1-1-2015

Shape and Other Properties of 1324-Avoiding Permutations

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Abstract
Of the three Wilf classes of permutations avoiding a single pattern of length 4, the exact enumerations for two of them were found by Gessel (1990) and Bona (1997). More recently, the Stanley-Wilf conjecture was proved by Marcus and Tardos (2004) relying on work by Furedi and Hajnal (1992), and work by Klazar (2000). Work by Arratia (1999) shows that this implies the existence of an exponential growth rate for any of these classes of permutations. 1324-avoiding permutations belong to the final Wilf class of permutations avoiding a single pattern of length 4. Unlike the other two, not only is the exact enumeration yet to be found, the growth rate is also unknown. We explore the known bounds to the growth rate of this class as well as discuss possible approaches to improving them. There has also been recent work done by Dokos and Pak (2014) and Miner and Pak (2014) regarding the shape of other classes of permutations. In the second part of this thesis, we explore the shape of 1324-avoiding permutations, and show that there are two regions that decay to 0 exponentially, which has a size that depends on the growth rate of the class.

Degree Type
Dissertation

Degree Name
Doctor of Philosophy (PhD)

Graduate Group
Mathematics

First Advisor
Robin Pemantle

Keywords
1324, avoiding, pattern, permutations, shape

Subject Categories
Mathematics

SHAPE AND OTHER PROPERTIES OF 1324-AVOIDING PERMUTATIONS

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A DISSERTATION

in

Mathematics

Presented to the Faculties of the University of Pennsylvania

in

Partial Fulfillment of the Requirements for the

Degree of Doctor of Philosophy

2015

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Acknowledgments

My advisor's advice was critical in selecting a problem to work on, as well as in finishing the writing of this dissertation. Thus, I am deeply grateful to Robin Pemantle for his invaluable help. From the start of my grad student years, when he taught the probability course I took, to the end, I have learned much through his guidance.

I would also like to thank the other two members of my dissertation committee, Henry Towsner and Nicolas Fraiman, for checking my work. I am also thankful for the efforts of the faculty who taught me throughout my years at Penn, particularly James Haglund, who taught me combinatorics. I am also grateful to our department staff, Janet Burns (who retired last year), Reshma Tanna, Monica Pallanti, Robin Toney, and Paula Scarborough, as well as the building staff, who have made sure that things functioned smoothly in the department. I will also fondly remember my time with the other grad students, in particular Jonathan Kariv, Torin Greenwood, and Sneha Subramanian. I found our discussions on various topics enriching, and they helped me to keep my morale up.

Finally, I would like to thank my parents, Poh Guan Seng and Choo Ee Khoon, who despite not having the opportunity for higher education themselves, made tremendous sacrifices in order to allow me the opportunity to complete my undergraduate education at Cornell University, without which I would not be here today.

ABSTRACT

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Wei Quan Julius Poh

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Of the three Wilf classes of permutations avoiding a single pattern of length 4, the exact enumerations for two of them were found by Gessel (1990) and Bona (1997). More recently, the Stanley-Wilf conjecture was proved by Marcus and Tardos (2004) relying on work by Furedi and Hajnal (1992), and work by Klazar (2000). Work by Arratia (1999) shows that this implies the existence of an exponential growth rate for any of these classes of permutations. 1324-avoiding permutations belong to the final Wilf class of permutations avoiding a single pattern of length 4. Unlike the other two, not only is the exact enumeration yet to be found, the growth rate is also unknown. We explore the known bounds to the growth rate of this class as well as discuss possible approaches to improving them. There has also been recent work done by Dokos and Pak (2014) and Miner and Pak (2014) regarding the shape of other classes of permutations. In the second part of this thesis, we explore the shape of 1324-avoiding permutations, and show that there are two regions that decay to 0 exponentially, which has a size that depends on the growth rate of the class.

Contents

Chapter 1

Introduction

1324-avoiding permutations are a type of pattern avoiding permutation that avoids the pattern 1324. A permutation pattern is a subpermutation.

Definition 1. *A permutation σ contains the* **pattern** π *of length k if there exist $x_1 < x_2 < ... < x_k$ such that $\sigma(x_i) < \sigma(x_j)$ if and only if $\pi(i) < \pi(j)$.*

When written in one-line notation, this amounts to picking out some subset of the numbers, then "collapsing" the selected numbers so that they range from 1 to k, while preserving the order. Unless otherwise stated, permutations will be assumed to be in one-line notation, and will behave very much like a length n word in n letters from 1 to n.

Definition 2. *A* **π-avoiding permutation** *is a permutation that does not contain the pattern π.*

In the case of permutations avoiding the 1324 pattern, if we represent a permutation of length n, σ, as a string of n integers $\sigma(1), \sigma(2), ..., \sigma(n)$, and if we were to pick any 4 of them $0 < k_1 < k_2 < k_3 < k_4 < n+1$, then write 1 for the smallest of $\{\sigma(k_1), \sigma(k_2), \sigma(k_3), \sigma(k_4)\}$, 2 for the second smallest, etc., $\sigma(k_1), \sigma(k_2), \sigma(k_3), \sigma(k_4)$ would not become 1,3,2,4. In other words, we do not have $\sigma(k_1) < \sigma(k_3) < \sigma(k_2) < \sigma(k_4)$.

For a pattern π, let $s_n(\pi)$ be the number of π-avoiding permutations of length n, and $S_n(\pi)$ be the set of such permutations.

There are a couple of things that we will explore here about these permutations, their enumeration and their "shape". In this introduction, we will cover the basic results known about pattern avoiding permutations, and touch on recent work done on shape. First, we address enumeration. What is the number of such permutations of length n? Failing that, how fast does it grow? We know from the Stanley-Wilf conjecture, proved by Marcus and Tardos [13], that it grows at an exponential rate. That is, $\lim_{n\to\infty} s_n(1324)^{1/n}$ converges on some positive finite real, where $s_n(1324)$ is the number of 1324-avoiding permutations of length n.

The Stanley-Wilf conjecture was formulated in the late 1980s or early 1990s, as the guess that for any pattern π, there exists some constant C_π such that

$\lim_{n\to\infty} s_n(\pi) \leq C_\pi^n$ for all n. Arratia [2] then showed in 1999 that this was equivalent to the convergence of $\lim_{n\to\infty} s_n(\pi)^{1/n}$, so that there exists a constant c_π such that $\lim_{n\to\infty} s_n(\pi)^{1/n} = c_\pi$. This was done by showing that $\log(s_n(\pi))$ is superadditive, via construction of an injection from pairs of π-avoiding permutations of length n and length m to the set of π-avoiding permutations of length $n+m$, then applying Fekete's lemma on subadditive sequences. c_π is the *growth rate* of π-avoiding permutations.

In more detail, in order to get such an injection, we can assume without loss of generality that for our length k pattern π, if $\pi(i') = 1$ and $\pi(j') = k$, then $j' < i'$. Then for two π-avoiding permutations of length n and length m, σ_n and σ_m, we can create a unique π-avoiding permutation of length $n+m$ simply by concatenating them as follows. We construct σ_{n+m} such that for $i \leq n$, $\sigma_{n+m}(i) = \sigma_n(i)$, and $\sigma_{n+m}(i) = n + \sigma_m(i-n)$ otherwise. Since the pattern occurs in neither σ_n nor σ_m, if it is to occur in σ_{n+m}, $\pi(i')$ must occur in the "left" part that came from σ_n while $\pi(j')$ must occur in the "right" part that came from σ_m. But this is not possible since $j' < i'$. Thus the new permutation avoids π as well.

In 1992, the Füredi-Hajnal conjecture was posed in [10], regarding 0-1 matrices avoiding permutation matrices. 0-1 matrices are matrices whose entries are all either 0 or 1. In this context, avoidance is a similar concept, with a permutation matrix P being said to be avoided by a 0-1 matrix if there does not exist a submatrix the same size as P which has 1s wherever P has 1s. The number of 1s in any n by n 0-1 matrix is at most quadratic, and the conjecture was that for any given permutation matrix P, the largest number of 1s possible while avoiding P is in fact only $O(n)$. That is to say, if $f_n(P)$ is the largest number of 1s possible in an n by n 0-1 matrix avoiding P, then $f_n(P) = O(n)$.

This conjecture was then shown by Klazar in 2000 [12] to imply the Stanley-Wilf conjecture. A summary of the argument is as follows. Let $g_n(P)$ be the number of n by n 0-1 matrices that avoid P. Consider such a matrix of size $2n$ by $2n$. Divide this into 2 by 2 blocks and replace each block with only 0s with a single 0, and all others with a 1, obtaining an n by n matrix. This matrix then avoids P as well, since any pattern here can be found in the original matrix. Now this n by n matrix can be mapped to by such a method by at most 15^t $2n$ by $2n$ matrices, where t is the number of 1s in the n by n matrix, since each 1 entry can come from at most 15 different 2 by 2 blocks. Since t is at most $f_n(P)$, we have $g_{2n}(P) \leq 15^{f_n(P)} g_n(P)$. Thus the Füredi-Hajnal conjecture $f_n(P) = O(n)$ implies that $g_{2n}(P)/g_n(P)$ can be bounded by some c^n, which in turn gives the Stanley-Wilf conjecture as a consequence.

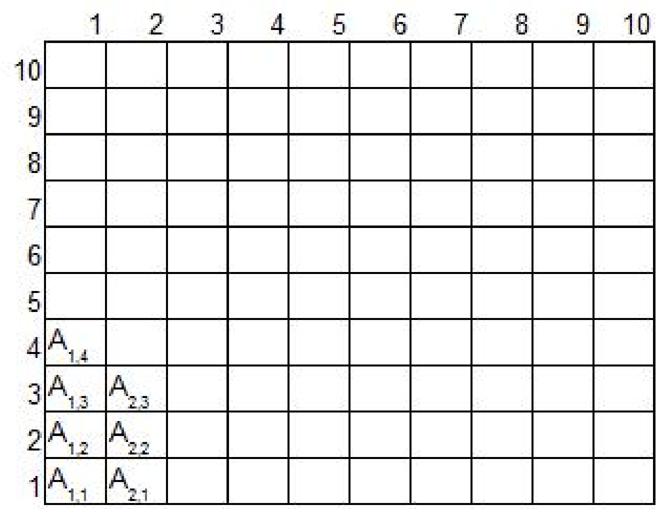
This conjecture was then proved by Marcus and Tardos, completing the proof of the Stanley-Wilf conjecture. A brief sketch of their proof is as follows. Suppose A is an n by n 0-1 matrix avoiding P with the maximum possible number of 1 entries, $f_n(P)$, where P is k by k. We may assume that k^2 divides n, since this will not affect the desired result that $f_n(P) = O(n)$. A is then divided into k^2 by k^2 blocks, and a n/k^2 by n/k^2 matrix B is created by writing a 0 for an entry if the corresponding block in A contains only 0s, and 1 otherwise. They then prove an upper bound of $k\binom{k^2}{k}$ on the number of blocks with 1 entries in at least k different columns in each column of blocks, via the pigeonhole principle. This also applies to the number of blocks with 1 entries in at least k different rows, in each row of blocks. Then by counting the number of 1s in both types of blocks, as well as blocks that neither have 1s in at least k different rows nor in k different columns, it was shown that $f_n(P) \leq 2k^4\binom{k^2}{k}n$, thereby yielding the desired result.

While we know from all this that we have a growth rate $c_\pi := \lim_{n\to\infty} s_n(\pi)^{1/n}$ for every pattern π, this does not tell us what that rate is in general - there is a great deal of looseness in the bounds used in those proofs. A brief discussion of the work done on specific cases is left to the next chapter, and some approaches and work on bounds on the growth rate for the 1324 pattern with some partial results will also be discussed.

