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Comments

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Coherence and Consistency in Domains *
(Extended Outline)

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Abstract

Almost all of the categories normally used as a mathematical foundation for denotational semantics satisfy a condition known as *consistent completeness.* The goal of this paper is to explore the possibility of using a different condition—that of *coherence*—which has its origins in topology and logic. In particular, we concentrate on those posets whose principal ideals are algebraic lattices and whose topologies are coherent. These form a cartesian closed category which has fixed points for domain equations. It is shown that a "universal domain" exists. Since the construction of this domain seems to be of general significance, a categorical treatment is provided and its relationship to other applications discussed.

1 Introduction.

The first structures used as a mathematical foundation for the denotational semantics of programming languages were lattices. With lattices it was possible to solve the necessary recursive equations and an elegant mathematical theory could be developed using the familiar category of (countably based) algebraic lattices [Sco76] (although it was necessary to take some care to choose the right notion of morphism). As experience with denotational semantics grew, deeper computational intuitions were developed and new categories were introduced in attempts to match these intuitions to the mathematical constructs. For example, it was desirable to have a class of domains which included such structures as the partial functions from natural numbers to natural numbers which—under their usual ordering—do not form a lattice. Such theories were proposed by Plotkin [Plo78], Berry [Ber78] and also Scott [Sco81, Sco82a, Sco82b].

The category which Scott proposed was very similar to the algebraic lattices: a cpo D is said to be a *Scott domain* if the cpo $D^\top$ obtained by adding a top to D is an algebraic lattice

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(with a countable basis). The arrows of the category are continuous functions, *i.e.* monotone functions which preserve joins of directed collections of elements. The category of Scott domains is easy to work with and has an intuitive logical character which has been the subject of several investigations (see, in particular, [Sco82b, Abr87]). One central feature of these treatments is the concept of *consistency* of data. One may think of a Scott domain as a collection of propositions or data elements under an ordering of *partial information.* An element x is ordered below an element y in a domain D if x is "more partial" than y. The element x is a kind of partial description of y. Now, given two data elements x_1 and x_2, there may or may not be a third element y which they describe. If there is such a y, then x_1 and x_2 are said to be *consistent,* otherwise they are *inconsistent.* A crucial feature of a Scott domain is the following fact: *if two elements of a Scott domain D are consistent, then they have a join in D.* This property is commonly referred to as *consistent completeness.*

The use of consistent complete domains for modeling the semantics of types in programming languages has become the general practice. However, we would like to note in this paper that *it is not the only reasonable direction the theory could have taken* at the point that consistency was recognized as a central concept. Up until the time we are writing this paper, almost all of the categories of domains that have been proposed as a possible foundation for the semantics of programming languages have been (essentially equivalent to) cpo's which satisfy the consistent completeness condition. This includes those categories which use stable continuous functions [Ber78, Gir86] as well as categories related to the Scott domains (such as the continuous lattices).[1] The one noteworthy exception is the category of strongly algebraic domains which was introduced by Plotkin [Plo76] (where it is called **SFP**). These will be discussed below.

One might apply the following line of reasoning in an attempt to deal with the concept of consistency of data. A domain is a collection of propositions providing partial descriptions of elements (which may also be propositions describing further elements); a given element dominates a collection of data elements which provide partial descriptions of it. We propose the following condition on the structure of the partial descriptions of an element: *the partial descriptions of an element must form an algebraic lattice.* Let us refer to this condition as *local algebraicity.* But a locally algebraic cpo (with a countable basis) is just a Scott domain right? No, not at all! Aside from the fact that such a domain need not have a least element (an infinite discrete domain is locally algebraic for example) it is even possible that a consistent pair of elements have no join! (See Figure 1.) One can show, however, that almost all of the essential features needed to provide semantics for programming languages are satisfied by locally algebraic domains.

