^{c-a-b}F(c-a,c-b;c;z)\ , \qquad |z|<1\ , \tag{6.4.72}$$

which is applicable because $\operatorname{sech} l < 1$. Replacing $\mu \to (1-d)/2$ and $\nu \to -1/2+\nu$ we obtain (6.4.53).

de Sitter

The Bunch-Davies de Sitter two-point function can be obtained by analytic continuation from the sphere, $\theta = it + \pi/2$. Under this continuation the geodesic distance Θ defined by

$\cos\Theta = \vec{n}\cdot\vec{n}' = \cos\theta_1\cos\theta_1 + \cos\alpha\sin\theta_1\sin\theta_2$ becomes

$$\cosh\ell = -\sinh t_1\sinh t_1 + \cos\alpha\cos t_1\cosh t_2\ , \tag{6.4.73}$$

where we recall that α is the angular separation in the sphere $\mathrm{dS}_{d+1} \cong \mathbb{R}_t \times \mathbb{S}^d$. The de Sitter Green's function can now be expressed in arbitrary coordinates by realizing that $\cosh\ell = \eta^{\mu\nu}n_\mu n'_\nu$. where $\eta^{\mu\nu}$ is the (mostly plus) metric for $\mathbb{R}^{1,d+1}$ and n, n' label two points on the single-sheeted hyperboloid defined by $n_\mu n^\mu = 1$. The geodesic distance l can be either real (for timelike separated points) or imaginary (for spacelike separations). In the flat slicing of de Sitter space we have

$$\cosh\ell = \frac{\eta^2 + \eta'^2 - (\vec{x}-\vec{x}')^2}{2\eta\eta'}\ . \tag{6.4.74}$$

The Bunch-Davies propagator has the following asymptotics for large ℓ

$$G_{\mathrm{BD}}(\ell) \sim Ae^{-\delta\ell} + Be^{-(d-\delta)\ell}\ . \tag{6.4.75}$$

The observation that the Bunch-Davies propagator contains two asymptotic components has been used to argue that it cannot be defined as a sum over trajectories in de Sitter space. The alternative proposal is to take [154]

$$G_{\mathrm{dS}}(n,n';m^2) = G_{\mathrm{AdS}}(n,n';-m^2) \sim Ae^{-\delta\ell}\ . \tag{6.4.76}$$

6.4.4 Differential regularization

We have seen in Section (6.1.1) that the hard-cut off regulator introduces both power law and logarithmic divergences in the CFT two-point function on flat space. There should exist a regularization scheme in which only the logarithmic divergences appear, since the only unambiguous information present in the divergences is contained in the coefficients of these logarithms. The differential regularization of [131] is such a scheme. In this appendix we review the calculation of these logarithmic divergences which first appeared in [132]. Consider the two-point function of an operator of dimension Δ. We begin by expressing $1/|x|^{2\Delta}$ in terms of an arbitrary number of Laplacians $\Box^{k+1}$,

$$\frac{1}{|x|^{2\Delta}} = \frac{1}{2^{2k+2}} \frac{\Gamma(\Delta-k-1)}{\Gamma(\Delta)} \frac{\Gamma(\Delta-k-d/2)}{\Gamma(\Delta-d/2+1)} \Box^{k+1} \frac{1}{|x|^{2(\Delta-k-1)}}. \tag{6.4.77}$$

We notice that the coefficient diverges whenever $\Delta = d/2+k$ where $k=0,1,2,\ldots$. Letting $\Delta = d/2+k+\epsilon$ we obtain

$$\frac{1}{|x|^{2\Delta}} = \frac{1}{\epsilon(d+2\epsilon-2)} \frac{1}{2^{2k+1}} \frac{\Gamma(d/2+\epsilon)}{\Gamma(d/2+k+\epsilon)} \frac{\Gamma(1+\epsilon)}{\Gamma(k+1+\epsilon)} \Box^k \left(\Box \frac{1}{|x|^{d-2+2\epsilon}} \right) . \tag{6.4.78}$$

As $\epsilon \to 0$ two things happen; the object in parentheses approaches a delta function and the coefficient diverges. Expanding in ϵ and keeping only the leading divergent term we find

$$\frac{1}{|x|^{2\Delta}} \sim -V_{\mathbb{S}^{d-1}} \frac{1}{\epsilon} \frac{1}{2^{2k+1}} \frac{\Gamma(d/2)}{\Gamma(d/2+k)} \frac{1}{k!} \Box^k \delta^{(d)}(x) . \tag{6.4.79}$$