Coming to the second question, what is the "shape" of such permutations? What can we say about the probability of, say, $\sigma(k)$ being l for a (uniform) randomly chosen permutation σ? How likely is it that $\sigma(k)$ falls within a certain range of values? Similar questions have been asked and answered in considerable detail for patterns of length 3. [14]

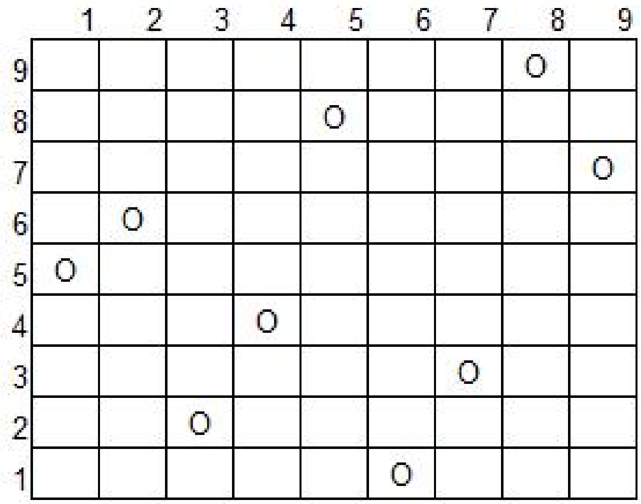
As a visual aid, consider an n by n grid, A. The columns run from 1 to n going from left to right, and the rows run from 1 to n going from the bottom to the top. Let the square in the ith column and jth row be $A_{i,j}$, as seen in Figure 1.1.

Figure 1.1: Grid for n=10



We represent a length n permutation σ by placing n points in the squares of this grid. For each $\sigma(i)$, place a point in $A_{i,\sigma(i)}$, leaving the other squares empty. This is really a permutation matrix rotated anticlockwise by $\pi/2$, with points instead of 1s and empty squares instead of 0s. An example is in figure 1.2. We can formally associate this grid with the corresponding permutation.

Figure 1.2: Grid corresponding to the permutation 562481397



Definition 3. *The* **associated grid** *of σ is the matrix $A(\sigma)$ with $A(\sigma)_{i,\sigma(i)} = 1$ for each i, and all other entries 0.*

For ease of visualisation, we will continue to think of the matrix as a grid with column and row ordering as stated earlier, and with the 1s as points and 0s as empty cells. This is the visualisation that is used in the figures in section 3.1.

Suppose we took all the π-avoiding permutations of length n, and added their grids together. Then each entry tells us the number of such permutations that has a 1 there. Then dividing by $s_n(\pi)$, we get the proportion of such permutations that have a 1 at that point. In other words, if we took a π-avoiding permutation of length n uniformly at random, we would find a 1 there with that probability. We can call this our "shape".

Definition 4. *The length n* **shape** *of π is the matrix*

$$\mathrm{SH}_\pi(n) = (1/s_n(\pi)) \sum_{\sigma \in S_n(\pi)} A(\sigma)$$

.

In the next chapter we start by following the progress made by a number of researchers working on the growth rate of 1324-avoiding permutations, and sketch certain results. In the second section, we will look at the transfer matrix, which is a well known method that is applicable to this problem. I will then show how this works with 213-avoiding permutations, and in the third section, how it may apply to the upper and lower bounds in the 1324-avoiding case along with a standard "top insertion" mechanism.

In section 2.4 I introduce the concept of "local" growth rates, deriving some relations between this and the location of the top element, using top insertion in the 1324-avoiding case. I will also then show that a conjecture about the convergence of this local growth rate implies a result on the distribution of the top element. Finally, in section 2.5, I conclude our discussion of bounds with an approach to the lower bound inspired by data from my simulations, which will be discussed in section 3.1. While the bound of 8 obtained is worse than known bounds, the approach currently contains a lot of looseness in the bounds that might be open to refinement.

In chapter 3, we first start by discussing normalised shape, mentioning work done by others on the shape of other classes of permutations. I then briefly explain the method, suggested by my advisor, of estimating the shape. We will look at why it may look the way it does and its relation to top insertion, as well as making conjectures about the normalised shape. Finally, in section 3.2, I will prove the main result, that we have exponential decay of entries in at least two regions around corners of the normalised shape, as n increases. This region is dependent on the growth rate, and increases in size as the lower bound is improved.

Chapter 2

Growth rates

2.1 Introduction and background

We first restate the definition of the growth rate.

Definition 5. *The* **growth rate** *of 1324-avoiding permutations is*

$c_{1324} := \lim_{n\to\infty} s_n(1324)^{1/n}.$

Definition 6. *For two given patterns σ and π of length k, the σ-avoiding permutations and π-avoiding permutations are said to be in the same* **Wilf class** *if $s_n(\sigma) = s_n(\pi)$ for all n.*

The exact enumeration of length n permutations avoiding a pattern of length 3 has been known for a long time [15] to be the Catalan numbers, and hence are all in the same Wilf class, with a growth rate of 4. Patterns of length 4 is where it gets more interesting.

While there are 24 possible patterns of length 4, many of them are symmetric. If we take their associated grid defined in the introduction, we can rotate or flip them to form another pattern. For example, 1243 can be rotated to form 4312, 2134, and 3421. These belong in the same symmetric class, and since a similar transformation of the corresponding pattern avoiding permutations preserves the avoidance, they are in the same Wilf class as well. There are 7 symmetry classes. Work done by Stankova [16] [17] and Babson and West [4] showed that the symmetry classes can be grouped into only three Wilf classes. The Wilf class that the 1324-avoiding permutations belong to has only one symmetry class, while the other two Wilf classes contain two and four symmetry classes.

Out of these three Wilf classes of length 4, the 1324-avoiding permutations belongs to the only one for which the exact enumeration has not been found. The other two were settled by Gessel [11] and Bona [5], and have growth rates of 8 and 9. For 1324, however, not even the growth rate is known. Recent work by a number of people has been done on the upper and lower bounds, successively narrowing them. We first look at the upper bound and summarise some of the arguments.

In 2012, Claesson, Jelinek, and Steingrimsson [8] showed that the growth rate is at most 16. They used the idea of a *merge* of permutations, where a permutation is a merge of two smaller permutations if it can be divided into two patterns that match those two smaller permutations. For example, 15243 can be divided into 124 and 53, corresponding to the patterns 123 and 21, so 15243 is a merge of 123 and 21. They then defined two special types of merges. First, a direct sum, where the two constituent permutations are concatenated with the second being assigned higher numbers than any of the first. For example, the direct sum $123 \oplus 21$ would give 12354. Second, a skew sum, which is the same except that the first part is assigned higher numbers than any of the second. Thus the skew sum $123 \ominus 21$ would become 34521.

Next, they showed that for three, possibly empty permutations σ, τ, and ρ, a permutation that avoids $\sigma \oplus (\tau \ominus 1) \oplus \rho$ can be written as a merge of one permutation avoiding $\sigma \oplus (\tau \ominus 1)$, and another avoiding $(\tau \ominus 1) \oplus \rho$. This was done by carefully dividing the original permutation into two parts that avoid the given patterns, which they coloured red and blue. Going from left to right, they coloured elements red unless it would complete a red $\sigma \oplus (\tau \ominus 1)$ pattern or if there is already a blue element smaller than the one being coloured. It was then shown after some work that the red and blue parts do indeed avoid the required patterns. At this point, we can see that this applies to 1324 with σ, τ, and ρ all 1, with each 1324-avoiding permutation expressible as a merge of a 132-avoiding permutation, and a 213-avoiding permutation.

Finally, by counting the number of possible ways to merge permutations, they found that if every permutation of a certain class of pattern avoiding permutations can be expressed as a merge of two other classes, then the square root of its growth rate is at most the sum of the square roots of the other two classes. Since permutations avoiding patterns of length 3 have a growth rate of 4, an upper bound of 16 is obtained.

Later in 2012, Bona [6] improved this bound to $7 + 4\sqrt{3} \approx 13.93$ by refining the above argument. Using the same rules for colouring the elements of a 1324-avoiding permutation, the red and blue elements are further split into two categories each, so that the elements are labelled A, B, C, and D. The elements which are left-to-right minima among the red entries are labelled A, while the other red entries are labelled B. Symmetrically, the elements which are right-to-left maxima among the blue entries are labelled D, while the other blue entries are labelled C.

This labelling can be used to produce two words in these four letters for each 1324-avoiding permutation σ. One word is just the labels of σ going from left to right, or in other words, it has its ith letter having the same label as the element $(i, \sigma(i))$. The other word is the labels going from bottom to top, or in other words, it has its ith letter having the same label as the element with second coordinate

i. Through reconstruction of a length n 1324-avoiding permutation from the pair of words associated with it, it was shown that each pair of such words of length n may be produced by at most one 1324-avoiding permutation. That is, the map from such a permutation to pairs of words is injective. This alone clearly gives an upper bound of 16, but the improvement lay in the observation that in neither of the two words can B immediately follow C. We see that if a B is directly to the right of a C, it has to be lower than the C by the rules of the colouring. This implies that there has to be an A to the lower left of both of them, and a D to the upper right, which forms a 1324 pattern. A similar argument holds when we consider a B directly above a C.

Now, counting the number of all possible pairs of such words that do not have a B directly following a C, it was shown to be bounded by $(7+4\sqrt{3})^n$ using standard generating function techniques.

In 2015, Bona [7] again improved this bound to 13.73718, by further refining the argument.

Turning to the lower bound, Albert, Elder, Rechnitzer, Westcott, and Zabrocki [1] showed in 2006 with the aid of heavily computational methods that the growth rate is at least 9.35. The general idea was to use a certain insertion encoding, a variant of the simple top insertion, to build up 1324-avoiding permutations. Structures of allowable insertions are thought of as states, and a 1324-avoiding permutation can be described as a series of transitions between these states. The growth of the number of such possibilities then corresponds to an eigenvalue of what is known as a transfer matrix. As a similar idea applies with the usual top-element insertion, this technique is discussed in the next section.

In early 2015, Bevan [3] further improved this bound to 9.81, by showing that a subset of 1324-avoiding permutations can be built by alternating sets of points that form trees in a "Hasse graph", then counting these to obtain a minimum growth rate.

2.2 The transfer matrix

Suppose we have a class of objects that can be built up from a base state or states through a series of transitions from one state to another, where each state has a certain range of allowable transitions, regardless of what happened before. We can then create a matrix T that encodes this information nicely, and allow us to count the objects by repeated multiplication. We now look at this formally.

Consider the set of words whose letters are positive integers.

Definition 7. *A* **transfer rule** *S is a set of pairs of positive integers.*

Definition 8. *A word is* **valid** *with respect to S if for any adjacent letters x and y, $(x, y) \in S$.*

Definition 9. *A* **transfer matrix** *$T(S)$ is a 0-1 matrix such that $T(S)_{y,x}$ is 1 if $(x, y) \in S$ and 0 otherwise.*

More generally, we could use a multiset for S and allow positive integer entries greater than 1 for the transfer matrix, but this is unnecessary for the insertion we will be looking at.

Definition 10. *An* **initial state set** *B is a set of positive integers.*

Definition 11. ***b*** *is an* **initial state vector** *corresponding to B such that $b_i = 1$ if $i \in B$ and 0 otherwise.*

Now we can perform a straightforward induction. Notice that b_i is the number of valid words of length 1 which end in the letter i, and starting with a letter from B. If we have a vector $\mathbf{v}$ such that v_i is the number of valid words of length n which end in a letter of the form i, and starting with a letter from B, then $T(S)\mathbf{v}$ has the same properties, for valid words of length $n+1$. This is because $T(S)_{j,i}v_i$ corresponds to taking the words counted by v_i, and appending j to them iff it is valid. That gives us the following.