The concept of a locally algebraic domain was formulated by the second author who came across the concept in the course of his investigations into extensions of Smyth's Theorem [Jun88]. We refer to locally algebraic domains as *L-domains* to keep the terminology short. They were independently discovered by Thierry Coquand as a special instance of his categories of embeddings [Coq88]. We will discuss some basic properties of L-domains in the next section—for a more detailed discus-

[1]We omit from discussion categories of cpo's with no assumptions about the existence of a basis.

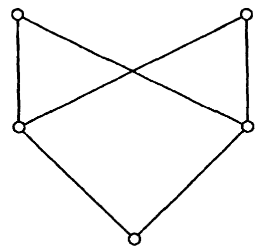


Figure 1: A locally algebraic domain which is not consistent complete.

sion, the reader can examine [Coq88, Jun88]. The bulk of the paper will focus on the properties of a subcategory of the L-domains which were introduced in the first author's doctoral dissertation [Gun85]. The category which was investigated there (the objects were called *short* domains) consisted of those L-domains which were strongly algebraic. It was proved there that such domains formed a cartesian closed category in which one could solve recursive domain equations. However, we would like to demonstrate a further fact about them below. Namely, that *there is a "universal" domain in this category.* Our construction is similar to that which appears in [Gun87] for the strongly algebraic domains, but a more subtle ordering is needed to make things work properly.

The paper is divided into five sections. Section two provides some definitions and establishes notation. A few basic propositions are also remarked. The third section discusses the coherence condition on the topology of a domain. We show how this condition translates into an order-theoretic one and discuss some important properties of domains with coherent topologies. The fourth section discusses the universal domain construction. Since this construction seems to have a general significance, we have attempted to provide a categorical treatment of it. This categorical treatment makes it possible to see the construction in this paper and the one that was presented in [Gun87] as instances of a more general theory which may have applications in other cases. The fifth and final section contains some concluding remarks.

In order to make the discussion as succinct as possible, we have omitted almost all of the proofs for this extended outline version. A fuller version of the paper will contain all of the non-trivial proofs.

2 Basic definitions and facts.

For the purposes of this paper a cpo (complete poset) is a poset $\langle D, \sqsubseteq \rangle$ with joins $\bigsqcup M$ for all directed subsets M. (Sometimes a cpo is required to have a least element $\bot$, but this is not being required here.) A function $f : D \to E$ between cpo's D and E is *continuous* if it is monotone and preserves joins of directed subsets of D. An element x of a cpo D is said to be *compact* if, whenever M is a directed subset of D and $x \sqsubseteq \bigsqcup M$, then there is a $y \in M$ such that $x \sqsubseteq y$. Let D^0 be the collection of compact elements of a cpo D. A cpo D is said to be *algebraic* if, for every $x \in D$, the set M of elements $x_0 \in D^0$ such that $x_0 \sqsubseteq x$ is directed and $\bigsqcup M = x$. D is said to be ω-algebraic if it is algebraic and D^0 is countable. An *algebraic lattice* is an algebraic cpo which is a lattice.

Definition: A cpo D is *locally algebraic* if, for every $x \in D$, the principal ideal

$$\downarrow x = \{y \in D \mid y \sqsubseteq x\}$$

generated by x is an algebraic lattice. ∎

Proposition 1 *If D is locally algebraic, then it is algebraic.* ∎

To keep the terminology short, we will refer to locally algebraic cpo's as *L-domains*. The category of L-domains properly contains the class of Scott-domains: Figure 1 shows an example. The difference between the two concepts is illustrated by the following characterizations:

Proposition 2 *Let D be an algebraic cpo.*

1. *D is a Scott-domain, if and only if every nonempty subset has a meet in D.*
2. *D is an L-domain, if and only if every bounded nonempty subset has a meet in D.* ∎

The difference may seem a slight one but it has some important consequences. The basis of the function space of a Scott-domain D has always the same cardinality as D^0, whereas the cardinality may increase if D is an L-domain. However, the following (which was found independently by Thierry Coquand) remains true:

Theorem 3 *The category of L-domains and continuous functions is cartesian closed.* ∎

(In fact, the category of L-domains forms a *bicartesian closed category* since it is possible to define a coproduct functor on it.)

