1-1-2015

Essays on Search and Matching Equilibria

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Abstract
This dissertation considers three separate applications of the theory of search and matching equilibria. The first chapter considers a partnership formation game, where agents on two sides of a market need to find a partner before a deadline, and search frictions make it difficult to find an acceptable partner. I characterize agents acceptance decisions – those with whom they would be willing to match – show existence, and provide a condition for uniqueness of equilibrium. This study provides a step towards a better understanding of matching behavior in non-stationary environments where agents have persistent type. The second chapter in this dissertation considers the import of adverse selection in a modern model of directed search in labor markets. Competition in this market drives firms to offer contracts that increase over time, limiting turnover. Adverse selection does not perturb contracts for less attractive types, but leads more attractive workers to accept initially low wages that grow faster than they would under full information. The final chapter of this dissertation explores the import of sequential search behavior in a model of equilibrium price setting by multi-product firms. On the one hand, the market produces results which affirm the common empirical focus on marginal distributions of individual goods' prices across firms. On the other, when some firms do not offer every good, search behavior leads to interesting pricing patterns which would not occur in single-product markets.

Degree Type
Dissertation

Degree Name
Doctor of Philosophy (PhD)

Graduate Group
Economics

First Advisor
Kenneth Burdett

Keywords
Deadlines, Matching, Search

Subject Categories
Economics

ESSAYS ON SEARCH AND MATCHING EQUILIBRIA

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A DISSERTATION

in

Economics

Presented to the Faculties of the University of Pennsylvania in Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy

2015

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Acknowledgments

I would like to give special thanks to my advisor Kenneth Burdett for his knowledge, patience, and support, along with the rest of my committee, Guido Menzio and Andrew Postlewaite for their guidance. Of course, much appreciation goes to my family, especially my mother Garnet for helping me get here. For their comments on the first chapter, I would also like to acknowledge Andrew Clausen, Rudy Henkel, Philip Kircher, Tymofiy Mylovanov, and Mallesh Pai along with seminar participants at Edinburgh University, the New York University Micro Student Lunch, and the University of Pennsylvania. I would like to further acknowledge Kenneth Burdett again for coming up with the idea for the last chapter, and I note that he is co-author on that chapter.

ABSTRACT

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This dissertation considers three separate applications of the theory of search and matching equilibria. The first chapter considers a partnership formation game, where agents on two sides of a market need to find a partner before a deadline, and search frictions make it difficult to find an acceptable partner. I characterize agents acceptance decisions – those with whom they would be willing to match – show existence, and provide a condition for uniqueness of equilibrium. This study provides a step towards a better understanding of matching behavior in non-stationary environments where agents have persistent type. The second chapter in this dissertation considers the import of adverse selection in a modern model of directed search in labor markets. Competition in this market drives firms to offer contracts that increase over time, limiting turnover. Adverse selection does not perturb contracts for less attractive types, but leads more attractive workers to accept initially low wages that grow faster than they would under full information. The final chapter of this dissertation explores the import of sequential search behavior in a model of equilibrium price setting by multi-product firms. On the one hand, the market produces results which affirm the common empirical focus on marginal distributions

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Chapter 1

Introduction

Search and matching is the exploration of two related observations. First, before one can engage in most any economic activity, one must first identify that activity. One must search. Second, once an opportunity has been identified, all of the parties to that activity must agree to the terms, and ultimately to participate. One must match. This basic observation has spawned a large literature, with applications from the theory of contracts, the analysis of labor markets, to international economics and trade, or even financial markets and urban and real estate economics. Simply, in any environment where information is diffuse, good opportunities are rare, and several agents must come together for success, one must consider search and matching. This dissertation considers three different applications of the modern theory of search and matching. The first chapter explores the import of search and matching in a market for partnership in the presence of a deadline, exploring the interaction of an evolving market and individual decisions. The second focuses on the role of information in a modern model of the labor market. Finally, the third chapter considers retail markets, exploring the interaction of consumers' search behavior and firms' pricing decisions.

Deadlines and fixed end dates are pervasive in matching markets including school choice, the market for new graduates, and even financial markets such as the market for federal funds. Deadlines drive fundamental non-stationarity and complexity in behavior, generating significant departures from the steady-state equilibria usually studied in the search and matching literature. In the second chapter, I consider a two-sided matching market with search frictions where vertically differentiated agents attempt to form bilateral matches before a deadline. I give conditions for existence and uniqueness of equilibria, and show that all equilibria exhibit an "anticipation effect" where less attractive agents become increasingly choosy over time, preferring to wait for the opportunity to match with attractive agents who, in turn, become less selective as the deadline approaches. When payoffs accrue after the deadline, or agents do not discount, a sharp characterization is available: at any point in time, the market is segmented into a first class of matching agents and a second class of waiting agents. This points to a different interpretation of unraveling observed in some markets and provides a benchmark for other studies of non-stationary matching. A simple intervention – a small participation cost – can dramatically improve efficiency.

The second chapter considers a dynamic labor market where workers are privately informed about their attachment to the labor force and firms competitively post contracts to direct workers' search. This extends the static results on adverse selection in competitive search markets of Guerrieri et al. (2010) to a dynamic environment with on the job search à la Shi (2009). Characterizing the dynamic contracting problem of firms and the search problem of workers, I show that equilibria feature full separation, increasing wage profiles, and "job lock" for committed (long duration) workers, reducing their frequency of transitions relative to a full information benchmark.

Finally, almost all retailers offer multiple products, and consumers search for low prices on a basket of goods. Kaplan and Menzio (2014) document a great deal of price dispersion both within and across stores offering multiple products. The third chapter extends Burdett and Judd (1983), a canonical model of equilibrium price dispersion, to the case of multiple products. As shown in Burdett and Malueg (1981), when sequentially searching for multiple products, consumers (a) face a lower cost of search per good and (b) may capitalize on low prices for one good while continuing to search for an acceptable price on the others. This leads multi-product consumers to set one reservation price for a basket of goods, and a higher (per good) reservation price for each good alone. This chapter characterizes firms' pricing decisions in light of this search behavior. In a simple version of the model where all firms offer every good, the marginal distribution of each price is unique and of the same form as would obtain in a simple single product model, and any joint distribution with support contained in the acceptance set of consumers satisfies equilibrium. This provides theoretical foundation for the common empirical focus on marginal price distributions – as only these are determined in equilibrium. While the structure of equilibrium is unaffected by the addition of single good demanders, the addition of single good firms can lead to one of several pricing patterns depending on parameters. A consistent prediction is that, if enough firms can offer only a single good, these single product firms crowd out the bottom of the price distribution, with the interesting equilibrium effect of also lowering the highest prices charged by multi-product firms – an effect which would not obtain in the single product case.

Chapter 2

Deadlines and Matching

In this paper, I analyze the impact of a deadline, a fixed end date when the market closes, on equilibrium dynamics in a canonical model of frictional matching. In the model, search frictions limit the rate at which vertically differentiated agents meet potential partners. When two agents meet, they each learn the type of their prospective partner, and hence their payoff from matching. If both agree, the pair match and leave the market. If not, they continue searching. These exits cause the distribution of available partners to evolve over time. At the deadline, unmatched agents receive some outside option and the game ends. I establish existence of equilibria, provide a condition ensuring uniqueness, and characterize behavior.

Many matching markets feature a deadline. In education, students must find a seat before the start of the school year. In the market for entry level professionals, new graduates want to find a job before graduation. In the market for federal funds, banks must meet their reserve requirements before the monitoring deadline every evening. When present, deadlines and the consequent cyclical nature of these markets allows for the implementation of centralized, static mechanisms. Prominent examples include the medical resident matching program and the school choice mechanisms in New York and Boston, in addition to somewhat less structured systems like the signaling mechanism provided by the American Economic Association's JOE program.[1]

The design and analysis of such systems derive from the now prominent literature on centralized matching, which studies what may obtain when agents come together to form matches through a common marketplace or clearinghouse.[2] A dual literature, usually termed search and matching, studies incentives and equilibria when agents must seek out matches in a decentralized fashion, lacking ready access to relevant partners. This study applies the decentralized paradigm to markets with deadlines, providing a positive theory of dynamic behavior in the absence of clearinghouses – a model of the *status quo ante* that one can compare to the successes of centralization.

Consider a decision maker facing a simple search decision problem with a deadline after which continued search is impossible. Over time, the decision maker encounters opportunities that she can either accept, ending search, or reject, giving

[1] See Roth and Peranson (1999) on medical residents; Abdulkadiroğlu et al. (2005), Abdulkadiroğlu et al. (2006), and Pathak and Sönmez (2008) on school choice; and Coles et al. (2010) on the market for new economists.

[2] The authoritative introduction being Roth et al. (1992); see Sotomayor and Özak (2012) for a more recent and very concise summary.

up the opportunity in hopes of finding a better one in the future. As the deadline approaches, she has less time remaining to search, and therefore will encounter fewer opportunities in the future. This leads her to be less selective over time. If the distribution worsens as time goes on, making good opportunities rarer, this should further drive her to adopt a declining reservation level, and also to accept early opportunities. Finally, if she is impatient, with a positive discount rate, pure preference induces her to accept early opportunities.

This intuitive strategy – where one both accepts some selection of early opportunities and becomes less choosy over time – holds exactly for the most attractive agents in a matching market with deadline. Everyone will always accept the most attractive type, so the most attractive agents need not concern themselves with the possibility of being rejected by a potential partner; they exactly face the simple decision problem outlined above. Less attractive agents, however, are not so lucky. They may be refused by desirable partners, and so must formulate their strategies in light of the acceptance decisions of others.

In a steady state version of the model, Burdett and Coles (1997) show that matching sets partition agents into a finite number of classes, disjoint sets of mutually acceptable types.[3] When there is a deadline, one might conjecture that some flavor of a class system persists. Perhaps some finite number of temporary, time-

[3]This result was developed across a series of papers each with subtly differing assumptions including Bloch and Ryder (2000), Burdett and Coles (1997), Chade (2001), Eeckhout (1999), andMcNamara and Collins (1990). The framework of Burdett and Coles (1997) is the most similar to mine.

varying classes obtain. Indeed, a first class exists by exactly the same logic as in steady state – once one becomes acceptable to the highest type, one is universally acceptable, so one chooses the same strategy as the highest type. But the dynamics in the model destroy any hope of summarizing less attractive agents so simply.

The complication derives from an "anticipation effect." When agents join the first class, their opportunity sets jump discretely. As different agents anticipate that they will receive this dramatic improvement in opportunities at different times, they each follow different strategies, destroying the class system. When impatient, agents become increasingly choosy as they get close to joining the first class, further complicating behavior. If there is no discounting, however, the behavior of agents outside the first is easily described; they do not match at all, preferring to wait for the opportunity to match with high types later. At each point in time, the market segments into a first class of matching agents and a second class of waiting agents.

This partitioning has a number of implications. The first concerns sorting. In the unravelling literature, agents rush the market. Early matching prevents sorting. Here, because of search frictions, early matching improves efficiency and sorting. The second implication is that a small flow cost of search is Pareto improving, as it drives low types out of the market until it is their time to match. This eliminates the search externality low types exert on high types, and all meetings result in a match. High types obviously appreciate this, but low types do not mind as a higher match probability compensates low types for a lower quality of partner, in expectation.

The next section considers some important predecessors in the literature. The following section lays out the basic framework. Section 2.4 presents general results and is followed by analysis and discussion of the case of patient agents in Section 2.3. Section 2.5 considers the effect of costs on search behavior for patient agents. The paper then concludes with some discussion.

2.1 Context in the Literature

The current study is a direct extension of Burdett and Coles (1997) as I impose a deadline on their steady state model. This simple change generates substantially different behavior than previously analyzed in the literature; specifically, almost no work considers non-stationary dynamics in a rich search and matching model. Early predecessors of my paper studied search-theoretic decision problems in a changing world. These include Van Den Berg (1990) and Smith (1999).[4] These studies hint at the anticipation effect – that one should be willing to wait for promising opportunities in the future – but these are decision theoretic studies, and the strong equilibrium implications of anticipation are obscured.

Two other studies are closely related to mine. The first, Afonso and Lagos (2012), considers a model of decentralized trade before a deadline, and is applied to the market for federal funds. In their model, all agents hold some quantity

[4]To the author's knowledge, the first paper which describes the Bellman equation faced by a decision maker in a model of non-stationary search was Mortensen (1986). But he immediately specializes to the stationary case.

of federal funds and search for a partner with whom to trade, after which they continue to search for profitable trades until a deadline. They obtain the remarkable result that, if agents share concave values over final holdings, all meetings result in trade. In that they characterize the case of repeated trade with transferable utility, while the current study considers nontransferable utility with only a single trade – partnership formation – Afonso and Lagos (2012) provides a valuable counterpoint to the results developed below. The second predecessor, Damiano et al. (2005), considers a model of partnership formation with nontransferable utility as in the current study, but differs in that, instead of randomly encountering partners over time, agents encounter one another over a finite number of discrete rounds. This leads to dramatically different results when search costs are incorporated, and so I leave further discussion of this paper to section 2.5.[5]

Generally, the search literature related to this study can be broken into two strands. One considers non-trivial matching decisions, but in steady state, and the other explores non-stationary dynamics, but without meaningful matching decisions. The non-stationary literature is concerned primarily with macroeconomic fluctuations, and employs search frictions as a means of explaining labor market dynamics.[6] In order to keep the state space small, heterogeneity is either completely idiosyncratic, or absent. In steady state, there is a large literature addressing equi-

[5]This discrete time matching framework has also been considered in the theoretical biology literature, see Alpern et al. (2005) for results which expand upon the Damiano et al. (2005) framework, and summarize previous work in that other literature.

[6]Rogerson et al. (2005) and Rogerson and Shimer (2011) survey the literature.

librium matching behavior. Prominent examples include Burdett and Coles (1997) and Shimer and Smith (2000). The restriction to steady state allows for a careful consideration of the matching decisions of heterogeneous agents, but that restriction precludes analysis of the effect of a changing environment on equilibrium interactions at the heart of the current study.

There are but a handful of recent advances towards reconciling non-stationarity and heterogeneity. Rudanko (2011) and Menzio and Shi (2011) assume agents can direct their search, only meeting the partners for whom they actively search. This, coupled with a free entry condition, dramatically simplifies the firms' side of the market, allowing for a clean characterization of behavior. Coles and Mortensen (2012), Moscarini and Postel-Vinay (2013), and Robin (2011) take a different tack, each showing that a different restriction on the contracting space can simplify the movements of individuals across jobs, affording sharp results. Instead, the current study makes a stark assumption on the nature of non-stationarity – the deadline – and focuses on matching decisions exclusively, eliminating the complications of contracting by instead assuming non-transferable utility. This allows the current study to offer a clean description of matching behavior, highlighting the equilibrium forces underlying non-stationary matching problems more broadly.

2.2 The Framework

The framework is a non-stationary extension of Burdett and Coles (1997). Two groups of agents, say workers and firms, attempt to find a partner from the other side. At time zero, the market is populated with equal masses of workers and firms measuring size N_0. Instead of explicitly modeling the process by which the two sides evaluate each other, assume that individuals can be characterized by a fixed real number which, following Burdett and Coles (1997), is termed pizazz. This is a vertically differentiated market. Agents' pizazz are initially distributed according to $G^0(z)$ with support $X = [\underline{x}, \overline{x}] \subset (0, \infty)$. Time flows continuously from zero up to $T > 0$. During this time, agents search for partners from the other group. Each agent encounters a potential partner at a constant rate $\alpha > 0$.[7] Upon meeting, two agents observe each other's pizazz and simultaneously decide whether or not to propose a match. For a match to occur, both agents in a meet must propose. Utility is non-transferable; the value to an agent with pizazz y of matching with an agent of pizazz x is exactly equal to x, irrespective of y.[8] Once matched, agents leave the market (there is no recall or divorce).

If, upon reaching time T, an agent remains unmatched, they receive utility from

[7] Which one could rationalize with a constant returns to scale meeting function.

[8] It is not clear whether this is a restriction above and beyond the requirement of identical time-valued VNM preferences. Indeed the analysis goes through equally well if agents receive a general payoff $f(x, y)$ so long as this is multiplicatively separable, increasing, and strictly positive. Additive separability may also be accommodated when agents are patient and do not discount. Eeckhout (1999) and Smith (2006) allow for type-dependent preferences and show that all that is required for a class system to obtain in a stationary framework is identical static VNM preferences across agents, which implicitly allows different discount factors. This paper will not allow for differences in discount rates, and so assumes identical cardinal preferences from the outset.

an outside option, the value of which is 0. That all agents share a uniform outside option is not without loss of generality and represents a significant simplification. The strongest implication is that all agents prefer matching with even the least attractive agent to taking the outside option. In addition to a declining probability of meeting (because time is running out), agents may be impatient and discount the future at a rate $r \geq 0$.

Suppose that agents flow into the market at a rate $\zeta(t) \geq 0$ which is bounded above by some $\bar{\zeta}$ and that the distribution of the inflowing agents is $H(z, t)$ with support contained in X. Let $G(z, t)$ be the distribution of pizazz at time t (reflecting changes due to both inflows and outflows). Further, write $N(t)$ for the mass of agents at time t so that $N(t)G(z, t)$ is the mass of agents of pizazz less than z at time t.

Since an agent x may not receive a proposal from every meeting, write $\alpha(x, t)$ for the (possibly time varying) arrival rate of proposals and $G_x(z, t)$ for the distribution of agents who would propose to x upon meeting. Write

$$\Omega(x, t) = \{y | y \text{ is willing to propose to } x\}$$

and

$$\mathcal{A}(x, t) = \{y | x \text{ is willing to propose to } y\}$$

and call these the opportunity and acceptance sets, respectively.

With the basic elements in hand, write $U(x,t)$ as the (Bellman) value at time t for an agent of pizazz x. Focus on symmetric cutoff strategies where agents accept any partner with pizazz greater than or equal to his or her current value.[9] Standard arguments then yield the following Hamilton-Jacobi-Bellman (HJB) equation for the agent's reservation value.[10]

$$\dot{U}(x,t) = rU(x,t) - \alpha(x,t)\int_{U(x,t)}^{\overline{x}} (z - U(x,t))G_x(dz,t)$$

with boundary condition $U(x,T) = 0$. This states that, as agents wait for a match, the change in their reservation value is given by the asset value of their future opportunities, less the excess value of current matches which did not materialize. Integration by parts gives a more convenient formulation:

$$\dot{U}(x,t) = rU(x,t) - \alpha(x,t)\int_{U(x,t)}^{\overline{x}} (1 - G_x(z,t))dz. \qquad (2.2.1)$$

Given that agents use cutoff strategies, we have the following.

Remark 2.2.1. Since x will accept any $y \geq U(x,t)$ we have $\mathcal{A}(x,t) = \{y|y \geq U(x,t)\}$, $\Omega(x,t) = \{y|x \geq U(y,t)\}$, $\alpha(x,t) = \alpha \int_{\Omega(x,t)} G(dz,t)$ and

[9] Cutoff strategies are the only weakly undominated ones, and restricting attention to cutoff strategies removes pathological equilibria such as 'everyone always rejects.' Moreover, it is a strong symmetry assumption – all x type firms play the same strategy as all x type workers. Symmetry within a group is not binding. While I prove existence of equilibria with symmetry across groups, there may exist asymmetric equilibria even with symmetric initial data, but this is left for future work.

[10] This equation was first derived in search theory work by Mortensen (1986). His analysis was later expanded to consider more general kinds of time variation by Van Den Berg (1990).

$$G_x(z,t) = \frac{\int_{\Omega(x,t)} \mathbb{1}\{y \leq z\}\, G(dy,t)}{\int_{\Omega(x,t)} G(dy,t)}.$$

This allows one to write $\alpha(x,t)\int f(z)G_x(dz,t) = \alpha\int_{\Omega(x,t)} f(z)G(dz,t)$, for any integrable f, which will be used extensively. In particular, it implies that one's decision problem depends only on the time path of one's opportunity set.

With the individual's problem defined, the last step in the setup of the model is to derive the dynamic for G. Write $\theta(x,t)$ for the probability that a meeting will result in a match for an agent with pizazz x,

$$\theta(x,t) = \int\limits_{\mathcal{A}(x,t)\cap\Omega(x,t)} G(dy,t),$$

so that the exit rate for an agent of pizazz x is $\alpha\theta(x,t)$. Supposing, momentarily, that $G(z,t)$ and $H(z,t)$ possess densities $g(z,t)$ and $h(z,t)$, the number of agents with pizazz z in the market at time t is $n(z,t) = N(t)g(z,t)$. The number of agents with pizazz z leaving the market is $\alpha g(z,t)\theta(z,t)N(t)$ and the number entering is $\zeta(t)h(z,t)$. This gives $\dot{n}(z,t) = -\alpha\theta(z,t)g(z,t)N(t) + \zeta(t)h(z,t)$, and, after integrating, $\dot{N}(t) = -\alpha N(t)\,\mathbb{E}(\theta(x,t)) + \zeta(t)$. Writing $\eta(t) = \zeta(t)/N(t)$, and noting that $\dot{g} = [\dot{n}N - n\dot{N}]/N^2$, one observes

$$\dot{g}(z,t) = \alpha g(z,t)[\mathbb{E}(\theta(x)) - \theta(z)] - \eta(t)[g(z,t) - h(z,t)].$$

This can be read as saying that, if a given agent's probability of being matched is greater than average, their relative numbers tend to decline (the first term) unless the entrance of new agents more than compensates (the second term). Integrating again gives the dynamic for G.[11]

$$\dot{G}(z,t) = \alpha G(z,t)[\mathbb{E}(\theta(x)) - \mathbb{E}(\theta(x)|x \leq z)] - \eta(t)[G(z,t) - H(z,t)]. \qquad (2.2.2)$$

With the framework in hand, consider now the general properties of the model.

2.3 Patient Agents

In our motivating applications, agents receive their payoff after the market closes, so it is appropriate to assume no discounting, $r = 0$. For example, an academic economist does not start working until several months after the end of the search process, and universities do not receive services until that time. Moreover, the case of $r = 0$ strongly highlights the anticipation effect and produces a tractable equilibrium characterization: highly attractive agents, following the intuitive strategy alluded to in the introduction, become less selective as time ticks on and low type agents prefer not to match early in the market, instead waiting until highly attractive agents will accept them.

Since this case is relatively uncomplicated, I keep the analysis in this section informal, leaving most formal results for the next section. The first step in the

[11]Which holds whether or not G and H possess densities, the above derivation being only for the purposes of exposition.

characterization is to notice that when there is no discounting, reservation values can never rise over time. If there is a high value available in the future, patient agents will simply wait for it rather than accepting less attractive options today.

Lemma 2.3.1. *$U(x,t)$ is weakly decreasing in t when $r = 0$.*

Proof. Recall equation (2.2.1) and substitute $r = 0$,

$$\dot{U}(x,t) = -\alpha(x,t)\int_{U(x,t)}^{\overline{x}}(1 - G_x(z,t))dz \leq 0.$$

□

Next, a bound on the value of the highest type obtains. Suppose $\overline{x}$ were alone in a market exclusively populated with the most attractive agents who are all willing to match. The value in this market is simply equal to the probability of matching $(1 - \exp\{-\alpha(T-t)\})$ times the value of matching with the highest type $(\overline{x})$. This rosy scenario gives a bound on the reservation value of the highest type in any equilibrium:

$$U(\overline{x},t) < \hat{U}(t) \equiv \overline{x}(1 - \exp\{-\alpha(T-t)\}).$$

This implies that, at time zero, at least all agents with $x \geq \overline{x}(1 - \exp\{-\alpha T\})$ are acceptable to $\overline{x}$. Further, all agents become acceptable to $\overline{x}$ at some point (because $\underline{x} > 0 = U(\overline{x},T)$). Define the set acceptable to $\overline{x}$ as the first class: $\mathcal{F}(t) = \{x \geq U(\overline{x},t)\}$. The time when one joins the first class is important. Define these hitting times as $\tau(x) = \min\{t \in [0,T] | U(\overline{x},t) \leq x\}$, so that $\tau(x)$ is the time when x becomes acceptable to $\overline{x}$ (and they remain acceptable because of Lemma 2.3.1).

Being acceptable to $\overline{x}$ has an important implication. If $t \geq \tau(x)$, so that x is acceptable to $\overline{x}$, then $U(x,t) = U(\overline{x},t)$: If one is acceptable to $\overline{x}$ for all future time, one is acceptable to all other agents into the future.[12] Then, since values depend only on opportunity sets, one's expected value from search is exactly the same as $\overline{x}$.

This has a strong equilibrium implication: no one outside the first class matches. At $\tau(x)$, x gets a partner of his or her own pizazz in expectation: $U(x,\tau(x)) = x$ because $U(x,\tau(x)) = U(\overline{x},\tau(x)) = x$. Moreover, $U(x,t) \geq x$ for $t < \tau(x)$ by Lemma 2.3.1. Finally, it can also be shown that $U(x,t) \leq x$. That is, one is always willing to accept a partner of equal pizazz.[13] These, then, give $U(x,t) = x$ for $t < \tau(x)$, and all behavior is driven by the value of the highest type. This is summarized in the following proposition and illustrated in figure 2.1.

Proposition 2.3.2. *When $r = 0$, $U(\overline{x},t)$ wholly determines the equilibrium as*

$$U(x,t) = \begin{cases} x & \text{if } t < \tau(x) \\ U(\overline{x},t) & \text{if } t \geq \tau(x). \end{cases}$$

Suppress time arguments and write $\bar{U} = U(\overline{x},t)$, the dynamic for G simplifies to

[12] Which assumes monotone reservation values, proved in by Corollary 2.4.5 below

[13] Which is proved formally in Corollary 2.4.6 below.