Proposition 1. *The sum of the entries of $T(S)^{n-1}\boldsymbol{b}$ is the number of valid words of length n that start with a letter from B.*

Now we discuss how this relates to pattern avoiding permutations. Consider the easier case of 213-avoiding permutations. We may build any such permutation through the well known process of "top insertion", starting with 1 and successively adding higher numbers until we obtain the desired permutation. For example, 4231 can be obtained by starting with 1, inserting 2 to the left to form 21, inserting 3 in the middle to obtain 231, then inserting 4 to the left to form 4231. The sequence of insertions exists and is unique for each permutation. This is more rigorously defined in the following section.

We look at which insertions do not create a 213 pattern. Consider the first 21 pattern in such a string. The next insertion cannot occur after this, or it would create a 213 pattern. Therefore the insertions can only happen immediately to the left or right, or anywhere in between, the longest initial 21-avoiding substring. Conveniently for this case, this is just the longest initial ascending substring, just before the first descent, if any. If this is length k, then there are $k+1$ places to make the next top insertion. The valid places for insertion only depend on this initial string, so we may divide possible strings into "states" based on their longest initial 21-avoiding substring. Since there is only one possible such substring for each positive length k, we may label the state with that number.

Definition 12. *A 213-avoiding permutation is in the* **state** k *if its longest initial 21-avoiding substring is of length* k.

If we insert the next element in the ith possible spot counting from the left, so that there are $i-1$ elements of the initial 21-avoiding substring to its left, the new permutation has a longest initial 21-avoiding substring of length i, since there must be a descent after the new top element if there are any more elements to the right. Thus it is in state i. Associate this with the letter i. Then let S_{213} be the set that include all pairs (i, j) where i is any positive integer and j is at most $i+1$. We can associate the word 1 as the initial string of length 1, which is in state 1, and then any word starting with 1 that is valid with respect to S_{213} corresponds to a series of top insertions that results in a unique 213-avoiding permutation. This bijection means that given the initial state vector $\mathbf{e_1}$, we can use the above work.

Proposition 2. *The sum of the entries of* $T(S_{213})^{n-1}\boldsymbol{e_1}$ *is the number of 213-avoiding permutations of length* n.

In order to illustrate how this can be used in the 1324 case, we will look at bounds on the growth of 213-avoiding permutations. Consider the vector $\mathbf{u}$ where $u_i = 1/2^{i-1}$. Looking at each entry, we see that $T(S_{213})\mathbf{u} \leq 4\mathbf{u}$. We use the following lemmas.

Lemma 3. *If a matrix T and vectors $\boldsymbol{w}$ and $\boldsymbol{v}$ have only nonnegative entries, then $T\boldsymbol{v} \geq c\boldsymbol{v}$ and $\boldsymbol{w} \geq \boldsymbol{v}$ implies that $T^n\boldsymbol{w} \geq T^n\boldsymbol{v} \geq c^n\boldsymbol{v}$.*

Again, it is straightforward from considering each entry. Likewise,

Lemma 4. *If a matrix T and vectors $\boldsymbol{w}$ and $\boldsymbol{v}$ have only nonnegative entries, then $T\boldsymbol{v} \leq c\boldsymbol{v}$ and $\boldsymbol{w} \leq \boldsymbol{v}$ implies that $T^n\boldsymbol{w} \leq T^n\boldsymbol{v} \leq c^n\boldsymbol{v}$.*

This gives us the following proposition almost directly.

Proposition 5. *If $T(S_{213})\boldsymbol{v} \leq c\boldsymbol{v}$ for a vector $\boldsymbol{v}$ with positive first entry, no negative entries, and which has a sum of entries that is finite, then the growth rate of the 213-avoiding permutations is at most c.*

Proof. We may multiply $\mathbf{v}$ by a suitable amount so that its first entry is greater than 1. Then $T(S_{213})^{n-1}\mathbf{e_1} \leq T(S_{213})^{n-1}\mathbf{v} \leq c^{n-1}\mathbf{v}$.

□

This gives us an upper bound of 4 on the growth rate. Of course, we already knew that since they are enumerated by the Catalan numbers, but the same approach applies to 1324-avoiding permutations as we will see in the next section.

Now we wish to search for a lower bound by finding a suitable vector $\mathbf{u'}$. Consider $u'_1 = 3$, $u'_2 = 3$, $u'_3 = 2$, $u'_4 = 1$, and $u'_i = 0$ for $i > 4$. This vector has a finite number of positive entries, and satisfies $T(S_{213})\mathbf{u'} \geq 3\mathbf{u'}$. Now we use a similar proposition.

Proposition 6. *If $T(S_{213})\boldsymbol{v} \geq c\boldsymbol{v}$ for a vector $\boldsymbol{v}$ with at least one positive entry, no negative entries, and which has a finite number of positive entries, then the growth rate of the 213-avoiding permutations is at least c.*

Proof. Since every state can be reached from our initial state, there exists k such that $T(S_{213})^k\mathbf{e_1} \geq \mathbf{v}$. Then $T(S_{213})^{n+k}\mathbf{e_1} \geq T(S_{213})^n\mathbf{v} \geq c^n\mathbf{v}$. Taking limits, the result follows.

□

Thus we get a lower bound of 3 on the growth rate. In fact, we will see that we can get arbitrarily close to 4, which means that the growth rate is 4, as expected given that it is counted by the Catalan numbers.

For any $1 > \epsilon > 0$, consider the sequence $a_0 = 1$, $a_1 = 3-\epsilon$, and $a_{k+1} = (4-\epsilon)a_k - \sum_{i\leq k} a_i$ for $k > 0$. Then $a_{k+2} - a_{k+1} = (4-\epsilon)a_{k+1} - (4-\epsilon)a_k - \sum_{i\leq k+1} a_i + \sum_{i\leq k} a_i$. This simplifies to $a_{k+2} = (4-\epsilon)a_{k+1} - (4-\epsilon)a_k$. Solving, we obtain the complex characteristic roots $\lambda = \frac{4-\epsilon\pm\sqrt{-(4-\epsilon)\epsilon}}{2}$. This means that the sequence cannot be strictly increasing, and there is a smallest positive integer m such that $a_m \leq a_{m-1}$.

From the recurrence relation, we see that a_m is nonnegative.

Now let $v_i = a_{m+1-i}$ for $1 \leq i \leq m+1$, and 0 otherwise. For $0 < k \leq m$, the $(m+1-k)$th entry of $T(S_{213})\mathbf{v}$ is of the form $a_{k+1} + \sum_{i \leq k} a_i = (4-\epsilon)a_k$, which is $(4-\epsilon)$ times the corresponding entry of $\mathbf{v}$. Since $a_m \leq a_{m-1}$, the first entry is also at least $(4-\epsilon)$ times the corresponding entry of $\mathbf{v}$. Likewise, this is also true for the $(m+1)$th entry. Thus we can apply Proposition 6, giving us a lower bound of $(4-\epsilon)$ on the growth rate.

We then see how this adapts to the 1324-avoiding case in the section that follows.

2.3 Top insertion for 1324-avoiding permutations

Definition 13. *For a positive integer* $k \leq n$*, the permutation* $\mathrm{Top}_k(\sigma)$ *is formed by deleting elements greater than* k *from a permutation* σ *of length* n*.*

For example, $\mathrm{Top}_4(534162) = 3412$.

Definition 14. *Given a permutation* σ *of length* n*, a new permutation* σ' *of length* $n+1$ *is formed by* **top insertion** *in the* j*th* **slot** *of* σ *if* $\sigma'(i) = \sigma(i)$ *for* $i < j$*,* $\sigma'(i) = \sigma(i-1)$ *for* $i > j$*, and* $\sigma'(j) = n+1$*.* σ' *formed in this way is also written as* $\mathrm{INS}(\sigma, j)$*.*

For example, the 4th step in creating 534162 by top insertion is in slot 2 because $\mathrm{Top}_4(534162) = 3412$ has 1 element before the "4". The idea is that given $\mathrm{Top}_3(534162) = 312$, the 4 could be inserted in the space or "slot" between any of the two elements, or to the right or left of the whole string. These slots are then numbered from the left.

Much as in the 213-avoiding case, we are interested in the initial 132-avoiding substring, since any top insertion after a 132 pattern creates a 1324 pattern.

Definition 15. *The* **initial 132-avoiding part** $\sigma_{\mathrm{initial}}$ *of a 1324-avoiding permutation* σ *is its longest possible initial string that avoids a 132 pattern. That is, either* σ *is 132-avoiding, in which case* $\sigma_{\mathrm{initial}} = \sigma$*, or there exists* k *such that* $\sigma_{\mathrm{initial}}(i) = \sigma(i)$ *for* $1 \leq i \leq k$*, and* $\sigma_{\mathrm{initial}}$ *is a length* k *132-avoiding string, and the length* $k+1$

string σ' contains a 132 pattern, where $\sigma'(i) = \sigma(i)$ for $1 \leq i \leq k+1$.

That is, the first 132 pattern of a 1324-avoiding permutation includes the element right after the initial segment, if it occurs at all.

Definition 16. *The* **initial segment** *σ_I of a 1324-avoiding permutation σ is its initial 132-avoiding part reordered so that σ_I is a 132-avoiding permutation. That is, for each i, if $\sigma_{\text{initial}}(i)$ is the jth largest element among the elements of σ_{initial}, then $\sigma_I(i) = j$.*

For example, 54128367 has initial 132-avoiding part 54128, and initial segment 43125. One can think of the initial segment as a string which is the "collapsed" version of the initial 132-avoiding part, that includes elements from 1 to some k. Now we note a few easy facts.

Lemma 7. *Any permutation formed by top insertion in slot j of a 1324-avoiding permutation σ is also 1324-avoiding, iff $j \leq k+1$, where k is the length of the initial segment of σ.*

Proof. If the new permutation has a 1324 pattern, it must include the inserted element, otherwise that pattern is also present in the old permutation, which is 1324-avoiding. Since the inserted element is also the largest, it must be the "4" of the pattern, so there is a 132 pattern before it. But all the elements before it lie in the initial segment of the old permutation, which is 132-avoiding. On the other hand, if $j > k+1$, then it forms a 1324 pattern.

□

Lemma 8. *Any 1324-avoiding permutation σ of length $n \geq 2$ is formed from top insertion of exactly one 1324-avoiding permutation of length $n-1$, in only one slot.*

Proof. Since σ is 1324-avoiding, so is $\mathrm{Top}_{n-1}(\sigma)$. If the element n is jth from the left in σ, then σ can only be formed from top insertion in the jth slot of $\mathrm{Top}_{n-1}(\sigma)$.

□

Putting the two together gives the following.

Proposition 9. *Every 1324-avoiding permutation is formed by a unique series of top insertions starting from the length 1 permutation.*

In other words, every 1324-avoiding permutation can be written as a sequence of which slots are inserted.

Also note that when we form a new permutation by top insertion, the new initial segment depends only on the old initial segment with the inserted element. The rest of the permutation does not affect whether a new 132 pattern appears that includes the inserted element and is closer to the start of the permutation (and so must appear within where the old initial segment was).

Proposition 10. *The initial segment of* $\mathrm{INS}(\sigma, j)$ *is the same as that of* $\mathrm{INS}(\sigma_I, j)$.

Definition 17. *1324-avoiding permutations with the same initial segment are said* *to be in the same* **insertion state***. The number of 1324-avoiding permutations with length n and initial segment σ is denoted $St_n(\sigma)$.*

Definition 18. *A* **state descendant** *of an insertion state σ are the insertion states corresponding to the initial segments obtained when top insertion is performed on σ. Formally, σ' is a state descendant of an insertion state σ of length k if there exists a positive integer $j \leq k+1$ such that $\mathrm{INS}(\sigma, j)_I = \sigma'$. Likewise, a* **state ancestor** *of an insertion state is an insertion state of which it is a state descendant.*

Note that a state can be formed by top insertion in exactly one slot from each state ancestor, since its largest element corresponds to the slot that must have taken the insertion. Thus, each 1324-avoiding permutation in a given state is formed by top insertion from a permutation in a state ancestor, and each such permutation only forms one such permutation in that given state. So we have the following.