We therefore define the renormalized two-point function as

$$\langle \mathcal{O}_\Delta(x) \mathcal{O}_\Delta(0) \rangle \equiv \lim_{\epsilon\to 0} \left[\frac{1}{|x|^{2\Delta}} + V_{\mathbb{S}^{d-1}} \frac{\mu^{2\epsilon}}{\epsilon} \frac{1}{2^{2k+1}} \frac{\Gamma(d/2)}{\Gamma(d/2+k)} \frac{1}{k!} \Box^k \delta^{(d)}(x) \right] , \tag{6.4.80}$$

where we have introduced the mass scale μ to keep the equation dimensionally correct. As an example, consider $d=4$ and $\Delta=2$ ($k=0$),

$$\langle \mathcal{O}_2(x) \mathcal{O}_2(0) \rangle = \lim_{\epsilon\to 0} \left[\frac{1}{x^4} + \frac{\pi^2 \mu^{2\epsilon}}{\epsilon} \delta^4(x) \right] \tag{6.4.81}$$

$$= \lim_{\epsilon\to 0} \left[\frac{1}{x^4} + \left(\frac{\pi^2}{\epsilon} + 2\pi^2 \log\mu \right) \delta^4(x) \right] \tag{6.4.82}$$

so

$$\mu \frac{\partial}{\partial\mu} \langle \mathcal{O}_2(x) \mathcal{O}_2(0) \rangle = 2\pi^2 \delta^4(x), \tag{6.4.83}$$

in agreement with (6.1.13) obtained using the cutoff method.

We can re-express the delta function as a derivative to obtain an alternative expression for the two-point correlator

$$\langle \mathcal{O}_\Delta(x)\mathcal{O}_\Delta(0)\rangle = -\frac{1}{d-2}\frac{1}{2^{2k+1}}\frac{\Gamma(d/2)}{\Gamma(d/2+k)}\frac{1}{k!}\Box^{k+1}\frac{1}{|x|^{d-2}}\log(x^2\mu^2)\ . \tag{6.4.84}$$

To compute the Fourier transform we use[14] (see Eq. (A.2) of [131]),

$$\int d^d x\, e^{ipx}\frac{1}{|x|^{d-2}}\log(\mu^2 x^2) = -\frac{4\pi^{d/2}}{\Gamma(d/2-1)}\frac{1}{p^2}\log(p^2/\bar{\mu}^2), \qquad \bar{\mu} \equiv 2\mu/\gamma. \tag{6.4.85}$$

It follows that

$$\langle \mathcal{O}_\Delta(k)\mathcal{O}_\Delta(-k)\rangle \propto p^{2k}\log(p^2/\bar{\mu}^2). \tag{6.4.86}$$

[14] This formula requires differential regularization to make sense.

Chapter 7

Conclusion

There exists [40, 41, 55] a wide range of novel scalar field theories with interesting properties such as Vainshtein screening and non-renormalization theorems in common with the original Galileon models of [4, 160]. These properties hold out the hope that such models may be of use both in particle physics, and as a possible way to modify gravity in the infrared. However, coupling such fields to General Relativity in a way that preserves their symmetries and second order equations of motion seems to be impossible [5]. Instead, Galileon-like scalar fields seem to most naturally couple to dRGT massive gravity [7].

The consistency of such a proposal rests on the preservation of the hard-won ghost-free structure of the dRGT theory. We have shown that a theory of nonlinear massive gravity coupled to DBI scalars in such a way as to preserve the generalized Galileon shift symmetry and the property of having second order equations of motion is ghost free. Our proof is based on the vierbein formulation of massive gravity, in which the Hamiltonian analysis simplifies. Our analysis shows that the dRGT-DBI system provides a consistent framework in which models of interest to cosmology [53] may be developed.

Motivated by the theoretical consistency of the massive gravity Galileon theory, we have examined the nature of cosmological perturbations around the self-accelerating branch. One of the more striking results of dRGT massive gravity is that the kinetic terms for both vector and scalar perturbations vanish around the phenomenologically interesting self-accelerating branch of the theory. The main result of our analysis is that the vanishing of these kinetic terms is preserved around the analogous de Sitter branch in the more general class of theories, suggesting that this is a generic result tied to the existence of self-accelerating solutions in theories with this general structure. Furthermore, we have verified that the tensor perturbations are ghost-free, and that while the details of the analysis of their mass terms differs from that in pure massive gravity, any tachyonic modes are similarly unstable on Hubble timescales.

