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Some Applications of Natural Motor Control

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Abstract
This paper presents two setpoint regulation problems that may be distinguished from the traditional preview of feedback design by the a priori impossibility of building a smooth bounded controller whose closed loop yields asymptotic stability while preserving configuration constraints. An appeal to the theoretical ideas introduced in [13] yields a solution to each of these problems in the form of a navigation function thatserves as an instance of the natural control philosophy. That is to say, the intrinsic dynamics of the mechanical system, when properly "programmed" are capable of "solving" what have often been cast as planning problems. The resulting closed loop behavior demonstrates a kind of autonomy in that the goal is achieved with probability one and with no further intervention on the part of a "higher level planner."

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Some Applications of Natural Motion Control[1]

This paper offers two engineering applications of the theoretical ideas presented in [13]. Each case concerns a generalized setpoint problem for a mechanical system whose configuration space does not permit globally asymptotically stable closed loops. By presenting a navigation function for each case, the paper shows how "natural control" may provide a degree of autonomy of execution without the appeal to a higher level planning system.

1 Introduction

This paper presents two setpoint regulation problems that may be distinguished from the traditional purview of feedback design by the à priori impossibility of building a smooth bounded controller whose closed loop yields asymptotic stability while preserving configuration constraints. An appeal to the theoretical ideas introduced in [13] yields a solution to each of these problems in the form of a navigation function that serves as an instance of the natural control philosophy. That is to say, the intrinsic dynamics of the mechanical system, when properly "programmed" are capable of "solving" what have often been cast as planning problems. The resulting closed loop behavior demonstrates a kind of autonomy in that the goal is achieved with probability one and with no further intervention on the part of a "higher level planner."

1.1 Satellite Attitude Control. The control of large angle maneuvers for rigid spacecraft has been studied extensively by numerous authors. Crouch [5] in an interesting and relatively recent contribution, considers the controllability properties of a rigid satellite both locally and globally, in cases where there are less than three degrees of actuator freedom. Dwyer [8], in a paper appearing at the same time, considers the problem of large motion control with full actuation and linearizes exactly the equations of motion (expressed in the quaternions) by appealing to the geometric methodology of Hunt, Su, and Meyer [7]. Of course, the spatial rotations, SO(3), cannot be globally identified with a Euclidean vector space.[2] In a very nice series of reports issued more than a decade ago, Meyer [15] attempted to generalize PD techniques to the global control of spacecraft attitude. His point of view is very close to the spirit of this paper, and, in some sense, the application presented here might be seen as a continuation and extension of that earlier work.

The problem of global satellite attitude tracking can be cast as a setpoint regulation problem by introduction of the appropriate error coordinates. The results of [13] are used to produce the first (to the best of the author's knowledge) smooth feedback controller for a fully actuated satellite which is well defined on the entirety of the rotations and which achieves asymptotically exact attitude tracking around an arbitrary reference trajectory with probability one.[3]

1.2 Artificial Potential Fields. The idea of using "potential functions" for the specification of robot tasks was pioneered by Khatib [9] in the context of obstacle avoidance, and further advanced by fundamental work of Hogan [6] in the context of force control. The methodology was developed independently by Arimoto in Japan [16], and by Soviet investigators as well [18]. Of course, the possibility of solving complex problems by resort to analog (or iterated discrete approximations of analog) methods of computation has a much older history, and may be found in many engineering papers. All of these applications have historically been plagued by the appearance of spurious minima.

It is this problem that the methods presented in Section 3 solve. In contrast to the previous case, the configuration space is embedded in a Euclidean vector space, however, the complicated constraints introduced by the obstacles give rise to boundaries demarcating forbidden regions. Here, the construction of a navigation function results in a robot which is guaranteed to approach a desired destination point in a cluttered space without hitting any of the clutter (from every zero velocity initial condition excepting a set of zero measure). This represents the first (to the best of the author's knowledge) smooth bounded feedback controller which solves the global robot obstacle avoidance problem on nontrivial spaces of arbitary dimension.[4]

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[2]Dwyer prohibits the purely imaginary quaternions (all symmetric rotations besides the identity) and works in the open three-disk (i.e., the upper hemisphere of the unit sphere in $\mathbb{R}^4$) – a homeomorph of $\mathbb{R}^3$ (a tutorial sketch of the geometry of the spatial rotations is presented in the Appendix).