Proposition 11. *Let A be the set of state ancestors of σ. Then $St_{n+1}(\sigma) = \sum_{\tau \in A} St_n(\tau)$.*

If we enumerate the insertion states, which are countable, then we can have the transfer rule S_{1324} consisting of pairs (x, y) where y is a state descendant of x. This gives us the corresponding propositions to those in the previous section.

Proposition 12. *If $T(S_{1324})\boldsymbol{v} \leq c\boldsymbol{v}$ for a vector $\boldsymbol{v}$ with positive first entry, no negative entries, and which has a sum of entries that is finite, then the growth rate of the 1324-avoiding permutations is at most c.*

Proposition 13. *If $T(S_{1324})\boldsymbol{v} \geq c\boldsymbol{v}$ for a vector $\boldsymbol{v}$ with at least one positive entry, no negative entries, and which has a finite number of positive entries, then the growth rate of the 1324-avoiding permutations is at least c.*

It is difficult to obtain any results without some clever way to group the states into a more manageable form. For instance, suppose we could group states into classes such that for each state the number of ancestor states from higher class is zero except for a bounded number of classes above it, and the number of such ancestor states for each class is bounded by a constant. Then, as in the 213 case, we can choose a $\mathbf{v}$ that decays exponentially by a fixed factor, and we would be able to find c such that $T(S_{1324})\mathbf{v} \leq c\mathbf{v}$. By adjusting $\mathbf{v}$ and the classes of states, c might be improved. Even if found, such bounds could be very loose, unfortunately.

For the lower bound, another approach to this could eventually yield results. Suppose $T'(S_{1324})$ is a "truncated" version of $T(S_{1324})$, with some 1s replaced by 0s. If $T'(S_{1324})\mathbf{v} \geq c\mathbf{v}$, then $T(S_{1324})\mathbf{v} \geq T'(S_{1324})\mathbf{v} \geq c\mathbf{v}$, which satisfies the above proposition.

Corollary 14. *Suppose $T'(S_{1324})$ is equal to $T(S_{1324})$ except for some 1-entries changed to 0. If $T'(S_{1324})\boldsymbol{v} \geq c\boldsymbol{v}$ for a vector $\boldsymbol{v}$ with at least one positive entry, no negative entries, and which has a finite number of positive entries, then the growth rate of the 1324-avoiding permutations is at least c.*

This suggests that we may truncate $T(S_{1324})$ and search for suitable eigenvectors. Unfortunately, while a computer may be able to obtain some results, trying to surpass known bounds with such a proof using calculations by hand is nearly impossible without further simplifying ideas. This truncation is actually what was used by Albert et al [1] in the paper mentioned in the previous section. The insertion encoding they used is also be truncated in a natural way, by omitting states above a certain length.

As a small example of how this could work, consider the states of length 3 or less. Note that the descending permutations 1, 21, and 321 can only be state descendants to the descending permutations of length one less than their own, so we remove these as well and consider the five states 12, 123, 213, 231, and 312. Label these 1 to 5 in that order. We find the state descendants of each of these by insertion into all their available slots and computing the resulting initial segments, discarding those that are not among these five. The resultant matrix is $T_3 =$

$$\begin{pmatrix} 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Solving for eigenvalues, we get $(\lambda - 1)(1 - \lambda^4 + 2\lambda^3)$, which shows that the largest eigenvalue is slightly greater than 2. Once we have an eigenvalue c and a corresponding eigenvector $\mathbf{v}$ with no negative entries, we can apply Corollary 14. Alternatively, we may find a vector satisfying Corollary 14 through other means, such as $\mathbf{v} = (1, 1, 0, 0, 0)^{\mathrm{T}}$, which gives us $T_3\mathbf{v} \geq 2\mathbf{v}$, giving a lower bound of 2. This is far lower than the known lower bound, but in principle the lower bound can be improved arbitrarily close to the actual growth rate, depending on the efficiency of the insertion scheme used and the amount of computational power. In practice the amount of computational power limits how far this can go.

Taking another approach, if we look at the proportion of length n permutations in each insertion state, it seems like this should converge as n grows.

Conjecture 1. *For each σ, $\frac{St_n(\sigma)}{s_n(1324)}$ converges as n goes to infinity.*

If this is true, then $\frac{1}{s_n(1324)}T(S_{1324})^{n-1}\mathbf{e_1}$ has entries which are all convergent, which suggests that it has an eigenvector with the actual growth rate as an eigenvalue.

Alternatively, it would also imply that $St_{n+1}(\sigma)/St_n(\sigma)$ approaches the actual growth rate $\lim_{n\to\infty} s_n(1324)^{1/n}$.

In the next section, we will look at what I will call a "local" growth rate and its relation with top insertion.

2.4 Local growth rates

Definition 19. *The* **local growth rate** *at step n is $g_n = \frac{s_{n+1}(1324)}{s_n(1324)}$.*

As mentioned in the introduction, Arratia [2] showed that $\lim_{n\to\infty} s_n(\pi)^{1/n}$ converges, via superadditivity of $\log(s_n(\pi))$. Unfortunately not even this superadditivity would imply that the local growth rate converges as n increases. It seems likely that it is true, based on actual growth rates.

Conjecture 2. *g_n converges to c_{1324} as n goes to infinity.*

One may also consider the two weaker related conjectures on either side.

Conjecture 3. $\limsup_{n\to\infty} g_n = c_{1324}$

Conjecture 4. $\liminf_{n\to\infty} g_n = c_{1324}$

For each $\epsilon > 0$ we cannot have all the g_n after some point m exceed $c_{1324} + \epsilon$, or $\lim_{n\to\infty} s_n(1324)^{1/n}$ cannot converge to c_{1324}. Likewise they cannot all be below $c_{1324} - \epsilon$. Thus, immediately,

Proposition 15. $\limsup_{n\to\infty} g_n \geq c_{1324} \geq \liminf_{n\to\infty} g_n$

Thus the two weaker conjectures are really that there does not exist an infinite sequence of a_i and $\epsilon > 0$ such that $g_{a_i} \geq c_{1324} + \epsilon$, and similarly for $g_{a_i} \leq c_{1324} - \epsilon$.

Even the following weaker conjecture would be helpful in obtaining some results on the location of top elements, and hence the shape of 1324-avoiding permutations.

Conjecture 5. $\{g_n\}$ *is bounded.*

With this weakest conjecture, one might be tempted to think that surely the superadditivity of $\log(s_n(1324))$ must somehow imply this. However, it is not sufficient. To see this, one can construct pathological sequences that satisfy this superadditivity, yet have unbounded g_n. Note that $\log(s_n(1324))$ is $\sum_{i=1}^{n-1} \log g_i$. If we first set all $\log g_i$ to $c-1$, then their sums are superadditive. If we then pick some positive integer a_1 and then increased $\log g_i$ by 1 for all i such that 2^{a_1} divides i, then $\sum_{i=1}^{n-1} \log g_i$ is still less than nc, and the sequence is still superadditive. At each successive step we can continue to find $a_k > a_{k-1}$ sufficiently large such that if we raise $\log g_i$ by 1 for all i such that 2^{a_k} divides i, then $\sum_{i=1}^{n-1} \log g_i$ is still less than nc. The sequence $\sum_{i=1}^{n-1} \log g_i$ will still remain superadditive, and we will end up with a sequence of $\log g_i$ that is unbounded from above. We must therefore use some other properties of 1324-avoiding permutations in order to show that this sort of thing cannot happen.

Consider the relation of the top insertion to the local growth rate. The number of 1324-avoiding permutations of length $n+1$ that are created by top insertion of a length n 1324-avoiding permutation is the number of slots of its initial segment, which is 1 plus the length of the initial segment. So we have a straightforward relation between the average length of the initial segment and the local growth rate.

Proposition 16. *g_n is exactly 1 more than the mean length of the initial segments of length n 1324-avoiding permutations.*

So if the previous conjecture is true, then this would also be bounded.

Proposition 17. *If there exist constants b and m such that $g_n \leq b$ for all $n > m$, then the mean length of the initial segments of length n 1324-avoiding permutations is at most $b - 1$.*

From this, one gets an even stronger intuitive sense that that conjecture should be true, as long initial segments should quickly spawn many more permutations with shorter initial segments, bring down the mean length. On the other hand, the permutations with longer initial segments are those with more descendants. Indeed, in the sections on shape we will look at some figures including one of a typical 1324-avoiding permutation, and one thing that we will notice is that permutations with long initial segments do indeed tend to be ancestors of most 1324-avoiding permutations of higher length, even though they are outnumbered compared to those of the same length.

Consider where the top element is distributed. A permutation with a length k initial segment has $k + 1$ descendants, with top elements in positions 1 to $k + 1$ counting from the left. The mean position is therefore $1 + k/2$. The distribution of the lengths of the initial segments is therefore directly connected to the distribution of the position of the top elements.

Finally, if g_n is indeed bounded, we see that as n increases, the top element is distributed closer and closer to the left, as a proportion of n, as follows.

Theorem 18. *Suppose there exist constants b and m such that $g_n \leq b$ for all $n > m$. Then for any $f(n)$ that approaches infinity as n approaches infinity, the proportion of length n 1324-avoiding permutations with the top element in position less than $f(n)$ approaches 1 as n goes to infinity. The position of the top element of a uniformly chosen length n 1324-avoiding permutation is tight.*

Proof. Let the mean position of the top element be p_n for the permutations of length n. Let x_n be the proportion of those permutations for which the top element is at least in position $f(n)$. Since $x_n f(n)$ is at least the contribution of those permutations to the mean position, $x_n f(n) \leq p_n$. The position of the top element cannot be greater than the length of the initial segment, since there cannot be a 132 pattern before it. By the previous proposition, $x_n f(n) \leq p_n \leq b - 1$. Then $x_n \leq (b-1)/f(n)$, so it goes to 0 as n goes to infinity.

Given any $\epsilon > 0$, we can replace $f(n)$ in the above argument by $(b-1)/\epsilon$. Then $x_n \leq \epsilon$ for all $n > m$. Thus the position of the top element is at most $(b-1)/\epsilon$ for all but ϵ of the 1324-avoiding permutations of length n, and there are a finite number of n not more than m for which such a bound may be obtained individually. Then there is some α where the position of the top element is at most $(b-1)/\epsilon$ for all but ϵ of the 1324-avoiding permutations of length n, whatever n is, so it is tight.

$\square$

This means that if you were to normalise the shape by plotting its points to fit in a unit square, the points in the first row would cluster closer to the top left corner as n increases.

Corollary 19. *Suppose there exist constants b and m such that $g_n \leq b$ for all $n > m$. Then for any $\epsilon > 0$, the proportion of length n 1324-avoiding permutations with the top element in position less than ϵn approaches 1 as n goes to infinity.*

In the next section, we will look at one more combinatorial approach to the lower bound, before focussing on the issue of shape.

2.5 Another approach towards the lower bound

Most 1324-avoiding permutations seem to have their points clustered somewhat close to the anti-diagonal as we will see in following sections regarding shape. If we can build a large subset of such permutations, the growth of this subset would provide a good lower bound to the growth rate. The following is one approach that gives a lower bound of 8, which is not better than the known bounds thus far, but has the possibility of improvement.

