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Learning Trajectories in Mathematics: A Foundation for Standards, Curriculum, Assessment, and Instruction

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Learning Trajectories in Mathematics: A Foundation for Standards, Curriculum, Assessment, and Instruction

Abstract
Learning Trajectories in Mathematics: A Foundation for Standards, Curriculum, Assessment, and Instruction aims to provide:

- A useful introduction to current work and thinking about learning trajectories for mathematics education
- An explanation for why we should care about these questions
- A strategy for how to think about what is being attempted in the field, casting some light on the varying, and perhaps confusing, ways in which the terms trajectory, progression, learning, teaching, and so on, are being used by the education community.

Specifically, the report builds on arguments published elsewhere to offer a working definition of the concept of learning trajectories in mathematics and to reflect on the intellectual status of the concept and its usefulness for policy and practice. It considers the potential of trajectories and progressions for informing the development of more useful assessments and supporting more effective formative assessment practices, for informing the on-going redesign of mathematics content and performance standards, and for supporting teachers' understanding of students' learning in ways that can strengthen their capability for providing adaptive instruction. The authors conclude with a set of recommended next steps for research and development, and for policy.

Disciplines
Curriculum and Instruction | Educational Administration and Supervision | Educational Assessment, Evaluation, and Research | Education Policy | Science and Mathematics Education

Comments
With Jeffrey Barrett, Michael Battista, Douglas Clements, Jere Confrey, Vinci Daro, Alan Maloney, Wakasa Nagakura, Marge Petit, and Julie Sarama.

CPRE

Consortium for Policy Research in Education

JANUARY 2011

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PREPARED BY

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Established in 1985, CPRE unites researchers from seven of the nation's leading research institutions in efforts to improve elementary and secondary education through practical research on policy, finance, school reform, and school governance. CPRE studies alternative approaches to education reform to determine how state and local policies can promote student learning. The Consortium's member institutions are the University of Pennsylvania, Teachers College-Columbia University, Harvard University, Stanford University, the University of Michigan, University of Wisconsin-Madison, and Northwestern University.

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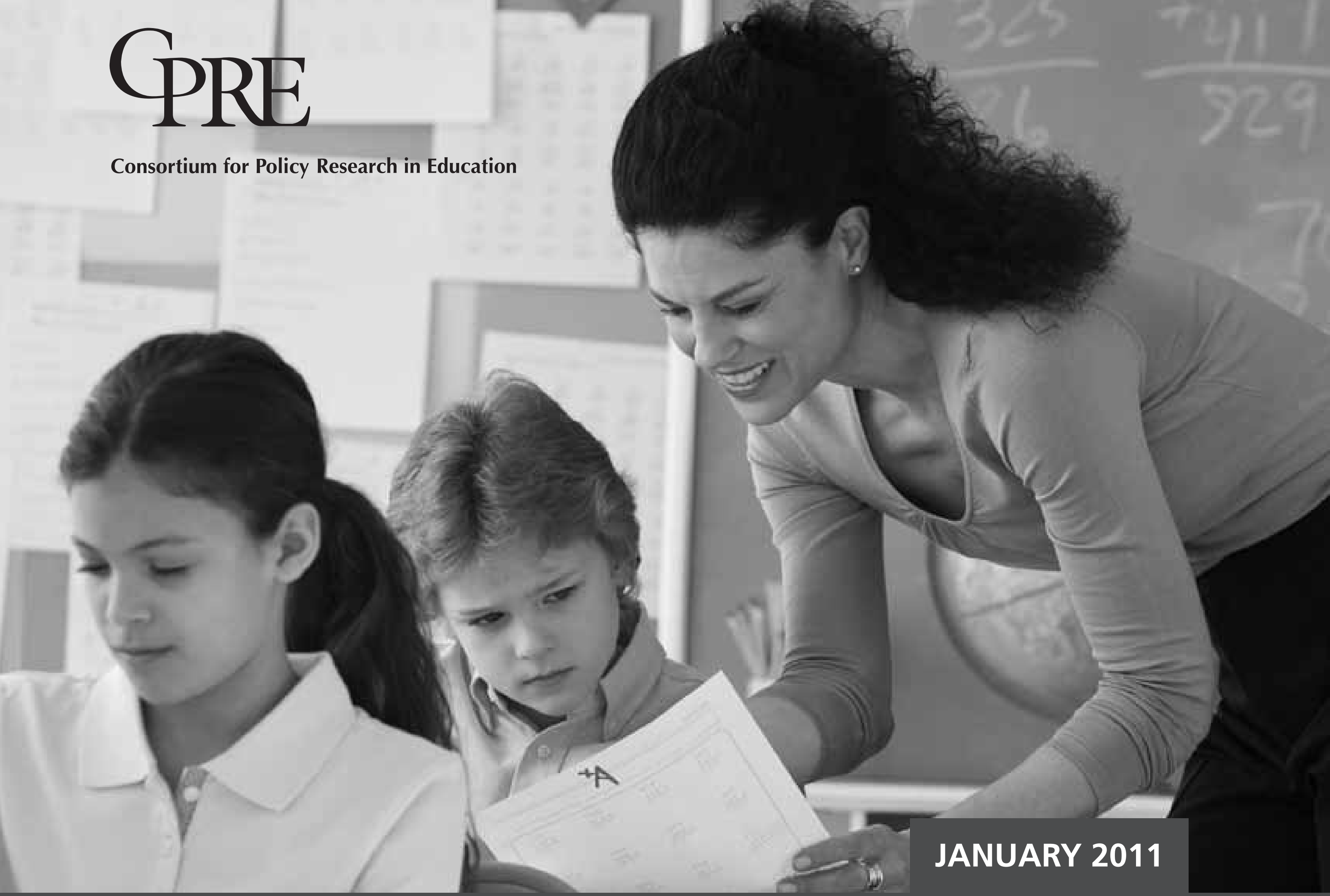

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FOREWORD

A major goal of the Center on Continuous Instructional Improvement (CCII) is to promote the use of research to improve teaching and learning. In pursuit of that goal, CCII is assessing, synthesizing and disseminating findings from research on learning progressions, or trajectories, in mathematics, science, and literacy, and promoting and supporting further development of progressions as well as research on their use and effects. CCII views learning progressions as potentially important, but as yet unproven, tools for improving teaching and learning, and recognizes that developing and utilizing this potential poses some challenges. This is the Center's second report; the first, *Learning Progressions in Science: An Evidence-based Approach to Reform*, by Tom Corcoran, Frederic A. Mosher, and Aaron Rogat was released in May, 2009.

First and foremost, we would like to thank Pearson Education and the William and Flora Hewlett Foundation for their generous support of CCII's work on learning progressions and trajectories in mathematics, science, and literacy. Through their continued support, CCII has been able to facilitate and extend communication among the groups that have an interest in the development and testing of learning trajectories in mathematics.

CCII initiated its work on learning trajectories in mathematics in 2008 by convening a working group of scholars with experience in research and development related to learning trajectories in mathematics to review the current status of thinking about the concept and to assess its potential usefulness for instructional improvement. The initial intention was to try to identify or develop a few strong examples of trajectories in key domains of learning in school mathematics and use these examples as a basis for discussion with a wider group of experts, practitioners, and policymakers about whether this idea has promise, and, if so, what actions would be required to realize that promise. However, as we progressed, our work on learning progressions intersected with the activities surrounding the initiative of the Council of Chief State School Officers (CCSSO), and the National Governors Association (NGA) to recruit most of the states, territories, and the District of Columbia to agree to develop and seriously consider adopting new national "Common Core College and Career Ready" secondary school leaving standards in mathematics and English language arts. This process then moved on to the work of mapping those standards back to what students should master at each of the grades K through 12 if they were to be on track to meeting those standards at the end of secondary school. The chair of CCII's working group and co-author of this report, Phil Daro, was recruited to play a lead role in the writing of the new CCSS, and subsequently in writing the related K-12 year-by-year standards.

Given differences in perspective, Daro thought it would be helpful for some of the key people leading and making decisions about how to draft the CCSS for K-12 mathematics to meet with researchers who have been active in developing learning trajectories that cover significant elements of the school mathematics curriculum to discuss the implications of the latter work for the standards writing effort.

This led to a timely and pivotal workshop attended by scholars working on trajectories and representatives of the Common Core Standards effort in August, 2009. The workshop was co-sponsored by CCII and the DELTA (Diagnostic E-Learning Trajectories Approach) Group, led by North Carolina State University (NCSU) Professors Jere Confrey and Alan Maloney, and hosted and skillfully organized by the William and Ida Friday Institute for Educational Innovation at NCSU The meeting focused on how research on learning trajectories could inform the design of the Common Core Standards being developed under the auspices of the Council of Chief State School Officers (CCSSO) and the National Governor's Association (NGA).

One result of the meeting was that the participants who had responsibility for the development of the CCSS came away with deeper understanding of the research on trajectories and a conviction that they had promise as a way of helping to inform the structure of the standards they were charged with producing. Another result was that many of the members of the CCII working group who participated in the meeting then became directly involved in working on and commenting on drafts of the proposed standards. Nevertheless we found the time needed for further deliberation and writing sufficient to enable us to put together this overview of the current understanding of trajectories and of the level of warrant for their use.

We are deeply indebted to the CCII working group members for their thoughtful input and constructive feedback, chapter contributions, and thorough reviews to earlier drafts of this report. The other working group members (in alphabetical order) include:

Michael Battista, Ohio State University

Jeffrey Barrett, Illinois State University

Douglas Clements, SUNY Buffalo

Jere Confrey, NCSU

Vinci Daro, Mathematics Education Consultant

Alan Maloney, NCSU

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Julie Sarama, SUNY Buffalo

Yan Liu, Consultant

We would also like to thank the key leaders and developers who participated in the co-sponsored August 2009 workshop. Participants, in alphabetical order, include:

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Martin Simon, NYU

Doug Sovde, Achieve

Paola Sztajn, NCSU

Pat Thompson, Arizona State University

Jason Zimba, Bennington College

We also would like to express our gratitude to Martin Simon, New York University; Leslie Steffe, University of Georgia; and Karen Fuson, Northwestern University, for their responses to a request for input we sent out to researchers in this field, and in the case of Simon, for his extended exchange of views on these issues. They were extremely helpful to us in clarifying our thinking on important issues, even though they may not fully accept where we came out on them.

Last but not least, we must recognize the steadfast support and dedication from our colleagues in producing this report. Special thanks to Vinci Daro and Wakasa Nagakura for their skillful editing and invaluable feedback throughout the writing process. Special thanks to Kelly Fair, CPRE's Communication Manager, for her masterful oversight of all stages of the report's production.

FOREWORD

This report aims to provide a useful introduction to current work and thinking about learning trajectories for mathematics education; why we should care about these questions; and how to think about what is being attempted, casting some light on the varying, and perhaps confusing, ways in which the terms trajectory, progression, learning, teaching, and so on, are being used by us and our colleagues in this work.

Phil Daro, Frederic A. Mosher, and Tom Corcoran

AUTHOR BIOGRAPHIES

Phil Daro is a member of the lead writing team for the K-12 Common Core State Standards, senior fellow for Mathematics of America's Choice, and director of the San Francisco Strategic Education Research Partnership (SERP)—a partnership of UC Berkeley, Stanford and the San Francisco Unified School District. He previously served as executive director of The Public Forum on School Accountability, as director of the New Standards Project (leader in standards and standards-based test development), and as director of Research and Development for the National Center for Education and the Economy (NCEE). He also directed large-scale teacher professional development programs for the University of California including the California Mathematics Project and the American Mathematics Project, and has held leadership positions within the California Department of Education. Phil has been a Trustee of the Noyce Foundation since 2005.

Frederic A. (Fritz) Mosher is senior research consultant to the Consortium for Policy Research in Education (CPRE). Mosher is a cognitive/social psychologist and knowledgeable about the development and use of learning progressions. He has worked with CPRE on the Center on Continuous Instructional Improvement (CCII) since its inception, helping to design the Center and taking a lead role in the Center's work on learning progressions. Mosher also has extensive knowledge of, and connections with the philanthropic community, reform organizations, and federal agencies. He has been advisor to the Spencer Foundation, a RAND Corporation adjunct staff member, advisor to the Assistant Secretary for Research and Improvement in the U.S. Department of Education, and a consultant to Achieve, Inc. For 36 years he was a program specialist with varying responsibilities at Carnegie Corporation of New York.

Tom Corcoran is co-director of the Consortium for Policy Research in Education (CPRE) at Teachers College, Columbia University and principal investigator of the Center on Continuous Instructional Improvement (CCII). Corcoran's research interests include the promotion of evidence-based practice, the effectiveness of various strategies for improving instruction, the use of research findings and clinical expertise to inform instructional policy and practice, knowledge management systems for schools, and the impact of changes in work environments on the productivity of teachers and students. Previously, Corcoran served as policy advisor for education for New Jersey Governor Jim Florio, director of school improvement for Research for Better Schools, and director of evaluation and chief of staff of the New Jersey Department of Education. He has designed and currently manages instructional improvement projects in Jordan and Thailand, and has served as a consultant to urban school districts and national foundations on improving school effectiveness and equity. He served as a member of the National Research Council's K–8 Science Learning Study and serves on the NRC Committee to Develop a Conceptual Framework for New Science Standards.

EXECUTIVE SUMMARY

There is a leading school of thought in American education reform circles that basically is agnostic about instruction and practice. In its purest form, it holds that government agencies shouldn't try to prescribe classroom practice to frontline educators. Rather, the system should specify the student outcomes it expects and hold teachers and schools accountable for achieving those outcomes, but leave them free to figure out the best ways to accomplish those results. This is sometimes framed as a trade off of increased autonomy or empowerment in return for greater accountability. A variation on this approach focuses on making structural and governance modifications that devolve authority for instructional decisions to local levels, reduce bureaucratic rules and constraints—including the constraints of collective bargaining contracts with teachers' unions—and provide more choice to parents and students, opening the system to market forces and incentives, also constrained only by accountability for students' success. A different version of the argument seems to be premised on the idea that good teachers are born not made, or taught, and that the system can be improved by selecting and keeping those teachers whose students do well on assessments, and by weeding out those whose students do less well, without trying to determine in detail what the successful teachers do, as one basis for learning how to help the less successful teachers do better.

This agnosticism has legitimate roots in a recognition that our current knowledge of effective instructional practices is insufficient to prescribe precisely the teaching that would ensure that substantially all students could reach the levels of success in the core school subjects and skills called for in the slogan "college and career ready." CCII doesn't, however, accept the ideas that we know nothing about effective instruction, or that it will not be possible over time to develop empirical evidence concerning instructional approaches that are much more likely to help most students succeed at the hoped-for levels. It seems to us that it would be foolish not to provide strong incentives or even requirements for teachers to use approaches based on that knowledge, perhaps with provisions for waivers to allow experimentation to find even better approaches. Conversely, it is not reasonable, or professional, to expect each teacher totally to invent or re-invent his or her own approach to instruction for the students he or she is given to teach.

To illustrate the scope of the problem facing American schools, a recent study by ACT Inc. (2010) looked at how 11th-grade students in five states that now require all students to take ACT's assessments (as opposed to including only students who are applying to college) did on the elements of their assessments that they consider to be indicative of readiness to perform effectively in college. They offer this as a rough baseline estimate of how the full range of American students might perform on new assessments based on the common core standards being developed by the two "race to the top" state assessment consortia. The results were that the percentage of all students who met ACT's proxy for college ready standards ranged from just over 30% to just over 50% for key subjects, and for African-American students it fell to as low as under 10% on some of the standards. The percentages for mathematics tended to be the lowest for any of the subjects tested. And these results are based on rather conventional assessments of college readiness, not performance items that require open-ended and extended effort, or transfer of knowledge to the solution of new and wide-ranging problems, which would be even more challenging reflections of the larger ambitions of common core reforms.

This study is useful in forcing us to attend to another of our education "gaps"—the gap between the ambitious goals of the reform rhetoric and the actual levels of knowledge and skill acquired by a very large proportion of American secondary school students—and the problem is not limited to poor and minority students, though it has chronically been more serious for them. Closing this gap will not be a trivial undertaking, and it will not happen in just a few years, or in response to arbitrary timetables such as those set by the NCLB legislation or envisioned by the Obama administration. A great many things will have to happen, both inside and outside of schools, if there is to be any hope of widespread success in meeting these goals. Certainly that should include policies that improve the social and economic conditions for children and families outside of school, and in particular, families' ability to support their children's learning and to contribute directly to it. Nevertheless, it also is clear that instruction within schools will have to become much more responsive to the particular needs of the students they serve. If substantially all students are to succeed at the hoped-for levels, it will not be sufficient just to meet the "opportunity to learn" standard of equitably delivering high- quality curricular content to all students, though that of course is a necessary step. Since students' learning, and their ability to meet ambitious standards in high school, builds over time—and takes time—if they are to have a reasonable chance to make it, their progress along the path to meeting those standards really has to be monitored purposefully, and action has to be taken whenever it is clear that they are not making adequate progress. When students go off track early, it is hard to bet on their succeeding later, unless there is timely intervention.

The concept of ***learning progressions*** offers one promising approach to developing the knowledge needed to define the "track" that students may be on, or should be on. Learning progressions can inform teachers about what to expect from their students. They provide an empirical basis for choices about when to teach what to whom. Learning progressions identify key waypoints along the path in which students' knowledge and skills are likely to grow and develop in school subjects (Corcoran, Mosher, & Rogat, 2009). Such waypoints could form the backbone for curriculum and instructionally meaningful assessments and performance standards. In mathematics education, these progressions are more commonly labeled ***learning trajectories***. These trajectories are empirically supported hypotheses about the levels or waypoints of thinking, knowledge, and skill in using knowledge, that students are likely to go through as they learn mathematics and, one hopes, reach or exceed the common goals set for their learning. Trajectories involve hypotheses both about the order and nature of the steps in the growth of students' mathematical understanding, and about the nature of the instructional experiences that might support them in moving step by step toward the goals of school mathematics.

The discussions among mathematics educators that led up to this report made it clear that trajectories are not a totally new idea, nor are they a magic solution to all of the problems of mathematics education. They represent another recognition that learning takes place and builds over time, and that instruction has to take account of what has gone before and what will come next. They share this with more traditional "scope and sequence" approaches to curriculum development. Where they differ is in the extent to which their hypotheses are rooted in actual empirical study of the ways in which students' thinking grows in response to relatively well specified instructional experiences, as opposed to being grounded mostly in the disciplinary logic of mathematics and the conventional wisdom of practice. By focusing on the identification of significant and recognizable clusters of concepts and connections in students' thinking that represent key steps forward, trajectories offer a stronger basis for describing the interim goals that students should meet if they are to reach the common core college and career ready high school standards. In addition, they provide understandable points of reference for designing assessments for both summative and formative uses that can report where students are in terms of those steps, rather than reporting only in terms of where students stand in comparison with their peers. Reporting in terms of scale scores or percentiles does not really provide much instructionally useful feedback.

By focusing on the identification of significant and recognizable clusters of concepts and connections in students' thinking that represent key steps forward, trajectories offer a stronger basis for describing the interim goals that students should meet if they are to reach the common core college and career ready high school standards. In addition, they provide understandable points of reference for designing assessments for both summative and formative uses that can report where students are in terms of those steps, rather than reporting only in terms of where students stand in comparison with their peers.

However, in sometimes using the language of development, descriptions of trajectories can give the impression that they are somehow tapping natural or inevitable orders of learning. It became clear in our discussions that this impression would be mistaken. There may be some truth to the idea that in the very early years, children's attention to number and quantity may develop in fairly universal ways (though it still will depend heavily on common experiences and vary in response to cultural variations in experience), but the influence of variations in experience, in the affordances of culture, and, particularly, in instructional environments, grows rapidly with age. While this influence makes clear that there are no single or universal trajectories of mathematics learning, trajectories are useful as modal descriptions of the development of student thinking over shorter ranges of specific mathematical topics and instruction, and within particular cultural and curricular contexts—useful as a basis for informing teachers about the (sometimes wide) range of student understanding they are likely to encounter, and the kinds of pedagogical responses that are likely to help students move along.

Most of the current work on trajectories, as described in this report, has this shorter term topical character. That is, they focus on a particular mathematical content area—such as number sense or measurement—and how learning in these areas develops over a few grades. These identified trajectories typically are treated somewhat in isolation from the influence of what everyone recognizes are parallel and ongoing trajectories for other mathematical content and practices that surely interact with any particular trajectory of immediate concern. The hope is that these delimited trajectories will prove to be useful to teachers in their day-to-day work, and that the interactions with parallel trajectories will prove to be productive, if arranged well in the curriculum. From the perspective of policy and the system, it should eventually be possible to string together the growing number of specific trajectories where careful empirical work is being done, and couple them with curriculum designs based on the best combinations of disciplinary knowledge, practical experience, and ongoing attention to students' thinking that we can currently muster, to produce descriptions of the key steps in students' thinking to be expected across all of the school mathematics curriculum. These in turn can then be used to improve current standards and assessments and develop better ones over time as our empirical knowledge also improves.

The CCII Panel has discussed these issues, and the potential of learning trajectories in mathematics, the work that has been done on them, the gaps that exist in this work, and some of the challenges facing developers and potential users. We have concluded that learning trajectories hold great promise as tools for improving instruction in mathematics, and they hold promise for guiding the development of better curriculum and assessments as well. We are agreed that it is important to advance the development of learning trajectories to provide new tools for teachers who are under increasing pressure to bring every child to high levels of proficiency.

With this goal in mind, we offer the following recommendations:

- **Mathematics educators and funding agencies should recognize research on learning trajectories in mathematics as a respected and important field of work.**

- **Funding agencies and foundations should initiate new research and development projects to fill critical knowledge gaps.** There are major gaps in our understanding of learning trajectories in mathematics. These include topics such as:
  - Algebra
  - Geometry
  - Measurement
  - Ratio, proportion and rate
  - Development of mathematical reasoning

  An immediate national initiative is needed to support work in these and other critical areas in order to fill in the gaps in our understanding.

- **Work should be undertaken to consolidate learning trajectories.** For topics such as counting, or multiplicative thinking, for example, different researchers in mathematics education have developed their own learning trajectories and these should be tested and integrated.

- **Mathematics educators should initiate work on integrating and connecting across trajectories.**

- **Studies should be undertaken of the development of students from different cultural backgrounds and with differing initial skill levels.**

- **The available learning trajectories should be shared broadly within the mathematics education and broader R & D communities.**

- **The available learning trajectories should be translated into usable tools for teachers.**

- **Funding agencies should provide additional support for research groups to validate the learning trajectories they have developed so they can test them in classroom settings and demonstrate their utility.**

- **Investments should be made in the development of assessment tools based on learning trajectories for use by teachers and schools.**

- **There should be more collaboration among mathematics education researchers, assessment experts, cognitive scientists, curriculum and assessment developers, and classroom teachers.**

- **And, finally as we undertake this work, it is important to remember that it is the knowledge of the mathematics education research that will empower teachers, not just the data from the results of assessments.**

I. INTRODUCTION

It is a staple of reports on American students' mathematics learning to run through a litany of comparisons with the performance of their peers from around the world, or to the standards of proficiency set for our own national or state assessments, and to conclude that we are doing at best a mediocre job of teaching mathematics. Our average performance falls in the mid range among nations; the proportion of high performers is lower than it is in many countries that are our strongest economic competitors; and we have wide gaps in performance among variously advantaged and disadvantaged groups, while the proportion of the latter groups in our population is growing.

