1-1-2016

Universal Memory Architectures for Autonomous Machines

Dan P. Guralnik
University of Pennsylvania, guraldan@seas.upenn.edu

Daniel E. Koditschek
University of Pennsylvania, kod@seas.upenn.edu

Universal Memory Architectures for Autonomous Machines

Abstract
We propose a self-organizing memory architecture (UMA) for perceptual experience provably capable of supporting autonomous learning and goal-directed problem solving in the absence of any prior information about the agent's environment. The architecture is simple enough to ensure (1) a quadratic bound (in the number of available sensors) on space requirements, and (2) a quadratic bound on the time-complexity of the update-execute cycle. At the same time, it is sufficiently complex to provide the agent with an internal representation which is (3) minimal among all representations which account for every sensory equivalence class consistent with the agent's belief state; (4) capable, in principle, of recovering a topological model of the problem space; and (5) learnable with arbitrary precision through a random application of the available actions. These provable properties — both the trainability and the operational efficacy of an effectively trained memory structure — exploit a duality between weak poc sets — a symbolic (discrete) representation of subset nesting relations — and non-positively curved cubical complexes, whose rich convexity theory underlies the planning cycle of the proposed architecture.

Keywords
general agent, self-organizing memory, universal representation, belief update and revision, non-positively curved cubical complex, weak poc set

Disciplines
Artificial Intelligence and Robotics

Universal Memory Architectures for Autonomous Machines

Dan P. Guralnik, Daniel E. Koditschek

Electrical & Systems Engineering, School of Engineering and Applied Sciences, University of Pennsylvania, 200 South 33rd Street, Moore bldg. #203, Philadelphia, PA 19104, United States

Abstract

We propose a self-organizing memory architecture (UMA) for perceptual experience provably capable of supporting autonomous learning and goal-directed problem solving in the absence of any prior information about the agent's environment. The architecture is simple enough to ensure (1) a quadratic bound (in the number of available sensors) on space requirements, and (2) a quadratic bound on the time-complexity of the update-execute cycle. At the same time, it is sufficiently complex to provide the agent with an internal representation which is (3) minimal among all representations which account for every sensory equivalence class consistent with the agent's belief state; (4) capable, in principle, of recovering a topological model of the problem space; and (5) learnable with arbitrary precision through a random application of the available actions. These provable properties — both the trainability and the operational efficacy of an effectively trained memory structure — exploit a duality between *weak poc sets* — a symbolic (discrete) representation of subset nesting relations — and *non-positively curved cubical complexes*, whose rich convexity theory underlies the planning cycle of the proposed architecture.

Keywords: general agent, self-organizing memory, universal representation, belief update, belief revision, non-positively curved cubical complex, weak poc set.

1. Introduction

1.1. Motivation

A major obstacle to autonomous systems synthesis is the absence of a capacious but efficient memory architecture. In humans, memory influences behaviour over a wide range of time scales, leading to the emergence of what seems to be a functional hierarchy of sub-systems [80]: from non-declarative vs. declarative through the split of declarative memory into semantic and episodic [93]; and on to theories of attention and recall [3]. This variety of scales is mirrored in the collection of problems addressed by the synthetic sciences: from learning dependable actions/motion primitives [57, 86]; through learning objects and their affordances [42, 38] to demonstration-driven task execution [84, 13]; through exploring and mapping an unknown environment (SLAM) [47, 43, 88, 55] and

Email addresses: `guraldan@seas.upenn.edu` (Dan P. Guralnik), `kod@seas.upenn.edu` (Daniel E. Koditschek)

motion planning [75, 69, 24]; and on to general problem solving [61] using artificial general intelligence architectures [46, 27, 63].

One idea stands out as common to all these approaches, beginning with the formal notion of a problem space introduced by Newell and Simon [61, 59]: the purpose of a memory architecture is to learn the transition structure (however deep) of the state space $\mathbf{X}$ of the system comprised of the agent and its environment $\mathbf{E}$ while processing the history of observations into a very general model $\mathbf{M}$ which facilitates future control even in the face of fairly radical changes in the environment. It is often argued (e.g. [14, 81, 53]) that memory architectures for general agents should enjoy a high degree of domain- and task-independence. However, clear definitions of notions such as 'domain' and 'task' are not readily forthcoming across the vast breadth of literatures discussing memory, agents and autonomy. Notions of 'universal learners' have been proposed [73] based on optimizing gain in estimators of predictive entropy ('curiosity surfing'), but there is also evidence to suggest that the resulting generality may still be insufficient for learning and retaining commonly considered highly repetitive tasks such as locomotion [51].

Absent broadly recognized formal foundations, we return to the most literal representation of information to study how perceptual bits might give rise to self-organizing internal representations capable of facilitating efficient control.

We introduce and characterize a very general class of representations supported by an architecture provably satisfying intuitive universality properties, including, most centrally: (1) interactions with the environment are encoded in the most generic, yet minimal, manner possible, while requiring no prior semantic information; and (2) learning obtains from direct binary sensory input, automatically developing appropriate contextual links between sensations of arbitrary modality. A key improvement over state of the art architectures is that an UMA provably encodes observation history in a geometry, or *model space*, whose convexity theory allows the agent's problem solving to take the form of reactive motion planning realized through following nearest point projection paths to the designated target.

1.2. Contributions and Challenges

We consider a generic *discrete binary agent* (DBA): a machine sensing and interacting with its environment in discrete time, equipped with a finite collection $\mathbf{\Sigma}$ of Boolean-valued sensors, some of which serve as triggers for actions/behaviors (switched on and off at will). Our formalism for a DBA may be viewed as a PSR [48] stripped of all probabilistic data. In that, it most resembles a discrete-time, non-deterministic version of a *diversity automaton* [70] allowing for an infinite/continuous environment. However, the internal representation developed by the agent differs significantly.

Given an instance of a DBA interacting with an environment $\mathbf{E}$, it is natural to view the set $\mathbf{\Xi}$ of perceptual classes of the associated dynamical system $\mathbf{X}$ as a subset of the power set $\{0,1\}^{\mathbf{\Sigma}}$. It has been proposed [22, 91] that a memory architecture must be capable of supporting an internal representation $\mathbf{M}$ rich enough to account for the diversity [70] of $\mathbf{X}$: Exact problem solving, when construed as abstract motion planning, requires an internal representation capable, eventually, of accounting for all the classes in $\mathbf{\Xi}$ and the transitions between them. Unfortunately, as expressed forcefully in [70] and as we review below, the task of obtaining an exact description of $\mathbf{\Xi}$ becomes intractable in the absence of strong simplifying assumptions about $\mathbf{X}$, as the number of sensors grows.

To circumvent this obstacle, rather than imposing any specific structure on $\mathbf{X}$, we propose to relax the requirement for precise reconstruction by introducing an approximation whose discrepancy from Ξ we characterize exactly and show to be the smallest possible given the information recorded by the agent.

The new memory and control architecture we propose here consists of two layers:

- A data structure $\mathbf{S}$ – called a *snapshot* – keeping track of the current state and summarizing observations in terms of a collection of real-valued registers, of size quadratic in the number of sensors, summarizing the history of observations made by the agents.
- A reactive planner, built on a *weak poc set structure* $\mathbf{P}$ ([31, 71] and defn. 3.3) constituting a record of pairwise implications among the atomic sensations as observed by the agent; $\mathbf{P}$ is computed from $\mathbf{S}$ in each control cycle.

A crucial property of our architecture is that $\mathbf{P}$ and $\mathbf{M}$ are formally reconstructible from each other. The model space $\mathbf{M}$ takes the form of a CAT(0) cubical complex, or *cubing*[1], whose 0-skeleton is contained in $\{0,1\}^{\mathbf{\Sigma}}$. As the snapshot $\mathbf{S}$ is updated by incoming observations, the space $\mathbf{M}$, as encoded by $\mathbf{P}$, is transformed along with it. We can state our main contributions – albeit, necessarily, informally at this point – in terms of provable properties of the architecture and its model spaces:

(i) Universality of Representation. $\mathbf{M}$ is the minimal model guaranteed to represent all the perceptual classes of *any* sensorium $\mathbf{\Sigma}$ satisfying the record $\mathbf{P}$ (Section 3.3, Theorem 3.15). In particular, given only the information encoded in $\mathbf{P}$, it is impossible to distinguish the 0-skeleton of $\mathbf{M}$ from the set of perceptual classes, Ξ.

(ii) Topological Approximation. As a topological space, $\mathbf{M}$ is always contractible[2]. Provided a sufficiently rich sensorium, the sub-complex $\mathbf{M}^{\times} \subset \mathbf{M}$ of faces all of whose vertices lie in Ξ inherits from $\mathbf{M}$ the topology[3] of the observed space $\mathbf{X}$ (Section 3.4, Theorem 3.19).

(iii) Low-complexity, Effective Learning. The proposed architecture requires quadratic space (in the number of sensors) for storage, and no more than quadratic time for updating. Furthermore, an agent picking actions at random learns an approximation of the resulting walk's limiting distribution on $\mathbf{X}$ (see 4.4.1).

(iv) Efficiency of Planning. Planning the next action given a target sensation takes quadratic time in the number of sensors, while eliminating the need for searching in the model space. With sufficient parallel processing power, this bound may be reduced to a constant multiple of the height — the maximum length of a chain of implications — of the record $\mathbf{P}$ (see 5.2).

---

[1] For a good introduction CAT(0) cubical complexes, see [96]. For a tutorial on cell complexes see [33], chapter 0 and appendix.

[2] The formal notion of being 'hole-free' — see [33], chapter 0.

[3] Up to homotopy equivalence — see definition in [33], chapter 0.

We note that implementing UMA on a truly parallel neural architecture featuring an appropriately modified Drescher "neural crossbar" [23], will reduce maintenance costs to $O(1)$ and planning costs to sub-linear in the number of sensors. To the best of our knowledge, this combination of provable properties has not previously appeared in the literature.

Caveats. It is crucial to remark here that, at this early stage, the reasoning capabilities of UMAs are limited by the following factors:

- **(a)** The computational advantages of UMAs come at a significant cost, driven largely by the topological complexity introduced into the problem by the set of *essential obstacles*[4], $\mathbf{M} \smallsetminus \mathbf{M}^{\times}$. The points of this set serve as obstructions to an UMA's reactive planning mechanism which capitalizes mainly on the contractibility of the model space (see examples in Section 5.3).
- **(b)** More generally, the lack of a principled mechanism for formulating strategically parsimonious new queries out of the available ones (e.g., through the formation of Boolean and LTL predicates) prevents agents from improving the resolution of the constructed model.

Consequently, at present, we only consider a narrow class of 'toy' examples in which the essential obstacles do not act as obstructions to the agent's planning mechanism in the context of a particular task. Of course, this contradicts the advocated goal of achieving generality in a synthetic agent. We discuss directions in our ongoing research meant to address this problem in Section 7.

1.3. Relation to Past Literature

1.3.1. Learning and Problem Solving as Abstracted Mapping and Navigation

Our point of departure is a general reduction of *any* discrete time problem (in the sense of Newell and Simon [61]) to a navigation and mapping problem of a point agent moving through a homotopically trivial ambient space, the model space $\mathbf{M}$, while avoiding a collection of obstacle regions corresponding to forbidden states, $\mathbf{M} \smallsetminus \mathbf{M}^{\times}$, as stated in **(i)** and **(ii)** above. Problems of this kind are fundamental to motion planning [75, 69] and mapping [47, 88, 66]. The ubiquity of obstacles in these settings introduces topological considerations whose primacy is well established in the algorithmic literature [66, 90, 44, 65, 92, 19], governing the complexity of not only motion planning [25] but even set membership [97]. The topological point of view has been shown to be well warranted in the discrete setting as well [64, 29, 40], and compatible with current ideas regarding localization based on estimating the nerve[5] of a system of place fields [17, 55, 53, 54]. Moreover, the idea of leveraging containment relations among sensor fields — the information used for encoding the model $\mathbf{M}$ — to represent causal and contextual information is a well-recognized tool across literatures, e.g. [66, 81, 57, 56], UMAs simply being the first to apply it uniformly and systematically to the agent's entire sensorium, regardless of modality, again see **(i)** above.

---

[4] As in the classical setting of navigation and planning problems [75, 69], computational complexity is driven by homotopy invariants of the problem domain [25].

[5] See *nerve of a covering* in [33], section 3.3.

1.3.2. Reinforcement Learning (RL) and Predictive State Representations (PSRs)

Similarly to the classical mapping and planning settings, the necessity to maintain and explore high-dimensional representations of the dynamics of $\mathbf{X}$ poses a major challenge for all current approaches (e.g. POMDP, SMDP) to RL [6, 7]. Modern ideas on constructing more compact representations — e.g. "object-focused" [14] and limited temporal horizon [56] — can be traced back to Rivest and Schapire [70], who proposed to replace the orthodox approach based on direct exploration of $\mathbf{X}$ with an approach based on learning the dynamics (e.g. "diversity automaton" structure) induced by the agent's actions on $\boldsymbol{\Xi}$, as sensed by the agent through a collection of binary 'tests'.

This line of thought also germinated the notion of a *predictive state representation*, or PSR [48]. A far-reaching generalization of POMDPs [79], a PSR is a high-order probabilistic model of the dynamics in $\boldsymbol{\Xi}$ (as opposed to an automaton), and much effort was invested in learning linear approximations of PSRs. For example, [8] demonstrates an impressive level of generality, with the agent reasoning about motion in a continuous, topologically non-trivial environment (an annulus), based only on simulated visual snapshots of the environment, without feature extraction. Still, the savings in representation costs obtained in this way, though significant, do not alter the very nature of the representation, which, *in the general case*, still requires a high-dimensional database instantiated in memory (which is not the case for UMAs — see contribution **iii**), with each individual task requiring a search (value optimization) through the space of action sequences. This is where such representations differ sharply from UMA representations, the latter integrating planning information directly into the geometry of the model space — see **(iv)** above.

1.3.3. Cognitive Architectures (CAs)

On the cognitive AI front, the curse of dimensionality led to a state of affairs where, typically, representations with guaranteed tractable performance come at the expense of generality, whereas the truly general architectures we know of eschew rigorous performance guarantees [34, 82], relying instead on *functional* modeling of problem-solving processes in the human brain [1] from a "systems perspective", as proposed by Newell [58]. The approaches range from "constructionist" hierarchical [30] architectures (GPS [60], SOAR [45], ACT-R [2], LIDA [27]), to "constructivist" architectures[6] such as Drescher's "Schema Mechanism" (SM) [23] or Rieger's [68] frames, aiming to achieve some of the functions of a problem-solving CA as emergent properties of a self-organizing network of simple, low-level computational components.

Of the above, Drescher's architecture SM is closest in spirit and structure to UMAs, but stops short of presenting a mathematical toolkit enabling a rigorous discussion of the architecture's capabilities. While currently somewhat ahead of UMAs in terms of its capacity for principled introduction of new computational elements (see our caveat **(b)** in Section 1.2 above), SM lacks an efficient navigation mechanism, as its model of the agent's interactions with the environment is, essentially, agglomerative. The fundamental building blocks of the two architectures being closely related (UMA is based on estimation of reliable implications, while SM is based on estimation of reliable causal descriptions

[6] Here we make use of terms Thorisson used in his criticism [87] of the approach to cognitive architectures prevalent at the time.

of actions[7]), it is one of our goals to seek the development of a "common refinement" of the two (see discussion in Section 7).

1.3.4. Belief Update and Revision, Situation Calculus

An UMA agent may be thought of as reasoning over a set of literals, one for each Boolean query from the agent's sensorium $\mathbf{\Sigma}$ (which is assumed closed under Boolean complementation), while continually updating its belief state, represented by (1) a collection of formulae — the weak poc-set structure $\mathbf{P}$ — of the form $a \to b$, $a, b \in \mathbf{\Sigma}$, and by (2) a term over $\mathbf{\Sigma}$ describing the current state of the world. The restricted nature of this representation precludes applying the generally accepted updating/revision operators [39, 9, 35, 50, 89] to $\mathbf{P}$, motivating our use of *snapshots*: the latter keep track of observation statistics and maintain a flexible Boolean network that encodes a belief state of the required form, facilitating internal deliberation based on the encoded belief.

Thus, the rigidity of belief state representation in UMAs is offset by the computational efficiency of the updating mechanism and the planning cycle, — see **(iii)** and **(iv)** above — exposing a rigorous mathematical connection between low-level connectionist computation and high-level symbolic problem solving.

1.4. Organization of the Paper

Section 2 formalizes the notion of a DBA, which may be seen as an non-deterministic abstraction of a PSR. Section 3 reviews weak poc set structures and the model spaces they encode, anticipating some of their basic uses by an UMA agent, including its formal properties expressed in contributions **(i)** and **(ii)** above. Additional technical details regarding weak poc sets are relegated to Appendix 8 for the sake of completeness of the exposition. Section 4 addresses contribution **(iii)**, characterizing the properties of a family of snapshots sufficient for learning. Section 5 is dedicated to planning and control **(iv)**, interpreting their algorithmic expression in terms of the geometry of convex sets in the model space. Proofs are relegated to Appendix 9 (that also offers a table of mathematical notation). Section 6 discusses the results of a variety of simulation studies. Finally, in Section 7 we offer a brief conclusion with a summary of forthcoming work now in progress.

2. Discrete Binary Agents

In this section we review and extend Sageev-Roller duality in parallel with the development of the notion of a *discrete binary agent*, or DBA. This overview of the preliminary meterial is meant to extend the initial discussion provided in [31] as well as to illustrate it with examples, intended as bridges to our current application. In keeping with tradition, we will develop a running example illustrating the various formal constructions.

This work hinges on a duality between poc sets and median algebras, going back to [36]. This duality was thoroughly studied by Martin Roller in [71], in a very successful attempt at constructing a rich and widely applicable theory of actions of discrete groups on simply connected non-positively curved cubical complexes — henceforth referred to as *cubings* — which was pioneered by Michah Sageev in [72]. In the end, an extension of

[7]Also known as "production rules" [45] and "contingencies", see [57].

this duality theory to *weak poc sets* will be called upon to provide the necessary formal guarantees that the proposed memory and control architectures actually do their job. We will mainly rely on [71] as a source of theoretical results, though sometimes it will be easier to use results from the elegant exposition in [62].

2.1. Environment and State

We place an agent in an environment $\mathbf{E}$. The state space of the system will be denoted by $\mathbf{X}$, where we assume there is a map $\mathtt{pos} : \mathbf{X} \to \mathbf{E}$, unknown to the agent, producing the location $\mathtt{pos}(x)$ of the agent in $\mathbf{E}$, given the state $x \in \mathbf{X}$ of the system as a whole. No further restrictions are placed on $\mathbf{E}$ or $\mathbf{X}$. Time $\mathbb{T}$ is modeled as the set of integers (the subjective time of the agent), with $t = 0$ corresponding to the initial time.

Definition 2.1. An element of the $(n+1)$-fold Cartesian power $\mathbf{X}^{n+1}$ is an *n-transition*. A map $\varphi = (\varphi|_t)_{t\in\mathbb{T}}$ from $\mathbb{T}$ to $\mathbf{X}$ is called a *trajectory through* $\mathbf{X}$, and gives rise to a trajectory $\mathrm{d}^n\varphi$ through $\mathbf{X}^{n+1}$ via $\mathrm{d}^n\varphi|_t = \varphi|_{t-n} \times \cdots \times \varphi|_t$. 0-transitions are *states*, and to 1-transitions are just *transitions*. □

With a mind toward inviting the broadest range of applications, we impose no additional requirements on either $\mathbf{X}$ or $\mathbf{E}$ at this point, much in the spirit of the way situation space is introduced in [52].

2.2. Binary Sensorium

We seek a language for discussing situated general agents observing their environment through binary input streams, or *sensors*. We start with:

Definition 2.2. A *complemented set* is a pair $(A, *)$ where A is a non-empty set equipped with a self-map $a \mapsto a^*$, satisfying $a^{**} = a$ and $a^* \neq a$, for all $a \in A$. □

Complemented sets provide the scaffolding for our formal notion of a *sensorium*, the sensory suite provided to an agent.

Definition 2.3. A *binary sensorium* (hereafter simply *sensorium*) is a tuple $(\mathbf{\Sigma}, *, \mathbf{0}, \rho)$ where $(\mathbf{\Sigma}, *)$ is a complemented set with a distinguished element $\mathbf{0}$, and each $a \in \mathbf{\Sigma}$ is assigned a non-negative integer *order* n_a, and a *realization* $\rho(a) \subseteq \mathbf{X}^{\mathrm{n}_a+1}$ such that:

1. $\mathrm{n}_{\mathbf{0}} = 0$ and $\rho(\mathbf{0}) = \varnothing$;
2. $\mathrm{n}_{a^*} = \mathrm{n}_a$ and $\rho(a^*) = \rho(a)^c := \mathbf{X}^{n_a+1} \smallsetminus \rho(a)$ for all $a \in \mathbf{\Sigma}$.

We refer to each $a \in \mathbf{\Sigma}$ as a n_a*-sensor*. For $A \subseteq \mathbf{\Sigma}$ we also denote $A^* := \{a^* \,|a \in A\}$ and, when relevant, $\rho_A := \bigcap_{a\in A} \rho(a)$. □

In other words, sensors are *evaluated* according to the rule:

Definition 2.4. Let $(\mathbf{\Sigma}, *, \mathbf{0}, \rho)$ be a sensorium. For $a \in \mathbf{\Sigma}$, the *value* $\langle a : \varphi\rangle \in \{0, 1\}$ of a on a trajectory φ at time $t \in \mathbb{T}$ is defined by $\langle a : \varphi\rangle|_t = 1 \Leftrightarrow \mathrm{d}^{n_a}\varphi|_t \in \rho(a)$. □

Remark 2.5 (Notational conventions for evaluation)**.** To avoid a proliferation of parentheses we will use the bracket notation $\langle g : s\rangle := g(s)$ to denote the evaluation of Boolean- and scalar-valued functions. We will often abuse notation and write $\langle a : x\rangle := \left\langle \mathbb{1}_{\rho(a)} : x\right\rangle$ when $a \in \mathbf{\Sigma}$ and $x \in \mathbf{X}^{\mathrm{n}_a+1}$. The symbol $\mathbb{1}_A$ will denote the indicator function of a set A with respect to the appropriate super-set. Also, note how the identity $\langle a^* : x\rangle \equiv 1-\langle a : x\rangle$ follows from $\rho(a^*) \equiv \rho(a)^c$; any Boolean function f (on any set) has a "complement" f^* defined through $\langle f^* : x\rangle = 1 - \langle f : x\rangle$.

2.3. Binary Observations

At any time $t \geq 0$, a sensorium $(\mathbf{\Sigma}, *, \mathbf{0}, \rho)$ is assumed to produce an observation:

Definition 2.6. The *unprocessed observation at time t* along a trajectory φ is the set $O|_t := \{a \in \mathbf{\Sigma} \,|\, \langle a : \varphi \rangle\, |_t = 1\}$. □

More generally, we need the following notions:

Definition 2.7. Let $(\mathbf{\Sigma}, *)$ be a complemented set. A $*$-*selection* on $\mathbf{\Sigma}$ is a subset $O \subset \mathbf{\Sigma}$ satisfying $O \cap O^* = \varnothing$. A $*$-selection O is *complete*[8] if $O \cup O^* = \mathbf{\Sigma}$. In anticipation of definition 2.9, the set of all complete $*$-selections on $\mathbf{\Sigma}$ will be denoted $S(\mathbf{\Sigma})^0$. □

Clearly, for a sensorium $(\mathbf{\Sigma}, *, \mathbf{0}, \rho)$, the unprocessed observation $O|_t$ is a complete $*$-selection on $\mathbf{\Sigma}$. It is time to introduce our running example.

Example 2.8. Setting $\mathbf{E} = \{0, \ldots, L\}$, L a positive integer, endow an agent with position sensors $a_1, \ldots, a_L \in \mathbf{\Sigma}$ realized as:

$$\langle a_k : x \rangle = 1 \Leftrightarrow \mathtt{pos}(x) < k\,, \quad \langle a_\ell^* : x \rangle = 1 \Leftrightarrow \mathtt{pos}(x) \geq \ell \tag{1}$$

Given a trajectory φ for the agent with $p = \mathtt{pos}(\varphi|_t)$ we must then have:

$$O|_t \cap \{a_1, \ldots, a_L, a_1^*, \ldots, a_L^*\} = \{a_k \,|k > p\} \cup \{a_k^* \,|k \leq p\} \tag{2}$$

We will keep expanding the sensory endowment of this agent in future examples. □

It is well-known [62] that the following is a metric (i.e. distance function) on $S(\mathbf{\Sigma})^0$:

$$\mathbf{\Delta}(A, B) = |A \smallsetminus B| = |B \smallsetminus A| = \tfrac{1}{2}\,|A \bigtriangleup B| \tag{3}$$

Indeed, fixing $A_0 \in S(\mathbf{\Sigma})^0$, an explicit isometry of the metric space $\left(S(\mathbf{\Sigma})^0, \mathbf{\Delta}\right)$ onto $\mathbf{2}^{A_0}$ endowed with the Hamming distance is constructed by sending $A \in S(\mathbf{\Sigma})^0$ to the [indicator function of the] set $A_0 \smallsetminus A$. We then see that $S(\mathbf{\Sigma})^0$ may be thought of as the vertex set, or 0-*skeleton*, of a $(\frac{|\mathbf{\Sigma}|}{2} - 1)$-dimensional standard unit cube; the edges of this cube, forming its 1-*skeleton*, are pairs A, B of vertices with $\mathbf{\Delta}(A, B) = 1$; the higher dimensional faces are given by:

Definition 2.9. Let $S(\mathbf{\Sigma})$ denote the cubical complex[9] whose vertices are the complete $*$-selections on $\mathbf{\Sigma}$ and with faces F_A defined as follows: for any $*$-selection A on $\mathbf{\Sigma}$, possibly incomplete, F_A is the set of all complete $*$-selections which, as subsets of $\mathbf{\Sigma}$, contain A. □

It is easy to verify that for any $0 \leq d \leq \frac{|\mathbf{\Sigma}|}{2} - 1$, the d-dimensional faces of $S(\mathbf{\Sigma})$ are in one-to-one correspondence with $*$-selections $A \subset \mathbf{\Sigma}$ satisfying $2(|A| + d) = |\mathbf{\Sigma}|$.

