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## Recommended Citation

Cohen, A., Moore, G., Nelson, P. C., & Polchinski, J. (1987). Semi-Off-Shell String Amplitudes. *Nuclear Physics B, 281* (1), 127-144. http://dx.doi.org/10.1016/0550-3213(87)90250-1

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**Disciplines**
Physical Sciences and Mathematics | Physics

**Comments**
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4

HUTP-86/A028
UTTG-10-86

# Semi-Off-Shell String Amplitudes

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4/86

## 1. Introduction

S-matrix elements in string theory can be calculated as the matrix elements of local vertex operators in the two-dimensional field theory of the string world sheet[1]. Unfortunately this procedure does not extend off the mass shell in any simple way. The light cone string path integral, or equivalently the light cone string field theory, defines off-shell quantities in a natural way as transition amplitudes between initial and final string configurations at finite times [2]. More recently it has been observed that such string to string amplitudes can be defined in an invariant way without first fixing the world sheet gauge invariances [3][4][5]. This is accomplished with the Polyakov path integral, by summing over all world sheets which have specified boundaries in spacetime. These amplitudes were introduced by Alvarez [3]. The explicit evaluation of the two-string amplitude, the propagator, was carried out in [4][5] for closed bosonic strings. The result is physically satisfying in that the singularities of the propagator are all poles which can be identified with physical particles.

The natural extension of the above work is to consider $n$-string amplitudes in which the world surface has $n$ boundary curves. Various aspects of these amplitudes have been considered in [6][7][8][9]. This is substantially more complicated than the cylindrical topology relevant to the propagator. In this paper we would like to consider an extension of the propagator calculation which is simpler than the $n$-holed sphere but still quite interesting. This is the "semi-off-shell amplitude," involving one or two closed bosonic strings at finite locations plus $m$ string states on-shell. The computation of these amplitudes, which is an easy extension of [4][5], is described in section two. In section three we discuss the extent to which the formal expressions we derive are well-defined. In fact, the amplitudes we find suffer from a new Weyl anomaly. We can avoid this problem by forming appropriate combinations of amplitudes, an alternative which does not exist for off-shell vertex operators.



In section four we show that our well-defined amplitudes are indeed the off-shell continuations of known scattering amplitudes. We illustrate in several cases the general rule that when combinations of momenta approach the mass-shell the Polyakov amplitude is dominated by the integration over an asymptotic region of moduli space. The reason for this is that the string integrand is never singular inside moduli space, so singularities can only come from integration over asymptotic regions. These regions have a clear physical interpretation which we will exploit to find the pole structure of amplitudes when momenta go on-shell.

In section five we describe some of the features of these amplitudes related to gauge invariance. We find that a principle discovered for the propagator continues to apply: quantities which are invariant under all local symmetries of the two-dimensional world sheet action are in fact invariant under the $d$-dimensional gauge symmetries.

In this paper we work by explicit calculation, and examine in detail only the simplest interesting cases: the off-shell states are the "punctual" states introduced in [4] rather than the general states constructed in [5]; the on-shell states are all tachyons or gravitons.

## 2. Semi-Off-Shell Amplitudes

In field theory, Green functions are functions of $n$ spacetime points: $G(x_1, \cdots, x_n)$. One might expect that in string theory the Green functions will be functions of $n$ spacetime loops $\ell_i$. A sensible prescription for $G(\ell_1, \cdots, \ell_n)$ is given by Polyakov's path integral for surfaces with $n$ boundaries:

$$G(\ell_1, \cdots, \ell_n) \equiv \int \frac{[dX][dg]}{vol_{GC\times W}} e^{-S} \quad , \tag{2.1}$$

where $X^\mu$ is the embedding of the world sheet in spacetime, $g_{ab}$ is the worldsheet metric, and $T$ is the string tension. $S$ is the action

$$S[g,X] = \frac{T}{2}\int \sqrt{g} g^{ab}\partial_a X^\mu \partial_b X^\mu \quad ,$$

and we have divided by the volume of the gauge group. The fields in (2.1) are subject to certain boundary conditions. We demand that the embedding map the boundaries of parameter space to the specified loops $\ell_i$; further technical boundary conditions on the metrics and diffeomorphisms are described in [3][4].

The standard gauge-fixing procedure reduces the expression (2.1) to a finite-dimensional integral over moduli space and a functional integral over boundary reparametrisations $\Sigma_i$. Specifically, choosing parametrised curves $X_i(\sigma)$ describing the loops $\ell_i$ we form the solution $\overline{X}$ to the Laplace equation with these boundary values and write

$$G(\ell_1, \cdots \ell_n) = \prod_{boundaries} \int_{Diff(S^1)} d\Sigma_i \int [d\tau](det' P_1^\dagger P_1)^{\frac{1}{2}}(det_D - \nabla^2)^{-13} e^{-S[\partial,\overline{X}]} \tag{2.2}$$

The measure for moduli $[d\tau]$ and the boundary conditions for the ghost determinant have been described in [3][10][11][12]. The scalar determinant is evaluated with Dirichlet boundary conditions. Although the reparametrization integrals can be worked out explicitly [5], they are complicated and nonlocal. We therefore replace the boundary loops $\ell_i$ by points $X_i$ in spacetime so that the reparametrization integrals become trivial. As discussed in [5], these "punctual states" have pathologies at short distances, but these difficulties will not be a problem here.