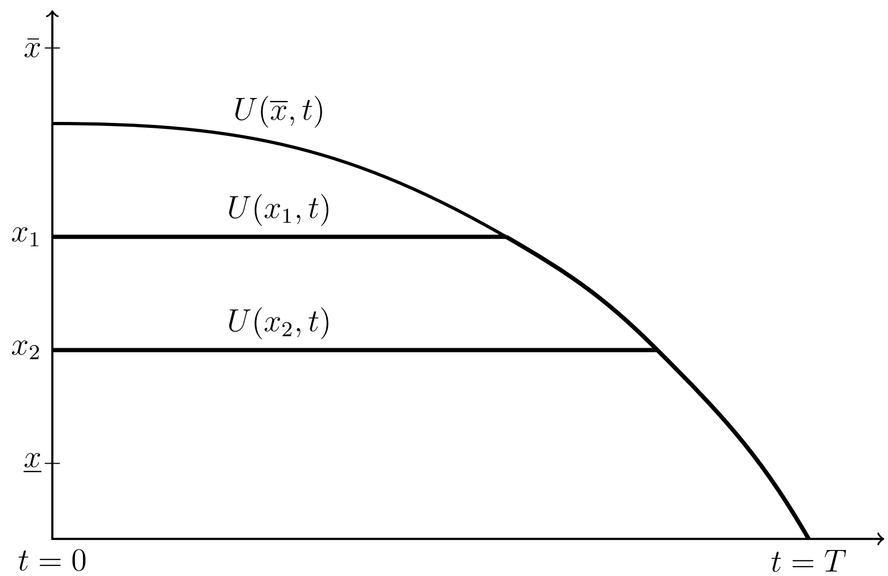

Figure 2.1: Reservation Values when $r = 0$

$$\dot{G}(z) = \begin{cases} \alpha G(z)[1 - G(\bar{U})]^2 - \eta(t)(G(z) - H(z)) & \text{if } z < \bar{U} \\ \\ \alpha G(\bar{U})[1 - G(\bar{U})][1 - G(z)] - \eta(t)(G(z) - H(z)) & \text{if } z \geq \bar{U}. \end{cases} \tag{2.3.1}$$

Proof. The specification of U derives from the discussion above. The relatively explicit form for $\dot{G}$ derives from the fact that $\theta(x, t)$, the probability of a meeting resulting in a match, collapses to a step function:[14]

[14] There are other possible dynamics if G contains atoms. In this case, the agents with positive mass are indifferent between matching with each other or not before $\tau(x)$. This dynamic assumes that they do not. This form would dissolve otherwise. Indeed, if there were some finite set of pizazz levels, then the anticipation result dissolves to some extent, as one equilibrium would be for all agents to match with equal pizazz agents before joining the first class. This is resolved by the introduction of avoidable search costs, which induce second class agents to stay home as described below.

$$\theta(x) = \begin{cases} (1 - G(\bar{U})) & \text{if } x \geq \bar{U} \\ \\ 0 & \text{if } x < \bar{U} \end{cases}$$

□

To reiterate, low types wait, with reservation value equal to their own type, until they become acceptable to the highest type, after which they share a value function with the highest type. The notion that patient agents should only match with their own type is perhaps not surprising. If one were to consider the limit of the Burdett-Coles economy as the discount rate goes to zero, the classes shrink to the point where each type is in their own class. That the introduction of a deadline leads to growing desperation is also unsurprising. The unobvious contribution is that that the interaction of these two considerations leads to equilibrium behavior that admits such a straightforward summary. Straightforward, however, should not be mistaken for simple, as the reservation value for $\overline{x}$ encodes all of the subtleties of an evolving distribution, weighing off the value of matching today against the possibility of remaining unmatched or facing poor opportunities in the future.

Because of the clear characterization available when agents are patient, another important result obtains:

Proposition 2.3.3 (Uniqueness)**.** *If there is no entry ($\eta = 0$), agents are patient ($r = 0$), and G^0 is continuous, then the equilibrium is unique.*

The proof is relegated to the appendix, but derives mostly from a careful consideration of the dynamics of the distribution in light of the equilibrium characterization from Proposition 2.3.2. Briefly, if one increases the initial reservation value, high types filter out for some period before the reservation falls back to the original level. This leads to a relatively flat path in the future. Hence, a high initial value leads to a high terminal value – only one path can satisfy the boundary condition.

In the context of the job market, that the best candidates match earliest fits common experience, is alluded to in Roth and Xing (1997) in the context of the market for clinical psychologists, and is a model prediction in Damiano et al. (2005) (when there are no costs) and Burdett and Coles (1997) (because higher agents are in larger classes). That low pizazz agents have no strict incentive to match early in the market reflects optimal waiting. At $\tau(x)$, the fact that many high type agents may have left is irrelevant. $U(\overline{x}, t)$ hits x exactly when the value of being in the first class equals x. The (possibly small) probability of matching with very attractive agents offsets the probability of only meeting agents without much pizazz, or having no future meetings at all.

2.4 General Results

This section provides results concerning existence and characterization of equilibria for any discount rate $r \geq 0$. In the job market for entry-level professionals, one might think of $r > 0$ as pure impatience, wanting to know sooner rather than later. Alternatively, r might represent the flow probability of a tragic event – the death of a relative, say – which would cause an agent to quit searching and abandon the market. One has a preference for securing an early match because it resolves this risk. When $r > 0$, the model exhibits rich behavior. But, before exploring this, note that behavior in the presence of discounting limits to the simpler behavior described above as $r \to 0$.

Proposition 2.4.1. *As $r \to 0$, the discounting equilibrium converges to the no-discounting equilibrium.*

The complication when $r > 0$ derives from early matching among less attractive agents. But as $r \to 0$, this early matching dissolves, and so even if agents are impatient, so long as the duration of the market is short and matching rates are high, early matching has little impact on equilibrium.

Turning now to existence, given the focus on cutoff strategies, an equilibrium is any pair U, G which simultaneously solve (2.2.1), the Bellman equation, and (2.2.2), the differential equation for G, subject to $U(x, T) = 0$ and $G(z, 0) = G^0(z)$. No restrictions are required on the initial distribution of pizazz in order to obtain existence. This derives from the fact that equilibrium is not required to exist in steady state; the only requirement is that agents correctly predict the time path of the distribution of pizazz when making matching decisions, and that these matching decisions generate the predicted time path. All omitted proofs can be found in the appendix.

Proposition 2.4.2 (Existence)**.** *There exists an equilibrium for any $r \geq 0$.*

The proof is closely related to that in Smith (2006) with the exception that one instead solves for a whole time path for each object. This leads to significant alteration of the “Fundamental Matching Lemma” which instead relies on arguments from the theory of Banach ODE.

When agents discount, expected present values can rise or fall over time – Lemma 2.3.1 does not hold. Specifically, the reservation value of the highest type can rise over time if the distribution improves sufficiently. This can occur either because high types enter or because low types match and exit. Hence, an agent who is acceptable to the highest type at a point in time need not be in the future, and so need not share the highest type's reservation. As in the case of $r = 0$, equilibrium revolves around the existence of a first class of agents who share the same reservation. Now, however, the first class does not consist of those acceptable to $\overline{x}$ at a point in time. Instead, say an agent is in the first class if they are universally acceptable now and forever. That is:

Definition 2.4.3. Let $\mathcal{F}(t) = \{x | \forall s \geq t, \Omega(x, s) = X\}$, and call this set the *First Class.*

Before we can characterize the first class and the behavior of first class agents, some intermediate results are required. The first states that higher types have more opportunities, which follows from cutoff strategies.

Lemma 2.4.4 (Monotone Opportunity Sets)**.** *If $x_1 \leq x_2$ then $\Omega(x_1, t) \subseteq \Omega(x_2, t)$, and $\alpha(x_1, t) \leq \alpha(x_2, t)$ for all t.*

This observation yields another intermediate result towards characterizing the first class. Because opportunity sets are increasing in type, so are reservation values.

Corollary 2.4.5 (Monotone Values)**.** *For all t, $U(x,t)$ is increasing in x, and $\Omega(x,t)$ is connected.*

Given monotone values, a simple upper bound obtains, yielding the intuitive result that agents are always willing to accept their equals:

Corollary 2.4.6. *$U(x,t) \leq x$ for all x, t.*

Proof. If an agent, x, has a value higher than his own pizazz, some other agent with higher pizazz $y > x$ must be willing to match with him (if not today then at some point in the future). But that would imply $x \geq U(y,t) \geq U(x,t)$. Discounting this observation backwards yields the result. $\square$

From these points one notices what is a general property of models with non-transferable utility and common preferences.

Remark 2.4.7. The model delivers *Positive Assortative Matching* at each point in time in the set-valued sense of Shimer and Smith (2000): the upper and lower bounds on the matching set are weakly increasing everywhere.

Because of monotonicity in opportunity sets, the time when one is universally acceptable going forward is exactly the same as the time when one is acceptable to the highest type. This allows for the first class to be formulated in a manner similar to the last section, but allowing for the possibility of non-monotonicity. One does not join the first class immediately upon becoming acceptable to the highest type. Instead, one joins the first class when one becomes acceptable to the highest type forever.

Remark 2.4.8. $\mathcal{F}(t) = \{x | x \geq \sup_{s \geq t} U(\overline{x}, s)\}$ by Lemma 2.4.4.

Not only is one always acceptable to one's equal, the assumption that $U(x, T) = 0$ implies that every agent is eventually universally acceptable. As in the no discounting case, all agents eventually join the first class.

Lemma 2.4.9. *For every agent, x, there exists $\tau(x) < T$ with $\tau(x) = \inf\{t | x \in \mathcal{F}(t)\}$.*

Proof. At time T, everyone is willing to match with everyone else because $\underline{x} > 0 = U(x, T)$. That there exists $\varepsilon > 0$ such that the same holds for all $t > T - \varepsilon$ follows from boundedness of $\dot{U}$. And, as one's value depends only on the future path of one's opportunity set, if $\Omega(x, t) = X = \Omega(\overline{x}, t)$ for all $t \geq \tau(x)$, then $U(x, t) = U(\overline{x}, t)$ for all $t \geq \tau(x)$. But $\tau(x)$ is precisely the moment when $\overline{x}$ joins $\Omega(x, t)$. Hence, it is the precise time when $x = U(\overline{x}, t)$. Thus, $U(x, \tau(x)) = x$. $\square$

These all together complete the description of the first class. The first class consists exactly of those who are permanently acceptable to the highest type, and all agents join the first class before the deadline. This leads to an analogue of Proposition 2.3.2 for the case of discounting.

Lemma 2.4.10 (First Class Values)**.** *All first class agents share the same value: If $t \geq \tau(x)$, $U(x,t) = U(\overline{x},t)$ and, specifically, $U(x,\tau(x)) = x$.*

Proof. That $U(x,t) = U(\overline{x},t)$ for $t \geq \tau(x)$ follows from simple inspection of the Bellman equation given that $G_x(\cdot,t) = G(\cdot,t) = G_{\bar{x}}(\cdot,t)$ and $\alpha(x,t) = \alpha = \alpha(\overline{x},t)$. And then, that $U(x,\tau(x)) = x$ follows from Remark 2.4.8. □

The intuition is the same as in the case of no discounting. Once one has joined the first class, one is universally acceptable going forward, by definition. One's problem is wholly defined by the time path of one's opportunity set. If two agents share the same opportunity set going forward, as they have the same preferences, they must make the same decisions and have the same value. Since all agents are eventually universally acceptable, they eventually all share the same value. Moreover, agents smoothly filter into the first class as the deadline approaches and the highest type becomes less and less selective. The fact that all agents eventually share a value function dramatically simplifies the analysis.

Note that it is here where the joint assumptions of common preferences and a common outside option truly bind. If one were to dispense with either of these, this sharp result would dissolve. Indeed, even with these, equilibrium still fails to admit any simple representation with some finite number of classes:

Remark 2.4.11. There do not exist persistent coincidences of matching sets outside the first class. Second class agents become increasingly selective before they join the first class: $\lim_{t \nearrow \tau(x)} \dot{U}(x,t) = rx$.

Because different agents expect to be able to get their own pizazz at some point in the future, there can be no persistent coincidence of matching sets for different pizazz levels with $\tau(x) > 0$. Indeed, the only class in the sense of Burdett and Coles (1997) consists of exactly those agents with $\tau(x) = 0$. If x has $\tau(x) = 0$, then x expects to be able to match with all agents at any point in the future. Hence, their problem is identical to that of $\overline{x}$. These agents all share the same value, $U(\overline{x}, t)$, across the whole time path; share the same matching set; and are always willing to match with each other. But, unless all agents fall into this class, one can not capture equilibrium behavior with any finite set of reservation values.

One might infer from the proof of Lemma 2.4.9 that low pizazz agents join the first class only ε-time before T. This is not the case as one can see from a bound on the reservation value of the highest type.

Lemma 2.4.12.

$$U(\overline{x}, t) \leq \hat{U}(\overline{x}, t) = \frac{\alpha}{r+\alpha}\overline{x}(1 - \exp\{-(r+\alpha)(T-t)\}),$$

and so

$$\tau(x) \leq \hat{\tau}(x) = T + \left(\frac{1}{r+\alpha}\right) \log\left[1 - \frac{x}{\overline{x}}\left(1 + \frac{r}{\alpha}\right)\right].$$

Proof. The bound on U derives from considering the value obtained if $\overline{x}$ were in a market with only other $\overline{x}$ pizazz agents: solve $\dot{\hat{U}}(\overline{x}, t) = (r+\alpha)\hat{U}(\overline{x}, t) - \alpha\overline{x}$, with $\hat{U}(\overline{x}, T) = 0$. The bound on $\tau(x)$ comes from solving $\hat{U}(\overline{x}, \hat{\tau}(x)) = x$ for $\hat{\tau}(x)$. $\square$

This implies that the first class consists of at least all agents with $\hat{\tau}(x) = 0$, those agents with $x \geq \hat{U}(\overline{x}, 0)$. Moreover, one can say (independent of T) that all agents are in the first class from time zero whenever

$$\frac{\overline{x}}{\underline{x}} < 1 + \frac{r}{\alpha}.$$

For matching not to be universal, the ratio between the highest and lowest pizazz levels can not be too tight compared to the matching friction, as measured by r/α.

As mentioned in Remark 2.4.11, reservations are increasing for agents just before they enter the first class. And, since $\tau(x)$ is continuous in x, agents who expect to join the first class near time zero have increasing reservations from the very beginning. Hence, lower agents have decreasing matching opportunities before they enter the first class as more attractive agents become increasingly selective before they join the first class. This, on the one hand, tends to drag down less attractive agents' reservations as their early matching opportunities dry up. On the other hand, as time goes on, agents move closer to joining the first class, which pushes up reservations. An integral of U makes this clear:

Remark 2.4.13. If one writes $y(x,t) = \sup\{y \in \Omega(x,t)\}$, then $\Omega(x,t) = [\underline{x}, y(x,t)]$

and

$$\dot{U}(x,t) = r\left(\underbrace{xe^{-r(\tau(x)-t)}}_{\text{A}} + \underbrace{\alpha\int_t^{\tau(x)} e^{-r(s-t)}\int_{U(x,s)}^{y(x,s)}(G(y(x,s)) - G(z,s))\,dz\,ds}_{\text{B}}\right) - \underbrace{\alpha\int_{U(x,t)}^{y(x,t)}(G(y(x,t)) - G(z,t))\,dz}_{\text{C}} \quad (2.4.1)$$

The expression derives from substituting $U(x,\tau(x)) = x$ into an integral of the Bellman equation and then substituting the result into the definition of $\dot{U}$. The first term, A, is the discounted contribution of the expectation that x will join the first class at time $\tau(x)$. The second, B, is the discounted contribution of future excess value of matching opportunities to current utility. The last, C, is the current excess match value. So, the change in reservation is given by the asset value of not matching, r times A plus B, less the expected value of the missed opportunity today, C. This is illustrated in Figure 2.2.

Suppose there is some agent x with $\tau(x) > 0$ and for all agents $z > x$ and times $t < \tau(z)$, $\dot{U}(z,t) > 0$. Then $y(x,t)$ is strictly decreasing over time.[15] Hence, matching opportunities are declining for x. This is reflected in C being large relative to B. So, if $\tau(x)$ is far off, A might also be small and so values would be declining. Or, with $\tau(x)$ close, A might be large relative to C, yielding increasing values. In general, values might be increasing or decreasing for different agents before they join

[15]And there exists some such x because for all z, $\dot{U}(z,\tau(z)) = rz > 0$.

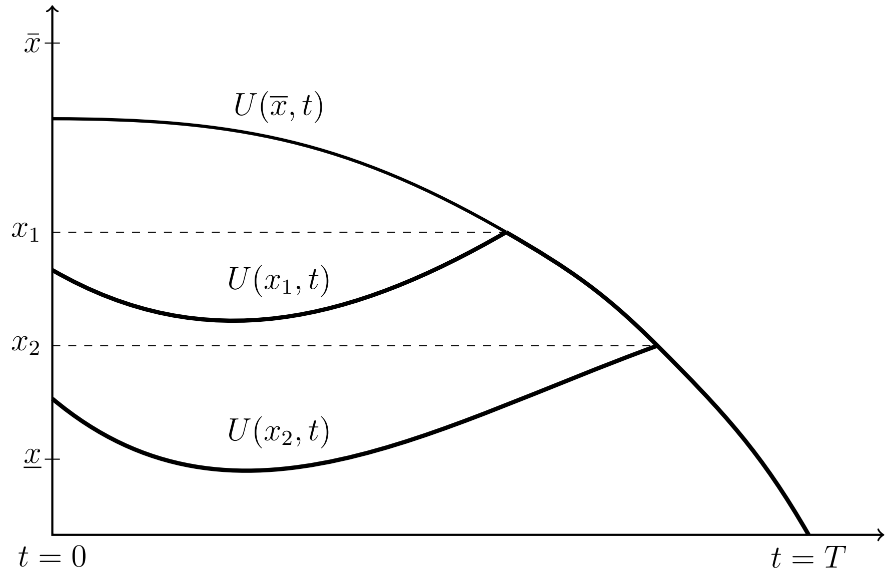

Figure 2.2: Value when $r > 0$

the first class (and then either increasing or decreasing thereafter). A condition, however, is available which guarantees that even the least attractive agents have increasing reservations over the whole period.

Lemma 2.4.14. *Write*

$$\lambda(\sigma) = \left[1 - \frac{\underline{x}}{\overline{x}}(1 + \sigma)\right]^{\left(-\frac{\sigma}{1+\sigma}\right)} e^{-rT}.$$

If

$$\left(1 + \frac{r}{\alpha}\right) \lambda \left(\frac{r}{\alpha}\right)^2 > 1$$

then for all x with $\tau(x) > 0$, $\dot{U}(x, t) > 0$ whenever $t \leq \tau(x)$.

While the proof is left for the appendix, it relies on using the bound on $\tau(x)$

from Lemma 2.4.12 to give an upper bound for $y(x,t)$ and evaluating the matching opportunities if x could match with $y(x,t)$ with rate α; hence the bound does not depend on the distribution of agents and is relatively weak.

Note that the result holds vacuously if $(\overline{x}/\underline{x}) < 1 + (r/\alpha)$ where all agents are always in the first class. But, there do exist parameters for which the result holds meaningfully because, for example, $\lim_{r\to 0}(1+(r/\alpha))\lambda(r/\alpha)^2 = 1$ and

$$\lim_{r\to 0}\frac{\partial}{\partial r}\left(1+\frac{r}{\alpha}\right)\lambda\left(\frac{r}{\alpha}\right)^2 = \frac{1}{\alpha}\left(1-2\alpha T - \log\left(1-\frac{\underline{x}}{\overline{x}}\right)\right) > 0$$

for $\underline{x}/\overline{x}$ large relative to T. For some parameter values, unattractive agents should all become more choosy over time before joining the first class.

Also, note that the definition of $\tau(x)$ can not be simplified: the reservation value of the most attractive agent need not be monotone. As the model allows for arbitrary inflows, this is somewhat obvious. What may be less obvious is that the highest types may become more selective even without inflows because matching behavior of lower types can improve the aggregate distribution. If, for instance, there is a relatively large population of low types, then they match out relatively quickly. This improves the distribution over time. If match rates are high and agents relatively impatient, this leads to an increasing value for the highest types. This is closely related to non-uniqueness in the $r > 0$ case.

The intuition for multiplicity is as follows: If a high pizazz agent, x, expects that other highly attractive agents will match quickly, leading to a poor distribution in the future, then x will lower his reservation value in the present, leading to a higher rate of exit. Alternately, if x expects the distribution to stay relatively stable, he is more patient, yielding a stable distribution.[16] This kind of multiplicity seems closely related to the thick markets externality described in Burdett and Coles (1997) which dates back to Diamond (1982), but the non-stationarity of the current environment adds a different flavor.

2.5 Unravelling and Costly Search with Patient Agents

In the market for entry-level professionals, many studies describe unravelling – an incentive to rush the market (e.g. Roth et al. (1992), Roth and Xing (1997), Li and Suen (2004)). The equilibria presented above do not feature this rushing of the market. Instead, agents wait patiently, smoothly filtering into the first class. To some extent, this is purely technological. The matching technology prevents a complete rushing of the market, as agents only occasionally meet a potential partner. But it is the strategic implications of search frictions that prevent unravelling more than the technology itself. When meetings are only occasional, everyone forecasts that at least a few attractive agents will have failed to match today, and so will be available to match in the future. This, then, allows for selectivity and so for

[16] The author has had no success in applying standard assumptions, such as log-concavity. These kinds of conditions do not seem to bite because, as $t \to T$, the entire shape of the distribution is important, so small initial changes in strategy may have large impacts in the future.

smoothly decreasing reservation values. High types, of course, would prefer to match with other high types, and the matching friction combined with a limited duration prevents them from doing so. Indeed, high types have a strict incentive to start searching earlier. What is less obvious, however, is that low types are either indifferent or prefer a longer duration.

Lemma 2.5.1. *When agents are patient, if the deadline is extended (or, equivalently, the market starts earlier), the extended market time-zero Pareto dominates the shorter market.*

That high types benefit from having more time to search for each other is clear. That low types do not mind the fact that they wait longer derives from patience. But if high types spend more time matching with each other, then when a low type does join the first class he or she samples from a worse distribution. They are exactly compensated for this by the higher probability of matching given the longer duration of the market.

To reduce the effect of search frictions, everyone would prefer that the market started earlier. Indeed, if agents could coordinate, the market would start at time minus infinity and would deliver perfect sorting. In the presence of search frictions, early matching serves to improve sorting rather than diminish it.

Moreover, it is exactly the anticipation effect which allows for this result. If meetings are too uniform and high types match out too quickly, then unravelling obtains. To this point, Damiano et al. (2005) consider a discrete-time version of the model here. In each period, each agent meets a partner randomly drawn from the set of unmatched agents. They show that, when there are participation costs and fewer rounds than types, the unique equilibrium involves complete unravelling – everyone accepts their first partner. This result derives from the uniformity of meetings. When all of the agents are paired in each period, one equilibrium is that everyone accepts their first partner, forecasting that the market will be empty next period. That no other equilibria exist derives from avoidable, costly search.

When search is costly and avoidable, low type agents opt out until they join the first class. That is, if one does not expect to match in a given period, one should wait outside of the market. This implies that, at any point in time, only first class agents participate. If meetings are uniform, if in each round every agent meets a partner, and all participating agents are mutually acceptable, then all will match and exit. Perforce, in the model with discrete and uniform meeting rounds, all of the first class agents at any time match out of the market. But the first class consists of exactly those types better than the expected type searching tomorrow less the search cost, and all of these exit today. So the best type left tomorrow must be worse than the average type tomorrow. No distribution has this property, everyone must have left today. The only equilibrium is complete unravelling.

If meeting rounds are not uniform and enough first class agents fail to meet a partner, this result breaks. Sorting can take place. Consider the continuous time model with random meeting times and patient agents, but suppose that in order to receive meetings at any time t, agents must incur a flow cost of c. This yields the following HJB equation:

$$\dot{U}(x,t) = -\max\left\{0, -c + \alpha(x,t)\int_{U(x,t)}^{\overline{x}} (z - U(x,t))G_x(dz,t)\right\}.$$

Proposition 2.5.2. *The equilibrium with $c > 0$ is totally determined by the reservation value of the highest type as in Proposition 2.3.2. Moreover, agents outside the first class do not participate, preferring to wait until they become acceptable to the highest type.*

Proof. Inspection of the HJB reveals non-increasing reservation values. A similar argument as above implies that $U(x,t) = x$ for $t < \tau(x)$. Hence, agents outside the first class find it unprofitable to search. □

As a point of clarification, the equilibrium does depend on costs. The characterization here is the same as in Proposition 2.3.2: all behavior can be summarized in terms of the reservation value of the highest type. This reservation value, however, is significantly affected both directly as it now includes costs but also indirectly because of the different population operating in the market. The important difference relative to the market without costs is that low types stay out of the market until they match. Since, when there are costs, all agents in at a given time are first class, all meetings result in matches. This tends to increase reservation values. On the other hand, costs have a direct negative effect on reservation values as they mimic impatience (as previously described in a steady state framework by Adachi (2003)).

In contrast to Damiano et al. (2005), notice that agents smoothly filter into the market no matter the magnitude of α (unless α is so small that it is not profitable to search at all). Hence, it is not a small expected number of meetings which leads to unravelling. Instead, the harsh strategic interaction induced by simultaneous and costly rounds of search leads to the stark results obtained in Damiano et al. (2005).

A final distinction is interesting. Far from destroying sorting, small search costs improve it. Even for vanishing search costs, less attractive types wait outside the market. This removes the search externality that low types exert on high types – without costs, the two meet although they are not do not match. With search costs, every meeting results in a match, thus increasing efficiency of the matching process. Costly search induces agents to "wait their turn," greatly improving the probability of a match for every single type, and also the sorting of types. When search costs are small, that the highest types prefer this arrangement is obvious – they trade a small flow cost for a discrete jump in match efficiency. That low types are indifferent or better off follows from the same logic as Lemma 2.5.1. The very lowest types are indifferent, receiving their own pizazz in expectation either way. That they match with a lower type in expectation (because high types match out faster) is exactly compensated for by an increased probability of matching. Medium types – those who are in the first class at time zero without search costs but not with them – are better off because, although they have to wait to join the first class, they receive a higher value when they do. Hence, small flow costs lead to a Pareto improvement over the no-cost model.

2.6 Conclusion

In this paper I explored the impact of a particularly harsh form of non-stationarity – a deadline – on a canonical matching model. I showed existence and characterized equilibria. Attractive individuals form a first class segment of the market whose members are all mutually acceptable. As the deadline approaches and the expected number of future meetings declines, this class expands. The model exhibits an "anticipation effect" for low types as they anticipate that their opportunity set will jump discretely when they join the first class. This drives less attractive agents either not to match at all before they join the first class or to become more selective, with increasing reservations before they join the first class. The two cases obtain when agents are patient or impatient, respectively. When agents are patient, the equilibrium is unique and a small cost of search both improves efficiency and sorting. The randomness of meeting opportunities prevents complete unravelling of the market as in Damiano et al. (2005) but still generates an incentive for early matching.

2.7 Omitted Proofs

Proof of Proposition 2.3.3 (Uniqueness). Suppose there are two equilibria ($U^L(t)$, $G^L(z,t)$) and ($U^H(t)$, $G^H(z,t)$) with $U^H(0) \geq U^L(0)$. The proof proceeds in three major steps. First, a likelihood ratio across the two equilibria is evaluated. From this one derives a mean life remaining ordering. This ordering, combined with the first step, implies a monotone likelihood ratio property which is used to show that the lower equilibrium is always flatter than the higher. Concluding, we find that the two equilibria can not both satisfy the terminal condition, so not both in fact satisfy equilibrium.

A word on notation: throughout, superscripts index the equilibrium from which the relevant object derives so that $\tau^L(x)$ solves $U^L(\tau^L(x)) = x$. Additionally subscripts indicate that $t = \tau(x)$ as $G^i_x(z) = G^i(z, \tau^i(x))$. Further, denote hazard rates with $r^i_x(z) = g^i_x(z)/(1 - G^i_x(z))$ and mean life remaining as $m^i_x(z) = \left(\int_z^{\overline{x}} (1 - G^i_x(y))\,dy\right) / (1 - G^i_x(z))$.