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[3]Wen and Kreutz [22] have independently introduced similar ideas, however their solution cannot be smooth since it is claimed to have global convergence properties.

[4]An important contribution to the construction of bounded controllers for obstacle avoidance has been made by Newman and Hogan [17] who use logical combinations of C^0 potentials that result in bang-bang controllers which guarantee safety from collisions. In simple settings they may be shown to guarantee convergence to the desired destination, and in very special environments they have time optimal properties as well. The methods here yield smooth (in fact, analytic) rather than discontinuous controllers which always guarantee both safety and global convergence with probability one (stronger results are precluded by the topology of the problem). They result in "satisficing" trajectories that are unlikely to meet any criterion of optimality.

2 Satellite Attitude Tracking

We now pose a classical control problem—asymptotically exact tracking—in a nonclassical setting—the group of spatial rotations, SO(3). Although the controller will necessarily involve a feedforward forcing term in this case, the application nevertheless highlights the utility of the natural control philosophy because the design of the closed loop error equations involves a navigation function on the configuration space. This section is abstracted from a more extended (and rigorous) treatment [11].

Suppose there is a single rigid body actuated by three independent gas jets operating outside of the earth's gravitational field: the only forces on the body are the controlled inputs from the actuators which are capable of delivering any desired force in the "wrench space" of the body. Both the position and the velocity of the body are available from sensors. It is desired to force the body to track an arbitrary but entirely known reference trajectory. Since the system is completely actuated, there is perfect state information, and all derivatives of the reference trajectory are known, the velocity tracking problem is trivial. Namely, all nonlinearities due to the kinetic energy may be exactly cancelled, leaving a completely decoupled linear time invariant system. This procedure may be recognized as a trivial implementation of the global exact linearization techniques which have become so popular in the nonlinear control literature.

Consider, instead, the problem of attitude tracking. Namely, given a desired motion, $D \in C^2[\mathbb{R}, SO(3)]$, construct a time invariant memoryless controller which causes the actual attitude to asymptotically approach $D(t)$ from any initial configuration, $A \in SO(3)$. So different is this from the trivial linear problem to which velocity tracking reduces that it is unsolvable as posed in that context. For, consider the particular case that $D(t) = D^*$ is some constant configuration. We seek a controller which makes that point (at zero angular velocity) a global attractor of the closed loop dynamics. Now the domain of attraction of an attracting point is homeomorphic to some Euclidean vector space [3]. But the state space of our mechanical system—the tangent bundle over the rotation group—is clearly not homeomorphic to any Euclidean vector space. Thus, it would be impossible for our closed loop system to bring all initial conditions to the desired attitude. Evidently, the control system arising from a single rigid body is *not* globally linearizable by any technique since its state space is not a vector space. Our problem statement must be refined.

2.1 Inverse Dynamics Controllers Obtain from Error Dynamics. Arriving at a refinement will be easier if we first review classical tracking theory. In the linear time invariant setting, inverse dynamics amounts to the use of a precompensator to make the errors between the feedback stabilized plant state and reference derivatives satisfy an asymptotically stable linear time invariant dynamical system. Not surprisingly, when the linear plant is also a mechanical control system, the asymptotically stable error dynamics may be interpreted as a particular dissipative mechanical system expressed in the error coordinates.

For example, consider the mechanical control system, Σ_P, introduced in Example 2.1.1 of [13]. After stabilizing with $u = -K_1 q - K_2 \dot{q} + v$, we may cause the plant to track an arbitrary desired reference signal, d, by pre-filtering, $v = M\ddot{d} + K_2\dot{d} + K_1 d$, since this results in the globally asymptotically stable error dynamics, Δ_{HR} reviewed in Example 2.3.1 of [13]. Lord Kelvin's results apply, and we are guaranteed of asymptotically exact tracking.