An obvious extension of this work is to study the behavior of the same perturbations around the other branch of cosmological solutions identified in [53]. It will also be interesting to ask whether the fluctuations of the Galileon around these accelerating solutions can be kept subluminal, in contrast to the situation around flat space [56].

We have also explored the cosmology of variable-mass massive gravity, which results from promoting the fixed mass term of the dRGT model to the vacuum expectation value of a dynamical scalar field, as suggested in [20]. In dRGT theory, there is a constraint equation that forbids non-trivial flat FRW solutions, though self-accelerating open solutions exist. In the variable mass theory, the form of the constraint is different, and it no longer forbids flat FRW solutions. The constraint, however, makes it difficult to realize the idea of self-inflation, i.e. using the self-acceleration properties of massive gravity in the early universe to drive inflation. Furthermore, we have demonstrated for the first time that a large class of cosmologies within these models exhibit a future curvature singularity of the "big brake" form.

In remainder of the thesis we explored alternatives to inflation which do not reply upon modifications of gravity, nor on the effects of gravitational dynamics itself. Instead, the proposal relies upon conformal field theory dynamics in the early universe. Intriguingly, however, we were able to show that the conformal universe scenario can be realized in the context of Einstein gravity coupled to a scalar field using the techniques of holography.

The holography of flat domain walls, and more recently their AdS-sliced counterparts, has been the subject of much study over the last decade. Here we have studied de Sitter sliced domain walls, arguing that the natural dual theory can be identified with the pseudo-conformal universe. We have calculated scalar and tensor one-point functions in a specific geometry analogous to the Janus solution of [95], which realizes the spontaneous symmetry breaking pattern $\mathrm{so}(2,4) \to \mathrm{so}(1,4)$.

The domain-wall background in which we calculated correlation functions is singular in the interior of the spacetime. It would be desirable to find a background that is regular throughout (see however [125]). Another pressing problem is to check stability of dS-sliced domain walls. There exist positive-energy arguments which guarantee the classical stability of flat and AdS-sliced domain walls [98], however these arguments rely on the existence of 'fake supersymmetry' and do not directly apply to dS-sliced domain walls [92].

An issue with the conformal universe scenario is that it requires weight-0 operators in order to generate a scale invariant spectrum. Naively there is a problem because the unitarity bound $\Delta \geq 1$ in four dimensions [128] prevents the existence of such operators. One of the assumptions underlying this bound is that there is a conformal vacuum. This corresponds to the existence of an AdS vacuum solution in the bulk theory. Bulk theories which are AdS invariant and yet have no AdS vacuum could be dual to conformal theories with no conformal vacuum and have a chance to allow dimension 0 operators. As far as we are aware there are no known examples of such theories, but it would be interesting to explore if there exist theories which allow solutions with AdS asymptotics and yet do not allow an exact AdS solution.

It would be interesting to study higher-point correlation functions. These satisfy Ward identities for the spontaneously broken symmetries, known in cosmology as consistency relations [100]. The consistency relations for pseudo-conformal correlators have been studied in [101], and it should be possible to understand these relations holographically.

In addition our AdS construction does not describe the eventual reheating phase of the early universe. It would be interesting to extend it along the lines described in [72].

Finally, it would be interesting to consider the holographic dual to turning on 4D gravity in the boundary theory. In the pseudo-conformal scenario, gravity becomes important at late times. For example, in the main example of [35] where a weight 1 field drives the scenario, the approximation of ignoring gravity is good until $\langle \mathcal{O} \rangle \sim M_{\mathrm{P}}$, at which point the effect of higher-dimension, Planck-suppressed operators becomes important. Dynamical gravity can be incorporated in the boundary by moving the boundary slightly into the bulk, to a cut-off surface at $z = \epsilon$. Local counter-terms which previously diverged in the $\epsilon \to 0$ limit are now regarded as finite kinetic terms for the graviton [102–104]. We leave these problems to future work.

Our successful implementation of the pseudo-conformal universe in AdS/CFT has motivated the development of a formalism that relates the spectral decomposition of correlation functions to the renormalized correlation functions in position space. In highly symmetric situations the spacetime representation of the two-point function can be deduced from spacetime symmetries. This is true of both the pure CFT on flat and curved backgrounds as well as CFTs with spacelike or timelike boundaries. We have checked in all these cases that both representations are related by integral transforms.

In other situations such as the pseudo-conformal CFT where the exact spacetime form of two-point function is not known, our formalism allows it to be computed implicitly from a knowledge of the spectral representation obtained via holography. It would be interesting to try to compare the results of that calculation with the low-momentum expansion of the two-point functions computed from effective field theory considerations in [35].

