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Spherical Sampling by Archimedes' Theorem

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In this paper we present a simple and efficient algorithm for generating uniformaly distributed samples on the unit sphere based on an Archimedes' theorem. The implementation is straightforward and may be easily extended to include stratified sampling for variance reduction. Applications in image synthesis include solid angle measurement, irradiance computation, and rendering equation solution for geometrically complex environments.

Keywords

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Disciplines

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Comments

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Spherical Sampling by Archimedes' Theorem

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Abstract

In this paper we present a simple and efficient algorithm for generating uniformly distributed samples on the unit sphere based on an Archimedes' theorem. The implementation is straightforward and may be easily extended to include stratified sampling for variance reduction. Applications in image synthesis include solid angle measurement, irradiance computation, and rendering equation solution for geometrically complex environments.

CR Categories and Subject Descriptors: I.3.5 [Computational Geometry and Object Modeling]: Geometric Algorithms; I.3.7 [Computer Graphics]: Three-Dimensional Graphics and Realism.

Additional Key Words and Phrases: Archimedes' theorem, axial projection, Monte Carlo, solid angle, spherical sampling.

1 Introduction

In computer graphics, the Monte Carlo method [4] has been used in solving the rendering equations, evaluating the form factors, and elsewhere [5, p. 182]. For Monte Carlo algorithms, random sampling is a fundamental operation. In image synthesis, an important sampling domain is the solid angles, i.e., regions on the unit sphere. Despite a significant amount of research in this area, to our knowledge, a uniformly distributed random sampling algorithm is not available for arbitrary measurable regions on the unit sphere. Recently, Arvo [2] presented such an algorithm for the spherical triangles.

In this paper, we present a general spherical sampling algorithm based on an Archimedes' theorem on the sphere and cylinder.

2 Archimedes' Theorem

In his lifetime, the great Greek mathematician Archimedes (287–212 B.C.) had made many important discoveries. Among them are the calculation of π, the mechanics law of lever, and the hydrostatics law of floating body. The following is a theorem that in accordance with Archimedes' wishes was inscribed on his tombstone. It comes from a corollary in his work *On the Sphere and Cylinder* [3, p. 107].

Archimedes' Theorem (Global). *The area of a sphere equals the area of every right circular cylinder circumscribed about the sphere excluding the bases.*

By the method of exhaustion, i.e., using inscribed and circumscribed rectilinear figures to "exhaust" the area, Archimedes proved that both areas are four times the area of the great circle on the sphere. We know that it is $4\pi r^2$ where r is the radius of the sphere. By more powerful techniques of calculus, however, this elegant global property turns out to be also local. That is, the same property holds for any point and its immediate neighborhood on the sphere. It is stated in the next theorem.

Archimedes' Theorem (Local). *The axial projection of any measurable region on a sphere on the right circular cylinder circumscribed about the sphere preserves area.*

Suppose P is a point on the sphere and M is the closest point to P on the axis of the circumscribed cylinder, the axial projection P' of P is the point at which ray MP intersects the cylinder; see Figure 1. The axial projection S' of a region S on the sphere is the set of axial projections of all points of the region. The above theorem asserts that the areas of S and S' equal.

It should be noted that the axial projection is a bijection for all points on the sphere except the two axial poles. For these two points, according to our definition, their axial projections are undetermined. We may, however, assume that they are projected to the entire corresponding base circles of the cylinder. Since the area of a circle is zero, the above theorem should hold if it holds anywhere else on the sphere.

The proof of the theorem is elementary. We begin with a differential area dS on the sphere and its axial projection dS' on the cylinder; See Figure 2. The differential area dS on the sphere is enclosed by differential arcs of two small circles with colatitudes θ and $\theta + d\theta$ and two great semicircles with longitudes ϕ and $\phi + d\phi$, where $0 < \theta < \pi$ and $0 < \phi < 2\pi$. Easily, its axial projection dS' on the cylinder is enclosed by two differential circular arcs with heights $r\cos(\theta)$ and $r\cos(\theta + d\theta)$ and two differential vertical line segments with longitudes ϕ and $\phi + d\phi$, where r is the radius of the sphere.