When we pull a point $X_i$ off to infinity, (*e.g.* to get the S-matrix) the Polyakov integral becomes dominated by configurations which in spacetime look like those in Fig. 1. Furthermore, in the intrinsic metric $g_{ab}$ these dominant surfaces also have a long narrow tube, or, what is conformally the same thing, a very small boundary curve (Fig. 2). Intuitively we expect that in the limit such surfaces can be replaced by propagators times the insertion of local operators on



a *punctured* Riemann surface (Fig. 3). We may test this intuition by replacing all holes but one with tachyon vertex operators:

$$g\int d^2\sigma\sqrt{g(\sigma)} : e^{ip_i\cdot X(\sigma)} : \quad , \tag{2.3}$$

thus defining the disk amplitude $D(X_0^\mu; p_1, ..., p_m)$ with a punctual state $X_0$ on the rim and $m$ on-shell tachyon vertex operators inserted on the interior. Since the solution $\overline{X}$ to the boundary value problem is simply $\overline{X} = X_0$ and since the disk has no moduli we have:

$$D(X^\mu; p_1, ..., p_m) = e^{i\sum p_k\cdot X_0} g^m \int_\Delta \prod_1^m d^2\rho_i \int \frac{[dX]}{vol SL(2,R)} \exp(i\sum p_i\cdot X(\rho_i) - S[X]) \tag{2.4}$$

where $\Delta$ is the unit disk and the functional integral on $X$ is to be computed with Dirichlet boundary conditions. The volume now reflects the residual conformal Killing group [3][11], which we fix by moving the first vertex operator to $\rho_1 = 0$ and then rotating the second to the positive real axis. The effect of this gauge fixing is to replace

$$d^2\rho_1 d^2\rho_2 \to |\rho_2| d|\rho_2| \quad . \tag{2.5}$$

It is now straightforward to evaluate (2.4). We compute the Green function using the method of images:

$$G(\rho,\rho') = \frac{1}{4\pi}\log\left|\frac{\rho-\rho'}{1-\rho\bar{\rho}'}\right|^2 \quad . \tag{2.6}$$

The normal ordering in (2.3) means that the self-contractions are to be evaluated with the normal-ordered Green function

$$\begin{aligned} :G(\rho,\rho):_\rho &\equiv lim_{\rho'\to\rho}[G(\rho',\rho) - \frac{1}{4\pi}\log\ |\rho-\rho'|^2] \\ &= -\frac{1}{4\pi}\log(1-|\rho|^2)^2 \end{aligned} \tag{2.7}$$

where the subscript $\rho$ on the normal-ordering symbol denotes the coordinate used in the subtraction. [1] An alternative procedure would have been to define

[1] This subtraction procedure introduces a coordinate dependence which, however, cancels for physical vertex operators [13].

the Green function directly as a $\varsigma$-function regulated sum over the eigenfunctions of the laplacian. This sum can be simply evaluated, the pole in $\varsigma$ being subtracted to yield the normal-ordered Green function, as in [10]. The prescription (2.7) for the self-contraction is equivalent to the one used in [10] in that it corresponds to renormalizing the vertex operator by a topology-independent factor. Putting these factors together and Fourier transforming on the variable $X^\mu$ we get the disk amplitude represented by Fig. 4

$$\mathcal{D}(p_0;p_1,...,p_m)=(2\pi)^{26}\delta(\sum p_i)g^{m-1}\int_\Delta\prod_2^m\frac{d^2\rho_i}{(1-|\rho_i|^2)^2}\prod_{1\le i\ne j\le m}\left|\frac{\rho_i-\rho_j}{1-\rho_i\overline{\rho}_j}\right|^{\frac{p_i\cdot p_j}{4\pi T}} \tag{2.8}$$

Note that we have reinstated the harmless integral of the phase of $\rho_2$. The irrelevance of this phase is due to $L_0-\overline{L}_0$ invariance, *i.e.* invariance under rotating the disk. About normalization: in proceeding from the path integral to (2.8) we have dropped various finite and infinite factors such as the volume of the group $Diff(S^1)$. We absorb all such factors into the normalization of the punctual state and take (2.8) as defining the normalization. With this definition the punctual state couples to the tachyon with unit amplitude, as can be seen for example from eq.(4.3) below. For the normalization of vertex operators and S-matrix elements we follow the paper of Weinberg [14].

Later it will be convenient to have the amplitude for a disk with one graviton and $m$ tachyon vertex operators inserted. Let the graviton have polarization $\varsigma_{\mu\nu}$. The answer is then easily found to be

$$\varsigma^{\mu\nu}\mathcal{D}_{\mu\nu}(p_0;p_1,...,p_m)=\frac{1}{4\pi T}\langle\sum\varsigma_{\mu\nu}p_i^\mu p_j^\nu(\rho_i-\overline{\rho}_j^{-1})(\overline{\rho}_j-\rho_j^{-1})\rangle\quad, \tag{2.9}$$

where the average of an operator $\mathcal{O}$ is defined by setting $\langle\mathcal{O}\rangle$ equal to the expression (2.8) with $\mathcal{O}$ under the integral.