In [Jun88] it is proved that, in the category of algebraic cpo's with least element, there are exactly two maximal cartesian closed subcategories: the category of L-domains and the category of profinite domains, which we now proceed to define.

A continuous function $f^L : D \to E$ between cpo's D and E is said to be an *embedding* if there is a continuous function $f^R : E \to D$ such that $f^R \circ f^L = id_D$ and $f^L \circ f^R \sqsubseteq id_E$ where id_D and id_E are the identity functions on D and E respectively. If there is such a function f^R, then it is uniquely determined by f^L and is said to be the *projection* corresponding to f^L. Pairs $f = \langle f^L, f^R \rangle : D \to E$, where f^L is an embedding and f^R the corresponding projection, form the arrows of a category $\mathbf{CPO}^{ep}$ which has cpo's as its objects. Composition is given by

$$\langle f^L, f^R \rangle \circ \langle g^L, g^R \rangle = \langle f^L \circ g^L, g^R \circ f^R \rangle.$$

It is a basic fact in the theory of domains that $\mathbf{CPO}^{ep}$ has directed colimits.

Theorem 4 *The category of L-domains and embedding-projection pairs has directed colimits.* ∎

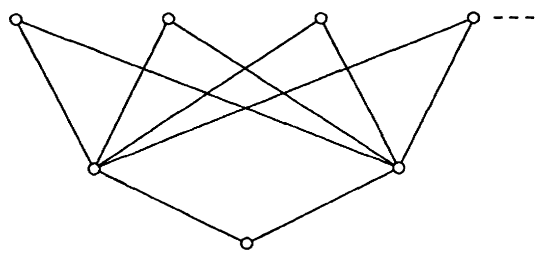


Figure 2: K has a countable basis, but $K \to K$ does not.

If a cpo is a directed colimit in $\mathbf{CPO}^{ep}$ of a family of finite posets, then it is said to be a *profinite domain.*[2] It is possible to show that profinite domains must be algebraic. Let $\mathbf{P}$ and $\mathbf{P}^{ep}$ be the categories of profinite domains with continuous functions and embedding-projection pairs respectively. It is possible to show that $\mathbf{P}$ is a bicartesian closed category and $\mathbf{P}^{ep}$ has colimits of directed families [Gun85, Gun87]. Profinite domains with a countable basis and least element are the "**SFP**-objects" of Plotkin [Plo76]. We will follow Smyth's terminology [Smy83] and refer to them as *strongly algebraic* domains. We write $\mathbf{SA}$ for the category with continuous functions and $\mathbf{SA}^{ep}$ for the category with embedding-projection pairs. The category $\mathbf{SA}$ is a cartesian closed and $\mathbf{SA}^{ep}$ has colimits for countable directed families [Plo76].

3 Coherence.

In order to get a satisfactory class of spaces as domains for denotational semantics it is desirable to impose a more restrictive condition than local algebraicity. Suppose one wished to define a notion of *computability* on L-domains. It might be possible to do this for the L-domains with a countable basis. So why not restrict oneself to these? The problem is that the L-domains with countable basis are not closed under the exponential! Consider the L-domain $\mathbf{N}$ of natural numbers (ordered discretely). The continuous functions from $\mathbf{N}$ to $\mathbf{N}$ are an L-domain, but there is obviously no countable basis. This may seem like a superficial problem, but it is not. Suppose, for example, that we try to fix things by requiring that there be a bottom element. The L-domain of continuous functions from $\mathbf{N}_\perp$ to $\mathbf{N}_\perp$ *does* have a countable basis. But consider the poset K pictured in Figure 2. This is an L-domain with a countable basis but $K \to K$ has a basis with continuum many members.

Since M. Smyth [Smy83] has proved that any domain which has an ω-algebraic function space is in fact profinite, it is reasonable to investigate the category of profinite L-domains which have

[2] Actually, *indfinite* domain might be a better name with this definition. One can show, however, that the domains which are directed colimits of finite elements of $\mathbf{CPO}^{ep}$ are exactly those domains which are codirected limits of finites in the dual category.

countable bases and least elements, *i.e.* the *strongly algebraic L-domains.* The poset in Figure 2 is a typical example of an L-domain that fails to be profinite.