Bibliography

[1] D. Baumann, doi:10.1142/9789814327183_0010 arXiv:0907.5424 [hep-th].

[2] C. de Rham and G. Gabadadze, Phys. Rev. D **82**, 044020 (2010) [arXiv:1007.0443 [hep-th]].

[3] C. de Rham, G. Gabadadze and A. J. Tolley, Phys. Rev. Lett. **106**, 231101 (2011) [arXiv:1011.1232 [hep-th]].

[4] A. Nicolis, R. Rattazzi and E. Trincherini, Phys. Rev. D **79**, 064036 (2009) [arXiv:0811.2197 [hep-th]].

[5] C. Deffayet, G. Esposito-Farese and A. Vikman, Phys. Rev. D **79**, 084003 (2009) [arXiv:0901.1314 [hep-th]].

[6] C. Deffayet, S. Deser and G. Esposito-Farese, Phys. Rev. D **80**, 064015 (2009) [arXiv:0906.1967 [gr-qc]].

[7] G. Gabadadze, K. Hinterbichler, J. Khoury, D. Pirtskhalava and M. Trodden, Phys. Rev. D **86**, 124004 (2012) [arXiv:1208.5773 [hep-th]].

[8] M. Fierz and W. Pauli, Proc. Roy. Soc. Lond. A **173**, 211 (1939).

[9] D. G. Boulware and S. Deser, Phys. Rev. D **6**, 3368 (1972).

[10] S. F. Hassan and R. A. Rosen, Phys. Rev. Lett. **108**, 041101 (2012) [arXiv:1106.3344 [hep-th]].

[11] S. F. Hassan and R. A. Rosen, JHEP **1204**, 123 (2012) [arXiv:1111.2070 [hep-th]].

[12] C. de Rham, G. Gabadadze and A. J. Tolley, JHEP **1111**, 093 (2011) [arXiv:1108.4521 [hep-th]].

[13] C. de Rham, G. Gabadadze and A. J. Tolley, Phys. Lett. B **711**, 190 (2012) [arXiv:1107.3820 [hep-th]].

[14] M. Mirbabayi, Phys. Rev. D **86**, 084006 (2012) [arXiv:1112.1435 [hep-th]].

[15] K. Hinterbichler and R. A. Rosen, JHEP **1207**, 047 (2012) [arXiv:1203.5783 [hep-th]].

[16] C. de Rham, G. Gabadadze, L. Heisenberg and D. Pirtskhalava, Phys. Rev. D **83**, 103516 (2011) [arXiv:1010.1780 [hep-th]].

[17] K. Koyama, G. Niz and G. Tasinato, Phys. Rev. Lett. **107**, 131101 (2011) [arXiv:1103.4708 [hep-th]].

[18] T. M. Nieuwenhuizen, Phys. Rev. D **84**, 024038 (2011) [arXiv:1103.5912 [gr-qc]].

[19] A. H. Chamseddine and M. S. Volkov, Phys. Lett. B **704**, 652 (2011) [arXiv:1107.5504 [hep-th]].

[20] G. D'Amico, C. de Rham, S. Dubovsky, G. Gabadadze, D. Pirtskhalava, A. J. Tolley, *Massive Cosmologies*, [arXiv:1108.5231].

[21] A. E. Gumrukcuoglu, C. Lin and S. Mukohyama, JCAP **1111**, 030 (2011) [arXiv:1109.3845 [hep-th]].

[22] N. Arkani-Hamed, H. Georgi and M. D. Schwartz, Annals Phys. **305**, 96 (2003) [hep-th/0210184].

[23] C. de Rham, G. Gabadadze, L. Heisenberg and D. Pirtskhalava, arXiv:1212.4128 [hep-th].

[24] A. E. Gumrukcuoglu, C. Lin and S. Mukohyama, JCAP **1203**, 006 (2012) [arXiv:1111.4107 [hep-th]].

[25] A. E. Gumrukcuoglu, C. Lin and S. Mukohyama, Phys. Lett. B **717**, 295 (2012) [arXiv:1206.2723 [hep-th]].

[26] A. De Felice, A. E. Gumrukcuoglu, C. Lin and S. Mukohyama, arXiv:1303.4154 [hep-th].

[27] G. D'Amico, G. Gabadadze, L. Hui and D. Pirtskhalava, arXiv:1206.4253 [hep-th].

[28] Z. Haghani, H. R. Sepangi and S. Shahidi, arXiv:1303.2843 [gr-qc].