Also note that indeed we must have $U^H(0) > U^L(0)$, otherwise $U^H(t) = U^L(t)$ for all t as the dynamic for U is Lipshitz. Since G^0 posesses a density, so does $G^i(z,t)$ and we may write

$$\dot{g}^i(z,t) = \begin{cases} \alpha g^i(z,t)(1 - G^i(U^i(t),t))^2 & \text{if } z < U^i(t) \\ -\alpha g^i(z,t) G^i(U^i(t),t)(1 - G^i(U^i(t),t)) & \text{if } U^i(t) \leq z. \end{cases}$$

Integrating this yields

$$g^i(z,t) = g^0(z)\exp\left\{\alpha\left[\int\limits_0^{\min\{\tau^i(z),t\}} [1-G^i(U^i(s),s)]\,ds \right.\right.$$
$$\left.\left. - \int_0^t G^i(U^i(s),s)[1-G^i(U^i(s),s)]\,ds\right]\right\}.$$

Hence,

$$\frac{g_x^L(z)}{g_x^H(z)} = \frac{\exp\left\{\alpha\left[\begin{array}{l}\int\limits_0^{\min\{\tau^L(z),\tau^L(x)\}} [1-G^L(U^L(s),s)]\,ds \\ -\int\limits_0^{\tau^L(x)} G^L(U^L(s),s)[1-G^L(U^L(s),s)]\,ds\end{array}\right]\right\}}{\exp\left\{\alpha\left[\begin{array}{l}\int\limits_0^{\min\{\tau^H(z),\tau^H(x)\}} [1-G^H(U^H(s),s)]\,ds \\ -\int\limits_0^{\tau^H(x)} G^H(U^H(s),s)[1-G^H(U^H(s),s)]\,ds\end{array}\right]\right\}}.$$

This expression is continuous everywhere and differentiable except at $U^L(0)$, $U^H(0)$, and x. Noting that $d\tau^i(z)/dz = 1/\dot{U}^i(\tau^i(z))$ by the inverse function theorem, some algebra gives

$$\frac{d}{dz}\left[\frac{g_x^L(z)}{g_x^H(z)}\right] = \begin{cases} 0 & \text{if } z < x, \\ \alpha\left(\frac{1-G_z^L(z)}{\dot{U}_z^L} - \frac{1-G_z^H(z)}{\dot{U}_z^H}\right)\left(\frac{g_x^L(z)}{g_x^H(z)}\right) & \text{if } x < z < U^L(0), \\ -\alpha\left(\frac{1-G_z^H(z)}{\dot{U}_z^H}\right)\left(\frac{g_x^L(z)}{g_x^H(z)}\right) & \text{if } U^L(0) < z < U^H(0), \\ 0 & \text{if } z < U^H(0). \end{cases}$$

Further recalling that $\dot{U}_x^i = -\alpha \int_x^{\overline{x}} (1 - G^i(z,t))dz$, we see that this can be written as

$$\frac{d}{dz}\left[\frac{g_x^L(z)}{g_x^H(z)}\right] = \begin{cases} 0 & \text{if } z < x, \\ \left(\frac{1}{m_z^H(z)} - \frac{1}{m_z^L(z)}\right)\left(\frac{g_x^L(z)}{g_x^H(z)}\right) & \text{if } x < z < U^L(0), \\ \left(\frac{1}{m_z^H(z)}\right)\left(\frac{g_x^L(z)}{g_x^H(z)}\right) & \text{if } U^L(0) < z < U^H(0), \\ 0 & \text{if } U^H(0) < z. \end{cases}$$

Hence, we have a monotone likelihood ratio at $\tau(x)$ if $m_z^L(z) \geq m_z^H(z)$ for $z \in (x, U^L(0))$. Monotone likelihood ratios implies monotone hazard rates. And, in particular, if we set $x = U^L(0)$, then

$$\frac{d}{dz}\left[\frac{g_x^L(z)}{g_x^H(z)}\right] = \begin{cases} 0 & \text{if } z < U^L(0), \\ \left(\frac{1}{m_z^H(z)}\right)\left(\frac{g_x^L(z)}{g_x^H(z)}\right) & \text{if } U^L(0) < z < U^H(0), \\ 0 & \text{if } U^H(0) < z. \end{cases}$$

This implies that $r^H_{U^L(0)}(z) = r^L_{U^L(0)}(z)$ for $z > U^H(0)$ and $r^H_{U^L(0)}(z) > r^L_{U^L(0)}(z)$ for $z < U^H(0)$. Also, noting that $dm^i/dz = r^i m^i - 1$, it straightforward to derive that

$$m_x^i(z) = \int_z^{\overline{x}} \exp\left\{-\int_z^y r_x^i(s)ds\right\} dy.$$

This, combined with our inequality on r^i above, yields

$$m^L_{U^L(0)}(U^L(0)) > m^H_{U^L(0)}(U^L(0)).$$

On the way to a contradiction, suppose there exists some $x < U^L(0)$ such that $m^L_x(x) = m^H_x(x)$ and let $\tilde{x}$ denote the largest such crossing point. Because $\tilde{x}$ is the largest such x, $m^L_x(x)$ is continuous in x, and $m^L_{U^L(0)}(U^L(0)) > m^H_{U^L(0)}(U^L(0))$, we must have $m^L_x(x) > m^H_x(x)$ for all $x > \tilde{x}$. Hence, $g^L_x(z)/g^H_x(z)$ is increasing in z, and strictly so for $z \in (x, U^H(0))$ and $x \geq \tilde{x}$. This implies, for $x \in (\tilde{x}, U^L(0)]$, that $r^H_x(z) = r^L_x(z)$ for $z \in (U^H(0), \overline{x})$, and $r^H_x(z) > r^L_x(z)$ for $z \in [x, U^H(0))$. So, from our equation for m^i above, we must also have $m^L_{\tilde{x}}(\tilde{x}) > m^H_{\tilde{x}}(\tilde{x})$, our desired contradiction. We conclude that $m^L_x(x) > m^H_x(x)$ for all $x \in [0, U^L(0)]$, so that the likelihood ratio $g^L_x(z)/g^H_x(z)$ is increasing in z for all $x \in [0, U^L(0)]$. This, then, implies that $1 - G^L_x(z) \geq 1 - G^H_x(z)$ for all $x \in [0, U^L(0)]$ and $z \in X$, so that $\int_x^{\overline{x}}(1 - G^L_x(z))dz > \int_x^{\overline{x}}(1 - G^H_x(z))dz$ and $\dot{U}^L_x < \dot{U}^H_x$. Thus, since $T = \tau^i(0)$, we have

$$\begin{aligned} T = \int_{U^L(0)}^{0} \frac{d}{dz}\tau^L(z)dz &= \int_{U^L(0)}^{0} \frac{1}{\dot{U}^L_z}dz \\ &< \int_{U^L(0)}^{0} \frac{1}{\dot{U}^H_z}dz = \int_{U^L(0)}^{0} \frac{d}{dz}\tau^L(z)dz = T - \tau^H(U^L(0)) < T, \end{aligned} \tag{2.7.1}$$

a contradiction. We conclude that the equilibrium is unique. □

Proof of Lemma 2.4.4. Suppose $x_1 < x_2$ and fix t. Suppose $y \in \Omega(x_1, t)$ so that $x_1 \geq U(y, t)$. Then $x_2 \geq U(y, t)$, so $y \in \Omega(x_2, t)$. Hence $\Omega(x_1, t) \subset \Omega(x_2, t)$. The rest follows by Remark 2.2.1. $\square$

Proof of Corollary 2.4.5. For monotone U, note that if $x_1 \leq x_2$, then x_2 could simply choose $\mathcal{A}(x_2, t) = \mathcal{A}(x_1, t) \cap \Omega(x_1, t)$ and receive the same value as x_1. Hence, $U(x_2, t) \geq U(x_1, t)$. That Ω is connected follows from $x \geq U(z, t) \Rightarrow x \geq U(z', t)$ for all $z' < z$. $\square$

Proof of Proposition 2.4.2 (Existence). Without loss of generality, suppose $T = 1$. Further, write $\hat{m}(x, y, t)$ for the acceptability function: $\hat{m}(x, y, t) = 1$ if $y \in \Omega(x, t)$ and 0 otherwise. Next, write $m(x, y, t)$ for the matching function: $m(x, y, t) = \hat{m}(x, y, t)\hat{m}(y, x, t)$ which equals one if (x, y) are mutually acceptable at time t and zero otherwise. In what follows some function arguments, subscripts, etc. are dropped to save space when it does not cause confusion.

The proof is in several steps and closely follows Smith (2006). Given value functions $U(x, t)$, a continuous map $U \to m_U$ is defined (Lemma 2.7.1). Next, we show that $m \to G_m$ exists and is continuous (Lemma 2.7.2). Finally, closing the circle, define an operator, T, from the HJB equation, substituting in m_U and G_{m_U}, prove the existence of a fixed point for $U = TU$ – which is an equilibrium – using Schauder's fixed point theorem.

First, let $B \geq \max\{\overline{x}, \alpha\overline{x}\}$ be some fixed number and let

$$B_t = B \exp\left\{(r+\alpha)(1-t)\right\}.$$

Let $\mathcal{V}_t = \{f : X \to \mathbb{R} | 0 \leq f \leq \overline{x}, \|f\| \leq B_t\}$ where the norm is the total variation norm. I.e., $\mathcal{V}_t$ is a subset of the functions of bounded variation on X. Equip $\mathcal{V}_t$ with the weak-* topology.[17] Then, by Alaoglu's theorem, $\mathcal{V}_t$ is weak-* compact. And, by Tychonoff's theorem, $\mathcal{V} = \prod_{t\in[0,1]} \mathcal{V}_t$ is compact in the product topology. Since $\mathcal{V}_t$ is convex, $\mathcal{V}$ is convex under pointwise operations. $\mathcal{V}$ will be the space of candidate U used in the application of Schauder's Fixed point theorem.

Define $T : \mathcal{V} \to \mathcal{V}$ by

$$T(U)(x,t) = \int_t^1 \left(-rU(x,s) + \alpha \int_{\Omega_U(x,s)} \max\{0, z - U(x,s)\} G_U(dz,s) \right) ds$$

By Lemma 2.7.4, T is continuous. By Lemma 2.7.3, $T\mathcal{V} \subset \mathcal{V}$. Hence there exists a fixed point $U^* = TU^*$ by Schauder's Fixed Point theorem. $\square$

Lemma 2.7.1. *There exists a continuous map $U \to \hat{m}_U$ and a continuous map $U \to m_U$ both essentially unique.*

[17] To clarify, Let C be the set of continuous functions on X and BV be the set of functions of bounded variation on X. Of course, BV is isometrically isomorphic to the set of measures of bounded variation on X which is the dual of C by the Riesz Representation Theorem. The weak-* topology on BV is, then, the weakest topology where if $f \in C$ and $\mu \in BV$ then $\mu \to \int f d\mu$ is a continuous function for every f (this is also sometimes called the vague topology). Then, the weak-* topology on $\mathcal{V}_t$ is just the relative topology inherited from BV equipped with the weak-* topology.

Proof. Let $U_n \to U$ in $\mathcal{V}$. Smith (2006), in his Lemma 8(a), proves that, for fixed t, there exists a continuous map $U(\cdot, t) \to \hat{m}(\cdot, \cdot, t)$ and that this yields a continuous map $U(\cdot, t) \to m(\cdot, \cdot, t)$. Since $\mathcal{V}$ is equipped with the product topology in t, continuity for each t implies joint continuity of $U \to \hat{m}_U$ and $U \to m_U$. That these maps are only essentially unique follows from the fact that agents are indifferent over measure zero differences. But, as shown in Smith (2006), there exists but one m_U such that $U_n \to U$ implies $m_{U_n} \to m_U$ pointwise and it is this map which is selected. □

Lemma 2.7.2 (Fundamental Matching Lemma)**.** *There exists a continuous map $m \to G_m$ and it is unique.*

Proof. The Cauchy problem[18] is to find a G solving

$$\begin{aligned}\dot{G}(z,t) &= \alpha G(z,t)\left(\mathbb{E}_x[\theta(x,t)]\right.\\ &\qquad \left.- \mathbb{E}_x[\theta(x,t)|x \leq z]\right) - \eta(t)\left[G(z,t) - H(z,t)\right]\\ &\qquad \equiv F(t, G(\cdot,t))(z) \qquad (2.7.2)\end{aligned}$$

and $G(z,0) = G^0(z)$ where $\theta(x,t) = \int m(x,z,t)G(dz,t)$ is the probability that a meet will result in a match for x at time t. Existence and uniqueness follow from the Cauchy-Lipshitz theorem for which we need to check that F is bounded, measurable in t, and Lipshitz in G.

[18]This proof relies heavily the theory of ODE in Banach spaces. Statements and proofs of the relevant theorems can be found, for example, in Driver (2003).

Notice that since $m(x, y, t)$ is bounded and measurable, then both $\theta(x, t)$ and $\mathbb{E}(\theta, x, t))$ are bounded and measurable as well. If we equip $G(\cdot, t)$ with the weak-* topology (i.e. Lévy metric), then θ is continuous as a function of G and so F is continuous in G. Given that we are using the weak-* topology for G, it suffices to show that $F(t, G)$ has uniformly bounded variation. So, fix G and let $z_1, z_2 \in X$. Then $|F(t, G)(z_1) - F(t, G)(z_2)| =$

$$
\begin{aligned}
&|\alpha G(z_1, t) \left(\mathbb{E}_x[\theta(x, t)] - \mathbb{E}_x[\theta(x, t)|x \leq z_1]\right) - \eta(t) \left[G(z_1, t) - H(z_1, t)\right] \\
&- \left(\alpha G(z_2, t) \left(\mathbb{E}_x[\theta(x, t)] - \mathbb{E}_x[\theta(x, t)|x \leq z_2]\right) - \eta(t) \left[G(z_2, t) - H(z_2, t)\right]\right)| \\
&= |\alpha(G(z_1, t) - G(z_2, t))\, \mathbb{E}_x[\theta(x, t)] \\
&- \alpha(G(z_1, t)\, \mathbb{E}_x[\theta(x, t)|x \leq z_1] - G(z_2, t)\, \mathbb{E}_x[\theta(x, t)|x \leq z_2]) \\
&- \eta(t)(G(z_1, t) - G(z_2, t) - (H(z_1, t) - H(z_2, t)))| \\
&\leq |2\alpha + \eta(t)||G(z_1, t) - G(z_2, t)| + |\eta(t)||H(z_1, t) - H(z_2, t)|.
\end{aligned}
$$

where the last inequality follows because $|G|, |\theta| \leq 1$, and

$$
\begin{aligned}
&|G(z_1, t)\, \mathbb{E}(\theta|x \leq z_1) - G(z_2, t)\, \mathbb{E}(\theta|x \leq z_2)| \\
&\qquad = \left| \int_{z_1 \geq x \geq z_2} \left(\int m(x, y, t) G(dy, t) \right) G(dx, t) \right| \\
&\qquad\qquad \leq \left| \int_{z_2}^{z_1} G(dx, t) \right| \leq |G(z_1, t) - G(z_2, t)|.
\end{aligned}
$$

Since G and H are probability distributions, their total variation is one. So, if

$\bar{\eta}(t) = \sup_t \eta(t) \leq \bar{\zeta} N_0 \exp(\alpha)$, then $\|F\| \leq 2(\alpha + \bar{\eta})$. Thus, there exists a solution. For uniqueness, consider the following: Fix two distributions, G_1 and G_2. Given the calculation on θ above, we have

$$\|G_1(\cdot,t)\,\mathbb{E}(\theta_{G_1}(x)|x \leq \cdot) - G_2(\cdot,t)\,\mathbb{E}(\theta_{G_2}(x)|x \leq \cdot)\| \leq \|G_1 - G_2\|$$

and note that θ is Lipshitz in G: $\|\theta_{g_1}(x) - \theta_{g_2}(x)\| = \|\int m(x,y,t)(G_1(dy,t) - G_2(dy,t))\| \leq \|G_1 - G_2\|$, hence any definite integral of θ is Lipshitz in G, and so is any other Lipshitz function of θ. Hence,

$$N_G(t) = \int_0^t \exp\left(\alpha \int_s^t \mathbb{E}_G\,\theta_G(x,\tau)\,d\tau\right)\zeta(s)\,ds$$

is Lipshitz in G and, finally, $\eta_G(t) = \zeta(t)/N_G(t)$ is Lipshitz in G because $N_G(t) \geq N_0 \exp(-\alpha T)$ (I.e. there are always more people in the economy than if all matches were accepted over all time). Thus, since F is a composition of Lipshitz functions, it is Lipshitz. Hence, the solution is unique and continuous in m. $\square$

Lemma 2.7.3 (Uniform Boundedness)**.** *If $U \in \mathcal{V}$, then $TU \in \mathcal{V}$.*

Proof. We need $0 \leq TU(x,t) \leq \overline{x}$ and $TU(\cdot,t)$ to have total variation less than B_t. Simple boundedness is obvious, so focus on bounding the total variation. Let $U \in \mathcal{V}$, $t \in [0,1]$, and $x_1 < x_2 \in X$ be arbitrary but fixed. We will bound $|TU(x_1) - TU(x_2)|$ and then sum over all partitions to obtain a bound for the total variation of TU. Write $\Delta_{x_1,x_2} = \Omega(x_1) \setminus \Omega(x_1)$ (recall $x_1 < x_2 \Longrightarrow \Omega(x_1) \subseteq \Omega(x_2)$).

Break up the second integral in TU into two pieces $Q_1(x_1, x_2)$ and $Q_2(x_1, x_2)$ as follows.

$$\int_{\Omega(x_2)} \max\{0, z - U(x_2)\} G(dz) - \int_{\Omega(x_1)} \max\{0, z - U(x_1)\} G(dz)$$
$$= \underbrace{\int_{\Omega(x_1)} \max\{0, z - U(x_2)\} - \max\{0, z - U(x_1)\} G(dz)}_{\equiv Q_1(x_1, x_2)}$$
$$+ \underbrace{\int_{\Delta} \max\{0, z - U(x_2)\} G(dz)}_{\equiv Q_2(x_1, x_2)}$$

Now, because $|\max(a_1, b_1) - \max(a_2, b_2)| \leq |a_1 - a_2| + |b_1 - b_2|$, we have

$$|Q_1(x_1, x_2)| \leq \int_{\Omega(x_1)} |0 - 0| + |U(x_1) - U(x_2)| G(dz) \leq |U(x_1) - U(x_2)|$$
$$\leq B_t |x_1 - x_2|.$$

And,

$$|Q_2(x_1, x_2)| \leq \int_{\Delta} |\max\{0, z - U(x_2)\}| G(dz) \leq \overline{x} \int_{\Delta} G(dz).$$

Continuing,

$$|TU(x_2) - TU(x_1)|$$
$$= \left| \int_t^1 \left(r(U(x_2) - U(x_1)) - \alpha \left(\int_{\Omega(x_2)} \max\{0, z - U(x_2)\} dG \right.\right.\right.$$
$$\left.\left.\left. - \int_{\Omega(x_1)} \max\{0, z - U(x_1)\} dG \right) \right) ds \right|$$

$$\leq \int_t^1 r|U(x_2) - U(x_1)| + \alpha\left(|Q_1(x_1,x_2)| + |Q_2(x_1,x_2)|\right) ds$$

$$\leq \int_t^1 \left(rB_s|x_1 - x_2| + \alpha B_s|x_1 - x_2| + \alpha\overline{x}\int_\Delta G(dz) \right) ds$$

Substituting in for B_t, one obtains

$$\int_t^1 \left((r+\alpha)Be^{(r+\alpha)(1-s)}|x_1 - x_2| + \alpha\overline{x}\int_\Delta G(dz) \right) ds$$

$$= -B|x_1 - x_2|\left(1 - e^{(r+\alpha)(1-t)}\right) + \alpha\overline{x}(1-t)\int_\Delta G(dz).$$

Hence, summing over all possible partitions of X,

$$\|TU\| = \sup_{\{x_i \in X\}} \sum_{x_i} |TU(x_i) - TU(x_{i-1})| \leq B|\overline{x} - \underline{x}|\left(e^{(r+\alpha)(1-t)} - 1\right) + \alpha\overline{x}(1-t) \leq B_t.$$

□

Lemma 2.7.4 (Continuity)**.** *T is continuous.*

Proof. Fix $U, U_n \in \mathcal{V}$ with $U_n \to U$. Recall that $\mathcal{V}$ has the product topology in the t dimension and the weak-* topology in the x dimension. Hence, $U_n(x,t) \to U(x,t)$ pointwise in t and a.e. pointwise in x. And, because $0 \leq U_n, U \leq \overline{x}$, the dominated convergence theorem gives convergence in L^1 in both x and t. To show continuity, we need $TU_n \to TU$ weak-* for each t. A sufficient condition for convergence is that, for each t, $\int_I |TU_n(x,t) - TU(x,t)|dx$ for every measurable $I \subset X$. But, since $0 \leq TU \leq \overline{x}$, we need only show a.e. pointwise convergence (again by the dominated convergence theorem). We will divide $|TU_n - TU|$ into several pieces and apply the triangle inequality. While there are many expressions, the division looks at the two terms of T and decomposes the change in each into (1) a part from the change in Ω, (2) a part from the direct change in U, and (3) a part from the change in G. Define the following:

$$Q_1(x,s,n) = \left(\alpha \int\limits_{\Omega_U(x,s)} G_U(dz,s) + r\right) U(x,s) - \left(\alpha \int\limits_{\Omega_{U_n}(x,s)} G_U(dz,s) + r\right) U(x,s),$$

$$Q_2(x,s,n) = \left(\alpha \int\limits_{\Omega_{U_n}(x,s)} G_U(dz,s) + r\right) U(x,s) - \left(\alpha \int\limits_{\Omega_{U_n}(x,s)} G_U(dz,s) + r\right) U_n(x,s),$$

$$Q_3(x,s,n) = \left(\alpha \int\limits_{\Omega_{U_n}(x,s)} G_U(dz,s) + r\right) U_n(x,s) - \left(\alpha \int\limits_{\Omega_{U_n}(x,s)} G_{U_n}(dz,s) + r\right) U_n(x,s),$$

$$W_1(x,s,n) = \int\limits_{\Omega_U(x,s)} \max\{z, U(x,s)\} G_U(dz,s)$$

$$- \int\limits_{\Omega_{U_n}(x,s)} \max\{z, U(x,s)\} G_U(dz,s),$$

$$W_2(x,s,n) = \int\limits_{\Omega_{U_n}(x,s)} \max\{z, U(x,s)\} G_U(dz,s)$$

$$- \int\limits_{\Omega_{U_n}(x,s)} \max\{z, U_n(x,s)\} G_U(dz,s),$$

$$W_3(x,s,n) = \int\limits_{\Omega_{U_n}(x,s)} \max\{z, U_n(x,s)\} G_U(dz,s)$$

$$- \int\limits_{\Omega_{U_n}(x,s)} \max\{z, U_n(x,s)\} G_{U_n}(dz,s).$$

Note, then, that

$$TU(x,t) - TU_n(x,t) = \int_t^1 \left(\sum_i Q_i(x,s,n) - \alpha \sum_i W_i(x,s,n) \right) ds.$$

Consider each term in turn. Because $\hat{m}_{U_n} \to \hat{m}_U$ pointwise almost everywhere,

$$|Q_1(x,s,n)| = \alpha U(x,s) \left| \int \hat{m}_U(x,z,s) - \hat{m}_{U_n}(x,z,s) G(dz,s) \right| \to 0$$

for a.e. (x,s). Because $U_n(x,s) \to U(x,s)$ pointwise a.e.,

$$|Q_2(x,s,n)| = |U(x,s) - U_n(x,s)| \left| \alpha \int_{\Omega_U(x,s)} G_U(dz,s) + r \right| \to 0$$

for a.e. (x,s). Next, because $G_{U_n}(z,s) \to G_U(z,s)$ weak-* for a.e. s, we have

$$|Q_3(x,s,n)| = \alpha U_n(x,s) \int_{\Omega_{U_n}(x,s)} |G_{U_n}(dz,s) - G_U(dz,s)| \to 0 \text{ for a.e. } (x,s).$$

The same arguments apply for $|W_i|$, $i = 1,2,3$. Hence,

$$\int_t^1 \sum_i |Q_i(x,s,n)| + \alpha \sum_i |W_i(x,s,n)| ds \to 0$$

for a.e. x again by dominated convergence, so that $|TU(x,t) - TU_n(x,t)| \to 0$ for a.e. x. This, then, gives $\int_I |TU(x,t) - TU_n(x,t)| dx \to 0$ for every t. □

Lemma 2.7.5. *For fixed G and $r = 0$ the dynamic for U is Lipshitz continuous.*

Proof. When $r = 0$, we need only consider the dynamic for $\overline{x}$ which we will write as $\dot{U} = L(U) = -\alpha \int_U^{\overline{x}} (1 - G(z)) dz$. Then, fixing U_1 and U_2, we have

$$\|LU_1 - LU_2\| = \alpha \left\| \int_{U_1}^{U_2} (1 - G(z)) dz \right\| \leq \alpha \|U_1 - U_2\|.$$

So the dynamic has a Lipshitz constant of α.

□

Proof of Proposition 2.4.1. Because $U(x, \tau(x)) = x$, agent's utility is bounded below by $xe^{-r(\tau(x)-t)}$ (i.e. agents can do no worse at any time than deciding not to match, instead waiting to join the first class). Hence, for all t where an agent is not in the first class, his utility is bounded below by $x\exp\{-r(T-t)\}$. Letting $r \to 0$, $U(x,t) \geq x$ for all t when x is not in the first class. I.e. as $r \to 0$, we obtain an equilibrium where low agents wait to join the first class. $\square$

Proof of Lemma 2.4.14. Since, for $t < \tau(x)$, we can write

$$U(x,t) = xe^{-r(\tau(x)-t)} + \alpha \int_t^{\tau(x)} e^{-r(s-t)} \int_{U(x,t)}^{y(x,t)} (G(y(x,t)) - G(z))\,dz\,ds$$

we have

$$\begin{aligned} U(x,t) &\geq xe^{-r(\tau(x)-t)} \\ &\geq xe^{-r(\hat{\tau}(x)-t)} \\ &= x\left[1 - \frac{x}{\underline{x}}\left(1 + \frac{r}{\alpha}\right)\right]^{-\frac{r}{r+\alpha}} e^{-r(T-t)} \equiv \hat{U}(x,t) \end{aligned}$$

where the last line comes from substituting in for $\hat{\tau}(x)$ as defined in Lemma 2.4.12. Since $U(y(x,t),t) = x$, we have

$$\begin{aligned} y(x,t) \leq x\left[1 - \frac{y(x,t)}{\underline{x}}\left(1 + \frac{r}{\alpha}\right)\right]^{\frac{r}{r+\alpha}} e^{r(T-t)} &\leq x\left[1 - \frac{x}{\underline{x}}\left(1 + \frac{r}{\alpha}\right)\right]^{\frac{r}{r+\alpha}} e^{r(T-t)} \\ &\equiv \hat{y}(x,t). \end{aligned}$$

Write $P(x,t) = G(y(x,t)) - G(U(x,t))$ and $V(x,t) = \mathbb{E}(z|y \geq z > U)$ so that

$$\dot{U}(x,t) = (r + \alpha P(x,t))U(x,t) - \alpha P(x,t)V(x,t).$$

Now, $V(x,t) < y(x,t)$ so that

$$\begin{aligned}\dot{U}(x,t) &\geq (r + \alpha P(x,t))\hat{U}(x,t) - \alpha P(x,t)\hat{y}(x,t) \\ &\geq (r + \alpha P(x,t))x \left[1 - \frac{x}{\underline{x}}\left(1 + \frac{r}{\alpha}\right)\right]^{-\frac{r}{r+\alpha}} e^{-r(T-t)} \\ &- \alpha P(x,t)x \left[1 - \frac{x}{\underline{x}}\left(1 + \frac{r}{\alpha}\right)\right]^{\frac{r}{r+\alpha}} e^{r(T-t)} \\ &= x(r\hat{\lambda} + \alpha P(x,t)(\hat{\lambda} - \hat{\lambda}^{-1}))\end{aligned}$$

if one writes

$$\hat{\lambda} \equiv \left[1 - \frac{x}{\underline{x}}\left(1 + \frac{r}{\alpha}\right)\right]^{\frac{r}{r+\alpha}} e^{-r(T-t)}.$$

Then, since $t < \tau(x)$, $\hat{\lambda} \leq 1$, and so $\dot{U}(x,t) \geq 0$ if

$$\frac{(r/\alpha)\hat{\lambda}^2}{1 - \hat{\lambda}^2} \geq P(x,t).$$

And, since $P(x,t) < 1$, the result obtains.