[8] This is identical to the notion of a *term* (respectively, a *complete term*), when $\mathbf{\Sigma}$ is viewed as the set of literals used to maintain the belief state of the agent, see e.g. [50].

[9] Please see [41], Chapter 2, for a very quick introduction to polyhedral complexes.

2.4. Action Model and definition of a DBA

We model the decisions available to our agents as transition sensors (1-sensors, see Def. 2.3). Transitions have *outcomes*:

Definition 2.10. Let $(\mathbf{\Sigma}, *, \mathbf{0}, \rho)$ be a sensorium and let $A \subset \mathbf{\Sigma}$ be a set of transition sensors. The *set of outcomes* of A is defined to be $\mathtt{out}_A(x) := \{y \in \mathbf{X} \,|\, x \times y \in \rho_A\}$ (see Def. 2.3). □

Embedding decisions in the sensorium reflects the viewpoint that (1) an action taken at a state $x \in \mathbf{X}$ may be seen as imposing a time-independent restriction on the set of states the system could enter in the following moment, and (2) the agent is capable of observing its own decisions as they are being invoked. This leads to the following formal and very broad definition of an agent (compare with Sec. 3 of [70]):

Definition 2.11. A discrete binary agent is a tuple $(\mathbf{X}, \mathbf{E}, \mathtt{pos}, \mathbf{\Sigma}, *, \mathbf{0}, \rho, \mathbf{\Sigma}_{act})$ such that $(\mathbf{\Sigma}, *, \mathbf{0}, \rho)$ is a sensorium on $\mathbf{X}$ and $\mathbf{\Sigma}_{act} \subset \mathbf{\Sigma}$ satisfies the following requirements:

(a) **Actions are binary.** $\mathbf{\Sigma}_{act} \cap \mathbf{\Sigma}^*_{act} = \varnothing$, and denote $\mathbf{Act} := \mathbf{\Sigma}_{act} \cup \mathbf{\Sigma}^*_{act} \cup \{\mathbf{0}, \mathbf{0}^*\}$. Note that $\mathbf{Act}$ is itself a sensorium.

(b) **Every action has outcomes.** For all $x \in \mathbf{X}$ and any complete $*$-selection A on $\mathbf{Act}$, the set $\mathtt{out}_A(x)$ is non-empty. □

In summary, a DBA occupying the state $x \in \mathbf{X}$ at time t makes an observation $O = O|_t \in S(\mathbf{\Sigma})^0$, and is then tasked with producing a decision encoded as $A|_t \in S(\mathbf{Act})^0$. The agent's decision imposes the constraint $A|_t \subseteq O|_{t+1}$ on the next state of the system.

Remark 2.12. Our model departs from the ubiquitous practice of including possible state-dependent restrictions on the executability of actions — see e.g. [24, 83, 35]. Here we interpret actions as mere control signals sent by the agent's 'mind' to the agent's 'body' in an attempt to invoke one or more of a fixed set of available behaviors. The signals may be sent — and will therefore appear in the next observation — regardless of whether or not they result in producing a meaningful interaction with the environment.[10] The 'mind' should be tasked with identifying, over time, whether or not a control signal produces meaningful outcomes. ■

Example 2.13. Continuing Example 2.8, provide the agent with the elementary actions enabling motion from any vertex $k \in \mathbf{E}$ to the adjacent $k+1$ using $\{\mathtt{fd}, \mathtt{bk}^*\}$, and to $k-1$ using $\{\mathtt{fd}^*, \mathtt{bk}\}$; standing still corresponds to $\{\mathtt{fd}^*, \mathtt{bk}^*\}$:

$$\begin{aligned} y \in \mathtt{fd}(x) \quad &\Leftrightarrow \quad \mathtt{pos}(y) = \min\{L, \mathtt{pos}(x)+1\} \;\vee\; (\mathtt{pos}(y) = \mathtt{pos}(x) \;\wedge\; \mathtt{flt}(y) = 1) \\ y \in \mathtt{bk}(x) \quad &\Leftrightarrow \quad \mathtt{pos}(y) = \max\{0, \mathtt{pos}(x)-1\} \;\vee\; (\mathtt{pos}(y) = \mathtt{pos}(x) \;\wedge\; \mathtt{flt}(y) = 1) \end{aligned}$$

where $\mathtt{flt}(x) \in \{0,1\}$ is an auxiliary state variable whose existence is necessitated by the requirements of Defn. 2.11(b). In addition to the necessary expansion of $\mathbf{X}$, its function

[10] It is easy to imagine more restrictive settings, where engaging in one set of elementary actions might preclude an agent from engaging in others (in fact, the example we consider is one such natural setting). While our formalism in this general case will have to be amended, our impression is that *at present* there is little to be gained in practice from extending it — see related discussion in Sec. 5.1.

is to declare a "fault" following any attempt to invoke the action $\{\mathtt{fd},\mathtt{bk}\}$; note that no tangible outcome arises for this action: we did not even provide the agent with a sensor reporting the value of $\mathtt{flt}(x)$. It is critical to see though, that such synthetic augmentations of $\mathbf{X}$ are only required in simulated scenarios: for any robotic agent in a physical environment, physics mandates (and creates) outcomes in all contexts. □

3. Overview: Memory Architecture and Model Spaces

3.1. Perceptual Classes

Since a sensorium $(\mathbf{\Sigma},*,\mathbf{0},\rho)$ may contain sensors of different orders, we need to formalize the notion of a perceptual class with some care. Set $N=\max_{a\in\mathbf{\Sigma}}\mathrm{n}_a$. Then, for any $a\in\mathbf{\Sigma}$ consider the set $\tilde{\rho}(a):=\left\{x\times y\,\middle|\,y\in\rho(a)\,,\ x\in\mathbf{X}^{N+1-\mathrm{n}_a}\right\}$. This gives rise to a new sensorium $(\mathbf{\Sigma},*,\mathbf{0},\tilde{\rho})$ where (1) all $a\in\mathbf{\Sigma}$ have the same order N, and (2) the value of a on any trajectory φ at any time t coincides with its value as given by the original sensorium.

Definition 3.1. Let $(\mathbf{\Sigma},*,\mathbf{0},\rho)$ be a sensorium and let $\tilde{\rho}$ be as above. The map $\rho^*:\mathbf{X}^{N+1}\to S(\mathbf{\Sigma})^0$ is then defined by $\mathbf{x}\mapsto\{\mathbf{a}\in\mathbf{\Sigma}\,|\mathbf{x}\in\tilde{\rho}(\mathbf{a})\}$, and its fibers[11] are referred to as the *perceptual classes* of the sensorium (compare with [22]). □

From the point of view of a DBA, the system is only observable through the map ρ^*, and the agent is only able to reason over the perceptual classes in their symbolic form, as observations belonging to the image of ρ^*:

Definition 3.2. Let $(\mathbf{\Sigma},*,\mathbf{0},\rho)$ be a sensorium. We define $\mathbf{\Xi}_\rho$ to be the image of ρ^*. □

3.2. An Approximate Record of Implications: Weak Poc Sets

Informally, by a "record of implications in $\mathbf{\Sigma}$" we mean a partial ordering on $\mathbf{\Sigma}$ intended to serve as the agent's belief regarding Boolean implications holding among the sensations and their complements. Formally, let us recall a definition from [31]:

Definition 3.3. A *weak poc set* is a tuple $\mathbf{P}=(\mathbf{\Sigma},\leq,\mathbf{0},*)$ where $(\mathbf{\Sigma},*)$ is a complemented set and $(\mathbf{\Sigma},\leq)$ is a poset with minimum $\mathbf{0}\in\mathbf{\Sigma}$, with the two structures connected by the requirement that $a\leq b\Rightarrow b^*\leq a^*$ for all $a,b\in\mathbf{\Sigma}$. □

Remark 3.4 (Notation for Weak Poc Sets)**.** For $\mathbf{P}$ as above we will often write $a\in\mathbf{P}$ meaning $a\in\mathbf{\Sigma}$, by abuse of notation. Furthermore, for $A\subseteq\mathbf{\Sigma}$ we will use the notation

$$A{\uparrow}=\bigcup_{a\in A}\{b\in\mathbf{\Sigma}\,|b\geq a\}\ ,\quad A{\downarrow}=\bigcup_{a\in A}\{b\in\mathbf{\Sigma}\,|b\leq a\}\ . \tag{4}$$

Where always $A^*{\uparrow}=A{\downarrow}^*$ and $A^*{\downarrow}=A{\uparrow}^*$, due to the order-reversal property of $*$.

We would like our agents to maintain their belief in the form of a weak poc set structure $\mathbf{P}|_t$ over the sensorium $(\mathbf{\Sigma},*,\mathbf{0},\rho)$. As the map ρ is unknown to the agent, we intend for the agent to interpret a relation of the form $a\leq b$ in $\mathbf{P}|_t$ as $\langle a:\varphi\rangle|_{t'}\leq\langle b:\varphi\rangle|_{t'}$ holding for all $t'\in\mathbb{T}$ and all trajectories φ. Back to our running example:

[11] That is, the point-preimages of ρ^*.

Example 3.5. Continuing example 2.13, it would make sense for our agent to learn the relations $a_k < a_{k+1}$ for $k \le L-1$ ("standing to the left of position k implies standing to the left of $k+1$"), as they provide information about the geometric structure of $\mathbf{E}$, seen as a discretized interval. The same applies to the relations $\texttt{fd} < a_1^*$ and $\texttt{bk} < a_L$ which specify the special role of the endpoints $0, L \in \mathbf{E}$ with respect to the available actions.□

Some additional terminology will be useful:

Definition 3.6. In a weak poc set $\mathbf{P}$, an element $a \in \mathbf{P}$ is said to be *negligible* if $a \le a^*$; a is *proper* if neither a nor a^* are negligible. If $\mathbf{0}$ is the only negligible element, then P is said to be a *(true) poc set.* □

Weak poc sets form a category[12], with the following notion of map, or *morphism*:

Definition 3.7. A function $f : \mathbf{P} \to \mathbf{Q}$ between two weak poc sets is a *poc morphism* if $f(\mathbf{0}) = \mathbf{0}$ and $f(a^*) = f(a)^*$, $a \le b \Rightarrow f(a) \le f(b)$ are satisfied for all $a, b \in \mathbf{P}$. The set of all poc morphisms as above will be denoted $\mathrm{Hom}(P,\, Q)$. □

Example 3.8 (The Minimal Weak Poc Set)**.** The set $\{0, 1\}$ with the relations $0 < 1$ and $1 = 0^*$ is a poc set, and it is denoted by $\mathbf{2}$. Clearly, there is only one poc morphism of $\mathbf{2}$ into any weak poc set P, but then there may be many poc morphisms of a weak poc set P onto $\mathbf{2}$. □

Example 3.9 (The Orthogonal Poc Set)**.** Any complemented set $(\mathbf{\Sigma}, *)$ with distinguished element $\mathbf{0}$ gives rise to a poc set with minimum $\mathbf{0}$ and where no two elements in $\mathbf{\Sigma} \smallsetminus \{\mathbf{0}, \mathbf{0}^*\}$ are comparable. □

Example 3.10 (σ-Algebras as poc sets)**.** Let $\mathscr{B}$ be a σ-algebra on a non-empty (possibly infinite) set $\mathbf{X}$. Then $(\mathscr{B}, \subseteq, F \mapsto \mathbf{X} \smallsetminus F)$ is a poc set. In particular, the power set of $\mathbf{X}$, denoted $\mathbf{2}^{\mathbf{X}}$, obtains the structure of a poc set in this way. It is standard to identify $\mathbf{2}^{\mathbf{X}}$ with the space of functions $f : \mathbf{X} \to \mathbf{2}$: any such f will be identified with the subset $f^{-1}(1) \in \mathbf{2}^{\mathbf{X}}$. Recalling our notation for the evaluation of functions, the order structure on $\mathbf{2}^{\mathbf{X}}$ may be written as $f \le g \Leftrightarrow \forall_{x \in \mathbf{X}} \langle f : x \rangle \le \langle g : x \rangle \Leftrightarrow fg = f$. Also, $\mathbf{2}^{\mathbf{X}}$ is a true poc set, that is: $\mathbf{2}^{\mathbf{X}}$ contains no negligible elements save for the zero function $\mathbf{0}$. □

Deferring additional examples we briefly turn to an important relationship between weak poc sets, true poc sets and the learning goals of DBAs:

Definition 3.11. Let $\mathbf{P}$ be a weak poc set and let $\mathbf{X}$ be a non-empty set. A *realization of* $\mathbf{P}$ *in* $\mathbf{X}$ is a poc morphism of $\mathbf{P}$ into $\mathbf{2}^{\mathbf{X}}$. □

A realization $r : \mathbf{P} \to \mathbf{2}^{\mathbf{X}}$ provides a consistent way of regarding each $a \in \mathbf{P}$ as a binary query over $\mathbf{X}$, so that the set of all $x \in \mathbf{X}$ with $\langle r(a) : x \rangle = 1$ is the set of all points where the question is answered affirmatively. Thus, given a DBA with sensorium $(\mathbf{\Sigma}, *, \mathbf{0}, \rho)$, one way for the DBA to obtain a useful representation of the unknown sets $\rho(a)$, $a \in \mathbf{\Sigma}$, is to make use of the observations $O|_s$, $0 \le s \le t$ for evolving a weak poc set structure $\mathbf{P}|_t$ over $(\mathbf{\Sigma}, *, \mathbf{0})$ — possibly beginning with $\mathbf{P}|_0$ as the orthogonal poc set structure[13] — such that $\mathbf{P}|_t$ is as rich as possible and such that the extended map $\tilde{\rho}$ of Section 3.1 comes as close as possible to being a realization of $\mathbf{P}|_t$ in $\mathbf{X}^{N+1}$, as t progresses. This is the *Learning Objective of an UMA agent*, which we further substantiate in the next section.

[12] See [41], chapter 4, for a quick reference on the elements and basic uses of Category Theory.

[13] Note that any map of the orthogonal poc set into $\mathbf{2}^Z$ is a realization, for any Z

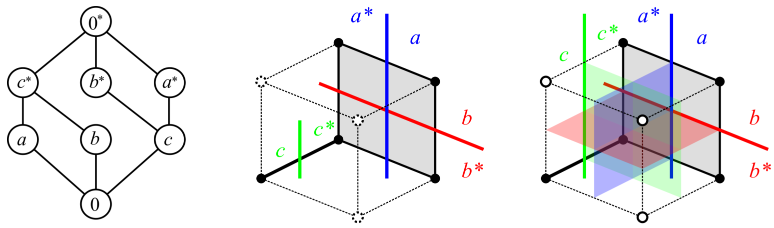

Figure 1: (left) A simple poc set $\mathbf{P}$ over the complemented set $\mathbf{\Sigma} = \{\mathbf{0}, \mathbf{0}^*, a, a^*, b, b^*, c, c^*\}$ and the resulting cube complex $\mathtt{Cube}(\mathbf{P})$ (center), obtained by deleting all incoherent vertices from the cube $S(\mathbf{\Sigma})$ (right).

3.3. Model Spaces and Universality

Similarly to the situation in propositional belief updating, we would like a DBA with sensorium $(\mathbf{\Sigma}, *, \mathbf{0}, \rho)$ to reason over the collection $S(\mathbf{\Sigma})^0$ of all complete $*$-selections on $\mathbf{\Sigma}$. However, instead of a "possible worlds" interpretation, we see $S(\mathbf{\Sigma})^0$ as enumerating the set $\mathbf{\Xi}_\rho$ of *possible* perceptual classes of the system. Clearly, it is to the advantage of a DBA with this sensorium to be aware which $O \in S(\mathbf{\Sigma})^0$ are inconsistent (in other words, will never be observed). However, distilling an explicit list thereof may require prohibitive amounts of storage (exponential in $|\mathbf{\Sigma}|$), not to mention the computational costs. We propose a tractable alternative based on the following construction, due to Sageev [72] and Roller [71]:

Definition 3.12. Let $\mathbf{P} = (\mathbf{\Sigma}, \leq, \mathbf{0}, *)$ be a finite poc set. A pair of elements $a, b \in \mathbf{P}$ is said to be *incoherent* if $a \leq b^*$. A subset A of a poc set $\mathbf{P}$ is said to be *coherent* if it contains no incoherent pair[14]. Furthermore:

(a) **The dual[15] cubing of P**, denoted $\mathtt{Cube}(\mathbf{P})$, is the (cubical) sub-complex of $S(\mathbf{\Sigma})$ induced by the set of coherent vertices (see Figure 1);

(b) **The set dual P**, denoted $\mathbf{P}^\circ$, is the vertex set (or 0-*skeleton*) of $\mathtt{Cube}(\mathbf{P})$;

(c) **The dual graph of** P, denoted $\mathtt{Dual}(\mathbf{P})$, is the union of the vertex and edge sets (or 1-*skeleton*) of $\mathtt{Cube}(\mathbf{P})$. □

Example 3.13. Let us set $\mathbf{\Sigma} = \{\mathbf{0}, \mathbf{0}^*, a_1, a_1^*, \ldots, a_L, a_L^*\}$ with two different poc set structures, $\mathbf{P}$ and $\mathbf{Q}$, defined by the relations $a_k < a_{k+1}$, $1 \leq k < L$ in $\mathbf{P}$ and $a_i < a_j^*$, $1 \leq i < j \leq L$ in $\mathbf{Q}$ (and the necessary consequences required by the axioms of a weak poc set). These may be regarded as abstractions of two sensoria constructed as follows. Let $p_1 < \ldots < p_L$ in $[0,1]$ be points that are pairwise at least ϵ apart, $\epsilon > 0$. Then $\mathbf{P}$ may be realized by setting $\langle a_k : x \rangle = 1 \Leftrightarrow \mathtt{pos}(x) < p_k$ ("threshold sensors"), while $\mathbf{Q}$ may be realized, for example, by $\langle a_k : x \rangle = 1 \Leftrightarrow \mathtt{dist}(\mathtt{pos}(x), p_k) < \epsilon$ ("beacon sensors").

The vertices of $\mathtt{Cube}(\mathbf{P})$ have the form $V_k = \{\mathbf{0}^*\} \cup \{a_j^*\}_{j>k} \cup \{a_i\}_{i \geq k}$, $0 \leq k \leq L$, with an edge joining V_k to V_{k+1} for all $k < L$ (recall that edges in $\mathtt{Cube}(\mathbf{P})$ are edges

[14] We have chosen the term *coherent subset* over Roller's *filter base* to better fit the context of our application.

[15] Appendix 8.2 discusses the category-theoretical context within which duality should be understood.

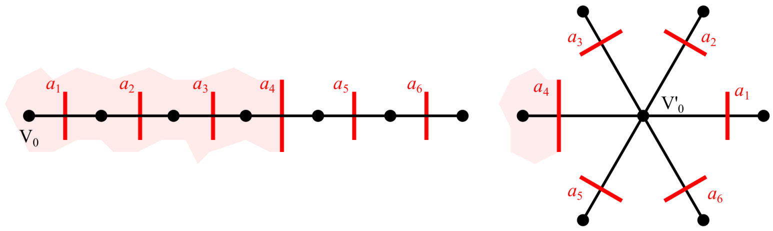

Figure 2: Dual graphs for two arrangements of sensors along the real line (see Example 3.13): 'threshold' sensors encoding a path (left), and 'beacon' sensors encoding a starfish (right).

of the cube $S(\mathbf{P})$). The complex $\texttt{Cube}(\mathbf{Q})$ has a different collection of vertices, dictated by the fact that all pairs $\{a_i, a_j\}$ with $i \neq j$ are incoherent: there is a 'special' vertex $V_0' = \{\mathbf{0}^*, a_1^*, \ldots, a_L^*\}$ and a collection of 'generic' ones, $V_k' = \{\mathbf{0}^*, a_k\} \cup \{a_j^*\}_{j \neq k}$; all the V_k', $k > 0$, are adjacent to V_0', and no other pair of vertices are adjacent. Figure 2 shows $\texttt{Cube}(\mathbf{P})$ (left), which is an L-path, and $\texttt{Cube}(\mathbf{Q})$ (right), which we will refer to in the future as a *starfish*. Note how, of the two model spaces, $\texttt{Cube}(\mathbf{P})$ seems to provide the better discretization of $[0, 1]$. □

Definition 3.14. The model space $\mathbf{M}|_t$ maintained by an UMA agent is derived from $\mathbf{P}|_t$ through $\mathbf{M}|_t := \texttt{Cube}(\mathbf{P}|_t)$.

At any time t, our agents will reach decisions based on the assumption that they are navigating in the space $\mathbf{M}|_t$. A compelling reason for choosing $\mathbf{P}|_t^\circ$ as the vertex set of our model is the following simple extension of an observation from [31]:

Theorem 3.15 (Universality of Representation)**.** *Let* $\mathbf{P}$ *be a weak poc set structure on the complemented set* $(\mathbf{\Sigma}, *)$ *with minimum element* $\mathbf{0}$*. Then* $\mathbf{P}^\circ$ *contains* $\mathbf{\Xi}_\rho$ *for any non-empty set* $\mathbf{X}$ *and any sensorium* $(\mathbf{\Sigma}, *, \mathbf{0}, \rho)$*, provided the map* $\tilde{\rho}$ *(as defined in Section 3.1) is a realization of* $\mathbf{P}$*. Moreover, no proper subset of* $\mathbf{P}^\circ$ *has this property.*

The proof is a standard argument from Sageev-Roller duality theory:

Proof. Pick any point $\mathbf{x} \in \mathbf{X}^{N+1}$. By definition, $\xi = \rho^*(\mathbf{x})$ lies in $\mathbf{P}^\circ$ if and only if no $a, b \in \xi$ satisfy $a \leq b^*$ in $\mathbf{P}$. However, if $\tilde{\rho}$ is order-preserving and $a \leq b^*$ for $a, b \in \xi$ then $\tilde{\rho}(a) \cap \tilde{\rho}(b) = \varnothing$ and $\mathbf{x} \in \tilde{\rho}(a) \cap \tilde{\rho}(b)$ at the same time — contradiction.

Now, consider the space $\mathbf{X} = \mathbf{P}^\circ$ with $\rho : \mathbf{\Sigma} \to \mathbf{2}^{\mathbf{X}}$ given by $\rho(a) = \{U \in \mathbf{P}^\circ \,|a \in U\}$. It is easily verified that ρ is a poc morphism and that $\rho^* : \mathbf{X} \to \mathbf{P}^\circ$ is the identity map (and hence surjective), finishing the proof. ■

Thus, $\mathbf{P}|_t^\circ$ is the "least biased" and minimalist choice of structure representing the possible perceptual classes given the belief state $\mathbf{P}|_t$. The last theorem may also be restated as follows: Given $\mathbf{P}$ and *any* realization ρ of $\mathbf{P}$, $\texttt{Cube}(\mathbf{P})$ is the smallest cubical sub-complex of $S(\Sigma)^0$ accounting for all the perceptual classes of the sensorium, *no matter the particular choice of* $\mathbf{X}$ *or the particular realization* ρ. In the case of an embodied agent[16] this result — in fact, its proof — demonstrates how implications learned in

[16]See Ziemke [98] on the role of situatedness and embodiment in the emergence of radical constructivism in AI.

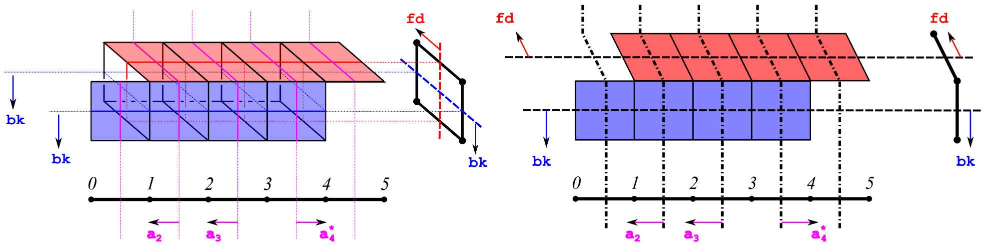

Figure 3: Model space for a DBA placed in a discrete path and endowed with "GPS" sensors and a capability for a back and forth stepwise traversal of the path (Example 3.16 with $L = 5$). On left: agent does not have $\texttt{fd} < \texttt{bk}^*$ on record, which gives rise to 3-dimensional cubes. On right: agent has $\texttt{fd} < \texttt{bk}^*$ on record.

one environment may serve an UMA agent in another environment satisfying a similar collection of rules. To close this section, let us return to our running example:

Example 3.16. With the sensorium and poc set structure of Example 3.5, what is the model space $\texttt{Cube}(\mathbf{P})$? Since $\texttt{Cube}(\mathbf{P})$ is constructed from $S(\mathbf{P})$ by erasing vertices, $\texttt{Cube}(\mathbf{P})$ may be obtained by splitting $\mathbf{\Sigma}$ as the union of two subsets, $A = \{\mathbf{0}, \mathbf{0}^*, \texttt{fd}, \texttt{fd}^*, \texttt{bk}, \texttt{bk}^*\}$ and $B = \{\mathbf{0}, \mathbf{0}^*, a_1, a_1^*, \ldots, a_L, a_L^*\}$, and executing the following steps:

1. Compute B° and A° where B and A are viewed as poc sets with respect to the ordering inherited from $\mathbf{P}$;
2. Observe that $\mathbf{P}^\circ \subseteq B^\circ \times A^\circ$: any coherent $*$-selection on $\mathbf{\Sigma}$ restricts to a coherent $*$-selection on either of A, B.
3. Obtain $\mathbf{P}^\circ$ by removing the vertices of $B^\circ \times A^\circ$ containing any incoherent pairs $\{p, a\}$ with $p \in B$ and $a \in A$.