We now consider the case with *two* particles off-shell. Thus we have a tube with vertex operators inserted, as in Fig. 5. We may parametrize the gauge-



inequivalent tubes by choosing the standard metric $ds^2=(d\sigma^1)^2+(d\sigma^2)^2$ on the region $(\sigma^1,\sigma^2)\in[0,1]\times[0,\lambda]$ with $\sigma^1$ identified with $\sigma^1+1$. Since we have chosen punctual states we must shift by the classical solution $\overline{X}=X_i+\frac{\sigma^2}{\lambda}(\Delta X)$. In the case with no vertex operators we found in [4] that the tube amplitude is

$$\mathcal{T}(X_i,X_f)=\frac{1}{2T}(\frac{T}{2\pi})^{13}\int_0^\infty\frac{d\lambda}{\lambda^{13}}\prod_{n>0}(1-e^{-4\pi n\lambda})^{-24}e^{4\pi\lambda}e^{-T(X_i-X_f)^2/2\pi\lambda}\quad. \tag{2.10}$$

Ignoring the (irrelevant) short-distance singularity at $\lambda\to0$ and going to momentum space we find

$$\mathcal{T}(p,-p)=\frac{1}{2T}\int_0^\infty d\lambda\, e^{4\pi\lambda}\prod_{n>0}(1-e^{-4\pi n\lambda})^{-24}e^{-\lambda p^2/2T} \tag{2.11}$$

We can follow the same procedure for computing semi-off-shell amplitudes by inserting vertex operators. The only new feature in the computation is the Dirichlet Green function which can be expressed in terms of the variables $z=\sigma^1+i\sigma^2$ and $\rho=e^{2\pi iz}$ as

$$G_D(\rho,\rho')=\frac{\log|\rho|\log|\rho'|}{4\pi^2\lambda}+\frac{1}{4\pi}\lim_{N\to\infty}\sum_{n=-N}^{+N}\log\left|\frac{\rho-\rho'e^{4\pi n\lambda}}{1-\rho\overline{\rho}'e^{4\pi n\lambda}}\right|^2 \tag{2.12}$$

Equation (2.12) is obtained by the method of images. The first term can be thought of as arising from a "charge at infinity;" it arises because the sum is not absolutely convergent. The self-contractions are defined analogously to (2.7). Thus in momentum space the tube amplitude $\mathcal{T}$ becomes

$$\begin{aligned}\mathcal{T}(p_0,p_{m+1};p_1,\dots,p_m)&=(2\pi)^{26}\delta^{26}(\sum p_i)\frac{g^m}{2T}\int_0^\infty d\lambda\int d^2z_1\cdots d^2z_m\, e^{4\pi\lambda}\\&\times\prod_{n>0}(1-e^{-4\pi n\lambda})^{-24}\exp\Big\{-\frac{\lambda}{8T}\Big(p_{m+1}-p_0+\sum_j\frac{2\sigma_j^2-\lambda}{\lambda}p_j\Big)^2\\&+\frac{1}{T}\sum_{1\le i\ne j\le m+1}p_i\cdot p_j\,G_D(\sigma_i,\sigma_j)+\frac{1}{2T}\sum_{i=1}^m p_i^2\,:G_D(\sigma_i,\sigma_i):_z\Big\}\end{aligned} \tag{2.13}$$

The normalization of the punctual state in this expression agrees with that in (2.8) . This may be shown either by considering the coupling of the punctual state to the tachyon or by a careful treatment of the normalisation of the path integral following that of [10] .

We close this section with a conjecture. Note that the Polyakov integrand in (2.13) can be written entirely in terms of classical functions. First, we have that

$$G_D(\rho,\rho') = \frac{\sigma^2\sigma^{2\prime}}{\lambda} + \frac{1}{4\pi}\log\left|\frac{\vartheta\left[\begin{smallmatrix}\frac{1}{2}\\ \frac{1}{2}\end{smallmatrix}\right](z_1 - z_2|\tau)}{\vartheta\left[\begin{smallmatrix}\frac{1}{2}\\ \frac{1}{2}\end{smallmatrix}\right](z_1 - \bar{z}_2|\tau)}\right|^2 \quad \text{and}$$

$$:G_D(\rho,\rho):_s = \frac{(\sigma^2)^2}{\lambda} + \frac{1}{4\pi}\log\left|\frac{\vartheta'\left[\begin{smallmatrix}\frac{1}{2}\\ \frac{1}{2}\end{smallmatrix}\right](0|\tau)}{\vartheta\left[\begin{smallmatrix}\frac{1}{2}\\ \frac{1}{2}\end{smallmatrix}\right](z - \bar{z}|\tau)}\right|^2 ,$$

where $\tau = 2i\lambda$. Define the Dedekind eta function by

$$\eta = q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^n)$$

where $q = e^{2\pi i\tau}$. Then (2.13) becomes

$$\mathcal{T}(p_0, p_{m+1}; p_1, \ldots, p_m) = (2\pi)^{26}\delta^{26}(\sum p_i)\frac{g^m}{2T}\int_0^\infty d\lambda \int d^2z_1\cdots d^2z_m\,\eta(\tau)^{-24}$$

$$\times e^{-\lambda p_{m+1}^2/2T} \prod_{i=1}^{m}\left|\frac{\vartheta'\left[\begin{smallmatrix}\frac{1}{2}\\ \frac{1}{2}\end{smallmatrix}\right](0|\tau)}{\vartheta\left[\begin{smallmatrix}\frac{1}{2}\\ \frac{1}{2}\end{smallmatrix}\right](z_i - \bar{z}_i|\tau)}\right|^4 \prod_{1\leq i\neq j\leq m+1}\left|\frac{\vartheta\left[\begin{smallmatrix}\frac{1}{2}\\ \frac{1}{2}\end{smallmatrix}\right](z_i - z_j|\tau)}{\vartheta\left[\begin{smallmatrix}\frac{1}{2}\\ \frac{1}{2}\end{smallmatrix}\right](z_i - \bar{z}_j|\tau)}\right|^{\frac{p_i\cdot p_j}{4\pi T}} \tag{2.14}$$

where $p_0, p_{m+1}$ are off shell, and $z_{m+1} = i\lambda$. The notation for theta functions follows [15]. Our conjecture is that for *arbitrary* manifolds with boundary the Polyakov integral will be an integral over the real slice of some complex moduli space of an integrand expressed in terms of theta functions, similarly to [16].