We have the differential area on the sphere

$$\begin{aligned} dS &= rd\theta \cdot r\sin(\theta)d\phi + O(d\theta d\phi) \\ &= r^2\sin(\theta)d\theta d\phi + O(d\theta d\phi), \end{aligned} \tag{1}$$

and the differential area of its axial projection on the cylinder

$$\begin{aligned} dS' &= r(\cos(\theta) - \cos(\theta + d\theta)) \cdot rd\phi \\ &= r^2(\cos(\theta) - \cos(\theta)\cos(d\theta) + \sin(\theta)\sin(d\theta))d\phi \\ &= r^2\sin(\theta)d\theta d\phi + O(d\theta d\phi). \end{aligned} \tag{2}$$

For any measurable region S on the sphere and its axial projection S' on the cylinder, their areas are Riemann-integrable [1, p. 389]. Hence, the theorem immediately follows by Equations (1) and (2).

It is noted that the global theorem is a special case of the local theorem where the region S is the entire sphere and its axial projection S' is the entire cylinder excluding the bases.

3 Spherical Sampling

By the Archimedes' theorem, for any two measurable regions S_1 and S_2 on the unit sphere with equal areas, their axial projections S_1' and S_2' on the circumscribed cylinder will have equal areas. The reverse is also true because the axial projection is a bijection except for the two poles and their corresponding base circles which all have zero area. While the sphere is not developable, the cylinder is. This naturally leads to a spherical sampling algorithm which can be described in one sentence: Generate a random point on the cylinder $[-1, 1] \times [0, 2\pi]$ and then find its inverse axial projection on the unit sphere. If a random point is uniformly distributed on the cylinder, by the above argument, its inverse axial projection will be uniformly distributed on the sphere.

Some final remarks may be in order.

1. The algorithm may be used to sample or estimate the area of any measurable region on the unit sphere to any precision provided a point-in-region test algorithm exists. Similarly, it may be used to sample or estimate the solid angle subtended by any

measurable arbitrarily-shaped object to any precision provided a line-object stabbing algorithm exists. For efficiency, bounding boxes may be constructed in the domain $[-1, 1] \times [0, 2\pi]$ for subsets on the unit sphere.

2. Stratified sampling is trivial over domain $[-1, 1] \times [0, 2\pi]$. It is recommended since stratification usually reduces variance and generates more evenly distributed samples.

3. Numerical truncation errors are not uniformly distributed over the unit sphere under inverse axial projection. They tend to be greater around the pole areas. One solution is to use a random axis for every sample. For stratified sampling under this scheme, we may multiply a stratified axial sampling with a stratified cylindrical sampling over possibly coarser partition grids.

Figures 3 and 4 show examples of uniform and stratified samplings on a sphere and a spherical triangle. Figure 5 shows samples over the solid angle subtended by a teapot. A pseudo-random double-precision floating-point number generator has been used in the implementation.

4 Conclusions

Based on an Archimedes' theorem, we have presented a simple and efficient algorithm for generating uniformly distributed samples on the unit sphere. The implementation is straightforward and may be easily extended to include stratified sampling for variance reduction. Applications in image synthesis include solid angle measurement, irradiance computation, and rendering equation solution for geometrically complex environments.