From the preceding example we already know that $\texttt{Cube}(B)$ is the L-path, whereas $\texttt{Cube}(A)$ is the complete 2-dimensional cube as all the $*$-selections on A are coherent (no relations between $\texttt{fd}$ and $\texttt{bk}$, as these signals may be set arbitrarily). Therefore, $\texttt{Cube}(\mathbf{P})$ needs to be "excavated" from a $1 \times 1 \times L$ stack of unit cubes. Figure 3(left) shows the result.

Note, however, that a frustrated designer might want to supply the agent with the information $\texttt{fd} < \texttt{bk}^*$ beforehand, since this relation may be regarded less a characteristic of the environment and more as one of the "motor suite" provided to the agent. The corresponding model space immediately simplifies to the one depicted in Figure 3(right), through erasing all the vertices containing the now incoherent pair $\{\texttt{fd}, \texttt{bk}\}$. □

Another illustration of universality is provided by Example 3.18.

3.4. Model Spaces, Topology and Control

Given the preceding results, why even consider the rest of the dual structure (the vertices and edges of $\texttt{Dual}(\mathbf{P})$ forming the 1-skeleton of $\texttt{Cube}(\mathbf{P})$; the higher-dimensional cubical cells of $\texttt{Cube}(\mathbf{P})$)?

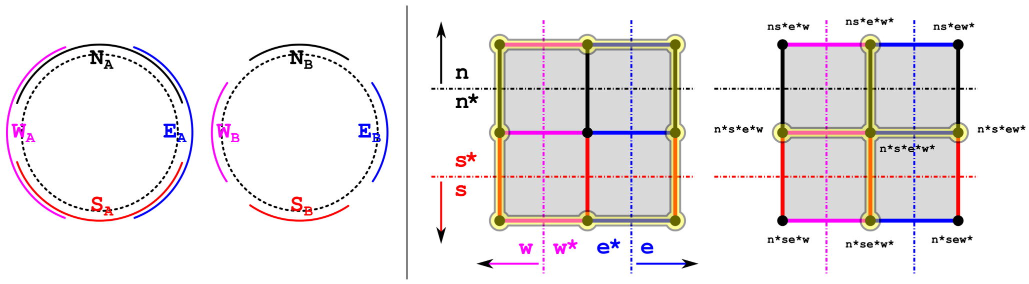

Figure 4: Realizations (left) in $\mathbb{S}^1$ (black, dashed) for the sensors of the two search party members of Example 3.18. The corresponding punctured models are highlighted in yellow as sub-complexes of the common model space (right). Note how the subset of $\mathbb{S}^1$ realizing the vertex $\mathtt{n}^*\mathtt{s}^*\mathtt{e}^*\mathtt{w}^*$ is empty in one case and disconnected in the other.

Definition 3.17. Let $\mathbf{P}$ be a weak poc set structure on the complemented set $(\mathbf{\Sigma}, *)$ with minimum element $\mathbf{0}$ and let $(\mathbf{\Sigma}, *, \mathbf{0}, \rho)$ be a sensorium. The associated *punctured model space*, denoted $\mathtt{Cube}^{\times}(\mathbf{P}, \rho)$, is the sub-complex of $\mathtt{Cube}(\mathbf{P})$ induced by $\mathbf{\Xi}_\rho$, that is: a cube $C \in \mathtt{Cube}(\mathbf{P})$ belongs in $\mathtt{Cube}^{\times}(\mathbf{P}, \rho)$ if and only if all its vertices lie in $\mathbf{\Xi}_\rho$. Faces of $\mathtt{Cube}(\mathbf{P}) \smallsetminus \mathtt{Cube}(\mathbf{P}, \rho)$ will be referred to as the *essential obstacles* in this setting. $\square$

Example 3.18. Consider the poc set $\mathbf{P}$ over $\mathbf{\Sigma} = \{\mathbf{0}, \mathbf{0}^*, \mathtt{n}, \mathtt{n}^*, \mathtt{s}, \mathtt{s}^*, \mathtt{e}, \mathtt{e}^*, \mathtt{w}, \mathtt{w}^*\}$ with the relations $\mathtt{n} < \mathtt{s}^*$ and $\mathtt{e} < \mathtt{w}^*$. This may be thought of as representing the "least common denominator" among, say, members of a search party, discussing the source direction of a radio-locator signal. The state space for their common problem is the unit circle $\mathbf{X} = \mathbb{S}^1$, but their criteria for identifying the four basic directions may differ, for example: suppose members A and B in the search party both have $\rho_\sigma : \mathbf{\Sigma} \to \mathbf{2}^{\mathbf{X}}$, $\sigma \in \{A, B\}$ specified via $\rho_\sigma(\mathtt{n}) = N_\sigma$, $\rho_\sigma(\mathtt{s}) = S_\sigma$ etc., as described in Figure 4 (left). Then both ρ_A and ρ_B are legitimate realizations despite the significant differences between $\mathtt{Cube}^{\times}(\mathbf{P}, \rho_A)$ and $\mathtt{Cube}^{\times}(\mathbf{P}, \rho_B)$, as shown in Figure 4 (right). We see how $\mathtt{Cube}(\mathbf{P})$ provides a model space just large enough to accommodate both 'viewpoints' (universality), while $\mathtt{Cube}^{\times}(\mathbf{P}, \rho_A)$ is a much better model of a circle (the state space $\mathbf{X}$) than $\mathtt{Cube}^{\times}(\mathbf{P}, \rho_B)$. $\square$

Again returning to the notation of Section 3.1 we may apply Theorem 3.1 of [31] to our setting *verbatim* to obtain:

Theorem 3.19. *Let $\mathbf{P}$ be a weak poc set structure on the complemented set $(\mathbf{\Sigma}, *)$ with minimum element $\mathbf{0}$ and let $(\mathbf{\Sigma}, *, \mathbf{0}, \rho)$ be a sensorium. Let $\varnothing \neq Z \subset \mathbf{X}^{N+1}$ be a subspace, and let $\tilde{\rho}_Z : \mathbf{\Sigma} \to \mathbf{2}^Z$ be defined by $\tilde{\rho}_Z(a) = Z \cap \tilde{\rho}(a)$. Finally, for each cube $C \in \mathtt{Cube}^{\times}(\mathbf{P}, \rho)$ let $Z_C = Z \cap (\rho^*)^{-1}(C)$ be the set of points in Z witnessing C.*

Assume now that, for each $C \in \mathtt{Cube}^{\times}(\mathbf{P}, \rho)$, Z_C has a contractible open neighbourhood N_C in Z such that the map from the nerve of the covering $\{Z_C \,|C \in \mathtt{Cube}^{\times}(\mathbf{P}, \rho)\}$ to the nerve of the covering $\{N_C \,|C \in \mathtt{Cube}^{\times}(\mathbf{P}, \rho)\}$ induced by $Z_C \mapsto N_C$ is an isomorphism. Then, if $\tilde{\rho}_Z$ is a realization, $\mathtt{Cube}^{\times}(\mathbf{P}, \rho)$ is homotopy-equivalent to Z. $\square$

Example 3.18 provides a simple but powerful illustration of this theorem: observe how $\mathtt{Cube}^{\times}(\mathbf{P}, \rho_A)$ replicates the homotopy type of the circle, while $\mathtt{Cube}^{\times}(\mathbf{P}, \rho_B)$ fails to do so; at the same time we observe that the set of points witnessing the vertex $\mathtt{n}^*\mathtt{s}^*\mathtt{e}^*\mathtt{w}^*$ has four connected components and thus fails to be contractible[17].

The implications of the above theorem in our discussion are as follows. Since, in general, one cannot expect an agent to be capable of exploring the entirety of $\mathbf{X}^{N+1}$ from a given initial condition, pick Z to be the corresponding reachable set; the agent's actions may be seen as providing an approximation to the connectivity structure in Z. The theorem then states that, *given a sufficiently rich and tame sensorium*, if the agent manages to learn a correct model $\mathbf{P}$ of the implication structure among the sensors then knowledge of the *essential obstacles* allows the recovery of the "topological shape" of Z by computing $\mathtt{Cube}^{\times}(\mathbf{P}, \rho)$. Adding to this the fact (see Theorems 3.28,3.29 below) that $\mathtt{Cube}(\mathbf{P})$ is always contractible, we find that the role of $\mathtt{Cube}(\mathbf{P})$ in the agent's exploration of Z is analogous to that of an occupancy grid in SLAM[18]: a discretized model of the state space of the system, where one of the objectives of the robot is to "black out" the grid points corresponding to obstacles to identify the space in which it can move freely. Obtaining an understanding of the homotopy type of Z is crucial to controlling embodied agents, due to tame attractors (one possible representation of a desired task) inheriting the homotopy type of their basins of attraction.

3.5. Interlude: Geometry and Convexity in the Model Spaces

We will now review the geometry of the dual graphs of weak poc sets. A feature of poc set duals — perhaps *the* feature in our context — is their extremely strong convexity theory. This theory was, historically, shown to accommodate only true poc sets. However, the authors in [31] have pointed out the need for an extended theory encompassing the weaker version of poc sets for the purpose of supporting the learning of poc set representations. There, the observation was made that every *weak* poc set $\mathbf{P}$ has a *canonical quotient map* $\pi : a \mapsto \hat{a}$ onto a *true* poc set $\hat{\mathbf{P}}$ inducing a canonical isomorphism of $\mathtt{Cube}(\hat{\mathbf{P}})$ onto $\mathtt{Cube}(\mathbf{P})$ (see also Appendix 8.1.2). Thus, all the results of "classical" Sageev-Roller duality theory apply equally well to weak poc sets as they do to true poc sets, enabling us to state them in the more general context of weak poc sets.

We briefly recall the graph-theoretic notion of convexity:

Definition 3.20. Let $G = (V, E)$ be a connected simple graph[19] and let $u, v \in V$. The *hop distance* $d_G(u, v)$ is defined to be the minimum length of an edge-path in G joining u with v. The interval $I(u, v)$ is defined to be the set of all vertices $w \in V$ satisfying the equality $d_G(u, v) = d_G(u, w) + d_G(w, v)$. □

Definition 3.21. Let $G = (V, E)$ be a connected simple graph. A set $C \subseteq V$ is said to be *convex*, if $I(u, v) \subseteq C$ holds for all $u, v \in C$. A set $H \subseteq V$ is a *half-space of* G, if both H and $H^c = V \smallsetminus H$ are convex sets in G. Let $\mathcal{H}(G)$ denote the poc set whose elements are half-spaces of G, ordered by inclusion, and with $H^* = H^c$. □

---

[17] Recall that a contractible space is continuously deformable to a point (within itself!), and therefore must be connected.

[18] Simultaneous Localization and Mapping

[19] By a simple graph we mean a graph with no loops and at most one undirected edge joining any pair of vertices.

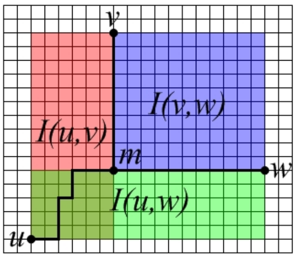

Figure 5: Computing a median in a rectangle G cut out of the integer grid (all vertices of the form $m \times n$, $m, n \in \mathbb{Z}$, with edges joining a vertex $m \times n$ to the vertices $(m \pm 1) \times n$ and $m \times (n \pm 1)$).

We refer the reader to [62], section 4, for the (very elegant and much more general) proofs of the following results:

Lemma 3.22. *Let $G = \texttt{Dual}(\mathbf{P})$ for a finite weak poc set $\mathbf{P}$. Then the metric $\mathbf{\Delta}$ coincides with the hop metric on G.* □

Lemma 3.23. *Let $\mathbf{P}$ be a weak poc set. Then the half-spaces of $\texttt{Dual}(\mathbf{P})$ are precisely the subsets of $\mathbf{P}^\circ$ of the form*[20] $\mathfrak{h}(a) := \{u \in \mathbf{P}^\circ \,|a \in u\}$, $a \in \mathbf{P}$. *In particular, subsets of $\mathbf{P}^\circ$ of the form $\mathfrak{h}(K) := \{u \in \mathbf{P}^\circ \,|K \subseteq u\} = \bigcap_{a\in K} \mathfrak{h}(a)$ are convex in $\texttt{Dual}(\mathbf{P})$.* □

The above are largely due to $\texttt{Dual}(\mathbf{P})$ being a *median graph* [12, 94]:

Definition 3.24. A connected simple graph $G = (V, E)$ is said to be a *median graph*, if the set $I(u, v) \cap I(v, w) \cap I(u, w)$ contains exactly one vertex for each $u, v, w \in V$. This vertex is the *median* of the triple (u, v, w) and denoted by $\mathrm{med}(u, v, w)$ – see Figure 5.□

Median graphs are a special subfamily of *median algebras*, [77, 78, 37, 4]. Some modern generalizations and applications may be found in [11].

A central result in Sageev-Roller duality, specialized here to the finite case, is:

Theorem 3.25. *The dual $G = \texttt{Dual}(P)$ of a finite poc set $\mathbf{P}$ is a finite median graph, with the median calculated according to $\mathrm{med}(u, v, w) = (u\cap v) \cup (u\cap w) \cup (v\cap w)$, for all $u, v, w \in \mathbf{P}^\circ$. Conversely, if G is a finite median graph then G is naturally isomorphic to $\texttt{Dual}(\mathcal{H}(G))$.*

In fact, much more can be said in general:

Theorem 3.26 (Properties of median graphs, [71], section 2)**.** *Let $G = (V, E)$ be a finite median graph. Then:*

1. *Every convex set is an intersection of halfspaces;*
2. *Any family of pairwise-intersecting convex sets has a common vertex (1-dimensional Helly property);*
3. *For any convex subset $K \subset V$, the subgraph of G induced by K is a median graph;*

[20]Note that $\mathfrak{h}(a^*) = \mathbf{P}^\circ \smallsetminus \mathfrak{h}(a)$ for all $a \in \mathbf{P}$.

4. *For any convex $K \subset V$ and any vertex $v \notin K$ there exists a unique vertex $\mathtt{proj}_K v \in K$ at minimum hop distance from v.*
5. *For any convex $K \subset V$, the closest-point projection $\mathtt{proj}_K(\bullet)$ is a median-preserving, distance non-increasing map of G onto the subgraph of G induced by K.* □

To see how all this connects with the higher-dimensional notions of a dual (in particular, with our model spaces), we recall a definition from [72]:

Definition 3.27. A *cubing* is a simply connected, non-positively curved cubical complex.

We point the reader to [10] for a detailed account of non-positively curved metric spaces. For the purpose of this paper it will suffice to quote a corollary of the well-known Cartan-Hadamard theorem ([10], II.4.1):

Theorem 3.28. *Cubings are contractible.*

We owe the following theorem in its full generality (finite and infinite cases) to the collective efforts of Michah Sageev [72], Martin Roller [71] and Victor Chepoi [12].

Theorem 3.29. *The following are equivalent for a finite simple graph G:*

1. *G is the 1-dimensional skeleton of a cubing;*
2. *G is a median graph;*
3. *G is isomorphic to $\mathtt{Dual}(\mathbf{P})$ for some poc set $\mathbf{P}$;*
4. *G is the 1-dimensional skeleton of $\mathtt{Cube}(\mathbf{P})$ for some poc set $\mathbf{P}$.*

Summarizing all the above from the point of view of the internal representation of an UMA agent we obtain the following: *observations $O \subset \mathbf{\Sigma}$ (complete or incomplete) that are coherent with respect to $\mathbf{P}|_t$ stand in one-to-one correspondence with convex subsets of the model space $\mathbf{M}|_t$.* This leads us to a clear cut answer to the question of how the agent's belief regarding the current state should be maintained, to be addressed next.

3.6. Belief Update and Convexity

It is conceivable that an agent's belief state approaching time $t \in \mathbb{T}$ is contradicted by the incoming observation $O|_t$. Methods of the Belief Update and Revision literature have focused on maintaining the consistency of the belief state while obeying "minimal change" constraints, that is: the incoming observation triggers certain transformations in the agent's theory (the collection of formulae kept by the agent as its representation of the "possible worlds" it might occupy) so as to obtain a new theory within which this observation is possible, but differing as little as possible from the preceding one [9, 50].

For any DBA, an obvious choice for representing the perceived current state of the system at time $t \in \mathbb{T}$ is the unprocessed observation $O|_t$. In an UMA agent, this observation triggers an updating cycle of the algorithm in charge of learning the weak poc set presentation (Section 4), which produces $\mathbf{P}|_t$. However, with the agent's internal description of the world given away to $\mathbf{P}|_t$ (through $\mathbf{M}|_t = \mathtt{Cube}(\mathbf{P}|_t)$), the problem may arise of $O|_t$ turning out incoherent. In other words, the current state literally "falls off the map", as $O|_t \notin \mathbf{M}|_t$.

The solution of the problem is to admit a relaxed current state representation, denoted by $S|_t := \mathtt{coh}(O|_t)$, where the operation $\mathtt{coh}(\bullet)$ yields the "best approximation" of $O|_t$ by a convex subset of $\mathbf{M}|_t$, echoing the principle of minimal change as seen through Dalal's way [18] of quantifying the distance between theories:

Proposition 3.30 (Coherent Approximation). *Let $\mathbf{P} = (\mathbf{\Sigma}, *, \min, \leq)$ be a weak poc set. For any $A \subseteq \mathbf{\Sigma}$ define $\mathtt{coh}(A) := A\!\uparrow \smallsetminus A^*\!\downarrow = A\!\uparrow \smallsetminus A\!\uparrow^*$. For any $A \in S(\mathbf{\Sigma})^0$, if $B \in \mathbf{P}^\circ$ minimizes the distance of A to $\mathbf{P}^\circ$, then $B \in \mathfrak{h}(\mathtt{coh}(A))$.*

Proof. See Appendix 9.1. ■

The mapping $A \mapsto \mathtt{coh}(A)$ enjoys additional properties standardly viewed as desirable in the context of belief update:

Proposition 3.31 (Coherent Projection). *Let $\mathbf{P} = (\mathbf{\Sigma}, *, \min, \leq)$ be a weak poc set. Then the following hold for all $A \subseteq \mathbf{\Sigma}$:*

(a) $\mathtt{coh}(A)$ *is coherent and* $\mathtt{coh}(A)\!\uparrow = \mathtt{coh}(A)$*;*

(b) $\mathtt{coh}(\mathtt{coh}(A)) = \mathtt{coh}(A)$*;*

(c) $A \subseteq \mathtt{coh}(A)$ *whenever A is coherent;*

(d) $\mathtt{coh}(A) = A$ *if and only if A is coherent and* $A\!\uparrow = A$.

We will refer to the map $A \mapsto \mathtt{coh}(A)$ defined on $\mathbf{2}^{\mathbf{\Sigma}}$ as the coherent projection *associated with* $\mathbf{P}$. □

Note how (a) and (c) turn $\mathtt{coh}(\bullet)$ into a closure operator with respect to inference (implication). At the same time, (b) and (d) characterize the set of terms that are closed under inference.

Proof. See Appendix 9.2. ■

Thus, an UMA agent naturally resolves possible contradictions at the price of introducing uncertainty/ambiguity into its record of the current state: instead of marking a single vertex of $\mathbf{M}|_t$ as the current state, any vertex of the convex set $\mathfrak{h}(S|_t)$ may turn out to be the correct current state. We find that this is an intriguingly natural way of maintaining an internal model with a built-in degree of resilience to observations that fail to make immediate sense to the agent. We will see in the sequel the learning mechanism of a snapshot could be engineered so that repeated observations of this kind trigger a revision of the model space after which the same unprocessed observation will no longer require relaxation to be coherent.

The complexity of computing the coherent projection (lemma 5.3) and its role in the agent's reasoning processes, its interplay with the convexity theory of the model space $\mathbf{M}|_t$ and its interpretation as the basis for viewing our architecture as a connectionist model (albeit a very limited one) of cognition will all be discussed in section 5.2.2.

4. Snapshot Structures: From Observation Sequences to a Memory Structure

In [31] we have introduced the rather loose notion of a *snapshot*, aiming to outline a class of database structures for dynamically maintaining weak poc-set structures from a sequence of observations. This section provides a rigorous development of this tool.

4.1. Snapshots

Definition 4.1. Denote by $\mathbf{K_\Sigma}$ the undirected graph obtained from the complete graph over the vertex set $\mathbf{\Sigma}$ by removing all edges of the form aa^*, $a \in \mathbf{\Sigma}$. Edges of $\mathbf{K_\Sigma}$ will be referred to as *proper pairs* in $\mathbf{\Sigma}$. We will abuse notation and write $ab \in \mathbf{K_\Sigma}$ for the edge $\{a, b\}$ of $\mathbf{K_\Sigma}$. □

Definition 4.2. A snapshot $\mathbf{S}$ over $\mathbf{\Sigma}$ consists of the following:

(a) Each vertex $a \in \mathbf{\Sigma}$ of $\mathbf{K_\Sigma}$ is assigned a binary state $\#_a\mathbf{S} \in \{0, 1\}$. The set

$$\#\mathbf{S} = \{a \in \mathbf{\Sigma} \,|\, \#_a\mathbf{S} = 1\} \tag{5}$$

is called the *state of the snapshot* $\mathbf{S}$ and is required to be a $*$-selection on $\mathbf{\Sigma}$.

(b) Each edge $ab \in \mathbf{K_\Sigma}$ is assigned a *weight* $w_{ab} = w_{ab}(\mathbf{S}) \in \mathbb{R}_{\geq 0}$.

(c) Each edge $ab \in \mathbf{K_\Sigma}$ is assigned a *learning threshold* $\tau_{ab} = \tau_{ab}(\mathbf{S}) \in \mathbb{R}_{\geq 0}$ satisfying

$$\tau_{ab} = \tau_{a^*b} = \tau_{ab^*} = \tau_{a^*b^*} \leq \tfrac{1}{4} \tag{6}$$

For every $ab \in \mathbf{K_\Sigma}$, the restriction of $\mathbf{S}$ to the subgraph induced by the vertices a, a^*, b and b^* will be denoted by $\mathbf{S}|_{ab}$ and referred to as a *square* in $\mathbf{S}$. □

An UMA agent maintains snapshots $\mathbf{S}|_t$, $t \geq 0$, whose role in the control loop, at time t, is as follows:

Step 1. Apply $O|_t$ to $\mathbf{S}|_{t-1}$ to obtain new values for $w^t_{ab} := w_{ab}(\mathbf{S}|_t)$ and $\tau^t_{ab} := \tau_{ab}(\mathbf{S}|_t)$;

Step 2. From the new weights, compute the weak poc set structure $\mathbf{P}|_t$;

Step 3. Complete the update by computing $\#\mathbf{S}|_t := \mathtt{coh}(O|_t)$ from $\mathbf{P}|_t$;

Step 4. Use $\mathbf{P}|_t$ to reach a decision $A|_t \in S(\mathbf{Act})^0$;

Step 5. Invoke the action $A|_t$ to generate the observation $O|_{t+1}$.

In this work, we will keep the learning thresholds τ^t_{ab} fixed throughout the lifetime of an agent, setting the problem of controlling them aside for future research.

4.2. Weak Poc Set Structures from Snapshots

The original purpose [31] for the edge weights w_{ab} in a snapshot was to quantify the relevance (e.g. frequency) of the event $(a \,\mathtt{and}\, b)$, allowing one to obtain a graphical representation of $\mathbf{P}|_t$ from $\mathbf{K_\Sigma}$, first by *partially* orienting the edges of $\mathbf{K_\Sigma}$ according to the rule of thumb illustrated in Figure 6(a), and then by removing all the unoriented edges. The resulting directed graph will then have the following properties:

Definition 4.3. A *poc graph* $\mathbf{\Gamma}$ over $\mathbf{\Sigma}$ is a directed graph with vertex set $\mathbf{\Sigma}$, and edges satisfying:

- For every proper pair $a, b \in \mathbf{\Sigma}$, there is at most one directed edge ab from a to b, and at most one directed edge ba from b to a.