## 3. An Off-Shell Weyl Anomaly

The amplitudes (2.8) and (2.14) will contain divergences for values of the external momenta in the physical region; these divergences are a result of the



expected analytic structure of the amplitude at this order. We can avoid these physical singularities in the usual way by restricting the domain of $\mathcal{D}$ and $\mathcal{T}$ to a portion of the Euclidean region. Since the tachyon has a (Euclidean) mass squared equal to $8\pi T$, this region is defined by[2] $q^2 > 8\pi T$, where $q$ is $p_0$ or any internal momentum formed by summing two or more of $p_0, p_1, \ldots, p_m$. The amplitude in the physical region is then defined by analytic continuation. This immediately eliminates two potential divergences:

i.) When two or more vertex operators approach each other.

ii.) When the tube length $\lambda \to \infty$.

There remain two more potential problems:

iii.) When $\lambda \to 0$. The integral (2.10) diverges as $\lambda \to 0$ if $(X_i - X_f)^2 < \frac{2\pi}{T}$. As discussed in [5] this is an artifact of the highly singular punctual state, and not a general property of off-shell amplitudes. The divergence as $\lambda \to 0$ does not appear in the channels we will consider in sections 4 and 5. This is why we have chosen to use the convenient punctual states.

iv.) When vertex operators approach the boundary. This is a new feature of semi-off-shell amplitudes. As we will see in section six, there is a corresponding region of moduli space for a fully off-shell amplitude, so this is not an artifact of our approach. To isolate the divergence we consider a vertex operator : $e^{ip_j\cdot X(\sigma_j^1,\sigma_j^2)}$ : as it approaches the boundary $\sigma_j^2 = 0$.(Since locally any boundary looks the same, this applies equally well to the disk and the tube.) The principal effect of the boundary can be accounted for by including a negative image charge at $(\sigma_j^1, -\sigma_j^2)$. This gives effectively the operator

[2] This is true since we only have single particle intermediate states. For contributions to the amplitude that include two or more particles in the intermediate state, the amplitude will have singularities for all real values of the external momenta, due to the tachyon in the theory.

$$
\begin{aligned}
:e^{ip_j\cdot x(\sigma_j^1,\sigma_j^2)}::e^{-ip_j\cdot x(\sigma_j^1,-\sigma_j^2)}: &= :e^{ip_j\cdot x(\sigma_j^1,\sigma_j^2)-ip_j\cdot x(\sigma_j^1,-\sigma_j^2)}:(2\sigma_j^2)^{-2} \\
&= (2\sigma_j^2)^{-2} + (2\sigma_j^2)^{-1} ip_j\cdot\partial_2 x(\sigma_j^1,0) + O((\sigma_j^2)^0)
\end{aligned}
\tag{3.1}
$$

The $\sigma_j$ integral thus produces a linear and a logarithmic divergence. This is evident, for example, in (2.8). We can study these divergences by cutting off the $\sigma^2$ integral, *i.e.*

$$\sigma^2 > e^{-\phi/2}/\Lambda \quad . \tag{3.2}$$

The metric scale factor $\phi$ ( where $ds^2 = e^{\phi}[(d\sigma^1)^2 + (d\sigma^2)^2]$ ) appears so that the cutoff is coordinate-invariant, *i.e.* expressed in terms of the proper distance. The integral over $\sigma_j^2$ then produces

$$\int d\sigma_j^1 \left(\frac{1}{4}\Lambda e^{\phi/2} - \frac{i}{2} p_j\cdot\partial_2 x(\sigma_j^1,0)\log(\Lambda e^{\phi/2})\right) \quad . \tag{3.3}$$

The divergences can be removed by an *additive* renormalization of the operator product:

$$
\begin{aligned}
P(p_0)V(p_j) \mapsto & P(p_0)V(p_j) - \frac{1}{4}\Lambda\left(\int_{\partial M} ds\right) P(p_0+p_j) \\
& + \frac{i}{2}\log\Lambda \int_{\partial M} ds\, n\cdot\partial(p_j\cdot x) P(p_0+p_j) \quad ,
\end{aligned}
\tag{3.4}
$$

where $P(p_0)$ denotes the operator which creates a punctual state. However, when this is done we see that we have re-introduced a $\phi$-dependence: the choice of counterterm that preserves coordinate invariance necessarily breaks Weyl invariance. This is a new Weyl anomaly that afflicts off-shell string amplitudes.