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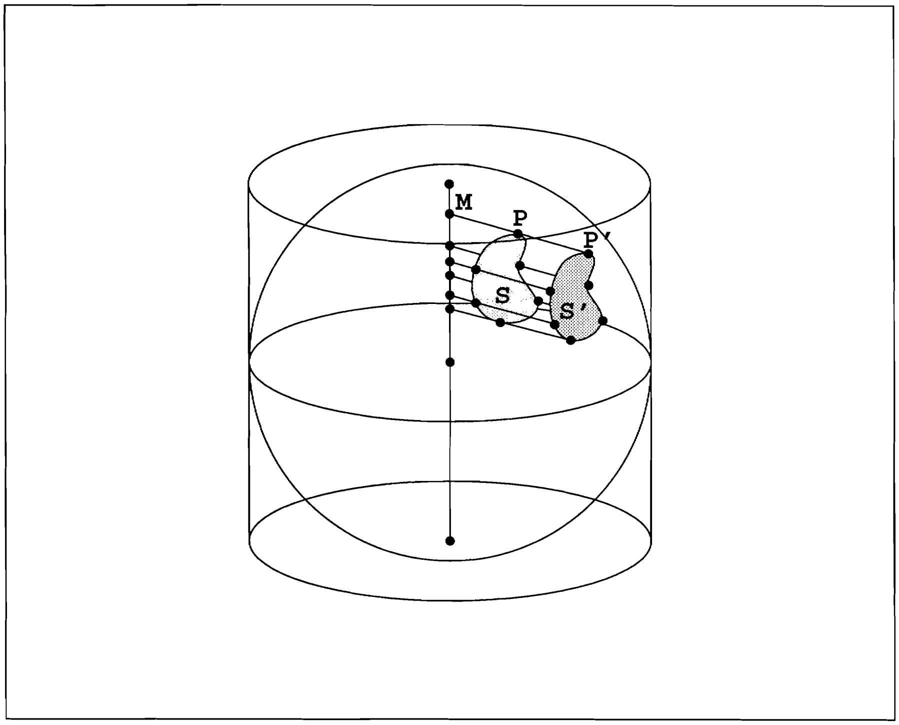


Figure 1: A sphere and a right circular cylinder circumscribed about the sphere. Spherical point P and region S and their axial projections P' and S' on the cylinder.