An unfortunate drawback to the profiniteness condition is the fact that it is not very easy to understand. Although intrinsic descriptions are possible and these do help in reasoning about profinite domains, it would still be nice to work with a simpler class of structures. However, it turns out that the strongly algebraic domains which are L-domains may be somewhat more easily characterized than strongly algebraic domains in general. In particular, they may be identified as those L-domains which have a "nice" Scott topology.

We will follow the definitions and notation in Johnstone [Joh82]. A cpo D can be given a topology as follows. The open subsets of the topology are those which satisfy:

1. whenever $x \in U$ and $x \sqsubseteq y$, then $y \in U$, and

2. whenever $M \subseteq D$ is directed and $\bigsqcup M \in U$, then $M \cap U \neq \emptyset$.

This is usually called the *Scott topology* on D and it will be denoted ΣD. It is possible to show that a function $f : D \to E$ between cpo's D and E is continuous in the sense that $f(\bigsqcup M) = \bigsqcup f(M)$, for any directed $M \subseteq D$, if and only if it is continuous in the usual topological sense—with respect to the Scott topology.

Definition: Let D be an algebraic cpo. The topology ΣD is said to be *coherent* if the compact open subsets of D are closed under finite intersections. ▮

We would like to make two brief remarks about this terminology. First, to keep things simple, we have restricted the definition to algebraic cpo's; the definition above would not correspond to the usual notion of a coherent topology if D were allowed to be an arbitrary cpo. Second, we would like to comment that the meaning for the term "coherent" which we have given should not be confused with other meanings from the domain theory literature. In particular, a poset is sometimes said to be coherent if any pairwise consistent set has a least upper bound. This condition is *stronger* than consistent completeness and certainly does not correspond to the condition we are using here!

Coherence is an elegant condition on the topology of a domain D which has an important significance for the order structure of D. Let us say that a poset P has the *strong minimal upper bounds property* (or *property M* for short) if, for every finite subset $u \subseteq P$, the set v of minimal upper bounds of u satisfies the following properties:

1. v has only finitely many elements and

2. v is complete in the sense that for every $p \in P$, if $x \sqsubseteq p$ for every $x \in u$, then $y \sqsubseteq p$ for some $y \in v$.

We have the following:

Proposition 5 *Let D be an algebraic cpo. Then ΣD is coherent if and only if the basis D^0 of D has property M.* ▮

The central theorem of this section states that a profinite L-domain may be characterized using the coherence condition:

Theorem 6 *Let D be an L-domain. Then ΣD is coherent if and only if D is profinite.* ▮

Moreover, since the profinite L-domains lie at the intersection of two nice categories, they inherit some of that niceness themselves:

Proposition 7 *The category of profinite L-domains and continuous functions is a bicartesian closed category.* ▮

Proposition 8 *The category of profinite L-domains and embedding-projection pairs has colimits for directed collections.* ▮

4 Building universal domains.

The concept of a "universal domain" dates back at least to Scott's paper [Sco76] on $\mathbf{P}\omega$ and is widely used in the current literature. The term "universal domain" is somewhat vaguely defined, however. We see basically two uses as being the most common. The easiest of these to understand is what one might call a "poor man's universal domain". Typically it is a domain which satisfies an isomorphism

$$V \cong (V \to V) + F_1(V) + \cdots + F_n(V) \tag{1}$$

where $F_1, \ldots, F_n$ are operators over which domain equations must be solved. One often sees such universal domains being used in the type theory literature [MPS84, Car84]. The theory of domains provides us with all of the mathematical tools generally needed for solving equations like (1) so that we may employ such definitions quite freely and confidently. On the other hand, the poor man's universal domain depends on the choice of the functors F_i (what if we want to add another one?—the universal domain would need to be changed) and it would be nice to know more facts about the order structure of the solution than the existence result for the solution tells us. It is therefore appealing to have a *single* universal domain $\mathcal{U}$ which has *all* domains of interest as retracts. Of course, this is subject to one's interpretation of "domains of interest", but it is not dependent on a commitment to some finite list of functors. We refer the reader to Taylor [Tay87] for a full discussion of universal domains (which he calls "saturated domains"). For the purpose of clarity, let us propose a crude definition of "universal domain" which will give the reader some idea what we are after.