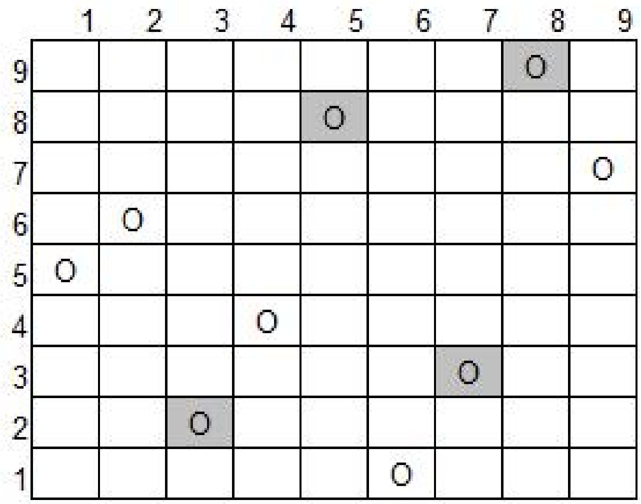
We first note that counting the number of permutations that avoid 1324 corresponds to counting the number of such grids that do not have four points that form a 1324 pattern: the first point counting from left to right being the lowest of the four, the second being second-highest, the third being second-lowest and the last being highest among the four. Such a pattern is shown in Figure 2.1. We count a subset of such permutations in the following way.

Choose some positive integers k and $m > 2$, and let $n = 2km$. Divide the n by n grid evenly into "boxes" of side length $2k$. Formally, we have:

Definition 20. *A* **box** *of the matrix $A(\sigma)$ is a submatrix $B[x, y]$ consisting of the entries $A(\sigma)_{i,j}$ for $2k(x-1)+1 \leq i \leq 2kx$ and $2km-2ky+1 \leq j \leq 2km-2k(y-1)$, where $B[x, y]_{i,j} = A_{2k(x-1)+i,2km-2ky+j}$.*

For future ease of manipulation, I have labelled the top left box with the co-

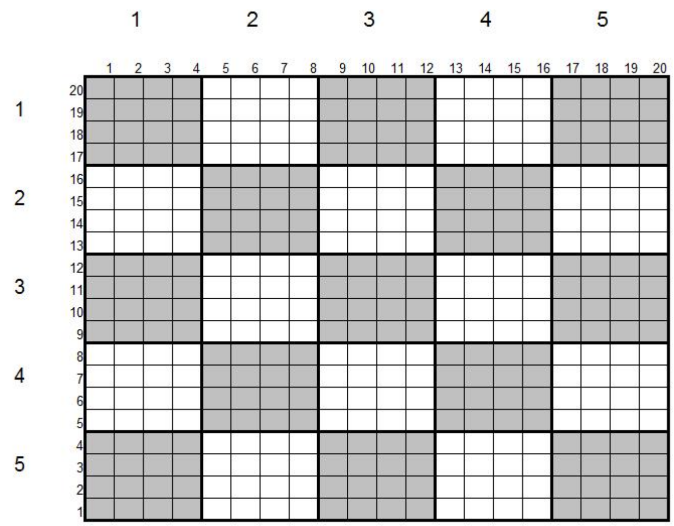
Figure 2.1: A 1324 pattern. The leftmost grey square is the "1" of the pattern, the next is the "3", etc.



ordinates [1,1], and the square directly to the right of that as [2,1], etc. so that the box labelled $[i,j]$ occupies the part of the grid with column numbers between $2k(i-1)+1$ and $2ki$, and row numbers between $2km-2kj+1$ and $2km-2k(j-1)$. That is, the second coordinate of our boxes go from top to bottom, rather than bottom to top like how our rows are labelled, as shown in Figure 2.2.

Now we will generate a unique 1324-avoiding permutation from a set of choices. Pick $m-1$ 132-avoiding permutations of length k, σ_i for $i=1$ to $m-1$. Also pick $2m$ length $2k$ vectors with k 0s and k 1s as its entries, and label them v_i and w_j for i and j ranging from 1 to m. We will use the vectors v_i and w_j as our choice of rows and columns to fill certain boxes with k points each.

Figure 2.2: Boxes for k = 2 and m = 5. The alternating grey colours are for visibility.



Now for each v_i, let $v_i(j)$ be the position of the jth 1 and $v_i'(j)$ be the position of the jth 0. That is to say, they are a subset of 1 to $2k$ and its complement. Define $w_i(j)$ and $w_i'(j)$ likewise. We then place points in boxes using these choices.

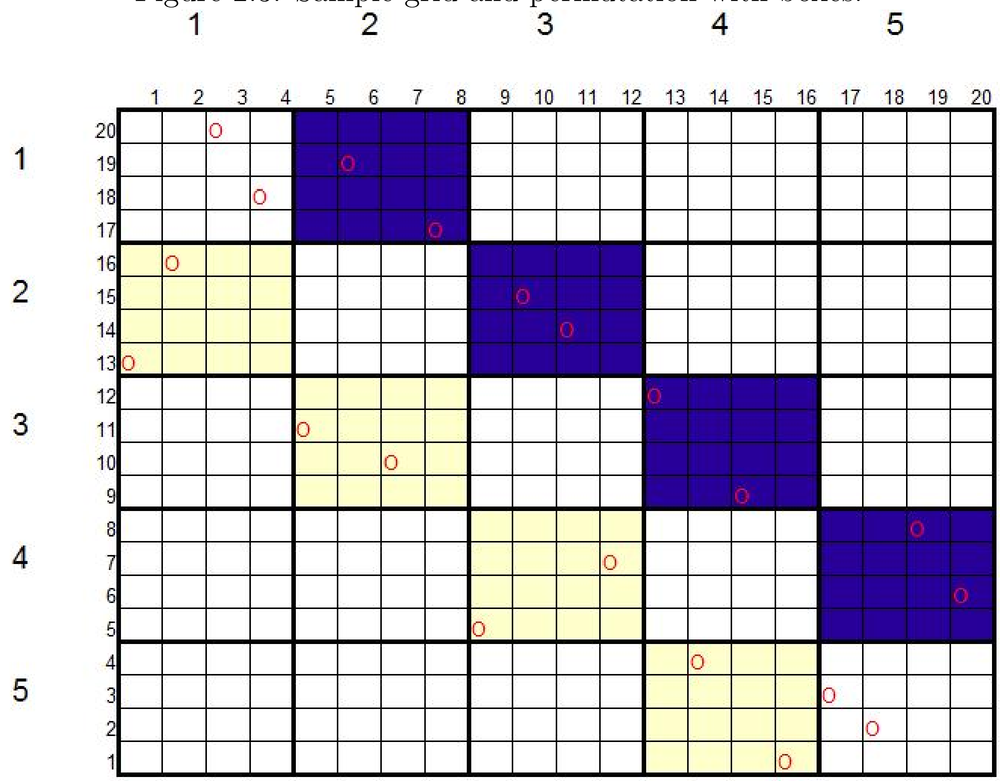
For each box $[i{,}i+1]$ for i from 1 to $m-1$, we fill it with our choice of σ_i using the $v_i(j)$ and $w_i(j)$ as our choice of rows and columns in the following way.

For each j from 1 to k, place a point in $[i{,}i+1]_{v_i(j),w_{i+1}(\sigma_i(j))}$.

Now we fill each $[i+1{,}i]$ for i from 1 to $m-1$ by placing a point in $[i+1{,}i]_{v_{i+1}'(j),w_i'(k-j+1)}$. This means that the points in this box are all descending, and has no ascents.

Finally, we fill [1,1] by placing a point in $[1,1]_{v'_1(j),w_1(k-j+1)}$ for each j from 1 to k, and $[m,m]$ by placing a point in $[m,m]_{v_m(j),w'_m(k-j+1)}$ for each j from 1 to k.

Figure 2.3: Sample grid and permutation with boxes.



Since we have filled in $n = 2km$ points, one for each column and row, this corresponds to a permutation, σ.

Figure 2.3 shows a small example of a permutation formed by this process. In this simple case for $k = 2$, $m = 5$, our σ_i are 12, 21, 12, 21, respectively. Our v_i are (1,1,0,0), (1,0,1,0), (1,0,0,1), (0,1,0,1), and (1,1,0,0), and our w_j are (1,0,1,0), (1,0,0,1), (0,1,1,0), (0,1,0,1), and (1,0,0,1). Consider the leftmost yellow box [1,2]. Since v_1 is (1,1,0,0) and w_2 is (1,0,0,1), the $k = 2$ points in this box are placed in the subgrid formed by the first two columns and the first and last rows. Since σ_1 is 12, they are placed in that pattern within that subgrid as shown. The rest of the yellow boxes are filled in similar fashion. The top-left white box, [1,1], is filled by descending points within the subgrid determined by the rows indicated by w_1, and the columns left unused in [1,2]. The bottom-right corner box is filled similarly, and the blue boxes are then filled by descending points in the subgrids formed from rows and columns left unused by the other boxes.

We will now see that the permutation formed is 1324-avoiding.

Proposition 20. *σ is 1324-avoiding.*

Proof. Suppose that we have a 1324 pattern, $a' < c' < b' < d'$, with $\sigma(a') < \sigma(b') < \sigma(c') < \sigma(d')$. This corresponds to the points $A_{a',\sigma(a')}$, $A_{b',\sigma(b')}$, $A_{c',\sigma(c')}$, and $A_{d',\sigma(d')}$, which we call a, b, c, and d respectively. The a, c, b, and d correspond to the 1, 3, 2, and 4 respectively, in the 1324 pattern.

If d is in any of the boxes $[i,i+1]$, the a, c, and b have to be as well, as there are no other nonempty boxes that are to the bottom left of the d. This is impossible since that box is 132-avoiding and a, c, and b form a 132 pattern.

By similar argument, the d cannot be in [1,1] or [m,m], as that restricts the a, c, and b to a single box that avoids 132.

Thus the d must be in one of the boxes of the form $[i+1,i]$. Since $[i+1,i]$ contains no ascents, the a, c, and b are again restricted to those boxes of the form $[j,j+1]$ that lie to the bottom left of the d. The a of the pattern must then lie in some $[l,l+1]$. But since the c and b must be to the top right of the a, they cannot lie in another box of the form $[j,j+1]$. Thus they must all be in $[l,l+1]$, which forms a 132 pattern, a contradiction.

$\square$

The number of 1324-avoiding permutations of length $n = 2km$ is then at least the number of choices of $m-1$ 132-avoiding permutations of length k, and $2m$ length $2k$ vectors with k 0s and k 1s as its entries. We can then count these.

Let the kth catalan number be C_k, which is the number of 132-avoiding permutations of length k. Then $s_{2km}(1324)$ is at least $C_k^{m-1}\binom{2k}{k}^{2m}$.

Pick any small real $1 > \epsilon' > 0$. From Stirling's formula, we see that we can find K large enough such that for any $k > K$, both C_k and $\binom{2k}{k}$ are greater than $(2-\epsilon')^{2k}$.

Now we can pick M large enough that for $m > M$, $(2-\epsilon')^{m-1} > (2-2\epsilon')^m$.

Then we have that

$$C_k^{m-1}\binom{2k}{k}^{2m} > (2-2\epsilon')^{2km}(2-\epsilon')^{4km}$$

which means

$$s_{2km}(1324) > (8-16\epsilon'+10\epsilon'^2-2\epsilon'^3)^{2km}$$

Now, for any real $\epsilon > 0$, we can pick ϵ' small enough such that $\epsilon > 16\epsilon' - 10\epsilon'^2 + 2\epsilon'^3$.

This gives $s_{2km}(1324) > (8-\epsilon)^{2km}$ when $k > K$ and $m > M$ for some sufficiently large K and M.

Then $\lim_{n\to\infty} \frac{L^n}{s_n(1324)}$ is 0 for any $L < 8$.

Since we know that $\lim_{n\to\infty} s_n(1324)^{1/n}$ exists, we have

Theorem 21. $\lim_{n\to\infty} s_n(1324)^{1/n} \geq 8$.