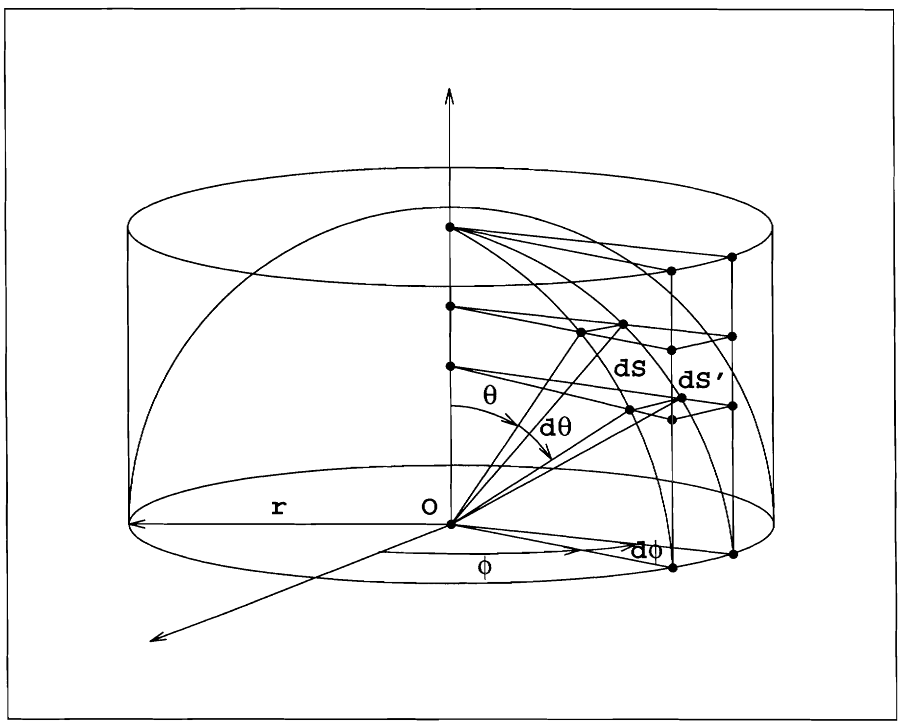
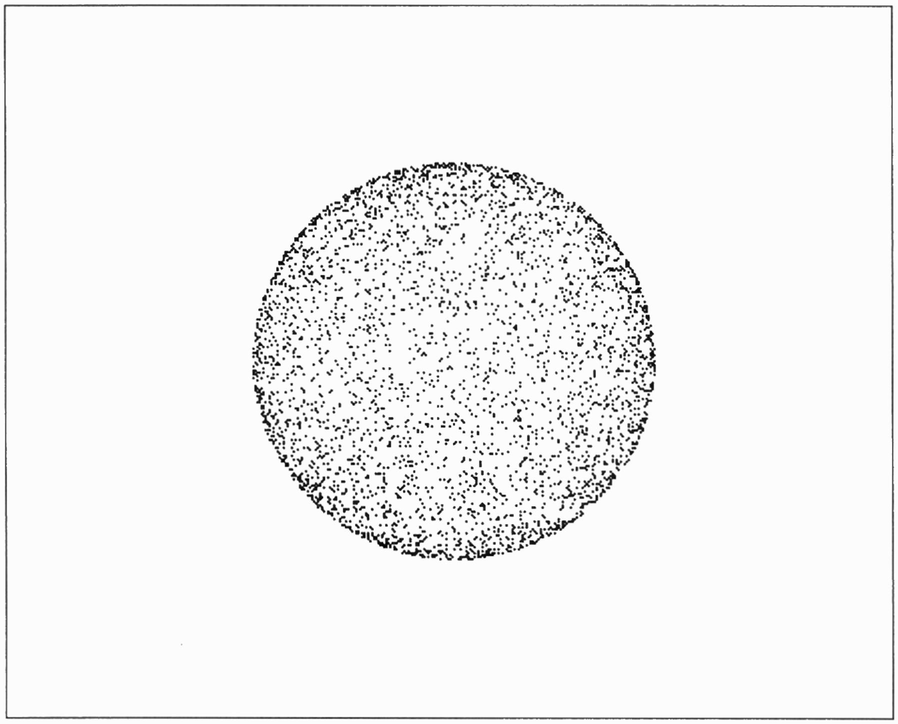
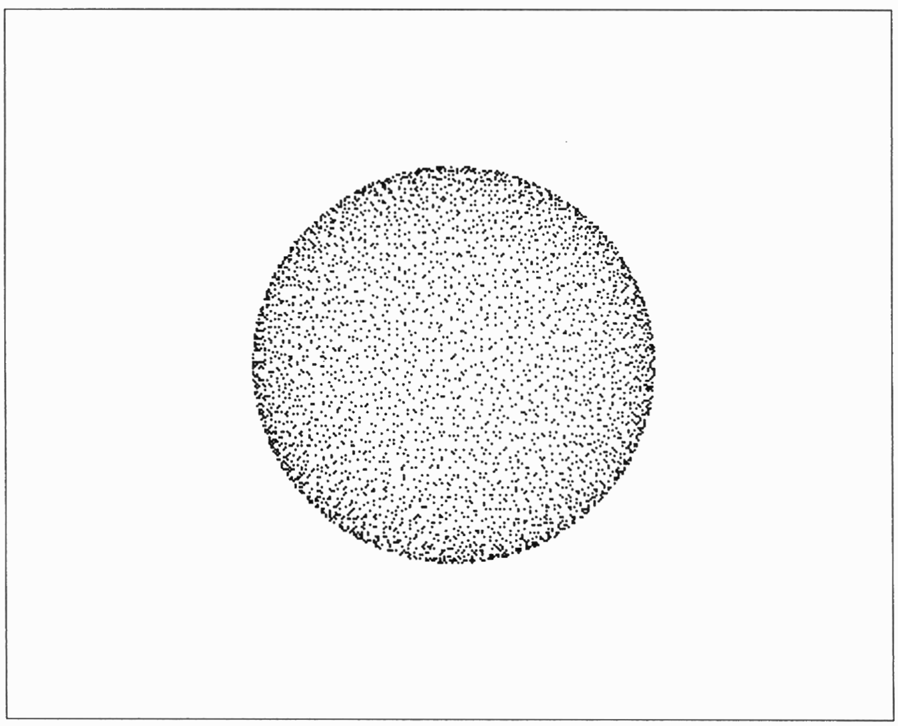


Figure 2: Differential area dS on the sphere and its axial projection dS' on the cylinder.