It is possible, however, to form certain special off-shell amplitudes which are fully invariant under the world-sheet symmetries. Consider

$$\mathcal{D}_f \equiv \int d^{26}p_0 Dp_1 \dots Dp_m\, f(p_0,p_1,\dots,p_m)\mathcal{D}(p_0;p_1,\dots,p_m) \tag{3.5}$$



for arbitrary weight function $f$, where $Dp_k \equiv d^{26}p_k\,\delta(p_k^2 - m^2)$. Inspection of the form of the divergences (3.4) shows that provided

$$\int d^{26}p_0\, Dp_j\, f(p_0,p_1,\dots,p_m)\delta\Big(\sum_0^m p_i\Big) = 0 \quad \text{and} \tag{3.6}$$

$$\int d^{26}p_0\, Dp_j\, p_j^{\mu} f(p_0,p_1,\dots,p_m)\delta\Big(\sum_0^m p_i\Big) = 0 \tag{3.7}$$

for each $j$ then the amplitude $\mathcal{D}_f$ will be finite (and hence Weyl invariant) without subtraction. For the tube amplitude we must add corresponding conditions with $p_0 \to p_{m+1}$. Note that these conditions are nonlocal; because of the mass-shell condition they have no simple coordinate space interpretation. [3]

## 4. Boundaries of Moduli Space

We must now show that the well-defined amplitudes we have found are indeed off-shell continuations of known on-shell scattering amplitudes. We will see that when combinations of momenta go on-shell an asymptotic region of moduli space dominates the integration and produces the pole. This principle is already well-known in the case of the Shapiro-Virasoro amplitude $S(p_1,\dots,p_m)$ for the scattering of $m$ on-shell tachyons:

$$S(p_1,\dots,p_m) = (2\pi)^{26}\delta^{26}\Big(\sum_1^m p_i\Big) 8\pi^2 T g^{m-2} \int_C \prod_3^{m-1} d^2 z_i \prod_{1\le i\ne j\le m-1} |z_i - z_j|^{\frac{p_i\cdot p_j}{4\pi T}} \tag{4.1}$$

where $z_1 = 0$ and $z_2 = 1$. The integration can (and should) be thought of as the integration over the moduli space of the m-times punctured sphere. The boundaries of moduli space correspond to the places where punctures approach each

[3] We could make a subtraction in place of the first condition (3.5), but it is simpler and equivalent to treat the linear and logarithmic divergences the same way.

other. As is well-known, these are the regions of integration which dominate when combinations of momenta go on-shell.

Similarly, examination of an asymptotic region of the moduli space of the $m$-times punctured disk allows us to recover $\mathcal{S}$ from the disk amplitude $\mathcal{D}$. We expect that as $p_0$ goes on-shell the dominant region of the moduli space of the $m$-times punctured disk will be that in which the boundary moves far away from all the vertex operators (Fig. 6). This is conformally equivalent to a situation where the boundary stays fixed, while all the vertex operators crowd towards the center of the disk (Fig. 7). To show that this region dominates the integral we integrate the phase of $\rho_2$ and define a collective scale factor $x$ by $\rho_k = x\varsigma_k$ (so $\varsigma_2 = 1$). In terms of these variables the disk amplitude becomes

$$2\pi\int_0^1 \frac{dx}{(1-x^2)^2} x^{-3+p_0^2/4\pi T} \int_{0\le|\varsigma_i|\le x^{-1}} \prod_{i=3}^m \frac{d^2\varsigma_i}{(1-x^2|\varsigma_i|^2)^2} \prod_{1\le k\ne l\le m} \left|\frac{\varsigma_k-\varsigma_l}{1-x^2\varsigma_k\bar{\varsigma}_l}\right|^{\frac{p_k\cdot p_l}{4\pi T}} \tag{4.2}$$

We see that, neglecting the boundary divergence, as $p_0^2 \to 8\pi T$ the dominant region of integration is that of small $x$ (we may expand the various terms in a power series in $x$ and analytically continue to get the higher mass poles) and that the leading pole is just

$$\frac{1}{p_0^2 - 8\pi T}\mathcal{S}(p_0, p_1, \ldots, p_m) \quad . \tag{4.3}$$

Thus the disk amplitude is indeed an off-shell continuation of the Shapiro-Virasoro amplitude.

Let us now consider the tube amplitude $\mathcal{T}$ and study its behavior as $p_{m+1}$ goes on-shell. We expect that the region of moduli space in which $\lambda \to \infty$ and the vertex operators stay to the left will dominate, as shown in Fig. 8. To demonstrate this it is useful to map the tube to an annulus of outer radius one and inner radius $e^{-2\pi\lambda}$ by

$$\rho = e^{2\pi i(\sigma^1 + i\sigma^2)}. \tag{4.4}$$



As $\lambda \to \infty$ the annulus becomes a disk and the Green functions satisfy

$$\begin{aligned} G(\sigma_i,\sigma_j) &= \frac{\sigma_i^2\sigma_j^2}{\lambda} + \frac{1}{4\pi}\log\left|\frac{\rho_i-\rho_j}{1-\rho_i\bar{\rho}_j}\right|^2 + O(e^{-4\pi\lambda}) \\ :G(\sigma,\sigma):_\rho &= \frac{\sigma^2(\sigma^2-\lambda)}{\lambda} - \frac{1}{4\pi}\log(1-|\rho|^2)^2 + O(e^{-4\pi\lambda}) \end{aligned} \tag{4.5}$$