In order to improve the bound given by this approach, one might try expanding the subset of 1324-avoiding permutations counted by allowing more boxes to be filled, or by counting more of the permutations that can fit within those boxes. For example, notice that the blue boxes in Figure 2.3 contain only descending points. This omits many possible 1324-avoiding permutations. If we found a way to put in a 213-avoiding permutation instead, in such a way as to avoid ascents causing a 1324 pattern to form, the bound might be improved.

Chapter 3

Shape

3.1 Normalised shape

Suppose we normalised the length n shape of a pattern π to fit in a unit square. What will the pattern of points look like? Will they converge to a limiting shape in some sense? We will make such notions more precise later on.

Recent work has studied this for certain classes of permutations. In 2014, Dokos and Pak [9] studied the shape of doubly alternating Baxter permutations, and Miner and Pak [14] studied the shape of 123-avoiding permutations and 132-avoiding permutations.

The methods used involve a good understanding of the enumeration of the permutations under consideration, so the techniques do not appear to transfer to the 1324-avoiding case, where we are still very far from an exact enumeration. Nonetheless, the questions they answered about the shape of those permutation classes are also those that we would like to ask for 1324-avoiding permutations.

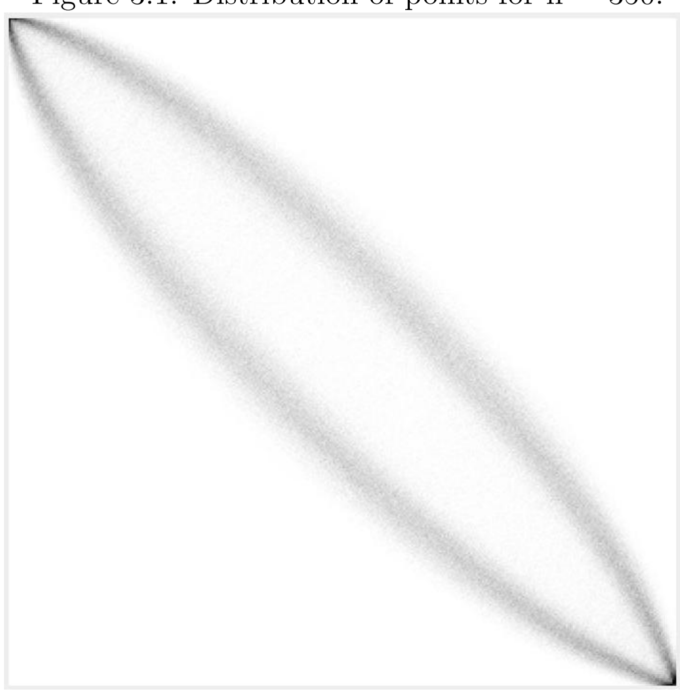
In the case of 123-avoiding permutations, for example, what Miner and Pak showed was that the points on the shape decay exponentially as you move away from the anti-diagonal, and can describe in some detail what happens close to the anti-diagonal as well. Not knowing the enumeration for the 1324-avoiding case, we will nonetheless be able to prove that there is exponential decay in a region close to the corners.

Estimates of shapes corresponding with various lengths, obtained using a Monte Carlo Markov Chain (MCMC) method, are very suggestive of what a limiting shape would look like, as well as where the points would be concentrated in a typical 1324-avoiding permutation.

The idea of the MCMC method is to start with some 1324-avoiding permutation of length n, then pick an integer between 1 and n uniformly at random, and swap the element at that position with another one, also chosen uniformly at random. The new permutation is checked for 1324 patterns and reverted to the previous one if one is found. The process is then repeated. The probability of picking a permutation converges to the uniform distribution. After a certain large number of such steps, a permutation is saved, and after that permutations are saved at regular intervals of steps.

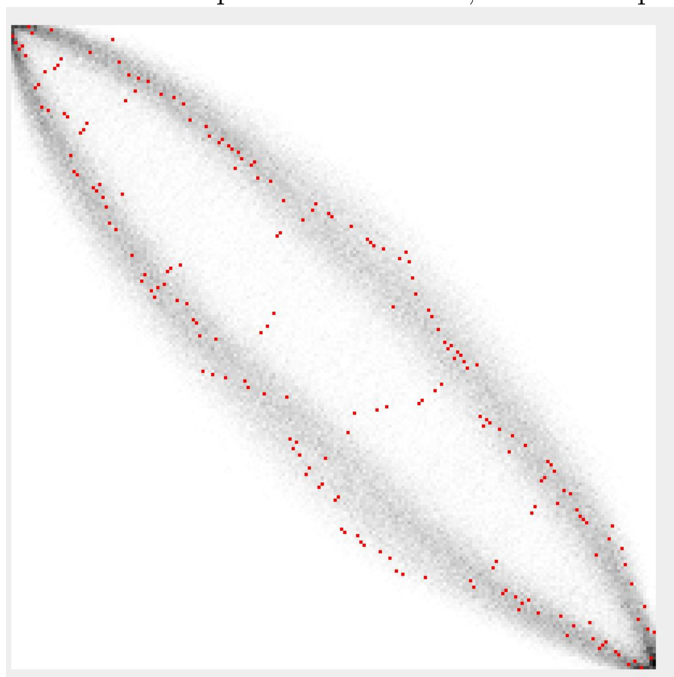
The saved permutations are then combined to form a picture that approximates the shape of length n 1324-avoiding permutations, with darker spots representing greater numbers of permutations with a point at that location. Figure 3.6 shows one such picture for $n = 350$.

Figure 3.1: Distribution of points for n = 350.



Figures 3.3, 3.4, 3.5, and 3.6, show the midpoints of the arcs when viewed diag-

Figure 3.2: Distribution of points for n = 200, with a sample permutation



onally, as red lines. The green lines in those figures show the limits of the middle 80 percent of the points of the lower arc. The arcs appear that they might be converging slowly as n increases. This raises a few questions. First, does this converge to the anti-diagonal? That is to say, if we look at a point on the normalised shape, does the density of points in that neighbourhood go to 0 as n increases? Since the shape $\mathrm{SH}_{1324}(n)$ has entries that sum to n, and area increases quadratically, we may make this conjecture more precise as follows.

Conjecture 6. *For any real x and y in $(0, 1]$, $x + y \neq 1$,*

$\lim_{n\to\infty} n\mathrm{SH}_{1324}(n)_{\lceil xn\rceil,\lceil yn\rceil} = 0.$

Even if this is not true, we may instead ask for which x and y is the above true.

Figure 3.3: Distribution of points for n = 50, with arcs

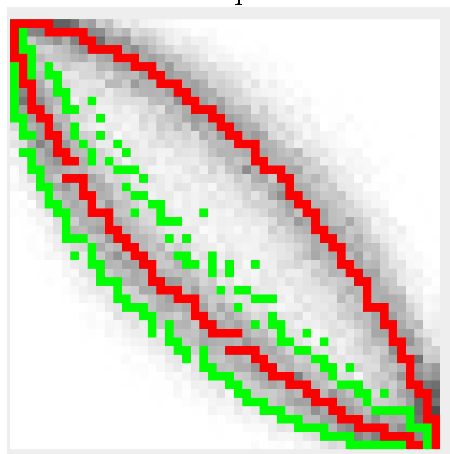
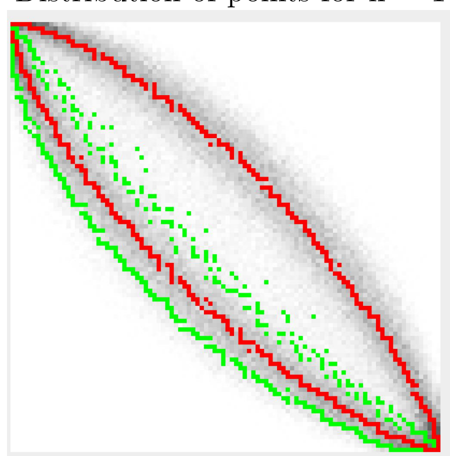


Figure 3.4: Distribution of points for n = 100, with arcs



The main result, shown in the next section, will answer this for areas around two corners.

If it does not converge to the anti-diagonal, does it nonetheless converge to some limit shape?

Conjecture 7. *For any real x and y in $(0,1]$, $\lim_{n\to\infty} n\mathrm{SH}_{1324}(n)_{\lceil xn\rceil,\lceil yn\rceil}$ exists.*

If they exist, what are they? An even harder question is: How fast does

Figure 3.5: Distribution of points for n = 200, with arcs

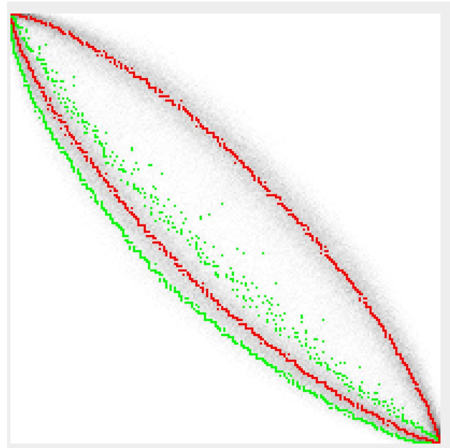
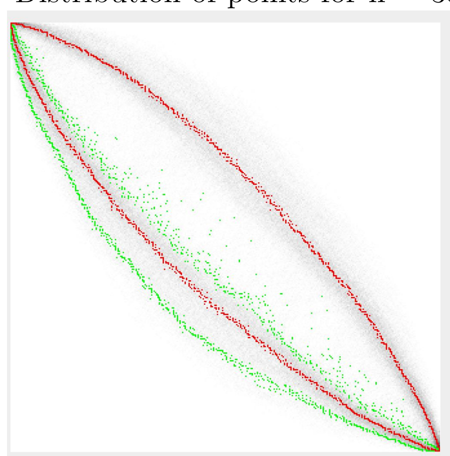


Figure 3.6: Distribution of points for n = 333, with arcs



$n\mathrm{SH}_{1324}(n)_{\lceil xn \rceil, \lceil yn \rceil}$ decay away from those points (x, y) for which the limit is nonzero, with respect to x and y?

Looking at Figure 3.2, it appears that typical 1324-avoiding permutations are made up of a lower 132-avoiding arc, an upper 213-avoiding arc, and a scattering of points in between. This is not entirely surprising given the way that 1324-avoiding permutations can be separated into a 132-avoiding part and a 213-avoiding part, as Claesson, Jelinek, and Steingrimsson showed [8]. The points between the arcs are arranged in a few thin ascending lines. This makes sense, since if you had a descending pair close to each other in the middle, it would be easy to point a point on each of the arcs to complete a 1324 pattern.

This can also be connected to the top insertion scheme. When building up a given permutation of length n using top insertion, the location of the kth insertion can be seen by looking at where the element in the kth row is relative to the bottom $k-1$ rows. We notice that as the gap between the arcs grow, it means that we have many more valid slots to insert. Then our initial segment must be growing well beyond the growth rate.

This may seem counterintuitive at first, but this whole process is conditioned on ending up with a 1324-avoiding permutation of length n uniformly at random. While those permutations of length n or length about $n/2$ are likely to have short initial segments if we pick them uniformly among 1324-avoiding permutations of the same length, the permutations with long initial segments have more children. This means that the ancestors of the typical length n permutation are more likely to have longer initial segments.

This could present another angle of attack to this problem. If we can discover some relation between how small a proportion of permutations have longer initial segments versus the children they can have, and make it precise, it may be possible to bound the distance between the upper and lower arcs.

We now turn to the main result on shape.

3.2 Exponential decay at the corners

In this section I will consider the normalised shape, consisting at each point (i, j) of the proportions of the set of 1324-avoiding permutations of length n that have $\pi(i) = j$.