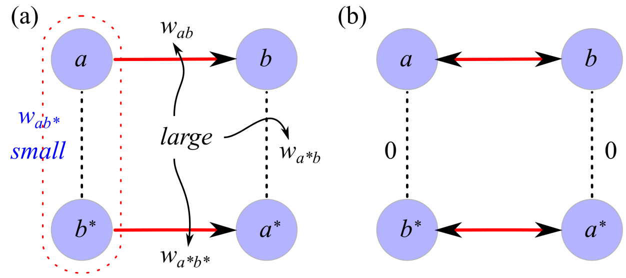

Figure 6: determining edge orientations in a snapshot to determine (a) implication, and (b) equivalence.

- For any edge $ab \in \mathbf{\Gamma}$ one also has $b^*a^* \in \mathbf{\Gamma}$;
- For any edge $ab \in \mathbf{\Gamma}$, the edges a^*b, ba^*, b^*a, ab^* do not lie in $\mathbf{\Gamma}$. □

The properties of $\mathbf{\Gamma}$ in this definition allow precisely for encoding a transitive relation on $\mathbf{\Sigma}$ by setting $a \leq b$ iff $\mathbf{\Gamma}$ contains a directed path from a to b.[21] The property that $a \leq b$ implies $b^* \leq a^*$ immediately follows from the second requirement. Of the axioms of a weak poc set (Definition 3.3) only one remains that is not automatic: the anti-symmetry requirement of a partial ordering.

Lemma 4.4 (derived poc set)**.** *The transitive closure of $\mathbf{\Gamma}$ over $\mathbf{\Sigma}$ is a weak poc set structure on $\mathbf{\Sigma}$ if and only if, as a directed graph, $\mathbf{\Gamma}$ is acyclic (that is, contains no directed cycles).*

Proof. This follows directly the standard fact that the transitive closure of a directed graph is a partial ordering if and only if the graph contains no directed cycles. ■

The implication record constructed from an acyclic poc graph cannot recognize possible equivalences among sensations — only irreversible implications. At the same time it makes perfect sense to interpret a relation of the form $w_{ab^*} = w_{a^*b} = 0$ in a snapshot $\mathbf{S}$ as encoding the logical equivalence $a \Leftrightarrow b$, see figure 6(b). This requires restricting attention to classes of snapshots which encode (in this way) poc graphs enjoying the following property:

Definition 4.5. A poc graph $\mathbf{\Gamma}$ is *weakly acyclic* if every proper pair $a, b \in \mathbf{\Sigma}$ sharing a strong component[22] of $\mathbf{\Gamma}$ also satisfies $ab, ba \in \mathbf{\Gamma}$. □

It is easy to see that contracting all strong components of a weakly acyclic poc graph yields an acyclic poc graph on the appropriate quotient of the sensorium. Appendix 9.4 proves the validity of this extension in the context of our application and discusses its impact on computations and model spaces.

[21] This relation is known [95] as the transitive closure of $\mathbf{\Gamma}$.

[22] Recall [95] that a pair of vertices in a directed graph $\mathbf{\Gamma}$ are said to lie in the same strong component if and only if there is a directed cycle containing them. The strong components of $\mathbf{\Gamma}$ form a partition of the vertex set.

4.3. A Natural Family of Snapshots

We now define the class of snapshots whose properties we intend to leverage for the purpose of learning in UMA agents.

4.3.1. Probabilistic Snapshots — Definition and Motivation

Definition 4.6. We say that a snapshot $\mathbf{S}$ is *probabilistic*, if $\#\mathbf{S}$ is a coherent $*$-selection and the edge weights satisfy the following:

(a) **Consistency constraint.** If $ab, ac \in \mathbf{K}_{\mathbf{\Sigma}}$ then $w_{ab} + w_{ab^*} = w_{ac} + w_{ac^*}$. This allows us to extend the function $w_\bullet$ via $w_{aa} := w_{ab} + w_{ab^*}$, as this quantity is independent of the choice of $b \in \mathbf{\Sigma}$.

(b) **Normalization constraint.** If $ab \in \mathbf{K}_{\mathbf{\Sigma}}$ then $w_{ab} + w_{a^*b} + w_{a^*b^*} + w_{ab^*} = 1$.

(c) **Orientation constraint.** $\omega(ab) + \omega(bc) = \omega(ac)$ for all $a, b, c \in \mathbf{\Sigma}$, where we define $\omega(ab) := w_{a^*b} - w_{ab^*}$.

(d) **Measure constraint.** $\delta(ac) \leq \delta(ab) + \delta(bc)$ for all $a, b, c \in \mathbf{\Sigma}$, where we define $\delta(ab) := w_{a^*b} + w_{ab^*} \geq 0$.

We denote the set of all probabilistic snapshots over $\mathbf{\Sigma}$ by $\mathscr{P}_{\mathbf{\Sigma}}$. □

Remark 4.7. Note the symmetries of $\omega(\bullet)$ and of $\delta(\bullet)$: one has $\omega(ab) = -\omega(ba)$ and $\omega(aa) = 0$, as well as $\delta(ab) = \delta(ba) \in [0, 1]$, $\delta(aa) = 0$ and $\delta(aa^*) = 1$, emerging from the consistency and normalization constraints.

In 3.2 we informally stated the learning goal for our agents to be that of identifying persistent time-independent implications within the sensorium. The formal restatement of this goal is as follows. Assume a probability measure μ is defined on the space of trajectories $\mathbf{X}^\omega := \prod_{t=0}^{\infty} \mathbf{X}$, supported in the set of trajectories achievable by the agent (given the initial conditions), representing a collection of desirable behaviors. For simplicity let us assume that the character of interactions between the agent and the environment does not change with time, implying μ is shift-invariant. Let $\mu_{A,t}$, $A \subseteq \mathbf{\Sigma}$ denote the probability measure on $S(\mathbf{\Sigma})^0$ obtained from the joint distribution of the random variables $f_a(s) := \langle a : \bullet \rangle |_s$, $a \in A$, restricted to the time t. The shift-invariance of μ implies the $\mu_{A,t}$ are independent of t, allowing us to suppress the time index. Thus, interpreting the weights in $\mathbf{S}|_t$ as approximations $w_{ab}^t \approx \Pr(\{a, b\} \subset O|_t)$ is consistent with setting the goal for the agent to learn the pairwise marginals μ_{ab} of the total joint probability $\mu_{\mathbf{\Sigma}}$. It must be noted that the requirements of Definition 4.6 were originally distilled for the sole purpose of characterizing weak acyclicity in derived poc graphs; we are grateful to Dr. Stephen Howard (DSTO, Melbourne, AU) for pointing out this probabilistic interpretation of our approach.

In addition to serving as direct motivation of the requirements (a) and (b) of a probabilistic snapshot, the notion that the weights w_{ab}^t should derive from a common probability function also serves to motivate (c) and (d). Indeed, if w_{ab}^t were to coincide with the probability $\Pr(\{a, b\} \subset O|_t)$ under $\mu_{\mathbf{\Sigma}}$ for all $a, b \in \mathbf{\Sigma}$, then:

$$\begin{aligned} \delta(ab) &= \Pr(E_a + E_b) \\ \omega(ab) &= \Pr(E_a \smallsetminus E_b) - \Pr(E_b \smallsetminus E_a) \end{aligned} \tag{7}$$

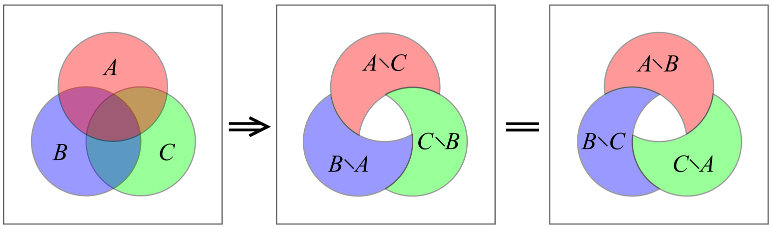

Figure 7: Illustrating the identity $(A \setminus B) \cup (B \setminus C) \cup (C \setminus A) = (B \setminus A) \cup (C \setminus B) \cup (A \setminus C)$ underlying the orientation constraint in Definition 4.6(c), as suggested by Equation (7).

where E_a denotes the event $a \in O|_t$ and the operator $+$ on sets denotes symmetric difference. We conclude that (d) holds by the well-known fact[23] that $d(A, B) := \nu(A + B)$ satisfies the triangle inequality for any measure ν. Finally, the orientation constraint becomes an easy consequence of the elementary set-theoretic identity illustrated in Figure 7, upon substituting $A = E_a$, $B = E_b$ and $C = E_c$.

In view of the above, it is reasonable to employ the *coincidence indicators* along a trajectory φ as building blocks for probabilistic snapshots:

$$c^t_{ab} := \langle a : \varphi \rangle\,|_t \cdot \langle b : \varphi \rangle\,|_t \tag{8}$$

Lemma 4.8. *Any convex combination of coincidence indicators (for varying values of t) satisfies requirements* (a)-(d) *of a probabilistic snapshot.*

Proof. For each fixed t, the indicators $(c^t_{ab})_{a,b \in \mathbf{\Sigma}}$ satisfy the demands (a)-(d), as c^t_{ab} coincides with the probability of the event $E_a \cap E_b$ under the atomic probability measure concentrated at the point $\varphi|_t \in \mathbf{X}$. The affine identities (a)-(d) then carry over to any combination of the form $w_{ab} \equiv \sum_i q_i c^{t_i}_{ab}$ with $\sum_i q_i = 1$ and $q_i \geq 0$. ∎

4.3.2. Weak Acyclicity of Probabilistic Snapshots

A fundamental observation regarding probabilistic snapshots is the following

Proposition 4.9 (Weak Acyclicity Lemma)**.** *Suppose* $\mathbf{S}$ *is a probabilistic snapshot over* $\mathbf{\Sigma}$ *and* $\mathbf{\Gamma}$ *is a poc graph satisfying the requirements:*

1. $\delta(ab) = 0 \Rightarrow ab \in \mathbf{\Gamma}$;
2. $ab \in \mathbf{\Gamma}\,,\ \delta(ab) > 0 \Rightarrow \omega(ab) > 0$.

Then $\mathbf{\Gamma}$ *is weakly acyclic.*

Proof. See appendix 9.3. ∎

This proposition puts the vague notion from figure 6 on how to derive implications and equivalences from a snapshot on a firm footing:

[23]See [20], Section 3.2.

Proposition 4.10. *Suppose* $\mathbf{S}$ *is a probabilistic snapshot. Construct a poc graph* $\mathtt{Dir}(\mathbf{S})$ *by setting* $ab \in \mathtt{Dir}(\mathbf{S})$ *if and only if either* $\delta(ab) = 0$ *or:*

$$w_{ab^*} < \min\{\, \tau_{ab}\,,\ w_{ab}\,,\ w_{a^*b}\,,\ w_{a^*b^*}\,\} \tag{9}$$

Then $\mathtt{Dir}(\mathbf{S})$ *is a weakly acyclic poc graph.*

Proof. The symmetries of $\tau_\bullet$ and $w_\bullet$ immediately imply $ab \in \mathtt{Dir}(\mathbf{S})$ iff $b^*a^* \in \mathtt{Dir}(\mathbf{S})$. The strict inequality in (9) implies the second condition of a poc graph holds as well. To finish the proof we apply the weak acyclicity lemma. ■

Following lemma 4.4 we may now safely define (see also appendix 9.4):

Definition 4.11. Let $\mathbf{S}$ be a probabilistic snapshot. Denote by $\mathtt{Poc}(\mathbf{S})$ the quotient weak poc set structure obtained by first identifying any pair $a, b \in \mathbf{\Sigma}$ having $\delta(ab) = 0$, and then setting $a \leq b$ iff there exists a directed path in $\mathtt{Dir}(\mathbf{S})$ from a to b. □

The fact that $\mathtt{Poc}(\mathbf{S})$ is indeed a weak poc set structure follows from lemma 9.3.

4.4. Examples of Snapshot Structures

4.4.1. Empirical Snapshots and Random Walks

The empirical snapshot structure maintains an empirical approximation of the relative frequencies of co-incidental occurrences of pairs $a, b \in \mathbf{\Sigma}$:

$$w^t_{ab} := \sum_{k=0}^{t} c^k_{ab} = w^{t-1}_{ab} + c^t_{ab}\,, \tag{10}$$

with $\mathbf{S}|_t$ trivial (that is, $\#\mathbf{S}|_t = \varnothing$ and $w^t_\bullet \equiv 0$) for $t < 0$. We refer to the snapshots $\mathbf{S}|_t$ as "empirical snapshots" and to the update rule above as the "empirical update", where c^t_{ab} are the coincidence indicators from (8). DBAs maintaining empirical snapshots are "empirical agents". An immediate corollary of Prop. 4.10 is:

Proposition 4.12 (empirical implies acyclic)**.** *Let* $\mathbf{S}|_t$ *be an empirical snapshot. Then the graph* $\mathbf{\Gamma}|_t = \mathtt{Dir}(\mathbf{S}|_t)$ *defined by setting* $ab \in \mathbf{\Gamma}$ *iff* $\delta(ab) = 0$ *or*

$$w_{ab^*} < \min\{\, t \cdot \tau_{ab}\,,\ w_{ab}\,,\ w_{a^*b}\,,\ w_{a^*b^*}\,\} \tag{11}$$

is a weakly acyclic poc graph, and $\mathtt{Poc}(\mathbf{S}|_t)$ *as defined in Defn. 4.11 is a weak poc set structure on* $\mathbf{\Sigma}$*.*

Proof. Normalizing the weights of $\mathbf{S}|_t$ yields a snapshot whose edge weights $\frac{w_{ab}}{t}$ coincide with the sample mean of the coincidence indicator c_{ab}. By lemma 4.8, such a snapshot is probabilistic. ■

An empirical agent starting out at time $t = 0$ with a trivial snapshot $\mathbf{S}|_0$, has no knowledge of its environment; it may therefore be directed to engage in random exploration, picking one of K decisions $A|_t \in S(\mathbf{Act})^0$ uniformly at random (or using some other weighting reflecting the designer's knowledge of the motor capabilities of the agent such as in Example 3.16) at each time $0 \leq t \in \mathbb{T}$ until actionable information becomes available.

Formally, suppose the pairing of the agent with the environment satisfies the requirement that any of the allowed decisions $\alpha \in S(\mathbf{Act})^0$ induces the structure of a Markov chain on $S(\mathbf{\Sigma})^0$ with transition probabilities

$$p_\alpha(u \to v) := \Pr\left(O|_{t+1} = v \,\middle|\, O|_t = u\,, A|_t = \alpha\right) \tag{12}$$

independent of the time t. Note that $p_\alpha(u \to v) = 0$ whenever $\alpha \not\subseteq v$. Averaging over all decisions α we obtain a Markov chain with transition probabilities

$$p(u \to v) = \frac{1}{K} \sum_\alpha p_\alpha(u \to v)\,, \tag{13}$$

and the problem of guaranteeing "good learning" by the agent becomes that of guaranteeing proper exposure to the environment: by the ergodic theorem for Markov chains [26] we have —

Proposition 4.13. *Suppose* (13) *defines an a-periodic, irreducible, positive-recurrent Markov chain with limiting distribution* π*. Then the empirical snapshot weight* w^t_{ab} *converges to the marginal* π_{ab}*, as defined above in 4.3.1, for all* $a, b \in \mathbf{\Sigma}$*.* □

It follows from the decomposition theorem for Markov chains [26] that the ergodicity assumption on (13) in the above proposition does not impose undue restrictions on our model, as we only expect an agent to learn implications from recurring observations anyway. We also note that the special case when (13) is a (lazy) random walk guarantees an exponential rate of convergence to the limiting distribution (see Theorem 5.1 of [49] and Theorem 9 of [70]).

4.4.2. Discounted Snapshots and Decaying Memories

A notable weakness of empirical agents is their dependence on the *entire* history of the agent's observations. Faulty decisions regarding the ordering in $\mathbf{P}|_t$ require an ever larger volume of evidence to contradict them as time progresses. Instead, we consider:

Definition 4.14. (discounted update) Let $q \in [0, 1]$ and let $\mathbf{S}$ be a probabilistic snapshot over $\mathbf{\Sigma}$. For any complete $*$-selection O on $\mathbf{\Sigma}$ define the *q-discounted update* of $\mathbf{S}$ to be the snapshot $O *_q \mathbf{S}$ with weights determined by

$$w_{ab}(O *_q \mathbf{S}) := q w_{ab}(\mathbf{S}) + (1 - q) \left\langle \mathbb{1}_O : a \right\rangle \cdot \left\langle \mathbb{1}_O : b \right\rangle \tag{14}$$

The state of $O *_q \mathbf{S}$ is set to $\texttt{coh}(O)$, the reduction being computed with respect to the weak poc set structure derived from the new weights. We refer to q as the *decay parameter.* □

A significant advantage of the discounted update is its applicability to arbitrary probabilistic snapshots:

Lemma 4.15. *The q-discounted update of a probabilistic snapshot by a complete $*$-selection is probabilistic.*

Proof. It is clear that the discounted update preserves the property of being probabilistic, as a convex combination of probabilistic snapshots is probabilisitic. ■

We consider the length of time (or the amount of evidence) it takes a discounted snapshot to acquire an implication, compared to the amount of evidence required for giving up an implication already on record, assuming a fixed value of the decay parameter.

It is easy to see that the shortest time Δt required for the acquisition of a relation $a \leq b$ corresponds to a sequence of consecutive observations satisfying $c_{ab^*} = 0$, with

$$\Delta t > \log_q \tau_{ab} \,. \tag{15}$$

Analogously, the shortest period Δt guaranteeing recovery from a false relation $a \leq b$ is realized by a sequence of Δt consecutive observations satisfying $c_{ab^*} = 1$, where, after some manipulation one obtains

$$\Delta t \geq \log_q(1 - \tau_{ab}) - \log_q(1 - w_{ab^*}) \tag{16}$$

Thus, $\log_q(1 - \tau_{ab})$ consecutive synchronous observations of a, b^* will result in recovery no matter how long the agent's record persisted in the error.

Since $\tau_{ab} < \frac{1}{4}$, the time for recovery from a false relation is significantly shorter than the time required for learning it. Pushing the learning threshold below $(1 - q)$ ensures recovery by observing a single counter-example!

As each of the τ_{ab} may be set independently of the others, one could attempt improving the quality/dependability of the model space by altering the flexibility of the learning process in a localized manner[24]. The simulation results in 6.1 emphasize the need for this kind of control, showing that a discounted agent is much more susceptible to changes in geometry and topology/combinatorics of the sensor fields than an empirical one. We conjecture that methods analogous to those of [51] and [14] may apply in this context.

5. Control with Snapshots

This section introduces the basic control function of a snapshot. For reasons of convenience we will assume all sensors are either of order 0 (state sensors) or of order 1 (transition sensors)[25]. Recall that state sensors and transition sensors (our DBA's actions among them, recall Section 2.4) may be viewed as Boolean and situational fluents over the situation space $\mathbf{X}$, which is sufficient for setting up a discussion of actions and competencies according to [52].

At the technical level, this section requires a more thorough understanding of the convexity theory of cubings. While an overview of the relevant classical results was provided in Section 3.5, the new technical results we had to derive in support of our use of snapshots for greedy navigation in cubings are covered in appendix 9.5.

[24] In fact, one could imagine lowering some thresholds so drastically as to preclude learning in the corresponding squares, thus providing means for pre-wiring agents, if necessary.

[25] A reduction to this case is easily achieved by replacing $\mathbf{X}$ with the "phase space" $\tilde{\mathbf{X}} = \mathbf{X}^{N+1}$ where $N = \max_{a \in \mathbf{\Sigma}} n_a$, in a manner analogous to the standard reformulation of a higher-order ODE in one dimension as a first order ODE in multiple dimensions.

5.1. Actions and the Model Spaces

Recall from definition 2.11 and the discussion preceding it that a DBA's decision at any time $t \geq 0$ is a complete $*$-selection $A|_t$ on $\mathbf{Act} = \mathbf{\Sigma}_{act} \cup \mathbf{\Sigma}^*_{act} \cup \{\mathbf{0}, \mathbf{0}^*\}$, satisfying the condition that the actions listed in $A|_t$ have common outcomes in $\mathbf{X}$. It is conceivable, however, that a specific problem setting places restrictions on the set of decisions: a motor cannot apply *both* a negative and a positive torque to its shaft (the torque values must be reconciled prior to feeding input to the motor); a chess player is only allowed to pick *one* move at a time.

The seeming contradiction between our formalism and reality may be resolved in two ways. The first solution is to extend $\mathbf{X}$ to accommodate for "failure states" and endow the DBA with a mechanism to sense failure modes and reason about them. The second is to restrict the DBA to decisions from a prescribed subset of $S(\mathbf{Act})^0$. Although, ideally, the first solution is preferable, we do not yet have a principled way of endowing a DBA with a mechanism for reasoning about failure and we are reluctant to introduce teachers into the discussion at this point. We therefore resort in all examples in this paper to the second solution, where some elements of $S(\mathbf{Act})^0$ may have no outcomes, but the controller is restricted to producing only decisions with outcomes.

5.2. Reactive Planning

5.2.1. Statement of the planning problem

In this section we consider a DBA at time $t > 0$, equipped with a snapshot $\mathbf{S}|_t$ with a derived poc graph $\mathbf{\Gamma}|_t = \mathtt{Dir}(\mathbf{S}|_t)$ and associated weak poc set $\mathbf{P}|_t$. The agent's tasks at hand are:

(T1) Predict the immediate outcome of any available action $A \in \mathtt{Cube}(\mathbf{Act}|_t)$;

(T2) Given a set $T \subset \mathbf{\Sigma}$ of target sensations to be achieved *jointly*, decide on an action $A|_t \in S(\mathbf{Act}|_t)^0$ for the agent to invoke in the next transition.

It is crucial to interpret tasks **(T1-2)** in terms of the model space $\mathbf{M}|_t = \mathtt{Cube}(\mathbf{P}|_t)$: recalling that the sets $\mathfrak{h}(B) := \{V \in \mathbf{P}|_t^\circ \,|B \subseteq V\}$ are *precisely* the convex subsets of the 1-skeleton of $\mathbf{M}|_t$ (Theorem 3.26), we observe that **(T2)** addresses the agent with the problem of reaching $\mathfrak{h}(T)$ from a (possibly unknown) position in the convex set $\mathfrak{h}(S|_t)$.

5.2.2. Signal Propagation over a Snapshot

To use $\mathbf{\Gamma}|_t$ in calculations, we "load" it with information about the current state. Formally:

Definition 5.1. Let $B \subset \mathbf{\Sigma}$. Denote by $[\mathbf{\Gamma}|_t, B]$ the weighted graph obtained from $\mathbf{\Gamma}|_t$ by attaching the Boolean weight $\langle \mathbb{1}_B : v\rangle$ to each vertex $v \in \mathbf{\Sigma}$, and refer to it as $\mathbf{\Gamma}|_t$ *being loaded with* B. □

Definition 5.2. A *propagation algorithm along* $\mathbf{\Gamma}|_t$ is any algorithm which, for any coherent load $B \subset \mathbf{\Sigma}$ and *any* $T \subseteq \mathbf{\Sigma}$ accepts $[\mathbf{\Gamma}|_t, B]$ and T as input and produces as its output the loaded graph $[\mathbf{\Gamma}|_t, R]$ where $a \in R$ if and only if:

1. there is a directed path in $\mathbf{\Gamma}|_t$ from $B \cup T$ to a, or –
2. there is no directed path in $\mathbf{\Gamma}|_t$ from a into T^*.

Algorithm 1 Snapshot updating procedure, given $O|_t$.

```
procedure UPDATE_SNAPSHOT(O|_t)
    Update weights w_•^{t-1} to w_•^t using O|_t
    Compute the derived graph Γ|_t from w_•^t
    S|_t ←PROPAGATE(Γ|_t, ∅, O|_t)
end procedure
```

The set $R \subset \mathbf{\Sigma}$ is said to be the *result of propagating the signal T along* $[\mathbf{\Gamma}|_t, B]$. □

Lemma 5.3 (Implementing the State Update)**.** *For any propagation algorithm, propagating the signal $O|_t$ along $[\mathbf{\Gamma}|_t, \varnothing]$ produces $S|_t = \mathtt{coh}(O|_t)$, see Algorithm 1.* ■

The following result is the key tool for turning a propagation algorithm into a reactive planner:

Lemma 5.4 (Reasoning in Snapshots)**.** *Let $T \subset \mathbf{\Sigma}$ be any set. For any propagation algorithm, propagating the signal T along $[\mathbf{\Gamma}|_t, S|_t]$ produces the projection in $\mathbf{M}|_t$ of the current state $\mathfrak{h}(S|_t)$ to the reduced target $\mathfrak{h}(\mathtt{coh}(T)) \subset \mathbf{M}|_t$.* ■

Both lemmas are corollaries of the geometric interpretation of the planning tasks **(T1-2)** above and of the following new technical result:

Proposition 5.5. *Let $S, T \subset \mathbf{\Sigma}$ and suppose S is coherent in $\mathbf{P}|_t$. Let $L = \mathfrak{h}(S)$ and $K = \mathfrak{h}(\mathtt{coh}(T))$. Then:*

$$\mathtt{proj}_K L = (S\!\uparrow \cup T\!\uparrow) \smallsetminus T\!\uparrow^{*} = (S\!\uparrow \smallsetminus T\!\uparrow^{*}) \cup \mathtt{coh}(T) \tag{17}$$

where $\mathtt{proj}_K(\bullet)$ denotes the closest point projection to K in the model space $\mathbf{M}|_t$ and $\uparrow$ denotes forward closure in $\mathbf{\Gamma}|_t$ — see eqn. (34)*.*

Proof. Corollary 9.17 proves this result for weak poc sets. Proposition 9.6 interprets it in terms of propagation on weakly acyclic poc graphs. ■

In practice, one can implement propagation using a variant of depth-first search (DFS) on $\mathbf{\Gamma}|_t$, while maintaining an expanding record of vertices visited [15] — see Algorithm 2. This algorithm clearly has time complexity that is at most quadratic in the number of sensors, and we conclude:

Corollary 5.6 (Quadratic Snapshot Maintenance)**.** *Both the time and space complexity of updating the snapshot $\mathbf{S}|_{t-1}$ with an observation $O|_t$ to form $\mathbf{S}|_t$ are at most quadratic in $|\mathbf{\Sigma}|$.* ■

Implementation on a truly parallel machine, realizing each vertex of $\mathbf{\Gamma}|_t$ as an actor which responds to propagated signals as they arrive, will bring the complexity of propagation down to sub-linear in $|\mathbf{\Sigma}|$, namely to "big O" of the height of $\mathbf{P}|_t$. The challenge is, of course, implementing in hardware the extreme plasticity of $\mathbf{\Gamma}|_t$, observed as the snapshot structure adjusts itself to the observed reality of the agent.