Thus, after a little algebra we find

$$\begin{aligned} \mathcal{T}(p_0, p_{m+1}; p_1, \ldots, p_m) \to & (2\pi)^{26}\delta\left(\sum p_i\right)\frac{g^m}{2T}\int_0^\infty d\lambda\, e^{-\lambda(p_{m+1}^2 - 8\pi T)/2T} \\ & \int_{e^{-2\pi\lambda}\le|\rho|\le 1} \prod_{k=1}^m \frac{d^2\rho_k}{(1-|\rho_k|^2)^2} \prod_{1\le i\ne j\le m+1}\left|\frac{\rho_i-\rho_j}{1-\rho_i\bar{\rho}_j}\right|^{\frac{p_i\cdot p_j}{4\pi T}} \end{aligned} \tag{4.6}$$

with $\rho_{m+1} = 0$. That is

$$\mathcal{T}(p_0, p_{m+1}; p_1, \ldots, p_m) \to \frac{1}{p_{m+1}^2 - 8\pi T}\mathcal{D}(p_0; p_{m+1}, p_1, \ldots, p_m) \tag{4.7}$$

and the tube amplitude reproduces the disk amplitude as one of the momenta goes on-shell. Since the disk amplitude is an off-shell continuation of $\mathcal{S}$ so is the the tube amplitude. A nice consistency check on the theory is obtained by taking two ends of the tube to infinity simultaneously as in Fig. 9. We do this by defining $\lambda_1 = \sigma_{j=1}^2$ and $\eta_j = \rho_j/\rho_1$ and expanding the Green functions in power series as $\lambda_1$ and $\lambda_2 = \lambda - \lambda_1$ go to infinity while $\eta$ are held fixed. One recovers once again the Shapiro-Virasoro amplitude.

It is also interesting to see what happens when *internal* momenta go on-shell in the semi-off-shell amplitudes. Consider, for example the behavior of the tube amplitude when $k = p_0 + \cdots p_l$ goes on-shell. It is convenient to define

$$\varsigma_j = e^{-2\pi\lambda}\bar{\rho}_j^{-1} \qquad l+1 \le j \le m\,. \tag{4.8}$$

The singularities in $k$ come from the region of moduli space $\lambda \to \infty$ illustrated in Fig. 10

with $\rho_j$ and $\varsigma_j$ held fixed. Expanding the various factors in (2.14) in this limit and dropping terms suppressed by powers of $e^{-2\pi\lambda}$, we obtain

$$\mathcal{T}^{(0)}(p_0, p_{m+1}; p_1, \dots, p_m) = \frac{1}{2T}\int_0^\infty d\lambda e^{-\lambda(k^2-8\pi T)/2T}\mathcal{D}_I\mathcal{D}_F \quad , \tag{4.9}$$

where $\mathcal{D}_{I,F}$ are the disk amplitudes given by (2.8), so that the tube amplitude factorises into initial and final disk amplitudes. Working a little harder, and keeping all terms suppressed by $e^{-4\pi\lambda}$ (terms suppressed by $e^{-2\pi\lambda}$ average to zero by $L_0 - \overline{L}_0$ invariance) the first subleading term is

$$\mathcal{T}^{(1)} = \frac{1}{2T}\int_0^\infty d\lambda e^{-\lambda p^2/2T}[\mathcal{D}_I^{dil}\mathcal{D}_F^{dil} + \mathcal{D}_I^{\mu\nu}(g_{\mu\rho}g_{\nu\sigma} - \frac{1}{24}g_{\mu\nu}g_{\rho\sigma})\mathcal{D}_F^{\rho\sigma}] \, , \tag{4.10}$$

where $\mathcal{D}^{\mu\nu}$ is defined in (2.9) and $\mathcal{D}^{dil}$ is the disk amplitude with a dilaton vertex inserted

$$\mathcal{D}^{dil} = \sqrt{24}\langle 1 + \frac{1}{96\pi T}\sum_{i,j=1}^{l} p_i \cdot p_j(\rho_i - \overline{\rho}_i^{-1})(\overline{\rho}_j - \rho_j^{-1})\rangle \tag{4.11}$$

These results are shown pictorially in Fig. 11. The results of this section can be understood in a more general way from the operator product expansion, as will be explained in the conclusion.

## 5. Gauge Invariance

We now consider some issues related to the gauge invariance of the off-shell amplitudes. On-shell scattering amplitudes are gauge invariant because the longitudinal parts of vertex operators are total divergences which vanish when the parameter space of the string world sheet has no boundary. For example, if we replace the polarization tensor $\varsigma_{\mu\nu}$ of a graviton by $p_{(\mu}\epsilon_{\nu)}$ then the graviton vertex operator is replaced by

$$\epsilon_\nu \int \nabla^\beta(\nabla_\beta X^\nu e^{ip\cdot X}) \quad . \tag{5.1}$$



When the parameter space has a boundary the total divergence gives a vertex operator at the boundary. This is precisely where we found difficulties with divergences before, so we must be careful. Writing

$$\mathcal{D}^{\mu\nu} = \lim_{\rho_1\to 0}\langle : \partial_{\rho_1}X^\mu\partial_{\overline{\rho}_1}X^\nu e^{ip_1\cdot X} : V(p_2)\cdots V(p_m)\rangle \tag{5.2}$$

where the correlation function refers only to contractions of $X's$ and the normal ordering symbols indicate no self-contractions we find that

$$p_1^\mu \mathcal{D}_{\mu\nu} = \lim_{\rho_1\to 0}\langle : \partial_{\overline{\rho}_1}X^\nu\partial_{\rho_1}(e^{ip_1\cdot X(\rho_1)}) : V(p_2)\cdots V(p_m)\rangle \tag{5.3}$$

and, expressing the derivatives of the Green functions in terms of derivatives with respect to the other variable we can write

$$p_1^\mu \mathcal{D}_{\mu\nu} = \sum_{i=2}^{m}\int_\Delta d^2\rho_i(\frac{\partial}{\partial\overline{\rho}_i}\overline{\rho}_i^2 - \frac{\partial}{\partial\rho_i})\Big\langle : \partial_{\overline{\rho}_1}e^{ip_1\cdot X(\rho_1)} : e^{ip_i\cdot X(\rho_i)}\prod_{j\neq i}V(p_j)\Big\rangle \quad . \tag{5.4}$$

It follows immediately that $\mathcal{D}^{\mu\nu}$ is transverse when we pick out the residues of the poles in ${p_0}^2$, since these come from the integration region where the boundary runs off to infinity, so the surface integral vanishes.