[29] G. D'Amico, G. Gabadadze, L. Hui and D. Pirtskhalava, arXiv:1304.0723 [hep-th].

[30] A. E. Gumrukcuoglu, K. Hinterbichler, C. Lin, S. Mukohyama and M. Trodden, arXiv:1304.0449 [hep-th].

[31] Q. G. Huang, Y. S. Piao and S. Y. Zhou, Phys. Rev. D **86**, 124014 (2012) [arXiv:1206.5678 [hep-th]].

[32] A. S. Goldhaber and M. M. Nieto, Rev. Mod. Phys. **82**, 939 (2010) [arXiv:0809.1003 [hep-ph]].

[33] V. A. Rubakov, JCAP **0909**, 030 (2009) [arXiv:0906.3693 [hep-th]].

[34] P. Creminelli, A. Nicolis and E. Trincherini, JCAP **1011**, 021 (2010) [arXiv:1007.0027 [hep-th]].

[35] K. Hinterbichler and J. Khoury, JCAP **1204**, 023 (2012) [arXiv:1106.1428 [hep-th]].

[36] K. Hinterbichler, A. Joyce and J. Khoury, JCAP **1206**, 043 (2012) [arXiv:1202.6056 [hep-th]].

[37] M. Andrews, G. Goon, K. Hinterbichler, J. Stokes and M. Trodden, arXiv:1303.1177 [hep-th].

[38] C. de Rham and A. J. Tolley, JCAP **1005**, 015 (2010) [arXiv:1003.5917 [hep-th]].

[39] K. Hinterbichler, M. Trodden and D. Wesley, Phys. Rev. D **82**, 124018 (2010) [arXiv:1008.1305 [hep-th]].

[40] G. Goon, K. Hinterbichler and M. Trodden, JCAP **1107**, 017 (2011) [arXiv:1103.5745 [hep-th]].

[41] G. Goon, K. Hinterbichler and M. Trodden, Phys. Rev. Lett. **106**, 231102 (2011) [arXiv:1103.6029 [hep-th]].

[42] C. Burrage, C. de Rham and L. Heisenberg, JCAP **1105**, 025 (2011) [arXiv:1104.0155 [hep-th]].

[43] G. Goon, K. Hinterbichler and M. Trodden, to appear

[44] S. F. Hassan, R. A. Rosen and A. Schmidt-May, JHEP **1202**, 026 (2012) [arXiv:1109.3230 [hep-th]].

[45] S. F. Hassan and R. A. Rosen, JHEP **1202**, 126 (2012) [arXiv:1109.3515 [hep-th]].

[46] C. Deffayet, J. Mourad and G. Zahariade, JCAP **1301**, 032 (2013) [arXiv:1207.6338 [hep-th]].

[47] S. Deser and A. Waldron, arXiv:1212.5835 [hep-th].

[48] C. Deffayet, J. Mourad and G. Zahariade, arXiv:1208.4493 [gr-qc].

[49] S. Deser and C. J. Isham, Phys. Rev. D **14**, 2505 (1976).

[50] J. Kluson, arXiv:1305.6751 [hep-th].

[51] S. -Y. Zhou, Phys. Rev. D **83**, 064005 (2011) [arXiv:1011.0863 [hep-th]].

[52] V. Sivanesan, Phys. Rev. D **85**, 084018 (2012) [arXiv:1111.3558 [hep-th]].

[53] K. Hinterbichler, J. Stokes and M. Trodden, arXiv:1301.4993 [astro-ph.CO].

[54] M. Andrews, K. Hinterbichler, J. Stokes and M. Trodden, Class. Quant. Grav. **30**, 184006 (2013) [arXiv:1306.5743 [hep-th]].

[55] G. Goon, K. Hinterbichler and M. Trodden, JCAP **1112**, 004 (2011) [arXiv:1109.3450 [hep-th]].

[56] G. L. Goon, K. Hinterbichler and M. Trodden, Phys. Rev. D **83**, 085015 (2011) [arXiv:1008.4580 [hep-th]].

[57] E. N. Saridakis, arXiv:1207.1800 [gr-qc].

[58] Y. -F. Cai, C. Gao and E. N. Saridakis, arXiv:1207.3786 [astro-ph.CO].

[59] D. -J. Wu, Y. -F. Cai and Y. -S. Piao, arXiv:1301.4326 [hep-th].

[60] J. D. Barrow, Class. Quant. Grav. **21**, L79 (2004) [gr-qc/0403084].

[61] M. P. Dabrowski and A. Balcerzak, gr-qc/0701056.