Algorithm 2 Propagating a signal T over a loaded poc graph $[\mathbf{\Gamma}, B]$ using depth-first search.

```
function PROPAGATE(Γ, B, T)
    visited ← ∅
    U ← CLOSURE(Γ, T)
    return (B ∪ U) ∖ U*
end function
function CLOSURE(Γ, T)                          ▷ Forward closure of T in Γ
    for all a ∈ T do
        EXPLORE(Γ, a)
    end for
    return visited
end function
procedure EXPLORE(Γ, v)                          ▷ Recursive step
    visited ← visited ∪ {v}
    for all w ∈ CHILDREN(Γ, v) ∖ visited do
        EXPLORE(Γ, w)
    end for
end procedure
function CHILDREN(Γ, v)                          ▷ Children of v in Γ
    return {w ∈ Σ | vw ∈ Γ}
end function
```

5.2.3. Evaluating a Decision

Planning of any kind requires an ability to sense the context of an action. We impart this ability to the agent by introducing sensors of the form

$$\langle \alpha \wedge s : \varphi \rangle |_t = \langle \alpha : \varphi \rangle |_t \cdot \#_s \mathbf{S}|_{t-1} \tag{18}$$

where α is an action and $s \in \mathbf{\Sigma}$ is any sensor.

The construction of a judicious process enriching the sensorium with a minimal and effective collection of introspective sensors of this kind is set aside for future research[26]. In this paper we have, instead, committed to a sensorium containing an over-abundance of such sensors, finally clarifying to some degree the distinction we make between the state space $\mathbf{X}$ and the environment $\mathbf{E}$.

- **"Position" Sensors.** We assume $\mathbf{E}$ is given as the union of a collection $\mathscr{U}$ satisfying (1) $U \subset \mathbf{E}$ for all $U \in \mathscr{U}$, and (2) $\mathbf{E} \smallsetminus U \in \mathscr{U}$ for all $U \in \mathscr{U}$, with the agent having a state sensor $\mathtt{loc}[U]$ for each $U \in \mathscr{U}$ defined by $\langle \mathtt{loc}[U] : x \rangle = \langle \mathbb{1}_U : \mathtt{pos}(x) \rangle$.
- **Actions.** A collection of actions (in the form of 1-sensors) is provided.
- **Contextualized actions.** For each $U \in \mathscr{U}$ and $\alpha \in \mathbf{Act}$ the agent is given the sensors $\alpha \wedge \mathtt{loc}[U]$ and $\alpha^* \wedge \mathtt{loc}[U]$.

[26]Though note that a self-enrichment mechanism similar to the one proposed by Drescher [23] may be used in the context of empirical snapshots.

Algorithm 3 Evaluation of an action A by a snapshot-driven DBA.

```
function HALUCINATE(A)
    Signal ← ∅
    for all α ∈ A do
        Signal ← Signal ∪ {α ∧ loc[U] | loc[U] ∈ S|_t }
    end for
    return PROPAGATE(Γ|_t, S|_t, Signal)
end function
```

Algorithm 4 Greedy Reactive Planning (GRP) for a snapshot-driven DBA.

```
function GRP(T)
    Route ←PROPAGATE(Γ|_t, S|_t, T)
    Best ← arg min_{A∈Act|_t} |Route∖HALUCINATE(A)|
    return a random element from Best
end function
```

Under these assumptions, the following result yields a mechanism allowing the agent to 'hallucinate' the broadest consequences of an action for its position in the environment within the context of its current model space $\mathbf{M}|_t$:

Corollary 5.7 (Computing the Consequences of an Action)**.** *For any decision $A|_t \in \mathbf{Act}|_t$, the result of applying $A|_t$ in the transition from time t to time $(t+1)$ is computed by Algorithm 3.* ■

Thus, propagation provides a provably correct and computationally efficient mechanism for predicting the immediate outcomes of an action, provided a sensorium of the above form and a snapshot faithfully recording the nesting relations among the sensors.

5.2.4. Algorithm: Greedy Reactive Planning (GRP)

The ability to compute the immediate consequences of any available action and the convexity theory of $\mathbf{M}|_t$ underlie the greedy algorithm, Algorithm 4, used to decide on an action to be taken for the purpose of achieving a *long-term* goal.

By lemma 5.4, Algorithm 4 is directly analogous to motion planning in the Euclidean plane in the absence of obstacles: the agent selects an action which, to the best of its knowledge, best approximates the greedy path towards the closest point of the indicated target. The next section will consider difficulties arising in the presence of obstacles in the model space. Let us return to our running example one last time:

Example 5.8. To illustrate the above, we continue example 3.16. Recalling $\mathbf{E} = \{0, \ldots, L\}$ we see that the sensors a_k defined in (1) may be rewritten as:

$$a_k = \mathtt{loc}[U_k]\,, \quad U_k = \{i \in \mathbf{E}\,|0 \leq i < k\} \tag{19}$$

Thus, for example, adjoining the two sensors $\mathtt{fd}\wedge a_2^*$ and $\mathtt{bk}\wedge a_4$ to $\mathbf{\Sigma}$ implies the relations

$$\mathtt{fd} \wedge a_2^* < a_3^*\,, \quad \mathtt{bk} \wedge a_4 < a_3 \tag{20}$$

whose effect on $\texttt{Cube}(\mathbf{P})$, once they are learned by the agent, is shown in Figure 8 (left).

Further expanding $\mathbf{\Sigma}$ to include all the sensors

$$\begin{array}{ll} \texttt{fd} \wedge a_k^*, & k = 1, \ldots, L-1 \\ \texttt{bk} \wedge a_k, & k = 2, \ldots, L \end{array} \tag{21}$$

turns $\texttt{Cube}(\mathbf{P})$ into the complex illustrated in figure 8 (right). The order structure on $\mathbf{P}$ encodes both large-scale geometry (the agent may use propagation to conclude "in order to reach $\mathfrak{h}(a_5^*)$, I need to to reach $\mathfrak{h}(a_2^*)$"), and the actions required to negotiate this geometry ("I know that $\texttt{fd} \wedge a_1^*$ implies a_2^*, and I am currently in $\mathfrak{h}(a_1^*)$"). □

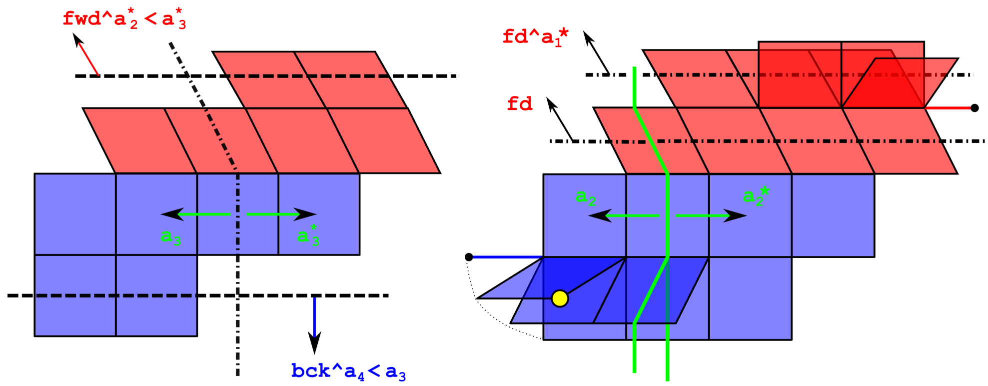

Figure 8: Left: model space for an agent on a discrete path, enriched with two contextualized action sensors of the form (21). Right: the model space arising with almost a full complement of contextualized action sensors (the full complement would be too cluttered if visualized), is now sufficiently rich to illustrate the geometry underlying planning by propagation in Example 5.8. For example, reaching a_5^* from the current state (yellow dot), it is necessary to cross over into $\mathfrak{h}(a_2^*)$; this can be done by deciding on $\{\texttt{fd}, \texttt{bk}^*\}$.

5.3. Some Obstructions to Greedy Reactive Planning.

The constraints of a particular setting prevent a DBA from ever experiencing the vertices of $S(\mathbf{\Sigma})^0$ not corresponding to perceptual classes. Given the internal representation of a DBA with sensorium $(\mathbf{\Sigma}, *, \mathbf{0}, \rho)$ at time t is $\mathbf{P}|_t$, the relevant space to consider is the punctured model space

$$\mathbf{M}^\times|_t := \texttt{Cube}^\times(\mathbf{P}|_t, \rho) \tag{22}$$

(see Definition 3.17). In addition to the risk of false implications in $\mathbf{P}|_t$ influencing the agent's reasoning, it is also possible for $\mathbf{M}|_t$ to contain obstacles to GRP in the form of vertices in $\mathbf{M}|_t \smallsetminus \mathbf{M}^\times|_t \neq \varnothing$. In fact, we recall that the presence of such obstacles is *guaranteed* by Theorem 3.19 — at least when the covering $\mathscr{U}$ of $\mathbf{E}$ by location fields satisfies the richness requirements placed on it by that theorem, and $\mathbf{E}$ fails to have the homotopy type of a point. Let us consider two of examples of this kind.

5.3.1. Example: A Punctured Grid

We compare empirical agents in the square grid $G_N = \{0, \dots, N\} \times \{0, \dots, N\}$, as described in Section 6 setting (c) — also see Figure 9 — with agents living in the *punctured grid* $G_N^\times$, obtained from G_N by removing an interior vertex v_0. An agent in $G_N^\times$ attempting an action which would have resulted in it occupying v_0 had it lived in G_N is assumed to retain its original position. For N sufficiently large, a random-walking empirical agent is then guaranteed to learn the *same* weak poc set structure for *either* environment. This results in the sensory equivalence class of v_0 obstructing GRP in $\mathbf{E} = G_N^\times$ whenever v_0 belongs to a shortest path in G_N joining the current position to the prescribed target.

5.3.2. Example: Agent on a Circular Rail

Consider setting (b) of Sec. 6. We specify a target $T = \{U_p\}$ where $p \in \mathbf{E}$ is sufficiently removed from the current position $q \in \mathbf{E}$ of the agent to accommodate a pair U_i, U_j with the property that $U_i \cup U_j$ separates the set U_p from the set U_q. Both the current state and the target region then satisfy the constraints $\texttt{loc}[U_i]^*$ and $\texttt{loc}[U_j]^*$, which implies that *any* geodesic in the model space joining the current model state with the target set passes through $\mathfrak{h}(\texttt{loc}[U_i]^*, \texttt{loc}[U_j]^*)$, yet it is impossible to guarantee these constraints by *any* of the available actions.

5.4. Closing the Loop with Excitation-Driven Navigation

The examples of section 5.3 demonstrate the necessity of sensory enrichment for overcoming the obstructions to GRP. In particular, these examples seem to favor the introduction of an internal state variable evaluating success (and failure) of invoking a planned action. The need for closed-loop control suggests implementing local control mechanisms based on internally-defined *navigation functions* [69].

In the absence of tools for reactive replanning [67] (our current situation), we have chosen to study a simplified notion of target, allowing us to close the control loop with a motion command generated with the aim to guarantee an immediate decrease in the value of an internal excitation signal.

The simplest instance of such a controller, applied to the navigation setting, seems to be the following. In addition to a sensorium of the form described above in 5.2.3, we endow the DBA with a pair of sensors `better` and `worse`, responding to the decrease and increase, respectively, in a fixed measure of distance to a target point in the environment $\mathbf{E}$, over a single transition (think of this as a radically simplified sense of smell). This measure plays the role of a navigation function.

Starting out as a 'lazy' random-walking agent (the agent may choose not to act at all), the agent applies Algorithm 3 at each step to obtain an action resulting with `better` as its first priority. In the case of failure to produce such an action, the agent attempts to guarantee $\texttt{worse}^*$, periodically invoking a random action so as not to get stuck in place (upon having figured out that $\texttt{worse}^*$ may be brought about by not moving). Section 6.2 presents simulation results for agents of this form.

6. Simulation Results

Proposition 4.13 provides strong performance guarantees for learning done by empirical agents. In this section we examine, through numerical simulation, the effect of the

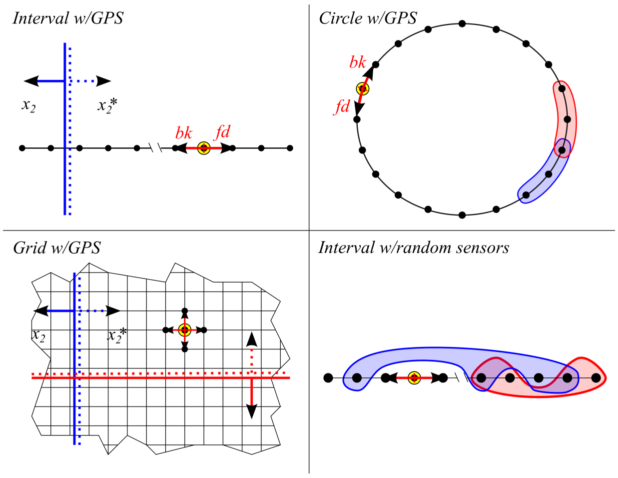

Figure 9: DBAs and environments considered in our simulations (a-d) in Section 6. Agents are colored yellow, with the available actions indicated by red arrows. Sensor fields are marked blue and red.

geometry and topology of the environment on (1) the performance of snapshot learning algorithms (empirical and discounted) applied to random walking DBAs, and (2) the performance of the simple excitation-driven agents from section 5.4.

Our simulated agents are equipped with a sensorium of the form described in 5.2.3, sometimes with additional sensors. We conduct comparisons between four settings with an equal number ($4N$) of location sensors:

(a) **Discrete Path.** Here $\mathbf{E} = \{0, \ldots, 2N\}$ and $\mathscr{U}$ is the collection of sub-intervals of the form $U_i = \{p \in \mathbf{E} \,|p < i\}$, $i = 1, \ldots, 2N$, and their complements.

(b) **Discretized Circle.** $\mathbf{E} = \{0, \ldots, 2N-1\}$, with an array of $4N$ location sensors with activation fields $U_i = \{i-1, i, i+1\}$ (operations modulo $2N$).

(c) **Square Grid.** $\mathbf{E} = \{0, \ldots, N\} \times \{0, \ldots, N\}$, with $\mathscr{U}$ containing all sets of the form $V_i = \{p \times q \,|p < i\}$, $H_j = \{p \times q \,|q < j\}$, $1 \leq i, j \leq N$, and their complements.

(d) **Discrete Path with Random Sensors.** $\mathbf{E} = \{0, \ldots, 2N\}$, with $2N$ randomly selected location sensors (and their $2N$ complements).

The set of location sensors in $\mathbf{\Sigma}$ will be denoted Λ. The available elementary actions in (a) and (d) are those of advancing (`fd`) or retreating (`bk`) a single step along the path, when possible (example 2.13). Analogously in (b), but with a wrap-around modulo $2N$, and in (c) where we provide the agent with the elementary actions `up`, `dn`, `lt` and `rt` as in section 5.3.1.

All the plots in this section are generated for environments with $N = 10$ (that is, 40 location sensors each), and depict averages over 50 distinct runs for each choice of

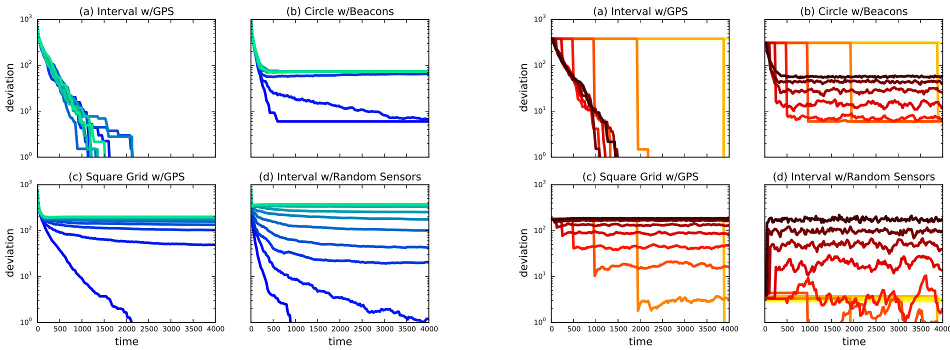

Figure 10: Logarithmic plots of the mean number of incorrect edges in the derived poc graph of a random-walking UMA agent in the settings of Section 6 (a-d) with $N = 20$ (40 sensors each), see Figure 9, averaged over 50 runs of random walks each. Left: empirical agent with learning thresholds varying linearly between $\frac{1}{4}$ (cyan/light) and $\frac{1}{20^3}$ (blue/dark). Right: discounted agent for varying values of the decay parameter, $q = 1 - \frac{1}{2^{k+2}}$, k from 0 (red/dark) to 9 (yellow/light).

parameters (learning thresholds, decay coefficients, etc.). The agent is provided with an "empty" snapshot[27] and occupies a random position in $\mathbf{E}$ at the start of each run.

6.1. Learning Implications from a Random Walk

6.1.1. Learning in Empirical Agents

Figure 10 (left) plots the number of incorrect recorded implications among the location sensors for a random-walking empirical agent as a function of time. More formally, we plot the mean, taken over a number of runs, of the function $Err(t)$ defined as follows:

$$Err(t) := \left\| \mathtt{Dir}^{\infty} - \mathtt{Dir}^{t} \right\|_1 \tag{23}$$

where $\mathtt{Dir}^t_{ab} \in \{0,1\}$ for $t \in \mathbb{T} \cup \{\infty\}$ and $a, b \in \Lambda$ are defined as[28]:

$$\begin{array}{lcl} \mathtt{Dir}^t_{ab} = 1 & \Leftrightarrow & ab \in \mathbf{\Gamma}|_t \\ \mathtt{Dir}^{\infty} = 1 & \Leftrightarrow & \rho(a) \subseteq \rho(b) \end{array} \tag{24}$$

We use a logarithmic plot due to the expected exponential convergence of the snapshot weights to the marginals of the limiting distribution — see remarks following Prop. 4.13.

The figures seem to suggest a dependency of the upper bound on "effective" learning thresholds on the geometry/topology of the environment[29].

---

[27] Assuming $w^t_{ab} \equiv 0$ for an empirical agent and $w^t_{ab} \equiv \frac{1}{4}$ for a discounted agent, for all $t < 0$.

[28] Recall that $\mathtt{Dir}(\mathbf{S})$ introduced in Prop. 4.10 is a directed graph. This new notation is intended to connote a matrix representation of such a graph.

[29] We refer the reader to the technical report [32] for a more developed discussion of these results.

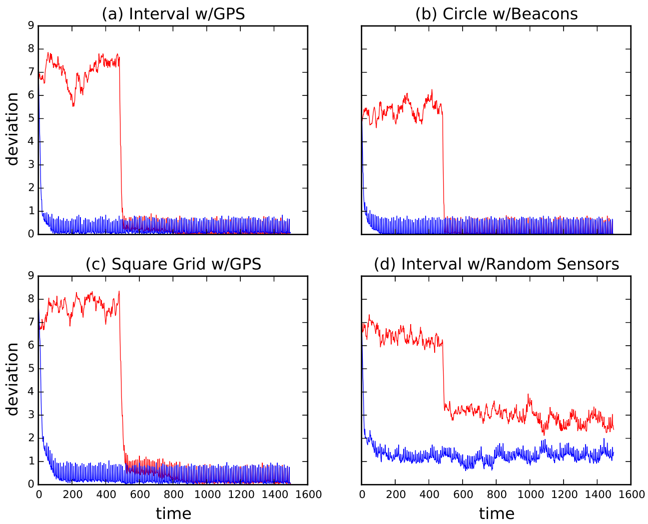

Figure 11: Mean deviation from target for empirical (blue) and a discounted (red) agents (40 sensors each), as a function of time in four different settings, averaging over 50 runs.

6.1.2. Learning in Discounted Agents

Figure 10 (right) compares the mean error, see(24), for a discounted snapshot learning from a random walk, for a learning threshold of $\tau = \frac{1}{20^3}$ and decay parameter q given by $q = 1 - \frac{1}{2^{k+2}}$, $0 \leq k \leq 9$. Note the dependence of the learning process on q is not monotone: $k = 5$ seems to work best in terms of minimizing the eventual error; a choice of $k = 4$ is more reasonable given the observed waiting time until meaningful learning occurs in the structured environments (a)-(c).

6.2. Excitation-Driven Agents

Figure 11 shows the average distance of an excitation-driven agent (section 5.4) to a randomly chosen target as a function of time. It is important to stress that, by the results of section 5, the guarantee of the agents in figure 11(a)-(c) finding their targets *provided sufficient exposure* is absolute. To see this, it suffices to verify for the *true* poc set structure on $\mathbf{\Sigma}$ that any position other than the target has associated with it a location sensor $a = \mathtt{loc}[U]$ and an action α such that every state x with $\langle a : x \rangle = 1$ has $\alpha(x)$ closer to the target than x is.

7. Conclusion

In this paper we introduce a new efficient architecture intended to endow a generic discrete binary agent with the capacity to build over time an actionable model, $\mathbf{M}|_t$, of its operations within a completely unknown and possibly dynamic environment, $\mathbf{E}$. The proposed architecture has a dual nature. On one hand, the agent maintains an evolving data structure, — the snapshot $\mathbf{S}|_t$ — of size quadratic in the number of sensors, encoding a planning mechanism based on propagation of excitation and inhibition signals through the highly plastic directed network $\mathtt{Dir}(\mathbf{S}|_t)$, and is, thus, in a very crude sense, a connectionist learning and control architecture. On the other hand, the rather specific ordering properties of networks arising in this way (the weak poc set structure $\mathbf{P}|_t$ derived from $\mathbf{S}|_t$) also characterize any such network as encoding a system of "half-spaces" in a geometric internal model $\mathbf{M}|_t$ that is *just rich enough* to account for all perceptual classes derivable from the agent's sensorium $\mathbf{\Sigma}$.

Recall that the entropy $H(\mathscr{P})$ of a partition $\mathscr{P}$ of a probability space equals the greatest lower bound on the expected number of *arbitrary* binary queries required for determining which block of $\mathscr{P}$ contains a random point of the space [76]. Historically, this gave rise to the paradigm of efficient coding: *given* a partition of a probability space, one should attempt to characterize it by a collection of binary queries yielding performance near the entropy bound. A general agent — a DBA in particular — may be thought of as being faced, among other tasks, with the inverse problem: given a fixed collection of repeatable binary queries (which may or may not include means for active exploration), produce a decent approximation of the true probability distribution over the partition of the observed space into perceptual classes. If the set of available queries — the agent's sensorium $\mathbf{\Sigma}$ — forms an efficient coding of this partition, then the agent cannot avoid maintaining a database of exponential size in $|\mathbf{\Sigma}|$, incurring super-exponential computational costs in belief update, reasoning and planning. On the other hand, if the agent's queries happen to provide a highly redundant coding of the set of perceptual classes, the agent might be able to leverage the redundancies to obtain savings in representational and computational costs.

UMAs are nothing but a formalization of this principle, where the meaning of the word 'reasoning' was limited *by design* to only the application of known implications and the negation operator. We find it surprising that despite these severe restrictions, snapshots are capable of encoding a high-level representation of the problem space.

Our simulation studies suggest that an UMA agent with sufficient sensing and actuation is capable of learning a useful approximation of the gradient field of a navigation function [69] despite the lack of prior semantic information. A sensorium reflective of the topology of the environment (in the sense of theorem 3.19) is beneficial for learning such fields. At the same time, it appears that — see Fig. 11d — a random sensorium may be almost just as useful. Granted a principled mechanism for self-enrichment (see below), this motivates asking whether an initial "well-behaved" sensorium is at all necessary for the eventual proper functioning of an UMA agent.

UMAs allow easy integration of motivational systems (such as, for example [16]) through introspective sensing of motivational signals. We have only considered very simple excitation mechanisms causing the agent to choose actions maximizing immediate excitation gain (to the extent measurable by the sensorium) but these mechanisms can be readily extended to a suite of sensors encoding tasks ranging from (a) maintaining internally available resources (e.g. battery charge); through (b) attraction/repulsion (either in the sense of navigation functions [69] or in the broader sense of RL [6]); and all the way to (d) dynamic replanning (frustration [67]) and curiousity-driven exploration [5, 74].