Definition: Let $\mathbf{C}$ be a category. An object $\mathcal{U}$ is *universal* in $\mathbf{C}$ if it is weakly terminal, *i.e.* for every object A of $\mathbf{C}$, there is a (not necessarily unique) arrow $f : A \to \mathcal{U}$. ▮

Of course, any category that has a terminal object has a universal domain. However, one typically has it in mind that the arrows of the category $\mathbf{C}$ are monics. In particular, we show that the category $\mathbf{SALdom}^{ep}$ of strongly algebraic L-domains with embedding-projection pairs has a universal domain.

Figure 3: A typical increment in $\mathbf{SALdom}^{ep}$. The poset on the left is embedded in the poset on the right. The open circles show the image of the embedding.

The proof uses techniques from [Gun87]. However, naively mimicing the construction which appears there will not work. We therefore begin by devising a general theory which can be applied to obtain the universal domain for both $\mathbf{SA}^{ep}$ (as described in [Gun87]) and $\mathbf{SALdom}^{ep}$.

In particular, we provide a categorical treatment of the essential ingredients that make the universal domain construction work. The construction is reminiscent of one from general model theory. For example, one can show that every countable model A has a countably homogenous elementary extension as follows. It is easy to see that A is elementarily embedded in a countable model A_1 which is homogeneous with respect to finite sequences taken from A. One can use a similar construction to build a sequence of models A_i such that, for each $j < i$, the model A_i is homogeneous with respect to finite sequences of elements from A_j and A_j is elementarily embedded in A_i. The colimit of this chain will be the desired homogeneous elementary extension of A. The reader can find many constructions that use this basic idea in a standard book on model theory such as [CK73] (where a more detailed description of the construction above appears on page 130).

We begin with the following concept:

Definition: An arrow $f : A \to B$ is an *increment* if, whenever $f = h \circ g$, then either h or g is an isomorphism. ▮

Perhaps the simplest example of an increment is the inclusion map $f : S \to T$ between finite sets S and T, such that $S = T \cup \{x\}$ for some x. If $\mathbf{C}$ is a poset (considered as a category), then an arrow $x \sqsubseteq y$ is an increment if and only if there is no element of $\mathbf{C}$ between x and y. If we consider the category of L-domains with embedding-projection pairs, then an arrow $s : A \to A'$ from a finite L-domain A into an L-domain A' is an increment if and only if A' has at most one more point than A. Figure 3 indicates a typical increment in this category. The increment embeds a four element poset into a poset with five elements; the closed circle indicates the "new" element.

An *ω-chain* in a category $\mathbf{C}$ is a functor $F : \omega \to \mathbf{C}$ from the ordinal ω (considered as a category) into $\mathbf{C}$. In essence, an ω-chain is a sequence of objects A_i where $i < \omega$ and a collection of arrows

$a_{ij} : A_i \to A_j$ where $i \leq j < \omega$. For each i, the arrow a_{ii} is the identity on A_i and, for any $i \leq j \leq k$, one has $a_{jk} \circ a_{ij} = a_{ik}$.