[62] R. Gregory, N. Kaloper, R. C. Myers and A. Padilla, JHEP **0710**, 069 (2007) [arXiv:0707.2666 [hep-th]].

[63] K. Hinterbichler, J. Stokes and M. Trodden, JHEP **1501**, 090 (2015) [arXiv:1408.1955 [hep-th]].

[64] J. M. Maldacena, Adv. Theor. Math. Phys. **2**, 231 (1998) [hep-th/9711200].

[65] A. M. Polyakov, Nucl. Phys. Proc. Suppl. **68**, 1 (1998) [hep-th/9711002].

[66] A. Strominger, JHEP **0110**, 034 (2001) [hep-th/0106113].

[67] J. M. Maldacena, JHEP **0305**, 013 (2003) [astro-ph/0210603].

[68] M. Cvetic and H. H. Soleng, Phys. Rev. D **51**, 5768 (1995) [hep-th/9411170].

[69] P. McFadden and K. Skenderis, Phys. Rev. D **81**, 021301 (2010) [arXiv:0907.5542 [hep-th]].

[70] P. McFadden and K. Skenderis, J. Phys. Conf. Ser. **222**, 012007 (2010) [arXiv:1001.2007 [hep-th]].

[71] A. Bzowski, P. McFadden and K. Skenderis, JHEP **1304**, 047 (2013) [arXiv:1211.4550 [hep-th]].

[72] Y. Wang and R. Brandenberger, JCAP **1210**, 021 (2012) [arXiv:1206.4309 [hep-th]].

[73] P. A. R. Ade *et al.* [BICEP2 Collaboration], Phys. Rev. Lett. **112**, 241101 (2014) [arXiv:1403.3985 [astro-ph.CO]].

[74] I. R. Klebanov and E. Witten, Nucl. Phys. B **556**, 89 (1999) [hep-th/9905104].

[75] R. Bousso, B. Freivogel, S. Leichenauer, V. Rosenhaus and C. Zukowski, Phys. Rev. D **88**, 064057 (2013) [arXiv:1209.4641 [hep-th]].

[76] R. Bousso, S. Leichenauer and V. Rosenhaus, Phys. Rev. D **86**, 046009 (2012) [arXiv:1203.6619 [hep-th]].

[77] B. Czech, J. L. Karczmarek, F. Nogueira and M. Van Raamsdonk, Class. Quant. Grav. **29**, 155009 (2012) [arXiv:1204.1330 [hep-th]].

[78] V. E. Hubeny and M. Rangamani, JHEP **1206**, 114 (2012) [arXiv:1204.1698 [hep-th]].

[79] S. Leichenauer and V. Rosenhaus, Phys. Rev. D **88**, no. 2, 026003 (2013) [arXiv:1304.6821 [hep-th]].

[80] P. Breitenlohner and D. Z. Freedman, Phys. Lett. B **115**, 197 (1982).

[81] G. W. Gibbons and S. W. Hawking, Phys. Rev. D **15**, 2752 (1977).

[82] J. W. York, Jr., Phys. Rev. Lett. **28**, 1082 (1972).

[83] T. Hertog and G. T. Horowitz, JHEP **0504**, 005 (2005) [hep-th/0503071].

[84] T. Hertog and G. T. Horowitz, JHEP **0407**, 073 (2004) [hep-th/0406134].

[85] B. Craps, T. Hertog and N. Turok, Phys. Rev. D **86**, 043513 (2012) [arXiv:0712.4180 [hep-th]].

[86] S. R. Das, J. Michelson, K. Narayan and S. P. Trivedi, Phys. Rev. D **74**, 026002 (2006) [hep-th/0602107].

[87] S. R. Das, J. Michelson, K. Narayan and S. P. Trivedi, Phys. Rev. D **75**, 026002 (2007) [hep-th/0610053].

[88] A. Awad, S. R. Das, K. Narayan and S. P. Trivedi, Phys. Rev. D **77**, 046008 (2008) [arXiv:0711.2994 [hep-th]].

[89] A. Awad, S. R. Das, S. Nampuri, K. Narayan and S. P. Trivedi, Phys. Rev. D **79**, 046004 (2009) [arXiv:0807.1517 [hep-th]].

[90] A. Awad, S. R. Das, A. Ghosh, J. H. Oh and S. P. Trivedi, Phys. Rev. D **80**, 126011 (2009) [arXiv:0906.3275 [hep-th]].

[91] A. B. Clark, D. Z. Freedman, A. Karch and M. Schnabl, Phys. Rev. D **71**, 066003 (2005) [hep-th/0407073].