$\square$

Proof of Lemma 2.5.1. Suppose $T' > T$ are two deadlines, and that $\bar{U}'$ and $\bar{U}$ are the equilibrium reservation values of the highest type under each deadline. The same logic as in the proof of Proposition 2.3.3 shows that $\bar{U}'(0) > \bar{U}(0)$ (whichever reservation value starts lower must hit 0 earlier, and so it must be $\bar{U}$).

Those in the first class under the extended duration get $\bar{U}'(0)$ instead of $\bar{U}(0)$, an improvement. Moreover, all $x < \bar{U}(0)$ are indifferent between the two equilibria, because they get their own pizazz in expectation under both. Those who were first class in the short duration market but are not in the long duration instead get their own pizazz. This is an improvement, as they were getting only $\bar{U}(0)$ with the short duration – the definition of being in the first class at time zero. $\square$

Chapter 3

Directed Search for Wage-Tenure Contracts with Adverse Selection

This study considers a dynamic labor market where workers are privately informed about their attachment to the labor force and firms competitively post contracts to direct workers' search. This extends the static results on adverse selection in competitive search markets of Guerrieri et al. (2010) to a dynamic environment with on the job search à la Shi (2009). Characterizing the dynamic contracting problem of firms and the search problem of workers, I show that equilibria feature full separation, increasing wage profiles, and "job lock" for committed (long duration) workers, reducing their frequency of transitions relative to a full information benchmark.

Workers differ in their commitment to the labor market. This can stem from a variety of sources such as fluctuations in household income, the demands of home life, or changing career interests. One prominent example is the difference between those who plan to stay in the labor force after having a child and those who plan to exit. One modeling approach, when group differences exist but individual preference is unobserved, leans on statistical discrimination to explain differences in labor market outcomes. If firms pay only flat wages or lack commitment power, this is, perhaps, the only approach. But if firms can promise long term contracts, heterogeneity in expected tenure induces heterogeneity over tenure-wage profiles. Separation may obtain, negating the need for statistical discrimination. Moreover, if this separation is sufficiently costly in terms of efficiency, the same empirical group differences in wages which motivate models of statistical discrimination in the labor market may obtain.

If firms indeed were to pursue this strategy, what sort of effects can a search equilibrium produce with regards to relative wage levels, their trajectories, job-to-job transitions, and the career ladder? What are the effects of anti-discrimination policies? I provide a first attempt to address these questions in the context of a modern model of the labor market featuring competitive, directed search for wage-tenure contracts under adverse selection. An interesting result is that anti-discrimination rules have no impact on women, but instead serve only to distort the market for men. Another is that career-oriented women suffer from a form of job-lock. The efficiency cost of offering contracts which separate long and short duration agents is amortized over the life of the employment contract. A job-to-job transition, then, involves incurring the this cost again, which reduces the frequency of turnovers.

Beyond the current application, the methods developed below are more generally applicable to competitive search for dynamic contracts with adverse selection. First, equilibria can be found as the solution to a recursive social planner's problem as in Guerrieri et al. (2010). Second, it maintains block-recursivity as in Shi (2009). And, third, while differences in labor market histories induce continuous heterogeneity, and hence a continuum of incentive compatibility conditions, I show that, in equilibrium, only one binds for each firm. When formulating contracts, firms need only need only consider the incentives of workers who face job queues of equal length.

The next section outlines related literature, followed by a description of the framework, a benchmark full information equilibrium, the equilibrium with adverse selection, and some concluding remarks.

3.1 Some Related Models

As oceans of ink have been spilled on the subject of gender differences in the labor market, I restrict myself here to discussing some direct antecedents of the present model so as to highlight the implications of the various modeling choices.

Salop and Salop (1976) develops a prototypical version of the model I consider. Turnover is assumed costly because of up-front training costs on the part of the firm. If prospective employees know their probability of quitting, the firm would want to elicit this information because low-quitters are more valuable. The authors show that if workers are risk neutral and have access to capital markets, an employment fee is optimal, and that the efficient allocation can be recovered in a competitive equilibrium.

Stevens (2004) models homogeneous and risk neutral agents in a labor market with random search where firms write tenure-based contracts to limit turnover, workers are ex-ante identical who search on the job. The equilibrium wage profile is a step contract paying the minimum wage for a period and then jumping up to marginal product. The result can be thought of as an extension of Salop and Salop (1976) where worker cash flows are restricted to be non-negative – a lump sum payment up front is infeasible.

Burdett and Coles (2003) considers the framework of Stevens (2004) but with risk averse agents. Risk aversion induces firms to provide smooth contracts, but, so as to reduce turnover, firms pay wages that increase steadily over time. This result highlights the moral-hazard quality of job-to-job moving. A similar flavor is provided by Lazear (1981), albeit a rather different environment, which shows that firms should offer increasing wage profiles to encourage high effort over a career. Burdett and Coles (2003) has two unfortunate properties both stemming from the assumption of random search. First, individual decisions depend upon the entire distribution of offers which makes the equilibrium somewhat difficult to analyze. More damming, as all jobs are posted in the same market, the distribution of job-to-job transitions includes one-step jumps from near the very bottom to near the very top of the market. This is perfectly acceptable if one understands the model as a single, narrowly defined, corner of the labor market, but it seems unreasonable that there exist positive probability of stepping out of McDonald's and into the board room.[1]

The basic framework for my model was developed in Shi (2009) – henceforth Shi – who adapts the theory of Burdett and Coles (2003) to an environment with directed search. This results in bounded job-to-job transitions, and simplifies individual decisions.

The next most important ingredient for my analysis comes from Guerrieri et al. (2010) – henceforth GSW – who develop a theory of competitive search in the presence of adverse selection. As first illustrated by Rothschild and Stiglitz (1976), competitive equilibria in the presence of adverse selection may suffer from non-existence. I adopt the notion developed by GSW who extend the notion of competitive search equilibrium, first developed by Moen (1997), to the case of adverse selection in a static model. They show that equilibria can be computed as the solution of a sequence of principal-agent problems. This program produces the directed

[1]Although the model I follow, Shi (2009) has something of the opposite problem: all job transitions are a deterministic function of a worker's current wage.

search analogue of a least cost separating contract. Moreover, it maintains the usual property of least-cost separating contracts, that the lowest type receives the same allocation as he would have had there been no information friction. The analogue with Rothschild and Stiglitz (1976) is especially strong in the current environment, as exogenous exit in this market mimics the probability of a loss. Unfortunately, the static notion of GSW does not generalize in a straightforward manner to dynamic problems. The key insight developed below is that the sequence of problems derived by GSW do not require a fully static environment, but only static preferences over contracts on both sides of the market. And while, in the current model, preferences over contracts depend upon individual dynamics, that dependence can be summarized by the agent's current value. Hence, the program of GSW can be used to generate optimal individual decisions contingent upon a continuation, which is then embedded into a fixed point problem, producing a solution to the dynamic model. Given a current value, the worker's search decision is static. The firm's dynamic problem can be decomposed into a promise keeping part and a promise making part, where the solution of the former produces static preferences over the latter. This last observation was first made in the context of wage-tenure contracts by Stevens (2004), but the optimal control argument necessary in the current environment was developed by Burdett and Coles (2003) in an environment with random search.

3.2 Framework

This is a model of the labor market with directed search on the job and wage-tenure contracts. The primitives follow Shi with one exception. Workers differ in their labor market attachment, which I model as differences in exogenous quit rates. I reproduce the details of the model for convenience and to fix notation. A longer discussion can be found in Shi.

3.2.1 Model Primitives

A mass of workers derive flow utility from wealth according to $u(w)$ which is assumed $C^2[0, \infty]$, increasing, and concave with $\lim_{w \to 0} u'(w) = \infty$. Workers may neither borrow nor save.[2] Employed workers produce flow output y for their employer and consume their wages; unemployed workers receive $b < y$ from home production or some other source outside the model. Workers, whether employed or unemployed, continuously search for better employment, as described below. Workers can not commit to contracts. More specifically, a worker will always quit to any better job and may at any time quit to unemployment.

There are two types of workers, men and women. Denote these M and W; i will index types, $i \in \{M, W\}$, throughout. The two types differ only in their Poisson intensity of exogenously exiting the labor market: $\delta_W > \delta_M$. Exogenous exit leaves a worker with continuation value normalized to 0 and exiting workers are replaced

[2] Although equilibrium will imply that saving would be sub-optimal, the no borrowing constraint is significant.

by inflows of equal mass to maintain constant population. Exogenous exit should not be interpreted simply as death. It acts as a stand in for the arrival of any life event that would cause a worker to abandon the market. The payoffs from such events need not be modeled provided that they are sufficiently large that no one would ever continue working after the arrival of a δ event. Of course, there are many intermediate life events which would only cause some workers to exit. If a worker's parent becomes ill and requires elder care, high wage earners may choose to hire help and continue to work while low wage earners might exit. This dependence would complicate the model and since the main thrust is to explore the role of adverse selection in a dynamic context, I maintain the simpler assumption that differences are over *exogenous* exits. I discuss the possibility of enriching the environment at greater length, along with other extensions, below.

On the other side of the market, a large mass of infinitely lived, risk neutral, and identical prospective firms compete for workers. To obtain a worker, a firm must advertise a job, incurring a flow cost of k. Advertisements consist of a wage-tenure contract, a function $w(t)$ indicating the wage to be paid at every tenure $t \in [0, \infty)$ so long as the worker stays at the firm and, if discrimination is allowed, the required gender of an applicant. Write x for a contract and $w_x(t)$ for the wage paid by that contract at tenure t. A filled job with contract x provides flow value $y - w_x(t)$ to the firm. Both workers and firms discount the future at a common rate $\rho > 0$. Crucially, assume that firms possess perfect commitment power; that workers outside offers are assumed unobservable, so can not be contracted on; and that there can be no renegotiation. Whether firms can or would discriminate is also of central concern. I will consider both the case where firms can perfectly discriminate – by detailing the required gender of a worker in advertisements – and the case where such discrimination is disallowed either because the information is unavailable (blind offers) or because it is non-contractible (no-discrimination laws).

Matching between workers and advertised jobs operates in a continuum of sub-markets. Each market is characterized by the wage contract advertised in that market, its tightness, and the characteristics of its applicants. Write θ_x for the tightness in the market offering contract x. This is the density of searching workers divided by the density of advertised jobs. Given a tightness, an individual worker meets a firm with Poisson intensity $\tilde{p}(\theta)$ while firms receive meetings with Poisson intensity $\tilde{q}(\theta) = \theta\tilde{p}(\theta)$. It will be convenient to eliminate θ and write simply $q(p)$ or $p(q)$. While we refer the reader to Shi for a detailed enumeration of assumptions on various model primitives, it suffices for now to require $p(q)$ be strictly decreasing and concave, and that q is bounded above by $\bar{q} = q(0)$. The grand market, the collection of all the sub-markets, is characterized by the set of contracts, $\mathcal{X}$, the job-finding rate $p(x)$ for each contract, and the relative proportion of M and W searching in each market, γ_x^i. At any point in time, a worker receives offers exclusively in the market in which they search. Each worker may only search in a single market at any time. And firms may only advertise a job with contract x in the appropriate market. As I only consider equilibrium in steady state, no market or aggregate variables depend on time.

3.2.2 Worker's Problem

Given these preliminaries, workers choose where to direct their search, whether to accept a contract offered to them in a meeting with a firm, and whether to quit into unemployment if employed. Note that, no matter their employment status, a worker's current decisions depend only on their current value in that state. Write V_u^i for the value of unemployment, and $V_x^i(t)$ for the value of employment at tenure t under contract x for $i = M, W$. Workers consider the set of available markets in which to search and choose optimally given their current state. Given a set of contracts $\mathcal{X}$ and associated job finding rate $p(x)$, a worker with current value V^i searches to maximize the excess value of a new offer:

$$S_i(V^i) = \max_{x \in \mathcal{X}} p(x)(V_x^i(0) - V^i).$$

For now, suppose that a solution exists and write $x = F_i(V^i)$ for the maximizing choice. Searching for x generates meetings at a rate $p(x)$. Given a worker searching for x meets with a firm advertising x, the worker will accept the offer – otherwise they would have searched for a different contract. Further, suppose that if a worker searches in a market, firms in that market accept (which will be the case in equilibrium). Workers would never search in a market where they expect to be rejected by all firms. And since firms are identical, the statement that all firms accept is simply the statement that firms do not randomize in their acceptance decisions.[3]

Because it will be used extensively below, write $r_i = \delta_i + \rho$. Given optimal search decisions, the value of employment under x at t is given by

$$r_i V_x^i(t) = u(w(t)) + S(V_x^i(t)) + \dot{V}_x^i(t).$$

The (mortality adjusted) asset-value of employment, $r_i V_x^i$, must equal the dividend $u(w(t))$ plus the flow value of changing jobs plus the change in the value. Since $w(t)$ need not be constant, there will indeed be non-zero change in a worker's value. But when unemployed, workers receive a constant benefit, so that the value of unemployment is constant. Specifically,

$$r_i V_u^i = u(b) + S(V).$$

Turn now to the firm's primitive problem.

3.2.3 Firm's Values

Firms who have a type i worker with tenure t under contract x have no decisions to make going forward as they are comitted to the contract. They may not end or

[3]This does not preclude discrimination. Rather, it requires that firms include any discriminatory policy in their advertisements. If the matching technology had increasing returns to scale, it might be optimal to induce workers to search in a market, but then reject their application. I ignore this possibility, and other related considerations, by simply assuming that the advertisement is binding on firms.

amend the relationship. They anticipate, however, that their worker will end the relationship at some point, either exogenously or because of a competing offer. A worker with current value $V_x^i(t)$ exits at a constant rate δ_i and quits to a competing offer at a rate $p(F(V_x^i(t)))$. The value of the worker to the firm, $J_x(t)$, must solve

$$\rho J_x^i(t) = y - w(t) + \dot{J}_x^i(t) - (\delta_i + p(F(V_x^i(t))))J_x^i(t).$$

This equation simply states that the asset value of an employee is equal to the flow product, y, less wages, $w(t)$, plus the change in the asset value less the probability of loosing the worker times the magnitude of the loss. Write $R_i(V) = \rho + \delta_i + p(F_i(V))$ for the interest-rate adjusted Poisson intensity of separation. Suppressing the contract, the probability a worker survives to tenure t is given by

$$\psi_i(t) = \exp\left\{-\int_0^t R_i(V_x^i(s))\,ds\right\}.$$

Conditional on survival at tenure t, the probability of surviving to $t+\tau$ is simply $\psi_i(t+\tau)/\psi_i(t)$. Assuming bounded wages, the firm's value function can be integrated as

$$J_x^i(t) = \int_t^\infty \frac{\psi_i(s)}{\psi_i(t)}\left[y - w(s)\right] ds.$$

The profit to a firm of hiring a worker under contract x is $J_x(0)$. But before a worker can be hired, the contract must be advertised and a worker found. Offering a contract x yields an applicant of type i at rate $\gamma_i q(p(x))$ and then payoff of $J_x^i(0)$, together yielding a flow of $\sum_i \gamma_i q(p(x)) J_x^i(0)$ which must be greater than or equal to the flow cost of advertisement, k. Competition will drive this to equality.

3.3 Equilibrium with Perfect Discrimination

When perfect discrimination is allowed, so that a worker's type is contractible, there is perfect separation and the presence of multiple types is somewhat irrelevant. The equilibrium for each type is the same as if there were only one. To see that the equilibrium can be made fully separating, note that if both types were to search in a single market, firms must make equal profits from both. Otherwise, the firm could announce a slightly more generous contract restricted to the more profitable type, increasing profits. Given workers provide equal profit, and the matching technology is assumed homogeneous, simply splitting the market in accordance with relative populations will provide the same matching rate, but be separating.

Shi solves the model when all workers share a common δ.[4] The following summarizes part of his argument. The market with two types is, then, simply two independent markets alongside each other. Suppose for the moment that firms, instead of full wage contracts, promise a value to workers and then optimally choose the wage to provide that value; this yields $V_x(0) = x$. Write $p(x)$ and $q(x)$ for the matching rates for worker and firm in the market providing value x. Suppose that $p'(x) < 0, p''(x) < 0$ so that $S(V) = \max_x p(x)(x - V)$ has a unique maximizing

[4]As I expand on his equilibrium, I maintain his assumptions 1 and 2 which ensure existence and regularity of equilibrium.

value $F(V)$. Let $[\underline{x}, \overline{x}]$ be the set of offered utilities and suppose that $\bar{q} = q(\overline{x})$ – the maximum matching rate for firms is given to the firm promising the highest utility. Notice that if a firm promises $\overline{x}$, he faces no voluntary exits: $R(\overline{x}) = \rho + \delta = r$. As is easy to see, then, the profit maximizing contract offered by the highest firm is a flat wage, $\bar{w}$ which yields the firm a value of

$$\underline{J} \equiv J_{\bar{w}}(0) = \frac{y - \bar{w}}{r}.$$

Since even the most generous employer must cover their advertising costs, zero profit implies that $\bar{q}\frac{y-\bar{w}}{r} = k$ so that $\bar{w} = y - rk/\bar{q}$. A worker employed at $\bar{w}$ will then enjoy a value $\overline{x} = u(\bar{w})/r$. And as a worker can always quit to unemployment and stay forever, a lower bound on possible utilities is $\underline{x} = u(b)/r$.

Given these bounds, for each $x \in [\underline{x}, \overline{x}]$ consider the firm's problem conditional on having just matched with a worker at a promised utility for the worker of x.

$$\max_{w(\cdot)} \int_0^\infty \phi(s)[y - w(s)]\, ds \qquad \text{s.t. } x = V_w(0).$$

This is an optimal control problem with state variable V with dynamic defined by the worker's Bellman equation. Shi shows that $\lim_{t\to\infty} w(t) = \bar{w}$ in equilibrium. This implies that for every $x \in [\underline{x}, \overline{x}]$, there is some τ_x such that $V_w(\tau_x) = x$. Next, by the principal of optimality, it must be the case that if $w(\cdot)$ solves the firm's problem for some x_0, then for all $x \in [x_0, \overline{x})$, there exists some τ_x such that $\tilde{w}(t) = w(t + \tau_x)$

solves the firm's problem given a promise of $x > x_0$. In other words, there is a baseline wage profile. And given any promised value, the firm provides this value by offering the baseline wage at some starting time. This observation significantly simplifies the structure of equilibrium. One need only solve for a single wage profile, and then promised utilities are given by starting times along this profile.

As derived by Shi, optimality conditions for the firm yield the following differential equations which, together with the Bellman equations for the worker, yield w, J and V.

$$\dot{w} = \frac{[u'(w)]^2}{u''(w)} J(t) \left[\frac{dp(F(V))}{dV} \right], \qquad \dot{J}(t) = -\frac{\dot{V}}{u'(w)} \tag{3.3.1}$$

with boundary $J(\infty) = \underline{J}$, $V(\infty) = \bar{V}$, and $w(\infty) = \bar{w}$. These, along with zero profit for the firm and optimal search by the agent define the conditions of equilibrium for Shi. Off equilibrium beliefs are easy to derive given the structure of firms' optimal offers. For $x \geq \overline{x}$, firms make negative profit and so these are not offered or, more specifically, workers accept an infinite tightness for those contracts. Similarly, no worker would ever accept any $x < V_u$ so these are not searched for. The only question concerns contracts in $[V_u, F(V_u))$ as these contracts are not part of equilibrium but can not be ruled out for promising negative value. Instead, equilibrium specifies (common) beliefs concerning the tightness that would obtain if those markets were to operate. This is given by the zero profit condition for the firm. Specifically, the system which produced the wage contract can be run backward giving w and hence J for any promised utility in $[V_u, F(V_u))$. So, if $x \in [V_u, F(V_u))$, let $q(x) = k/J_x$. Competition guarantees that if a market were to operate it would give zero profit. Hence, the ratio of workers to jobs is set such that exactly that obtains. Existence derives from construction of a fixed point on w for this system.

Note that although I have presented the equilibrium with $\delta_i = \delta$, the case where $\delta_W > \delta_M$ will produce an equilibrium consisting of two parallel ladders, one for M and one for W, which satisfy the same conditions with the small adjustment that advertised contracts specify the type of applicants. As the relationship between model parameters is complicated, and the equilibrium can not be exhibited constructively, comparative statics are challenging. Some points, however, are obvious. First, with $\delta_W > \delta_M$, it must be that $\bar{w}_W < \bar{w}_M$. Simply, since M are expected to be attached for longer, they have a higher expected lifetime product and hence a taller wage ladder. Second, zero profit implies that, for any contract x_i, $q(x_i)J_{x_i} = k$, so the job filling rate exactly pins down firms' values. But then, since M are more productive, if $q(x_M) = q(x_W)$ then the expected present value wage bill under x_M must be greater than under x_W.

As briefly mentioned before, pooling can not be a part of any equilibrium where discrimination is allowed, because every wage profile produces strictly greater profit for firms employing M than W. Hence, offering a wage slightly higher and specifying M would attract every M participant in the pooling contract and produce higher profit. When discrimination is allowed, off-equilibrium contracts are easy to rule out. But when the composition of agents who would search for a deviating contract, if offered, is endogenous, the problem is more delicate. GSW provide an algorithm which both detects equilibrium offers and specifies deviation payoffs precisely.

3.4 No Discrimination: The Case of Adverse Selection

Suppose now that firms do not observe an agent's type before entering into the relationship. Our equilibrium definition almost exactly follows GSW. Let $\mathcal{W} = \{w : [0, \infty) \to [0, y] | \dot{w} > 0\}$ be the set of admissible wage functions and $\mathcal{W}^* \subset \mathcal{W}$ for the set of contracts offered in equilibrium with a measure H on $\mathcal{W}^*$ specifying the proportion of the population searching for each w. For each $w \in \mathcal{W}$, write $\gamma_M(w) = 1 - \gamma_W(w)$ for the proportion of i types searching in the market offering w. Let $\Theta : \mathcal{W} \to [0, \infty]$ describe the tightness in each market. For any value V, the equilibrium specifies a search strategy $F_i(V)$ which is individually rational for workers. Given Θ and F_i, firms make non-positive profit at any w and exactly zero profit in any contract in $\mathcal{W}^*$. Static adding up: $\int_{\mathcal{W}^*} \gamma_i(w)/\Theta(w)\,dH(w) = n_i$ where n_i is the number of workers of type i. Dynamic adding up: transitions implied by worker turnover, exit, and entry, together with optimal search, generate H.

The last condition does not appear in GSW as theirs is a static framework. But, as is common in directed search equilibria with free entry, the aggregate distribution of agents plays no role in individual decisions. Hence, once equilibrium decision rules have been calculated, the aggregate distributions are generated mechanically. Shi refers to this property as *Block Recursivity*. Moreover, the above statement suppresses the dependence of search decisions on a worker's current state, but this comes in through the distribution H: a worker's current state implies a search decision which is reflected in H.

Here, I describe the solution method derived by GSW and proceed to apply the idea to the current environment. To be concrete, suppose the two types are M and W with static utilities $U^i(y)$ over a set of abstract contracts $y \in Y$ with firm's profits $v(y|i)$. Given a matching function $q(p)$ as above, GSW suggest the following program. First, solve

$$\bar{S}^W = \max_{p,y \in Y} \left\{ pU^W(y) \quad \text{s.t.} \quad q(p)L(y) \geq k \right\}$$

and then

$$\bar{S}^M = \max_{p,y \in Y} \left\{ pU^M(y) \quad \text{s.t} \quad q(p)L(y) \geq k, \quad \text{and} \quad pU^W(y) \leq \bar{S}^W \right\}.$$

They show, given assumptions detailed in their paper, that this program will produce contracts and matching rates that satisfy free entry and optimal search. They go on to use this solution to specify matching rates for off-equilibrium contracts which make deviations unprofitable, so that the equilibrium is indeed profit maximizing.

The problem solves for the optimal static value of search. In the current model, the value of search to a worker is not static, but a flow: $p(x)(x-V)$. But the search decision does not depend upon dynamic considerations beyond the current state, V. Moreover, in the section above, while a firm's optimal wage contract depends upon the dynamic flow of quitters, $\delta_i + p_i(F_i(V))$, there exists a fixed wage contract and the firm merely choose which segment as a function of the initial promised utility. Hence, the firm has static preferences over promised utility: $J_i(x)$. So long as the function $J_i(x)$ can be identified outside of the problem, the GSW program can be solved for any initial utility V.

The first cornerstone of my construction rests on the observation that one can select an equilibrium such that the low type, W, remains untouched by the introduction of a higher type. This is done by construction. Given the equilibrium for W alone, I construct markets for M which do not attract any W. Because there were no deviations when W were alone, and the new contracts added to the market are selected exactly to exclude W, there are still no deviations. Hence, the solution of the first problem in GSW's program, the problem for W, was solved in the last section. Let $p_W(x)$, $F_W(x)$, and $w_x^W(\cdot)$ be the matching function (as a function of promised value), optimal search decision, and wage profile derived by Shi given $\delta = \delta_W$, as described above. This yields $S_W(V) = p_W(F_W(V))(F_W(V) - V)$.

Now consider the problem of finding the optimal contract and matching rate for

M. The incentive compatibility constraint represents a significant problem. As W and M value contracts differently, the firm can not solve for a single wage profile independently. The difference in preferences between M and W is fundamental for separation. Contracts intended for M can not simply be indexed by their value, because the entire wage profile is required to satisfy incentive compatibility on the part of W. Moreover, there is now a continuum of W at different scales in the wage ladder, so, in principle, a continuum of conditions might need to be checked.