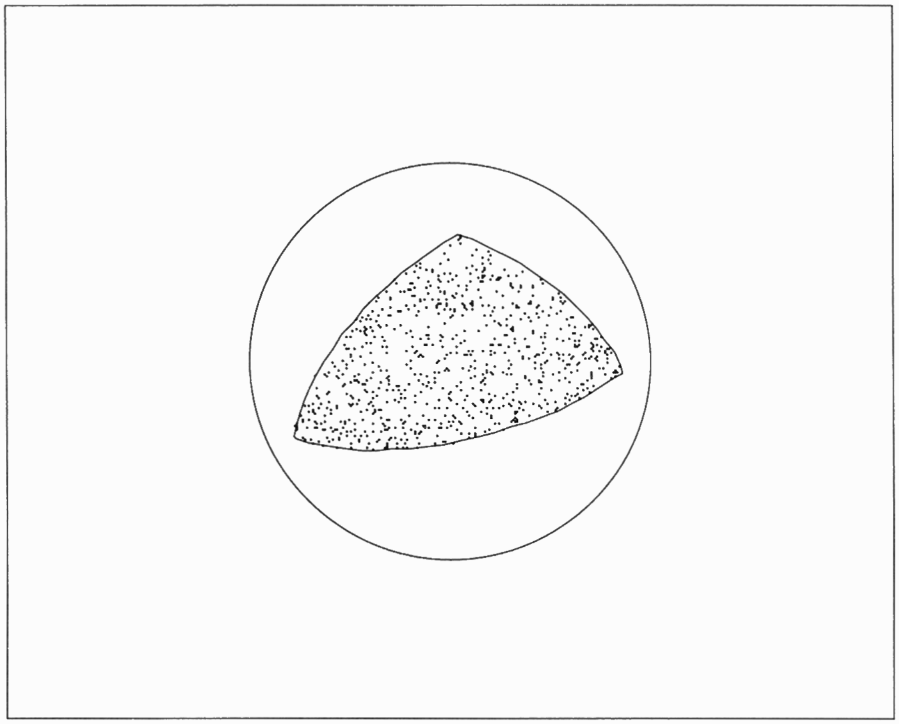


(a) uniform sampling

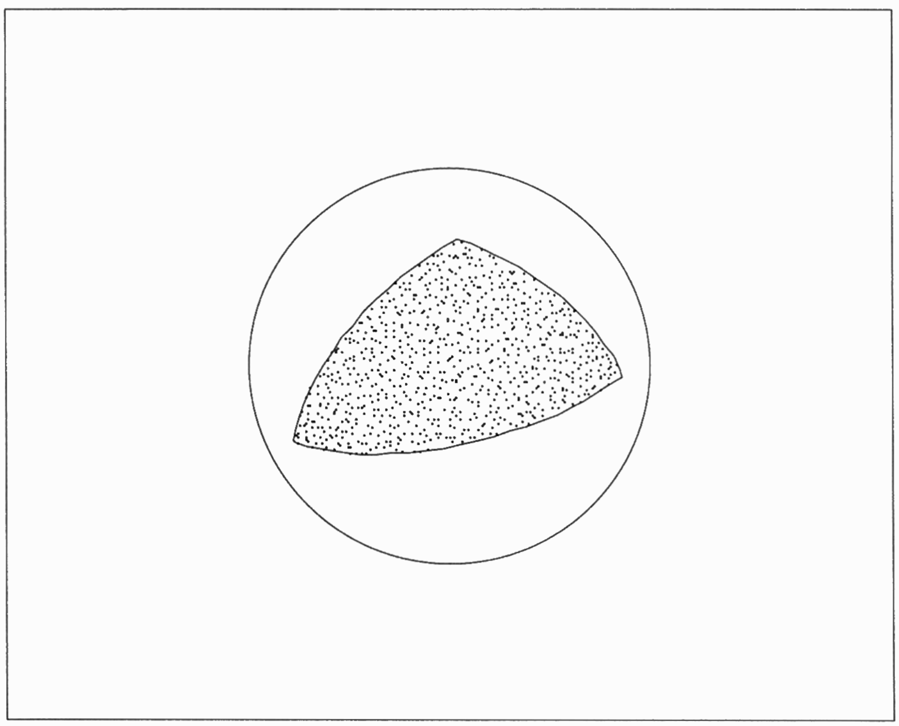


(b) stratified sampling

Figure 3: Samples on a sphere.