[92] K. Skenderis and P. K. Townsend, Phys. Rev. Lett. **96**, 191301 (2006) [hep-th/0602260].

[93] P. Kraus, F. Larsen and S. P. Trivedi, JHEP **9903**, 003 (1999) [hep-th/9811120].

[94] S. S. Gubser, Adv. Theor. Math. Phys. **4**, 679 (2000) [hep-th/0002160].

[95] D. Bak, M. Gutperle and S. Hirano, JHEP **0305**, 072 (2003) [hep-th/0304129].

[96] C. Fefferman and C. R. Graham, arXiv:0710.0919 [math.DG].

[97] I. Papadimitriou and K. Skenderis, JHEP **0410**, 075 (2004) [hep-th/0407071].

[98] D. Z. Freedman, C. Nunez, M. Schnabl and K. Skenderis, Phys. Rev. D **69**, 104027 (2004) [hep-th/0312055].

[99] G. Mack, Commun. Math. Phys. **55**, 1 (1977).

[100] K. Hinterbichler, L. Hui and J. Khoury, JCAP **1401**, 039 (2014) [arXiv:1304.5527 [hep-th]].

[101] P. Creminelli, A. Joyce, J. Khoury and M. Simonovic, JCAP **1304**, 020 (2013) [arXiv:1212.3329].

[102] S. W. Hawking, T. Hertog and H. S. Reall, Phys. Rev. D **62**, 043501 (2000) [hep-th/0003052].

[103] N. Arkani-Hamed, M. Porrati and L. Randall, JHEP **0108**, 017 (2001) [hep-th/0012148].

[104] M. Perez-Victoria, JHEP **0105**, 064 (2001) [hep-th/0105048].

[105] M. Henningson and K. Skenderis, JHEP **9807**, 023 (1998) [hep-th/9806087].

[106] S. de Haro, S. N. Solodukhin and K. Skenderis, Commun. Math. Phys. **217**, 595 (2001) [hep-th/0002230].

[107] M. Bianchi, D. Z. Freedman and K. Skenderis, Nucl. Phys. B **631**, 159 (2002) [hep-th/0112119].

[108] K. Skenderis, Class. Quant. Grav. **19**, 5849 (2002) [hep-th/0209067].

[109] S. Nojiri and S. D. Odintsov, Phys. Lett. B **444**, 92 (1998) [hep-th/9810008].

[110] S. Nojiri, S. D. Odintsov and S. Ogushi, Phys. Rev. D **62**, 124002 (2000) [hep-th/0001122].

[111] S. Nojiri, S. D. Odintsov and S. Ogushi, Prog. Theor. Phys. **104**, 867 (2000) [hep-th/0005197].

[112] E. Witten, Adv. Theor. Math. Phys. **2**, 253 (1998) [hep-th/9802150].

[113] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, Phys. Lett. B **428**, 105 (1998) [hep-th/9802109].

[114] R. L. Arnowitt, S. Deser and C. W. Misner, Gen. Rel. Grav. **40**, 1997 (2008) [gr-qc/0405109].

[115] D. M. McAvity and H. Osborn, Nucl. Phys. B **455**, 522 (1995) [cond-mat/9505127].

[116] K. Hinterbichler, J. Stokes and M. Trodden, arXiv:1505.05513 [hep-th].

[117] C. Closset, T. T. Dumitrescu, G. Festuccia, Z. Komargodski and N. Seiberg, JHEP **1209**, 091 (2012) [arXiv:1206.5218 [hep-th]].

[118] T. Takayanagi, Phys. Rev. Lett. **107**, 101602 (2011) [arXiv:1105.5165 [hep-th]].

[119] M. Fujita, T. Takayanagi and E. Tonni, JHEP **1111**, 043 (2011) [arXiv:1108.5152 [hep-th]].

[120] M. Nozaki, T. Takayanagi and T. Ugajin, JHEP **1206**, 066 (2012) [arXiv:1205.1573 [hep-th]].

[121] J. L. Cardy, hep-th/0411189.

[122] L. Randall and R. Sundrum, Phys. Rev. Lett. **83**, 3370 (1999) [hep-ph/9905221].

[123] J. Erlich, E. Katz, D. T. Son and M. A. Stephanov, Phys. Rev. Lett. **95**, 261602 (2005) [hep-ph/0501128].

[124] A. Karch and L. Randall, JHEP **0105**, 008 (2001) [hep-th/0011156].

[125] M. Libanov, V. Rubakov and S. Sibiryakov, Phys. Lett. B **741**, 239 (2015) [arXiv:1409.4363 [hep-th]].