Definition: A concrete category **C** is *incremental* if

1. **C** has an initial object,
2. **C** has colimits of ω-chains,
3. every object A of **C** is a colimit of an ω-chain (A_i, a_{ij}) where A_0 is initial, each A_i is finite (in the category **C**) and each arrow $a_{i,i+1} : A_i \to A_{i+1}$ is an increment. ▮

For example, the category of countable sets and injections is incremental. However, we are interested in a more subtle example:

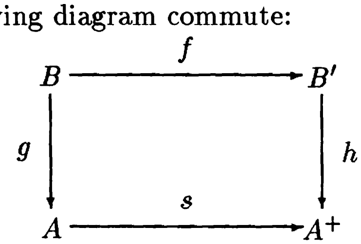
Theorem 9 *The category* $\mathbf{SA}^{ep}$ *of strongly algebraic domains and embedding-projection pairs is incremental.*

Proof: This is Theorem 22 (the Enumeration Theorem) of [Gun87]. ▮

Corollary 10 $\mathbf{SALdom}^{ep}$ *is incremental.*

Proof: Let A and B be strongly algebraic domains. If B is an L-domain and there is a projection-embedding pair $f : A \to B$, then A is an L-domain. Since $\mathbf{SA}^{ep}$ is incremental, it immediately follows that $\mathbf{SALdom}^{ep}$ is. ▮

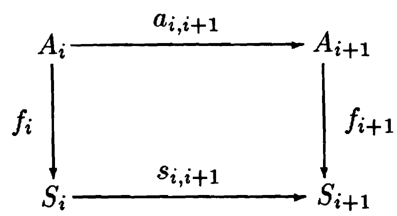
Let **C** be an incremental category and let A be an object of **C**. An object A^+ and arrow $s : A \to A^+$ is a *saturation* of A if, for every increment $f : B \to B'$ and arrow $g : B \to A$, there is an arrow h which makes the following diagram commute:



Let us say that an incremental category **C** *has finite saturations* if, for every finite object A of **C**, there is a saturation $s : A \to A^+$ where A^+ is finite.

Theorem 11 *If an incremental category has finite saturations, then it has a universal object.*

Proof: Suppose **C** is an incremental category with finite saturations. Let S_0 be any initial object of **C**. Build the chain $S_0, S_1 = S_0^+, \ldots, S_{i+1} = S_i^+, \ldots$ where $s_{i,i+1}$ is a saturation for each i. Let $\mathcal{U}$ be a colimit for this chain. We claim that $\mathcal{U}$ is universal. To see this, suppose A is any object of **C** and we will demonstrate an arrow $f : A \to \mathcal{U}$. Since **C** is incremental, A is the colimit of a chain (A_i, a_{ij}) of finite objects where A_0 is initial and each arrow $a_{i,i+1} : A_i \to A_{i+1}$ is an increment. Now, there is an arrow $f_0 : A_0 \to S_0$ since A_0 is initial. Suppose an arrow $f_i : A_i \to S_i$ is given. Since $a_{i,i+1}$ is an increment and $s_{i,i+1}$ is a saturation, there is an arrow f_{i+1} such that the following diagram commutes:



This collection of arrows f_i gives rise to a cocone with vertex $\mathcal{U}$ over the chain (A_i, a_{ij}). Since A is a colimit of this chain, there must consequently be a mediating arrow $f : A \to \mathcal{U}$ as desired. ▮

Thus, to prove that there is a universal object in the category of strongly algebraic domains (as was done in [Gun87]) or that of strongly algebraic L-domains, it suffices to demonstrate that the category in question has finite saturations. The fact that $\mathbf{SA}^{ep}$ has finite saturations is proved in [Gun87]. To get the result for $\mathbf{SALdom}^{ep}$ requires a trick which we outline in the proof of the following

Lemma 12 $\mathbf{SALdom}^{ep}$ *has finite saturations.*

Proof: (Sketch.) Let A be a finite L-domain. First, note that an L-domain has meets of bounded subsets. We define A^+ to be the set of pairs (u, U) where $u \in A$ and $U \subseteq A$ such that

- $U = \uparrow U$,
- $u \sqsubseteq U$, and
- for every $a, b, c \in U$, if $a, b \sqsubseteq c$, then $a \sqcup b \in U \cup \{u\}$.

We define the order relation on A^+ by taking $(u, U) \leq (v, V)$ iff

- $v \in U$ or
- $u = v$ and $U \supseteq V$.