Definition 21. *The* **normalised shape** *of π-avoiding permutations of length n is the function* $\mathrm{NSH}_{\pi,n}$ *on* $(0, 1]X(0, 1]$ *that satisfies* $\mathrm{NSH}_{\pi,n}(x, y) = \mathrm{SH}_{1324}(n)_{\lceil xn \rceil, \lceil yn \rceil}$.

I will show that at least a certain part of this normalised shape near the bottom-left and top-right corners is "empty" in the limit, in the sense that the proportion of permutations with a point at a particular position in that region decays to 0 exponentially. This also implies that the proportion of points over such a region in the normalised grid decays exponentially, as the corresponding area in the un-normalised shape increases only quadratically. This will be shown for the bottom-left corner, with the other corner following by symmetry.

Definition 22. *A* **corner point** *of a permutation is a point that does not have any other points to its bottom-left.*

That is to say, $(i, \pi(i))$ is a corner point if and only if there does not exist any $j < i$ s.t. $\pi(j) < \pi(i)$. In other words, corner points are left-to-right minima.

Now we want to rigorously pin down which permutations a position in the normalised shape is associated with, so that we can count them and examine what happens in the limit.

For positive integers a, b, let $f_n(a,b)$ be the number of length n 1324-avoiding permutations with a corner point at (a,b), and let $F_n(a,b) = f_n(a,b)/s_n(1324)$ be the proportion of such permutations.

Let $g_n(a,b)$ be the number of length n 1324-avoiding permutations with a point at (a,b), and likewise let $G_n(a,b) = g_n(a,b)/s_n(1324)$. This is just the shape in function form, for notational convenience for the purposes of this section.

Let $F'_n(a',b')$ be the "normalised" version of $F_n(a,b)$. That is to say, $F'_n(a',b') = F_n(a,b)$, where a', b' are in $(0,1]$ and $a = \lceil a'n \rceil$ and likewise for b'. Normalised versions of the others are defined similarly. In summary,

Definition 23. *For positive integers a, b, and a', b' in $(0,1]$,*

$f_n(a,b) = |\{\sigma \in S_n(1324) | (a,b) \text{ is a corner point of } \sigma\}|.$

$g_n(a,b) = |\{\sigma \in S_n(1324) | \sigma(a) = b\}|.$

$F_n(a,b) = f_n(a,b)/s_n(1324).$

$G_n(a,b) = g_n(a,b)/s_n(1324).$

$F'_n(a',b') = F_n(\lceil a'n \rceil, \lceil b'n \rceil).$

$G'_n(a',b') = G_n(\lceil a'n \rceil, \lceil b'n \rceil).$

Notice that G'_n is $\mathrm{NSH}_{1324,n}$. I first bound $g_n(a,b)$ in terms of functions of the form $f_n(i, j)$.

Lemma 22. $g_n(a, b) \leq f_n(a, b) + \sum_{i<a,j<b} f_n(i, j)$

Proof. Suppose a length n 1324-avoiding permutation π has a point at (a, b). Then either (a, b) is a corner point, or there is at least one point (c, d) with $c < a$ and $d < b$. Pick the point with the smallest first coordinate, and this is a corner point of π.

Every permutation counted by $g_n(a, b)$ is then counted by either $f_n(a, b)$, or at least one of $f_n(i, j)$ for $i < a$, $j < b$.

□

Since we don't know the exact growth rate of 1324-avoiding permutations, known lower bounds can be used.

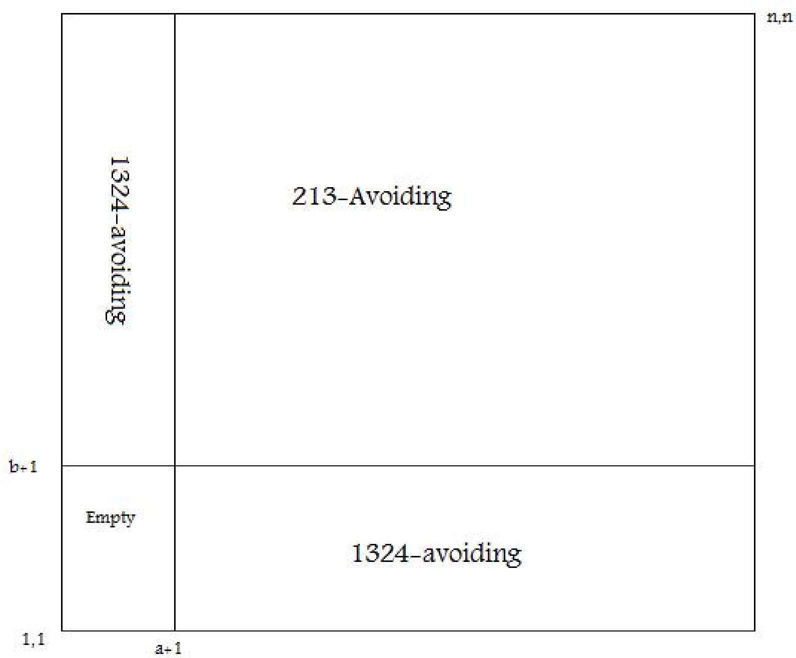
Lemma 23. *If* $\lim_{n\to\infty} s_n(1324)^{1/n} = c$*, and* $H(n)$ *in* $o(C^n)$ *exists such that* $C < c$ *and* $f_n(i, j) < H(n)$ *for all* $i \leq a'n + 1$*,* $j \leq b'n + 1$*, then* $G'_n(a', b')$ *decays exponentially to 0.*

Proof. The number of pairs (i, j) with $i < a'n + 1$, $j < b'n + 1$ is less than n^2. Let $a = \lceil a'n \rceil$ and $b = \lceil b'n \rceil$. Then from Lemma 22, $g_n(a, b) < n^2 H(n)$. Since $n^2 H(n)$ is in $o((C + \epsilon)^n)$ for any $\epsilon > 0$, there exists positive $d < 1$ such that $G'_n(a', b') = g_n(a, b)/s_n(1324)$ is in $o(d^n)$.

□

So the better the lower bound to the "growth rate" C is known, the higher the bound $H(n)$ can be. This would yield a larger area over which G' decays exponentially, using this method.

Figure 3.7: Counting $f_{n+1}(a+1, b+1)$



Now we proceed to bound $f_{n+1}(a+1, b+1)$.

Lemma 24. *$f_{n+1}(a+1, b+1) \leq d_a d_b \frac{(n-b)!(n-a)!(2n-2a-2b)!}{a!b!(n-a-b+1)!(n-a-b)!^3}$, where d_k denotes $s_k(1324)$.*

Proof. We divide the grid into four quadrants as shown in Figure 3.7, the part that is to the bottom-left of $(a+1, b+1)$, the part that is to the top-right, etc. Suppose a permutation has a corner point at $(a+1, b+1)$. Then the bottom-left quadrant must be empty, the top-left and bottom-right quadrants must each be 1324-avoiding, and the top-right quadrant must be 213-avoiding, since any 213 pattern there would form a 1324 pattern with the point at $(a+1, b+1)$. So $f_{n+1}(a+1, b+1)$ is at most the number of permutations that satisfy these conditions.

Note that since there are no points in the bottom-left quadrant, there must be a points in the top-left, and b points in the bottom-right.

There are $\binom{n-b}{a}$ choices for which rows in the top-left quadrant contain the a points, and d_a choices of permutations restricted to those rows. Thus there are $\binom{n-b}{a} d_a$ possible states for the top-left quadrant. Likewise we have $\binom{n-a}{b} d_b$ for the bottom-right quadrant. The rows and columns of the top-right quadrant that have points in them are determined entirely by the choices of rows and columns in the adjacent quadrants, so there are only the Catalan number $C_{n-a-b} = \frac{(2n-2a-2b)!}{(n-a-b+1)!(n-a-b)!}$ of choices [15]. Each such set of choices yields a unique permutation with the above properties, and vice versa. Multiplication yields the desired result.

$\square$

This bound is likely to be loose, whereas the following bounding is relatively sharp, at least for the purposes of the main theorem of this section.

Given that f is 0 if $a+b \geq n$, we restrict our attention to $n-a-b>0$.

We now turn this bound into a form more suited for our purposes by using well-known bounds to convert the factorials into powers as follows.

Lemma 25. *$f_{n+1}(a+1,b+1) \leq n^{3/2}\left(c^{a'+b'}4^{(1-a'-b')}\left(\frac{(1-b')^{(1-b')}(1-a')^{(1-a')}}{a'^{a'}b'^{b'}(1-a'-b')^{2(1-a'-b')}}\right)\right)^n$, where $a' = a/n$ and $b' = b/n$.*

Proof. From Lemma 24, using $a' = a/n$ and $b' = b/n$ we get

$$f_{n+1}(a+1,b+1) \leq d_a d_b \frac{((1-b')n)!((1-a')n)!((2-2a'-2b')n)!}{(a'n)!(b'n)!((1-a'-b')n)!((1-a'-b')n)!^3}$$

Using the relation $\sqrt{2\pi}n^{n+1/2}e^{-n} \leq n! \leq en^{n+1/2}e^{-n}$,

$$\leq \frac{e^3}{8\pi^3}d_a d_b \frac{(1-b')^{(1-b')n+1/2}(1-a')^{(1-a')n+1/2}(2-2a'-2b')^{(2-2a'-2b')n+1/2}}{a'^{a'n+1/2}b'^{b'n+1/2}(1-a'-b')^{4(1-a'-b')n+2}}$$

$$\frac{n^{3/2}}{n^3}\frac{(en)^{(1-b')n}(en)^{(1-a')n}(en)^{(2-2a'-2b')n}}{(en)^{a'n}(en)^{b'n}(en)^{4(1-a'-b')n}}$$

which after cancellation and using loose bounds $e<3$ and $\pi>3$ for the constant term at the front which won't affect sharpness of the result, becomes

$$\leq (1/2)d_a d_b n^{-3/2}\frac{(1-b')^{(1-b')n+1/2}(1-a')^{(1-a')n+1/2}(2-2a'-2b')^{(2-2a'-2b')n+1/2}}{a'^{a'n+1/2}b'^{b'n+1/2}(1-a'-b')^{4(1-a'-b')n+2}}$$

now taking care of the extra "+1/2" and "+2" powers which again don't affect sharpness, by using the relations $1 \geq a', b' \geq 1/n$ and $1 \geq 1 - a' - b' \geq 1/n$, we see that

$$\leq (1/2) d_a d_b n^{-3/2} \frac{(1-b')^{1/2}(1-a')^{1/2}(2-2a'-2b')^{1/2}}{a'^{1/2} b'^{1/2} (1-a'-b')^2}$$

$$\frac{(1-b')^{(1-b')n}(1-a')^{(1-a')n}(2-2a'-2b')^{(2-2a'-2b')n}}{a'^{a'n} b'^{b'n} (1-a'-b')^{4(1-a'-b')n}}$$

$$\leq (1/2) d_a d_b n^{-3/2} \frac{2}{n^{-1/2} n^{-1/2} n^{-2}} \frac{(1-b')^{(1-b')n}(1-a')^{(1-a')n}(2-2a'-2b')^{(2-2a'-2b')n}}{a'^{a'n} b'^{b'n} (1-a'-b')^{4(1-a'-b')n}}$$

$$\leq d_a d_b n^{3/2} \left(\frac{(1-b')^{(1-b')}(1-a')^{(1-a')} 4^{(1-a'-b')}}{a'^{a'} b'^{b'} (1-a'-b')^{2(1-a'-b')}} \right)^n$$

Since $d_a \leq c^a$ as also shown by Arratia [2], and with a bit of rearranging, we have the desired result.

$\square$

Now we are ready for the main result of this section.