We expect such complex motivational mechanisms — especially ones including curiosity and frustration — to facilitate the control of structural parameters of the agent's snapshot architecture. A 'frustration' signal could be used to control learning thresholds and to facilitate chunking by driving the creation of new introspective sensors detecting essential obstacles in the model space, while curiosity could drive the learning of useful complex actions (as has already been proposed for many other architectures [73, 57, 14, 51]), improving the connectivity of the punctured model space.

In contrast to some AGI architectures such as Drescher's "Schema Mechanism" (SM), The current snapshot architectures (Section 4) still lack a mechanism for enriching the set of available queries with, for example, general Boolean predicates (or, even better, some limited LTL predicates) composed of the original atomic sensations, including actions. Such "compound" sensors are required for facilitating chunking and the learning of useful motor primitives. In fact, the task of characterizing the essential obstacles in $\mathbf{M}$ may be seen as an application of a chunking mechanism; finding a snapshot-based mechanism facilitating this function of the memory architecture is therefore a high priority for further research on UMAs.

Another required feature is a capacity for symbolic abstraction, that is: relating problem spaces via symbolic substitution. While the duality theory of weak poc sets and their model spaces (appendix 8.2) enables a rigorous discussion of symbolic abstraction, it is not yet clear how to engineer an enlarged snapshot-like architecture realizing such meta-extensions.

Of course, the problem lies not in proposing intuitively attractive approaches (there are many) but rather doing so in a principled, economical way that maintains the present combination of analytical and computational tractability. For example, the closely related SM architecture of Drescher [23] uses an empirical estimate of the dependability of schema outcomes to determine the need for enriching the system with more specialized/detailed schemata; however Drescher readily admits that the approach is lacking in rigor, and concedes that garbage collection is one of the major challenges for his architecture. A similar problem occurs with the more recent QLAP architecture by Mugan and Kuipers [57], also based on a mechanism for the distillation of schema-like entities, where arbitrary choices have to be made to prune an otherwise unmanageable population of computational units.

In contrast, the added power of understanding the relationship between the geometry of the model spaces and snapshot plasticity in UMAs provide a novel direction of inquiry into the problem of judicious self-enrichment by introspective queries. For example, enriching an agent with sensors characterizing newly discovered failure modes of the navigation algorithm (GRP, Section 5.2.4) should be possible; this will require the introduction of intrinsic motivation mechanisms as discussed above, to steer the agent away from obstacle states in $\mathbf{M}$ and towards desirable behaviors (that is, not necessarily states of $\mathbf{X}$, but reference dynamical systems over subsets of $\mathbf{X}$). It seems plausible that a compromise can be reached between the simplicity of representation and learning in UMAs and the versatility of state-of-the-art knowledge representations (e.g. [50, 28, 85, 89]) — especially those using prime forms, — which would allow for navigation and problem solving in the presence of broad classes of essential obstacles.

Acknowledgements

This work was funded in part by the Air Force Office of Science Research under the MURI FA9550-10-1-0567 and in part by the National Science Foundation under CDI-II-1028237.

References

[1] Anderson, J.R., Lebiere, C., 2003. The newell test for a theory of cognition. Behavioral and Brain Sciences 26, 587–601.

[2] Anderson, J.R., Matessa, M., Lebiere, C., 1997. Act-r: A theory of higher level cognition and its relation to visual attention. Human-Computer Interaction 12, 439–462.

[3] Baars, B.J., 1993. A cognitive theory of consciousness. Cambridge University Press.

[4] Bandelt, H.J., Hedlíková, J., 1983. Median algebras. Discrete Math. 45, 1–30. URL: `http://dx.doi.org/10.1016/0012-365X(83)90173-5`, doi:`10.1016/0012-365X(83)90173-5`.

[5] Barto, A., Mirolli, M., Baldassarre, G., 2013. Novelty or surprise? Frontiers in psychology 4.

[6] Barto, A.G., 1998. Reinforcement learning: An introduction. MIT press.

[7] Barto, A.G., Mahadevan, S., 2003. Recent advances in hierarchical reinforcement learning. Discrete Event Dynamic Systems 13, 341–379.

[8] Boots, B., Siddiqi, S.M., Gordon, G.J., 2011. Closing the learning-planning loop with predictive state representations. The International Journal of Robotics Research 30, 954–966.

[9] Boutilier, C., 1998. A unified model of qualitative belief change: A dynamical systems perspective. Artificial Intelligence 98, 281–316.

[10] Bridson, M.R., Haefliger, A., 1999. Metric spaces of non-positive curvature. volume 319 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin.

[11] Chatterji, I., Druţu, C., Haglund, F., 2010. Kazhdan and Haagerup properties from the median viewpoint. Adv. Math. 225, 882–921. URL: `http://dx.doi.org/10.1016/j.aim.2010.03.012`, doi:`10.1016/j.aim.2010.03.012`.

[12] Chepoi, V., 2000. Graphs of some CAT(0) complexes. Adv. in Appl. Math. 24, 125–179. URL: `http://dx.doi.org/10.1006/aama.1999.0677`, doi:`10.1006/aama.1999.0677`.

[13] Chernova, S., Thomaz, A.L., 2014. Robot learning from human teachers. Synthesis Lectures on Artificial Intelligence and Machine Learning 8, 1–121.

[14] Cobo, L.C., Isbell, C.L., Thomaz, A.L., 2013. Object focused q-learning for autonomous agents, in: Proceedings of the 2013 international conference on Autonomous agents and multi-agent systems, International Foundation for Autonomous Agents and Multiagent Systems. pp. 1061–1068.

[15] Cormen, T., Leiserson, C., Rivest, R., Stein, C., 2003. Introduction to algorithms mit press. Cambridge, MA, .

[16] Cox, B., Krichmar, J.L., 2009. Neuromodulation as a robot controller. Robotics & Automation Magazine, IEEE 16, 72–80.

[17] Curto, C., Itskov, V., 2008. Cell groups reveal structure of stimulus space. PLoS Computational Biology 4. doi:`10.1371/journal.pcbi.1000205`.

[18] Dalal, M., 1988. Investigations into a theory of knowledge base revision: preliminary report, in: Proceedings of the Seventh National Conference on Artificial Intelligence, Citeseer. pp. 475–479.

[19] Derenick, J., Speranzon, A., Ghrist, R., 2013. Homological sensing for mobile robot localization, in: Robotics and Automation (ICRA), 2013 IEEE International Conference on, IEEE. pp. 572–579.

[20] Deza, M.M., Laurent, M., 1997. Geometry of cuts and metrics. volume 15 of *Algorithms and Combinatorics*. Springer-Verlag, Berlin. URL: `http://dx.doi.org/10.1007/978-3-642-04295-9`, doi:`10.1007/978-3-642-04295-9`.

[21] Dicks, W., Dunwoody, M.J., 1989. Groups acting on graphs. volume 17. Cambridge University Press.

[22] Donald, B., Jennings, J., 1991. Sensor interpretation and task-directed planning using perceptual equivalence classes, in: Robotics and Automation, 1991. Proceedings., 1991 IEEE International Conference on, IEEE. pp. 190–197.

[23] Drescher, G.L., 1991. Made-up minds: a constructivist approach to artificial intelligence. MIT press.

[24] Erdmann, M., 2009. On the topology of discrete strategies. The International Journal of Robotics Research .

[25] Farber, M., 2003. Topological complexity of motion planning. Discrete Comput. Geom. 29, 211–221. URL: `http://dx.doi.org/10.1007/s00454-002-0760-9`, doi:`10.1007/s00454-002-0760-9`.

[26] Feller, W., 1967. An introduction to the theory of probability and its applications.

[27] Franklin, S., Madl, T., D'Mello, S., Snaider, J., 2014. Lida: A systems-level architecture for cognition, emotion, and learning. Autonomous Mental Development, IEEE Transactions on 6.

[28] Gabaldon, A., 2011. Non-markovian control in the situation calculus. Artificial Intelligence 175, 25–48.

[29] Ghrist, R., Peterson, V., 2007. The geometry and topology of reconfiguration. Advances in applied mathematics 38, 302–323.

[30] Goerick, C., 2011. Towards an understanding of hierarchical architectures. Autonomous Mental Development, IEEE Transactions on 3, 54–63.

[31] Guralnik, D., Koditschek, D., 2012. Toward a memory model for autonomous topological mapping and navigation: The case of binary sensors and discrete actions, in: Communication, Control, and Computing (Allerton), 2012 50th Annual Allerton Conference on, pp. 936–945. doi:`10.1109/Allerton.2012.6483319`.

[32] Guralnik, D.P., Koditschek, D.E., 2015. Universal memory architectures for autonomous agents (technical report). preprint URL: `http://arxiv.org/abs/1502.06132`.

[33] Hatcher, A., 2002. Algebraic topology. Cambridge University Press, Cambridge.

[34] Helgason, H.P., 2013. General Attention Mechanism for Artificial Intelligence Systems. Ph.D. thesis. Reykjavik University. URL: `http://skemman.is/en/item/view/1946/16163`.

[35] Herzig, A., Varzinczak, I., 2007. Metatheory of actions: beyond consistency. Artificial Intelligence 171, 951–984.

[36] Isbell, J.R., 1980a. Median algebra. Trans. Amer. Math. Soc. 260, 319–362. URL: `http://dx.doi.org/10.2307/1998007`, doi:`10.2307/1998007`.

[37] Isbell, J.R., 1980b. Median algebra. Trans. Amer. Math. Soc. 260, 319–362. URL: `http://dx.doi.org/10.2307/1998007`, doi:`10.2307/1998007`.

[38] Ivaldi, S., Nguyen, S., Lyubova, N., Droniou, A., Padois, V., Filliat, D., Oudeyer, P.Y., Sigaud, O., 2014. Object learning through active exploration. Autonomous Mental Development, IEEE Transactions on 6.

[39] Katsuno, H., Mendelzon, A.O., 1991. Propositional knowledge base revision and minimal change. Artificial Intelligence 52, 263–294.

[40] Klavins, E., Ghrist, R., Lipsky, D., 2006. A grammatical approach to self-organizing robotic systems. IEEE Trans. Automat. Control 51, 949–962. URL: `http://dx.doi.org/10.1109/TAC.2006.876950`, doi:`10.1109/TAC.2006.876950`.

[41] Kozlov, D., 2008. Combinatorial algebraic topology. volume 21 of *Algorithms and Computation in Mathematics*. Springer, Berlin. URL: `http://dx.doi.org/10.1007/978-3-540-71962-5`, doi:`10.1007/978-3-540-71962-5`.

[42] Kraft, D., Detry, R., Pugeault, N., Baseski, E., Guerin, F., Piater, J.H., Kruger, N., 2010. Development of object and grasping knowledge by robot exploration. Autonomous Mental Development, IEEE Transactions on 2, 368–383.

[43] Kuipers, B., 2008. An intellectual history of the spatial semantic hierarchy. Robotics and cognitive approaches to spatial mapping 38, 243264.

[44] Kuipers, B., Modayil, J., Beeson, P., MacMahon, M., Savelli, F., 2004. Local metrical and global topological maps in the hybrid spatial semantic hierarchy, in: Robotics and Automation, 2004. Proceedings. ICRA04. 2004 IEEE International Conference on, IEEE. pp. 4845–4851.

[45] Laird, J., 2012. The Soar cognitive architecture. MIT Press.

[46] Langley, P., Laird, J.E., Rogers, S., 2009. Cognitive architectures: Research issues and challenges. Cognitive Systems Research 10, 141–160.

[47] Leonard, J., Durrant-Whyte, H., 1991. Mobile robot localization by tracking geometric beacons. IEEE Transactions on Robotics and Automation 7, 376382. doi:`10.1109/70.88147`.

[48] Littman, M.L., Sutton, R.S., Singh, S.P., 2001. Predictive representations of state, in: NIPS, pp. 1555–1561.

[49] Lovász, L., 1993. Random walks on graphs: A survey. Combinatorics, Paul erdos is eighty 2, 1–46.

[50] Marchi, J., Bittencourt, G., Perrussel, L., 2010. Prime forms and minimal change in propositional belief bases. Annals of Mathematics and Artificial Intelligence 59, 1–45.

[51] Martius, G., Der, R., Ay, N., 2013. Information driven self-organization of complex robotic behaviors. PloS one 8, e63400.

[52] McCarthy, J., Hayes, P., 1968. Some philosophical problems from the standpoint of artificial intelligence. Stanford University USA.

[53] Milford, M., Wyeth, G., 2010. Persistent navigation and mapping using a biologically inspired slam system. The International Journal of Robotics Research 29, 1131–1153.

[54] Milford, M.J., Jacobson, A., 2013. Brain-inspired sensor fusion for navigating robots, in: Robotics and Automation (ICRA), 2013 IEEE International Conference on, IEEE. pp. 2906–2913.

[55] Milford, M.J., Wyeth, G.F., Prasser, D., 2004. Ratslam: a hippocampal model for simultaneous localization and mapping, in: Robotics and Automation, 2004. Proceedings. ICRA'04. 2004 IEEE International Conference on, IEEE. pp. 403–408.

[56] Modayil, J., White, A., Sutton, R.S., 2014. Multi-timescale nexting in a reinforcement learning robot. Adaptive Behavior 22, 146–160.

[57] Mugan, J., Kuipers, B., 2012. Autonomous learning of high-level states and actions in continuous environments. Autonomous Mental Development, IEEE Transactions on 4, 70–86.

[58] Newell, A., 1973. You can't play 20 questions with nature and win: Projective comments on the papers of this symposium. Carnegie Mellon University Research Showcase CMU URL: `http://repository.cmu.edu/cgi/viewcontent.cgi?article=3032&context=compsci`.

[59] Newell, A., 1979. Reasoning, problem solving and decision processes : the problem space as a fundamental category. Carnegie Mellon University Research Showcase CMU URL: `http://repository.cmu.edu/cgi/viewcontent.cgi?article=2574&context=compsci`.

[60] Newell, A., Simon, H.A., 1961. GPS, a program that simulates human thought. Defense Technical Information Center.

[61] Newell, A., Simon, H.A., 1972. Human Problem Solving. Prentice-Hall, Inc., New Jersey.

[62] Nica, B., 2004. Cubulating spaces with walls. Algebr. Geom. Topol. 4, 297–309 (electronic). URL: `http://dx.doi.org/10.2140/agt.2004.4.297`, doi:`10.2140/agt.2004.4.297`.

[63] Nivel, E., Thórisson, K.R., 2013. Towards a programming paradigm for control systems with high levels of existential autonomy, in: Artificial General Intelligence. Springer, pp. 78–87.

[64] Pratt, V., 1991. Modeling concurrency with geometry, in: Proceedings of the 18th ACM SIGPLAN-SIGACT symposium on Principles of programming languages, ACM. pp. 311–322.

[65] Ranganathan, A., Dellaert, F., 2011. Online probabilistic topological mapping. The International Journal of Robotics Research , 0278364910393287.

[66] Remolina, E., Kuipers, B., 2004. Towards a general theory of topological maps. Artificial Intelligence 152, 47–104.

[67] Revzen, S., Ilhan, B., Koditschek, D., 2012. Dynamical trajectory replanning for uncertain environments, in: Decision and Control (CDC), 2012 IEEE 51st Annual Conference on, pp. 3476–3483. doi:`10.1109/CDC.2012.6425897`.

[68] Rieger, C., 1976. An organization of knowledge for problem solving and language comprehension. Artificial Intelligence 7, 89–127.

[69] Rimon, E., Koditschek, D.E., 1992. Exact robot navigation using artificial potential functions. Robotics and Automation, IEEE Transactions on 8, 501–518.

[70] Rivest, R.L., Schapire, R.E., 1994. Diversity-based inference of finite automata. Journal of the ACM (JACM) 41, 555–589.

[71] Roller, M., 1998. Poc sets, median algebras and group actions. University of Southampton, Faculty of Math. stud., preprint series .

[72] Sageev, M., 1995. Ends of groups pairs and non-positively curved cube complexes. Proc. London Math. Soc. 3, 586–617.

[73] Schmidhuber, J., 2009. Ultimate cognition à la gödel. Cognitive Computation 1, 177–193.

[74] Schmidhuber, J., 2010. Formal theory of creativity, fun, and intrinsic motivation (1990–2010). Autonomous Mental Development, IEEE Transactions on 2, 230–247.

[75] Schwartz, J.T., Sharir, M., 1983. On the piano movers problem: Ii. general techniques for computing topological properties of real algebraic manifolds. Advances in applied Mathematics 4, 298351.

[76] Shannon, C., 1948. A mathematical theory of communication. The Bell System Technical Journal 27, 379–423, 623–656.

[77] Sholander, M., 1954a. Medians and betweenness. Proc. Amer. Math. Soc. 5, 801–807.

[78] Sholander, M., 1954b. Medians, lattices, and trees. Proc. Amer. Math. Soc. 5, 808–812.

[79] Singh, S., James, M.R., Rudary, M.R., 2004. Predictive state representations: A new theory for modeling dynamical systems, in: Proceedings of the 20th conference on Uncertainty in artificial intelligence, AUAI Press. pp. 512–519.

[80] Squire, L.R., 2004. Memory systems of the brain: a brief history and current perspective. Neurobiology of learning and memory 82, 171–177.

[81] Stachowicz, D., Kruijff, G., 2012. Episodic-like memory for cognitive robots. Autonomous Mental Development, IEEE Transactions on 4, 1–16.
[82] Steunebrink, B.R., Koutník, J., Thórisson, K.R., Nivel, E., Schmidhuber, J., 2013. Resource-bounded machines are motivated to be effective, efficient, and curious, in: Artificial General Intelligence. Springer, pp. 119–129.
[83] Sutton, R.S., Precup, D., Singh, S., 1999. Between mdps and semi-mdps: A framework for temporal abstraction in reinforcement learning. Artificial intelligence 112, 181–211.
[84] Tellex, S., Thaker, P., Joseph, J., Roy, N., 2014. Learning perceptually grounded word meanings from unaligned parallel data. Machine Learning 94, 151–167.
[85] Thielscher, M., 2011. A unifying action calculus. Artificial Intelligence 175, 120–141.
[86] Thomas, P.S., Barto, A.G., 2012. Motor primitive discovery., in: ICDL-EPIROB, pp. 1–8.
[87] Thórisson, K.R., 2012. A new constructivist ai: from manual methods to self-constructive systems, in: Theoretical Foundations of Artificial General Intelligence. Springer, pp. 145–171.
[88] Thrun, S., Burgard, W., Fox, D., 2005. Probabilistic robotics. MIT press.
[89] To, S.T., Son, T.C., Pontelli, E., 2015. A generic approach to planning in the presence of incomplete information: Theory and implementation. Artificial Intelligence 227, 1–51.
[90] Tomatis, N., Nourbakhsh, I., Siegwart, R., 2003. Hybrid simultaneous localization and map building: a natural integration of topological and metric. Robotics and Autonomous systems 44, 3–14.
[91] Tovar, B., Yershova, A., O'Kane, J., LaValle, S.M., 2005. Information spaces for mobile robots, in: Robot Motion and Control, 2005. RoMoCo'05. Proceedings of the Fifth International Workshop on, IEEE. pp. 11–20.
[92] Tully, S., Kantor, G., Choset, H., 2012. A unified bayesian framework for global localization and slam in hybrid metric/topological maps. The International Journal of Robotics Research .
[93] Tulving, E., 2002. Episodic memory: From mind to brain. Annual Review of Psychology 53, 125. doi:`10.1146/annurev.psych.53.100901.135114`.
[94] van de Vel, M.L.J., 1993. Theory of convex structures. volume 50 of *North-Holland Mathematical Library*. North-Holland Publishing Co., Amsterdam.
[95] West, D.B., 2001. Introduction to Graph Theory. Prentice-Hall.
[96] Wise, D.T., 2012. From riches to raags: 3-manifolds, right-angled Artin groups, and cubical geometry. volume 117 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC.
[97] Yao, A.C.C., 1997. Decision tree complexity and Betti numbers. J. Comput. System Sci. 55, 36–43. URL: `http://dx.doi.org/10.1006/jcss.1997.1495`, doi:`10.1006/jcss.1997.1495`. 26th Annual ACM Symposium on the Theory of Computing (STOC '94) (Montreal, PQ, 1994).
[98] Ziemke, T., 2001. The construction of 'reality' in the robot: Constructivist perspectives on situated artificial intelligence and adaptive robotics. Foundations of Science 6, 163–233.

8. Appendix: Poc Sets and Sageev-Roller Duality

The duality between poc sets and median algebras, going back to Isbell's work [36], was thoroughly studied by Martin Roller in [71] as part of a very successful program to push the envelope on a theory of actions of discrete groups on simply connected non-positively curved cubical complexes – henceforth reffered to as *cubings* – pioneered by Michah Sageev in [72] and by Victor Chepoi [12], who characterized such complexes in terms of the convexity theory of their 1-dimensional skeleta.

This appendix provides additional details of this theory required to support the memory architecture proposed in this paper. This overview of the preliminary meterial is meant to extend the initial discussion provided in [31] and in Section 2, to provide additional examples and to prepare the necessary technical background for the proofs of the results of this paper. We will mainly rely on [71] as a source of theoretical results, though sometimes it will be easier to use results from the elegant exposition in [62].

8.1. More on Poc Sets

We start out with a compact way of constructing and representing poc sets using "generators and relations". The reader might want to skip the details at first reading.

8.1.1. Generators and Relations

A weak poc set $\mathbf{P} = \langle S \,|R\rangle$ may be specified using a set S of generators and a set of relations R of the form $a < b$ or $a^* < b$ or $a < b^*$ for $a, b \in S$. One may also use weak inequalities ($\le$) to specify relations in R.

Formally, $\mathbf{P}$ is constructed as follows. Assume that the symbol $\mathbf{0}$ is not contained in S. First, set $S_\pm := (\{\mathbf{0}\} \sqcup S) \times \{+, -\}$ and define $(s, +)^* = (s, -)$ and $(s, -)^* = (s, +)$. Thus, $S_\pm$ obtains the structure of a complemented set. For simplicity, for each $s \in \{\mathbf{0}\} \cup S$ we identify $(s, +)$ with s. The relation set R is required to be a subset of $S_\pm \times S_\pm$. We then define an extension R_{poc} of R to be the intersection of all relations $W \subseteq S_\pm \times S_\pm$ that are reflexive, transitive and, in addition, satisfy (1) $(\mathbf{0}, a) \in W$ holds for all $a \in S_\pm$; and (2) For all $a, b \in S_\pm$, if $(a, b) \in W$ then $(b^*, a^*) \in W$. We set $\mathbf{P}$ to be the quotient of $S_\pm$ modulo $x \sim y \Leftrightarrow (x, y) \in R_{poc} \wedge (y, x) \in R_{poc}$, with the induced partial ordering $[x] \le [y] \Leftrightarrow (x, y) \in R_{poc}$.

For example, the notation $\langle a, b, c \,|a < c,\, b < c\rangle$ stands for the poc set with elements $\mathbf{0}$, $\mathbf{0}^*$, a, b, c, a^*, b^* and c^* having the order relations $\mathbf{0} < a < c < \mathbf{0}^*$, $\mathbf{0} < c^* < a^* < \mathbf{0}^*$, $\mathbf{0} < b < c < \mathbf{0}^*$ and $\mathbf{0} < c^* < b^* < \mathbf{0}^*$, as well as the ones derived from these by transitivity. Thus, generators and relations provide a compact way of representing a (weak) poc set explicitly.

As another example, consider the poc sets $\mathbf{P} = \langle a, b \,|a < b\rangle$, $\mathbf{Q} = \langle a, b \,|a^* < b\rangle$. The partial assignment $f : \mathbf{P} \to \mathbf{Q}$ satisfying $f(a) = a^*$, $f(b) = b$ has one and only one extension to a poc morphism of $\mathbf{P}$ into $\mathbf{Q}$.

Another, more general, example is provided by seeing the weak poc set $\mathbf{P} = \mathtt{Poc}(\mathbf{\Gamma})$ derived from a weakly acyclic poc graph $\mathbf{\Gamma}$ over a complemented set $\mathbf{\Sigma}$ (Definition 9.4) as $\mathbf{P} = \langle \mathbf{\Sigma} \,|a \le b \text{ iff } ab \in \mathbf{\Gamma}\rangle$.

8.1.2. The Canonical Quotient of a Weak Poc Set

We have already mentioned in Section 3.5 that every weak poc set $\mathbf{P}$ has a canonical *true* poc set quotient, $\hat{\mathbf{P}}$. It is obtained as the quotient of $\mathbf{P}$ by the equivalence relation

$$a \sim b \Leftrightarrow a = b \text{ or } a, b \in N \text{ or } a, b \in N^*\,, \tag{25}$$

where N is the set of negligible elements in $\mathbf{P}$.

Definition 8.1. Let $\mathbf{P}$ be a weak poc set and let $\hat{\mathbf{P}}$ denote its canonical poc quotient. For every $a \in \mathbf{P}$, we denote the equivalence class of a in $\hat{\mathbf{P}}$ with $\hat{a}$. The map $a \mapsto \hat{a}$ will be denoted by π.