I resolve the issue in two steps. First, for an arbitrary wage profile, $w(\cdot)$, write $V^W(t|w)$ for the value to W of the contract w at tenure t. For any (p, w) pair, incentive compatibility requires $S_W(V) \geq p(V^W(0|w) - V)$ for every $V \in \mathcal{X}^W$. Write this as

$$V^W(0|w) \leq \min_{V \in \mathcal{X}^W} \left\{ V + \frac{S_W(V)}{p} \right\} \equiv \bar{U}^W(p). \tag{3.4.1}$$

$\bar{U}^W(p)$ defines the maximum value any wage contract can deliver to a W and still satisfy incentive compatibility. The equilibrium value of search for W, $S_W(V)$, is decreasing and convex (see Shi) hence

Lemma 3.4.1. *Given p, the maximum value that can be delivered to W solves $p_W(\bar{U}^W(p)) = p$. Further,*

$$\frac{\partial \bar{U}^W(p)}{\partial p} = \frac{1}{p_W'(\bar{U}^W(p))} < 0 \quad \textit{and} \quad \frac{\partial^2 \bar{U}^W(p)}{\partial p^2} = -\frac{p_W''(\bar{U}^W(p))}{[p_W'(\bar{U}^W(p))]^3} < 0$$

In particular, for $p = 0$, $\bar{U}^W(0) = u(\bar{w}_W)/r_W$

Proof. Observe that

$$\frac{\partial}{\partial V}\left[V + \frac{S_W(V)}{p}\right] = 1 - \frac{p_W(F(V))}{p} = 0 \Longrightarrow p_W(F(V)) = p.$$

The derivatives are given by the Inverse Function Theorem. □

Although the correct constraint has been identified, it still depends on the entire wage contract. The intuition was that we could solve for the wage contract first, and then solve the optimal search problem over promised values. This leads me to include the incentive compatibility constraint in the firm's problem directly. Suppose that, in equilibrium, M can secure values in $[\underline{x}_M, \overline{x}_M] = \mathcal{X}^M$ with match rate $p_M(x)$, and corresponding optimal search $F_M(V)$ yielding excess value of search $S_M(V) = p_M(F_M(V))(F_M(V) - V)$. Consider a firm's problem when constrained to offer a value of x to M and no more than $\bar{U}^W(p)$ to W.

$$J^*(x,p) = \max_{w(\cdot)} \int_0^\infty \psi^M(s)[y - w(s)]ds$$

$$\text{s.t} \qquad \dot{\psi}_M = -\left[r_M + p_M(F_M(V))\right]\psi_M \tag{3.4.2}$$

$$\dot{V}^M = r_M V^M - u(w) - S_M(V^M) \tag{3.4.3}$$

$$\dot{V}^W = r_W V^W - u(w) - S_W(V^W) \tag{3.4.4}$$

$$V^M(0) = x, \qquad V^W(0) \leq \bar{U}^W(p), \qquad \psi(0) = 1. \tag{3.4.5}$$

To each state ψ^M, V^M, V^W associate co-state Λ^ψ, Λ^M, Λ^W. The Hamiltonian is

$$H = \psi^M(y-w) - \Lambda^\psi \psi^M \left[r_m + p_m(F_m(V^M))\right] + \sum_i \Lambda^i \left[r_i V^i - u(w) - S_i(V^i)\right] \tag{3.4.6}$$

Write $\Gamma^i = \Lambda^i/\psi^M$ and note that $\dot{\Lambda}^i = \dot{\Gamma}^i \psi_M - \left[r_M + p_M(F_M(V))\right]\psi_M \Gamma^i$. Optimality implies

$$-1 - u'(w)\left[\sum_i \Gamma^i\right] \leq 0 \quad \text{and } w \geq 0 \quad \text{(comp. slack)} \tag{3.4.7}$$

Noting that $S_i'(V) = -p_i(F_i(V))$,

$$-\dot{\Lambda}^M = -\Lambda^\psi p_M'(F_M(V))F_M'(V)\psi^M + \Lambda^M(r_M + p_M(F_M(V)))$$

which, after substitution of Γ^M, gives

$$\dot{\Gamma}^M = \Lambda^\psi p_M'(F_M(V))F_M'(V). \tag{3.4.8}$$

$$-\dot{\Lambda}^W = \Lambda^W \left[r_W + p_W(F_W(V))\right]$$

which we can integrate and obtain

$$\Gamma^W(t) = \Gamma^W(0)\frac{\psi^W(t)}{\psi^M(t)} \quad \text{with} \quad \Gamma^W(0) \geq 0, \quad \Gamma^W(0)(V^W(0) - \bar{U}^W(p)) = 0.$$

Here, $\Gamma^W(0)$ measures the shadow value of deterring W so that $\Gamma^W(0) > 0$ if the incentive compatibility constraint is binding. Continuing,

$$-\dot{\Lambda}^{\psi} = (y - w) - \Lambda^{\psi}[r_M + p_M(F_M(V))]$$

$$\implies (y - w)\psi^M = \Lambda^{\psi}\psi^M[r_M + p_M(F_M)] - \dot{\Lambda}^{\psi}\psi^M = -\frac{d}{dt}\left[\Lambda^{\psi}\psi^M\right].$$

Given the transversality condition $\lim_{t\to\infty}\psi^M(t)\Lambda^{\psi}(t) = 0$, and $\lim_{t\to\infty} J(t) = 0$, integrating both sides yields $J(t) = \Lambda^{\psi}$. And since $u'(0) = \infty$, $w > 0$ so (3.4.7) implies $\Gamma^M = -1/u'(w) - \Gamma^W$ and so $\dot{\Gamma}^M = -\dot{w}u''(w)/(u'(w))^2 - \dot{\Gamma}^W$. Substituting these into (3.4.8) and recognizing that $d(p_M(F_M(V)))/dV = -S''_M$ yields

$$-JS''_M = [\delta_W - \delta_M + p_W - p_M]\,\Gamma^W + \frac{u''(w)}{(u'(w))^2}\dot{w}. \tag{3.4.9}$$

Note, this condition differs from (3.3.1) by the term involving Γ^W. Γ^W is the shadow value on the incentive compatibility constraint, and the term in the bracket is the difference in quit rates between W and M. Where it is positive, the wages for M are steeper than optimal. In particular, by definition of $\bar{U}^W(p)$, we have $p_W = p_M$ at tenure zero in equilibrium. So whenever incentive compatibility binds, the wages for M are steeper than optimal, at least at tenure zero.

It can be shown that $H = 0$ for all t. Given this, substituting (3.4.2) into H and then J for Λ^{ψ} yields $H = \psi^M\left[\sum_i\left(\Gamma^i\dot{V}^i\right) - \dot{J}\right]$. Together with (3.4.7), this yields

$$\dot{J} = -\frac{\dot{V}^M}{u'(w)} + \Gamma^W\left[\dot{V}^W - \dot{V}^M\right]. \tag{3.4.10}$$

Compare this with (3.3.1). When $\Gamma^W = 0$, the equation carries the interpretation that an increase in the worker's value with tenure exactly matched the decrease in the firm's value, where u' serves to adjust units. When incentive compatibility binds, there is a wedge between the two.

Equations (3.4.10), (3.4.9), the Bellman equations for workers and firm, along with the solution for Γ^W provide a system of differential equations which, along with boundary and transversality, determine the system. Let $J^*(x,p)$ be the maximized value for the firm.

The most valuable payoff to M, $\overline{x}^M$, is given by $J^*(\overline{x}^M, 0) = k/\bar{q}$. This value depends only on model primitives because a worker in the best job does not search. Notice, this will not be a flat contract because it must satisfy incentive compatibility. Further, define the lowest value as $\underline{x}^M = u(b)/r_m$, and write $\mathcal{X}^M = [\underline{x}^M, \overline{x}^M]$. Equilibrium requires that each $x \in \mathcal{X}^M$ be assigned a wage profile and a matching rate $p_M(x)$ that satisfy incentive compatibility and are optimal for both workers and firms.

Having embedded the incentive compatibility constraint into the firm's problem, I provide first order conditions governing the optimal choice of p, x for every $V \in \mathcal{X}^M$ given a matching function p_m to be used in the firm's problem. The GSW program:

$$\mathcal{P}_{\text{GSW}}(V|p_m, F_m): \qquad S_M(V) = \max_{p,x} \{p(x - V) \quad \text{s.t.} \quad q(p)J^*(x,p) = k\}.$$

Letting μ be the multiplier on the zero profit constraint, the first order conditions are

$$x - V = -\mu \left[q'(p) J^*(x,p) + \Gamma^W(0) q(p) (d\bar{U}^W(p)/dp) \right],$$

$$p = \mu q(p) \left[\Gamma^W(0) + \frac{1}{u'(w(0))} \right],$$

$$q(p) J^*(x,p) = k.$$

Where I have substituted $\partial J^*/\partial x = \Gamma^M(0)$ and $\partial J^*/\partial U^W = \Gamma^W(0)$.

For each V and given p_m and F_m, $\mathcal{P}_{\text{GSW}}(V|p_m, F_m)$ yields promised utility, $x = \hat{F}(V)$ and job finding rates $\hat{p}(x)$. The equilibrium will then be a fixed point in this mapping. Let $\bar{p}$ solve $q(\bar{p})(y-b)/r_M = k$; $\bar{p}$ is upper bound on p: a firm making the maximum feasible payoff must suffer a compensating amount of competition which induces high job-finding.

Let

$$\mathfrak{P} = \left\{ p \in C[\mathcal{X}^M] | 0 \leq p \leq \bar{p}, p(\overline{x}^M) = 0, p \text{ weakly decreasing, weakly concave} \right\},$$

$$\mathfrak{F} = \left\{ F \in C[\mathcal{X}^M] | \underline{x}^M \leq F \leq \overline{x}^M, F(x) \geq x, \text{ weakly increasing} \right\}.$$

And define $\Omega : \mathfrak{P} \times \mathfrak{F} \to \mathfrak{P} \times \mathfrak{F}$ by $(\hat{p}, \hat{F}) = \Omega(p, F) = \left\{ (\hat{p}, \hat{F}) \text{ solves } \mathcal{P}_{\text{GSW}}(\cdot | p, F) \right\}$. Our equilibrium for M will be a fixed point of Ω. As of yet, however, an existence proof eludes me. In the discrete time analogue of this model, the proof of Menzio and Shi (2010) applies with little alteration. I am in the process of formulating a limiting argument to obtain existence in the continuous time model where the convenient characterization of the wage contracts is available.

3.5 Discussion and Plans for Future Work

The analysis above remains somewhat incomplete. In this section, I provide some discussion, conjecture, and plans to improve upon this work.

While the model at hand is possessed of some elegance and simplicity, its solution is not. Moreover, any econometric or serious calibration study would require the model be expanded to allow for a great deal more flexibility, only increasing its intractability. A relatively straightforward extension would allow for n different separation rates. This kind of equilibrium could be solved iteratively in the same way that the problem for M was solved conditional on the outcome for W in this paper. Other extensions – such as heterogeneous productivity or values from unemployment – seem like they would be more complex, but have been accommodated in similar models such as Burdett et al. (2011).

As already mentioned, I have yet to prove existence of the equilibrium but that hurdle seems surmountable. Somewhat more frustratingly, I have had surprising difficulty in deriving results comparing the two wage ladders in the no-discrimination case. I am specifically interested in the relative levels and slopes of wage profiles by gender. The original idea for this project was that, since separation probabilities induce a single crossing over wage-profiles, self-selection should be supported in a directed search environment. And, hence, one should expect different contracts to separate types, and these separating contracts to generate heterogeneous labor market outcomes. Specifically, one should expect steeper contracts for M. As of yet, I have only proved this for the best contract offered.

That low turnover workers might want to self-select into steep wage contracts is far from novel, having been discussed in Salop and Salop (1976), and in a more general discussion of the kinds and nature of discrimination, by Stiglitz (1973). But modern equilibrium search models provide for rich equilibrium effects even with seemingly vanilla primitives. Re-casting old intuition in these new models has the possibility of bringing more realistic features without resort to more contrived examples.

While I glossed over this fact in the discussion above, the fixed point problem will only yield strategies and beliefs within equilibrium. But beliefs need to be specified for any feasible contract, not just those acted upon in equilibrium. Constructing these beliefs for the discriminatory economy was much easier because agents shared preferences and deviations could be restricted to one-dimensional set. To be well defined, equilibria in directed search environments require beliefs over the tightness and composition not just of markets operating in equilibrium, but all possible markets. This prevents incompleteness and coordination issues described in Delacroix and Shi (2006) (i.e. workers do not search for a desirable contract x because they believe no firm will offer it, and no firm offers it because every firm believes that no worker would search for it). As pointed out by GSW, specifying out of equilibrium tightness serves to refine the set of equilibria. My equilibrium construction only provides tightness for contracts offered in equilibrium. I have yet to solve for a supporting set of off-the-path beliefs. GSW provide an algorithm, but theirs is a finite context; I have yet to find a suitable generalization.

Finally, as I mentioned above while introducing the nature of gender heterogeneity, the analysis could be extended to allow for endogenous quits. Indeed, it can. If, for example, a worker faces an α arrival of a “life event” that would yield flow utility z drawn from some distribution $G(z)$, but would require the worker to quit. Then the exit rate the firm faces changes to $R = r + p(F(V)) + \alpha[1 - G(V)]$ and would affect worker’s value functions in an analogous way. This opens up the interesting possibility that a group with, ex-ante, a relatively high probability of exit, might receive steeper wage profiles than others so as to encourage staying. And then, if the population consisted of individuals differing in the distribution of outside options they receive, one would have a qualitatively similar adverse selection problem, but with a richer moral hazard problem attached. This would seem to be the correct model in which to address the differences between self-selection and statistical discrimination if some women, for example, were more or less career oriented, as described by different distributions for the exogenous outside option.

Perhaps the most important direction for further research is an examination of the vast empirical literature. While certain facts seem clear – the higher quit rates for women in aggregate – there are subtleties – when sufficient controls are included, differential quit rates may disappear. But since my model should deliver predictions linking labor force attachment and the trajectory of wages and careers, testing it will require I gather stylized facts regarding those variables.

3.6 Conclusion

While the analysis remains preliminary, the model I develop seems capable of producing a variety of predictions regarding various labor market outcomes with relative parsimony: a single parameter difference, when combined with equilibrium search, produces rich equilibrium objects. Until I develop more concrete results with the model, however, the most significant contribution is methodological. I illustrate an example of how, in some situations, the purely static framework of GSW can be extended to dynamic economies of adverse selection.

Chapter 4

Noisy Search for Multiple Products[1]

Firms offer a variety of goods and consumers search for low prices on a basket of different goods. As a consumer visits different stores in a search for multiple products, at any individual firm, they can chose to purchase all of their desired goods, or just to purchase one, planning to purchase the rest elsewhere. Hence, consumers purchasing decisions depend both on the sum of prices in a basket, but also on the individual prices themselves. The resultant purchasing behavior differs from simple single product search. The focus of this study is on the implications of this behavior on equilibrium pricing decisions, and the dispersion of prices both within and across firms.

Prices are disperse. The law of one price mostly never holds. *Prima facia*, one

[1]Kenneth Burdett is a co-author on this project.

might suppose that this is easily explained by the fact that information is costly and consumers do not know where the best prices are and so firms may set different ones. Search is, however, not suffcient to deliver dispersed prices – this is the so-called Diamond (1971) paradox. If a firm knows a consumers' reservation value, they can charge it, leaving consumers with no surplus. If consumers must pay to search, they know they will simply be charged their reservation price and so do not search in the first place. Simple, homogeneous, costly sequential search leads firms to post equal prices, and to no sales.

This negative result can be broken in a variety of ways, but most all require sacrificing the homogeneity that induced firms to know, and be willing to charge, consumer's reservation values. One plausible and tractable model requires no ex-ante heterogeneity across consumers, but instead supposes that some consumers can be lucky in their search, and simultaneously receive more than one price offer. This is the noisy search model of Burdett and Judd (1983). In this model, the monopoly power of search highlighted by Diamond is tempered by possibility that consumers may have another offer. So long as some consumers have only one offer while some others have more than one, firms must act as though they are in an auction with an unknown number of bidders, and so randomize their prices. Price dispersion is an equilibrium object, and so rationalizes consumer's need to search. It is this search protocol we embrace in this study.

If instead of retail markets, one considers the labor market, that a job has many dimensions is largely unimportant. Insofar as consumers must take all of the characteristics at one job, only the derived utility of the bundle matters – that one has high pay and another good benefits is inconsequential. Similarly, if a consumer must purchase, for example, all of the elements for an audio system from one retailer because of compatibility, only the bundle price matters, and so single-product models are appropriate. Similarly, if each individual retailer offered only a single good, so long as preferences are sufficiently separable there will be no interaction between the different goods. But when firms must post prices for each good, and consumers decide which they would like to buy, one must consider an explicitly multi-product model.

4.1 Some Literature

The first paper to consider the multi-product search, Burdett and Judd (1983), solved the consumer's decision problem given a distribution of prices, in extreme two cases – one where consumers have free recall of previous prices, and the other where there is no recall. While Burdett and Judd (1983) solved only the two-good case with no recall, Carlson and McAfee (1984) further characterize the N good case and derive various comparative statics. Consumers

These early studies focused on unit demands – as we will in this paper. In the single product setting this is something of a normalization. Whether consumers have unit demands or non-degenerate downward sloping curves, consumer will follow a reservation price strategy. Anglin and Baye (1987) and Anglin (1990) shows that, in general, in the multi-product case, this is no longer true. Gatti (1999) provides sufficient conditions on the utility function to regain the reservation price property.

Other studies avoid this issue by constraining the consumer's ability to sequentially search. Shelegia (2012) and Zhou (2014) consider a duopoly case, so that search histories have at most two entries. Zhou obtains results in a model of horizontal differentiation in a simlar spirit to Weitzman (1979) and characterizes firms' pricing decisions. He shows that, in this setting, there can be "joint search effect" which can induce firms to lower their prices in response to higher search costs. Shelegia (2012) is much closer to the current study, as consumers have unit demands and consumer engage in noisy search. Indeed the simplest case we describe here is reproduced almost exactly there. But that study is interested in the impact of consumers' joint valuation of goods, i.e. the impact of substitutes and complements on price distributions. And specifically, reservation values are taken as given. In the current study, we derive these as the result of strategic sequential search.

4.2 The Model

This is a model of a retail market for goods A and B. Generically, write i for a good and, when necessary, j for the other good. The market is populated by a unit mass of firms, and some mass of consumers. A mass $q \in [0, 1/2]$ of firms can produce good A at constant marginal cost c instantaneously on demand. Similarly, a mass q of firms can produce only good B at a constant marginal cost c instantaneously on demand. Call these one-good-firms (1f's). Finally, a mass $1 - 2q$ of firms can produce both A and B, each at constant marginal cost c instantaneously on demand. Call these two-good-firms (2f's). Firms must post prices, may not bundle, nor discriminate in any other way.

Each good is worth z to a consumer who demands it, and consumers demand at most one unit of each good. A mass n_1^0 of consumers enter the market demanding one unit of good A. Similarly, a mass n_1^0 of consumers enter the market demanding one unit of good B. Collectively call these one-good-buyers (1b's). Finally, a mass n_2^0 of consumers enter the market demanding a single unit of both goods. Call these two-good-buyers (2b's). The search protocol of consumers is a hybrid of standard costly sequential search without recall from, e.g., Lippman and McCall (1976) and the noisy search protocol of Burdett and Judd (1983). Specifically, some consumers initially have free contact with either one or two sellers (the noisy search protocol) but may reject these initial offers and search sequentially, paying a cost k to contact a new firm at random with out recall.

The model is essentially static, but consider the following timing for the purposes of exposition. Each day firms and consumers are born. At the end of the day they die, collecting payoffs. Consumers search to maximize expected utility given their belief about the joint distribution of prices – surplus from purchases less total search costs. Firms set prices to maximize profits over the course of the day – number of sales times markup over cost. Each day is divided into a countably infinite number of rounds, numbered $0, 1, 2, \ldots$. No party discounts during the day. In round zero, firms set prices which they can not change during the day. In round one, consumers begin their search. Some proportion $\beta^0 \in (0, 1)$ of the 1b's are freely in contact with one firm, each drawing at random from the set of firms. A complementary proportion of the 1b's, $1 - \beta^0$, are freely in contact with two firms and may choose to buy from either firm – there is no cost to purchasing one good from each firm. Similarly, a proportion $\alpha^0 \in (0, 1)$ of 2b's have free contact with one firm in round one, and $1 - \alpha^0$ of 2b's have free contact with two firms. All consumers who have satisfied their demand exit at the end of the period, and any consumer may also choose to exit even if they have not satisfied their demand.[2] In round two and all subsequent rounds, if a consumer has not previously exited, they must pay k, and then randomly contact a new firm, losing contact with previous firms (there is no recall), and may buy any good the firm offers at its posted price, and either exit or continue to the next round. Equilibrium, then, requires the distribution of posted prices equal beliefs, consumers purchase optimally, and firms profit maximize given consumers' purchasing behavior.

[2]An option which would be exercised if the expected value of continued search is negative.

4.2.1 The Consumers' Problems

Consumers solve an optimal stopping problem given their belief about the joint distribution of prices posted by firms, $F_{A,B}(p_A, p_B)$ with marginals F_A and F_B. For convenience, treat a 1f who does not sell a good i as selling it, but at an infinite price. As noisy search is assumed only to occur in the first round, a consumer who is lucky enough to see two sets of prices in the first round faces the same problem going forward as one who was not so lucky. Noisy search induces competition among firms, but does not change reservation values relative to sequential search.[3] Burdett and Malueg (1981) provide formal solution of the sequential search problem without recall in the case of two products. We summarize the result here for convenience.

Consumers follow an optimal stopping rule with three reservation values written W, R_A, and R_B, corresponding to the three possible states in which searching consumers may find themselves – searching for both goods (2b's), searching only for A (having already purchased B for the 2b's or never having demanded B in the first place), or searching only for B. R_i is the standard reservation value from single product search for good i (see, e.g. Lippman and McCall (1976)). Consumers pay at most their expected cost of continued search, or their reservation value, whichever is lower.

[3]In their original work, Burdett and Judd (1983) consider the case where consumers sequentially conduct noisy search, with the possibility of seeing one or two prices in each round of search. In the multi-product case, this would requires integration over of the joint distribution of the minimum of two price draws which is less tractable. This is left for future work.

$$R_i = \min\{z, k + \int_{\underline{p}_i}^{R_i} p_i dF_i(p_i) + (1 - F_i(R_i)R_i\}. \tag{4.2.1}$$

If $R_i = z$, then search is not profitable – consumers drop out after their first (free) search.

W is the highest amount a consumer would pay for a basket of both goods. That is, given a menu of prices (p_A, p_B), the consumer chooses to buy one, both, or neither goods, paying (in expectation, assuming continued search is profitable)

$$\min\{p_A + R_B, R_A + p_B, p_A + p_B, W\}$$

If one writes $Q_A = W - R_B$, $Q_B = W - R_A$, then 2b's will buy according to the following strategy:

1. Purchase A and B if $p_A + p_B \leq W$ and $p_i \leq R_i$.

2. Purchase A but not B if $p_A \leq Q_A$ and $p_B > R_B$.

3. Purchase B but not A if $p_B \leq Q_B$ and $p_A > R_A$.

4. Otherwise, purchase neither and search again.

Given this strategy, one derives W analogously to R_i as

$$W = \min \left\{ 2z, k + \int_{\substack{\{p_A+p_B \le W, \\ p_i \le R_i\}}} (p_A + p_B)\, dF_{A,B}(p_A, p_B) \right.$$
$$+ \int_{\substack{\{p_A \le Q_A, \\ p_B > R_B\}}} (p_A + R_B)\, dF_{A,B}(p_A, p_B) + \int_{\substack{\{p_B \le Q_B, \\ p_A > R_A\}}} (p_B + R_A)\, dF_{A,B}(p_A, p_B)$$
$$\left. + \int_{\substack{\{p_A+p_B > W, \\ p_i > Q_i\}}} W\, dF_{A,B}(p_A, p_B) \right\} \qquad (4.2.2)$$

As a practical matter, solving this equation is less straightforward than in the case of single product search. One can, however, derive most of the same comparative statics. These, along with the search problem for N goods, are explored by Carlson and McAfee (1984).

4.2.2 Firms' Problem

In their price setting decision, firms trade off price against sales. Noisy search induces a downward sloping demand curve – the higher a firms' price, the greater the probability that a noisy searcher is quoted a lower price from another firm. Further, firms must decide whether to target 1b's, requiring only that $p_i \le R_i$, or 2b's with the additional constraint that $p_A + p_B \le W$. This last is, however, not strictly true – as noisy searchers see two firms prices, they can make a basket costing less than W, allowing firms to sell to 2b's that would not buy without this other firm.

Firms must form beliefs concerning consumers' reservation values, and the number of consumers of each type who visit their store over the course of the day. Write N for the number of consumer-rounds per firm over the course of the day. That is, if, say, there were 1 consumer per firm in the first round and all consumers exit each period with probability r, then the total number of consumer-rounds per firm would be $1/r$. Write $n_1 = n_A = n_B$ for the proportion of these consumer-rounds with demand for only A and the proportion with demand for only B (which equal one another). Consumers who originally demanded both goods may buy one while continuing to search for the other, and so may become one-good-buyers. As it should not cause confusion, refer to these as 1b's. Write $n_2 = 1 - 2n_1$ for the proportion who demand both goods. Further, write $\alpha \in (0, 1)$ for the proportion of the $N \cdot n_2$ consumer-rounds of two-good-consumers in contact with only one firm, and $1 - \alpha$ for the proportion who simultaneously contact two firms. Similarly, write $\beta \in (0, 1)$ for the proportion of the $N \cdot n_1$ consumer-rounds of one-good-buyers in contact with only one firm, and $1 - \beta$ for the proportion in simultaneous contact with two firms.[4] It will be convenient to have some notation. Write

$$n_1^s \equiv (1 - \beta)n_1, \qquad n_2^s \equiv (1 - \alpha)n_2, \qquad \text{and} \qquad n^s \equiv n_1^s + n_2^s$$

[4]Notice, these are not the same as α^0 and β^0 as these are the proportion of first round searches with single contact. Whereas α and β are the proportion of all searches with single contact throughout the day (which is the relevant quantity for firms' pricing decisions).

for the one-good-consumers with simultaneous contact, two-good-consumers with simultaneous contact, and total proportion, respectively. Similarly, write

$$n_1^c \equiv \beta n_1, \qquad n_2^c \equiv \alpha n_2, \quad \text{and} \quad n^c \equiv n_1^c + n_2^c$$

for the proportion of consumes with single contact. As some consumers see two sets of prices, the total number of visitors a firm expects over the course of the day is $N(n^c + 2n^s)$.

Given consumers' reservation values, W, R_A, R_B, firms sell to different subsets of consumers as a function of their price levels. A 1f selling i with price less than Q_i can sell to both the 1b's demanding i and also to 2b's, but will only sell to noisy searchers whose alternate price offer is higher.[5] If this 1f prices between Q_i and R_i it will sell to any 1b desiring i, will never sell to 2b's seeing only one set of prices, but may sell to a 2b seeing another set of prices. This last occurs if the alternate firm prices i high, but j low enough to make a basket with its competitor costing less than W. That is, if a 1f offers p_i on good i, and a noisy searcher is in contact with an alternate firm pricing at (p_i', p_j') and $p_i < p_i'$ and $p_i + p_j \leq W$, the 1f will sell good i to that consumer. Write $X_A(p_A) \equiv F_B(W - p_A) - F_{A,B}(p_A, W - p_A)$ (similarly define $X_B(p_B)$), for the probability of selling in such a situation. These observations lead to the following profit function for the 1f.

[5] We'll ignore the possibility of the tie, as equilibria price distributions will be atomless.

Remark 4.2.1. A 1f selling good A faces a profit function

$$\pi_1(p_A) = N[n^c + 2n^s(1 - F_A(p_A))](p - c)$$

if $p \leq Q$. If, instead, $Q_A < p_A \leq R_A$, the profit function is given by

$$\pi_1(p_A) = N[n_1^c + \underbrace{2n_1^s(1 - F_A(p_A))}_{\text{Noisy 1b}} + 2\underbrace{n_2^s X_A(p_A)}_{\text{Noisy 2b}}](p - c).$$

Finally, profit is zero for $p_A > R_A$. A similar result holds for 1f selling good B.