Theorem 26. *If* $4^{(1-a'-b')} \left(\frac{(1-b')^{(1-b')}(1-a')^{(1-a')}}{a'^{a'} b'^{b'} (1-a'-b')^{2(1-a'-b')}} \right) < c^{1-a'-b'}$ *for all* $0 < a' < A$ *and* $0 < b' < B$*, where* $c = \lim_{n\to\infty} s_n(1324)^{1/n}$*, then* $G'_n(a', b')$ *decays exponentially to 0 for all* $0 < a' < A$ *and* $0 < b' < B$*.*

Proof. If $4^{(1-a'-b')}\left(\frac{(1-b')^{(1-b')}(1-a')^{(1-a')}}{a'^{a'}b'^{b'}(1-a'-b')^{2(1-a'-b')}}\right)$ is less than $c^{1-a'-b'}$ for all $a' < A$ and $b' < B$, then there exists C and C' such that

$$c^{a'+b'}4^{(1-a'-b')}\left(\frac{(1-b')^{(1-b')}(1-a')^{(1-a')}}{a'^{a'}b'^{b'}(1-a'-b')^{2(1-a'-b')}}\right) < C' < C < c.$$

$f_n(a+1, b+1) \leq f_{n+1}(a+1, b+1)$, since we can form an injection from 1324-avoiding permutations of length n with a corner point at $(a+1, b+1)$ to those of length $n+1$ by sending each σ to σ' where $\sigma'(a+2) = b+2$, $\sigma'(a+1) = b+1$, $\sigma'(i) = \sigma(i)+1$ for $i < a+1$, $\sigma'(i) = \sigma(i-1)+1$ for $i > a+2$ and $\sigma(i-1) > a+1$, and $\sigma'(i) = \sigma(i-1)$ for $i > a+2$ and $\sigma(i-1) < a+1$. Visually, this adds a point at $(a+2, b+2)$ and "pushes" points to the top and right accordingly by 1. The permutation thus formed is unique since it can only be formed from the permutation obtain by the reverse process, deleting the point at $(a+2, b+2)$. The permutation formed has a corner point at $(a+1, b+1)$ and remains 1324-avoiding, as the new point added at $(a+2, b+2)$ cannot form a 1324 pattern with $(a+1, b+1)$ in it, and if it formed a 1324 pattern with three other points, it would have to form a 1324 pattern with $(a+1, b+1)$ in the original σ.

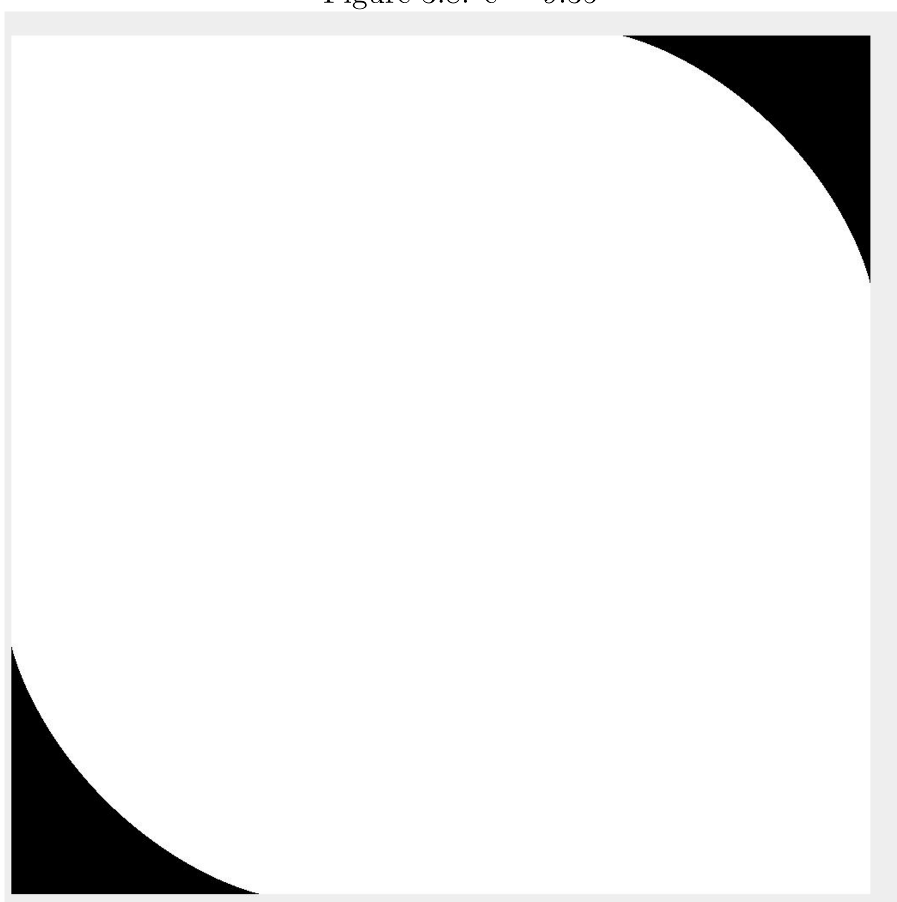
From Lemma 25, we see that $f_n(a+1, b+1) \leq f_{n+1}(a+1, b+1) < n^{3/2}C'^n$.

If we let $H(n) = n^{3/2}C'^n$ in $o(C^n)$, then $f_n(i, j) < H(n)$ for all $i \leq a'n+1$, $j \leq b'n+1$ where $a' < A$ and $b' < B$. Now apply Lemma 23, and all the points to the bottom-left of (A, B) decay exponentially to 0.

$\square$

For reasonable guesses of c, this amounts to the region below a certain curve, as shown by the black area in Figure 3.8.

Figure 3.8: c = 9.35



As a concrete example, I will prove that this does indeed enable us to prove that some areas decay exponentially to 0.

Lemma 27. *For x, a', b', $a'+b'$ in $(0,1)$,*

(1) x^x is decreasing on $(0,1/e)$ and increasing on $(1/e,1)$.

(2) $(1-b')^{(1-b')}(1-a')^{(1-a')} < (1-a'-b')^{(1-a'-b')}$.

Proof. (1) The first derivative is $(\log(x)+1)x^x$ which is negative on $(0,1/e)$ and positive on $(1/e,1)$.

(2) $x\log(x)$ has first derivative $(\log(x)+1)$ and second derivative $1/x$, which is positive on $(0,1)$. Since it is convex, $(1\log(1)-x\log(x))/(1-x)$ is increasing. Thus

$$\frac{-(1-a')\log(1-a')}{a'} > \frac{-(1-a'-b')\log(1-a'-b')}{a'+b'}$$

and hence

$$-(1-a')\log(1-a') > \frac{-a'(1-a'-b')\log(1-a'-b')}{a'+b'}$$

Doing the same for b' and combining them yields

$$-(1-a')\log(1-a')-(1-b')\log(1-b') > -(1-a'-b')\log(1-a'-b')$$

Now we have

$$(1-a')\log(1-a')+(1-b')\log(1-b') < (1-a'-b')\log(1-a'-b')$$

which gives us

$$(1-b')^{(1-b')}(1-a')^{(1-a')} < (1-a'-b')^{(1-a'-b')}$$

.

□

Corollary 28. *If* $0 < a', b' < 1/16$, $G'_n(a', b')$ *decays exponentially to 0.*

Proof. We could use the loose bound of $c \geq 8$ obtained in the last section, but a better bound was found by Albert, et al., of $c \geq 9.35$. We proceed with the round value of $c \geq 9$.

We wish to show that $\frac{(1-b')^{(1-b')}(1-a')^{(1-a')}}{a'^{a'}b'^{b'}(1-a'-b')^{2(1-a'-b')}} < (9/4)^{1-a'-b'}$. From Lemma 27 we see that $a'^{a'} > (1/16)^{1/16}$, and likewise for $b'^{b'}$. Similarly, $(1-a'-b')^{(1-a'-b')} > (7/8)^{7/8}$. Then also using (2) from Lemma 27 we get

$$\begin{aligned}\frac{(1-b')^{(1-b')}(1-a')^{(1-a')}}{a'^{a'}b'^{b'}(1-a'-b')^{2(1-a'-b')}} &< \frac{1}{a'^{a'}b'^{b'}(1-a'-b')^{(1-a'-b')}} \\ &< \frac{1}{(1/16)^{1/8}(7/8)^{7/8}}\end{aligned}$$

Now by brute force, we find that $0.7^8 = 0.05764801$, which is less than $1/16$. The above is then less than $\frac{1}{0.7(7/8)} < 80/49 < 2$.

On the other hand, $(9/4)^{1-a'-b'} > (9/4)^{7/8}$. Again using brute force calculation, $(2^8)(4^7) = 4194304 < 9^7 = 4782969$, which means that $(9/4)^{7/8} > 2$. Hence $\frac{(1-b')^{(1-b')}(1-a')^{(1-a')}}{a'^{a'}b'^{b'}(1-a'-b')^{2(1-a'-b')}} < (9/4)^{1-a'-b'}$, and by the theorem, $\lim_{n\to\infty} G'_n(a', b') = 0$.

$\square$

If one accepts the validity of computer calculation, then this can easily be further improved, to beyond 1/10. Such calculation yields

$$\frac{1}{a'^{a'} b'^{b'} (1 - a' - b')^{(1-a'-b')}} < \frac{1}{(1/10)^{1/5} (4/5)^{4/5}} = 1.89...$$

which is less than $(9.35/4)^{1-a'-b'} > (9.35/4)^{4/5} = 1.97....$

If c has a higher value than now known, the area where we know $G'_n(a', b')$ decays exponentially to 0 will likewise be increased. Figures 3.9 and 3.10 show the areas for $c = 10$ and $c = 11$ respectively.

Figure 3.9: c = 10

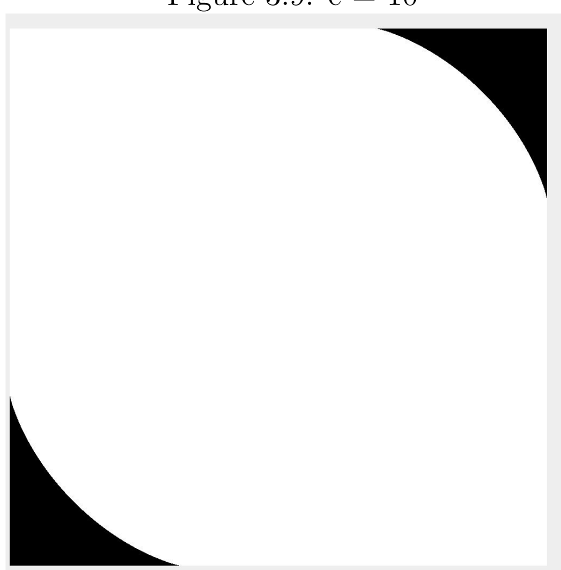
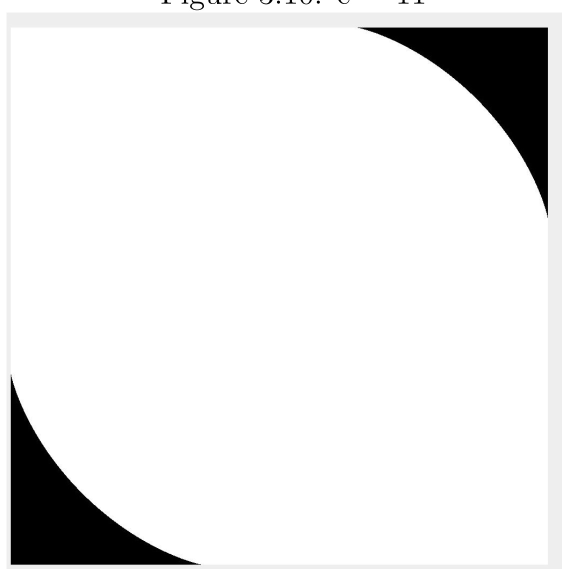


Figure 3.10: c = 11



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