All of this is true. But it also is true that long term NAEP mathematics results from 1978 to 2008 provide no evidence that American students' performance is getting worse, and the increasing numbers of students who take higher level mathematics courses in high school (Advanced Placement, International Baccalaureate, and so on) imply that the number of students with knowledge of more advanced mathematical content should be increasing (The College Board, n.d.; Rampey, Dion, & Donahue, 2009). With a large population, the absolute number of our high performers is probably still competitive with most of our rivals, but declines in the number of students entering mathematics and engineering programs require us to recruit abroad to meet the demand for science, mathematics, engineering, and technology graduates. Nevertheless, what has changed is that our rivals are succeeding with growing proportions of their populations, and we are now much more acutely aware of how the uneven quality of K-12 education and unevenly distributed opportunities among groups in our society betray our values and handicap us in economic competition. So our problems are real. We should simply stipulate that.

The prevalent approach to instruction in our schools will have to change in fairly fundamental ways, if we want "all" or much higher proportions of our students to meet or exceed standards of mathematical understanding and skill that would give them a good chance of succeeding in further education and in the economy and polity of the 21st century. The Common Core State Standards (CCSS) in mathematics provide us with standards that are higher, clearer, and more focused than those now set so varyingly by our states under No Child Left Behind (NCLB); if they are adopted and implemented by the states they will undoubtedly provide better guidance to education leaders, teachers, and students about where they should be heading. But such standards for content and performance are not in themselves sufficient to ensure that actions will be taken to help most students reach them. For that to happen, teachers are going to have to find ways to attend more closely and regularly to each of their students during instruction to determine where they are in their progress toward meeting the standards, and the kinds of problems they might be having along the way. Then teachers must use that information to decide what to do to help each student continue to progress, to provide students with feedback, and help them overcome their particular problems to get back on a path toward success. In other words, instruction will not only have to attend to students' particular needs but must also *adapt* to them to try to get—or keep—them on track to success, rather than simply *selecting* for success those who are easy to teach, and leaving the rest behind to find and settle into their particular niches on the normal grading "curve." This is what is known as adaptive instruction and it is what practice must look like in a standards-based system.

There are no panaceas, no canned programs, no technology that can replace careful attention and timely interventions by a well-trained teacher who understands how children learn mathematics, and also where they struggle and what to do about it.

But note that, to adapt, a teacher must know how to get students to reveal where they are in terms of what they understand and what their problems might be. They have to have specific ideas of how students are likely to progress, including what prerequisite knowledge and skill they should have mastered, and how they might be expected to go off track or have problems. And they would need to have, or develop, ideas about what to do to respond helpfully to the particular evidence of progress and problems they observe.

This report addresses the question of where these ideas and practices that teachers need might come from, and what forms they should take, if they are to support instruction in useful and effective ways.

Ideally, teachers would learn in their pre-service courses and clinical experiences most of what they need to know about how students learn mathematics.

It would help if those courses and experiences anticipated the textbooks, curriculum materials, and instructional units the teachers would likely be using in the schools where they will be teaching, so that explicit connections could be made between what they were learning about students' cognitive development and mathematics learning and the students they will be teaching and the instructional materials they will be using. This is how it is done in Singapore, Finland, and other high-performing countries. In America this is unlikely to happen, because of the fragmented governance and institutional structure, the norms of autonomy and academic freedom in teacher training institutions, and the "local control" bias in the American system. Few assumptions can be made ahead of time about the curriculum and materials teachers will be expected to use in the districts or schools where they will end up teaching, and if valid assumptions can be made, faculty may resist preparing teachers for a particular curriculum. Perhaps for these reasons, more attention is sometimes given in teacher training institutions to particular pedagogical styles or approaches than to the content and sequencing of what is to be taught. In addition, perhaps because of the emphasis on delivery of content without a concomitant focus on what to do if the content is not learned, little attention has been given to gathering empirical evidence, or collecting and warranting teacher lore, that could provide pre-service teachers with trustworthy suggestions about how they might tell how a student was progressing or what specific things might be going wrong; and, even less attention has been given to what teachers might do about those things if they spot them.

Given all this, novice teachers usually are left alone behind their closed classroom doors essentially to make up the details of their own curriculum—extrapolating from whatever the district-or school-adopted textbook or mathematics program might offer—and they are told that this opportunity for "creativity" reflects the essence of their responsibility as "professionals." [1]

But this is a distorted view of what being professional means. To be sure, professionals value (and vary in) creativity, but what they do—as doctors, lawyers, and, we should hope, teachers—is supposed to be rooted in a codified body of knowledge that provides them with pretty clear basic ideas of what to do in response to the typical situations that present themselves in their day to day practice. Also, what they do is supposed to be responsive to the particular needs of their clients. Our hypothesis is that in American education the modal practice of delivering the content and expecting the students to succeed or fail according to their talent or background and family support, without taking responsibility to track progress and intervene when students are known to be falling behind has undermined the development of a body of truly professional knowledge that could support more adaptive responses to students' needs. This problem has been aggravated by the fact that American education researchers tend to focus on the problems that interest them, not necessarily those that bother teachers, and have not focused on developing knowledge that could inform adaptive instructional practice.

Pieces of the necessary knowledge are nevertheless available, and the standards-based reform movement of the last few decades is shifting the norms of teaching away from just delivering the content and towards taking more responsibility for helping all students at least to achieve adequate levels of performance in core subjects. The state content standards, as they have been tied to grade levels, can be seen as a first approximation of the order in which students should learn the required content and skills. However, the current state standards are more prescriptive than they are descriptive. They define the order in which, and the time or grade by which, students *should* learn specific content and skills as evidenced by satisfactory performance levels. But typically state standards have not been deeply rooted in empirical studies of the ways children's thinking and understanding of mathematics actually develop in interaction with instruction.[2] Rather they usually have been compromises derived from the disciplinary logic of mathematics itself, experience with the ways mathematics has usually been taught, as reflected in textbooks and teachers' practical wisdom, and lobbying and special pleading on behalf of influential individuals and groups arguing for inclusion of particular topics, or particular ideas about "reform"

1 The recent emphasis on strict curricular "pacing" in many districts that are feeling "adequate annual progress" pressures from NCLB might seem to be an exception, because they do involve tighter control on teachers' choices of the content to be taught, but that content still varies district by district, and teachers still are usually left to choose how they will teach the content. In addition, whole-class pacing does limit teachers' options for responding to individual students' levels of progress.

2 This is also changing, and a number of states have recently used research on learning progressions in science and learning trajectories in mathematics to revise their standards.

or "the basics." Absent a strong grounding in research on student learning, and the efficacy of associated instructional responses, state standards tend at best to be lists of mathematics topics and some indication of when they should be taught grade by grade without explicit attention being paid to how those topics relate to each other and whether they offer students opportunities over time to develop a coherent understanding of core mathematical concepts and the nature of mathematical argument. The end result has been a structure of standards and loosely associated curricula that has been famously described as being "a mile wide and an inch deep" (Schmidt et al., 1997).

Of course some of the problems with current standards could be remedied by being even more mathematical—that is, by considering the structure of the discipline and being much clearer about which concepts are more central or "bigger," and about how they connect to each other in terms of disciplinary priority. A focus on what can be derived from what might yield a more coherent ordering of what should be taught. And recognizing the logic of that ordering might lead teachers to encourage learning of the central ideas more thoroughly when they are first encountered, so that those ideas don't spread so broadly and ineffectively through large swaths of the curriculum. But even with improved logical coherence, it is not necessarily the case that all or even most students will perceive and appreciate that coherence. So, there still is the issue of whether the standards should also reflect what is known about the ways in which students actually develop understanding or construe what they are supposedly being taught, and whether, if they did, such standards might come closer to providing the kind of knowledge and support we have suggested teachers will need if they are to be able to respond effectively to their students' needs.

Instruction, as Cohen, Raudenbush, and Ball (2003) have pointed out, can be described as a triangular relationship involving interactions among a teacher or teaching; a learner; and the content, skills, or material that instruction is focused on. Our point is that the current standards tend to focus primarily on the content side of the triangle. They would be more useful if they also took into account the ways in which students are likely to learn them and how that should influence teaching. Instruction is clearly a socially structured communicative interaction in which the purpose of one communicator, the teacher, obviously, is to tell, show, arrange experiences, and give feedback so that the students learn new things that are consistent with the goals of instruction.[3] As with all human beings, students are always learning in that they are trying to make sense of experience in ways that serve their purposes and interests. Their learning grows or progresses, at least in the sense of accretion—adding new connections, perceptions, and expectations—but whether it progresses in the direction of the goals of instruction as represented by standards, and at the pace the standards imply, is uncertain, and that is the fundamental problem of instruction in a standards-based world.

So, what might be done to help teachers coordinate their efforts more effectively with students' learning? What is needed to ensure that the CCSS move us toward the aspirations of the standards movement, an education system capable of achieving both excellence and equity?

Over the past 20 years or so the process of "formative assessment" has attracted attention as a promising way to connect teaching more closely and adaptively to students' thinking (Sadler, 1989; Black & Wiliam, 1998). Formative assessment involves a teacher in seeking evidence during instruction (evidence from student work, from classroom questions and dialog or one-on-one interviews, sometimes from using assessment tools designed specifically for the purpose, and so on) of whether students are understanding and progressing toward the goals of instruction, or whether they are having difficulties or falling off track in some way, and using that information to shape pedagogical responses designed to provide students with the feedback and experiences they may need to keep or get on track. This is not a new idea; it is what coaches in music, drama, and sports have always done. Studies of the use of formative assessment practices (Black & Wiliam, 1998; National Mathematics

[3] We favor the view that students are active participants in their learning, bringing to it their own theories or cognitive structures (sometimes called "schemes" or "schemata" in the cognitive science literature) on what they are learning and how it works, and assimilating new experience into those theories if they can, or modifying them to accommodate experiences that do not fit. Their theories also may evolve and generalize based on their recognition of and reflection on similarities and connections in their experiences, but just how these learning processes work is an issue that requires further research (Simon et al., 2010). We would not, however, carry this view so far as to say that students cannot be told things by teachers or learn things from books that will modify their learning (or their theories)—that they have to discover everything for themselves. A central function of telling and showing in instruction is presumably to help to direct attention to aspects of experience that students' theories can assimilate or accommodate to in constructive ways.

Advisory Panel, 2008) indicate that they can have quite promising effects on improving students' outcomes, but they also suggest that in order to work well they require that teachers have in mind theories or expectations about how students' thinking will change and develop, what problems they are likely to face, and what kinds of responses from the teacher are likely to help them progress. This in turn has led some to turn their attention to developing empirically testable and verifiable theories to increase our understanding, in detail, about the ways that students are most likely to progress in their learning of particular subjects that could provide the understanding teachers need to be able to interpret student performance and adapt their teaching in response.

This brings us to the idea of "learning progressions," or, as the concept more often is termed in the mathematics education literature—"learning trajectories." These are labels given to attempts to gather and characterize evidence about the paths children seem to follow as they learn mathematics. Hypotheses about the paths described by learning trajectories have roots in developmental and cognitive psychology and, more recently, developmental neuroscience. These include roots in, for instance, Piaget's genetic epistemology which tried to describe the ways children's actions, thinking, and logic move through characteristic stages in their understanding of the world (Piaget, 1970) and Vygotsky's description of the "Zone of Proximal Educational Development" that characterized the ways in which children's learning can be socially supported or "scaffolded" at its leading edge and addressed the extent to which individual learners may follow such supports and reach beyond their present level of thinking (Vygotsky, 1978).[4] These attempts to describe how children learn mathematics also are influenced by more conventional "scope and sequence" approaches to curriculum design, but in contrast to those approaches, they focus on seeking evidence that students' understanding and skill actually do develop in the ways they are hypothesized to, and on revising those hypotheses if they don't.

The first use of the term "learning trajectory" as applied to mathematics learning and teaching seems to have been by Martin Simon in his 1995 paper (*Reconstructing Mathematics Pedagogy from a Constructivist Perspective*) reporting his own work as a researcher/teacher with a class of prospective teachers. The paper is a quite subtle treatment of the issues we have tried to describe above, in that his concern is with how a teacher teaches if he does not expect simply to tell students how to think about a mathematical concept, but rather accepts responsibility for trying to check on whether they are in fact understanding it, and for arranging new experiences or problems designed to help them move toward understanding, if they are not. This engages him directly in the relationships among his goals for the students, what he thinks they already understand, his ideas about the kinds of tasks and problems that might bring them to attend to and comprehend the new concept, and an ongoing process of adjustment or revision and supplementation of these expectations and tasks as he tries them with his students and observes their responses. Simon used the term "hypothetical learning trajectory" to refer to the framing of a teacher's lesson plan based on his reasoned anticipation of how students' learning might be expected to develop towards the goal(s) of the lesson, based on his own understanding of the mathematics entailed in the goal(s), his knowledge of how other students have come to understand that mathematics, his estimates of his students' current (range of) understanding, and his choice of a mathematical task or sequence of tasks that, as students work on them, should lead them to a grounded understanding of the desired concept(s) or skill(s). In summary, for Simon a hypothetical learning trajectory for a lesson "is made up of three components: the learning goal that defines the direction, the learning activities, and the hypothetical learning processes—a prediction of how the students' thinking and understanding will evolve in the context of the learning activities" (Simon, 1995, p. 136). The hypothetical trajectory asserts the interdependence of the activities and the learning processes.[5]

[4] Infant studies suggest that very young children have an essentially inborn capacity to attend to quantitative differences and equivalences, and perhaps to discriminate among very small numbers (Xu, Spelke, & Goddard, 2005; Sophian, 2007), capacities that provide a grounding for future mathematics learning. Detailed clinical interviews and studies that describe characteristic ways in which children's understanding of number and ability to count and do simple arithmetic develop (Gelman & Gallistel, 1986; Ginsburg, 1983; Moss & Case, 1999). Hypotheses about trajectories also stem from the growing tradition of design experiments exploring the learning of other strands of mathematics (Clements, Swaminathan, Hannibal, & Sarama, 1999).

[5] It might have been clearer if Simon had used the term "hypothetical teaching or pedagogical trajectory," or perhaps, because of the need to anticipate the way the choices and sequence of teaching activities might interact with the development of students' thinking or understanding, they should have been called "teaching and learning trajectories," or even "instructional trajectories" (assuming "instruction" is understood to encompass both teaching and learning). There is a slight ambiguity in any case in talking about learning as having a trajectory. If learning is understood as being a process, with its own mechanisms, it isn't learning per se that develops and has a trajectory so much as the products of learning (thinking, or rather concepts, of increasing complexity or sophistication, skills, and so on) that do. But that is a minor quibble, reflecting the varying connotations of "learning" (we won't try to address ideas about "learning to learn" here).

While Simon's trajectories were hypotheses about the sequences of activities and tasks that might support the development of students' understanding of a specific instructional goal, many of the researchers and developers who have since adopted this language to describe aspects of their work have clearly wanted to apply the idea of trajectories to greater ranges of the mathematics curriculum, and to goals and sub-goals of varying grain size. In addition, as we have implied leading in to this discussion, there are many who have hopes that well-constructed and validated trajectories might provide better descriptions of how students' mathematical understanding and skill should develop over time. Such trajectories could be used as a basis for designing more coherent and instructionally useful standards, curricula, assessments, and approaches to teacher professional development.

It might help to look at an example. Clements and Sarama (2004) offer a rather carefully balanced view:

> we conceptualize learning trajectories as descriptions of children's thinking and learning in a specific mathematical domain, and a related conjectured route through a set of instructional tasks designed to engender those mental processes or actions hypothesized to move children through a developmental progression of levels of thinking, created with the intent of supporting children's achievement of specific goals in that mathematical domain. (p.83)

Brief characterizations like this inevitably require further specification and illustration before they communicate fully, as Clements and Sarama well know. Their definition highlights the concern with the "specific goals" of teaching in the domain but stresses that the problem of teaching is that it has to take into account children's current thinking, and how it is that they learn, in order to design tasks or experiences that will engage those processes of learning in ways that will support them in proceeding toward the goals the teachers set for them. Taking into account children's current thinking includes identifying where their thinking stands in terms of a developmental progression of levels and kinds of thinking. Introducing the word "developmental" doesn't at all imply that students' thinking could progress independently of experience, but it does suggest that teaching needs to take into account issues of timing and readiness ("maturation" is a word that once would have been used). Progress is not only or simply responsive to experience but will unfold over time in an ordered way based on internal factors, though this is likely to be contingent on the student's having appropriate experiences. The specific timing for particular students may vary for both internal and external reasons.

Clements and Sarama accept that one can legitimately focus solely on studying the development of students' thinking ***or*** on how to order instructional sequences, and that either focus can be useful, but for them it is clear that the two are inextricably related, at least in the context of schooling. They really should be studied, and understood, together.

At this point we can only question whether the right label for the focus of that joint study is "learning trajectories," or whether it should be something more compound and complex to encompass both learning and teaching, and whether there should be some separate label for the aspects of development that are significantly influenced by "internal" factors.[6] Others seem to have recognized this point. The recent National Research Council (NRC) report on early learning in mathematics (Cross, Woods, & Schweingruber, 2009) uses the term, "teaching-learning paths" for a related concept; and the Freudenthal program in Realistic Mathematics Education, which has had a fundamental impact on mathematics instruction and policy in the Netherlands, uses the term "learning-teaching trajectories," (Van den Heuvel-Panguizen, 2008) so the nomenclature catches up with the complexity of the concept in some places.

Organization of the Report. This report grew out of the efforts of a working group originally convened by the Center on Continuous Instructional Improvement (CCII) to review the current status of thinking about and development of the concept of learning progressions or trajectories in mathematics education. Our initial intention was to try to identify or develop a few strong examples of trajectories in key domains of learning in school mathematics, and to document the issues that we faced in doing that, particularly in terms of the kinds of warrant we could assert for the

[6] "Trajectory" as a metaphor has a ballistic connotation—something that has a target, or at least a track, and an anticipated point of impact. "Progression" is more agnostic about the end point—it just implies movement in a direction, and seems to fit a focus on something unfolding in the mind of the student, wherever it may end up, and thus it might be better reserved for use with respect to the more maturational, internal, and intuitive side of the equation of cognitive/thinking development. But it may well be too late to try to sort out such questions of nomenclature.

examples we chose. We intended to use these examples as a basis for discussion with a wider group of experts, practitioners, and policymakers about whether this idea has promise, and, if so, what else would be required to realize that promise.

As our work proceeded, it ran into, or perhaps fell into step with, the activities surrounding the initiative of the Council of Chief State School Officers (CCSSO), and the National Governors Association (NGA) to recruit most of the states, territories, and the District of Columbia to agree to develop and seriously consider adopting new national "Common Core College and Career Ready" secondary school leaving standards in mathematics and English language arts. This process then moved on to the work of mapping those standards back to what students should master at each of the grades K through 12 if they were to be on track to meeting those standards at the end of secondary school. The chair of our working group, Phil Daro, was recruited to play a lead role in the writing of the new CCSS, and subsequently in writing the related K-12 year-by-year standards. He reflects on that experience in Section V of this report.

It was clear that the concept of "mapping back" to the K-12 grades from the college and career-ready secondary standards implied some kind of progression or growth of knowledge and understanding over time, and that therefore, the work on learning trajectories ought to have something useful to say about the nature of those maps and what the important waypoints on them might be. Clearly there was a difference between the approach taken to developing learning trajectories, which begins with defining a starting point based on children's entering understandings and skills, and then working forward, as opposed to logically working backwards from a set of desired outcomes to define pathways or benchmarks. The latter approach poses a serious problem since we want to apply the new standards to all students. It is certainly possible to map backwards in a logical manner, but this may result in defining a pathway that is much too steep for many children given their entering skills, or that requires more instructional time and support than the schools are able to provide. It is also possible to work iteratively back and forth between the desired graduation target and children's varied entry points, and to try to build carefully scaffolded pathways that will help most children reach the desired target, but this probably would require multiple pathways and special attention to children who enter the system with lower levels of mathematical understanding.

Given these differences in perspective, Daro thought it would be helpful for some of the key people leading and making decisions about how to draft the CCSS for K-12 mathematics to meet with researchers who have been active in developing learning trajectories that cover significant elements of the school mathematics curriculum to discuss the implications of the latter work for the standards writing effort. Professors Jere Confrey and Alan Maloney at North Carolina State University (NCSU), who had recently joined our working group, suggested that their National Science Foundation-supported project on a learning trajectory for rational number reasoning and NCSU's Friday Institute had resources they could use to host and, with CPRE/CCII, co-sponsor a workshop that would include scholars working on trajectories along with representatives of the core standards effort. A two-day meeting was duly organized and carried out at the William and Ida Friday Institute for Educational Innovation, College of Education, at NCSU in August 2009.

That meeting was a success in that the participants who had responsibility for the development of the CCSS came away with deeper understanding of the research on trajectories or progressions and a conviction that they had great promise as a way of helping to inform the structure of the standards they were charged with producing. The downside of that success was that many of the researchers who participated in the meeting then became directly involved in working on drafts of the proposed standards which took time and attention away from the efforts of the CCII working group.

Nevertheless, we found the time needed for further deliberation, and writing, sufficient to enable us to put together this overview of the current understanding of trajectories and of the level of warrant for their use. The next section builds on work published elsewhere by Douglas Clements and Julie Sarama to offer a working definition of the concept of learning trajectories in mathematics and to reflect on the intellectual status of the concept and its usefulness for policy and practice. Section III, based in part on suggestions made by Jere Confrey and Alan Maloney and on the discussions within the working group, elaborates the implications of trajectories and progressions for the design of potentially more effective assessments and assessment practices. It is followed by a section (Section IV) written by Marge Petit that offers insights from her work on the Vermont Mathematics Partnership Ongoing Assessment Project (OGAP) about how teachers' understanding of learning trajectories can inform their practices of formative assessment and adaptive instruction. Section V, written by Phil Daro, is based on his key role in the development of the CCSS for mathematics, and reflects on the ways concepts of trajectories and progressions influenced that process and draws some implications for ways of approaching standards in general. Section VI, offers a set of recommended next steps for research and development, and for policy, based on the implications of the working group's discussions and writing. This report is supplemented by two appendices. First, Appendix A, developed by Wakasa Nagakura and Vinci Daro, provides summary descriptions of a number of efforts to describe learning trajectories in key domains of mathematics learning. Vinci Daro has written an analytic introduction to the appendix describing some of the important similarities and differences in the approaches taken to developing and describing trajectories. Her introduction has benefitted significantly from the perspectives offered by Jeffrey Barrett and Michael Battista[7], who drafted a joint paper based on comparing their differing approaches to describing the development of children's understanding of measurement, and their generalization from that comparison to a model of the ways in which approaches to trajectories might differ, while also showing some similarities and encompassing similar phenomena. Finally, to supplement the OGAP discussion in Section IV, Appendix B provides a Multiplicative Framework developed by the Vermont Mathematics Partnership Ongoing Assessment Project (OGAP) as a tool to analyze student work, to guide teacher instruction, and to engage students in self-assessment.

We hope readers will find this report a useful introduction to current work and thinking about learning trajectories for mathematics education. In this introduction to the report we have tried to show readers why we care, and they should care, about these questions, and we have tried to offer a perspective on how to think about what is being attempted that might cast some light on the varying, and sometimes confusing, ways in which the terms trajectory, progression, learning, teaching, and so on, are being used by us and our colleagues in this work.