A 2f firm faces a more complicated problem. They sell to all 1b's so long as $p_A \leq R_A$ and $p_B \leq R_B$, and the 1b does not have a better offer. If $p_A + p_B \leq W$, they additionally sell a good to all 2b without a better offer on that good. Finally, if $p_A + p_B \geq W$, they may sell one good to noisy searcher 2b who have an offer making a basket, as was the case with 1f's above. This yields the following:

Remark 4.2.2. Recalling that $X_A(p_A) \equiv F_B(W - p_A) - F_{A,B}(p_A, W - p_A)$, the profit function for a 2f is given by $\pi_2(p_A, p_B) =$

$$
\begin{cases}
\sum_{i\in\{A,B\}} N(n^c + 2n^s(1 - F_i(p_i))(p_i - c) & \text{if } p_A + p_B \le W, p_i \le R_i \\
N(n^c + 2n^s(1 - F_A(p_A))(p_A - c) & \text{if } p_A \le Q_A, p_B > R_B \\
N(n^c + 2n^s(1 - F_B(p_B))(p_B - c) & \text{if } p_B \le Q_B, p_A > R_A \\
\sum_{i\in\{A,B\}} N[n_1^c + 2n_1^s(1 - F_i(p_i)) + 2n_2^s X_i(p_i)](p_i - c) & \text{if } Q_i < p_i \le R_i, p_A + p_B > W \\
N[n_1^c + 2n_1^s(1 - F_A(p_A)) + 2n_2^s X_A(p_A)](p_A - c) & \text{if } Q_A < p_A \le R_A, p_B > R_B \\
N[n_1^c + 2n_1^s(1 - F_B(p_B)) + 2n_2^s X_B(p_B)](p_B - c) & \text{if } Q_B < p_B \le R_B, p_A > R_A \\
0 & \text{if } p_i > R_i.
\end{cases}
\tag{4.2.3}
$$

Notice, that in each of the seven cases above, profit is additively separable across goods. Also note that no firm will ever set a price greater than R_i on a good that they offer, as this yields no sales.

Given symmetric data, we will focus on symmetric equilibria (across goods) and define equilibrium as

Definition 4.2.3. A symmetric equilibrium is a list of reservation values (W, R), profit for both types of firms $(\bar{\pi}_1, \bar{\pi}_2)$, a joint distribution of prices $F_{A,B}$ with marginals F_A and F_B and pricing strategies of firms $(G^1(p), G^2(p_A, p_B))$ such that

1. Consumers formulate W and R according to equations (4.2.2) and (4.2.1).

2. Firms maximize: $\pi_1(p) = \bar{\pi}_1$ for all p in the support of G^1 with $\pi_1(p) \le \bar{\pi}_1$ elsewhere; $\pi_2(p_A, p_B) = \bar{\pi}_2$ for all (p_A, p_B) in the support of G^2 with $\pi_2(p_A, p_B) \le$

$\bar{\pi}_2$ elsewhere.

3. Consistency:[6]

$$F_{A,B}(p_A, p_B) = q(\mathbb{1}\{p_B = \infty\}G^1(p_A)+\mathbb{1}\{p_A = \infty\}G^1(p_B))+(1-2q)G^2(p_A, p_B).$$

Next, we formulate equilibrium in the simplest case.

4.3 Simplest Case

In this section we construct an equilibrium reminiscent of the single product equilibrium of Burdett and Judd (1983). Suppose $q = 0$ so that all firms offer both goods, and $n_1^0 = 0$ so that all consumers initially demand both goods. Recall from the consumers' problem that 2b's only transition to become 1b's after encountering a price pair (p_i, p_j) with $p_i \leq Q$ and $p_j > R$. As no firm prices above R on goods they offer, and all firms offer both goods when $q = 0$, we conclude that no 2b's ever become 1b's. In this simplest case, we have $n_1 = 0$. This then implies $n^c = \alpha n_2$ and $n^s = (1 - \alpha)n_2$. Because of this, we can derive our first result. No firm offers a basket costing more than W.

Lemma 4.3.1. *When $q = 0$ and $n_1 = 0$, then all firms' basket price is acceptable*

[6]Recall, this is an improper distribution if $q > 0$, as we follow the notational convention that firms who can not sell a good post an infinite price for that good.

so that

$$\operatorname{supp} F_{A,B} \subseteq \{p_A, p_B | p_i \leq R, p_A + p_B \leq W\}.$$

Proof. As no consumer will ever pay more than R for a good, no firm will ever set a price higher than R. Let θ_i be the probability a consumer purchases good i from a firm posting (p_A, p_B) (outside the acceptance set) conditional on not receiving a better offer for good i from another firm.[7] That is,

$$\theta_i = \frac{X_i(p_i)}{1 - F_i(p_i)}.$$

Let $\hat{p}_i$ be the expected price a consumer expects to pay for good i conditional on receiving the offer (p_A, p_B). That is

$$\hat{p}_i = \theta_i p_i + (1 - \theta_i)\,\mathbb{E}[P_i | P_i \leq p_i]$$

where the expectation is taken over expected acceptable prices. Then we have the

[7] This proof almost exactly follows the proof in McAfee (1995), except that without recall θ_i is defined differently.

following:

$$
\begin{aligned}
\pi(p_A, p_B) &= \sum_i 2n^s(1 - F_i(p_i))\theta_i(p_i - c) \\
&\leq \sum_i 2n^s(1 - F_i(p_i)) \left[\theta_i(p_i - c) + (1 - \theta_i)(\mathbb{E}[P_i | P_i \leq p_i] - c)\right] \\
&= \sum_i 2n^s(1 - F_i(p_i))(\hat{p}_i - c) \\
&< \sum_i [n^c + 2n^s(1 - F_i(\hat{p}_i))](\hat{p}_i - c) \\
&= \pi(\hat{p}_A, \hat{p}_B).
\end{aligned}
$$

Hence, deviating to $(\hat{p}_A, \hat{p}_B)$ is profitable. □

4.3.1 Deriving Equilibrium

Recall, if $p_i \leq R$ and $p_A + p_B \leq W$, then

$$\pi(p_A, p_B) = \sum_{i \in \{A,B\}} N[n^c + 2n^s(1 - F_i(p_i))](p_i - c). \tag{4.3.1}$$

Equilibrium requires constant profits: $\pi(p_A, p_B) = \bar{\pi}$ on the support of F and $\pi(p_A, p_B) \leq \bar{\pi}$ elsewhere. As in Burdett and Judd (1983), F must be continuous, and so at interior optima, first order conditions give

$$
\begin{aligned}
0 &= N[n^c + 2n^s(1 - F_i(p_i))] - 2Nn^s f_i(p_i)(p_i - c) \\
&= Nn_2\{[\alpha + 2(1-\alpha)(1 - F_i(p_i))] - 2(1-\alpha) f_i(p_i)(p_i - c)\} \\
\implies &\quad \frac{d}{dp}[F_i(p)(p-c)] = \frac{2-\alpha}{2(1-\alpha)} \\
\implies &\quad F_i(p) = \frac{2+\alpha}{2(1-\alpha)}\left(\frac{p}{p-c}\right) + \frac{Cons}{p-c}.
\end{aligned}
$$

To determine the constant of integration, substitute $F_i(\bar{p}_i) = 1$ to get

$$
Cons = -\frac{\alpha}{2(1-\alpha)}\bar{p}_i - c
$$

so

$$
F_i(p) = \frac{2-\alpha}{2(1-\alpha)} - \frac{\alpha}{2(1-\alpha)}\frac{\bar{p}_i - c}{p - c}.
$$

Solving for $F_i(\underline{p}_i) = 0$ yields

$$
\underline{p}_i = \frac{\alpha\bar{p}_i + 2(1-\alpha)c}{2-\alpha}.
$$

Hence, given the upper bound of the support $\bar{p}_i$, the marginal distributions of each price are determined. To obtain this upper bound, one simply notices that the highest price charge, $\bar{p}_i$ must simply be the reservation value of the consumer: $R = \bar{p}_i$. By Lemma 4.3.1, we must have $\bar{p}_i \leq R$. For the reverse inequality, suppose not, that $\bar{p}_i < R$. But then, a firm charging $p_i = R$ loses no customers (because it is already pricing at the top of the distribution) and strictly increases profits. Closing equilibrium then just requires solving for R in the consumer's problem given this distribution. The result is in three cases concerning the relation between costs, values, and competition.

Proposition 4.3.2. *If*

$$z > c + \frac{k}{1 + \frac{\alpha}{2(1-\alpha)} \log\left(\frac{2-\alpha}{\alpha}\right)},$$

then the marginal distribution of prices in any interior equilibrium are given by

$$F_i(p_i) = \frac{2-\alpha}{2(1-\alpha)} - \frac{1}{\frac{2(1-\alpha)}{\alpha} + \log\left(\frac{2-\alpha}{\alpha}\right)} \frac{k}{p_i - c}.$$

Reservation values are given by

$$R = c + \frac{k}{1 + \frac{\alpha}{2(1-\alpha)} \log\left(\frac{2-\alpha}{\alpha}\right)}, \qquad Q = c - k\frac{\frac{\alpha}{2(1-\alpha)} \log\left(\frac{2-\alpha}{\alpha}\right)}{1 + \frac{\alpha}{2(1-\alpha)} \log\left(\frac{2-\alpha}{\alpha}\right)},$$

$$W = 2c + k\frac{1 - \frac{\alpha}{2(1-\alpha)} \log\left(\frac{2-\alpha}{\alpha}\right)}{1 + \frac{\alpha}{2(1-\alpha)} \log\left(\frac{2-\alpha}{\alpha}\right)}.$$

If, instead

$$c < z < c + \frac{k}{1 + \frac{\alpha}{2(1-\alpha)} \log\left(\frac{2-\alpha}{\alpha}\right)}$$

then

$$F_i(p) = \frac{2-\alpha}{2(1-\alpha)} - \frac{\alpha}{2(1-\alpha)} \frac{z-c}{p-c}$$

and

$$R = k+c+\frac{\alpha}{2(1-\alpha)} \log\left(\frac{2-\alpha}{\alpha}\right)(z-c), \qquad Q = c+\frac{\alpha}{2(1-\alpha)} \log\left(\frac{2-\alpha}{\alpha}\right)(z-c),$$

$$W = k + 2\left(c + \frac{\alpha}{2(1-\alpha)} \log\left(\frac{2-\alpha}{\alpha}\right)(z-c)\right).$$

Finally, if $c \leq z$ there is no market.

Proof. Suppose we are in the first case. Given the above, we need to find a reservation value R where $R = k + \mathbb{E}[p_i]$, or

$$R = k + \int_{\underline{p}_i}^{\bar{p}_i} p dF_i(p).$$

Substituting in, we get

$$\begin{aligned} R &= k + \int\limits_{\frac{\alpha R + 2(1-\alpha)c}{2-\alpha}}^{R} p\left(\frac{\alpha}{2(1-\alpha)}\right)\left(\frac{R-c}{(p-c)^2}\right) dp \\ &= k + c + \frac{\alpha}{2(1-\alpha)} \log\left(\frac{2-\alpha}{\alpha}\right)(R-c) \\ \implies R &= c + \frac{k}{1 + \frac{\alpha}{2(1-\alpha)} \log\left(\frac{2-\alpha}{\alpha}\right)} \end{aligned}$$

Substituting into F gives the result above. For W and Q, recall that when all firms price in the acceptance set, $W = k + \mathbb{E}[P_A] + \mathbb{E}[P_B]$, and $Q = W - R = \mathbb{E}[P_i]$.

In the second case, the consumer's value limits prices, and is thus the upper bound of the distribution, but all else proceeds the same. And in the final case, there are no gains from trade. $\square$

These statements were made conditional on all firms pricing at an interior optimum. That is, firms are not constrained by the acceptance set in equilibrium. If consumers can freely recall previous offers, McAfee (1995) shows that boundary equilibria, where firms are constrained by the acceptance set and price on the boundary, may obtain. Without recall, this is not possible.

Lemma 4.3.3. *All equilibria are interior, of the form in Proposition 4.3.2. Firms are not constrained by the boundary.*

Proof. Suppose there were an interval $[p_A^0, p_A^1] \equiv I_A$ where firms priced on the boundary $\{(p_A, p_B) \mid p_A \in [p_A^0, p_A^1], p_B = W - p_A\}$ and were constrained:

$$\frac{\partial \pi(p_A, p_B)}{\partial p_A} = \frac{\partial \pi(p_A, p_B)}{\partial p_B} = \lambda > 0.$$

Without loss of generality, suppose $p_A^0 < W/2$ (otherwise $p_B^0 \equiv W - p_A^0 < W/2$ and so relabel).

Separability, then, implies that no other firms price below this interval, as increasing towards the boundary would increase profits. That is, if $p_A \in I_A$ and $(p_A, p_B) \in \operatorname{supp} F_{A,B}$ then $p_A + p_B = W$ and similarly for $p_B \in W - I_A$. As mass is distributed along a line with slope -1, the marginal distributions must move in lock step. Thus, $F_A(p_A) - F_A(p_A^0) = F_B(W - p_A^0) - F_B(W - p_A)$ and $f_A(p_A) = f_B(W - p_A)$.

Looking again at the first order condition at the boundary, we have

$$N[n^c + 2n^s(1 - F_A(p_A)) - 2n^s f_A(p_A)(p_A - c)] = N[n^c + 2n^s(1 - F_B(p_B)) - 2n^s f_B(p_B)(p_B - c)].$$

Simplifying gives

$$F_A(p_A) + f_A(p_A)(p_A - c) = F_B(p_B) + f_B(p_B)(p_B - c).$$

Substitutiong in $p_B = W - p_A$ and distributions gives

$$\begin{aligned} &F_A(p_A) + f_A(p_A)(p_A - c) \\ &\qquad = -F_A(p_A) + (F_A(p_A^0) + F_B(W - p_A^0)) + f_A(p_A)(W - p_A - c) \end{aligned}$$

Hence,

$$
\begin{aligned}
2F_A(p_A) + f_A(p_A)(2p_A - W) &= (F_A(p_A^0) + F_B(W - p_A^0)). \\
\frac{\partial}{\partial p_A}\left(F_A(p_A)(2p_A - W)\right) &= (F_A(p_A^0) + F_B(W - p_A^0)). \\
\implies F_A(p_A)(2p_A - W) &= (F_A(p_A^0) + F_B(W - p_A^0))p_A + \text{cons}. \\
\implies F_A(p_A) &= \frac{(F_A(p_A^0) + F_B(W - p_A^0))p_A + \text{cons}}{2p_A - W}.
\end{aligned}
$$

This must also hold at p_A^0 whereby one finds the constant from

$$
\begin{aligned}
\text{cons} &= F_A(p_A^0)(2p_A^0 - W) - p_A^0(F_B(W - p_A^0) + F_A(p_A^0)) \\
&= F_A(p_A^0)p_B^0 - F_B(p_B^0)p_A^0
\end{aligned}
$$

where we have written $p_B^0 \equiv W - p_A^0$. The final distribution required by equal profit is then given by

$$
F_A(p_A) = \frac{(F_A(p_A^0) + F_B(p_B^0))p_A + F_A(p_A^0)p_B^0 - F_B(p_B^0)p_A^0}{2p_A - W}.
$$

This is not a distribution function as

$$
\frac{dF}{dp_A} = \frac{\begin{bmatrix} (F_A(p_A^0) + F_B(p_B^0))(2p_A - W) \\ \quad - 2((F_A(p_A^0) + F_B(p_B^0))p_A + F_A(p_A^0)p_B^0 - F_B(p_B^0)p_A^0) \end{bmatrix}}{(2p_A - W)^2} < 0
$$

for $p_A^0 < p_A < W/2$. $\square$

As boundary equilibria do not obtain, the marginal distribution given in proposition 4.3.2 must obtain.

One should note that this is exactly the same marginal distribution derived in Burdett and Judd (1983). This result is somewhat astounding – consumers' ability to economize on their search for several goods has no effect on the equilibrium price distribution. As will prove clear below when $q > 0$ and $n_1 > 0$, this derives from two closely related artifacts of the homogeneous model: no consumer choose to search again, and all firms price in the joint acceptance set. When some consumers must search multiple times, their ability to economize on search costs via joint search will alter the firm's problem. There must, however, be some incentive driving multiple search, an obvious example being the case where some firms sell only one good.

Proposition 4.3.2 only specifies the marginal distribution of prices, and so does not exactly settle the question of existence, as we have yet to derive a joint distribution of prices. This is easily settled. There exist a huge multiplicity of equilibria. Indeed, any joint distribution with the given marginals and support contained in the acceptance set will suffice. In the two good case, one easily derived example is that of perfectly negative rank correlation in prices where firms price on the line $p_B(p_A)$ solving $F_A(p_A) = 1 - F_B(p_B(p_A))$. This yields

$$p_B(p_A) = c + k\left[\frac{2}{\alpha} + \frac{1}{1-\alpha}\log\left(\frac{2-\alpha}{\alpha}\right) - \frac{k}{p_A - c}\right]^{-1}.$$

In this case, the joint distribution of prices is given by

$$F_{A,B}(p_A, p_B) = \max\{0, F_A p_A + F_B p_B - 1\}.$$

A continuum of other possible equilibria exist.

One that generalizes to more than two goods (which the above perfect negative rank correlation equilibrium does not) has support

$$\left\{(p_A, p_B) \mid p_i \geq \underline{p} \text{ and } p_B \leq c + k\left[\frac{2}{\alpha} + \frac{1}{1-\alpha}\log\left(\frac{2-\alpha}{\alpha}\right) - \frac{k}{p_A - c}\right]^{-1}\right\}$$

which is just the set above the lowest price and below the line of support from the negative rank-correlation equilibrium above. The joint distribution is given by

$$\begin{aligned} F_{A,B}(p_A, p_B) = 2\left(1 - \sqrt{F(p_A)}\right)\left(1 - \sqrt{F(p_B)}\right) \\ - \left(\max\left\{1 - \sqrt{F(p_A)} - \sqrt{F(p_A)}, 0\right\}\right)^2. \quad (4.3.2) \end{aligned}$$

The huge multiplicity of equilibria is at once positive and negative. While the model gives no information about the joint distribution of prices – other than a limitation on its support – this provides a justification for the common empirical focus on marginal distribution of prices despite the presence of multiple products. Firms may play any of a continuum of joint distributions, but cost and other information can be determined from just marginal distributions. Indeed, in this simplest case, all of the parameters of the model except for N can be identified from the marginal distributions of posted prices and the marginal distribution of paid prices. Both of which are available from scanner data. We turn now to the more general case where some firms offer but one product and some consumers demand but one good.

4.4 General Case

Suppose now that not all firms offer both goods, that $q > 0$. In this case, with some firms offering only one good, some consumers will be forced to search again – the sequential search option will operate. Moreover, this will lead some 2b's to make a single-good purchase, transitioning to become 1b's so that $n_1 > 0$. Assume further that $n_1^0 > 0$ so that there are some consumers demanding one good with noisy search. Firms now have the option of targeting only captive single-product searchers, pricing outside the joint acceptance set. For one-good firms, this boils down to a decision between targeting both single and joint searchers by pricing below Q, or pricing higher and selling only to 1b's. It turns out, whether this option is profitable depends less on the proportion of 1b's than on the proportion of one-good firms. If there are sufficiently few 1f's so that there is "enough space" at the bottom of the price distribution from section 4.3.2 for these firms to price below Q. It's only as q increases that deviations become profitable. There are several cases to consider, and each is taken in turn.

Equilibria fall into four cases. In the first, the proportion of one-good firms is sufficiently small that are not directly constrained by the joint acceptance set of consumers. That is, there is “enough room” at the bottom of the equilibrium price distribution that one-good firms can price below Q and still make the same profit on the good they sell as two-good firms do from each good. The marginal distribution of prices is of the same form as in section 4.3.2. Two-good firms are affected by one-good firms only in that their presence increases the reservation value of consumers, as consumers are no longer able to satisfy their demand in the first round of search.

In the second case, the proportion of one-good firms is sufficiently high that they are constrained by Q, but not so high as to justify a jump to R. That is, there are enough one-good firms that the marginal distribution of prices below Q is higher than in the first case, and these firms would prefer to price above Q if joint searchers would buy from them at these prices. In this second case, however, these firms are not so constrained that it would be profitable to forsake joint searcher and price at R, targeting only one-searchers. The large proportion of firms pricing below Q has the effect of decreasing the highest price charged by two-good firms. There are two reasons. The first is the obvious one – when some firms are constrained to set low prices, the reservation value of consumers decreases. The second, and more important, reason is that as the one-good firms crowd-out the bottom of the price distribution, two-good firms must increase their minimum prices. To remain in the joint acceptance set, however, they must then decrease their maximum price, moving from (R, Q) on the reservation frontier down to some point $(\bar{p}, W - \bar{p})$.

In the third case, there is such a high proportion of one-good firms that not all can price below Q, some jump up to R to target single-product searchers. This has the effect of

4.4.1 Few One-Good-Firms.

In the equilibrium of section 4.3.2, firms make constant profits per good for any price between $\underline{p}$ and R. This was because a 2f pricing good A at R could price good B at $p \leq Q$ and so still fall in the joint acceptance set of the 2b's. 1f's are not so fortunate. As they sell but one good, and so implicitly have an infinite price for the second, 2b's will only buy from them at prices below Q.

If the proportion of one-sellers is low, equilibrium closely resembles the case where all firms sell both goods. One-sellers simply set prices less than Q in order to attract both one and two searchers. Both one-sellers and two-sellers make the same profit from each product line they offer.

Proposition 4.4.1. *Suppose $q < \bar{q}_1$ given below. Let $X \equiv (n^c + 2n^s)/2n^s$. Then* *the marginal distribution of prices is given by*

$$F(p) = \begin{cases} 1-q & \textit{if } R \leq p \\ X - (X-(1-q))\left(\frac{R-c}{p-c}\right) & \textit{if } c + \frac{X-(1-q)}{X}(R-c) \leq p < R \\ 0 & \textit{if } p < c + \frac{X-(1-q)}{X}(R-c). \end{cases}$$

Notice, this is an improper distribution as q firms do not offer one good, and so are treated as offering a price at infinity. Given this, the expected price offered is

$$\mathbb{E}[P|P \leq R] = c + \frac{k}{1-q}\frac{Z}{1-Z}$$

where

$$Z \equiv \left(\frac{X}{1-q} - 1\right)\log\left(\frac{X}{X-(1-q)}\right).$$

The reservation values of the consumer are given by

$$R = \frac{k}{1-q} + \mathbb{E}[P|P \leq R],$$

$$W = \frac{1+q}{1-q}k + 2\,\mathbb{E}[P|P \leq R],$$

and

$$Q = W - R = \frac{q}{1-q}k + \mathbb{E}[P|P \leq R].$$

Finally, $\bar{q}_1$ *solves*

$$q = F(Q) = X - \frac{X - (1-q)}{q + (1-q)Z}$$

which has no algebraic solution, but a solution exists and is unique.

Proof. Beginning with a one-searcher's problem, the $(1-q)$ firms who can offer the good at a price less than R, and q firms can not offer the good at all so

$$R = k + (1-q)\,\mathbb{E}[P|P \leq R] + qR$$

which immediately gives our expression for R. The reservation value W is similar. $(1-2q)$ firms will sell both goods to a two-searcher, and q sell each good (implying that the consumer continues to search as a one-searcher with probability q). Moreover, any firm contacted will sell at least one good, so no two-searcher continues as a two-searcher. These considerations along with linearity of expectation imply

$$\begin{aligned} W &= k + 2(\mathbb{E}[P|P \leq R](1-q) + qR) \\ &= \left(\frac{1+q}{1-q}\right) k + 2\,\mathbb{E}[P|P \leq R] \end{aligned}$$

where the last line obtains by substituting in for R. Q is simply $W - R$.

The profit firms make on a given good when pricing in the acceptance set are

$$\pi(p) = [\underbrace{n_1 + \alpha n_2}_{n^c} + 2\underbrace{(1-\alpha)n_2}_{n^s}(1 - F(p))](p - c)$$

Differentiating gives first order condition

$$0 = [n^c + 2n^s(1 - F(p))] - 2n^s f(p)(p - c).$$

Re-arranging gives

$$\frac{d}{dp}[F(p)(p - c)] = X$$

so upon integrating

$$F(p)(p - c) = Xp + cons.$$

Noting that $F(R) = 1 - q$ gives the value of the constant as

$$cons = -(X - (1 - q))(R - c) - Xc$$

so that

$$F(p) = X - (X - (1 - q))\frac{R - c}{p - c}.$$

The bottom of the support solves $F(\underline{p}) = 0$ so that

$$\underline{p} = c + \frac{X - (1 - q)}{X}(R - c).$$

Continuing, given this distribution we calculate

$$F[p|P \leq R] = \frac{X}{1 - q} - \left(\frac{X}{1 - q} - 1\right)\frac{R - c}{p - c}$$

so that

$$\begin{aligned}
\mathbb{E}[P|P \leq R] &= R - \int_{\underline{p}}^{R} \frac{X}{1-q} - \left(\frac{X}{1-q} - 1\right)\frac{R-c}{p-c}\,dp \\
&= R - \frac{X}{1-q}(R-\underline{p}) + \left(\frac{X}{1-q} - 1\right)(R-c)\log\left(\frac{R-c}{\underline{p}-c}\right) \\
&= c + \left(\frac{X}{1-q} - 1\right)(R-c)\log\left(\frac{X}{X-(1-q)}\right)
\end{aligned}$$

Finally, closing equilibrium, one simply substitutes the value of R into this equation and solves for $\mathbb{E}[P|P \leq R]$.

The value for q_1 is the upper limit such that, given the above distribution, the proportion of one-sellers is sufficiently small so that their population fits in the given distribution below Q. For existence, one simply notes that at $q = 0$, $F(Q) > 0$ and at $q = 1$, $F(Q) < 1$. For uniqueness, some time with a computer algebra system yields

$$\frac{d}{dq}F(Q) = -\frac{4(\alpha-1)^2 n_2^2 q}{\left((n_2(-2\alpha q + \alpha + 2q) + n_1)\log\left(\frac{n_1-(\alpha-2)n_2}{n_2(-2\alpha q+\alpha+2q)+n_1}\right) - 2(\alpha-1)n_2 q\right)^2} < 0.$$

$\square$

4.4.2 Case 2: $\bar{q}_1 < q \leq \bar{q}_2$

If the proportion of one-sellers is higher than will "fit naturally" below Q, but not much higher, the one-sellers crowd each other below Q so as to be able to target both one and two searchers. This crowding reduces their profit relative to two-sellers. The crowding among one-sellers has the effect of displacing two-sellers – there are "too many" firms charging at or below Q, so it is no longer profitable for two-sellers to charge Q, higher prices are more profitable. But for a two-seller to charge a higher price than Q on one good, the other good must be priced less than R. Two-sellers move down along the reservation frontier, pricing in a range $[\underline{p}^2, \bar{p}^2]$ with $\underline{p}^2 + \bar{p}^2 = W$, moving in equilibrium to the point where equal profits are made from both goods.