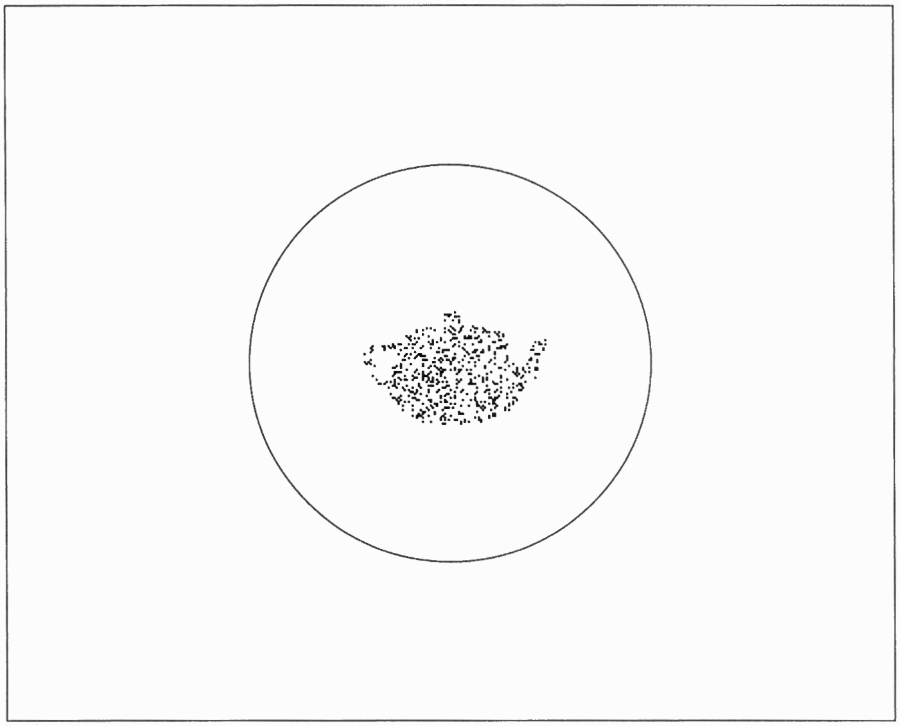


(a) uniform sampling

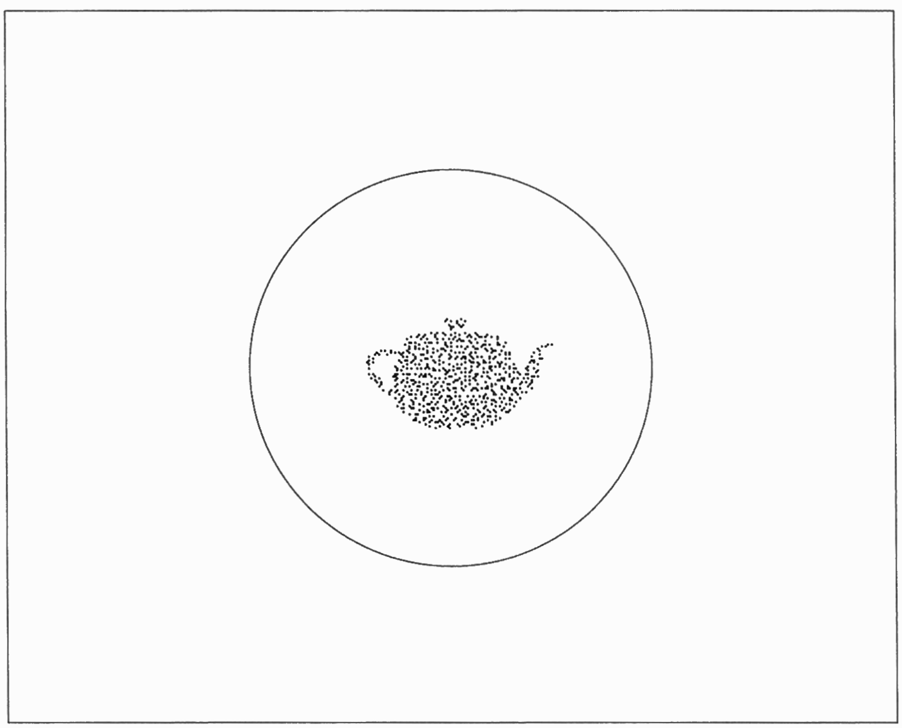


(b) stratified sampling

Figure 4: Samples on a spherical triangle.



(a) uniform sampling



(b) stratified sampling

Figure 5: Samples over the solid angle subtended by a teapot.