In general the surface integral is nonzero, and the expression in brackets in (5.4) diverges on the boundary. However, a careful evaluation of the integrals shows that the integrated boundary term is *finite* and of the form

$$\sum_{i=2}^{m} p_i^\mu \mathcal{A}_i^{\mu\nu} \tag{5.5}$$

where $\mathcal{A}_i^{\mu\nu}$ is independent of $p_i$. Hence when we specialise to the amplitudes (3.5) which are finite and free of world sheet anomalies, the conditions (3.6),(3.7) ensure that the surface term vanishes. Thus the conditions necessary to establish invariance under the local symmetries on the world-sheet (and hence, finiteness of the amplitudes) guarantee that the amplitudes are also spacetime gauge invariant. This seems to be a general principle.

A second, related issue is the behavior of the tube amplitude as the internal graviton line goes on-shell with external lines off-shell, (4.10). General reasoning suggests that negative norm states cannot appear in a theory defined by a sum over surfaces. Defining states by taking spacelike hypersurfaces, there is a positive inner product, the delta functional on loop space. As long as we consider only amplitudes invariant under all the two-dimensional gauge symmetries, we see only positive norm intermediate states. In lattice gauge theories, which can also be interpreted as sums over surfaces, such reasoning can be turned into a rigorous argument [17]. Note that this applies to off-shell quantities as well as on-shell, as long as they are invariant under all the world-sheet gauge symmetries.[4]

On the other hand, inspection of (4.10) seems to indicate that the graviton propagator is in Feynman gauge, with negative norm states contributing. Indeed, we usually expect the full set of off-shell amplitudes in a theory with gravity to be gauge dependent, since we cannot specify the space-time position of a local operator in an invariant way. Because our calculation has treated the 26 dimensions covariantly, and because all covariant gauges contain states of negative norm, one might expect such states to appear in our amplitudes. This appears to contradict the argument in the preceding paragraph.

In fact there is no contradiction. As we have seen, the Weyl-invariant sum over surfaces does *not* provide us with a full set of off-shell amplitudes; only the special combinations (3.5) are well-defined. Consider for example the tube amplitudes $\mathcal{T}_f$. These factorize on the disk amplitudes $\mathcal{D}^{\mu\nu}_f$, which we just showed were *transverse*. Thus all gauge-dependent terms in the graviton propagator drop out, longitudinal gravitons decouple, and negative-norm states do not contribute to the singularities of $\mathcal{T}_f$.

[4] As further evidence for this point of view we note that the normalization of the one-loop amplitude obtained from ultralocality [10] agrees with that from unitarity [18]and that in references [4][5] negative norm poles never appeared in the two-point correlation function of local invariant objects.



Thus the sum over surfaces (2.1) defines only those amplitudes which are perturbatively independent of the choice of spacetime gauge. One might try to go further and define string amplitudes corresponding to all off-shell amplitudes in gravity. This might simply be a matter of introducing arbitrary ghost boundary conditions. By the above discussion, however, we expect that any such modification will not correspond to a Weyl-invariant sum over surfaces. Moreover, the modified prescription for amplitudes will have to have a more complicated pole structure for gravitons than (4.11). For example, if we introduce the natural gauge-fixing term $\frac{1}{2}\xi(\partial_\alpha h_{\alpha\mu} - \frac{1}{2}\partial_\mu h_{\alpha\alpha})^2$ for the graviton $h_{\mu\nu}$, then the graviton propagator becomes

$$D^{\alpha\rho,\beta\gamma} = \frac{1}{p^2}\left[\tfrac{1}{2}(g^{\alpha\beta}g^{\rho\gamma} + g^{\alpha\gamma}g^{\beta\rho}) - \frac{1}{24}g^{\alpha\rho}g^{\beta\gamma}\right] - \frac{1}{p^4}\frac{1+\xi}{2\xi}\left(p^\beta p^\rho g^{\alpha\gamma} + p^\beta p^\alpha g^{\rho\gamma} + p^\gamma p^\rho g^{\alpha\beta} + p^\gamma p^\alpha g^{\rho\beta}\right)$$

A gauge-dependent formulation of off-shell quantities will have to provide a way to generate the second order pole.

## 6. Conclusions

Our results give further indication that the Polyakov integral defines a sensible off-shell continuation of string theory. The amplitudes are, under the conditions discussed in section three, finite and Weyl invariant. The LSZ reduction gives the usual S-matrix elements. We have seen an interesting interplay between the two-dimensional and 26-dimensional gauge symmetries. The 26-dimensional invariance is not evident in the original path integral, but all well-defined off-shell quantities given by the path integral have turned out to be 26-dimensionally gauge invariant.