[7] We would like to acknowledge the input of Jeffrey Barrett and Michael Battista to this report; elaborations of their contributions will be available in 2011 in a volume edited by Confrey, Maloney, and Nguyen (forthcoming).

II. WHAT ARE LEARNING TRAJECTORIES? AND WHAT ARE THEY GOOD FOR?[8]

In the Introduction we referred to our colleagues', Julie Sarama and Douglas Clements', definition of mathematics learning trajectories and tried to parse it briefly. They define trajectories as:

> descriptions of children's thinking and learning in a specific mathematical domain, and a related conjectured route through a set of instructional tasks designed to engender those mental processes or actions hypothesized to move children through a developmental progression of levels of thinking, created with the intent of supporting children's achievement of specific goals in that mathematical domain. (Clements & Sarama, 2004, p. 83)

In this section we will continue our parsing in more detail, using their definition as a frame for exam-ining the concept of a trajectory and to consider the intellectual status and the usefulness of the idea. In this we rely heavily on the much more detailed discussions provided by Clements and Sarama in their two recent books on learning trajectories in early mathematics learning and teaching, one written for researchers and one for teachers and other educators (Clements & Sarama, 2009; Sarama & Clements, 2009a), and a long article drawn from those volumes, written as background for this report and scheduled to appear in a volume edited by Confrey, Maloney, and Nguyen (in press, 2011). We will not try here to repeat their closely reasoned and well documented arguments, available in those references, but rather we will try to summarize and reflect on them, consider their implications for current policy and practice, and suggest some limitations on the practical applicability of the concept of a trajectory, limitations that may be overcome with further research, design, and development.

All conceptions of trajectories or progressions have roots in the unsurprising observation that the amount and complexity of students' knowledge and skill in any domain starts out small and, with effective instruction, becomes much larger over time, and that the amount of growth clearly varies with experience and instruction but also seems to reflect factors associated with maturation, as well as significant individual differences in abilities, dispositions, and interests. Trajectories or progressions are ways of characterizing what happens in between any given set of beginning and endpoints and, in an educational context, describe what seems to be involved in helping students get to particular desired endpoints. Clements and Sarama build their definition from Marty Simon's original coinage, in which he said that a "hypothetical learning trajectory" contains "the learning goal, the learning activities, and the thinking and learning in which the students might engage" (1995, p. 133). Their amplification makes it more explicit that trajectories that are relevant to schools and instruction are concerned with specifying instructional targets—goals or standards—that should be framed both in terms of the way knowledge and skill are defined by the school subject or discipline, in this case mathematics, and in terms of the way the students actually apply the knowledge and skills.

In their formulation there actually are two or more closely related and interacting trajectories or ordered paths aimed at reaching the goal(s):

- Teachers use an ordered set of instructional experiences and tasks that are hypothesized to "engender the mental processes or actions" that develop or progress in the desired direction (or they use curricula and instructional materials that have been designed based on the same kinds of hypotheses, and on evidence supporting those hypotheses); and
- Students' "thinking and learning… in a specific mathematical domain" go through a "developmental progression of levels" which should lead to the desired goal if the choices of instructional experiences are successful.

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[8] Based on a paper prepared by Douglas Clements and Julie Sarama. The paper is based in part upon work supported by the Institute of Education Sciences, U.S. Department of Education, through Grant No. R305K05157 to the University at Buffalo, State University of New York, D. H. Clements, J. Sarama, and J. Lee, "Scaling Up TRIAD: Teaching Early Mathematics for Understanding with Trajectories and Technologies" and by the National Science Foundation Research Grants ESI-9730804, "Building Blocks--Foundations for Mathematical Thinking, Pre-Kindergarten to Grade 2: Research-based Materials Development." Any opinions, findings, and conclusions or recommendations expressed in this publication are those of the authors and do not necessarily reflect the views of the National Science Foundation.

The goals, and the trajectory of ordered instructional experiences, reflect the hopes of the school, and the society that supports the school, but if the students are actually to learn what is hoped, attention will have to be paid to whether in practice there is the expected correspondence between the trajectory of instructional experiences and the trajectory of students' thinking. The "conjectured" or hypothesized order of experiences that should engender progressive growth in the levels of students' thinking will need to be checked against actual evidence of progress, presumably to be revised and retried if the hypotheses prove false or faulty. While the two trajectories—of thinking and learning on the one hand, and teaching on the other—are analytically distinguishable, Clements and Sarama argue that they are inextricably connected and best understood as being so. Still, their stress on the active or constructive nature of students' learning does suggest that their learning may not just reflect the order of development that the tasks and experiences are expected to engender, but that learning may develop in ways that can sometimes be surprising and even new.

Clements and Sarama fit the concept of learning trajectories within a larger theoretical framework they call "Hierarchic Interactionalism" (HI). HI is a synthesis of contemporary approaches to understanding how human beings learn and develop. It holds that cognitive development, both general and domain specific, proceeds through a hierarchical sequence of levels of concepts and understanding, in which those levels grow within domains and in interaction with each other across domains, and their growth also reflects interaction between innate competencies and dispositions and internal resources, on the one hand, and experience, including the affordances of culture as well as deliberate instruction, on the other. Clements and Sarama say that "mathematical ideas are represented intuitively, then with language, then metacognitively, with the last indicating that the child possesses an understanding of the topic and can access and operate on those understandings to do useful and appropriate mathematical work." (Clements & Sarama, 2007b, p. 464)

HI would suggest, with respect to mathematics, at least, that the developmental levels described in trajectories are probably best understood and observed within specific mathematical domains or topics, though they also are influenced by more general, cross-domain development. The levels are seen as being qualitatively distinct cognitive structures of "increasing sophistication, complexity, abstraction, power, and generality."[9] For the most part they are thought to develop gradually out of the preceding level(s) rather than being sudden reconfigurations, and that means that students often can be considered to be partially at one level while showing some of the characteristics of the next, and "placing" them in order to assign challenging, but doable work becomes a matter of making probabilistic judgments that they are more likely to perform in ways characteristic of a particular level than those of levels that come before or after it. There is some suggestion that a "critical mass" of the elements at a new level have to be developed before a student will show a relatively high probability of responding in ways characteristic of that level, but HI does not suggest that ways of thinking or operating characteristics of earlier levels are abandoned—rather students may revert to them if conditions are stressful or particularly complex, or perhaps as they "regroup" before they move to an even higher level. Making the case for considering a student to be "at" a particular level requires observation and evidence about the student's probable responses in contexts where the level is relevant.

HI distinguishes its levels from developmental "stages" of the sort described by Piaget and others. Stages are thought to characterize cognitive performance across many substantive domains, whereas HI levels are considered to be domain specific, and the movement from one level to another can occur in varying time periods, but it generally will happen over a much shorter time than movement from one stage to the next. The latter can be measured in years. HI also adopts the skepticism of many students of development about the validity and generality of the stage concept.

In HI the levels and their order are considered to have a kind of "natural" quality, in that they are considered to have their beginnings in universal human dispositions to attend to particular aspects of experience, and, at least within a particular culture, to play out in roughly similar sequences given common experiences in that culture. And, while particular representations of mathematics knowledge certainly aren't thought to be inborn, HI cites evidence of the

[9] Clements and Sarama refer to the components of these structures as being "mental actions on objects" to indicate that the mental work is on or with the concepts, representations, and manipulations within specific mathematical domains.

Illustration of a portion of a learning trajectory describing the growth of children's understanding of linear measurement:

Trajectory Level	Conceptual Structures	Instructional Tasks	Example of Instructional Task
Age 6: **End-to-End** Length Measurer **(EE):** Lays units end-to-end. May not recognize the need for equal-length units. Needs a complete set of units to span a long object.	Expects that lengths can be composed as repetitions of shorter lengths. This initially only applies to small numbers of units. The scheme is enhanced by the growing conception of length measuring as sweeping through large units coordinated with composing a length with parts (unit sticks).	1) Provide incomplete sets of linear objects to span the length of an object (see next column) 2) Use relatively large objects as units and build a ruler with pen length units. 3) Compare two objects indirectly using only shorter objects. 4) Provide the student with a contiguous set of yellow strips taped in a row to find length for comparisons. 5) Draw a ruler and mark it with ticks and numerals to match units (in or cm).	Item 1 from prior column: How long is the blue strip, compared to one of the yellow strips? Can you find out without moving any more yellow strips? 
Age 7: **Length Unit Relater and Repeater (URR):** Measures by repeated use of a unit (*initially may be imprecise as with broken ruler tasks*). Relates size and number of units explicitly, but may use units of varying lengths. Can add lengths to obtain the length of a whole. Iterates a single unit to measure. Uses rulers with minimal guidance.	Action schemes include the ability to iterate a mental unit along an object. Cardinal values are connected to space units for small quantities but weaker beyond these. With the support of a perceptual context, scheme can predict that fewer larger units will be required.	1) Given a drawing of a 5-unit segment, ask students to draw a 3-unit length line segment (Cannon, 1992). 2) Have students create units of units, such as a "footstrip". 3) Repeat measures using different-sized units and relate them. 4) Broken ruler task. 5) Measure with a covered ruler section to prevent unit counts. 6) Compare wire around tile perimeter with tile edge as units. 7) Ask students who are counting points instead of intervals (over by one each time) to draw and measure decreasing sequences of segments. [next column illustrates this]	Item 7 from prior column: If the blue strip is reported to be 4 units long by a struggling student, have them find the length of the green and yellow strips. If the student reports 3 and 2 for these measures, ask them to draw a 1 unit long segment. Or, ask them how many 2 unit yellow strips would make up a 3 unit green strip. This should prompt them to re-measure and build up the yellow as 1 unit, the green as 2 units, and the blue as 3 units. 
Age 8: **Consistent Length Measurer (CLM)**: Finds length on a bent path as the sum of its parts. Measures consistently, knowing need for identical units, partitions of unit, zero point on rulers, and accumulation of distance. May coordinate units and subunits.	Scheme includes the ability simultaneously to imagine an object's length as a total extent and a composition of units. Only allows equal-length units. Can measure from starting points other than zero on a ruler. Units themselves can be partitioned to increase precision.	1) Use a physical unit and a ruler to measure line segments and objects that require both an iteration and subdivision of the unit. 2) Build sub-units to fourths and eighths. 3) Discuss how to deal with leftover space, to count it as a whole unit or as part of a unit when units do not iterate to an integer value for length. [next column illustrates an integration of all 3 points]	Draw 4 different paths that are shorter than 5 and one half inch and longer than 5 and one quarter inch. Put the paths in order, and describe the length of each one in inches.

Sources:
Barrett, J., Clements, D., Sarama, J., Cullen, C., McCool, J., Witkowski, C., & Klanderman, D. (in press). Evaluating and Improving a Learning Trajectory for Linear Measurement in Elementary Grades 2 and 3: A Longitudinal Study. *Mathematical Thinking and Learning.*

Sarama, J., & Clements, D. H. (2009). *Early childhood mathematics education research: Learning trajectories for young children.* New York: Routledge.

importance of "initial bootstraps" for developing mathematical understanding:

- Children have important, but often inchoate, pre-mathematical and general cognitive competencies and predispositions at birth or soon thereafter that support and constrain, but do not absolutely direct, subsequent development of mathematics knowledge. Some of these have been called "experience-expectant processes" (Greenough, Black, & Wallace,. 1987), in which universal experiences lead to an interaction of inborn capabilities and environmental inputs that guide development in similar ways across cultures and individuals. They are not built-in representations or knowledge, but predispositions and pathways to guide the development of knowledge (cf. Karmiloff-Smith, 1992). Other general cognitive and meta-cognitive competencies make children—from birth—active participants in their learning and development. (Tyler & McKenzie, 1990; Clements & Sarama, 2007b, p. 465)

However, HI also recognizes that the pace at which individuals' knowledge and skill develop, and the particular sub-paths they follow from level to level—

and certainly whether they reach later levels at all—can vary considerably with variations in experiences and probably according to individual differences as well. So, HI doesn't claim that any particular progression is inevitable, but rather asserts that some will be more likely than others, and that some will be more productive than others. In addition, HI makes a strong hypothetical claim that, with respect to the organization of instruction and the design of hypothetical learning trajectories, sequences of instructional experiences and tasks that follow and exploit the more likely developmental paths will prove to be more effective and efficient in helping most students move toward desired instructional goals, and do so in ways that leave them with deeper and more flexible understanding. Clements and Sarama cite some modest encouraging evidence that the number of short-term learning paths (or alternative solution strategies) likely to be seen in typical mathematics classes should normally be small enough for teachers to handle, and many of the variants will represent earlier or later points on the same trajectory (Murata & Fuson, 2006). However, they also stress that HI would postulate that the influence of more universal and internal factors relative to variations in external experience and instruction would become less and less as students get older and the mathematics becomes more advanced, and that the range of variation due to differences in experience will certainly increase.

So, what this boils down to is that close attention to developmental progressions and to the ways that students' thinking typically responds to instructional experiences should be particularly useful in designing teaching and learning trajectories—that is, in figuring out what kinds of tasks and experiences would model and require the kinds of cognitive action that would need to come next if a student were to be supported in moving from where his or her thinking now stands to levels that would be closer to matching the goals of instruction. HI makes clear that a lot of interacting and potentially compensating factors are normally at work in a student's response to an instructional experience, so instruction at any given time may relate to multiple levels of a learning trajectory for each student. A well-designed sequence of instructional tasks will develop robust competencies over the trajectory.

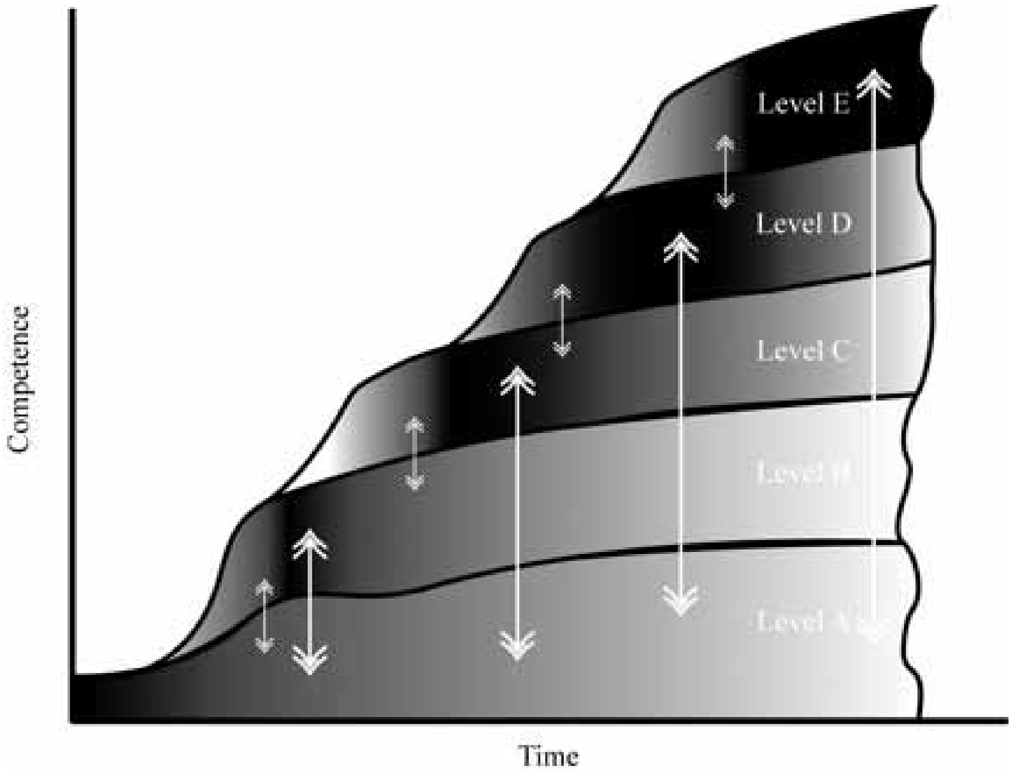
Researchers can use HI to frame an extended program of serious and iterative empirical work involving close observation of how students think as they learn mathematics, and of the particular circumstances in which they are learning, including what curriculum is being used and what the student's teacher and peers are actually doing, so that well grounded descriptions of likely teaching and learning trajectories, and their range of likely variation, can be developed. These descriptions can be used as a basis for designing even more effective trajectories and (adaptive) instructional regimes for use with other comparable populations of students.

See Illustration on page 27.

Clements and Sarama suggest that what distinguishes approaches to curriculum design based on learning trajectories and developmental progressions from other approaches, such as "scope and sequence," is not just that they order instructional experiences over time—because most past approaches have recognized the need to do that—but rather that the hypothesized order is based not only on the logic of the mathematics discipline or traditions of conventional practice but also on this close attention to evidence on students' thinking and how it actually develops in response to experience and instruction.

Whether this difference actually is significant or not depends on the rigor of the empirical work that supports the hypothetical trajectories, and curricula and instruction based on them. Elsewhere Clements and Sarama (2007c, 2008; Sarama & Clements, 2009b) have reported their own work on developing and testing learning trajectory-based instruction and curricula in early mathematics learning. Their "Building Blocks" curriculum (2007a) is supported by solid evidence, including evidence from random controlled trial experiments, that it performs significantly better than instruction based on curricula not rooted in trajectories—in the areas of early mathematics learning in understanding of number, operations, geometrical shapes, patterning, and measurement, among others. Our Appendix A lists a number of other examples of hypothesized trajectories that can offer some evidence to support the claim that they provide a basis for design of more effective instruction. While Clements and Sarama recognize that the model of development that would best fit the phenomena described by HI would probably require a complex web of interrelated progressions and contingencies, they argue that their practical work convinces them that it is useful to isolate and focus on domain- or topic-specific learning trajectories as the unit of analysis most relevant to instruction. Teachers find it difficult, and not particularly helpful, to focus on all of the factors that might be influencing their students' progress, but they seem to welcome guidance about the steps their students are likely to go through in developing their understanding of the current topic of instruction (as, for instance, multiplicative reasoning—see Section IV on OGAP).

Illustration of the theoretical account of developing competence over time, perhaps as short a timespan as 2 years, or as long as 10 years:



Source. Sarama & Clements, 2009a

NOTE: The layered figure illustrates the levels of developing competence as described by Hierarchic Interactionalism (Sarama & Clements, 2009a). The vertical axis describes conceptual and practical competence in a content domain. The horizontal axis represents developmental time. Several types of thinking develop at once, shown as various layers. Students may access them in varying ways over time. Darker shading indicates dominance of a type of thinking at some time. Students do not necessarily exhibit the most competent level of thinking they have achieved, but may fall back to simpler levels if practical. The small arrows show initial connections from one type of thinking to another, and the larger arrows show established connections, allowing for fall back or regaining a prior type of thinking.

The point of all this is that the proof is in the pudding. If it can be established that most students, at least within a particular society, within a wide range of ability, and with access to appropriate instruction, follow a similar sequence, or even a small finite range of sequences, of levels of learning of key concepts and skills, then it should be possible not only to devise instructional sequences to guide students in the desired directions, but it should also be possible to develop standards and expectations for students' performance that are referenced to those sequences; so that the standards, and derived assessments, report in terms that have educational meaning and relevance. The following sections suggest some of these implications, particularly for assessments and standards, but also for adaptive instruction.

III. TRAJECTORIES AND ASSESSMENT

In CPRE's report on Learning Progressions in Science (Corcoran, Mosher, & Rogat, 2009), we argued that one of the benefits of developing and testing progressions—well warranted hypotheses about the pathways students' learning of the core concepts and practices of science disciplines are likely to develop over time, given appropriate instruction—would be that the levels of learning identified in those progressions could serve as reference points for assessments designed to report where students are along the way to meeting the goals of instruction and perhaps something about the problems they might be having in moving ahead. Clearly, the related ideas about learning and teaching trajectories in mathematics hold out the same promise of providing a better grounding for designing assessments that can report in educationally meaningful terms.

What we are suggesting, however, is easier said than done. But we are not alone in suggesting it. The National Research Council's (NRC) 2001 report on the foundations of assessment, *Knowing what Students Know* (Pellegrino, Chudowsky, & Glaser, 2001), describes educational assessment as a triangular (and cyclical) process that ideally should relate:

- Scientifically grounded conceptions of the nature of children's and students' thinking, understanding, and skills, and how they develop; to
- The kinds of observations of students' and children's behavior and performance that might reflect where they are in the development of their thinking and understanding, and ability to use that knowledge; and to
- The kinds of reasoning from, or interpretation of, those observations that would support inferences about just where children and students were in the development of their thinking, understanding, and skill.

The vertices of the NRC report's assessment triangle were named cognition, observation, and interpretation.

What the NRC panel labeled 'cognition' involves a contemporary understanding of the ways in which sophisticated expertise in any field develops, with instruction and practice, out of earlier naïve conceptions. And they suggest that such expertise involves the development of coherent cognitive structures that organize understanding of a field in ways that make knowledge useful and go well beyond simple accumulation of facts or skills. In their view, the role of assessment should be to support inferences about the levels of these structures (they call them "schemas") that students have reached, along with the particular content they have learned and particular problems they might be having. That view seems to us to be completely consistent with our view of the role that learning progressions or trajectories should play (and at a number of points *Knowing what Students Know* in fact uses the term progressions to describe the content of the cognition vertex of their assessment triangle). Both their view and ours leave open to empirical investigation the question of how such progressions, or levels, should be further specified.

It is in this empirical work that the "easier said than done" aspect of these ideas comes into play. *Knowing what Students Know* makes it clear that assessment items or occasions to observe students' behavior should be derived from, and designed to reflect, the hypothesized cognitive model of students' learning, and then the results obtained when students perform the assessment tasks, or when their behavior is observed, should be subjected to rational examination and the application of statistical models to see whether the patterns of students' performance on the various tasks and observations look to be consistent with what one would expect if the cognitive theory is true and the items are related to it in the ways that one hoped. Mismatches should not in themselves invalidate the assessment or the related theory, but they do represent a challenge to move back through the chain of reasoning that was supposed to relate the assessment results to the underlying theory to see where in that chain the reasoning might have gone wrong. *Knowing what Students Know* provides a clear presentation of the case for this kind of evidence-based assessment design and then goes on to describe the considerations that go into the design of items and occasions for observation; so that they have a good chance of reflecting the ways knowledge and behavior are expected to grow based on cognitive theories and research; and so that the chances they also are reflecting unrelated factors and influences are reduced. Then in Chapter 4 (pp.111-172), authors Pellegrino, Chudowsky, and Glaser present a very useful overview of new approaches to psychometric and statistical modeling that can be used to test whether an assessment's items and observations behave in a way that would be predicted if the underlying theory of learning were true, and that also can frame the ways the results are reported and indicate the levels of confidence one should have in them.