It follows that $\hat{\mathbf{P}}$ inherits from $\mathbf{P}$ the structure of a complemented set (where $\mathbf{0} = N$ and $\hat{a}^* = \widehat{a^*}$). Moreover, observing that $N\!\downarrow = N$, one easily deduces that $\hat{\mathbf{P}}$ has an induced partial ordering given by $\hat{a} \le \hat{b}$ iff there exist $a' \sim a$ and $b' \sim b$ such that $a' \le b'$ in $\mathbf{P}$. Together these structure define a *true* poc set structure on $\hat{\mathbf{P}}$. The main characteristic of $\hat{\mathbf{P}}$ is the following elementary lemma:

Lemma 8.2. *Let* $\mathbf{P}$ *be a weak poc set. Then any poc morphism* $f : \mathbf{P} \to \mathbf{Q}$ *of* $\mathbf{P}$ *into a true poc set* $\mathbf{Q}$ *factors through* π*, that is: there exists one and only one poc morphism* $\hat{f} : \hat{\mathbf{P}} \to \mathbf{Q}$ *satisfying* $f = \hat{f} \circ \pi$.

Proof. Since $\mathbf{Q}$ is a true poc set, $f(n) = \mathbf{0} \in \mathbf{Q}$ for all negligible n in $\mathbf{P}$. In other words, $f(N) = \{\mathbf{0}\}$ and $f(N^*) = \mathbf{0}^*$, which makes the assignment $\hat{f}(\hat{a}) := f(a)$ a well-defined poc morphism from $\hat{\mathbf{P}}$ into $\mathbf{Q}$. It is evident that this is the only possible assignment for the job. ■

8.1.3. Nesting and Transversality

Sections 3.2–3.5 provide a bird's eye view of the geometry of $\texttt{Dual}(\mathbf{P})$ and $\texttt{Cube}(\mathbf{P})$, but the proofs of our new results require a slightly more detailed account. For this, we must consider the possible relations (if any) among elements a, b in a weak poc set $\mathbf{P}$:

$$a \leq b\,, \quad a^* \leq b\,, \quad a^* \leq b^*\,, \quad a \leq b^* \tag{26}$$

It is easy to see that a pair of distinct *proper* elements will never satisfy two of the above conditions at the same time, as $\texttt{Cube}(\mathbf{P})$ provides us with a realization of $\mathbf{P}$ inside $\mathbf{2}^{\mathbf{P}^\circ}$ – after all, $a \leq b$ if and only if $\mathfrak{h}(a) \subseteq \mathfrak{h}(b)$.

Definition 8.3. Suppose a, b are proper elements of a weak poc set $\mathbf{P}$. We say that they *cross* ($a \pitchfork b$), if none of (26) hold. Otherwise, we say they are *nested* ($a \| b$). A subset A of $\mathbf{P}$ is said to be *nested* if all its elements are pairwise nested, and *transverse* if its elements cross pairwise.

Thus, the half-spaces of $\texttt{Dual}(\mathbf{P})$ are nothing more than the restriction to $\mathbf{P}^\circ$ of the half-spaces of $S(\mathbf{P})^1$, with two of them nesting if and only if the corresponding elements of $\mathbf{P}$ are nested, that is, if and only if exactly one of the following holds:

$$\begin{array}{ll} \mathfrak{h}(a) \cap \mathfrak{h}(b) = \varnothing\,, & \mathfrak{h}(a^*) \cap \mathfrak{h}(b) = \varnothing\,, \\ \mathfrak{h}(a^*) \cap \mathfrak{h}(b^*) = \varnothing\,, & \mathfrak{h}(a) \cap \mathfrak{h}(b^*) = \varnothing \end{array} \tag{27}$$

We conclude that the more relations are on record in the order structure of $\mathbf{P}$ the fewer transverse sets there are to be found there. In other words, nesting relations are an obstruction to high-dimesional cubes in $\texttt{Cube}(\mathbf{P})$: each additional relation in $\mathbf{P}$ implies fewer faces of the original cube $S(\mathbf{P})$ survive the culling of incoherent vertices used for obtaining $\texttt{Cube}(\mathbf{P})$. At one extreme one finds $\texttt{Cube}(\mathbf{P}) = S(\mathbf{P})$ when $\mathbf{P}$ itself (up to removing improper elements) is transverse (the orthogonal poc set). At the other extreme, $\texttt{Cube}(P)$ forms a tree if and only if $\mathbf{P}$ is nested — a well-known result going back to Dunwoody's work on the almost-stability theorem, see [21] — which explains why both examples in Example 3.13 yield trees.

8.1.4. Example: direct sums of poc sets

The easiest way to join two poc sets together is to form their direct sum:

Definition 8.4. Let $\mathbf{P}$ and $\mathbf{Q}$ be poc sets. Their *direct sum* $\mathbf{P} \vee \mathbf{Q}$ is defined to be the quotient of their external disjoint union $P \sqcup Q$ by the identification $\mathbf{0_P} = \mathbf{0_Q}$ and $\mathbf{0}^*{}_{\mathbf{P}} = \mathbf{0}^*{}_{\mathbf{Q}}$, endowed with the following:

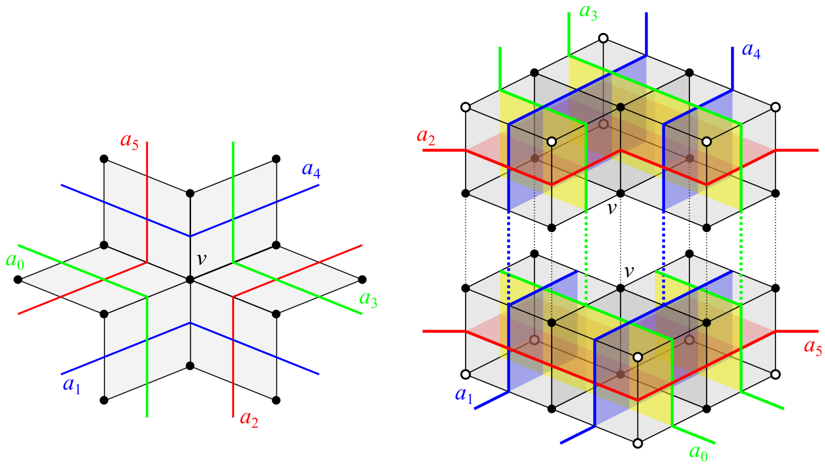

Figure 12: Cubical models for example 8.1.5 with poc relations $a_i < a^*_{i+x}$ where $x \in \{2, 3, 4\}$ and addition is modulo 6 (left), compared to the case when only the relations $a_i < a^*_{i+3}$ are present (right). Black vertices are those coherent in for both poc set structures. Vertices painted white are coherent vertices for agent #2 that are incoherent for agent #1. The vertex v corresponds to the shared coherent $*$-selection $\{a^*_0, \dots, a^*_5\}$.

- $a < b$ in $\mathbf{P} \vee \mathbf{Q}$ iff $a, b \in \mathbf{P}$ and $a < b$ or $a, b \in \mathbf{Q}$ and $a < b$;
- $b = a^*$ iff both $a, b \in \mathbf{P}$ and $b = a^*$ or $a, b \in \mathbf{Q}$ and $b = a^*$. □

We abuse notation by identifying each element of $\mathbf{P} \cup \mathbf{Q}$ with the equivalence class in $\mathbf{P} \vee \mathbf{Q}$ of its natural representative in $\mathbf{P} \sqcup \mathbf{Q}$. It is easy to verify, then, that

$$\texttt{Cube}(\mathbf{P} \vee \mathbf{Q}) \equiv \texttt{Cube}(\mathbf{P}) \times \texttt{Cube}(\mathbf{Q}) \tag{28}$$

where the isomorphism is that of cubical complexes. Intuitively, any proper elements $a \in \mathbf{P}$ and $b \in \mathbf{Q}$ satisfy $a \pitchfork b$, resulting in every cube in $\texttt{Cube}(\mathbf{P})$ and every cube in $\texttt{Cube}(\mathbf{Q})$ to form a product cube in $\texttt{Cube}(\mathbf{P} \vee \mathbf{Q})$. For example, the grid in Figure 5 may be thought of as the product of an N-path with an M-path (for the appropriate values of M and N) – hence the dual of the direct sum of two poc sets of the first type discussed in Example 3.13. This is also the principle formally underlying the computation in Example 3.16 of the cubings depicted in Figure 3.

8.1.5. Example: a cycle of length 6

Imagine an agent – call it #1 – living on the unit circle $\mathbf{E} = \mathbb{S}^1$. We mark six vertices, spread uniformly along the circle, with the digits $\{0, \dots, 5\}$. Suppose that agent #1 is capable, for each position it occupies on $\mathbf{E}$, of asking any of the binary questions

- A_j: *Am I positioned at arc length$<\frac{\pi}{3}$ from position j along* $\mathbf{E}$*?*

Agent #2 asks a slightly different set of questions:

- B_j: *Am I positioned at arc length* $< \frac{\pi}{2}$ *from position* j *along* $\mathbf{E}$*?*

The questions available to either agent have sufficient resolution to pinpoint the agent's position wherever it is, but we claim that the collection $\{A_j\}_{j=0}^5$ is, in a sense, more efficient than $\{B_j\}_{j=0}^5$ (this should be reminiscent of Example 3.18, and is a good illustration of Theorem 3.19). Let $\mathbf{\Sigma} = \{\mathbf{0}, \mathbf{0}^*\} \cup \{a_i, a_i^*\}_{i=0}^5$, where the a_i are symbols to represent the sensations corresponding to A_i for agent #1 and to B_i for agent #2. We compare the resulting embeddings $\rho_i : \mathbf{\Sigma} \hookrightarrow \mathbf{2^E}$ defined by

$$\rho_1(a_j) = A_j \ , \ \rho_1(a_j^*) = \mathbf{E} \smallsetminus A_j \ ,$$
$$\rho_2(a_j) = B_j \ , \ \rho_2(a_j^*) = \mathbf{E} \smallsetminus B_j \ ,$$

and with $\rho_i(\mathbf{0}) = \varnothing$ and $\rho_i(\mathbf{0}^*) = \mathbf{E}$, of course. We observe that both representations of P in $\mathbf{2^E}$ form injective poc morphisms of $\mathbf{P}$ into $\mathbf{2^E}$ if $\mathbf{P}$ is a poc set structure on $\mathbf{\Sigma}$ with relations of the form $a_i < a_{i+3}^*$ (addition modulo 6). However, only agent #1 can afford to also add the relations $a_i < a_{i+2}^*$ and $a_i < a_{i+4}^*$ to the record without losing the property of ρ_1 being a poc morphism. The difference between the duals (of the two different versions of $\mathbf{P}$) is significant – see figure 12 – clearly showing the advantage of the compact and simple world map that agent #1 could deduce over the cumbersome monstrosity agent #2 must deal with. Note how the complex (a) in the figure may be obtained from (b) through deleting the vertices painted white – those are precisely the vertices of (b) forming incoherent families for the poc set structure represented in (a).

8.2. Cubings and the Duality Theory of Weak Poc Sets

8.2.1. Sageev-Roller Duality from the Categorical Viewpoint

In the finite case, the duality theory of poc sets has a very clean formulation in category-theoretical terms. For a quick review of the basic notions of Category Theory we refer the reader to Chapter 4 of [41], while here we will stick to the specific categories of interest:

- $\mathbf{Poc}_f$, the category of finite true poc sets[30], where each $\mathbf{P}, \mathbf{Q} \in \mathbf{Poc}_f$ have assigned to them the set $\mathrm{Hom}(\mathbf{P},\, \mathbf{Q})$ of poc morphisms from $\mathbf{P}$ to $\mathbf{Q}$;

- $\mathbf{Med}_f$, the category of finite median graphs, where each $G, H \in \mathbf{Med}_f$ are assigned the set $\mathrm{Hom}(G,\, H)$ of median-preserving maps from the vertex set of G to the vertex set of H (such maps are called *median morphisms*).

What connects the two categories is the assignment of the graph $\mathtt{Dual}(\mathbf{P})$ to every poc set $\mathbf{P}$. The important bit here is that this assignment is not confined to the level of objects, but, rather, extends over the level of maps as well, and in a natural way:

Definition 8.5. Let $f : \mathbf{P} \to \mathbf{Q}$ be a morphism of weak poc sets. The dual map $f^\circ : \mathbf{Q}^\circ \to \mathbf{P}^\circ$ is defined to be the pullback map $f^\circ(A) = f^{-1}(A)$.

[30] One could work with the full category **Poc** of *all* poc sets (rather than just the finite ones) but this introduces major complications that seem unnecessary given the current application. Similarly for the case of median graphs/algebras.

It is easy to verify that $f^\circ : \mathbf{Q}^\circ \to \mathbf{P}^\circ$ is a median-preserving map, that is:

$$f^\circ\left(\mathrm{med}(u, v, w)\right) = \mathrm{med}(f^\circ(u), f^\circ(v), f^\circ(w)) \tag{29}$$

where the medians are computed in the appropriate duals. Thus, a map $f \in \mathrm{Hom}(\mathbf{P}, \mathbf{Q})$ yields a map $f^\circ \in \mathrm{Hom}(\mathtt{Dual}(\mathbf{Q}), \mathtt{Dual}(\mathbf{P}))$. Moreover, one easily checks that this is done in a way that respects composition, that is:

$$(g \circ f)^\circ = f^\circ \circ g^\circ \tag{30}$$

whenever the composition of the poc morphisms f, g is well-defined. This notion of map between categories is called a *functor*. The above constructions (of the dual graph and the dual map), together, are known as the *Sageev-Roller duality*.

Applying Theorem 3.25 we conclude that the above assignments form a *complete duality*, or *co-equivalence of categories*, between $\mathbf{Poc}_f$ and $\mathbf{Med}_f$. That is, there are:

- **A correspondence between $\mathbf{Poc}_f$ and $\mathbf{Med}_f$ at the level of objects:** $\mathbf{P} \mapsto \mathtt{Dual}(\mathbf{P})$ is a one-to-one correspondence between the collection of finite poc sets and the collection of median graphs;

- **A correspondence between $\mathbf{Poc}_f$ and $\mathbf{Med}_f$ at the level of maps:** $f \mapsto f^\circ$ is a composition-reversing one-to-one correspondence between poc morphisms and median morphisms.

Thus, Sageev-Roller duality is a dictionary, translating order-theoretic statements about finite poc sets into graph-theoretic statements about finite median graphs and vice-versa. Loosely speaking, the aspects of Boolean Algebra covered by poc sets may be conveniently interpreted in terms of the convex geometry of median graphs, reasoned about within this framework, and the conclusions may then be translated back into the Boolean Algebra setting for the purpose of dealing with applications.

8.2.2. Extending Sageev-Roller Duality to Weak Poc Sets

It is time to clarify the precise way in which Sageev-Roller duality extends to weak poc sets.

Lemma 8.2 is instrumental in this. A particularly interesting case of this lemma is that of $\mathbf{Q} = \mathbf{2}$. It is easy to verify that $f : \mathbf{P} \to \mathbf{2}$ is a poc morphism if and only if $f^{-1}(\mathbf{1})$ is a complete coherent $*$-selection. Thus, the set-dual $\mathbf{P}^\circ$ is in one-to-one correspondence with the set of all poc morphisms from $\mathbf{P}$ to $\mathbf{2}$ (which is what earns it the name of a 'dual'). But then the lemma states that this latter set is in one-to-one correspondence with the set of all poc morphisms $\hat{\mathbf{P}} \to \mathbf{2}$, which is the dual of the canonical quotient $\hat{\mathbf{P}}$. Carefully tracing through the definitions one obtains:

Corollary 8.6. *Let $\mathbf{P}$ be a weak poc set and let $\pi : \mathbf{P} \to \hat{\mathbf{P}}$ denote the canonical projection. Then $p^\circ : \hat{\mathbf{P}}^\circ \to \mathbf{P}^\circ$ is a median isomorphism. In particular,* $\mathtt{Cube}(\mathbf{P})$ *and* $\mathtt{Cube}(\hat{\mathbf{P}})$ *are naturally isomorphic cubical complexes.* ■

Thus, weak poc sets are indistinguishable from poc sets, as far as dual graphs are concerned. Applying Sageev-Roller duality (specifically, Theorems 3.25,3.29) one now obtains:

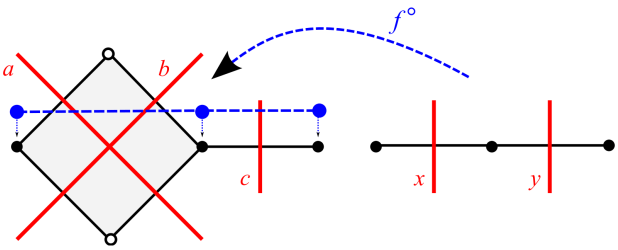

Figure 13: The dual of a poc morphism is not necessarily a graph morphism (details in 8.2.3).

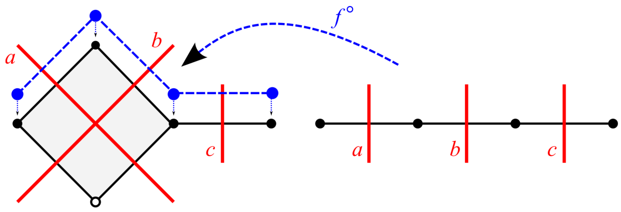

Figure 14: The dual of a degeneration is an embedding of median graphs (details in 8.2.4).

Corollary 8.7. *For any weak poc set* $\mathbf{P}$*,* $\hat{\mathbf{P}}$ *is naturally isomorphic to* $\mathcal{H}(\texttt{Dual}(\mathbf{P}))$*.* ∎

At the same time, weak poc sets form a more flexible class of objects. In particular, weak poc set structures are easier to represent and to evolve dynamically using snapshots.

8.2.3. Example: Higher-Dimensional Cubes and Duality

It is not true in general that the dual of a poc morphism $f : \mathbf{P} \to \mathbf{Q}$ extends to a morphism of graphs. For example, consider the situation

$$\mathbf{P} = \langle a, b, c \,|\, a < b,\ b < c \rangle\,, \quad \mathbf{Q} = \langle x, y \,|\, x < y \rangle \tag{31}$$

and $f : \mathbf{P} \to \mathbf{Q}$ is defined by $f(a) = f(b) = x$ and $f(c) = y$. The duals and dual map are illustrated in figure 13.

The absence of a canonical choice of extension for f° to a graph morphism of $\texttt{Dual}(\mathbf{Q})$ into $\texttt{Dual}(\mathbf{P})$ hints at a solution directly involving cubings: if one were to extend the range of f° to include the 2-dimensional cube shown in the figure, it would have been possible to extend f° to a cellular map taking the edge of $\texttt{Dual}(\mathbf{Q})$ crossed by x to an appropriately chosen diagonal of that cube. More generally, it *is* possible to extend f° to a continuous embedding of $\texttt{Cube}(\mathbf{Q})$ into $\texttt{Cube}(\mathbf{P})$ for *any* poc morphism $f : \mathbf{P} \to \mathbf{Q}$ by applying convexity properties of the canonical piecewise-Euclidean metrics on $\texttt{Cube}(\mathbf{P})$ and $\texttt{Cube}(\mathbf{Q})$ ([10], II.2.7). Thus, although median graphs are sufficient for describing the dual graphs of poc sets, describing the *dual morphisms* requires the higher dimensional geometry of cubings.

8.2.4. *Example: The Effect of Learning an Implication*

Snapshots maintain weak poc set structures on a sensorium $\mathbf{\Sigma}$ *dynamically*, updating the ordering on $\mathbf{\Sigma}$ in real time. The duality theory of poc sets provided the hint as to how such maintenance should be done. The learning methods of section 4 are motivated by an analogy between the following observations and the ideas underlying Hebbian learning, which we try to explain in the following example.

The kind of update we expect to see in a simplest instance of learning is captured in the following pair of poc sets:

$$\mathbf{P}_1 = \langle a, b, c \,| a < c,\, b < c \rangle \ , \quad \mathbf{P}_2 = \langle a, b, c \,| a < b < c \rangle \ ,$$

where the two poc set structures share their underlying set (denote it by $\mathbf{\Sigma}$), and the identity map $f = \mathtt{id}_{\mathbf{\Sigma}} : \mathbf{P}_1 \to \mathbf{P}_2$ is a morphism, while the inverse map $g = \mathtt{id}_{\mathbf{\Sigma}} : \mathbf{P}_2 \to \mathbf{P}_1$ is not (we say that f is a *degeneration*). Thinking of $\mathbf{P}_1$ as representing an agent yet undecided regarding the nature of nesting (if any) of the pair $\{a, b\}$ and therefore maintaining $a \pitchfork b$ in $\mathbf{P}_1$, we see poc set $\mathbf{P}_2$ as representing an observer with an identical set of beliefs except for the additional relation $a < b$. Figure 14 visualizes the dual map f°. In general, if $\mathbf{P}_1$ and $\mathbf{P}_2$ are poc sets with the same underlying set $\mathbf{\Sigma}$ and $f = \mathtt{id}_{\mathbf{\Sigma}} : \mathbf{P}_1 \to \mathbf{P}_2$ is a poc morphism, then the dual f° has the following properties (see e.g. [71]):

Proposition 8.8. *Suppose $f : \mathbf{P}_1 \to \mathbf{P}_2$ is a bijective poc morphism. Then:*

1. *f° is injective ([71], proposition 7.8);*
2. *f° extends to an injective cellular embedding of* $\mathtt{Cube}(\mathbf{P}_2)$ *in* $\mathtt{Cube}(\mathbf{P}_1)$*;*
3. *The image of* $\mathtt{Cube}(\mathbf{P}_2)$ *under this embedding is a strong deformation retract of* $\mathtt{Cube}(\mathbf{P}_1)$.

9. Appendix: Proofs of Technical Results

9.1. *Proof of Proposition 3.30*

Let $B \in \mathbf{P}^\circ$ be given such that $\mathtt{coh}(A)$ is not contained in B. We will find $B' \in \mathbf{P}^\circ$ such that $\mathbf{\Delta}(A, B') < \mathbf{\Delta}(A, B)$. Now find $a \in \mathtt{coh}(A) \smallsetminus B$. Then $a^* \in B$ and there is an element $b \in \min(B)$ with $b \leq a^*$. If $b \in A$ then $a \in A{\uparrow}^*$, contradicting $a \in \mathtt{coh}(A)$; hence, $b \in A^*$, which implies that $B' = (B \smallsetminus \{b\}) \cup \{b^*\}$ satisfies $\mathbf{\Delta}(A, B') = |B' \smallsetminus A| = |B \smallsetminus A| - 1 = \mathbf{\Delta}(A, B)$, as desired. ■

9.2. *Proof of Proposition 3.31*

Proof. Recall that $A \subseteq A\uparrow$, $A \uparrow\uparrow = A\uparrow$ and $A^* \downarrow = A \uparrow^*$ for all $A \subseteq \mathbf{\Sigma}$. We check that $\mathtt{coh}(A)$ **is coherent for all** A: for suppose that $b, c \in \mathtt{coh}(A)$ satisfy $b \leq c^*$; find $a \in A$ with $a \leq b$ to observe that $c^* \in A\uparrow$; equivalently, $c \in A\uparrow^*$, but that is impossible since $c \in \mathtt{coh}(A)$. Now we claim that $\mathtt{coh}(A)$ **is upwards closed**: to show that $\mathtt{coh}(A)\uparrow = \mathtt{coh}(A)$ it suffices to verify $\mathtt{coh}(A)\uparrow \subseteq \mathtt{coh}(A)$; since $\mathtt{coh}(A) \subseteq A\uparrow$ by definition, we have $\mathtt{coh}(A){\uparrow} \subseteq A{\uparrow}$ and it suffices to show no $b \in \mathtt{coh}(A){\uparrow}$ belongs to $A{\uparrow}^*$; were there such a b, there would have been $a \in \mathtt{coh}(A)$, $c \in A$ with $a \leq b$ and $c \leq b^*$, implying $a \leq c^*$ — a contradiction to $a \notin A^*{\downarrow} = A{\uparrow}^*$. This proves (a).

Now let us calculate: $\mathtt{coh}(\mathtt{coh}(A)) = \mathtt{coh}(A)\uparrow \smallsetminus \mathtt{coh}(A)\uparrow^* = \mathtt{coh}(A) \smallsetminus \mathtt{coh}(A)^* = \mathtt{coh}(A)$, the last equality due to $\mathtt{coh}(A)$ being coherent. At the same time, if A itself is coherent then $\mathtt{coh}(A) = A\uparrow \supseteq A$. Moreover, this shows $\mathtt{coh}(A) = A$ whenever $A \in \mathtt{coh}(\mathbf{P})$. Finally, if $A = \mathtt{coh}(A)$ then A is coherent and upwards closed because $\mathtt{coh}(A)$ is. This proves properties (b-d) for the map F. ■

9.3. Proof of proposition 4.9

Suppose $\mathbf{S}$ is a probabilistic snapshot, and let $\mathbf{\Gamma} = \mathtt{Dir}(\mathbf{S})$. To prove $\mathbf{\Gamma}$ is weakly acyclic, we consider a proper pair of sensors $a, b \in \mathbf{\Sigma}$ lying in the same strong component of $\mathbf{\Gamma}$, and we are required to show that $\delta(ab) = 0$ holds, demonstrating that $ab, ba \in \mathbf{\Gamma}$.