[126] M. Libanov and V. Rubakov, arXiv:1502.05897 [hep-th].

[127] H. Osborn and A. C. Petkou, Annals Phys. **231**, 311 (1994) [hep-th/9307010].

[128] G. Mack, Commun. Math. Phys. **55**, 1 (1977).

[129] I. Y. Aref'eva and I. V. Volovich, Phys. Lett. B **433**, 49 (1998) [hep-th/9804182].

[130] S. de Haro, S. N. Solodukhin and K. Skenderis, Commun. Math. Phys. **217**, 595 (2001) [hep-th/0002230].

[131] D. Z. Freedman, K. Johnson and J. I. Latorre, Nucl. Phys. B **371**, 353 (1992).

[132] A. Petkou and K. Skenderis, Nucl. Phys. B **561**, 100 (1999) [hep-th/9906030].

[133] N. D. Mermin and H. Wagner, Phys. Rev. Lett. **17**, 1133 (1966).

[134] P. C. Hohenberg, Phys. Rev. **158**, 383 (1967).

[135] S. R. Coleman, Commun. Math. Phys. **31**, 259 (1973).

[136] S. S. Gubser and I. R. Klebanov, Nucl. Phys. B **656**, 23 (2003) [hep-th/0212138].

[137] O. Aharony, D. Marolf and M. Rangamani, JHEP **1102**, 041 (2011) [arXiv:1011.6144 [hep-th]].

[138] A. Polyakov, Z. H. Saleem and J. Stokes, Nucl. Phys. B **893**, 54 (2015) [arXiv:1412.1437 [hep-th]].

[139] M. Bander and C. Itzykson, Rev. Mod. Phys., **38**, 330 (1966)

[140] O. Aharony, O. DeWolfe, D. Z. Freedman and A. Karch, JHEP **0307**, 030 (2003) [hep-th/0303249].

[141] N. Vilenkin, *Special functions and the theory of group representations*, American Mathematical Society, Rhode Island 1968.

[142] O. DeWolfe, D. Z. Freedman and H. Ooguri, Phys. Rev. D **66**, 025009 (2002) [hep-th/0111135].

[143] M. Alishahiha and R. Fareghbal, Phys. Rev. D **84**, 106002 (2011) [arXiv:1108.5607 [hep-th]].

[144] L. Durand1, P. M. Fishbane and L. M. Simmons Jr., J. Math. Phys. **17**, 1933 (1976)

[145] J. L. Cardy, Nucl. Phys. B **240**, 514 (1984).

[146] D. M. McAvity and H. Osborn, Nucl. Phys. B **455**, 522 (1995) [cond-mat/9505127].

[147] P. Breitenlohner and D. Z. Freedman, Annals Phys. **144**, 249 (1982).

[148] A. Passian, H. Simpson, S. Kouchekian, S. B. Yakubovic, On the orthogonality of the MacDonald's functions, J. Math. Anal. Appl. doi:10.1016/j.jmaa.2009.06.067

[149] C. Grosche and F. Steiner, Phys. Lett. A **123**, 319 (1987).

[150] C. Grosche and F. Steiner, Annals Phys. **182**, 120 (1988).

[151] G.S. Pogosyan and A. Yakhno, Physics of Atomic Nuclei, **74**, 1062 (2011).

[152] O. DeWolfe, D. Z. Freedman and H. Ooguri, Phys. Rev. D **66**, 025009 (2002) [hep-th/0111135].

[153] A. Folacci, Phys. Rev. D **46**, 2553 (1992) [arXiv:0911.2064 [gr-qc]].

[154] A. M. Polyakov, Nucl. Phys. B **797**, 199 (2008) [arXiv:0709.2899 [hep-th]].

[155] M. Porrati, JHEP **0204**, 058 (2002) [hep-th/0112166].

[156] C. P. Burgess and C. A. Lutken, Phys. Lett. B **153**, 137 (1985).

[157] Gradshteyn, I.S., Ryzhik, I.M.: Tables of Integrals, Series, and Products. 7th edn. Jeffrey, A. and Zwillinger, D. (eds.) Academic Press, London (2007)

[158] W. Mueck and K. S. Viswanathan, Phys. Rev. D **58**, 041901 (1998) [hep-th/9804035].

[159] H. Bateman, *Higher transcendental functions*, Vol. I, McGraw-Hill, New York, 1953.

[160] M. A. Luty, M. Porrati and R. Rattazzi, JHEP **0309**, 029 (2003) [hep-th/0303116].