As we have surveyed the work going on along these lines, we have concluded that these approaches are still pretty much in their infancy in terms of practical use. The bulk of large- and medium-scale assessment in this country is rooted in older psychometric models, or updated versions of them, which assume that the underlying trait that is the target of assessment arrays both students and assessment items along a single underlying dimension (such things as "mathematical ability," or "reading comprehension"). These models characterize a student's ability or skill with reference to his or her peers—to where they stand in the distribution of all students' performances (hence "norm-referenced")—and stress the ability of the assessment and its component items to distinguish or "discriminate" among students. The items in the assessment are written to be *about* the content of the school subject and to fit into a framework defining the elements of the content to be covered, but the fundamental characteristics that determine whether items get included in the assessment or not have as much or more to do with whether they "work" to discriminate among students and behave as though they are reflecting a single underlying dimension. Such assessments and the scales based on them (given assumptions about the nature of the underlying student performance distributions, the scale scores often are claimed to have "equal interval" properties—presumably useful for comparing such things as relative gains or losses for students at different locations on the scale) tend not to provide a lot of specific information about what students know and can do.[10] Nevertheless, in current practice the items that students who have particular scale scores tend to get right compared to students who are below them, and tend to fail compared to students who are above them, can be examined after the fact to try to infer something about what the scores at particular points on the scale imply about what students at those levels seem to know. It is these after the fact inferences, and then judgments based on what those inferences seem to describe, that are used to select the scale scores that are said to represent such things as "below basic, basic, proficient, and advanced" levels of performance on NAEP and on state assessments used for NCLB and accountability purposes. As teachers have found through hard experience, these scores and associated inferences are not of much help in designing instructional interventions to help students stay on track and continue to progress. This is one of the reasons that our various attempts at "data driven improvement" so often come up short.

Assessments designed in this way are not capable of reflecting more complex conceptions of the ways students' learning progresses, and at best they provide very crude feedback to teachers or to the system about what students actually are learning and what they can do. We don't need to look very far beyond the recent experience in New York in which the State Board of Regents asked a panel of experts to review the difficulty of the state's assessments of mathematics and English language arts and then responded to their report—that the assessments and performance standards had become too easy—by increasing the scale score levels on the assessments that would be considered to represent attainment of proficiency. That decision essentially wiped out much of the perceived performance gains and "gap-closing" touted by the current administration of the New York City Schools as the result of their tenure in office and has generated controversy about the effects of the city's reforms (Kemple, 2010). The real story behind this controversy is the essential arbitrariness of the assessment cut scores and the inability to offer any independent evidence about what students at any score level actually know or can do (or even evidence that changes in those scores are actually associated with changes in what they otherwise might be observed to know and do). It is dismaying that quite a bit of the commentary on this event seems to treat the increase in the percentages of students in various groups who now fall below proficiency as an indication that their actual capabilities have declined, rather than as just a necessary consequence of raising the score required for a student to be considered proficient, but that bit of ignorance really just reflects the degree of mystification that has been allowed to evolve around the design and meaning of state and national assessments.

10 The focus on reliability and on measuring an underlying dimension or trait, and selecting for use-only items that fit well with trait/dimensional assumptions, can mean that these assessments really mainly end up measuring something quite different from the specific things students know and can do, and their progress in learning such things. Rather, they may measure students' relative position on a scale of subject-specific aptitude and/or general aptitude (or I.Q.) and/or social class and family opportunity—things that make them fairly effective in predicting students' ability to learn new things but which give little specific information about what they have actually learned (and certainly not reliable information about the specifics). To be sure, because of ecological correlations, students who are high or low on these underlying traits, even when they have similar in-school exposure, are likely to have learned respectively more or less of the specific material, but the assessments will not give precise reports of the specifics, and the students' relative positions on the scales are not likely to change much even if they do in fact really learn quite a bit of the specifics—among other things because the assessments are often also designed to be curriculum-independent or -neutral.

The alternative, of course, is to design assessments so that they discriminate among, and report in terms of differences in, the levels or specific stages of knowledge and skill attained in particular school subjects; based on tested theories about how those subjects are learned by most students, as we and *Knowing what Students Know* (Pellegrino, Chudowsky, & Glaser, 2001) argue. One of the big questions here is whether one should think of the growth of student learning as being an essentially continuous process, albeit a multi-dimensional one, or whether it is more fruitful to conceive of it as looking like a series of relatively discrete, and at least temporarily stable, steps or cognitive structures that can be described and made the referents of assessment (even if the processes that go on in between as students move from one step to the next might actually have a more continuous, and certainly a probabilistic, character). Chapter 4 in *Knowing what Students Know* provides a helpful overview of the kinds of psychometric and statistical models that have been developed to reflect these different views of the underlying reality, and many of the issues involved in their use. To oversimplify, there are choices between "latent variable" and multivariable models, on the one hand, and latent class models on the other. "Latent" simply refers to the fact that the variables or classes represent hypotheses about what is going on and can't be observed directly. There are of course mixed cases. Rupp, Templin, and Henson (2010) provide a good treatment of the alternative models and relevant issues associated with what they call "Diagnostic Classification Models."

Some of the continuous models use psychometric assumptions similar to ones used in current assessments but focus more on discriminating among items than among students, and stress a more rigorous approach to item design to enhance the educational relevance and interpretability of the results, while allowing for increased complexity by assuming that there can be multiple underlying dimensions involved, even if each of them on its own has a linear character (see Wilson, 2005 for examples). The latent class models are in some ways even more exotic. Among the more interesting are those that rely on Bayesian inference and Bayesian networks (West et al., 2010) since those seem in principle to be able to model, and help to clarify, indefinitely complex ideas about the number of factors that might be involved in the growth of students' knowledge and skill. But for policymakers these models are more complex and even more obscure than more conventional psychometric models, and developing and implementing assessments based on them is likely to be more expensive. The relative promise and usefulness of the alternative models needs to be sorted out by use in practical settings, and it seems unlikely that there will be a significant shift toward the use of assessments designed in these ways until there have been some clear practical demonstrations that such assessments provide much better information for guiding practice and policy than current assessments are able to do.

In mathematics, a few investigators are developing assessments that reflect what we know or can hypothesize about students' learning trajectories. For example, our colleagues Jere Confrey and Alan Maloney at NCSU are working on assessments that reflect their conception of a learning trajectory for "equipartitioning" as part of the development of rational number reasoning (Confrey & Maloney in press, 2010; Maloney & Confrey 2010). They began with an extensive synthesis of the existing literature and supplemented it by conducting cross sectional clinical interviews and design studies to identify key levels of understanding along the trajectory. From these open-ended observations they developed a variety of assessment tasks designed to reflect the hypothesized levels. Students' performances on the tasks are being subjected to examination using Item Response Theory (IRT) models to see if the item difficulties and the results of alternative item selection procedures produce assessments that behave in the ways that would be predicted if the items in fact reflect the hypothesized trajectory and if that trajectory is a reasonable reflection of the ways students' understanding develops. They are working with Andre Rupp, a psychometrician at the University of Maryland, in carrying out this iterative approach that over time tests both the choices of items and the hypothesized trajectory. Finding lack of fit leads to further design, and the project has been open to the use of multiple models to see which of them seem to offer the most useful ways to represent the data. The work on this project is ongoing. Across the country, other researchers and assessment experts are working on the development of similar assessment tools.

A major development on the national horizon that may result in much more effort and resources being devoted to solving the problems of developing usable assessments based on more complex conceptions of how students actually learn, and produce results that can be more legitimately interpreted in terms of what students actually know and can do, is the result of the competition the U.S. Department of Education ran that will provide support to two consortia of states to develop assessments that can measure students' attainment of, and progress toward meeting, the new

Common Core State Standards (CCSS) for mathematics and English language arts. The consortia's proposals suggest that they will seek to develop measures that will report in terms of much more complex conceptions of student learning (not just facts and concrete skills, but understanding, and ability to use knowledge and to apply it in new situations, and so on) and also to determine whether students are "on track" over the earlier grades to be able to meet the "college-and career-ready" core standards by sometime during their high school years. The proposals vary in how clearly they recognize how much change in current methods will be required to reach these goals, and how long it may take to do it, but there is agreement about the importance of the task as well as its scope. The federal resources being made available should at least ensure that quite a bit of useful development and experimentation will be done—perhaps enough to set the practice of assessment design on a new path over the next few years.

With all this discussion of new and seemingly exotic psychometric models, however, we think there is something else to be kept in mind. In terms of everyday instruction, the application of latent variable or latent class models to the production of valid and reliable assessments that teachers might use to monitor student understanding is a bit like using a cannon to hunt ants. Adaptive instruction, as we have argued, involves systematic and continuous use of formative assessment, i.e. teachers' (and in many cases students themselves') reasoning from evidence in what they see in students' work, and their knowledge of what that implies about where the students are and what they might need to overcome obstacles or move to the next step, to respond appropriately and constructively to keep the process moving. That doesn't necessarily require the use of formal assessment tools, since well prepared teachers should know how to interpret the informal and ongoing flow of information generated by their students' interactions with classroom activities and the curriculum. That evidence doesn't have to meet the kinds of rigorous tests of reliability or validity that should be applied to high stakes and externally supplied assessments, because the teachers have the opportunity in the midst of instruction to test their interpretations by acting on them and seeing whether or not they get the expected response from the students—and by acting again if they don't. Also, if they are uncertain about the implications of what they see, they have the option simply of asking their student(s) to elaborate or explain, or of trying something else to gather additional evidence.[11] In the next section, our colleague Marge Petit provides a concrete example of what this process can look like in practice when it works well.

So we would argue that, while it is extremely important to apply the new approaches we have described briefly here to the design of much better large-scale assessments whose reports would be more informative because they are based on sound theories about how students' learning progresses, it also will be crucial to continue to focus on developing teachers' clinical understanding of students' learning in ways that can inform their interpretations of, and responses to, student progress and their implementation of the curricula they use. Teachers of course operate day to day on a different grain size of progress from the levels that large-scale assessments used for summative assessments are likely to target. The latter will tend to reference bigger intervals or significant stages of progress to inform policy and the larger system, as well as to inform more consequential decisions about students, teachers, and schools. Nevertheless, it would be crucial for there to be a correspondence between the conceptions of student progress teachers use in their classrooms and the conceptions that underlie the designs of large-scale assessments. The larger picture informing the assessment designs would help teachers to put their efforts in a context of where their students have been before and where they are heading.[12]

[11] Some scholars argue that another option to having research on learning trajectories directly influence practice through teacher knowledge is to develop diagnostic assessments that can be used more formally to support and enhance formative assessment practices (Confrey & Maloney, in press, 2010). In the latter work, the authors seek a means to develop measures and ways of documenting students' trajectories to track students' progress both quantitatively and qualitatively. A conference "Designing Technology-enabled Diagnostic Assessments for K-12 Mathematics," held November 16-17, 2010 at the Friday Institute, explored these ideas further (report is forthcoming). Some participants in the conference argued that such assessments certainly could be useful, but stressed their conviction that effective formative use would still require teachers to understand the research on mathematics learning that supports the conceptions of students' progress that provides the basis for the assessment designs, and also to know the evidence concerning the kinds of pedagogical responses that would help the students given what the assessments might indicate about their progress or problems. These perspectives represent a healthy tension, or at least a difference in emphasis, among researchers working on trajectories and formative and diagnostic assessment.

[12] Barrett, Clements, & Sarama are using clinical teaching cycles of assessment and instruction to check for the correspondence between claims about student progress and the cognitive schema collections that are used to describe children's thinking and ways of developing, or to design the large-scale assessments. This is being documented as a longitudinal account of eight students across a four-year span, at two different spans: Pre-K to Grade 2, and the other span from Grade 2 up through Grade 5 (Barrett, Clements, Cullen, McCool, Witkowski, & Klanderman, 2009).

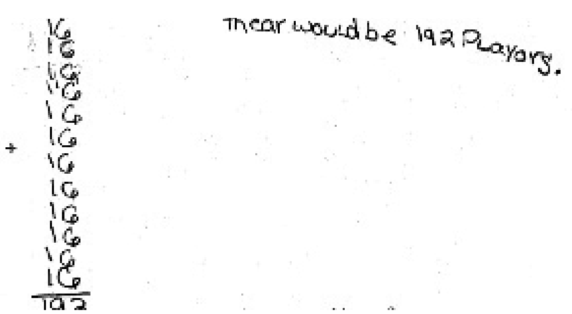
In addition, it should be helpful and reassuring to teachers if the assessments that others use to see how they and their students are doing are designed in ways that are consistent with the understandings of students' progress they are using in the classroom, so that they can have some confidence that there will be agreement between the progress they observe and progress, or lack of it, reported by these external assessments. Also, it would of course be desirable if those external reports were based on models that provide real assurance that the reports are valid and can be relied on.

IV. LEARNING TRAJECTORIES AND ADAPTIVE INSTRUCTION MEET THE REALITIES OF PRACTICE[13]

Imagine a 5th-grade teacher is analyzing evidence from student work on a whole number multiplication and division pre-assessment. The pre-assessment consisted of a mix of word problems from a range of contexts and some straight computation problems. She notices one student correctly answered 80% of the problems, but solved the problems using repeated addition or repeated subtraction (Example 1 below). In the past, the teacher might have been pleased that the student had 80% correct. However, she now knows that the use of repeated addition (subtraction) by a 5th-grade student is a long way from that student's attaining an efficient and generalizable multiplicative strategy such as the traditional algorithm (CCSSO/NGA, 2010). She also knows that this student is not ready to successfully engage in the use of new 5th-and 6th-grade concepts like multiplication of decimals (e.g., 2.5 x 0.78), or solving problems involving proportionality, which relies on strong multiplicative reasoning.

Example 1: Use of Repeated Addition (VMP OGAP, 2007)

> There are 16 players on a team in the Smithville Soccer League. How many players are in the league if there are 12 teams?



The teacher observes and records other evidence about the strategies or properties that her students have used to solve the problems (e.g., counting by ones, skip counting, area models, distributive property, the partial products algorithm, and the traditional algorithm); the multiplicative contexts that have caused her students difficulty (e.g., equal groups, multiplicative change, multiplicative comparisons, or measurement); and the types of errors that the students have made (e.g., place value, units, calculation, or equations). She will use this evidence to inform her instruction for the class as a whole, for individual students, and to identify students who could benefit with additional Response to Intervention (RTI) Tier II instruction—a school-wide data-driven system used to identify and support students at academic risk.[14]

This teacher and others like her who have participated in the Vermont Mathematics Partnership Ongoing Assessment Project (VMP OGAP) have used the OGAP Multiplicative Framework (See Appendix B) to analyze student work as briefly described above, to guide their instruction, and engage their students in self-assessment. In addition to administering pre-assessments, they administer formative assessment probes as their unit of instruction progresses. They use the OGAP Framework to identify where along the hypothesized trajectory (non-multiplicative – early additive – transitional – multiplicative) students are at any given time and in any given context, and to identify errors students make.

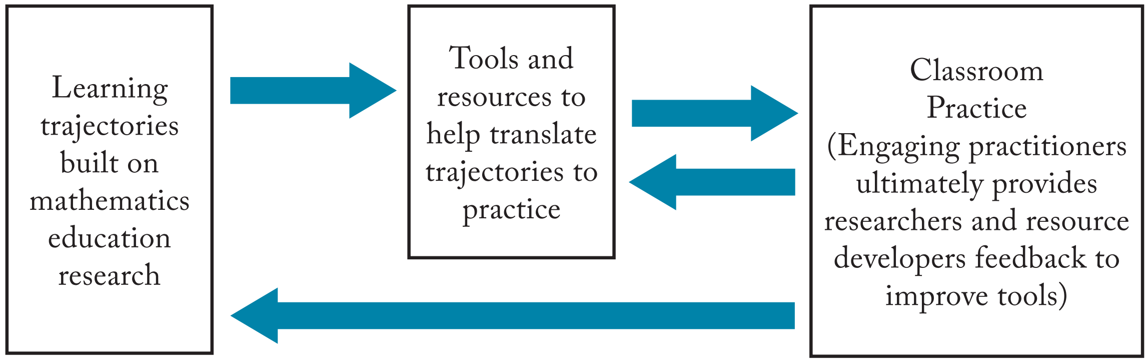
It is one thing to talk theoretically about learning trajectories and a whole other thing to understand how to transfer the knowledge from learning trajectory research to practice in a way that teachers can embrace it (see Figure 1 below). The latter involves designing tools and resources that serve as ways for classroom teachers to apply the trajectory in their instruction.

[13] Written by Marge Petit, educational consultant focusing on mathematics instruction and assessment. Petit's primary work is supporting the development and implementation of the Vermont Mathematics Partnership Ongoing Assessment Project (OGAP) formative assessment project.

[14] There are different levels of intervention. RTI Tier II provides students at academic risk focused instruction in addition to their regular classroom instruction. (http://www.rti4success.org/)

IV. LEARNING TRAJECTORIES AND ADAPTIVE INSTRUCTION MEET THE REALITIES OF PRACTICE

Figure 1. Transfer of Knowledge from Learning Trajectory Research into Classroom Practice



An example of a project that is developing tools and resources that bridge the gap between research and practice is the Vermont Mathematics Partnership Ongoing Assessment Project (OGAP), developed as one aspect of the Vermont Mathematics Partnership (VMP).[15] In 2003, a team of 18 Vermont mathematics educators (classroom teachers, school and district mathematics teacher leaders, an assessment specialist, and a mathematician) were charged with designing tools and resources for teachers to use to gather information about students' learning while they are learning, rather than just after their learning, for the sole purpose of informing instruction. Guided by findings of the NRC's expert panels (Pellegrino, Chudowsky, & Glaser, 2001; Kilpatrick, Swafford, & Findell, 2001), the design team adopted four principles that have guided their work through three studies (VMP OGAP, 2003, 2005, and 2007) involving over 100 teachers and thousands of students: 1) teach and assess for understanding (Kilpatrick, Swafford, & Findell, 2001; 2) use formative assessment intentionally and systematically (Pellegrino, Chudowsky, & Glaser, 2001; 3) build instruction on preexisting knowledge (Bransford, Brown, & Cocking, 2000); and, 4) build assessments on knowledge of how students learn concepts (Pellegrino, Chudowsky, & Glaser, 2001). Incorporating these elements into the tools and resources being developed provided a structure for helping OGAP teachers to engage in adaptive instruction as defined in the introduction to this report.

The fourth principle, build assessments on how students learn concepts, led, over time, to the development of item banks with hundreds of short, focused questions designed to elicit developing understandings, common errors, and preconceptions or misconceptions that may interfere with solving problems or learning new concepts. These questions can be embedded in instruction and used to gather evidence to inform instruction. Importantly, the OGAP design team developed tools and strategies for collecting evidence in student work. One of these tools is the OGAP Frameworks; for multiplication, division, proportionality, and fractions. Teachers use the frameworks to analyze student work and adapt instruction (See, for example, the OGAP Multiplicative Framework in Appendix B). Each OGAP Framework was designed to engage teachers and students in adaptive instruction and learning. Teachers studied the mathematics education research underlying the OGAP Frameworks, and put what they learned into practice. The OGAP Frameworks have three elements: 1) analysis of the structures of problems that influence how students solve them, 2) specification of a trajectory that describes how students develop understanding of concepts over time, and 3) identification of common errors and preconceptions or misconceptions that may interfere with students' understanding new concepts or solving problems.

From a policy perspective, an important finding from the Exploratory OGAP studies and the OGAP scale-up studies in Vermont and Alabama is that teachers reported that knowledge of mathematics education research and ultimately the OGAP Frameworks/trajectories helped them in a number of important ways. They reported that they are better able to understand evidence in student work, use the evidence to inform instruction, strengthen their first-wave instruction, and understand the purpose of the

[15] The Vermont Mathematics Partnership was funded by NSF (EHR-0227057) and the USDOE (S366A020002).

activities in the mathematics programs they use and in other instructional materials (VMP OGAP, 2005, 2007 cited in Petit, Laird, & Marsden, 2010).

The OGAP 2005 and 2007 studies present promising evidence that classroom teachers, when provided with the necessary knowledge, tools, and resources, will readily engage in adaptive instruction. However, other findings from the OGAP studies provide evidence that developing tools and providing the professional development and ongoing support necessary to make adaptive instruction a reality on a large scale will involve a considerable investment and many challenges.

To understand the challenges encountered in implementing adaptive instruction better we return to the teacher who observed a 5th-grade student using repeated addition as the primary strategy to solve multiplication problems. This teacher has made a major, but difficult transition from summative thinking to formative/adaptive thinking. She understands that looking at just the correctness of an answer may provide a "false positive" in regards to a 5th-grade student's multiplicative reasoning. She notices on the OGAP Multiplicative Framework that repeated addition is a beginning stage of development and that 5th-grade students should be using efficient and generalizable strategies like partial products or the traditional algorithm. On a large-scale assessment one cares if the answer is right or wrong. On the other hand, from a formative assessment/adaptive instruction lens, correctness is just one piece of information that is needed. A teacher also needs to know the strategies students are using, where they are on a learning trajectory in regards to where they should be, and the specifics about what errors they are making on which mathematics concepts or skills. This is the information that will help teachers adapt their instruction.

This transition from summative to formative/adaptive instruction was a major challenge for OGAP teachers who were well conditioned to administrating summative assessments ranging from classroom quizzes and tests to state assessments, all of which have very strict administration procedures. In formative assessment/adaptive instruction thinking your sole goal is to gather actionable information to inform instruction and student learning, not to grade or evaluate achievement. That means if the evidence on student work isn't clear—you can ask the student for clarification or ask the student another probing question.

OGAP studies showed that once a teacher became comfortable with looking at student work (e.g., classroom discussions, exit questions, class work, and homework) through this lens, their next question was—"Now that we know, what do we do about it?" As a case in point, one of the best documented fraction misconceptions is the treatment of a fraction as two whole numbers rather than as a quantity unto itself (Behr, Wachsmuth, Post, & Lesh, 1984; VMP OGAP, 2005, 2007; Petit, Laird, & Marsden, 2010; Saxe, Shaughnessy, Shannon, Langer-Osuna, Chinn, & Gearhart, 2007). This error results in students adding numerators and denominators when adding fractions, or comparing fractions by focusing on the numerators or denominators or on the differences between them. Example 2 below from a 5th-grade classroom is particularly troubling, and very informative. In the words of one teacher, "In the past I would have been excited that a beginning 5th-grade student could add fractions using a common denominator. I would have thought my work was done. It never occurred to me to ask the student the value of the sum." (VMP OGAP, 2005). When faced with evidence such as found in Example 2, OGAP teachers made the decision to place a greater instructional emphasis on the magnitude of fractions and the use of number lines, not as individual lessons as they found them in their text materials, but as a daily part of their instruction.

Example 2: Inappropriate Whole Number Reasoning Example

Added sums accurately and then used the magnitude of the denominator or numerator to determine that is closest to 20. (Petit, Laird, & Marsden, 2010)

The sum of $\frac{1}{12}$ and $\frac{7}{8}$ is closest to

A. 20
B. 8
C. ½
D. 1

Explain your answer.

$\frac{1}{12}+\frac{7}{8}=\frac{2}{24}+\frac{21}{24}=\frac{23}{24}$ is closest to 20.

This action is supported by mathematics education research that suggests that number lines can help to build understanding of the magnitude of fractions and build concepts of equivalence (Behr & Post, 1992; Saxe, Shaughnessy, Shannon, Langer-Osama, Chinn, & Gearhart, 2007; VMP OGAP, 2005 and 2007). Research also suggests the importance of focusing on the magnitude of fractions as students begin to operate with fractions (Bezuk & Bieck, 1993, p.127; VMP OGAP, 2005, 2007 cited in Petit, Laird, & Marsden, 2010).

This example has other implications for making adaptive instruction a reality in mathematics classrooms. Resources, like OGAP probes and frameworks, must be developed that are sensitive to the research. Teachers must receive extensive training in mathematics education research on the mathematics concepts that they teach so that they can better understand the evidence in student work (from OGAP-like probes or their mathematics program) and its implications for instruction. They need training and ongoing support to help capitalize on their mathematics program's materials, or supplement them as evidence suggests and help make research-based instructional decisions.