We claim that A^+ is again an L-domain: $(\bot, \uparrow\bot)$ is clearly the least element of A^+ and if (u, U) and (v, V) are bounded by (w, W), we form the join relative to (w, W) as follows:

1. If $w = u = v$, then $W \subseteq U, W \subseteq V$. The join is $(w, U \cap V)$.
2. If $w = u \neq v$, then $u = w \in V$ and $(v, V) \leq (u, U)$. The join is (u, U).
3. If $w \neq u$ and $w \neq v$, then $w \in U \cap V$. We have to distinguish three subcases:
   (a) If $(u, U) \leq (v, V)$, then (v, V) is the join.
   (b) If $u = v$, then $(u, U \cap V)$ is the join.
   (c) If cases a and b do not apply, then define x to be the meet of $(\downarrow w) \cap U \cap V$ and $(x, \uparrow x)$ is the desired join.

The mapping $a \mapsto (a, \uparrow a)$ embeds A into A^+ with the corresponding projection being $(u, U) \mapsto u$. If A is embedded in B then A^+ is embedded in B^+, because the corresponding projection preserves all existing meets.

Now let $f : A \rightarrow A'$ be an increment, that is, A' contains one more element a' than A. We show that A' is embedded in A^+. Let a be the largest element of A below a' and let V be the set of elements of A above a'. We define the embedding $h : A' \rightarrow A^+$ by

$$h(x) = \begin{cases} (x, \uparrow x), & \text{if } x \in A; \\ (a, V), & \text{if } x = a'. \end{cases}$$

Clearly, h is monotone and injective. The corresponding projection $p : A^+ \rightarrow A'$ is given by

$$p((u, U)) = \begin{cases} u, & \text{if } u \neq a'; \\ a', & \text{if } u = a \text{ and } U \subseteq V; \\ a, & \text{if } u = a \text{ and } U \not\subseteq V. \end{cases} \blacksquare$$

By Theorem 11 we therefore have the following:

Theorem 13 *The category of strongly algebraic L-domains with embedding-projection pairs has a universal domain.* ▮

Elsa Gunter has recently shown [Gun] that there is a continuous pseudo-retraction of $\mathbf{SA}^{ep}$ onto $\mathbf{SALdom}^{ep}$. That is,

Theorem 14 (E. Gunter) *There is a continuous functor $L : \mathbf{SA}^{ep} \rightarrow \mathbf{SALdom}^{ep}$ such that, for any strongly algebraic domain D, we have $L(D) \cong D$ if and only if D is an L-domain.* ▮

The proof of this result is non-trivial (finding an appropriate operator which is also a functor is the hard part) but it can be used to provide an easy proof of Theorem 13 using the existence of a universal domain for $\mathbf{SA}^{ep}$. Let $\mathcal{V}$ be the universal domain for $\mathbf{SA}^{ep}$ (the existence of such a domain was demonstrated in [Gun87]). Let $\mathcal{U} = L(\mathcal{V})$. If D is a strongly algebraic L-domain, then there is an arrow $f : D \rightarrow \mathcal{V}$. Now $L(f) : L(D) \rightarrow \mathcal{U}$, but $D \cong L(D)$ so $\mathcal{U}$ is the desired universal L-domain. The universal domain constructed in this way is probably not isomorphic to the one given by the use of saturations above.

5 Discussion.

We think we have now shown that strongly algebraic L-domains have many of the basic properties which one might want in a mathematical foundation for programming semantics. Moreover, the strongly algebraic L-domains have a simple description: they are countably based L-domains with bottoms and coherent topologies. Is there any sense in which the strongly algebraic L-domains are *better* than the Scott domains? For this we have no clear answer. The local algebraicity condition seems to extend very naturally to the categorical level whereas the proper categorical version of consistent completeness seems less clear. For the purposes of programming semantics, it might be worthwhile to investigate an extension of *profiniteness* to the categorical level. This would perhaps tie in the work of Coquand more tightly with what we have discussed above and might reveal an interesting category of embeddings. In particular, the coherence concept may be helpful in moving us toward a good theory.

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