Suppose, instead of the Hook's Law rule, φ_H [13, Example 2.2.1], we were to choose a different navigation function for the zero configuration, φ_T. We might now employ the controller

$$u = D\varphi_T(e) - K_2\dot{q} + v$$

$$v = M\ddot{d} + K_2\dot{d},$$

resulting in closed-loop error dynamics

$$\dot{e}_1 = e_2$$

$$\dot{e}_2 = -grad\varphi_T(e_1) - M^{-1}K_2 e_2,$$

specified by the dissipative mechanical system, $\Delta_{TR} = (\mathbb{R}^n, M, \varphi_T, K_2)$, also guaranteeing asymptotically exact tracking according to [13, Theorem 2]

Barring any further tracking criteria—for example, constraints on computational complexity; constraints on transient as well as limiting error behavior; and so on—there is no reason to prefer the familiar error system, Δ_{HR}, and its associated tracking controller over the new one, Δ_{TR}. In fact, in the present application, no Hook's Law spring can be defined over the configuration space SO(3) for exactly the same sort of topological reasons encountered in [13, Example 2.1.2]. Thus, if we seek to mimic classical tracking approaches on SO(3), there will be no choice but to find an alternative navigation function, φ_T.

2.2 Configuration Space, Phase Space, and Error Coordinates. As developed in the Appendix, the fully actuated rigid body gives rise to a mechanical control system, $\Sigma_R = (SO(3), M)$, whose internal dynamics f_{Σ_R} may be expressed in body coordinates as

$$\dot{R} = RJ(r)$$

$$\dot{r} = M^{-1}[u - J(r)Mr].$$

For the present application, we have assumed the à priori designation of a desired "reference trajectory," $p_d(t) \in \mathcal{P}$ which is "second order." That is to say, if $p_d(t) = (D,d)(t)$, then $\dot{D} \equiv DJ(d)$. Now if $p = (R,r)$ denotes the actual trajectory of the rigid body, we will find it useful to consider the "error coordinate system" obtained via left translation by p_d,

$$p_e = (E,e) \triangleq (D^T R, r - R^T D d),$$

preserving the second order property, $\dot{E}(t) = EJ(e(t))$.

2.3 A Navigation Function on SO(3). Meyer [15] chose for his potential law on SO(3) the distance from a reference point measured by the "natural distance function" based upon the trace of the rotation error, $E = D^T R$. Unfortunately, the trace function cannot offer navigation properties on SO(3) since its gradient vanishes at every symmetric rotation matrix—formally, it is not a Morse function. Instead, we will use a "modified trace" function according to the following result of Marsden and collaborators.

Lemma 1 (Chillingworth, Marsden, Wan[4]). If P is a symmetric 3×3 matrix with distinct eigenvalues, π_1, π_2, π_3, and

$$(\pi_1 + \pi_2)(\pi_1 + \pi_3)(\pi_3 + \pi_2) \neq 0,$$

then there are exactly four rotations, $R \in SO(3)$ at which PR is also symmetric. These are exactly the critical points of the "modified trace function,"

$$tr\{PR\},$$

whose Hessian matrix at each critical point has no zero eigenvalues.

We are thus led to define as a navigation function on SO(3)

$$\varphi_T(R) \triangleq \frac{1}{\pi'} tr\{P(I-R)\} \qquad (1)$$

where the quotient involving $\pi' \triangleq \pi_2 + \pi_3 - \pi_1$ is added to keep the image in the interval [0, 1] (assuming that $\pi_1 < \pi_2 < \pi_3$). If P, is chosen to be a positive definite symmetric matrix, which we now further assume, then the eigenvalue assumptions of the previous Lemma are assured. The Lemma says that the modified trace function has only four extrema, and its Hessian matrix is nonsingular—it is a Morse function. Since $D\varphi_T$ is a scalar multiple of Marsden's modified trace, φ_T is also a Morse function with four critical points specified in the same fashion. Moreover, φ_T takes its values on $\mathbb{R}^+$, vanishing only at $R = I$. Finally, since SO(3) has no boundary, φ_T is admissible. Thus, according to the criteria of [13], φ is indeed a navigation function. An algebraic function, its gradient may be readily computed as

$$\text{grad}\varphi_T(R) = 2M^{-1}J^{-1}(PR - R^TP).$$

via the isomorphism, J, [11] between skew symmetric matrices and vectors reviewed in the appendix of this paper.

According to the results of [13] together with Lemma 1 the negative flow of $\Gamma = (\text{SO}(3), M, \varphi_T)$ takes all points of SO(3) to one of four symmetric rotations—the identity, and the three orientations which are "180 deg away" along the x, y, z axes—and all points excepting a nowhere dense set to the identity.