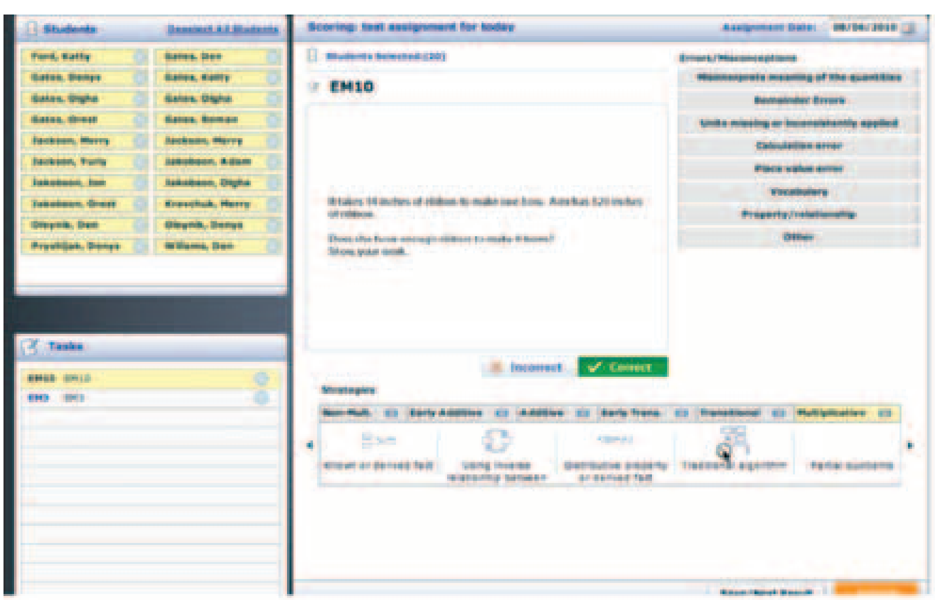
> I realized how valuable a well designed, research-based probe can be in finding evidence of students' understanding. Also, how this awareness of children's thinking helped me decide what they (students) knew versus what I thought they knew. (VMP OGAP, 2005 cited in Petit and Zawojewski, 2010, p. 73)

In addition, while it is true that formative assessment provides teachers the flexibility "to test their interpretations by acting on them and seeing whether or not they get the expected response from the students—and acting again if they don't" (see Section III of this report), OGAP studies show that teachers who understand the evidence in student work from a research perspective are looking for research-based interventions. Drawing on my own experience as a middle school teacher in the early 1990s when I was faced with students adding numerators and denominators (e.g., $\frac{3}{4} + \frac{7}{8} = \frac{10}{12}$), I would re-teach common denominators "louder and slower," never realizing that the problem was students' misunderstanding magnitude or that students did not have a mental model for addition of fractions as suggested in the research.

While there is research on actions to take based on evidence in student work, much more needs to be done if the potential of adaptive instruction is to be realized. Research resources need to be focused not only on validating trajectories as a research exercise, but on providing teachers with research-based instructional intervention choices.

OGAP teachers are now recording on paper a wealth of information on student learning as described earlier in this chapter. To help facilitate this process, OGAP is working closely with CPRE researchers from the University of Pennsylvania and Teachers College, Columbia University, and with the education technology company, Wireless Generation, in developing a technology-based data entry and reporting tool grounded on the OGAP Multiplicative Framework. The tool will be piloted in a small Vermont-based study during the 2010-2011 school year. It is designed to make the item bank easily accessible; it provides a data collection device based on the OGAP Multiplicative Framework linked to item selection (See Figure 2). The tool is designed

Figure 2: Draft Evidence Collection Tool that Uses Touch Screen Technology.



to provide reports that show where on the trajectory (OGAP Framework) each student is at any given time, with any given problem structure, and across time. It also provides results about accuracy and errors, and misconceptions by students and by the class.

Students' performances with respect to learning trajectories, like those in the OGAP Frameworks, do not simply increase monotonically. Rather students move back and forth along the trajectory as they interact with new contexts or more complex numbers until they have fully developed their multiplicative reasoning (VMP OGAP 2005, 2007; Clements, & Sarama, 2009). Development of tools, like the Wireless Generation tool being piloted, will need to account for this movement if they are to represent learning trajectories in a meaningful way.

A very important point here is that OGAP and the Wireless tool being developed is NOT taking the teacher out of the equation as some multiple choice-based diagnostic assessments are purporting to do, to make it easier for a teacher. Rather, the project has recognized the importance of empowering the teacher with knowledge of the research that they use when analyzing student work and making instructional decisions. These are the cornerstones of adaptive instruction. Our hypothesis continues to be that it is the knowledge of the mathematics education research that empowers teachers, not just the data from the results of assessments.

From a policy perspective, to accomplish implementation of adaptive instruction on a large scale our work has shown the importance of capitalizing on existing resources and strategies. In Vermont, this meant working with mathematics teacher leaders who were graduates of a three-year masters program in mathematics (the Vermont Mathematics Initiative). OGAP professional development was provided directly to teacher leaders in two stages. The first stage focused on teacher leader knowledge, and the second phase provided the teacher leaders with support as they worked with other teachers in their district.

Small pilots in Alabama have led to a decision by the Alabama Department of Education to make OGAP a major intervention strategy. Next June AMSTI (Alabama Mathematics and Science Teachers Initiative) leaders from across Alabama will receive OGAP training and support as they begin to engage Alabama teachers state wide. They recognize this is a multi-year effort, but they are setting the stage for it to begin.

Other district and state policies that value the use of formative assessment and adaptive instruction need to be put into place if these strategies are to be used at all by teachers. State standards that favor breadth over depth, or are not built on mathematics education research or districts' use of unrealistic pacing guides linked to quarterly assessments will all serve as formidable barriers to the use of formative assessment and adaptive instruction.

Our work indicates that it is possible to engage teachers in adaptive instruction and to use learning trajectories as described above, but it will take a commitment by policymakers, material developers, mathematics education researchers, and educators at all levels to accomplish the goal.

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> One sees the difficulty with this standards business. If they are taken too literally, they don't go far enough, unless you make them incredibly detailed. You might give a discussion of a couple of examples, to suggest how the standards should be interpreted in spirit rather than by the letter. But of course, this is a slippery slope.
>
> Roger Howe, Yale, March 15, 2010 input to Common Core State Standards

> … the "sequence of topics and performances" that is outlined in a body of mathematics standards must also respect what is known about how students learn. As Confrey (2007) points out, developing "sequenced obstacles and challenges for students…absent the insights about meaning that derive from careful study of learning, would be unfortunate and unwise." In recognition of this, the development of these Standards began with research-based learning progressions detailing what is known today about how students' mathematical knowledge, skill, and understanding develop over time.
>
> Common Core State Standards, 2010, p.4

Sequence, Coherence, and Focus in Standards

Standards, perforce, sequence as well as express priority. On what basis? By design, one hopes. I was a member of the small writing team for the Common Core State Standards (CCSS). As such, I was part of the design, deliberation and decision processes, including especially reviewing and making sense of diverse input, solicited and unsolicited. Among the solicited input were synthesized 'progressions' from learning progressions and learning trajectory researchers, and sequences proposed by mathematicians.

This section will look at the general issues of sequence, focus, and coherence in mathematics standards from the perspective of the CCSS for Mathematics.

Cognitive Development, Mathematical Coherence, and Pedagogic Pragmatics

Standards are pulled in three directions…cognitive development, mathematical coherence, and the pragmatics of instructional systems.

Decisions about sequence in standards must balance the pull of three important dimensions of progression: cognitive development, mathematical coherence, and the pragmatics of instructional systems. The situation differs for elementary, middle, and high school grades. In brief: elementary standards can be more determined by research in cognitive development, and high school more by the logical development of mathematics. Middle grades must bridge the two, by no means a trivial span.

Standards sequence for grade levels; that is, the granularity of the sequence is year-sized. Standards do not explicitly sequence within grade level, although they are presented in an order that makes some sense for this purpose.

Standards as a Design Project Informed by Evidence

The CCSS writing team had the unusual experience of working on the standards as a design project rather than as a political project. The charge was: design a good tool for improving mathematics achievement. Base it on evidence. Extensive input was organized from individuals, organizations and, especially, states. The states themselves organized input processes. Comprehending the variety of input and making use of it did, in fact, have several dimensions: finding good design suggestions at many levels, improvements in communication, and uncovering disagreements. Disagreements led the writers to balance clarity of focus, internal coherence and practical choices about the underlying issues that gave rise to the disagreements. Sometimes these choices felt political. Perhaps some were. Nonetheless, if one thinks of the CCSS as the quotient of the input that "goes into" a draft, politics was at most a remainder.

[16] Written by Philip Daro, Senior Fellow for Mathematics of America's Choice and Director, San Francisco Strategic Education Research Partnership (SERP), a partnership of UC Berkeley, Stanford, and the San Francisco Unified School District.

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Crucial to the design, and much more important than politics, was the evidence from learning trajectory research in the sequencing and content of the standards. The examples that follow show how learning trajectory research, mathematical coherence, and the constraints of instructional practice informed the design of the CCSS. Beyond their direct contribution, working with learning trajectories engendered a way of thinking about sequence and coherence that synthesized mathematical development with human development and learning. This way of thinking extended beyond the specifics of the research and was very fruitful.

The CCSS incorporate a progression for learning the arithmetic of the base ten number system from K through Grade 5. A mathematically coherent development is difficult because the simple ideas on which the base ten system is based are too advanced for these grades: sums of terms that are products of a single-digit number and a power of ten, including rational exponents for decimal fractions. Pragmatic design choices had to be made.

The CCSS for Grade 1 ask students to:

2. Understand that the two digits of a two-digit number represent amounts of tens and ones. Understand the following as special cases:

    a. 10 can be thought of as a bundle of ten ones—called a "ten."

    b. The numbers from 11 to 19 are composed of a ten and one, two, three, four, five, six, seven, eight, or nine ones.

(CCSS, 2010)

This approach takes advantage of what the researchers call "unitizing" (Glasersfeld 1995; Steffe & Cobb, 1988) by bundling ten ones into a ten to enable counting the tens and later adding and subtracting tens (as units).

The relative weight to give cognitive development vs. mathematical coherence gets more tangled with multiplication, the number line, and especially fractions. Understanding the arithmetic of fractions draws upon four prior progressions that informed the CCSS: equipartitioning, unitizing, number line, and operations.

The first two progressions, equipartitioning and unitizing, draw heavily from learning trajectory research. Confrey has established how children develop ideas of equipartitioning from early experiences with fair sharing and distributing. These developments have a life of their own apart from developing counting and adding. Clements and also Steffe have established the importance of children being able to see a group(s) of objects or an abstraction like 'tens' as a unit(s) that can be counted. Whatever can be counted can be added, and from there knowledge and expertise in whole number arithmetic can be applied to newly unitized objects; like counting tens in base 10, or adding standard lengths such as inches in measurement. The progression begins before school age with counting concrete objects and progresses up through the grades to counting groups of objects, groups of tens, units of measurement, unit fractions and onward, as illustrated in Table 1 below.

Table 1. Development of Equal Partitioning and Unitizing

Objects	3 objects + 5 objects = 8 objects
Pure numbers	3 ones + 5 ones = 8 ones
Groups of objects	3 groups of 10 objects + 5 groups of 10 objects = 8 groups of 10 objects = 80
Groups of 10 ones are tens	3 tens + 5 tens = 8 tens
Equal lengths are units	3 inches + 5 inches = 8 inches
A length can be equipartitioned into equal sized units.	¼ + ¼ + ¼ + ¼ = 1
A part of 1 inch, ¼ inch, can be counted, added, etc. as a unit	3 (¼ inches) + 5 (¼ inches) = 8 (¼ inches)
Unit fractions as pure numbers can be counted, added and multiplied	3(¼) + 5(¼) = 8(¼) = 8/4
Expressions with letters can be read as uncalculated numbers	3(x + 1) + 5(x+1) = 8(x+1)

The second two progressions feeding into fractions draw more heavily from the coherence of mathematics itself. The concept of number that includes rational numbers (and later, negative numbers and ordered pairs of numbers in a relationship between quantities) cannot be developed fully without the number line. In the CCSS, the number line is used to help define a unit fraction in third grade. In subsequent grades, operations with fractions are framed in part by interpreting them on the number line by building on whole number operations on the number line. The parallel between whole number operations and fraction operations depends on seeing unit fractions as something that can be counted.

Operations with whole numbers are the most reliable and robust mathematical resource for most children. In the CCSS, a distinction exists between calculating and reasoning algebraically with whole numbers. 'Calculating' is treated in the Cluster of standards, "Number and Operations in Base Ten" and 'reasoning algebraically with whole numbers' is treated in the Cluster, "Operations and Algebraic Thinking." It is this latter Cluster that develops concepts and fluency expressing operations as part of the language of mathematics: 3 + 5 is a phrase that refers to the sum of 3 and 5. The "+" is a conjunction in this phrase. Extending this basic language of operations from phrases with whole numbers (as nouns) to phrases with other units like 5(¼) + 3(¼) enables students to build on their most solid foundation: whole number arithmetic.

In third grade, the CCSS introduces two concepts of fractions:

1. Understand a fraction 1/*b* as the quantity formed by 1 part when a whole is partitioned into *b* equal parts; understand a fraction *a/b* as the quantity formed by *a* parts of size 1/*b*.

2. Understand a fraction as a number on the number line; represent fractions on a number line diagram.

   a. Represent a fraction 1/*b* on a number line diagram by defining the interval from 0 to 1 as the whole and partitioning it into *b* equal parts. Recognize that each part has size 1/*b* and that the endpoint of the part based at 0 locates the number 1/*b* on the number line.

   b. Represent a fraction *a/b* on a number line diagram by marking off *a* lengths 1/*b* from 0. Recognize that the resulting interval has size *a/b* and that its endpoint locates the number *a/b* on the number line.

(CCSS, 2010)

The first concept relies on student understanding of equipartitioning. Confrey (2008) and others have detailed the learning trajectory that establishes how young children build up this equipartitioning concept of fraction. Yet by itself, this concept is isolated from broader ideas of number that, for the sake of mathematical coherence, are needed early in the study of fractions. These ideas are established through the second standard that defines a fraction as a number on the number line. This definition has a lot of mathematical power and connects fractions in a simple way to whole numbers and, later, rational numbers including negatives (Wu, 2008). The role of the number line definition is not obvious coming to it from prior standards, let alone prior knowledge; its importance is evident in the standards that follow in subsequent grades. A teacher or test designer seeing exclusively within the grade level will miss the point. Multi-grade progression views of standards can avoid many misuses of standards.

The Writing Team of CCSS received wide and persistent input from teachers and mathematics educators that number lines were hard for young students to understand and, as an abstract metric, even harder to use in support of learning other concepts. Third grade, they said, is early for relying on the number line to help students understand fractions. We were warned that as important as number lines are as mathematical objects of study, number lines confused students when used to teach other ideas like operations and fractions. In other words, include the number line as something to learn, but don't rely on it to help students understand that a fraction is a number. We noted that this warning was based on present experience in the classroom and might be the result of poorly designed learning progressions related to learning the number line.

The difference in advice on fractions on the number line was not easy to sort through. In the end, we placed the cognitively sensible understanding first and the mathematical coherence with the number line second. We included both and used both to build understanding and proficiency with comparing fractions and operations with fractions.

Does the number line appear out of the blue in third grade? No. We looked to the research on learning trajectories for measurement and length to see how to build a foundation for number lines as metric objects (Clements, 1999; Nührenbörger, 2001; Nunes, Light, & Mason, 1993). The Standards from Asian countries like Singapore and Japan were also helpful in encouraging a deeper and richer development of measurement as a foundation for number and quantity.

Clements and Sarama (2009) emphasize the significance of measurement in connecting geometry and number, and in combining skills with foundational concepts such as conservation, transitivity, equipartitioning, unit, iteration of standard units, accumulation

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of distance, and origin. They have shown that by around age 8, children can use a ruler proficiently, create their own units, and estimate irregular lengths by mentally segmenting objects and counting the segments.

The CCSS foundation for the use of the number line with fractions in third grade can be found in the second grade Measurement standards (CCSS, 2010):

Measure and estimate lengths in standard units.

- Measure the length of an object by selecting and using appropriate tools such as rulers, yardsticks, meter sticks, and measuring tapes.
- Measure the length of an object twice, using length units of different lengths for the two measurements; describe how the two measurements relate to the size of the unit chosen.
- Estimate lengths using units of inches, feet, centimeters, and meters.
- Measure to determine how much longer one object is than another, expressing the length difference in terms of a standard length unit.

Relate addition and subtraction to length.

- Use addition and subtraction within 100 to solve word problems involving lengths that are given in the same units, e.g., by using drawings (such as drawings of rulers) and equations with a symbol for the unknown number to represent the problem.
- Represent whole numbers as lengths from 0 on a number line diagram with equally spaced points corresponding to the numbers 0, 1, 2, …, and represent whole-number sums and differences within 100 on a number line diagram.

This work in measurement in second grade is, in turn, supported by first grade standards (CCSS, 2010):

- Express the length of an object as a whole number of length units, by laying multiple copies of a shorter object (the length unit) end to end; understand that the length measurement of an object is the number of same-size length units that span it with no gaps or overlaps. *Limit to contexts where the object being measured is spanned by a whole number of length units with no gaps or overlaps.*

This sequence in the CCSS was guided by the learning trajectory research. This research informed the CCSS regarding essential constituent concepts and skills, appropriate age, and sequence. Yet the goal of having the number line available for fractions came from the need for mathematical coherence going forward from third grade. This example shows how pull along these two dimensions—empirical research on learning and mathematical coherence—can happen in concert to make standards a better tool for teaching and assessment on both counts.

Instructional Systems and Standards

Perhaps the most important consequence of standards is their impact on instruction and instructional systems. This impact is often mediated by high-stakes assessments, which will be dealt with later.

> *…the "immaculate progression" in standards contrasts with the spectacular variation of student readiness in real classrooms*

An issue arises at the outset from a problematic convention in the standards genre: they are written as though students in the middle of Grade 5, for example, have learned approximately 100% of what is in the standards for Grades K-4 and half of 5, in other words, they present an "immaculate progression." This is never close to possible in any real classroom. This difference between the genre convention of immaculate progression in standards and the wide variation of student readiness in real classrooms has important consequences. It means, for one thing, that standards are not a literal portrayal of where students are or can be at a given point in time. And, for me, the negation of 'can' negates 'should'. Standards serve a different purpose. They map stations through which students are lead from wherever they start.

Still, the convention seems a sensible approach to avoiding redundancy and excessive linguistic nuance. But how does this mere genre convention drive the management of instruction? Test construction? Instructional materials and their adoption? Teaching? Expectations and social justice? Ah…the letter or the spirit and the slippery slope.

The Rough Terrain of Prior Learning Where Lessons Lives

Standards imply a curriculum that would present a sequence of concepts and skills through the calendar: year to year, month to month, day to day. Textbooks and tests can be developed to match such a sequence at a surface level, but the underwater terrain of students' prior knowledge will persist in giving shape to students' engagement with this sequence. Each

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student arrives at the day's lesson with his or her own mathematical biography, including all of the particularities of how—and how well—the student learned the content of the curriculum on his or her path through mathematics so far. This section examines how standards, and the learning trajectories they are based on, relate to instructional programs that might more effectively work with the variety of what students' bring to the beginning of each lesson. This diversity of student thinking and knowledge is a natural condition teachers have always faced (see Murata & Fuson, 2006 for a related discussion).

…teaching is, and always has been, like riding a unicycle juggling balls you cannot see or count.

The teacher brings to this diversity an ambition for some mathematics to be learned. The mathematics has a location in another structure: the logical coherence of ideas that reflects the knowledge structure of mathematics (mathematical coherence). Thus, there is a manifold of three knowledge structures at play in the classroom: the variety of what students bring, mathematical coherence, and the learning trajectories developed by research. As real as these structures may be, none is in plain sight for the teacher in the classroom.

What is in plain sight are standards, tests, textbooks and students' responses to assigned work. What teachers know about the path a student has taken to the knowledge the student has at the time they first meet is likely to vary widely, depending on the quality of the assessments used in the school and district, the information systems in their school, and the time the student has been in the school or district. Nor have learning trajectory researchers fully mapped the territory of the mathematics standards with specific trajectories. And the full mathematical coherence of a particular topic is often beyond the mathematical education of the teacher. Under these conditions, standards can play a crucial mediating role. What is real may be hard to see, while standards flash brightly from every test, text and exhortation that comes the teacher's way. To the extent that the standards have been well designed to embody the critical knowledge structures in a form handy for teachers and the makers of tools for teachers, the sequence and focus of instruction can be coherent with respect to mathematics and with respect to how students think and learn. To the extent that standards fail to harmonize the knowledge structures, they can add to the dissonance.

Learning trajectory research develops evidence and evidence-based trajectories (learning trajectories). Evidence establishes that learning trajectories are real for some students, a possibility for any student and probably modal trajectories for the distribution of students. Learning trajectories are too complex and too conditional to serve as standards. Still, learning trajectories point the way to optimal learn-ing sequences and warn against the hazards that could lead to sequence errors (see below). The CCSS made substantial use of learning trajectories, but standards have to include the essential mathematics even when there has been little learning trajectory research on the topic. Standards have to function as a platform for instructional systems that can accommodate the variation in students, if not teachers, at each grade level.

Standards can tempt districts to simplistic mechanisms that mismanage student variety. One temptation to avoid is to impose strong standards-framed pressure in an accountability system that ignores student thinking on the principle that even the mention of student differences springs leaks of low expectations into the classrooms. It could name the territory between the knowledge students have and what standards demand the "achievement gap," a dark void that focuses attention on unexplored distances not traveled, rather than on the steps that need to be taken. It could tell teachers to keep turning the pages of the standards-based textbook according to the planned pace, and rely on the sheer force of expectation to pull students along. At least this would create the opportunity to learn, however fleeting and poorly prepared students might be to take advantage of it. While this is better than denial of opportunity, it is a feeble, if not cynical, response to the promise standards make to students. Shouldn't we do better?

What better options are there? Some nations, including some high-performing nations, assume in the structure of their instructional systems that students differ at the beginning of each lesson. Many Asian classrooms, K-8, follow a daily arc from the initial divergence of students' development (refracted through the day's mathematics problem(s) through classroom discourse about the different "ways of thinking") to a convergence of understanding a way of thinking that incorporates the mathematics to be learned. Each student is responsible for understanding each "way of thinking." The teacher leads a closing discussion, which begins with the way of thinking that depends on the least sophisticated mathematics. Students who use less sophisticated mathematics often rely on good problem solving and sense making skills, so other students can learn from their approach as well as from approaches involving more sophisticated mathematics. The discussion is then led through two or three other ways of thinking ordered by the sophistication of mathematics deployed in the way of thinking. Each way of thinking is explicitly related to each other through questions and discussion. This process approximates beginning the lesson at the diversity of student thinking found in the class and converging on the mathematical coherence. The actual content of the discussion—the illustrations, analogies, explanations, diagrams, narratives of student action—approximates a problem-specific slice of learning trajectories that connect the varied starting points to the mathematics to be learned. The arc of each lesson begins with divergence of prior experience and ends with convergence on mathematical understanding that belongs to a larger coherence framed by standards.