Let us discuss the results of section four, on the S-matrix, from a more general point of view. In terms of the variables $\rho$, equation (4.4) , the cylinder maps to the complex plane between $e^{-2\pi\lambda} < |\rho| < 1$. That is, as $\lambda \to \infty$, we have the disk minus a small circle centered at the origin. Now, any sufficiently localized disturbance in a field theory can be mimicked by a sum of local operators. The usual example is the operator product expansion, where the disturbance is a pair of local operators. Other familiar examples are the effects of virtual heavy particles and of small instantons, but the principle applies equally well to a small hole. Thus,

$$\text{small hole of radius } r \leftrightarrow \sum_i c_i(r) V_i \quad . \tag{6.1}$$

The sum runs over all suitably invariant operators – the vertex operators. The coefficients $c_i(r)$ are functions of the radius, which dimensionally are simply of the form $r^{(p^2+m_i^2)/2T}$. Integration with $dr/r$ then produces a pole times a vertex operator. Since this is an operator statement, it will apply to the corresponding asymptotic region of any off-shell amplitude.

A similar expansion, the representation of a long tube in terms of an operator sum, is seen in Fig. 11. In this case the topology at each end of the tube is a boundary plus some vertex operators, but the expansion is an operator statement and will apply with arbitrary topologies at the ends. The idea that asymptotic regions of moduli space can be studied using the operator product expansion has been raised in other contexts by Friedan, Fischler, Peskin, and Susskind[19]. A systematic treatment would be of great value; we are considering some aspects of this in [9].

Another idea worth attention, given the results here, is an attempt to make concrete the heuristic argument for unitarity along the lines of [17]. The natural object to consider would then be the transfer matrix, the amplitude to go from any state of string on a given spacelike hypersurface to a state on a slightly later hypersurface.



We would like to thank Luis Alvarez-Gaumé, Sidney Coleman, André LeClair, and Cumrun Vafa for valuable discussions. Recently G. Horowitz and A. Strominger have made progress towards a gauge-dependent Polyakov formalism [20].

This work was supported in part by NSF grants PHY-82-15249 and PHY-83-04629, the A. P. Sloane Foundation, the Robert Welch Foundation, and by the Harvard Society of Fellows. We also thank the Institute for Theoretical Physics, Santa Barbara for hospitality during the Workshop on String Unification, where this work was started.

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## Figure Captions

Fig. 1 World-sheet in spacetime as one end goes to infinity.

Fig. 2 A Riemann surface with boundary corresponding to fig.1.

Fig. 3 A punctured Riemann surface corresponding to fig.1.

Fig. 4 A disk with vertex operators.

Fig. 5 A tube with vertex operators .

Fig. 6 A configuration relevant when $p_0$ goes on-shell.

Fig. 7 A surface conformally equivalent to fig.6.

Fig. 8 A configuration relevant when $p_{m+1}$ goes on-shell.

Fig. 9 a.)$p_0$ and $p_{m+1}$ both go on-shell b.)the resulting amplitude.

Fig. 10 $q = p_0 + p_1 + p_2$ goes on-shell.

Fig. 11 The amplitude resulting from the limit of fig.10.

Figure 1

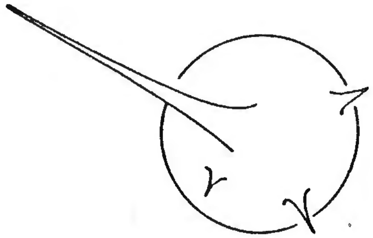


Figure 2

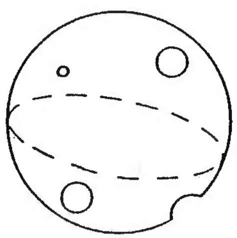

Figure 3

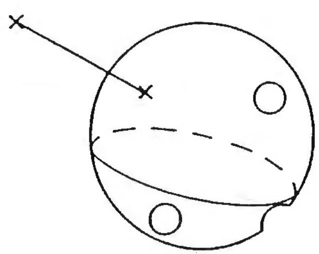

Figure 4

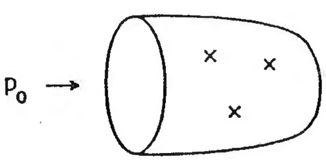


Figure 5

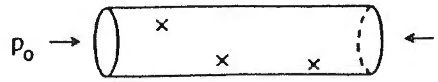


Figure 6

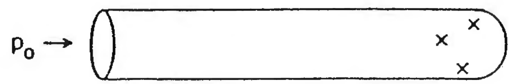


Figure 7

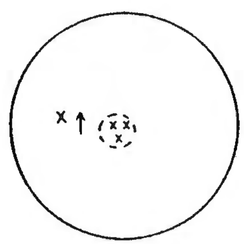

Figure 8

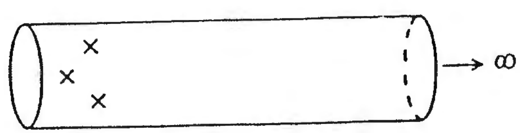


Figure 9a

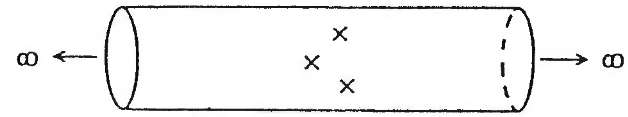


Figure 9b

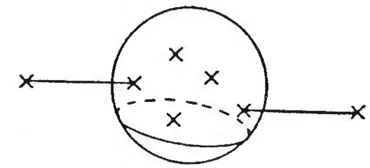

Figure 10


Figure 11