For *any* directed vertex path $p = (a_0, \ldots, a_m)$ in $\mathbf{\Gamma}$ from a to b, we apply the orientation constraint repeatedly to obtain:

$$\omega(ab) = \omega(a_0 a_1) + \ldots + \omega(a_{m-1} a_m)\,, \tag{32}$$

where we know that all the summands on the right-hand side are non-negative, and we conclude that $\omega(ab)$ is non-negative. Since $\mathbf{\Gamma}$ also contains a directed path from b to a, we must conclude $\omega(ab) = 0$, implying that $\omega(a_{i-1}a_i) = 0$ for all i. Now we apply the measure constraint repeatedly to obtain:

$$\delta(ab) \leq \delta(a_0 a_1) + \ldots + \delta(a_{m-1} a_m)\,, \tag{33}$$

For all i, since $a_{i-1}a_i \in \mathbf{\Gamma}$ with $\omega(a_{i-1}a_i) = 0$, we must also have $\delta(a_{i-1}a_i) = 0$ and we have $\delta(ab) = 0$, as desired.

9.4. Equivalences in probabilistic Snapshots

Throughout this section, let $\mathbf{\Gamma}$ denote a weakly acyclic poc graph on $\mathbf{\Sigma}$ (defn. 4.5). By assumption, each strong component of $\mathbf{\Gamma}$ is a strong clique. Let $\bar{\mathbf{\Sigma}}$ denote the partition of $\mathbf{\Sigma}$ into strong components of $\mathbf{\Gamma}$, and let $eq : \mathbf{\Sigma} \to \bar{\mathbf{\Sigma}}$ denote the quotient map sending each $a \in \mathbf{\Sigma}$ to its strong component in $\mathbf{\Gamma}$.

Recall the notion of *forward closure* in a directed graph (and, in particular, in a partially ordered set): for any set A of vertices in a directed graph $G = (V, E)$,

$$A\uparrow := \{v \in V\,|G \text{ contains a directed path from } A \text{ to } v\} \tag{34}$$

It is customary to write $a\uparrow := \{a\}\uparrow$. Thus,

$$eq(a) = eq(b) \Leftrightarrow a \in b\uparrow \text{ and } b \in a\uparrow\ . \tag{35}$$

We will consider and compare two ways in which $\mathbf{\Gamma}$ gives rise to a weak poc set structure. The first is as follows:

Lemma 9.1 (Deleted weakly acyclic is acyclic)**.** *Let $\mathbf{\Gamma}^{\times}$ denote the poc graph obtained from $\mathbf{\Gamma}$ by deleting all edges joining vertices of the same strong component of $\mathbf{\Gamma}$. Then $\mathbf{\Gamma}^{\times}$ is an acyclic poc graph.*

Proof. By definition, $\mathbf{\Gamma}^{\times}$ contains no edge-loops since $\mathbf{\Gamma}$ does not. Suppose $\mathbf{\Gamma}^{\times}$ contained a directed cycle γ. But then the vertices of γ all lie in the same strong component of $\mathbf{\Gamma}$, implying no edge of γ may lie in $\mathbf{\Gamma}^{\times}$ — a contradiction. ■

Another way $\mathbf{\Gamma}$ gives rise to an acyclic poc graph is by contracting its strong components. We verify:

Lemma 9.2. *For all* $a \in \mathbf{\Sigma}$ *one has* (1) $eq(a^*) = eq(a)^*$, *and* (2) $eq(a^*) \neq eq(a)$.

Proof. For (1), $b \in eq(a^*)$ iff $a^*b, ba^* \in \mathbf{\Gamma}$, iff $b^*a, ab^* \in \mathbf{\Gamma}$ (as $\mathbf{\Gamma}$ is a poc graph), iff $b^* \in eq(a)$, iff $b \in eq(a)^*$.

For (2), were it that $eq(a) = eq(a^*)$, then a^* would have belonged in $eq(a)$. This is impossible, since $aa^* \notin \mathbf{\Gamma}$. ■

We conclude that the operation $eq(a) \mapsto eq(a^*) = eq(a)^*$ satisfies the requirements a complemented set, as applied to $\bar{\mathbf{\Sigma}}$. Now we are able to state:

Lemma 9.3. *Let* $\bar{\mathbf{\Gamma}}$ *denote the directed graph with vertex set* $\bar{\mathbf{\Sigma}}$ *with* $eq(a)eq(b) \in \bar{\mathbf{\Gamma}}$ *if and only if* $eq(a) \neq eq(b)$ *and there is an edge* $a'b' \in \mathbf{\Gamma}$ *with* $a' \in eq(a)$ *and* $b' \in eq(b)$. *Then* $\bar{\mathbf{\Gamma}}$ *is an acyclic poc graph on* $\bar{\mathbf{\Sigma}}$.

Proof. A directed cycle in $\bar{\mathbf{\Gamma}}$ implies a directed cycle in $\mathbf{\Gamma}$ which is not contained in a strong component — contradiction. The other properties of a poc graph (over $\bar{\mathbf{\Sigma}}$) follow immediately by construction. ■

Recall that an acyclic poc graph yields a derived poc set (lemma 4.4). Consequently we may define:

Definition 9.4. Let the weak poc set derived from the acyclic poc graph $\bar{\mathbf{\Gamma}}$ be denoted by $\texttt{Poc}(\mathbf{\Gamma})$. □

Remark 9.5. Note that the weak poc set derived from $\mathbf{\Gamma}^\times$ coincides with $\texttt{Poc}(\mathbf{\Gamma}^\times)$, as the strong components of $\mathbf{\Gamma}^\times$ are all degenerate (singletons).

The following proposition list some important obvious corollaries of this construction.

Proposition 9.6. *Let* $\mathbf{\Gamma}$ *be a weakly acyclic poc graph over* $\mathbf{\Sigma}$. *Then:*

(a) *The map* $eq : \mathbf{\Sigma} \to \bar{\mathbf{\Sigma}}$ *is a poc morphism from* $\texttt{Poc}(\mathbf{\Gamma}^\times)$ *onto* $\texttt{Poc}(\mathbf{\Gamma})$.

(b) *The fibers* $\{eq(a)\}_{a \in \mathbf{\Sigma}}$ *of the map eq are transverse subsets of* $\texttt{Poc}(\mathbf{\Gamma})$.

(c) *For any subset* $A \subset \mathbf{\Sigma}$ *one has* $A{\uparrow} = eq^{-1}\left(eq(A){\uparrow}\right)$.

Statements (a),(b) of the proposition establish the precise relationship between the poc set — here denoted $\texttt{Poc}(\mathbf{\Gamma}^\times)$ — originally proposed in [31] and the "reduced" weak poc set $\texttt{Poc}(\mathbf{\Gamma})$ we have chosen to work with, obtained through the introduction of equivalences according to figure 6(b).

Statement (c) becomes important in the context of propagation, section 5.2.2, establishing the equivalence of propagation over $\mathbf{\Gamma}|_t = \texttt{Dir}(\mathbf{S}|_t)$ with closest point projection in the model space $\mathbf{M}|_t = \texttt{Cube}(\mathbf{P}|_t)$ where $\mathbf{P}|_t = \texttt{Poc}(\mathbf{\Gamma})$ (cor 9.17).

9.5. Local Structure of Duals and Greedy Navigation

In [31] we suggested exploring the link between the convexity theory of duals of weak poc sets and planning in DBAs, yet the formal results contained therein proved insufficient for supporting the planning algorithms proposed in this paper. This section fills in this gap.

Throughout this section we fix a finite weak poc set P and the median graph $\Gamma = \texttt{Dual}(P)$ (which is to say, Γ is an arbitrary finite median graph). We study the problem of computing the image of a non-empty convex subset $\mathfrak{h}(S)$ of Γ under the closest point projection of Γ to the convex subset $\mathfrak{h}(T)$.

9.5.1. Gates

We recall the following definitions and results from [71]:

Definition 9.7. Let $K, L \subseteq P^\circ$ be sets. The set

$$\texttt{sep}(K, L) = \{a \in P \,|\, K \subseteq \mathfrak{h}(a))\,,\ L \subseteq \mathfrak{h}(a^*)\,\} \tag{36}$$

is called the *s*eparator of K and L. □

The inequality $\mathbf{\Delta}(u, v) \geq |\texttt{sep}(K, L)|$ follows immediately for all $u \in K$ and $v \in L$. This motivates:

Definition 9.8. Let $K, L \subseteq P^\circ$. A *gate for* K, L is a pair of points $u \in K$, $v \in L$ such that $\mathbf{\Delta}(u, v) = |\texttt{sep}(K, L)|$. □

The following result is well known in our setting:

Proposition 9.9. *Let K, L be non-empty convex subsets of Γ and let $u \in K$ and $v \in L$. Then u, v form a gate for K, L if and only if $\texttt{proj}_K v = u$ and $\texttt{proj}_L u = v$. Moreover, any pair of non-empty convex subsets of Γ has a gate.*

We will apply this proposition without proof. An important consequence for us is the following:

Lemma 9.10. *Suppose $K = \mathfrak{h}(S)$ and $S \subset P$ is coherent. Then, for any $a \in P$, if $K \subseteq \mathfrak{h}(a)$ then there exists $s \in S$ such that $s \leq a$.*

Proof. Let $u \in K$ and $v \in L := \mathfrak{h}(a^*)$ form a gate. Since $v \notin A$, there exists $s \in S$ such that $v \in \mathfrak{h}(s^*)$.

Suppose there were a $w \in B$ with $w \in \mathfrak{h}(s)$, and consider $m = \text{med}(u, v, w)$. Then $a \in v, w$ implies $a \in m$, but the inequality

$$\mathbf{\Delta}(u, v) = \mathbf{\Delta}(u, m) + \mathbf{\Delta}(m, v) \geq \mathbf{\Delta}(u, m) \tag{37}$$

implies $m = v$, since $v = \texttt{proj}_L u$. On the other hand, $s \in u, w$ implies $s \in m$ – a contradiction.

Thus, we have shown that $L = \mathfrak{h}(a^*)$ is contained in $\mathfrak{h}(s^*)$. Equivalently, $a^* \leq s^*$, which is the same as $s \leq a$. ■

The same kind of reasoning yields:

Lemma 9.11. *Suppose K, L are non-empty convex subsets of* $\mathtt{Dual}(P)$. *If $K \cap L \neq \varnothing$, then $\mathtt{proj}_K L = \mathtt{proj}_L K = K \cap L$.*

Proof. Clearly, if $v \in K \cap L$ then $\mathtt{proj}_L(v) = v$, so $K \cap L \subset \mathtt{proj}_L K$. For the reverse inclusion, suppose $v \in \mathtt{proj}_L K$ and write $v = \mathtt{proj}_L u$, $u \in K$. Pick any point $w \in K \cap L$. Setting $m = \mathrm{med}(w, v, u)$ we note that $m \in L$ (because $w, v \in L$) and

$$\boldsymbol{\Delta}(u,v) = \boldsymbol{\Delta}(u,m) + \boldsymbol{\Delta}(m,v) \geq \boldsymbol{\Delta}(u,m) \ .$$

The uniqueness of projection forces $v = \mathtt{proj}_L u$ to coincide with m. However, since $w, u \in K$ we also have $m \in K$, showing $v \in K \cap L$. ■

9.5.2. Computing the Projection Maps

For a vertex $u \in P^\circ$ and any subset $A \subset u$, one defines:

$$[u]_A := (u \smallsetminus A) \cup A^* \tag{38}$$

Clearly, $[u]_A$ is a $*$-selection. It is easily verified that $[u]_A$ is coherent if and only if there exists no pair $a \in A$ and $b \in u \smallsetminus A$ satisfying $b < a$. This observation was first made in [72], leading to the following results in our setting:

Lemma 9.12. *Let P be a finite weak poc set and let $u \in P^\circ$ be any vertex. Then the set $N(u)$ of vertices adjacent to u in $\Gamma = \mathtt{Dual}(P)$ coincides with the set of all $[u]_a$, a ranging over the* minset *of u:*

$$\min(u) := \{a \in u \,|\, b < a \Rightarrow b \notin u\} \tag{39}$$

More generally, the cubes in $\mathtt{Cube}(P)$ are characterized as follows:

Lemma 9.13. *Let P be a finite weak poc set and $u \in P^\circ$ be a vertex. Then the cubes of $\mathtt{Cube}(P)$ incident to u are in one-to-one correspondence with the transverse subsets of* $\min(u)$.

A particular application of these observations is an explicit construction of a geodesic path in Γ emanating from a given vertex u and terminating at its unique closest point projection $\mathtt{proj}_{\mathfrak{h}(T)} u$:

Proposition 9.14. *Let P be a finite weak poc set and suppose $u \in P^\circ$ is a vertex. Let T be a coherent subset of P. Then the following algorithm constructs a shortest path in Γ from u to $K = \mathfrak{h}(T)$:*

1. *Find an element $b \in T \smallsetminus u$; if no such element, stop and output u.*
2. *Find an element $c \leq b^*$ with $c \in \min(u)$;*
3. *Replace u by $[u]_c$ and go to the first step.*

Proof. We have $u \in K$ iff $T \subset u$, which provides the stopping condition for the algorithm. Now, if $u \notin K$ and $b \in T \smallsetminus u$ then for all $v \in K$ one has $v \in \mathfrak{h}(b)$ and $u \in \mathfrak{h}(b^*)$. Since $c \leq b^*$, we have $u \in \mathfrak{h}(c) \subseteq \mathfrak{h}(b^*)$, implying $v \in \mathfrak{h}(c^*)$ and $c \in u \smallsetminus v$. As a result:

$$\boldsymbol{\Delta}\big(v, [u]_c\big) = \boldsymbol{\Delta}(v,u) - 1 \tag{40}$$

Having reduced $\boldsymbol{\Delta}(u,v)$ by a unit for all $v \in K$, we have reduced $\boldsymbol{\Delta}(u,K)$ by a unit as well. ■

Corollary 9.15 (Projection of a Point)**.** *Let P and T be as above. Then the closest point projection to $K = \mathfrak{h}(T)$ is given by the formula:*

$$\mathtt{proj}_K u = (u \smallsetminus T^*{\downarrow}) \cup T{\uparrow} = (u \cup T{\uparrow}) \smallsetminus T^*{\downarrow} \tag{41}$$

Proof. The second equality follows from the DeMorgan rules and the fact that $T{\uparrow} \cap T^*{\downarrow} = \varnothing$ (since T is coherent).

Set $K = \mathfrak{h}(T)$ and proceed by induction on $\mathbf{\Delta}(u, K)$. If $\mathbf{\Delta}(u, K) = 0$, then $u \in K$ and therefore $T \subset u$. In addition, u is coherent and we conclude $T^*{\downarrow} \cap u = \varnothing$, leaving us with

$$u \smallsetminus T^*{\downarrow} \cup T = u \cup T = u\,,$$

as desired. Now suppose $n := \mathbf{\Delta}(u, K) > 0$. By the preceding proposition, there is $a \in T^*{\downarrow} \cap u$ such that $v := [u]_a \in P^\circ$, $\mathbf{\Delta}(v, K) = n - 1$, and $\mathtt{proj}_K u = \mathtt{proj}_K v$. We thus have:

$$\mathtt{proj}_K u = \mathtt{proj}_K v = (v \smallsetminus T^*{\downarrow}) \cup T{\uparrow} = (u \smallsetminus T^*{\downarrow}) \cup T{\uparrow}\,,$$

the last equality being due to $a \in T^*$ and $a^* \in T$. Thus, the first identity has been proved. ■

9.5.3. Projecting a Convex Set to a Convex Set

Proposition 9.16. *Let K, L be non-empty convex subsets with $L = \mathfrak{h}(S)$ and $K = \mathfrak{h}(T)$. Then*

$$\begin{array}{rcl} \mathtt{proj}_K L & = & \mathfrak{h}((S{\uparrow} \cup T{\uparrow}) \smallsetminus T^*{\downarrow}) \\ & = & \mathfrak{h}(T) \cap \mathfrak{h}(S{\uparrow} \smallsetminus T{\uparrow}^*) \end{array} \tag{42}$$

Proof. Since T is coherent, $T{\uparrow}$ and $T^*{\downarrow} = T{\uparrow}^*$ are disjoint. This allows us to write:

$$\begin{array}{rcl} \mathfrak{h}((S{\uparrow} \cup T{\uparrow}) \smallsetminus T{\uparrow}^*) & = & \mathfrak{h}(T{\uparrow} \cup (S{\uparrow} \smallsetminus T{\uparrow}^*)) \\ & = & \mathfrak{h}(T{\uparrow}) \cap \mathfrak{h}(S{\uparrow} \smallsetminus T{\uparrow}^*) \end{array}$$

and the second equality in (42) follows from the identity $\mathfrak{h}(T) = \mathfrak{h}(T{\uparrow})$. Denote $R = S{\uparrow} \smallsetminus T{\uparrow}^*$ and $N = \mathfrak{h}(R)$.

For every $u \in L = \mathfrak{h}(S)$ we have $S{\uparrow} \subset u$, implying $\mathtt{proj}_K u$ contains $T{\uparrow} \cup R$, by corollary 9.15. Thus, $\mathtt{proj}_K L \subset K \cap N$, as required.

For the converse, observe that the case $K \cap L \neq \varnothing$ was already dealt with in lemma 9.11: if $K \cap L \neq \varnothing$, then

$$\mathtt{proj}_K L = K \cap L = \mathfrak{h}(S{\uparrow}) \cap \mathfrak{h}(T{\uparrow}) = \mathfrak{h}(S{\uparrow} \cup T{\uparrow})$$

In particular, $S{\uparrow} \cup T{\uparrow}$ is coherent, and hence does not intersect $T^*{\uparrow}$, and the formula (42) holds.

Thus we may henceforth assume $K \cap L = \varnothing$. Equivalently, $S{\uparrow} \cap T^*{\downarrow} \neq \varnothing$. In fact, by lemma 9.10 we have $S{\uparrow} \cap T^*{\downarrow} = \mathtt{sep}(A, B)$.

Starting with $v \in K \cap N$ we must show $v \in \mathtt{proj}_K L$. Set $u = \mathtt{proj}_L v$, $w = \mathtt{proj}_K u$, and $m = \mathrm{med}(u, v, w)$. Then $m \in K$ since $v, w \in K$. Since $K \cap L = \varnothing$, we have $\mathbf{\Delta}(u, v) > 0$ and $\mathbf{\Delta}(u, w) > 0$. Consider the point m: we have $m \in I(u, w)$ and $m \in K$; by the choice of w, m must equal w and therefore $w \in I(u,v)$. Thus, $w = \mathtt{proj}_K u \in I(u,v)$ and $u = \mathtt{proj}_L w$. By proposition 9.9, the pair u, w is a gate for K, L and we have

$$u \smallsetminus w = \mathtt{sep}(L,K) = S{\uparrow} \cap T^*{\downarrow} \ .$$

Consider an element $a \in v \smallsetminus u$. If $\mathfrak{h}(a) \cap L \neq \varnothing$, pick $u' \in \mathfrak{h}(a) \cap L$. Then $m = \mathrm{med}(u,v,u')$ will satisfy $m \in \mathfrak{h}(a) \cap L$ as well as

$$\mathbf{\Delta}(v,L) = \mathbf{\Delta}(v,u) = \mathbf{\Delta}(v,m) + \mathbf{\Delta}(m,u) \ .$$

Now, $\mathbf{\Delta}(u,m) > 0$ since $u \in \mathfrak{h}(a^*)$ and a contradiction to $u\mathtt{proj}_L v$ is obtained. Thus, $\mathfrak{h}(a) \cap L$ must be empty, which means $L \subseteq \mathfrak{h}(a^*)$. Applying lemma 9.10 we obtain $a^* \in S{\uparrow}$.

Overall, we have shown that $v \smallsetminus u \subseteq S{\uparrow}^*$. We will now verify that $v \smallsetminus w = \varnothing$, finishing the proof. Indeed, were it not so, there would have been $h \in v \smallsetminus w$. On one hand, $w \in I(u,v)$ implies $v \smallsetminus w \subset v \smallsetminus u$, and hence $h^* \in S{\uparrow}$. On the other hand, $h \notin w$ means $h^* \in w$ and therefore $h^* \notin \mathtt{sep}(L,K) = S{\uparrow} \cap T{\uparrow}^*$, which forces $h^* \in R$. Since $R \subset v$ (by choice of v), we have $h^* \in v$, contradicting our choice of h. ■

We will need the following technical corollary for the purposes of propagation:

Corollary 9.17. *Let $S, T \subset P$ be subsets and suppose S is coherent. Let $L = \mathfrak{h}(S)$ and $K = \mathfrak{h}(\mathtt{coh}(T))$. Then:*

$$\mathtt{proj}_K L = (S{\uparrow} \cup T{\uparrow}) \smallsetminus T{\uparrow}^* = (S{\uparrow} \smallsetminus T{\uparrow}^*) \cup \mathtt{coh}(T) \tag{43}$$

Proof. Recall that $\mathtt{coh}(T) = T{\uparrow} \smallsetminus T{\uparrow}^*$, and set $J = T{\uparrow} \cap T{\uparrow}^*$, so that $T{\uparrow} = \mathtt{coh}(T) + J$ and $T{\uparrow}^* = \mathtt{coh}(T)^* + J$. Then,

$$\begin{aligned}(S{\uparrow} \cup T{\uparrow}) \smallsetminus T{\uparrow}^* &= ((S{\uparrow} \cup \mathtt{coh}(T) \cup J) \smallsetminus \mathtt{coh}(T)^*) \smallsetminus J \\ &= (S{\uparrow} \cup \mathtt{coh}(T)) \smallsetminus \mathtt{coh}(T)^*\end{aligned}$$

Since $\mathtt{coh}(T){\uparrow} = \mathtt{coh}(T)$, the last expression equals $\mathtt{proj}_K L$, by the preceding proposition. The proof of the second equality is similar. ■

Table 1: Table of Mathematical Symbols

	Topic/Notation	Ref.
	General Notation	
$\langle f : x\rangle$	Evaluation of $f \in \mathbf{2}^X$ on $x \in X$	Remark 2.5
$S\uparrow$	Forward closure of a set S in a poset or in a directed graph	Eqn. (34)
	DBA Model (general)	
$\mathbf{E}, \mathbf{X}, \mathbb{T}$	Environment, State and Time	Sec.2.1
$\mathtt{pos}$	The position map $\mathbf{X} \to \mathbf{E}$	Sec.2.1
$\vert_t$	Reads as: "at time t"	Def.2.1
	DBA model (sensing)	
$\mathbf{\Sigma}$	Binary sensorium	Def.2.3
$S(\mathbf{\Sigma})^0$	The set of $*$-selections on $\mathbf{\Sigma}$	Def.2.7
ρ	Realization map of the sensorium $\mathbf{\Sigma}$	Def.2.3
$\langle a : x\rangle$	Evaluation, e.g. of $a \in \mathbf{\Sigma}$ on $x \in \mathbf{X}$	Remark 2.5
	DBA computational model (at time t)	
$\mathbf{S}\vert_t$	Agent's snapshot	Sec. 5.2.1
$\mathbf{\Gamma}\vert_t$	The derived poc graph, $\mathtt{Dir}(\mathbf{S}\vert_t)$	Sec. 5.2.1
$\mathbf{P}\vert_t$	Derived (weak) poc set structure on $\mathbf{\Sigma}$, $\mathtt{Poc}(\mathbf{S}\vert_t)$	Sec. 3.2
$\mathbf{M}\vert_t$	The model space $\mathtt{Cube}(\mathbf{P}\vert_t)$	Sec. 3.3
$\mathbf{M}^\times\vert_t$	The punctured model space $\mathtt{Cube}^\times(\mathbf{P}\vert_t, \rho)$	Def. 22
$O\vert_t$	Unprocessed observation	Def. 2.6
$S\vert_t$	Recorded observation	Sec. 3.6
$A\vert_t$	Decision (action) following the observation	Sec. 2.4
	Contents/parameters of a snapshot S	
$\mathbf{K}_{\mathbf{\Sigma}}$	The complete graph on $\mathbf{\Sigma}$ with all aa^* edges removed	Def.4.1
$\#\mathbf{S}$	State of the snapshot	Def.4.2(a)
w_{ab}	Weight on the edge ab	Def.4.2(b)
τ_{ab}	Learning threshold for the pair $a, b \in \mathbf{\Sigma}$	Def.4.2(c)
$\omega(ab)$	Orientation cocycle of $\mathbf{S}$	Prop.4.9
$\delta(ab)$	Dissimilarity measure of $\mathbf{S}$	App.9.4
	Objects derived from a snapshot S	
$\mathtt{Dir}(\mathbf{S})$	Derived poc graph	Prop.4.10
$\mathtt{Poc}(\mathbf{S})$	Derived weak poc set structure	Def.4.11
	Weak poc sets and their duals	
$\mathbf{P}, \mathbf{Q}, \ldots$	Poc sets (with and without indices)	Def.3.3
$S(\mathbf{\Sigma})$	The cubical complex of $*$-selections on $\mathbf{\Sigma}$	Def.2.9
$\mathbf{P}^\circ$	The set dual of $\mathbf{P}$, the 0-skeleton of $\mathtt{Cube}(\mathbf{P})$	Def. 3.12(b)
$\mathtt{Dual}(\mathbf{P})$	Dual graph of $\mathbf{P}$, the 1-skeleton of $\mathtt{Cube}(\mathbf{P})$	Def. 3.12(c)
$\mathtt{Cube}(\mathbf{P})$	Dual cubing of the poc set $\mathbf{P}$	Def. 3.12(a)
$\mathtt{Cube}^\times(\mathbf{P}, \rho)$	The punctured dual with respect to a realization ρ of $\mathbf{\Sigma}$	Def. 3.17
f°	The dual map $f^\circ : Q^\circ \to P^\circ$ of a poc morphism $f : P \to Q$	Defs. 3.7, 8.5