Such a system requires enough time to achieve convergence each day or two, which means enough time on a small number of problems that focus on a small number of topics. A hurried instructional system cannot 'wait' for students each day. To make time for daily convergence, standards must require less to learn rather than more each year. A fortunate irony revealed by the accomplishments of the high-performing Asian systems reveals that teaching less can result in learning more; that is, reaching a more advanced mathematical level by learning a more coherent and elegant body of knowledge rather than a sprawl of clutter and fragments. A system that optimizes daily convergence will be more robust and accumulate less debt in the form of students unprepared for the next lesson, and the next course. Unlike the national debt, this debt does not compound quietly, but makes all of the noises of childhood and adolescence scorned.

How can a system get from where it is to effective instruction using well-designed standards as a platform? Start by understanding the tasks and then preparing and supporting educators so they can accomplish the tasks. The core tasks are to assess the patterns of mathematical thinking, that is, the rough terrain of prior knowledge, that students bring to the classroom, and provide teachers with the curriculum and assessment tools needed to help students move along trajectories toward mathematical targets defined by the standards, give or take. We know enough to make learning trajectories and the mathematical coherence underlying the content and structure of the standards a top priority for teacher knowledge development.

With enough knowledge of relevant learning trajectories and enough understanding of how learning trajectories work, teachers will better anticipate and recognize the most common starting points they will find among their students (Murata & Fuson, 2006; Battista, 2010). They need knowledge of the relevant mathematical coherence so they can focus on the most valuable learning targets. And they need instructional tools (diagnostic lessons that make thinking visible rather than just "scoring" students) that illuminate rather than obscure student thinking. They need instructional programs, and lesson protocols that pose standards as the finish line, but accommodate variation of prior experience. They need instructionally embedded assessments that make student reasoning and conceptualization visible rather than hiding them behind a score that, in effect, pins the student to the donkey of failure. They need time within the lesson and across the unit to listen and respond to students with guidance on revising their thinking. This requires standards that are within reach of students and teachers.

The crucial issue in this situation is how well the standards-driven texts and tests improve the performance of the instructional system in moving students along the learning trajectories. It is quite possible for standards to be out of whack with learning trajectories and actual student thinking so that they mislead instruction and diminish performance. If the sequences in the standards conflict seriously with learning trajectories, are mathematically incoherent, or are too far removed from students' capacities, they can steer the instructional systems away from effective teaching and learning. Standards work best when they are reasonable targets for tested learning trajectories and when they illustrate a reasonable amount of mathematical coherence to help teachers respond effectively to real student thinking by moving them in the right directions.

The foregoing discussion might seem to suggest that each standard is situated in a single trajectory. This is not typically the case. A standard depends on many earlier standards that are often in several different trajectories. Likewise, standards in subsequent grades that depend on the standard can be in several trajectories. This web structure was illustrated by the fractions example earlier in this report. Another important example is the CCSS's Grade 7 standard for proportional relationships (CCSS, 2010).

2. Recognize and represent proportional relationships between covarying quantities.

    a. Decide whether two quantities are in a proportional relationship, e.g., by testing for equivalent ratios in a table or graphing on a coordinate plane and observing whether the graph is a straight line through the origin.

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b. Identify the constant of proportionality (unit rate) in tables, graphs, equations, diagrams, and verbal descriptions of proportional relationships.

c. Represent proportional relationships by equations. For example, total cost, *t*, is proportional to the number, *n*, purchased at a constant price, p; this relationship can be expressed as $t = pn$.

d. Explain what a point (x, y) on the graph of a proportional relationship means in terms of the situation, with special attention to the points (0, 0) and (1, r) where *r* is the unit rate.

This standard is the culmination of a manifold of learning trajectories and mathematical coherences that are reflected in progressions of standards in the CCSS, and is itself the beginning of subsequent progressions. Pat Thompson, Arizona State University, has remarked (2010, personal communication) that proportionality cannot be a single progression because it is a whole city of progressions (see also Clements and Sarama on "hierarchical interactionalism," 2009).

This standard, which stands along side other standards on ratios and rates, explicitly draws on prior knowledge of fractions, equivalence, quantitative relationships, the coordinate plane, unit rate, tables, ratios, rates, and equations. Implicitly, this prior knowledge grows from even broader prior knowledge. The sequence supporting this standard barely captures the peaks of a simplification of this knowledge structure. The complexity of the manifold of learning trajectories guarantees that teachers will encounter a wide variety of individual mathematical biographies involving proportional relationships in each class.

What can help teachers respond more effectively to the variety of readiness? Certainly not pressure to "cover" the standards in sequence, to keep moving along at a good pace to make sure all students have an 'opportunity' to see every standard flying by. Knowledge of relevant learning trajectories would help teachers manage the wide variety of individual learning paths by identifying a more limited range of specific types of reasoning to expect for a given type of problem (Battista, 2010). Even hypothetical learning trajectories can do more good than harm because they conceptualize the student as a competent knower and learner in the process of learning and knowing more (Clements & Sarama, 2004). The standards, based as much as possible on tested learning trajectories, identify what direction to lead the students from wherever they begin the lesson. A curriculum, based on the standards, with the diagnostic value of revealing how different students see the mathematics—how they think about it—and where they are along the learning trajectory would also help.

Even an instructional system with incomplete and imperfect knowledge of learning trajectories and actual student thinking treats students as works-in-progress and focuses teaching on making progress along illuminated paths. Such systems could easily function more effectively than a system that interpreted standards as direct descriptions of where students should be, and by implication characterizes real students as "unprepared" failures. Certainly, trajectory-informed instruction would be more motivating for teachers and students in its emphasis on the malleability of proficiency in mathematics in contrast to gap-informed systems that highlight the fixedness of proficiency (Dweck, 1999, 2002; Elliot & Murayama, 2008; Murayama & Elliot, 2009).

Do Standards Express the Form and Substance of What Students Learn?

What is the nature of the 'things' students learn? Sometimes what is wanted is a performance, as in learning to ride a bike. Standards, instruction, and assessment can happily focus on visible performances in such cases. But often, in mathematics anyway, what students learn are mental actions on mental objects, reasoning maneuvers and rules, representational systems and languages for mathematical objects and relations, cognitive schema and strategies, webs of structured knowledge, conventions, and so on. Many of these learned 'things' are not things, but systems that interact with other systems in thinking, knowing, and doing. Standards cannot express this kind of complexity; they refer to some observable surface of learning. But this linguistic convenience can lead to logical fallacies when we attribute unwarranted 'thinginess' properties to what we actually want students to learn.

The important point is that learned things are not thingy or topics. A sequence of topics or standards skims the surface and misses the substance—and even the form—of a subject. Compare, for example, the standard (CCSS, 2010), to what a student must actually know and do to "meet" the standard (for example, Steffe and Olive, 2009; Confrey, 2008; Confrey et al., 2009; Wu, 2008, Saxe et al., 2005).

- Add and subtract fractions with unlike denominators (including mixed numbers) by replacing given fractions with equivalent fractions in such a way as to produce an equivalent sum or difference of fractions with like denominators. *For example, 2/3 + 5/4 = 8/12 + 15/12 = 23/12. (In general, a/b + c/d = (ad + bc)/bd.)*

The standard gives a goal, but does not characterize the knowledge and competencies needed to achieve the goal. While this point may seem obvious, it gets lost in the compression chambers where systems are organized to manage instruction for school districts. Devices are installed to manage "pacing" and monitor progress with "benchmark assessments." These devices treat the grade-level standards as the form and substance of instruction. That is, students are taught grade-level "standards" instead of mathematics. And this nonsense is actually widespread, especially where pressures to "meet standards" are greatest.

Standards use conventional names and phrases for topics in a subject. To what do these refer? If the field had a well-understood corpus of cognitive actions, situations, knowledge, etc., then these names could refer to parts of this corpus. But the field, school mathematics, has no such widely understood corpus (indeed, it is an important hope that common standards will lead to common understandings of such things). What the names refer to, in effect, are the familiar conventions of what goes on in the classrooms. The reference degenerates to the old habits of teaching: assignments, grading, assessment, explanation, and discussion. The standards say, 'Do the usual assortment of classroom activities for some content that can be sorted into the familiar names in the standards.' We will call this "covering the standards" with instructional activity.

"Covering" has a very tenuous relationship with learning. First, there are many choices about focus within a topic, coherence within and between topics, what students should be learning to do with knowledge, how skillful they need to be at what, and so on. Teachers make these choices in many different ways. Too often, the choices are made in support of a classroom behavior management scheme. Meanwhile, different students will get very different learning from the same offered activity, and moreover, the quality of the discussion, the assigned and produced work, and the feedback given to students will vary widely from teacher to teacher working under the blessing of the same standard.

Covering is weak at best. When combined with standards that are too far from the prior knowledge of students, and too many for the time available, the chemistry gets nasty in a hurry. Teachers move on without the students; students accumulate debts of knowledge (knowledge owed to them); and the next chapter and the next course are undermined. But managing instruction by "meeting the needs of each student" is equally weak, because it opens the door to self-fulfilling low expectations. The way through this dilemma is to use standards that focus the use of time where it really matters so there is time to respond to students *thinking*, rather than their *needs*. (And what, really, is a "need"? Usually it is defined as something a student gets wrong or cannot do, or even more vaguely as a topic within which a student performs poorly. As such, "needs" are uninformative as a basis for teaching decisions or misinformative (when students are grouped by "needs").

… managing instruction with a system of "covering" standards that are too many and too far from students' prior knowledge is not management but posturing… the key is to respond to students' thinking, not their so-called "needs"… 'need' names a sled to low expectations

The starting point is the mathematics and thinking the student brings to the lesson, not the deficit of mathematics they do not bring. A standard defines a finish line, not the path. The path begins with the students' prior knowledge and finishes with the "standard" knowledge. The path itself is described by learning trajectories and mathematical coherences.

Errors in Sequence, Focus, and Coherence

The questions raised in the previous section are not only design choices, but potential sources of error with consequences for the viability of instruction. The next two subsections examine the types of errors that could menace a standards-based system.

Types of Sequence Errors

There are several types of errors with serious consequences for students and teachers in the way standards might be sequenced. A common type of sequence error occurs when a concept, B depends on A_2 version of concept A, more evolved than the A_1 version; Standards have only developed A_1. Student tries to learn B using A_1 instead of A_2. For example, rate, proportional relationships and linearity (B) depend on understanding multiplication as a scaling comparison (version A_2), but students may have only developed version A_1 concept of multiplication, the total of things in *a* groups of *b* each.

In the CCSS, multiplication is defined in Grade 3 as $a \times b = c$ means *a* groups of *b* things each is *c* things. In Grade 4, the concept of multiplication is extended

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to comparison where $c = a \times b$ means c is a times larger than b. In Grade 5, the CCSS has:

5. Interpret multiplication as scaling (resizing), by:

   a. Comparing the size of a product to the size of one factor on the basis of the size of the other factor, without performing the indicated multiplication.

   b. Explaining why multiplying a given number by a fraction greater than 1 results in a product greater than the given number (recognizing multiplication by whole numbers greater than 1 as a familiar case); explaining why multiplying a given number by a fraction less than 1 results in a product smaller than the given number; and relating the principle of fraction equivalence $a/b = (n \times a)/(n \times b)$ to the effect of multiplying a/b by 1.

(CCSS, 2010)

In Grades 6 and 7, rate, proportional relationships and linearity build upon this scalar extension of multiplication. Students who engage these concepts with the unextended version of multiplication (a groups of b things) will have prior knowledge that does not support the required mathematical coherences. This burdens the teacher and student with recovering through learning trajectories. This will be hard enough without ill sequenced standards causing instructional systems to neglect, in this case, extending multiplication to scaling.

Major types of sequence errors include:

1. Unrealistic:

   a. Too much, too fast leaving gaps in learning that create sequence issues for many students. The system cannot deliver students who are in sequence or handle so many students out of sequence. *Rushing past reasoning with operations on whole numbers to teach answer-getting calculations leaves huge gaps in the foundations of algebra.*

   b. Distribution of prior mathematics knowledge and proficiency in the student (and teacher) population is too far from the standards, and there is no practical way to get students close enough in time for sequence.

2. Missing ingredient:

   a. A is an essential ingredient of B, but standards sequence B before A. *Students try to learn fractions before essential concepts of number are available, for example, the number line.*

   b. Coherence requires progression ABC, but standards only have AC.

   c. Term is used that has insufficient definition for the intended use.

3. Cognitive prematurity:

   a. B depends on cognitive actions and structures that have not developed yet.

   b. B is a type of schema or reasoning system, and the learner has not developed that type of schema or system. *Base ten arithmetic, i.e. place value, depends on unitizing groups of ten, but some students have not acquired unitizing schema that can apply to "tens".*

   c. Student develops immature version of B and carries it forward (see also 6).

4. Contradiction:

   a. Cognitive development entails ABC, mathematical logic entails CBA.

5. Missing connection:

   a. B is about or depends on connection between X-Y, but X-Y connection is not established.

6. Interference:

   a. B depends on A_2 version of A, which is more evolved than A_1 version, standards have only developed A_1. Then student tries to learn B using A_1 instead of A_2.

   b. B belongs nestled between A and C, but D is already nestled there. When learning B is attempted, D interferes.

7. Cameo:

   a. B is learned but not used for a long time. There is far too much time before learning C such that C depends on B. B makes a cameo appearance and then gets lost in the land of free fragments. *Absolute value and scientific notation are often cameo topics long before they are useful. Properties of operations are treated as cameos when their routine use should be made explicit rather than hidden behind tricks and mnemonic devices for getting answers, e.g. "FOIL".*

8. Hard Way:

   a. C needs some ideas from B, but not all the difficult ideas and technical details that make B take more time than it is worth and make it hard for students to find the needed ideas from B, so C fails.

   b. There are multiple possible routes to get from A to E, and the standards take an unnecessarily difficult route.

9. Aimless:

   a. Standards presented as lists that lack comprehensible progression.

Types of Focus and Coherence Errors

The issues of focus and coherence in standards deserve more attention than we will give them here. Nonetheless, learning trajectories interact with coherence and focus in standards. The following are critical types of error of focus and coherence:

1. Sprawl:

   a. Mile wide, inch deep. Large collection of standards dilutes the importance of each one.

   b. Standards demand more than is possible in the available time for many students and teachers, so teachers and students are forced to edit on the fly; this is the opposite of focus.

   c. Standards are just lists without enough organizational cues for a hierarchy of concepts and skills.

2. Wrong grain size:

   a. The granularity is too specific or too general. The important understanding is at a certain level of specificity, where the structure and the cognitive handles are, and the grain size does not match up to prior knowledge. (As in Aristotle's *Ethics*, the choice of specificity is a claim that should be explicit as a claim and defended.)

   b. The granularity is too fine. Complex ideas are chopped up so the main idea is lost; the coherence may be evoked, but not illuminated. Alignment transactions in test construction and materials development miss the main point but 'cover' the incidentals (e.g., students can perform the vertical line test but do not know what a function is or how functions model phenomena.)

3. Wrong focus:

   a. Focus on answer-getting methods, often using mnemonic devices, rather than mathematics.

4. Narrow focus:

   a. Just skills, just concepts, just process, or just two out of the three.

5. Priorities do not cohere:

   a. Fragments have large gaps between them.

   b. Grain size too fine.

6. Congestion:

   a. Some grade levels are congested with too much to be learned; density precludes focus.

   b. B, C, D are all being learned at once, but cognitive actions needed for learning can only handle one or two at a time. Only BC and CD are learned, but the essential point is learning BCD and the system BC-BD-CD.

7. Inelegance:

   a. AXBYCZ is equivalent to ABC and time and cognition are wasted on X, Y, and Z.

8. Waste:

   a. Time and cognition are invested in B, and B is not important.

9. Resolution of hierarchy:

   a. The hierarchal relationship between standards is not explicated, and details are confused with main ideas.

   b. The hierarchy of standards does not explain relationships among ideas, it just collects standards into categories.

10. Excessively literal reading:

    a. This error is in the reading as much as the writing; it leads to fragmented interpretation of the subject, losing the coherence among the standards.

    b. Reading individual standards as individual ingredients of a test. When the explicit goal is to have the ingredients cook into a cake, tasting the uncooked ingredients is a poor measure of how the cake tastes (although it is related). The goal, as stated in the grade-level introductions and the practices standards, is for the students to cook.

Assessment

An assessment system designed to help *steer* the instructional system must give good information about *direction* as well as distance to travel. A system that keeps telling us we are not there yet is like a kid in the back seat whining "are we there yet? How much further?" In the U.S., our state assessment systems whine with scores that tell us how many students have met the year's standards in contrast to giving us a location in a map of trajectories. We measure failure and define success as "less failure."

Knowing an estimate for how many students are "proficient" serves a broad purpose that I hesitate to call by its customary name, 'accountability', because I cannot figure out exactly who is accountable to whom for what when it is said 'schools' must be accountable. Yet it may serve the broad and important purpose of enabling the interested to compare performance from place to place and time to time. Disaggregating scores by student sub-populations serves the critically important purpose of telling us how well we are achieving the social justice goals of public schooling. Yet the assessment results are used for many purposes in policy formation and management that go well beyond the design specifications of the assessment. How valid are these uses of the typical assessments? Do we need assessments designed for these uses?

Validity

Validity is a property of a use of an assessment, not of the assessment per se. The intended use of standards-based school accountability tests is to motivate and steer schools with carrots and sticks based on tests. Are the tests valid for this use? The empirical validity question is: are districts, schools, teachers, and students motivated and steered in the right directions? Hmmm…

How can we improve the validity of using formative and summative assessment for steering the system, at each level of the system? Too many "periodic" assessments at the district level are images of the state test, which is a fuzzy image of the standards, in part because the standards imply a full-year of instruction, not just an hour or two of testing. State tests were not designed with this use in mind. Perhaps we should design formative assessment that informs substantive feedback during the course of instruction first. The design of state tests could then be based on such formative assessment, rather than the other way around.

What makes an assessment formative? Its use to inform instruction, and to do this requires three things: 1) Timing: the assessment is available and used while instruction is still going on, while there is still time for instruction to respond to information; 2) Feedback: the assessment informs, that is the feedback has content (mathematical content) not just value judgment; and 3) Motivation: the assessment responds to learning (growth); the relationship between assessment and what should be studied is transparent and direct, not two different species of work that share a common topic or standard; test items look like the class work and homework implied by the standards, not like a psychological instrument.

Summative uses of assessment have their own issues. Summative assessment must:

- Focus on the priorities established in the standards;
- Report in categories (like "proficient") that mean what people rely on them to mean (ready for the next grade level, as validated by empirical studies);
- Detect the growth along progressions in the standards (progressions are the construct; the construct is not "difficulty");
- Fit to population: detect growth across the distribution;
- Fit within operational constraints: time, money, and schedule; and
- Be worthy of imitation for local periodic assessments: show thinking and knowledge.

There are trade-offs: optimizing score reliability vs. optimizing information about student knowledge, reasoning and how they get the wrong (or right) answer; optimizing ease of aggregating reports up the hierarchy vs. value for teachers inside instruction. The need for both implies the need to allocate time and resources to both.

There is duplicity in purpose for assessment that mirrors the two faces of instruction: facing *out* from instruction toward the system audiences (management, leadership, community, parents, students as clients); and facing *in* toward student cognition and student actions, which is where learning happens.

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Out-facing reports require summary and aggregation. To add up, we need common units: inches to inches, dust to dust. If we have apples and oranges, we need a common denominator: 3 apples + 5 oranges = 8 fruit. Adding up requires blurring distinctions. Reliability of scores is a measure of uni-dimensionality: apples to apples; average correlation of items to the total score adjusted to sample size (number of items). Assessments that only optimize reliable scores have small value for the kinds instructional choices teachers make, and encourage a view of intelligence as fixed.

Facing in, we need tools that make what the student is doing visible to the teacher, to the student's peers, and in the student's own reflection (metacognition). We need the misspellings themselves rather than a spelling score. Instruction should respond to the actual spellings and provide feedback to the student on their spelling, not on how far they are from being a "proficient speller." The teacher needs to know why a student is getting the fraction problem wrong (or right, for that matter), not just that he is. Knowing that a student scores low on fractions has even less value. A teacher needs to see the student working fraction problems, see where he goes wrong, and give feedback that responds to what that student actually does.

Too often, education efforts marked by confounding purposes invite the assumption of invalid models of learning. A common example that enfeebles many instructional systems stems from the illusion that poor performance is a trait of the student. This originates in the idea that we are measuring traits of students and a test score is a measure analogous to a student's height in inches. Differences in scores become differences in students. Some students are good at mathematics and some are weak at mathematics. Therefore, let's sort students by score so we can respond to the differences with differences in instruction. What's wrong with this analysis and decision process?

One-dimensional Tests for n-dimensional Constructs

Scores distribute students along one dimension, the trait "math," or perhaps the trait "fractions." There is a "gap" between where they are on the dimension and where their peers are. What can one do about this gap? The answer is usually "re-teach." In other words, repeat the process that left a gap in the first place.

Often, the 'gaps' are not gaps, but confusions that have their origins in instructional materials and classroom practice where long-term mathematics learning was swapped away for short-term answer getting. A well-known example is students not realizing that fractions are quantities, numbers with units (the unit fraction). Instead, they learned that fractions are two numbers, one on top and one on the bottom. Adding or dividing fractions is then a complicated procedure for doing arithmetic on tops and bottoms. The simple idea that adding or dividing two fractions is a case of adding or dividing two numbers is lost in the hard to remember procedures that make no sense.

The reality that teaching involves leading a variety of students through a web of trajectories needs to be reflected in the way the assessment system defines the goals in the system. Even though we cannot define the web of trajectories with any precision, or locate individual students precisely in the web; the assessments should be designed to incent the system to work in the web of trajectories, not ignore it by pretending students are scattered along a single track. This pretense rationalizes the view that mathematics achievement is a trait of the student rather than the work and responsibility of the system.

What are Standards?

The word "standards" as used in education has three quite different meanings that slip too easily from one to the other in the rhetoric of policy and decision-making. Standards mean:

1. The specification of content to be learned and proficiencies acquired (for example, CCSS).

2. The level to be reached, a cut score on a test, the passing grade (the score required to be "proficient" on a state test) — "performance standards," "achievement levels."

3. What people expect from themselves and each other as actual behavior and performance; what happens when a student does not do homework, disrupts class, submits sloppy work; when a teacher gives little feedback, allows poor work, teaches without preparation; in other words, the standards people live up to in the way they behave and take care of their responsibilities.

In this section, we are discussing the first meaning, but it is connected to the second when assessment comes into play. A score is not just a point on a scale, but a measure of how much of the content and proficiency in the standards a student has learned. What score range means a student is well-prepared to succeed in subsequent years?

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The reporting categories on which the public relies when weighing in on policy debates do not appear to be valid for the purpose that the pretensions of its vocabulary suggest: the meaning of "proficient" is taken to mean ready and prepared to succeed at the next grade level. But it is usually a quite arbitrary category, above a cut score established by judges in a dubious process that has been criticized for ignoring substantial evidence from social psychology.

It should be routine to analyze how well students at different score levels in one grade perform in subsequent grades. Longitudinal relationships of scores across grades should be a core part of how interpretation categories like "proficient" are defined and used. Yet we see little use of research evidence on how well 'proficient' students are prepared for future success in school. Questions of validity in standards-based assessment systems are serious, given the uses that standards-based management systems make of data.