2.4 Inverse Dynamics. We may now achieve $\Delta_{TR} = (\text{SO}(3), M, \varphi_T(E), K_2 e)$ closed loop error dynamics by building a controller according to the logic of Section 2.1. Namely, we will let

$$u = \text{grad}\varphi_T(E) + c(p, p_d) - K_2 r + v$$

$$v = -M(\dot{e} - \dot{r}) + K_2 d.$$

The additional term, c, in the feedback portion denotes a portion of the Coriolis and centripetal forces arising from the configuration dependent kinetic energy κ,

$$c(p, p_d) \triangleq J(r)M_e{}^T d + J(e^T d)M\dot{e}.$$

Similarly, expressing the feedforward reference acceleration term as the difference $\dot{e} - \dot{r}$ allows for proper cancellation of the remaining coriolis and centripetal forces,

$$\dot{e} = \dot{r} - E^T\dot{d} - [EJ(r - E^T d)]^T d$$

$$= \dot{r} - E^T\dot{d} + J(r)E^T d.$$

The resulting dynamics in the error coordinate system of $\mathcal{P}$ are given $\text{TSO}(3) = \text{SO}(3) \times \mathbb{R}^3$,

$$\dot{E} = EJ(e)$$

$$\dot{e} = -M^{-1}[J(e)Me + K_2 e + \text{grad}_\varphi(E)]. \tag{2}$$

Equation (2) is exactly the desired dissipative mechanical system, Δ_{TR} based upon the navigation function (1). The global version of Lord Kelvin's energy argument, [13, Theorem 2], applies directly. Since the configuration space has no boundary, there is no speed limit imposed upon the initial conditions. We immediately conclude:

Theorem 1. *All trajectories of (2) tend toward one of the four critical points of φ. A dense open set of initial conditions has its limit set at the desired point, $(E, e) = (I, 0)$.*

The satellite asymptotically attains the desired attitude trajectory, $D(t)$, except from a set of initial conditions of zero measure in the phase space. An adaptive variant of this algorithm has been presented in [11] in the case that M is not know in advance.

3 Robot Navigation

Consider the following problem in robotics. A kinematic chain—a sequence of mutually constrained actuated rigid bodies—is allowed to move in a cluttered workspace. Contained within the joint space—an analytic manifold which forms the configuration space of the kinematic chain—is the free space, $\mathfrak{F}$—the set of all configurations which do not involve intersection with any of the "obstacles" cluttering the workspace. Given any interior "destination point," $q_d \in \mathring{\mathfrak{F}}$ to which it is desired to move the robot, find a curve in $\mathfrak{F}$ from an arbitrary initial point to the desired destination.

The purely geometric problem of constructing a path between two points in a space obstructed by sets with arbitrary polynomial boundary (given perfect information) has already been completely solved [21]. The present formulation is motivated by the desire to incorporate explicitly aspects of the control problem—the construction of feedback compensators for a well characterized class of dynamical systems in the presence of well characterized constraints-in the planning phase of robot navigation problems.

3.1 Navigation Functions on Euclidean Sphere Worlds. A "Euclidean sphere world" is a compact connected subset of E^n whose boundary is the disjoint union of a finite number, say $M + 1$, of spheres. We suppose that perfect information about this space has been furnished in the form of $M + 1$ center points $\{q_i\}_{i=0}^M$ and radii $\{p_i\}_{i=0}^M$ for each of the bounding spheres. In our previous work [14], we have shown how to use this information to build a navigation function on the particular sphere work, $\mathfrak{M}$, considered as a simple freespace. Namely, letting γ denote the Euclidean distance to the destination, and $\beta = \Pi_{i=0}^M \beta_i$ denote the product of implicit representations, β_i, for each obstacle, it can be shown that

Theorem 2 ([14]). *If the free space, $\mathfrak{M}$, is a Euclidean sphere world then there exists a positive integer N such that for every $k \geq N$, for any finite number of obstacles, and for any destination point in the interior of $\mathfrak{M}$,*

$$\varphi = \left(\frac{\gamma^k}{\gamma^k + \beta}\right)^{1/\kappa}, \tag{3}$$

is a navigation function on $\mathfrak{M}$.

In the proof of this theorem (which comprises the central contribution of [14]) a constructive formula for N is given.