Proposition 4.4.2. *Suppose $\bar{q}_1 < q < \bar{q}_2$ given below. Let $X \equiv (n^c + 2n^s)/2n^s$. Then the marginal distribution of prices is given by*

$$F(p) = \begin{cases} 1 - q & \text{if } \bar{p}^2 \leq p \\ X - (X - (1-q))\left(\frac{\bar{p}^2 - c}{p - c}\right) & \text{if } \underline{p}^2 \leq p < \bar{p}^2 \\ q & \text{if } Q \leq p < \underline{p}^2 \\ X - (X - q)\left(\frac{Q - c}{p - c}\right) & \text{if } \underline{p}^1 \leq p < Q \\ 0 & \text{if } p < \underline{p}^1. \end{cases}$$

Notice, this is an improper distribution as q firms do not offer one good, and so are *treated as offering a price at infinity. The support for the two-sellers is* $[\underline{p}^2 < \bar{p}^2]$ *with*

$$\bar{p}^2 = \frac{W(n^c + 2n^s(1-q) - 2cn^s(1-2q)}{2(n^c+n^s)}$$

$$\underline{p}^2 = \frac{W(n^c + 2n^s q) + 2cn^s(1-2q)}{2(n^c+n^s)}.$$

The support for the one-sellers $[\underline{p}^1, Q]$ *with*

$$\underline{p}^1 = c + \left(\frac{X-q}{X}\right)(Q-c).$$

The expected cost can be written as $\mathbb{E}[P|P \leq R] = \frac{\Omega_k}{\Lambda_k}k + \frac{\Omega_c}{\Lambda_c}c$ *with*

$$\Omega_k = (n^c - 2n^s(q-1)) \cdot \left(2q(n^c+n^s)\log\left(\frac{n^c+2n^s}{n^c-2n^sq+2n^s}\right)\right.$$
$$\left. +(q+1)(n^c+2n^sq)\log\left(\frac{n^c-2n^sq+2n^s}{n^c+2n^sq}\right)\right)$$

$$\Lambda_k = 2(q-1)\left(\left((n^c)^2 + 2n^cn^s - 4(n^s)^2(q-1)q\right)\log\left(\frac{n^c-2n^sq+2n^s}{n^c+2n^sq}\right)\right.$$
$$\left. +2n^s(q-1)(n^c+n^s) + (n^c+n^s)(n^c-2n^s(q-1))\log\left(\frac{n^c+2n^s}{n^c-2n^sq+2n^s}\right)\right)$$

$$\Omega_c = \left((n^c)^2 + 2n^c n^s - 4(n^s)^2(q-1)q\right)\log\left(\frac{n^c - 2n^s q + 2n^s}{n^c + 2n^s q}\right)$$
$$+ 2n^s\left(-n^c q + 4n^s q^2 - 5n^s q + n^s\right) + (n^c + n^s)(n^c - 2n^s(q-1))$$
$$\cdot \log\left(\frac{n^c + 2n^s}{n^c - 2n^s q + 2n^s}\right)$$

$$\Lambda_c = \left[(n^c)^2 + 2n^c n^s - 4(n^s)^2(q-1)q\right]\log\left(\frac{n^c - 2n^s q + 2n^s}{n^c + 2n^s q}\right)$$
$$+ 2n^s(q-1)(n^c + n^s) + (n^c + n^s)(n^c - 2n^s(q-1))\log\left(\frac{n^c + 2n^s}{n^c - 2n^s q + 2n^s}\right)$$

$$Z \equiv \left(\frac{X}{1-q} - 1\right)\log\left(\frac{X}{X-(1-q)}\right).$$

The reservation values of the consumer are given by

$$R = \frac{k}{1-q} + \mathbb{E}[P|P \leq R],$$

$$W = \frac{1+q}{1-q}k + 2\,\mathbb{E}[P|P \leq R],$$

and

$$Q = W - R = \frac{q}{1-q}k + \mathbb{E}[P|P \leq R].$$

Finally, $\bar{q}_2$ solves

$$[n^c + 2n^s(1-q)](Q-c) = \max\{n_1(R-c), [n_1 + 2n^s(1-q)](\underline{p}^2 - c)\}.$$

The left hand side is profit for a one-seller pricing at Q. The first term in the maximand is the profit from pricing at R and selling only to (captive) one-searchers. The second term in the maximand is the profit from pricing at $\underline{p}^2$ and selling to one-searchers as well as all shopping two-searchers who also contact a two-seller or a one-seller of the other good – both of which cases yield a basket price below W resulting in a sale. This is precisely the condition that a 1f does not wish to deviate.

Proof. The reservation values follow from exactly the same argument as in Proposition 4.4.1. The values for $\bar{p}^2$ and $\underline{p}^2$ solve for equal profit along the line $W = \bar{p}^2 + \underline{p}^2$ and equal profit

$$[n^c + 2n^s(q)](\bar{p}^2 - c) = [n^c + 2n^s(1-q)](\underline{p}^2 - c)$$

as the highest price two-seller sells to a shopper only if that shopper faces another firm not selling the good (q) and the lowest price two-seller sells to a shopper if they have met either a firm not selling the good or another two-seller (who must have a higher price). To see that two-sellers do not wish to price along $p_A + p_B = W$ with $p_A > \bar{p}^2$ recall that $q < 1/2$ and so the profit of the firm on the frontier (which is the only relevant region) is

$$[n^c + 2n^s q](p - c) + [n^c + 2n^s(1 - q)](W - p - c)$$

is decreasing in p.

Given this distribution, we can find

$$F(p|P \leq R) = \begin{cases} 1 & \text{if } \bar{p}^2 \leq p \\ \frac{X}{1-q} - (\frac{X}{1-q} - 1)\left(\frac{\bar{p}^2 - c}{p-c}\right) & \text{if } \underline{p}^2 \leq p < \bar{p}^2 \\ \frac{q}{1-q} & \text{if } Q \leq p < \underline{p}^2 \\ \frac{X}{1-q} - (\frac{X}{1-q} - \frac{q}{1-q})\left(\frac{Q-c}{p-c}\right) & \text{if } \underline{p}^1 \leq p < Q \\ 0 & \text{if } p < \underline{p}^1. \end{cases}$$

From this one derives

$$\begin{aligned} \mathbb{E}[P|P \leq R] = \bar{p}^2 - \left(\frac{X}{1-q}\right)(\bar{p}^2 - \underline{p}^2) + \left(\frac{X}{1-q} - 1\right)(\bar{p}^2 - c) \\ \cdot \log\left(\frac{n^c + 2n^s(1-q)}{n^c + 2n^s q}\right) - \left(\frac{q}{1-q}\right)(\underline{p}^2 - Q) - \left(\frac{q}{1-q}\right)(Q - c) \\ + \left(\frac{X}{1-q} - \frac{q}{1-q}\right)(Q - c)\log\left(\frac{X}{X-q}\right) \end{aligned}$$

Where the insides of the logs derive from manipulation of the distribution function which gives

$$\frac{\bar{p}^2 - c}{\underline{p}^2 - c} = \frac{n^c + 2n^s(1-q)}{n^c + 2n^s q}, \text{ and } \frac{Q - c}{\underline{p}^1 - c} = \frac{X}{X - q}.$$

This expression for $\mathbb{E}[P|P \leq R]$ is linear in R, W, Q and price limits. As these are linear in $\mathbb{E}[P|P \leq R]$ themselves, one may substitutes in the values for these from above and solve. This produces the expression for the equilibrium value of $\mathbb{E}[P|P \leq R]$ given above in terms of parameters. $\square$

4.4.3 Case 3(a)

Which case obtains next may depend on parameters. As the proportion of one-sellers increases above $\bar{q}_2$, one of two regions become profitable for the 1f. Either pricing in a range contained in $[\bar{p}_2, R]$ or between $[Q, \underline{p}^2]$. Which is determined from whether pricing at R and making $n_1^c(R-c)$ is greater or less than pricing at $\underline{p}^2$ and making $[n_1^c + 2n^s(1-q)](\underline{p}^2 - c)$. Clearly, this depends on the relative values of n_1^c and n^s. In this subsection, suppose there are relatively many n_1^c so it is more profitable to deviate above $\bar{p}_2$. Let r be the proportion of 1f's pricing above $\bar{p}_2$ (with the complementary proportion $1-r$ still pricing below Q). The value of r is determined by equal profit:

$$[n^c + 2n^s(1-(1-r)q)](Q-c) = \beta n_1(R-c). \tag{4.4.1}$$

Of course, R and Q are endogenous and so this equation alone does not yield r. Recall, that a 1f selling A and pricing above $\bar{p}^2$ face a profit function of

$$\pi_1(p_A) = N[n_1^c + 2n_1^s(1 - F_A(p_A)) + 2n_2^s X_A(p_A)](p - c).$$

where, $X_A = F_B(W - p_A) - F_{A,B}(p_A, W - p_A)$. In this case, a 1f pricing in $[\bar{p}^2, R]$ can make a basket with any 1f pricing below Q. Hence, $X_A = (1 - r)q$ and the profit function is simply

$$\pi_1(p_A) = N[n_1^c + 2n_1^s(1 - F_A(p_A)) + 2n_2^s(1 - r)q](p - c).$$

Equal profit, requires, then, that the marginal distribution of prices follow a distribution of

$$F = Y - [Y - (1 - q)]\frac{R - c}{p - c}, \qquad \text{where} \quad Y \equiv \frac{n_1^c + 2n_2^s(1 - r)q + 2n_1^s}{2n_1^s}$$

on the interval $[\underline{p}^R, R]$ where $\underline{p}^R$ is defined similarly to $\underline{p}^1$ with

$$\underline{p}^R = c + \frac{Y - (1 - q)}{Y - (1 - (1 + r)q)}(R - c).$$

This implies that the 1f price according to

$$G^1(p) = \begin{cases} 1 & \text{if } R \leq p \\ \frac{1}{q}\left[Y - [Y - (1-q)]\frac{R-c}{p-c} - (1-2q)\right] & \text{if } \underline{p}^R \leq p < R \\ 1 - r & \text{if } Q \leq p < \underline{p}^R \\ \frac{1}{q}\left[X - (X - (1-r)q)\frac{Q-c}{p-c}\right] & \text{if } \underline{p}^1 \leq p < Q \\ 0 & \text{if } p < \underline{p}^1 \end{cases}$$

As before, two-sellers will not price at (R, Q), but instead at a higher minimum and lower maximum price solving $\bar{p}^2 + \underline{p}^2 = W$ and equal profit. The equal profit condition is

$$[n^c + 2n^s(1+r)q](\bar{p}^2 - c) = [n^c + 2n^s(1-(1-r)q)](\underline{p}^2 - c).$$

Solving gives

$$\bar{p}^2 = \frac{W(n^c + 2n^s[1-(1-r)q]) - 2n^s c(1-2q)}{2[n^c + n^s(1+2rq)]},$$

$$\underline{p}^2 = \frac{W(n^c + 2n^s(1+r)q) - 2n^s c(1-2q)}{2[n^c + n^s(1+2rq)]}.$$

So, 2f's price on the set $\{(p_A, p_B) | \underline{p}^2 \leq p_i \leq \bar{p}^2 \text{ and } p_A + p_B \leq W\}$ with marginal distribution

$$G^2(p, \infty) = \frac{1}{1-2q}\left[X - [X - (1-(1+r)q)]\frac{\bar{p}^2 - c}{p - c} - (1-r)q\right].$$

As above, any joint distribution on this set with the required marginals suffices as an equilibrium.

Adding these two gives the marginal distribution of prices in equilibrium, required by equal profit conditions:

$$F(p) = \begin{cases} 1-q & \text{if } R \leq p \\ Y - [Y-(1-q)]\frac{R-c}{p-c} & \text{if } \underline{p}^R \leq p < R \\ 1-(1+r)q & \text{if } \bar{p}^2 \leq p < \underline{p}^R \\ X - [X-(1-(1+r)q)]\frac{\bar{p}^2-c}{p-c} & \text{if } \underline{p}^2 \leq p < \bar{p}^2 \\ (1-r)q & \text{if } Q \leq p < \underline{p}^2 \\ X-(X-(1-r)q)\frac{Q-c}{p-c} & \text{if } \underline{p}^1 \leq p < Q \\ 0 & \text{if } p < \underline{p}^1 \end{cases}$$

From this we calculate

$$F(p|P \leq R) = \begin{cases} 1 & \text{if } R \leq p \\ \frac{Y}{1-q} - [\frac{Y}{1-q} - 1]\frac{R-c}{p-c} & \text{if } \underline{p}^1 \leq p < R \\ \frac{1-(1+r)q}{1-q} & \text{if } \bar{p}^2 \leq p < \underline{p}^1 \\ \frac{X}{1-q} - [\frac{X}{1-q} - \frac{1-(1+r)q}{1-q}]\frac{\bar{p}^2-c}{p-c} & \text{if } \underline{p}^2 \leq p < \bar{p}^2 \\ \frac{(1-r)q}{1-q} & \text{if } Q \leq p < \underline{p}^2 \\ \frac{X}{1-q} - (\frac{X}{1-q} - \frac{(1-r)q}{1-q})\frac{Q-c}{p-c} & \text{if } \underline{p}^1 \leq p < Q \\ 0 & \text{if } p < \underline{p}^1 \end{cases}$$

Next we calculate expected price in terms of endogenous variables:

$$\begin{aligned} \mathbb{E}[P|P \leq R] = R - \frac{Y}{1-q}(R - \underline{p}^R) \\ + \left[\frac{Y}{1-q} - 1\right](R-c)\log\left(\frac{Y-(1-(1+r)q)}{Y-(1-q)}\right) - \frac{1-(1+r)q}{1-q}(\underline{p}^R - \bar{p}^2) \\ - \frac{X}{1-q}(\bar{p}^2 - \underline{p}^2) + \left[\frac{X}{1-q} - \frac{1-(1+r)q}{1-q}\right](\bar{p}^2 - c)\log\left(\frac{X-(1-r)q}{X-(1-(1+r)q)}\right) \\ - \frac{(1-r)q}{1-q}(\underline{p}^2 - Q) - \frac{X}{1-q}(Q - \underline{p}^1) \\ + \left(\frac{X}{1-q} - \frac{(1-r)q}{1-q}\right)(Q-c)\log\left(\frac{X}{X-(1-r)q}\right). \end{aligned}$$

The reservation value of a one-searcher is as before (because $1-q$ price at or below R):

$$R = \frac{k}{1-q} + \mathbb{E}[P|P \leq R].$$

The reservation value of the 2b differs as they do not buy from the $2rq$ 1f's pricing above Q. Of firms that do sell, $1-2q$ sell both goods, and $2(1-r)q$ sell but one good, forcing the consumer to continue searching for the other. These considerations along with linearity of expectation gives

$$W = k + 2(\mathbb{E}[P|P < R]\,\mathrm{P}(P < R) + (1-r)qR) + 2rqW.$$

Noting that

$$\mathbb{E}[P|P < R]\,\mathrm{P}(P < R) = \mathbb{E}[P|P \leq R]\,\mathrm{P}(P \leq R) - R\,\mathrm{P}(P = R)$$

with $\mathrm{P}(P = R) = rq$ and $\mathrm{P}(P \leq R) = 1-q$ gives

$$\begin{aligned} W &= k + 2(\mathbb{E}[P|P \leq R](1-q) - rqR + (1-r)qR) + 2rqW \\ &= k + 2\left(\mathbb{E}[P|P \leq R](1-q) + (1-2r)q\left(\frac{k}{1-q} + \mathbb{E}[P|P \leq R]\right)\right) + 2rqW \end{aligned}$$

so

$$
\begin{aligned}
(1-2rq)W &= \left[1+2\frac{(1-2r)q}{1-q}\right]k + 2(1-2rq)\,\mathbb{E}[P|P\le R] \\
\implies W &= \left(\frac{1+q-4rq}{(1-q)(1-2rq)}\right)k + 2\,\mathbb{E}[P|P\le R].
\end{aligned}
$$

Finally,

$$
Q = W - R = \left(\frac{(1-2r)q}{(1-q)(1-2rq)}\right)k + \mathbb{E}[P|P\le R].
$$

Substituting these into the expression for $\mathbb{E}[P|P\le R]$ and solving gives

$$
\mathbb{E}[P|P\le R] = \frac{\Omega_k}{\Lambda_k}k + \frac{\Omega_c}{\Lambda_c}c
$$

where

$$
\begin{aligned}
\Omega_k = &-\frac{2(\beta(2q-3)-2q)\log\left(\frac{\beta(3-2q(r+1))+2q(r+1)}{3\beta}\right)}{\beta-1} - \frac{2(\beta+2)}{\beta-1} \\
&- \frac{2q(r-1)(q(4r-1)-1)(n^c+2n^sq(r+1))}{(2qr-1)(n^c+2n^sqr+n^s)} + \frac{2q(2r-1)(n^c+2n^s)}{n^s(2qr-1)} \\
&- \frac{2(n^c+2n^s)(q(4r-1)-1)(n^c+2n^sq(r+1))}{n^s(2qr-1)(n^c+2n^sqr+n^s)}
\end{aligned}
$$

$$+ \frac{(n^c + 2n^s)(q(4r-1)-1)(n^c + 2n^s(q(r-1)+1))}{n^s(2qr-1)(n^c + 2n^s qr + n^s)}$$

$$+ \frac{2(qr+q-1)(q(4r-1)-1)(n^c + 2n^s(q(r-1)+1))}{(2qr-1)(n^c + 2n^s qr + n^s)}$$

$$- \frac{2q(2r-1)(n^c + 2n^s(q(r-1)+1)) \log\left(\frac{n^c+2n^s}{n^c+2n^s(q(r-1)+1)}\right)}{n^s(2qr-1)}$$

$$- \frac{(q(4r-1)-1)(n^c + 2n^s(r+1))(n^c + 2n^s(q(r-1)+1)) \log\left(\frac{n^c+2n^s(q(r-1)+1)}{n^c+2n^s q(r+1)}\right)}{n^s(2qr-1)(n^c + 2n^s qr + n^s)}$$

$$+ \frac{4q^2(r-1)(2r-1)}{2qr-1} + 4(q-1)$$

$$\Lambda_k = 2(q-1)\left(-\frac{(\beta(2q-3)-2q)\log\left(\frac{\beta(3-2q(r+1))+2q(r+1)}{3\beta}\right)}{\beta-1} + \frac{\beta+2}{1-\beta}\right.$$

$$- \frac{2(n^c + 2n^s)(n^c + 2n^s q(r+1))}{n^s(n^c + 2n^s qr + n^s)} - \frac{2q(r-1)(n^c + 2n^s q(r+1))}{n^c + 2n^s qr + n^s}$$

$$+ \frac{(n^c + 2n^s)(n^c + 2n^s(q(r-1)+1))}{n^s(n^c + 2n^s qr + n^s)} + \frac{2(qr+q-1)(n^c + 2n^s(q(r-1)+1))}{n^c + 2n^s qr + n^s}$$

$$- \frac{(n^c + 2n^s(q(r-1)+1)) \log\left(\frac{n^c+2n^s}{n^c+2n^s(q(r-1)+1)}\right)}{n^s}$$

$$- \frac{(n^c + 2n^s(r+1))(n^c + 2n^s(q(r-1)+1)) \log\left(\frac{n^c+2n^s(q(r-1)+1)}{n^c+2n^s q(r+1)}\right)}{n^s(n^c + 2n^s qr + n^s)}$$

$$\left. + \frac{n^c}{n^s} + 2q(r-1) + 2\right)$$

$$
\begin{aligned}
\Omega_c = &\left(\frac{\beta+2}{2(1-\beta)(1-q)}-1\right)\log\left(\frac{\frac{\beta+2}{2(1-\beta)}+q(r+1)-1}{\frac{\beta+2}{2(1-\beta)}-1}\right) \\
&- \frac{n^s q(1-2q)(1-r)}{(1-q)(n^c+n^s(2qr+1))} + \frac{n^s(1-2q)(1-q(r+1))}{(1-q)(n^c+n^s(2qr+1))} \\
&+ \frac{(1-2q)(n^c+2n^s)}{2(1-q)(n^c+n^s(2qr+1))} + \frac{n^s(1-2q)\left(\frac{n^c+2n^s}{2n^s(1-q)}+\frac{r}{1-q}\right)\log\left(\frac{\frac{n^c+2n^s}{2n^s}-q(1-r)}{\frac{n^c+2n^s}{2n^s}+q(r+1)-1}\right)}{n^c+n^s(2qr+1)} \\
&+ \left(\frac{n^c+2n^s}{2n^s(1-q)}-\frac{q(1-r)}{1-q}\right)\log\left(\frac{n^c+2n^s}{2n^s\left(\frac{n^c+2n^s}{2n^s}-q(1-r)\right)}\right) \\
&+ \left(\frac{n^c+2n^s}{2n^s(1-q)}+\frac{r}{1-q}\right)\log\left(\frac{\frac{n^c+2n^s}{2n^s}-q(1-r)}{\frac{n^c+2n^s}{2n^s}+q(r+1)-1}\right)
\end{aligned}
$$

$$
\begin{aligned}
\Lambda_c = &\left(\frac{\beta+2}{2(1-\beta)(1-q)}-1\right)\log\left(\frac{\frac{\beta+2}{2(1-\beta)}+q(r+1)-1}{\frac{\beta+2}{2(1-\beta)}-1}\right) - \frac{\beta+2}{2(1-\beta)(1-q)} \\
&- \frac{(n^c+2n^s)(n^c+2n^s(1-q(1-r)))}{2n^s(1-q)(n^c+n^s(2qr+1))} + \frac{(1-q(r+1))(n^c+2n^s(1-q(1-r)))}{(1-q)(n^c+n^s(2qr+1))} \\
&- \frac{q(1-r)(n^c+2n^s q(r+1))}{(1-q)(n^c+n^s(2qr+1))} + \frac{(n^c+2n^s)(n^c+2n^s q(r+1))}{n^s(1-q)(n^c+n^s(2qr+1))} \\
&+ \left(\frac{n^c+2n^s}{2n^s(1-q)}-\frac{q(1-r)}{1-q}\right)\log\left(\frac{n^c+2n^s}{2n^s\left(\frac{n^c+2n^s}{2n^s}-q(1-r)\right)}\right) \\
&+ \frac{(n^c+2n^s(1-q(1-r)))\left(\frac{n^c+2n^s}{2n^s(1-q)}+\frac{r}{1-q}\right)\log\left(\frac{\frac{n^c+2n^s}{2n^s}-q(1-r)}{\frac{n^c+2n^s}{2n^s}+q(r+1)-1}\right)}{n^c+n^s(2qr+1)} \\
&- \frac{n^c+2n^s}{2n^s(1-q)} + \frac{q(1-r)}{1-q}.
\end{aligned}
$$

Substituting this back into R, W, and Q and then these into F gives the marginal distributions. Finally, the equal profit condition (4.4.1) gives r. Notice, there must exist an interior ($0 < r < 1$) solution. By assumption, $r = 0$ implies the RHS is strictly greater (which is why we consider this case). But $r = 1$ is not a solution, as then $Q = R$ so that the LHS is strictly greater.

4.4.4 Case 3(b)

In the previous case, $q > \bar{q}^2$ and $\pi_1(R) > \pi_1(\underline{p}^2)$ so that some 1f's priced above the 2f's. Suppose instead that $\pi^1(R) < \pi^1(\underline{p}^2)$ so that it is profitable for 1f's to price just below the 2f's. This occurs when n_1^c is relatively small, so that targeting only 1b's at R is not profitable. A 1f pricing at $\underline{p}^2$ sells to one-searchers, but also to two-searchers in contact with a 2f or a 1f selling the other good. That is, $X_i(p_i) > 0$ and specifically $X_i = (1-q)$ for $p \in [Q, \underline{p}^2]$. This gives a profit of

$$\pi_1(p_A) = N[n_1^c + 2n_1^s(1 - F_A(p_A)) + 2n_2^s(1-q)](p-c).$$

Write l for the proportion of one-firms pricing in $(Q, \underline{p}^2)$. Write

$$Y_l = \frac{n_1^c + 2n_2^s(1-q) + 2n_1^s}{2n_1^s}.$$

Equal profit for the 1f pricing in this region requires the aggregate marginal distribution of prices to be

$$F = Y_l - (Y_l - q)\frac{\underline{p}^2 - c}{p - c}$$

As above, one can derive the lower bound of the 1f's prices in this region as

$$\underline{p}^l = c + \frac{Y_l - q}{Y_l - (1-l)q}(\underline{p}^2 - c).$$

so that the 1f price according to

$$G^1(p) = \begin{cases} 1 & \text{if } \underline{p}^2 \leq p \\ \frac{1}{q}\left[Y_l - (Y_l - (1-r)q)\frac{\underline{p}^2 - c}{p-c}\right] & \text{if } \underline{p}^l \leq p < \underline{p}^2 \\ (1-l) & \text{if } Q \leq p < \underline{p}^l \\ \frac{1}{q}\left[X - (X - (1-r)q)\frac{Q-c}{p-c}\right] & \text{if } \underline{p}^1 \leq p < Q \\ 0 & \text{if } p < \underline{p}^1 \end{cases}$$

As in Case 2, the 2f's will shade down the reservation frontier. The equal profit condition determining $\underline{p}^2$ and $\bar{p}^2$ is exactly the same as in Case 2, as all 1f's price below $\underline{p}^2$. Further, the aggregate marginal distribution required for equal profit among the 2f's is the same. Hence, the marginal distribution of prices among the 2f is simply

$$G^2(p, \infty) = \begin{cases} 1 & \text{if } \bar{p}^2 \leq p \\ \frac{1}{1-2q}\left[X - (X - (1-q))\frac{\bar{p}^2 - c}{p-c} - q\right] & \text{if } \underline{p}^2 \leq p < p \\ 0 & \text{if } p < \underline{p}^2. \end{cases}$$

with

$$\bar{p}^2 = \frac{W(n^c + 2n^s(1-q) - 2cn^s(1-2q)}{2(n^c+n^s)}$$

$$\underline{p}^2 = \frac{W(n^c + 2n^s q) + 2cn^s(1-2q)}{2(n^c+n^s)}.$$

Again, any joint distribution with these marginals and support contained in

$$\{(p_A, p_B)|\ p_A + p_B \leq W \text{ and } \underline{p}^2 \leq p_i \leq \bar{p}^2\}$$

will suffice.