This nation has come closer than it ever has to building a coherent education system. Nearly 80% of the states have adopted the CCSS. What is the next step? People are the next step. If people just swap out the old standards and put the new ones in old boxes and power points, into old systems and procedures and relationships, then nothing will change.

The CCSS can be a new platform for better instructional systems and better ways of managing instruction. The CCSS build on achievements of the last 2 decades, but also build on lessons learned in last 2 decades, especially lessons about time and teachers. One of the old boxes that needs replacing is "alignment" and its offspring, "covering standards" and "pacing guides." These belong to a well intended, but weak concept for standards-based teaching and learning. Alignment is a bunt: lucky if you get to first base. We have to score. We need to swing at the ball. Covering standards is what a mile wide and an inch deep is called on the ground, in schools. Pacing means keep turning pages regardless of what students are learning: ignore student results. It is time to move on to something stronger, more effective. CCSS are designed as a tool to *raise* achievement, not just praise it.

What are standards? Standards are promises. Standards promise the student, "Study what is here, do your assignments and we promise you will learn what you need do well on the test." We need tests and examinations designed to keep that promise by rewarding that learning with a good score, and we need school systems designed to keep that promise. And beyond the tests which, after all, are part of the school system, we promise you that if you learn these things you will be well prepared to travel your life's trajectory in pursuit of your civic, economic, and personal goals. The CCSS are merely the promise. We hope it is a well made promise in three ways: it builds so that keeping one grade's promise makes keeping the next grade's within reach, the system is capable of keeping the promise, and the promise is worth keeping.

VI. NEXT STEPS

Our interest in learning trajectories arises from our belief that the normal/modal approach to instruction must change if we are to make progress toward our ambitious goal of preparing all of our children for success in postsecondary institutions and rewarding careers. We believe that teachers and students must take increased responsibility for monitoring students' learning and understanding, and responding to the results of that monitoring by taking steps to keep learning on track toward the goals of instruction, or to get it on track if it has gone off. This should occur continuously, and doing that requires two things: 1) Understanding what the "track" is, in some detail; and 2) knowing what is likely to help keep a student moving forward on it, or to get him or her back on it, if they are having problems. Our schools haven't been very good at either of these tasks, but in fairness, we have not given them the tools they need to do it.

We don't have a good description or understanding of the key steps in the development of mathematical knowledge and understanding, and we don't have a codified, warranted body of knowledge about what to do for students who manifest particular problems or misunderstandings at particular points along the path. Yet we have some knowledge in both of these domains, and we know what is required to deepen our knowledge. And we can make some reasonable guesses about what to do in the meantime in the areas where we lack knowledge. So our primary recommendation is that we should get on with this work, act on our best guesses where necessary, and keep track of how things work out; so that over time we can fill in the gaps.

Putting the knowledge we have into practice is a more challenging problem. There has been a lot of rhetoric about data-based decision-making in recent years, and too little attention to the usefulness of the data being provided to teachers. In general, the data are not fine-grained enough to be useful for diagnosis and not timely enough to support adaptive instruction. If we want teachers to understand student learning problems and respond quickly to help them, then we need to work harder to figure out what kinds of tools and support teachers need to observe and keep track of their students and what they might do to help them—and what happens because of that. We also have a lot to learn about the logistics of doing this in real classrooms with lots of students. Adaptive instruction may require re-organization of schools and classrooms. And we have a lot to learn about student motivation. It clearly is not the case that all students are particularly interested in learning what we would like them to learn. And if students do not increase their work effort, monitoring their progress and adapting instruction will not produce the gains we need to close achievement gaps and prepare our students to be competitive in the global economy.

Just to point out one serious problem that must be addressed, we really don't know how many different paths students are likely to take in mathematics and how that number may be influenced by curriculum or instructional choices, but we can hope that the number is finite and perhaps small, or that it can be effectively limited by choices of instructional trajectories without any harm to the students, or that we can identify a (small) set of common nodes through which almost all students will pass, even though their paths in between those nodes may be quite diverse.

The CCII Panel has discussed this issue and others, and the potential of learning trajectories in mathematics, the work that has been done on them, the gaps that exist in this work, and some of the challenges facing developers and potential users. We have concluded that learning trajectories hold great promise as tools for improving instruction in mathematics, and they hold promise for guiding the development of curriculum and assessment as well. We are agreed that it is important to advance the development of learning trajectories to provide new tools for teachers who are under increasing pressure to bring every child to high levels of proficiency. With this goal in mind, we offer the following recommendations:

- **Need to establish a respected research field on learning trajectories in mathematics.** Some researchers have told us that they have had trouble getting papers and articles on learning trajectory papers published. This is a problem for all new research paradigms which present models that do not fit well within the conventions of mathematics education research. Funding agencies and research organizations need to make the importance of this research clear. The discussions about sequence, solution methods, and learning supports are really about what is mathematically desirable. We need forums where these issues can be discussed in on-going ways to build and refine our knowledge of learning trajectories and effective learning supports.

- **Initiate new research and development projects to fill critical knowledge gaps.** There are major gaps in our understanding of learning trajectories in mathematics. These include topics such as:
  - » Algebra
  - » Geometry
  - » Measurement
  - » Ratio, proportion and rate
  - » Development of mathematical reasoning

An immediate national initiative is needed to support work in these and other critical areas and fill in the gaps in our understanding.

- **Consolidate learning trajectories.** For topics such as counting, or multiplicative thinking, for example, different researchers in mathematics education have developed their own learning trajectories. While there are a lot of similarities among these trajectories, there are also some differences, and researchers tend to defend and advance their own ideas. The field needs to come together to review this work and consolidate it. This sharing could take two forms: 1) Researchers could come to agree on common nomenclature or ideas when possible, or 2) they could explain why certain trajectories for the same topic are different and why they might need to co-exist while being tested (Barrett & Battista, forthcoming chapter scheduled to appear in a volume edited by Confrey, Maloney, and Nguyen, in press, 2011).

- **Initiate work on the integration and connections across trajectories.** Developers of trajectories, and those who support their work, should seek collaborations with other developers to examine the connectivity and interactions across trajectories, and to consider the implications of these interactions for curriculum. The work on trajectories within in a field like mathematics needs to be integrated for teachers as they cannot be expected to track students' progress on multiple trajectories simultaneously. Integration also would inform future work on learning trajectories and help standard-setters and curriculum developers determine what topics are most generative of student understanding of mathematics. Some will argue that it is too early to do this, but attempting to do it now will inform the development of the next generation of trajectories and help set priorities for that work.

- **Study development of students from different cultural backgrounds and with differing initial skill levels.** We desperately need to understand how to accelerate the learning of students who enter school with lower literacy levels and also to understand how cultural backgrounds and early experiences affect developmental paths in mathematics. Researchers recognize that the pathways described by trajectories are not developmentally inevitable and that there may be multiple pathways to learning a given idea or practice. They also recognize that prior experience, knowledge, and culture influence learning. Therefore, there is a need to explore how diversity affects the development and application of learning trajectories, and whether, and how trajectories can help us close achievement gaps in mathematics. This is particularly relevant in urban populations or schools with highly diverse groups of students.

- **Share the available learning trajectories broadly within the R & D community.** While the existing trajectories cover only parts of the K-12 mathematics curriculum, and most have not had extensive testing in classrooms, they can provide useful information for groups working on state and national standards as well as for developers working on curriculum and assessment. The use of research in trajectories in formulating the Common Core State Standards (CCSS) is evidence of their value. Although, the existing trajectories fill in only part of the picture, they provide clues about the structure and sequence of the missing parts of the curriculum. At this point, the work has not been widely shared; small groups of researchers have been working on completion or validation of learning trajectories they have developed. While some of this work has been shared through books, journal articles, or conference papers, most of it is not readily accessible to those who need access to it. In this regard, it is encouraging that there have been working conferences bringing researchers working on the development and testing of learning trajectories together to share their methods and findings. NSF or some other national organization should create a website where this work, including work in progress, can be displayed with all of the proper caveats. Even incomplete and untested work can be helpful to those who are working on standards, curricula, and assessments.

- **Translate the available learning trajectories into usable tools for teachers.** Bringing the research on learning trajectories to developers and teachers requires the development of new tools. Curriculum developers need versions of trajectories that stress the learning supports, the key mathematics ideas, and the key questions for students so that they can support classroom teachers and students through the learning paths. Classroom teachers need overviews so they can see the pathway clearly before they start it. Prototypes of trajectories could be developed in collaboration with developers and teachers to build transitional tools and procedures that encourage and support the use of adaptive instruction and the growth of teachers. Many current professional development programs attempt to support development of teachers' practical knowledge of students' ways of thinking in mathematics and foster new ways of conveying instructional ideas. The learning trajectories work is quite consistent with professional development projects focused on pedagogical content knowledge and needs to be incorporated into this mainstream work.

- **Validate the learning trajectories.** Funding agencies should provide additional support for research groups to validate the learning trajectories they have developed so they can test them in practice and demonstrate their utility. An effort should be made to collect evidence that using learning trajectories to inform curriculum, instruction, assessment design, professional development and/or education policy results in meaningful changes in instruction and gains in student achievement. This evidence is needed to defend the investments needed to continue the work and fill in gaps, and to respond to skepticism expressed by various stakeholders about the value and significance of learning trajectories.

- **Invest in the development of assessment tools based on learning trajectories for use by teachers and schools.** There are fundamental differences between assessments designed to distinguish how students perform compared to other students on general scales of "achievement" or ability, and assessments designed to distinguish among particular levels in the development of student knowledge and stages of sophistication in their understanding and ability to apply their knowledge of mathematics. Assessment tools of the latter type are needed to build and test trajectories and to provide teachers with the diagnostic information they need to adapt instruction to meet the needs of their students and also to give students themselves better information about where they stand with reference to their learning goals. Adequate development of assessments of this sort will require fundamental advances in psychometric methods and supporting technologies, and that too will deserve increased investment.

- **Encourage more collaboration among mathematics education researchers, assessment experts, cognitive scientists, curriculum and assessment developers, and classroom teachers.** Inadequate communication among the groups that have an interest in the development and testing of learning trajectories in mathematics is an obstacle to further progress. There is a need to build better understanding and more collaboration across these domains. Funding agencies should seek to foster better and more frequent communication among these communities. The National Science Foundation (NSF), the Hewlett Foundation, and the Pearson Foundation have supported several meetings of this type, but more needs to be done to foster collaborative work. NSF or another funder might consider sponsoring "state of the work" conferences annually, or convening various stakeholders at special sessions (e.g., organizations convening at NCTM's annual conference or other national meetings). They also might consider funding centers to work on learning trajectories and curriculum development where these different kinds of expertise and experience might be convened.

- **And, finally as we undertake this work, remember that it is the knowledge of the mathematics education research that will empower teachers, not just the data from the results of assessments.** This is extremely important and needs to be more clear from the outset and reinforced regularly. Otherwise some will seek to develop and promote tools —new technologies, new assessments, etc.— to substitute for teacher knowledge. Good tools, as we argue above, are needed, but they will have powerful effects only if they are placed in the hands of practitioners who have a strong command of their domain and an inclination to use the knowledge base in their field and learn from others to improve their practice. If we look at this problem as one of continuous improvement, then we will not expect practitioners to be expert in all parts of their domain, but we will expect them to develop expertise, and to work collaboratively with others whose knowledge complements their own. And we will expect them to contribute to the knowledge base by constructing and testing hypotheses in partnership with other teachers and with researchers.

VI. NEXT STEPS

Why Should These Steps Be Taken?

In CPRE's earlier publication on the development and use of learning progressions in science (Corcoran, Mosher, & Rogat, 2009), we argued that learning progressions could help us shape, test, and refine policies and practices in the areas of curriculum, assessment, teacher education, and professional development and improve coherence and alignment across these domains. Learning trajectories in mathematics have that same potential.

First and foremost, learning trajectories translated into usable tools can help teachers rethink instruction, assessment, and interventions for students who fall behind. They will make it possible to improve teacher diagnosis of student understanding, and enable teachers to practice adaptive instruction. Trajectories could provide teachers with the frameworks, tools, and resources needed to transform pedagogical content knowledge from a precious concept to an operational part of their practices.

Learning trajectories will help curriculum developers build mathematics programs that are more focused, better sequenced, and more coherent. New curricula should be consistent with established learning trajectories and their key features should be incorporated into instructional materials (e.g., a coherent developmental sequence based on research, specification of learning performances, and valid assessments that support diagnoses derived from evidence of the learning performances). New curricula also offer opportunities to test and revise hypothetical trajectories addressing gaps in our knowledge of mathematics.

Continued work on learning trajectories can help us revise and refine the CCSS over time, and rethink state assessments and professional development. By collaborating with researchers and developers on piloting materials and assessments linked to trajectories, and by inviting researchers to collaborate on tasks such as standards revision, curriculum selection, a ssessment selection, and professional development, we can advance the work and contribute to the growth of our knowledge.

Learning trajectories also could contribute to improvements in the design and operation of teacher education programs as well as the design of programs provided by local and regional professional development providers. For example, pre-service and professional development programs could help teachers develop deeper understanding of the central ideas in mathematics, how students typically master these ideas and develop more sophisticated understanding over time, and how to diagnose student progress and instructional needs. Trajectories could be used to help novice teachers understand how student understanding develops over time and how instruction affects their development.

Learning trajectories also can help to leverage the work done by mathematics education researchers and learning scientists by developing a body of work in the content areas that is immediately useful to policymakers and practitioners. Trajectories could bring focus to research; and instead of undertaking many small disconnected studies, the field could begin to build programs of research addressing the gaps in the progression work. This approach would build a stronger knowledge base for teaching and for the development of instructional tools and supports. The research on trajectories could highlight the areas where more research is badly needed (e.g., research on topics lacking trajectories, targeting specific age levels where our knowledge is thin, addressing the needs of culturally or linguistically diverse groups who do not perform well in mathematics, etc.).

Learning trajectories have enormous potential, but as the recommendations listed above make clear, there is a great deal of work to be done to realize this potential. If we are serious about making our students college- and career-ready, and about eliminating achievement gaps, this is the work that must be done. Pursuing quick fixes and structural solutions to the problems of public education will not do the job. As we have learned in medicine, agriculture, and other fields, there is no substitute for developing basic knowledge and translating it into tools that practitioners can use to solve the problems they face everyday. As we recommended for science education, a serious research and development effort in mathematics is needed to provide our teachers with the tools they need to do the job. Investing in learning trajectories would not solve all of our problems, but it would put us on the right path toward finding solutions.

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APPENDIX A: A SAMPLE OF MATHEMATICS LEARNING TRAJECTORIES

Introduction

This table below presents a sample of mathematics learning trajectories, along with a few examples of research that supports the development of such trajectories. It is not meant to be exhaustive. Our goal is to provide a sense of the range of topics for which learning trajectories have been developed, and enough detail about the examples to provide a glimpse into their content. The table includes the work of those who have contributed to this report, together with some other examples of significant mathematics learning trajectory research.

Some differences and similarities among learning trajectories are clear from the table, but others are less apparent; several similarities and differences not captured in the table are worth noting briefly here. For example, some researchers focus primarily on instructional tasks or activities that teachers and others can use to elicit and assess children's mathematical understandings (e.g., Battista, 2006, 2007) whereas others focus more on tasks designed to support learners' movement from one level of understanding to another in specific ways (Clements & Sarama, 2009; Barrett et al., 2009; Sherin & Fuson, 2005; Confrey et al., 2009). In many cases, this distinction is hard to make (and in some cases the distinction doesn't matter), but the purpose of a given learning trajectory does make a difference for the kinds of tasks or activities that get included in a trajectory. The result is that some present a continuum of tasks that are well-connected and build on each other in specific ways over time (e.g., Clements & Sarama 2009, Barrett et al., 2009), others present tasks that connect across topical areas of school mathematics (e.g., Confrey et al., 2009), and others offer more detailed guidance to teachers in understanding the capacities and misconceptions of their students at different points in their learning of a particular topic (e.g., Battista, 2006; Sherin & Fuson, 2005).

A related difference is the degree to which a strictly ordered sequence of understandings and abilities is to be expected or supported: in some research programs, learners' movements among levels are varied, with multiple routes or paths to higher levels (e.g., Battista, 2006, 2007), whereas in other programs, learners are expected and encouraged to move through levels in a certain order (e.g., Clements & Sarama, 2009, Barrett et al., 2009). Others have found evidence that supports a hybrid of strict sequencing for some abilities and tasks, with multiple possible pathways to others (e.g., Confrey et al., 2009). There is a resulting difference in emphasis—on developing descriptions of different levels of children's understanding and ability vs. identifying the most effective sequence for moving learners to higher levels of understanding of a given topic—but of course, most learning trajectory research does both of these, at least to some extent.

While all researchers in this field are aware of typical pre-coherent ways of thinking that are often labeled 'misconceptions', they focus on these in different ways in their work. Some of Battista's "levels of sophistication," for example, are themselves examples of "incorrect reasoning" that are precursors to important corrections in reasoning. Confrey et al. (2009) include "predictable patterns of errors" as components of trajectories, and include "obstacles" in their visual representation of trajectories as paths through a conceptual corridor. Clements and Sarama (2009) include fairly detailed accounts of typical misconceptions in their descriptions of the levels, along with direct quotes from children in their studies to illustrate these. The diagnostic value of these different ways of focusing on incorrect, immature, or provisional student thinking makes a difference both for teachers and for researchers interested in understanding and documenting progress from prior knowledge to new knowledge.

While there are also differences in grain size and level of detail included for each level or stage, and obvious differences in time span covered with each trajectory, the significance of these differences has more to do with the different purposes for which they have been developed—along with the constraints on resources available for the particular research programs—rather than a fundamental disagreement about ideal grain size or appropriate time span.[17] As with learning progressions in science, the trajectories share a common purpose in developing instructional sequences that are directly linked to empirical evidence of what 'works' (Corcoran, Mosher, & Rogat, 2009, p.8).

[17] Appropriate grain size was a point of contention during the August 2009 meetings held at the Friday Institute at NCSU, but the issue was framed by a larger discussion about the different purposes that learning trajectories serve.

APPENDIX A:
A SAMPLE OF MATHEMATICS LEARNING TRAJECTORIES

Because learning trajectories weave together what we know about cognitive development, instructional practice, and the coherence of mathematical ideas, most learning trajectory research aims to answer questions about what 'works' by evaluating empirical evidence through all three lenses, at least to some extent. That is, the aim is to develop trajectories that: 1) are chronologically *predictive*, in the sense that students do—or can, with appropriate instruction—move successfully from one level to the next more or less in the predicted sequence of levels; 2) *yield positive results*, for example, deepened conceptual understanding and transferability of knowledge and skills, as determined by external assessment or by assessment built into the learning trajectory; and 3) have learning goals that are *mathematically valuable*, that is align with broad agreement on what mathematics students *ought* to learn (now presumably reflected in the Common Core State Standards (CCSS).

At a more basic level, there is fundamental agreement among learning trajectory researchers on the focus on 1) mathematical thinking that is typical of students at different ages and grade levels; 2) major conceptual shifts that result from the coalescence of smaller shifts; and 3) getting the sequence right, based on (1) and (2), for teaching pivotal mathematical ideas and concepts.[18] Now that the CCSS for mathematics are out, they might serve to define more clearly the agreed upon goals for which specific learning trajectories must still be developed, insofar as they describe the pivotal ideas and concepts of school mathematics. Getting the sequence right, however, is not guaranteed by these descriptions. It involves testing the hypothesized dependency of one idea on another, with particular attention to areas where cognitive dependencies are potentially different from logical dependencies as a mathematician sees them.

As the empirical evidence grows for what works best to move students up the steepest slopes of learning, or most efficiently through a particular terrain of mathematical insights and potential misconceptions, learning trajectory researchers are answering questions about when instruction should follow a logical sequence of deduction from precise definitions and when instruction that builds on a more complex mixture of cognitive factors and prior knowledge is more effective. As stated above, this table does not represent all of the research working to answer these questions. It does, however, provide a sense of the kinds of answers that are already available, as well as some sense of the key areas of mathematics and age groups for which important questions remain.

[18] Developing a more precise way to talk about "major" conceptual shifts (and to distinguish them from not so major shifts) is one area for further theoretical and empirical investigation. Note that this issue shows up in the science work as well (see Wiser et al., 2009).

Appendix A. A Sample of Mathematics Learning Trajectories

TOPIC	GRADE/AGE	DEVELOPER/ PUBLISHER	FRAMEWORK
Number Core	Ages 2-7 yrs	NRC report (Cross, Woods, & Schweingruber, 2009)	The NRC number core focuses on four components: 1) seeing cardinality (seeing how many there are); 2) knowing the number word list (one, two, three, four, etc.); 3) 1-1 counting correspondences when counting; and 4) written number symbols (1, 2, 3, etc.). The number word list involves larger numbers than the other three components. Children's knowledge increases by learning larger numbers for each component and by making connections among them. First, they connect saying the number word list with 1-1 correspondences to begin counting objects. A crucial second step is connecting counting and cardinality so that the last count word tells how many there are. The third step connects counting and cardinality in the opposite direction: 4-year-olds/Pre-Kindergartners come to be able to count out a specified number of objects (e.g., six). Later steps involve relating groups of tens and ones to number words and place-value notation from 10 to 100 and then larger groups to each larger place value.
Recognition of Number and Subitizing	Ages 0-8 yrs	Clements & Sarama (2009)	Clements and Sarama define five separate, but highly interrelated, learning trajectories in the area of number and quantitative thinking. The trajectory for "recognition of number and subitizing" describes instructional tasks that support a developmental progression beginning with an infant's dishabituation to number without explicit, intentional knowledge of number. A child progresses to a stage of naming groups of one to two, and then becomes able to make one small collection with the same number as another collection. After that, tasks are designed to support a child's ability subitize perceptually--to instantly recognize the number of objects in a collection--up to four and to verbally name the number of items; later, a child becomes capable of subitizing larger numbers of items, up to five and then up to ten. A child can then learn to subitize conceptually--instantly seeing parts of a collection and combining them to know the total number--often still unconsciously. Later a child learns to skip count, and then begins to develop an understanding of place value. Finally, around age 8, a child can verbally label structured arrangements using groups (e.g., of tens, threes, etc.), multiplication, and place value.
Counting	Ages 1-7 yrs	Clements & Sarama (2009)	According to Clements and Sarama, counting is a child's main strategy for quantification. Around age 1, a child can name and sing/chant some numbers with no sequence, and can later verbally count with separate words, not necessarily in the correct order, up to five, and then up to ten with some correspondence with objects. The instructional tasks of the counting trajectory support children's progress in their ability to keep one-to-one correspondence between counting words and objects, and then in their ability to accurately count objects in a line and answer "how many?" question with the last number counted. A variety of instructional tasks then support a child's ability to count arrangements of objects to ten, to write numerals 1-10, and to count backward. Later, a child counts objects verbally beginning with numbers other than one, and can then skip count, using patterns to count. Then a child can count imagined items, can keep track of counting acts, and can count units with an understanding of base-ten and place value. Around age 7, a child can consistently conserve number even in the face of perceptual distractions such as spreading out objects in a collection, and can count forward and backward.