Continuing, these add to an aggregate marginal distribution of

$$F(p) = \begin{cases} 1-q & \text{if } \bar{p}^2 \leq p \\ X - (X-(1-q))\frac{\bar{p}^2-c}{p-c} & \text{if } \underline{p}^2 \leq p < \bar{p}^2 \\ Y_l - (Y_l - q)\frac{\underline{p}^2-c}{p-c} & \text{if } \underline{p}^l \leq p < \underline{p}^2 \\ (1-l)q & \text{if } Q \leq p < \underline{p}^l \\ X - (X-(1-l)q)\frac{Q-c}{p-c} & \text{if } \underline{p}^1 \leq p < Q. \end{cases}.$$

This gives a conditional price distribution

$$F(p|P \leq R) = \begin{cases} 1 & \text{if } \bar{p}^2 \leq p \\ \frac{X}{1-q} - (\frac{X}{1-q} - 1)\frac{\bar{p}^2-c}{p-c} & \text{if } \underline{p}^2 \leq p < \bar{p}^2 \\ \frac{Y_l}{1-q} - (\frac{Y_l}{1-q} - \frac{q}{1-q})\frac{\underline{p}^2-c}{p-c} & \text{if } \underline{p}^l \leq p < \underline{p}^2 \\ \frac{(1-l)q}{1-q} & \text{if } Q \leq p < \underline{p}^l \\ \frac{X}{1-q} - \left(\frac{X}{1-q} - \frac{(1-l)q)}{1-q}\right)\frac{Q-c}{p-c} & \text{if } \underline{p}^1 \leq p < Q. \end{cases} .$$

Integrating this gives expected acceptable prices for a 1b:

$$\begin{aligned} \mathbb{E}[p|P \leq R] = \bar{p}^2 - \left(\frac{X}{1-q}\right)(\bar{p}^2 - \underline{p}^2) + \left(\frac{X}{1-q} - 1\right)(\bar{p}^2 - c)\log\left(\frac{n^c + 2n^s(1-q)}{n^c + 2n^s q}\right) \\ - \left(\frac{Y_l}{1-q}\right)(\underline{p}^2 - \underline{p}^l) + \left(\frac{Y_l}{1-q} - \frac{q}{1-q}\right)(\underline{p}^2 - c)\log\left(\frac{Y_l - (1-l)q}{Y_l - q}\right) \\ - \left(\frac{(1-l)q}{1-q}\right)(\underline{p}^l - Q) - \left(\frac{X}{1-q}\right)(Q - \underline{p}^1) \\ + \left(\frac{X}{1-q} - \frac{(1-l)q}{1-q}\right)(Q-c)\log\left(\frac{X}{X-(1-l)q}\right) \end{aligned} \tag{4.4.2}$$

Reservation values are calculated similarly to case 3(a). The 1b buy from all firms offering their desired goods so that

$$R = \frac{k}{1-q} + \mathbb{E}[P|P \leq R].$$

The 2b's buy the offered good from 1f's pricing below Q, continuing to search for the other good, and buy both goods from 2f's

$$
\begin{aligned}
W &= k + 2\left(\int_{\underline{p}^1}^{Q} p\,dF(p) + (1-l)qR + \int_{\underline{p}^2}^{\bar{p}^2} p\,dF(p)\right) + 2lqW \\
&= k + 2\Bigg[Q(1-l)q - X(Q-\underline{p}^1) + (X-(1-l)q)(Q-c)\log\left(\frac{X}{X-(1-l)q}\right) \\
&\quad + (1-l)qR + \bar{p}^2(1-q) - \underline{p}^2 q - X(\bar{p}^2 - \underline{p}^2) \\
&\quad + (X-(1-q))(\bar{p}^2 - c)\log\left(\frac{X-q}{X-(1-q)}\right)\Bigg] + 2lqW.
\end{aligned}
$$

Substituting in $Q = W - R$, the various prices, and the value of R gives $W = \Omega_W/\Lambda_W$ where

$$
\begin{aligned}
&\Omega_W = \\
&-\frac{2((l-1)q+X)(c(q-1)+\mathbb{E}[P|P\leq R]q - \mathbb{E}[P|P\leq R] - k)\log\left(\frac{X}{(l-1)q+X}\right)}{q-1} \\
&\qquad - \frac{8cn^s q^2}{n^c+n^s} - \frac{2c(q+X-1)(n^c - 2n^s(q-1))\log\left(\frac{X-q}{q+X-1}\right)}{n^c+n^s} \\
&\qquad\quad + \frac{8cn^s q}{n^c+n^s} - \frac{2cn^s}{n^c+n^s} + 2cq + 2\,\mathbb{E}[P|P\leq R]q - \frac{2kq}{q-1} + k \qquad (4.4.3)
\end{aligned}
$$

$$
\begin{aligned}
\Lambda_W &= -2((l-1)q+X)\log\left(\frac{X}{(l-1)q+X}\right) + 2(l-1)q - 2lq \\
&\qquad - \frac{(q+X-1)(n^c+2n^s q)\log\left(\frac{X-q}{q+X-1}\right)}{n^c+n^s} + \frac{q(n^c+2n^s q)}{n^c+n^s} \\
&\qquad\quad + \frac{(q-1)(n^c+2n^s q)}{n^c+n^s} + 2(q-X) + 2X + 1. \qquad (4.4.4)
\end{aligned}
$$

Taking these and substituting in to $\mathbb{E}[P|P \leq R]$ and solving gives

$$\mathbb{E}[P|P \leq R] = \frac{\Omega_k}{\Lambda_k}k + \frac{\Omega_c}{\Lambda_c}c$$

where

$$\begin{aligned}
\Omega_k =& \\
&\left((q + Y_l - 1)(n^c + 2n^s q)\log\left(-\frac{lq - q + Y_l}{q - Y_l}\right) - q(n^c + n^s)\right) \\
&\cdot 2((l-1)q + X)\log\left(\frac{X}{(l-1)q + X}\right) \\
&+ ((2l-1)q - 1)(q + X - 1)(n^c + 2n^s q)\log\left(\frac{X - q}{q + X - 1}\right) - n^c q^2 \log\left(-\frac{lq - q + Y_l}{q - Y_l}\right) \\
&- 2ln^c q^2 - n^c q Y_l \log\left(-\frac{lq - q + Y_l}{q - Y_l}\right) + n^c \log\left(-\frac{lq - q + Y_l}{q - Y_l}\right) \\
&- n^c Y_l \log\left(-\frac{lq - q + Y_l}{q - Y_l}\right) + 2ln^c q - 2n^s q^3 \log\left(-\frac{lq - q + Y_l}{q - Y_l}\right) \\
&- 8ln^s q^3 - 2n^s q^2 Y_l \log\left(-\frac{lq - q + Y_l}{q - Y_l}\right) + 6ln^s q^2 + 2n^s q \log\left(-\frac{lq - q + Y_l}{q - Y_l}\right) \\
&- 2n^s q Y_l \log\left(-\frac{lq - q + Y_l}{q - Y_l}\right) + 2n^c q^2 + n^c q - n^c + 4n^s q^3 + 2n^s q^2 - 2n^s q
\end{aligned}$$

$$
\begin{aligned}
\Lambda_k = 2(q-1)\cdot& \\
&\left[((l-1)q+X)\log\left(\frac{X}{(l-1)q+X}\right)\right. \\
&\cdot\left((q+Y_l-1)(n^c+2n^s q)\log\left(-\frac{lq-q+Y_l}{q-Y_l}\right)-n^c-n^s\right) \\
&+(lq-1)(q+X-1)(n^c+2n^s q)\log\left(\frac{X-q}{q+X-1}\right)-n^c q^2\log\left(-\frac{lq-q+Y_l}{q-Y_l}\right) \\
&+n^c q\log\left(-\frac{lq-q+Y_l}{q-Y_l}\right)-n^c qY_l\log\left(-\frac{lq-q+Y_l}{q-Y_l}\right) \\
&-2n^s q^3\log\left(-\frac{lq-q+Y_l}{q-Y_l}\right)-4ln^s q^3+2n^s q^2\log\left(-\frac{lq-q+Y_l}{q-Y_l}\right) \\
&\left.-2n^s q^2 Y_l\log\left(-\frac{lq-q+Y_l}{q-Y_l}\right)+4ln^s q^2-ln^s q+n^c q+4n^s q^2-3n^s q+n^s\right]
\end{aligned}
$$

$$
\begin{aligned}
\Omega_c = (q+X-1)\log\left(\frac{X-q}{q+X-1}\right)&\left(n^c(lq-1)+2n^s\left(-3lq^2+2lq+q-1\right)\right) \\
&-((l-1)q+X)\log\left(\frac{X}{(l-1)q+X}\right) \\
&\cdot\left(-(q+Y_l-1)(n^c+n^s(4-6q))\log\left(-\frac{lq-q+Y_l}{q-Y_l}\right)+n^c+n^s\right) \\
&-n^c q^2\log\left(-\frac{lq-q+Y_l}{q-Y_l}\right)+n^c q\log\left(-\frac{lq-q+Y_l}{q-Y_l}\right)-n^c qY_l\log\left(-\frac{lq-q+Y_l}{q-Y_l}\right) \\
&-2n^s q^3\log\left(-\frac{lq-q+Y_l}{q-Y_l}\right)+12ln^s q^3+6n^s q^2\log\left(-\frac{lq-q+Y_l}{q-Y_l}\right) \\
&-2n^s q^2 Y_l\log\left(-\frac{lq-q+Y_l}{q-Y_l}\right)-12ln^s q^2-6n^s q\log\left(-\frac{lq-q+Y_l}{q-Y_l}\right) \\
&+4n^s qY_l\log\left(-\frac{lq-q+Y_l}{q-Y_l}\right)+2n^s\log\left(-\frac{lq-q+Y_l}{q-Y_l}\right) \\
&-2n^s Y_l\log\left(-\frac{lq-q+Y_l}{q-Y_l}\right)+3ln^s q+n^c q-4n^s q^2+5n^s q-n^s
\end{aligned}
$$

$$
\begin{aligned}
\Lambda_c = ((l-1)q + X) \log\left(\frac{X}{(l-1)q+X}\right) \\
\cdot \left((q+Y_l-1)(n^c+2n^sq)\log\left(-\frac{lq-q+Y_l}{q-Y_l}\right) - n^c - n^s\right) \\
+ (lq-1)(q+X-1)(n^c+2n^sq)\log\left(\frac{X-q}{q+X-1}\right) - n^cq^2\log\left(-\frac{lq-q+Y_l}{q-Y_l}\right) \\
+ n^cq\log\left(-\frac{lq-q+Y_l}{q-Y_l}\right) - n^cqY_l\log\left(-\frac{lq-q+Y_l}{q-Y_l}\right) - 2n^sq^3\log\left(-\frac{lq-q+Y_l}{q-Y_l}\right) \\
- 4ln^sq^3 + 2n^sq^2\log\left(-\frac{lq-q+Y_l}{q-Y_l}\right) - 2n^sq^2Y_l\log\left(-\frac{lq-q+Y_l}{q-Y_l}\right) + 4ln^sq^2 \\
- ln^sq + n^cq + 4n^sq^2 - 3n^sq + n^s.
\end{aligned}
$$

Substituting back $\mathbb{E}[P|P \leq R]$ gives all variables in terms of exogenous variables and l. As r did in case 3(a), l then solves the equal profit condition $\pi^1(Q) = \pi^1(\underline{p}^2)$ which has no algebraic solution. If there is no solution, then $l = 1$ so that no firm prices below Q which may be possible.

4.4.5 Case 4

In both case 3(a) and case 3(b), some 1f's priced above Q, either above the 2f's or below them. The next possibility is that both obtain. This would occur if, when calculating equilibrium profits in case 3(a) we have $\pi^1(\underline{p}^2) > \pi^1(R)$ or when calculating equilibrium profits in case 3(b) we have $\pi^1(R) > \pi^1(\underline{p}^2)$. Now, a proportion r of the 1f's will price in a range $[\underline{p}^r, R]$ with $\underline{p}^2 < \underline{p}^r$, a proportion l of the 1f's will price in a range $[\underline{p}^l, \underline{p}^2]$ with $Q < \underline{p}^l$, and a complementary proportion $1-l-r$ price below Q. Again, the 2f price in some range $[\underline{p}^2, \bar{p}^2]$ with $\underline{p}^2 + \bar{p}^2 = W$ and equal profit.

What will distinguish this case from case 5 below is that we assume that the ranges below R and $\underline{p}^2$ are sufficiently tight that the two groups do not complete baskets for each other. That is, we construct equilibrium under the assumption that $\underline{p}^r + \underline{p}^l > W$ so that a noisy searcher in contact with both an l firm and an r firm will not buy from either.

In this case profit for the 1f in the relevant regions is given by

$$\pi^1(p) = \begin{cases} N[n_1^c + 2n_2^s(1-l-r)q + 2n_1^s(1-F(p))](p-c) & \text{if } \bar{p}^2 \leq p \leq R \\ N[n_1^c + 2n_1^s(1-F(p)) + 2n_2^s(1-(1+r)q)](p-c) & \text{if } Q < p \leq \underline{p}^2 \\ N[n^c + 2n^s(1-F(p))](p-c) & \text{if } p \leq Q \end{cases}$$

where the second line reflects the fact that a 1f pricing between Q and $\underline{p}^2$ will sell to a 2b who is in contact with either a 2f or a 1f selling the other good and pricing below $\underline{p}^2$.

Similar considerations as in cases 3(a) and 3(b) above imply that the marginal distribution of prices is given by

$$F(p) = \begin{cases} 1-q & \text{if } R \leq p \\ Y - [Y-(1-q)]\frac{R-c}{p-c} & \text{if } \underline{p}^r \leq p < R \\ 1-(1+r)q & \text{if } \bar{p}^2 \leq p < \underline{p}^r \\ X - [X-(1-(1+r)q)]\frac{\bar{p}^2-c}{p-c} & \text{if } \underline{p}^2 \leq p < \bar{p}^2 \\ Y_l - [Y_l-(1-r)q]\frac{\underline{p}^2-c}{p-c} & \text{if } \underline{p}^l \leq p < \underline{p}^2 \\ (1-l-r)q & \text{if } Q \leq p < \underline{p}^l \\ X - [X-(1-l-r)q]\frac{Q-c}{p-c} & \text{if } \underline{p}^1 \leq p < Q \end{cases}$$

where now

$$Y_l = (n_1^c + 2n_2^s(1-(1+r)q) + 2n_1^s)/(2n_1^s)$$

and,

$$Y = (n_1^c + 2n_2^s(1-l-r)q + 2n_1^s)/(2n_1^s).$$

The price bounds for the 2f are the same as in case 3(a)

$$\bar{p}^2 = \frac{W(n^c + 2n^s[1-(1-r)q]) - 2n^s c(1-2q)}{2[n^c + n^s(1+2rq)]},$$

$$\underline{p}^2 = \frac{W(n^c + 2n^s(1+r)q) - 2n^s c(1-2q)}{2[n^c + n^s(1+2rq)]}.$$

2f's price with marginal distribution:

$$G^2(p,\infty) = \begin{cases} 1 & \text{if } \bar{p}^2 \leq p \\ \frac{1}{1-2q}\left[X - [X-(1-(1+r)q)]\frac{\bar{p}^2-c}{p-c} - (1-r)q\right] & \text{if } \underline{p}^2 \leq p < \bar{p}^2 \\ 0 & \text{if } p < \underline{p}^2 \end{cases}$$

on the set $\{(p_A, p_B) | \underline{p}^2 \leq p_i \leq \bar{p}^2 \text{ and } p_A + p_B \leq W\}$. As above, any joint distribution with support contained in this set with the required marginals suffices as an equilibrium.

The distribution played by the 1f's, is just $(F - (1-2q)G^2)/q$ which is given by

$$G^1(p) = \begin{cases} 1 & \text{if } R \leq p \\ \frac{1}{q}\left[Y - [Y-(1-q)]\frac{R-c}{p-c} - (1-2q)\right] & \text{if } \underline{p}^r \leq p < R \\ 1-r & \text{if } \underline{p}^2 \leq p < \underline{p}^r \\ \frac{1}{q}\left[Y_l - [Y_l - (1-r)q]\frac{\underline{p}^2-c}{p-c}\right] & \text{if } \underline{p}^l \leq p < \underline{p}^2 \\ 1-l-r & \text{if } Q \leq p < \underline{p}^l \\ \frac{1}{q}\left[X - [X-(1-l-r)q]\frac{Q-c}{p-c}\right] & \text{if } \underline{p}^1 \leq p < Q. \end{cases}$$

Where the price termini are given by

$$\underline{p}^r = c + \frac{Y-(1-q)}{Y-1-(1+r)q}(R-c),$$

$$\underline{p}^l = c + \frac{Y_l - (1-r)q}{Y_l - (1-l-r)q}(\underline{p}^2 - c),$$

and

$$\underline{p}^1 = c + \frac{X - (1-l-r)q}{X}(Q - c).$$

Turn now to the consumers' reservation values. As above, the reservation value for 1b's is straightforward.

$$R = \frac{k}{1-q} + \mathbb{E}[P|P \leq R].$$

To calculate $\mathbb{E}[P|P \leq R]$, first we derive the conditional price distribution:

$$F(p|P \leq R) = \begin{cases} 1 & \text{if } R \leq p \\ \frac{Y}{1-q} - [\frac{Y}{1-q} - 1]\frac{R-c}{p-c} & \text{if } \underline{p}^r \leq p < R \\ \frac{1-(1+r)q}{1-q} & \text{if } \bar{p}^2 \leq p < \underline{p}^r \\ \frac{X}{1-q} - [\frac{X}{1-q} - \frac{(1-(1+r)q)}{1-q}]\frac{\bar{p}^2-c}{p-c} & \text{if } \underline{p}^2 \leq p < \bar{p}^2 \\ \frac{Y_l}{1-q} - [\frac{Y_l}{1-q} - \frac{(1-r)q}{1-q}]\frac{\underline{p}^2-c}{p-c} & \text{if } \underline{p}^l \leq p < \underline{p}^2 \\ \frac{(1-l-r)q}{1-q} & \text{if } Q \leq p < \underline{p}^l \\ \frac{X}{1-q} - [\frac{X}{1-q} - \frac{(1-l-r)q}{1-q}]\frac{Q-c}{p-c} & \text{if } \underline{p}^1 \leq p < Q. \end{cases}$$

In terms of endogenous variables, the expected price is

$$
\begin{aligned}
\mathbb{E}[P|P \leq R] = R &- \frac{Y}{1-q}(R - \underline{p}^R) \\
&+ \left[\frac{Y}{1-q} - 1\right](R-c)\log\left(\frac{Y-(1-(1+r)q)}{Y-(1-q)}\right) \\
&- \frac{1-(1+r)q}{1-q}(\underline{p}^R - \bar{p}^2) - \frac{X}{1-q}(\bar{p}^2 - \underline{p}^2) \\
&+ \left[\frac{X}{1-q} - \frac{1-(1+r)q}{1-q}\right](\bar{p}^2 - c)\log\left(\frac{X-(1-r)q}{X-(1-(1+r)q)}\right) \\
&- \left(\frac{Y_l}{1-q}\right)(\underline{p}^2 - \underline{p}^l) + \left(\frac{Y_l}{1-q} - \frac{(1-r)q}{1-q}\right)(\underline{p}^2 - c)\log\left(\frac{Y_l - (1-l-r)q}{Y_l - (1-r)q}\right) \\
&- \left(\frac{(1-l)q}{1-q}\right)(\underline{p}^l - Q) - \frac{(1-l-r)q}{1-q}(\underline{p}^l - Q) - \left(\frac{X}{1-q}\right)(Q - \underline{p}^1) \\
&+ \left(\frac{X}{1-q} - \frac{(1-l-r)q}{1-q}\right)(Q-c)\log\left(\frac{X}{X-(1-l-r)q}\right).
\end{aligned}
\tag{4.4.5}
$$

As for W, the 2b's still buy only from 1f's with $p \leq Q$ (whom they meet with probability $(1-l-r)q$) whereupon they continue as a 1b, and also from 2f's. This yields a similar expression to case 3(b).

$$
\begin{aligned}
W &= k + 2\left(\int_{\underline{p}^1}^{Q} p\,dF(p) + (1-l-r)qR + \int_{\underline{p}^2}^{\bar{p}^2} p\,dF(p)\right) + 2(l+r)qW \\
&= k + 2\left[Q(1-l-r)q - X(Q-\underline{p}^1)\right. \\
&\qquad + (X-(1-l-r)q)(Q-c)\log\left(\frac{X}{X-(1-l-r)q}\right) \\
&\qquad + (1-l-r)qR + \bar{p}^2(1-(1+r)q) - \underline{p}^2(1-r)q - X(\bar{p}^2-\underline{p}^2) \\
&\qquad \left. + (X-(1-(1+r)q))(\bar{p}^2-c)\log\left(\frac{X-(1-r)q}{X-(1-(1+r)q)}\right)\right] + 2(l+r)qW.
\end{aligned}
$$

As above, one can plug in $Q = W - R$ and R and obtain an expression for W in terms of parameters and $\mathbb{E}[P|P \leq R]$. Plugging the resulting value of W, prices, etc. into $\mathbb{E}[P|P \leq R]$ leads to a linear equation in $\mathbb{E}[P|P \leq R]$, k, and c, ultimately resulting in an equation for $\mathbb{E}[P|P \leq R]$, W, and R which are linear functions of k and c in terms of parameters and l and r. The exact expressions are too long to print. Finally, as in previous cases, the values for l and r derive from equal profit:

$$
\bar{\pi}^1 = \pi^1(R), \quad \bar{\pi}^1 = \pi^1(\underline{p}^2), \quad \text{and} \quad \bar{\pi}^1 = \pi^1(Q).
$$

But these do not have algebraic solution.

4.4.6 Case 5

The last case to consider is similar to case 4, but now some 1f's above Q can make baskets with one another. In this case, the 1f's price in the same three ranges as in case 4: $[\underline{p}^1, Q]$, $[\underline{p}^l, \underline{p}^2]$, and $[\underline{p}^r, R]$. Here, now, $\underline{p}^r + \underline{p}^l < W$ so that $X_i(p)$ is not a constant. In this case, firms care not just about their rank in the distribution, but also the proportion of firms who can make a basket with them. Write

$$\hat{p}^l \equiv W - \underline{p}^r \quad \text{and} \quad \hat{p}^r \equiv W - \underline{p}^l.$$

The profit function for the 1f becomes

$$\pi^1(p) = \begin{cases} N[n_1^c + 2n_1^s(1 - F(p)) + 2n_2^s(1 - l - r)q](p - c) & \text{if } \hat{p}^r \leq p \leq R \\ N[n_1^c + 2n_1^s(1 - F(p)) + 2n_2^s F(W - p)](p - c) & \text{if } \underline{p}^r \leq p < \hat{p}^r \\ N[n_1^c + 2n_1^s(1 - F(p)) + 2n_2^s(1 - (1 + r)q)](p - c) & \text{if } \hat{p}^l < p \leq \underline{p}^2 \\ N[n_1^c + 2n_1^s(1 - F(p)) + 2n_2^s F(W - p)](p - c) & \text{if } \underline{p}^l \leq p < \hat{p}^l \\ N[n^c + 2n^s(1 - F(p))](p - c) & \text{if } p \leq Q. \end{cases}$$

The derivation of F is the same as above for the first, third, and fifth cases. But the second and third are interdependent and so we proceed slightly differently. Suppose $p \in [\underline{p}^r, \hat{p}^r]$ so that $p' \equiv W - p \in [\underline{p}^l, \hat{p}^l]$. Equal profit requires

$$(n_1^c + 2n_1^2(1 - F(p)) + 2n_2^s F(p'))(p - c)$$
$$= (n_1^c + 2n_1^2(1 - F(p')) + 2n_2^s F(p))(p' - c)$$

which gives

$$F(p') = F(p)\left(\frac{n_2^s W - (n_2^s + n_1^s)c + (n_1^s - n_2^s)p}{n_1^s W - (n_1^s + n_2^s)c + (n_2^s - n_1^s)p}\right) + \frac{(n_1^c + 2n_1^s)(W - 2p)}{2n_1^s W - 2(n_1^s + n_2^s)c + 2(n_2^s - n_1^s)p}.$$

Substituting this into $\pi^1(p) = \bar{\pi}^1$ and solving gives

$$F(p) = \frac{\bar{\pi}^1 - N\left(n_1^c + 2n_1^s + 2n_2^s\left(\frac{(n_1^c+2n_1^s)(W-2p)}{2n_1^s W - 2(n_1^s+n_2^s)c + 2(n_2^s - n_1^s)p}\right)\right)(p-c)}{N\left(2n_2^s\left(\frac{n_2^s W - (n_2^s + n_1^s)c + (n_1^s - n_2^s)p}{n_1^s W - (n_1^s + n_2^s)c + (n_2^s - n_1^s)p}\right) - 2n_1^s\right)(p-c)}.$$

Notice that the expression is exactly the same for $p \in [\underline{p}^l, \hat{p}^l]$. Finally, $\bar{\pi}^1$ can easily be read off from $\pi(R) = N[n_1^c + 2n_2^s(1-l-r)q](R-c)$ so that for $p \in [\underline{p}^l, \hat{p}^l] \cup [\underline{p}^r, \hat{p}^r]$

$$F(p) \quad = \quad H(p) \quad \equiv \quad \frac{[n_1^c + 2n_2^s(1-l-r)q](R-c) - \left(n_1^c + 2n_1^s + 2n_2^s\left(\frac{(n_1^c+2n_1^s)(W-2p)}{2n_1^s W - 2(n_1^s+n_2^s)c + 2(n_2^s - n_1^s)p}\right)\right)(p-c)}{\left(2n_2^s\left(\frac{n_2^s W - (n_2^s + n_1^s)c + (n_1^s - n_2^s)p}{n_1^s W - (n_1^s + n_2^s)c + (n_2^s - n_1^s)p}\right) - 2n_1^s\right)(p-c)}$$

Given this, we can solve for $\underline{p}^r$ and $\underline{p}^l$ as

$$F(\underline{p}^r) = 1 - (1 + r)q \quad \text{and} \quad F(\underline{p}^l) = (1 - l - r)q.$$

These equations are quadratic in p and so have two roots. It is not clear whether this delivers true multiplicity or whether one solution or another is false. I would conjecture that only the smaller roots satisfy equilibrium, but this requires more investigation. The other limits, $\underline{p}^1$, $\bar{p}^2$, and $\underline{p}^2$ are given as in case 4. Equal profit, then, requires the following marginal distribution of prices.

$$F(p) = \begin{cases} 1 - q & \text{if } R \leq p \\ Y - [Y - (1-q)]\frac{R-c}{p-c} & \text{if } \hat{p}^r \leq p < R \\ H(p) & \text{if } \underline{p}^r \leq p < \hat{p}^r \\ 1 - (1+r)q & \text{if } \bar{p}^2 \leq p < \underline{p}^r \\ X - [X - (1-r)q]\frac{\bar{p}^2-c}{p-c} & \text{if } \underline{p}^2 \leq p < \bar{p}^2 \\ Y_l - [Y_l - (1-r)q]\frac{\underline{p}^2-c}{p-c} & \text{if } \hat{p}^l \leq p < \underline{p}^2 \\ H(p) & \text{if } \underline{p}^l \leq p < \hat{p}^l \\ (1-l-r)q & \text{if } Q \leq p < \underline{p}^l \\ X - [X - (1-l-r)q]\frac{Q-c}{p-c} & \text{if } \underline{p}^1 \leq p < Q \end{cases}$$

The 2f's follow the same marginal distribution as in case 4, and the 1f's, then, set $G^1(p) = (F(p) - (1-2q)G^2(p,\infty))/q$. Reservation values take exactly the same form as in case 4, and $\mathbb{E}[P|P \leq R]$ integrates just as above, but it is no longer a linear equation in c and k because of the radicals involved in $\underline{p}^l$ and $\underline{p}^r$. An equilibrium, then, only requires solving for r and l which follows from equal profit as above. This case is, perhaps the most interesting and also the most bedevilling. More investigation is required.

4.5 Conclusion

This paper considered equilibrium in a model of multi-product retailing and sequential search. In a simple case where all firms can offer all goods, equilibrium very closely mirrors that of single product search. This obtains in equilibrium as a direct result of the fact that, while profit per good depends on one's rank in the price distribution for that good, overall profits are separable in price.

If not all firms offer both goods, but the number who do not is small, a similar result holds. The marginal distributions are of exactly the same form as would obtain in a single product model, except consumers reservation values are increased to reflect the extra search cost entailed in visiting multiple stores.

As the proportion of firms who can not sate the consumers' demand increases, an interesting effect obtains. Single-product firms must price at the bottom of the distribution for two-good buyers to demand from them. Bunching up at the bottom of the distribution induces two-good firms to raise prices – but prices can not be raised on both goods beyond the reservation curve of consumers. These firms, then, cut their price, reducing the highest price charged in equilibrium.

Several other cases are possible as fewer and fewer firms offer both goods. The most interesting concerns the case where single-product firms make baskets for each other. Two-good demanders have a maximum price they will pay for the basket of two goods which is higher than twice the price at which they will buy individual goods. As noisy search allows consumers to contact two firms at once, a one-good firm can set a price at which they could not sell by themselves, but at which they will sell if the consumer is in contact with another firm offering a sufficient low price. That is if two firms can together make a basket. This produces a truly global pricing problem different not just in form but also in kind from the standard model. This case requires further investigation.

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