Appendix A. A Sample of Mathematics Learning Trajectories

Relations Core	Ages 2-7 yrs	NRC report (Cross, Woods, & Schweingruber, 2009)	The "teaching-learning path" for the Relations Core focuses on the relations of "more than," "less than," and "equal to" on two sets of objects. Initially, children use general perceptual, length, or density strategies to decide whether one set is more than, less than or equal to another set, for sets of up to five entities. These are replaced by the more accurate strategies of matching entities in two sets to find out which has leftover entities, or counting both sets and using understandings of more than/less than for sets of up to five entities. Later, children compare situations with objects or drawings by counting or matching for numbers up to 10. By grade 1, children can answer the question "how many more (or less) is one group than another?" Thus, they can begin to see a third set potentially present in relational situations: the difference between the smaller set and the larger set. In this way, instruction can cultivate an understanding of relational situations as another kind of addition/subtraction situation.
Comparing, Ordering, and Estimating Numbers	Ages 0-8 yrs	Clements & Sarama (2009)	This trajectory builds on research that shows infants as young as 10 months begin developing the ability to construct equivalence relations between sets, possibly by intuitively establishing spatial temporal, or numerical correspondences. They can put objects, words, or actions in one-to-one or many-to-one correspondence, or a mixture of the two, and then develop an implicit sensitivity to the relation of more/less for very small numbers. Instructional tasks early in the trajectory support a child's developing ability to match groups by putting objects in one-to-one correspondence, to use the words "more," "less," or "same," and to compare collections very different in size perceptually and very small collections using number words. Later the tasks focus on the ability to identify the "first" and "second" objects in a sequence, and on verbally and non-verbally comparing collections of one to four of the same item, and later dissimilar items. A child then develops the ability to compare groups of one to five or six objects by matching and counting when objects are about the same size. Around the same time, a child can determine relative size and position with perceptual support for constructing a number line from zero to five, and then zero to ten. A child can then compare more accurately by counting how many more or less are in a collection, even when objects are different sizes; they also use ordinal numbers "first" through "tenth," and classify and compare sets with words like "little" and "big" based on spatial extent, not number. Instructional tasks then support ordering collections and numbers serially, including lengths marked with units, and estimating numerosity. Later, a child develops place value understanding, is able to construct a mental number line to 100, and can estimate quantities intuitively with fast scanning. Finally, around age 8, a child constructs a mental number line to 1000s, can estimate by counting a portion of the collection to use as a benchmark for estimating the size of the whole collection, estimates regularly and then irregularly arranged sets, and eventually can decompose or partition collections to determine numerosity by multiplication.

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Operations Core	Ages 2-7 yrs	NRC Report (Cross, Woods, & Schweingruber, 2009)	Children learn to see addition and subtraction by modeling or "mathematizing" aspects of real world situations, and they build language skills with word problems representing these situations. There are three types of addition/subtraction situations: change plus/change minus, put together/take apart (also called combine), and comparisons. Research supports three stages of understanding: Level 1 (primarily through kindergarten) - direct modeling of all quantities in a situation (counting things or counting with fingers); Level 2 (primarily in Grade 1) - using embedded number understanding to see the first addend within the total in order to "count on" to find a total and to find an unknown addend for subtraction; Level 3 - learning recomposing derived-fact methods that convert problems into simpler problems, such as "make-a-ten." Children become fluent early with particular sums and differences, such as plus one and minus one and doubles (e.g., 2 + 2, 3 + 3). They become fluent with others over time. At all levels, the solution methods require mathematizing the real-world situation (or later the word problem or the problem represented with numbers) to focus on only the mathematical aspects -- the numbers of things, the additive or subtractive operation in the situation, and the quantity that is unknown.
Addition and Subtraction (Emphasizing Counting Strategies)	Ages 1-7 yrs	Clements & Sarama (2009)	Clements and Sarama develop two separate trajectories for addition and subtraction: one for counting-based strategies, and one for composition of numbers and place value. The counting-based strategies trajectory begins with a child's pre-explicit sensitivity to adding and subtracting perceptually combined groups, and builds on their early ability to add and subtract very small collections nonverbally. Instructional tasks support a young child's ability to find sums up to 3+2 using a counting-all strategy with objects, and then to find sums for joining and part-part-whole problems by direct modeling or counting-all strategies using objects. Around the same time, children develop the ability to solve take-away problems by separating objects, and to add objects by making one number into another without needing to count from one, and can also find a missing addend by adding objects. Later, the trajectory supports a child's ability to find sums for joining and part-part-whole problems with fingers, by "counting on," and/or by "counting up to" to find a missing addend. Instructional tasks then focus on a child's developing understanding of part-whole relations, including "start unknown" problems, and then support their recognition of when a number is part of a whole and their ability to keep the part and the whole in mind simultaneously. Later the tasks focus on problem solving with derived combinations and flexible strategies, including moving part of a number to another with awareness of the increase/decrease and simple multidigit addition by incrementing tens or ones. Finally, around age 7, a child solves multidigit addition and subtraction and transitions to arithmetic based on composition of number, the next trajectory.

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Composing Number and Multidigit Addition and Subtraction	Ages 0-8 yrs	Clements & Sarama (2009)	This trajectory parallels the counting-based trajectory for addition and subtraction with a focus on doing arithmetic by composing and decomposing numbers. It begins with non-verbal recognition of parts and wholes between ages 0-2. Related tasks in the counting-based trajectory support a child's developing recognition that a whole is bigger than its parts, and then tasks in this trajectory support awareness of number combinations (e.g., 4 is 1 and 3) using fingers and objects, beginning with composition to four and building to composition to ten and higher. The trajectory then focuses on a child's ability to solve problems using derived combinations (e.g., 7+7=14, so 7+8=15) and "break-apart-to-make-ten." Around this time, a child can simultaneously think of three numbers within a sum, and can move part of a number to another with awareness of the increase/decrease. Later, a child can compose 2-digit numbers as tens and ones in the context of a variety of tasks involving cards and cubes, and can also solve all types of problems with flexible strategies, known combinations, and incrementing or combining tens and ones. Around age 7-8, a child uses composition of tens and all previous strategies to solve multidigit addition and subtraction problems in a wide range of tasks, including with number line contexts, calculators, and word problems.
Multiplication Strategies	3rd Grade	Sherin & Fuson (2005)	Sherin and Fuson build on their own and others' research to develop a taxonomy for multiplication computational strategies-- which they define as patterns in computational activity as opposed to knowledge possessed by individuals-- and they discuss learning progressions through this taxonomy. The taxonomy includes the following strategies for multiplication, in order of frequency of use from early to late in the cycle: count all, additive calculation, count by, pattern-based, learned products, and hybrids of these strategies. Each strategy progresses separately, and later computational resources merge into richer knowledge of the multiplicative structure of whole numbers up to eighty-one. Strategies may be difficult to differentiate early in the learning process for small multiplicands, and integration of strategies becomes more pervasive as expertise increases with instruction. While they agree with preceding research showing that strategy development for addition is largely driven by changes in how children conceptualize quantities, they find strategy development for single-digit multiplication to be heavily number-specific and driven by incremental mechanisms of number-specific computational knowledge, including learned products with certain operands. Because early learning of single-digit multiplication is so sensitive to the number-specific resources available to a child and these resources result from the timing and foci of instruction, including the order of particular numbers, variation across cultural and instructional contexts is to be expected.

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Multiplicative Thinking/ Rational Number Reasoning	K-Grade 8	Confrey et al. (2009)	Confrey and co-workers in the DELTA project (Diagnostic E-Learning Trajectories Approach) have conducted syntheses of mathematics education and cognitive psychology research pertaining to rational number reasoning. From these syntheses, they identified seven major learning trajectories, and combined them into a comprehensive framework for rational number reasoning, a complex set of constructs that spans elementary and middle school years. The learning trajectories are equipartitioning, Multiplication and Division, Fraction-as-Number, Ratio and Rate, Similarity and Scaling, Linear and Area Measurement, and Decimals and Percents. Confrey's structure further identifies three major meanings for a/b that encapsulate the major rational number understandings that children must distinguish and move fluently among in order to develop the strong rational number reasoning competencies necessary for algebraic reasoning: a/b as ratio, a/b as fraction, and "a/b of" as operator. Confrey has identified the construct of equipartitioning/splitting as foundational for these three rational number reasoning constructs. The research on equipartitioning, in particular, demonstrates that multiplicative reasoning based on rational number reasoning has cognitive roots distinct from counting and additive reasoning, and that children can develop early multiplicative reasoning independent of addition and subtraction. Current curricula that introduce young children to multiplication-as-repeated-addition, without parallel early development of the rational number constructs of equipartitioning, division and multiplication, ratio, and fraction, may actually inhibit development of multiplicative (and algebraic) reasoning by constraining children's multiplicative concepts solely to extensions of addition. Confrey et. al. are developing diagnostic measures based on these learning trajectories for charting student rational number learning progress profiles within and across academic years.
Shape	Ages 2-5 yrs (Kindergarten)	NRC Report (Cross, Woods, & Schweingruber, 2009)	The Geometry and Spatial Relations Core in the NRC report is divided into three areas--shape, spatial structure/spatial thinking, and measurement--and includes composition and decomposition as a category of learning in all three areas. The teaching-learning paths in this core describe children's competence on the basis of three levels of sophistication in thinking: thinking visually/holistically, thinking about parts, and relating parts and wholes. The teaching-learning path for shape is based on children's innate and implicit ability to recognize shapes; some of the goals in this path are as follows. Very young children do not reliably or explicitly distinguish circles, triangles, and squares from other shapes, but at ages 2 and 3 they begin to think visually or holistically about these shapes and can form schemes for these shape categories. Also at this time they can discriminate between 2D and 3D shapes by matching or naming, and can represent real world objects with blocks. Next, children learn to describe, then analyze, geometric figures at multiple orientations by thinking about the parts of shapes--e.g., the number of sides and/or corners--and can combine shapes and recognize new shapes they create. They can also compose building blocks to produce arches, enclosures, etc. systematically. They increasingly see relationships between parts of shapes and understand these relationships in terms of properties of the shapes. Finally, around age 5, they can name and describe the difference between 2D and 3D shapes, and can relate parts and wholes, including being able to identify faces of 3D shapes as 2D shapes and to describe congruent faces and, in certain activity contexts, parallel faces of blocks.

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Shapes; Composition and Decomposition of Shapes	Ages 0-8 yrs	Clements & Sarama (2009)	Clements and Sarama's learning trajectory for shapes incorporates four sub-trajectories that are related, but can develop somewhat independently: (a) Comparing (matching shapes by different criteria in the early levels and determining congruence); (b) Classifying (recognizing, identifying, analyzing and classifying shapes); (c) Parts (distinguishing, naming, describing, and quantifying the components of shapes, e.g., sides and angles); and (d) Representing (building and drawing shapes). Instruction to support the developmental progressions within these subtrajectories includes matching with blocks, manipulations in computer environments, "feely boxes," "shape hunts,"making shapes with straws, and "guess my rule" games for classifying shapes. Clements and Sarama also developed three separate but closely related trajectories for composition and decomposition of shapes: Composition of 3D shapes, which focuses on using building blocks to stack, line up, assemble, integrate, extend, and substitute shapes, including stairs, ramps, enclosures, bridges, and arches; Composition and decomposition of 2D shapes, which includes work with Pattern Block Puzzles and Geometry Snapshots; and Disembedding of geometric shapes, which is a tentative learning trajectory and focuses on identifying shapes within shapes.
Spatial Thinking	Ages 2-5 yrs (Kindergarten)	NRC Report (Cross, Woods, & Schweingruber, 2009)	In the NRC report, spatial thinking includes two main abilities: spatial orientation and spatial visualization and imagery. Vocabulary for spatial relationships are acquired in a consistent order (even across different languages): first, "in," "on," "under," "up," and "down"; then "beside" and "between"; later, "in front of" and "behind"; and much later, "left" and "right" (the source of enduring confusion). From as young as age 2, children can implicitly use knowledge of landmarks and distances between them to remember locations, and later begin to build mental representations and models of spatial relationships, including understanding what others need to hear in order to follow a route through a space. They also learn to apply spatial relationship vocabulary in 2D and 3D contexts, such as the "bottom" of a picture they are drawing. At the 'thinking about parts' level, preschoolers recognize "matching" shapes at different orientations, and can learn to check if pairs of 2D shapes are congruent using geometric motions intuitively, moving from less accurate strategies such as side-matching or using lengths, to superimposition; later they use geometric motions of slides, flips, and turns explicitly and intentionally in discussing solutions to puzzles or in computer environments. Predicting the effects of geometric motions lays the foundation for thinking at the relating parts and wholes level. Children also develop the ability to cover space with tiles, and represent their tilings with simple drawings, initially with gaps and misalignment; later they learn to count the squares in their tiling, with increasingly systematic strategies for keeping track. Their ability to build models of 3D objects is supported by stacking blocks and other tasks, and finally, kindergartners can understand and replicate the perspectives of different viewers.

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Spatial Thinking	Ages 0-8+ yrs	Clements & Sarama (2009)	Clements and Sarama describe separate trajectories for spatial orientation and spatial visualization/imagery; they include maps and coordinates in their trajectory for spatial orientation, which begins with children using landmarks to find objects. Building on this ability, children are able to find objects even after moving themselves relative to the landmark, and later even if the target is not specified ahead of time; they are then able to maintain in their head the overall shape of objects' arrangement in space, with some use of coordinate labels in simple situations like playground or classroom models or maps. Later children can read and plot coordinates on maps, and finally, at around age 8+, they can follow route maps and use frameworks that include observer and landmarks, with measurement precision dependent on instruction. The spatial visualization and imagery trajectory begins with duplicating and moving shapes to a specified location by sliding, and later, mentally turning them, and then increasingly by sliding, flipping, and turning horizontally, vertically, and then diagonally. Around age 8+, a child can predict the results of transformations using mental images of initial state, motion, and final state. Instruction supporting the development and integration of systems for coding spatial relations includes "feely boxes" that contain shapes to identify through touch, tangram puzzles, and computer environments involving snapshots with figures to match. The trajectory leads a child to view spatial figures from multiple perspectives by constructing mental representations of 2D and 3D space.
Measurement	Ages 2-5 yrs (Kindergarten)	NRC Report (Cross, Woods, & Schweingruber, 2009)	The NRC report discusses the importance of measurement for connecting the two crucial realms of geometry and number, and presents elements of teaching-learning paths for length, area, and volume measurement. The report provides significantly more detail for the teaching-learning path for length measurement, which is divided into the categories 'Objects and Spatial Relations' and 'Compositions and Decompositions' and organized into three levels of sophistication in thinking: thinking visually/holistically, thinking about parts, and relating parts and wholes. The report emphasizes experiences that give children opportunities to compare sizes of objects and to connect number to length, and opportunities to solve real measurement problems that help build their understanding of units, length-unit iteration, correct alignment (with a ruler) and the concept of the zero-point. Children's early competency in measurement is facilitated by play with structured manipulatives such as unit blocks, pattern blocks, and tiles, together with measurement of the same objects with rulers; this competency is strengthened with opportunities to reflect on and discuss these experiences. Children also need experience covering surfaces with appropriate measurement units, counting those units, and spatially structuring the object they are to measure, in order to build a foundation for eventual use of formulas. By around ages 4-5, most children can learn to reason about measurement, but before kindergarten, many children lack understanding of measurement ideas and procedures, such as lining up end points when comparing the lengths of two objects.

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Geometric Measurement	Ages 0-9 yrs	Clements & Sarama (2009)	Clements and Sarama emphasize the significance of measurement in early mathematics in relating to children's experience with the physical world, in connecting geometry and number, and in combining skills with foundational concepts such as conservation, transitivity, equal partitioning, unit, iteration of standard units, accumulation of distance, and origin. They develop separate trajectories for measurement of length (2-8 yrs), area (0-8 yrs), volume (0-9 yrs), and angle and turn (2-8+ yrs), and point out that some research suggests spatial structuring develops in order of one, then two, then three dimensions; they emphasize that comparisons and contrasts among unit structures in all areas should be explicit throughout. The trajectory for length measurement includes instruction that moves children from informal to formal measurement: from conversations about things that are "long," "tall," etc., to directly comparing heights and lengths, to using their arm as a unit of measurement, to measuring with toothpicks or other physical or drawn units. By around age 8, children can use a ruler proficiently, create their own units, and estimate irregular lengths by mentally segmenting objects and counting the segments. The trajectory for area measurement focuses on different ways of comparing, covering, and structuring space: from comparing sizes of paper, to comparing rectangles composed of unit squares, to comparing by counting rows of arrays; from covering by quilting and tiling, to covering by filling in missing rows or columns (or sections of rows or columns) of an array; from structuring by partitioning into subregions, to structuring by covering, to structuring by aligning units and creating rows and columns to create arrays. Around age 7, a child can conserve area and reason about additive composition of areas, and around age 8, they can multiplicatively iterate squares in a row or column to determine the area of a rectangle. The trajectory for volume measurement moves from directly and then indirectly comparing container capacities, to using unit cubes to fill boxes, to structuring 3D space using unit cubes and then using multiplicative thinking about rows and columns of units. By around age 9, a child can determine volume of a pictured box and then a box with only dimensions given by multiplicatively iterating units. The trajectory for angle and turn measurement, based on the division of a circle, moves from building and using angles, to matching congruent angles (including those that are part of different shapes), to comparing angles, and finally to measuring angles by age 8+.

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Length	Ages 3-12 yrs	Barrett et al. (2009)	Barrett et al. characterize each successive level of their length trajectory as increasingly sophisticated and integrative of prior levels, with fallback among levels to be expected. The higher layers indicate increasingly abstract patterns of reasoning that become dominant over time. However, the child retains even the earliest layers of reasoning about mathematical objects, though they are decreasingly engaged. The trajectory moves through a specific, idealized sequence of instructional activities based on empirical accounts of children's reasoning as they were guided through longitudinal teaching experiments; it describes a hierarchical sequence of 10 levels, and links observable actions, hypothesized internal actions on mental objects, and instructional tasks specific to each level. Initially, children acquire geometric concepts and language to discriminate length from area or volume dimensions as they make comparisons. Around age 6, children incorporate spatial operations such as sweeping or pointing along a line as part of their measuring scheme. Next, students gain unit operations and find ways of keeping an exact correspondence between counting and unit iteration as they extend indirect comparison into quantitative comparisons, gradually establishing a conviction about the conservation of quantity. Unit operations gain coherence as students coordinate point counting, interval counting, and the arithmetic operations on the successive number labels along a coordinate line (age 8-10), leading to the integration of measures for bent paths with the measures of each of the straight parts of those paths to reason about classes of paths (as with perimeters of several shapes having the same area) (age 11-12).
Length Measurement	Elementary School	Battista (2006)	Battista's cognition-based assessment system is not a learning trajectory per se, but traces students' understandings of measurement concepts as they develop without trajectory-targeted instruction from non-numerical to numerical understandings, from concrete and informal to abstract and formal understandings, and from less to more sophisticated, integrated, and coherent understandings. The model describes the cognitive terrain for learning about length and length measurement as a set of adjacent plateaus to represent different "levels of sophistication," and allows for multiple routes up from the bottom to the top levels rather than prescribing a single route or sequence. The levels of sophistication are arranged into two types of reasoning: (1) non-measurement reasoning (in which numbers are not used) begins with vague visual comparison, then progresses to correct comparison of straight paths, then systematic manipulation and comparison of parts of shapes, and finally to comparing lengths using transformations and geometric properties of shapes; (2) measurement reasoning (in which numbers are used) begins with uses of numbers unconnected or improperly connected to unit-length iterations, then progresses to correct unit-length iteration, then to operating on iterations numerically and logically, then operating on measurement numbers without iteration, and finally understanding and using formulas and variables to reason about length. These levels are described in order from least to most sophisticated, but students' paths through them may vary.

APPENDIX B: OGAP MULTIPLICATIVE REASONING FRAMEWORK—MULTIPLICATION

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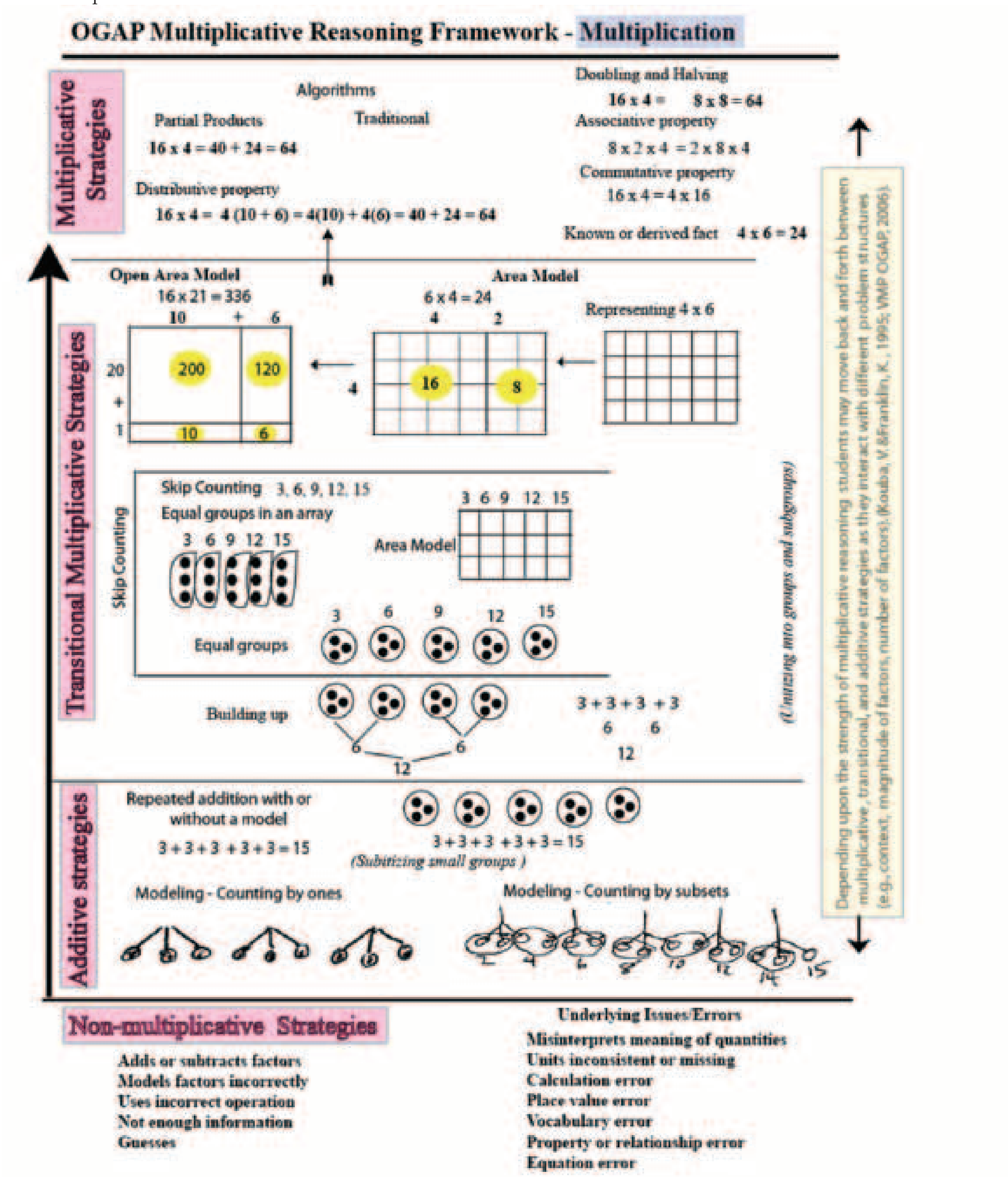

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