3.2 Navigation Functions Induced by Diffeomorphism. The Euclidean sphere world, of course, corresponds to a rather simplistic view of freespace. This section will describe how to extend the use of navigation functions to increasingly more realistic settings by recourse to a key attribute—their invariance under smooth change of coordinates. Simply put, suppose a geometrically simple environment, $\mathfrak{M}$ can be identified with a much more complicated obstacle course, $\mathfrak{Q}$, via a smooth one-to-one and onto function, h: $\mathfrak{Q} \rightarrow \mathfrak{M}$, whose inverse is also smooth—a diffeomorphism in mathematical parlance. Then, as the following result shows, once a navigation function has been constructed in the simple "model" case, an extension is automatically available to the realistic problem at hand.[5]

Proposition 2([14]). If φ is a navigation function on $\mathfrak{M}$ and h: $\mathfrak{F} \rightarrow \mathfrak{M}$ is a diffeomorphism then

$$\tilde{\varphi}(q) \triangleq \varphi(h(q))$$

is a navigation function on $\mathfrak{F}$.

In linear control problems, the appropriate change of coordinates is simply the familiar linear change of basis. In the present context it is not so easy to see how to identify a "topologically equivalent" but geometrically complicated" configuration space with a simple model in a constructive fashion. As an example of how this might be done, consider the following class of geometrically complex obstacle courses for which the Euclidean Sphere worlds, above constitute a suitable model.

A star shaped set is a deformed ball possessed of a distinguished interior center point from which all rays intersect its

[5]One begins to see in the robotics literature a growing awareness of the utility of coordinate changes in the obstacle avoidance problem [1, 10]. The work reported here provides a unified framework within which to prescribe the properties required of the transformations in order to guarantee correctness.

boundary in a unique point. A star world is a compact connected subset of E^n whose boundary is the disjoint union of a finite number of star shaped set boundaries. Now suppose the availability of an implicit representation for each boundary component, β_j, as well as knowledge of the obstacle center points, q_j. Further geometric information required in the construction to follow is detailed in the chief reference for this work [20]. A suitable Euclidean sphere world model, $\mathfrak{M}$, is explicitly constructed from this data. That is, we determine (p_j, ρ_j), the center and radius of a model j^{th} sphere, according to the center and minimum "radius" (the minimal distance from q_j to the jth obstacle) of the jth star shaped obstacle. This is turn determines the model space "obstacle functions," $\hat{\beta}_j$, as well as the navigation function on $\mathfrak{M}$, φ, as described above.

A transformation that identifies a star shape with a ball may be defined as

$$T_i(q) \triangleq (\nu_i \cdot [q - q_i]) + p_i. \tag{4}$$

This transformation first scales each ray passing through q_i by the amount ν_i and then translates along the vector p_i (the center of the target ball). Taking scaling factor ν_i to be

$$\nu_i(q) \triangleq [1 + \beta_i(q)]^{1/2} \frac{\rho}{\|q - q_i\|}, \tag{5}$$

note that the composition of the model ball obstacle function, $\hat{\beta}_i(p) = \|p - p_i\|^2 - \rho^2$, with T_i yields

$$(\hat{\beta}_i \circ T_i)(q) = \|\nu_i \cdot [q - q_i] + p_i - p_i\|^2 - \rho^2 = 1 + \beta_i(q) - 1 = \beta_i(q),$$

indicating that T_i maps the boundary of the star into the boundary of disk, its interior into the disc's interior, and its outside part into the disc's outside part as required.

A transformation, $h: \mathfrak{M} \to \mathfrak{F}$, may now be constructed in terms of the given star world and the derived model sphere world geometrical parameters as follows. Denote the "jth omitted product," $\Pi_{j=0}^{M} \beta_j$ as $\bar{\beta}_j$. The "jth analytic switch," $\sigma_j \in C^\omega[\mathfrak{F}, \mathbb{R}]$,

$$\sigma_j(q,\lambda) \triangleq \frac{x}{x+\lambda} \circ \frac{\gamma_d \bar{\beta}_j}{\beta_j} = \frac{\gamma_d \bar{\beta}_j b}{\gamma_d \bar{\beta}_j b + \lambda \bar{\beta}_j},$$

(where λ is a positive constant) attains the value one on the jth boundary and the value zero on every other boundary component of $\mathfrak{F}$. For the sake of notational consistency, denote the identity map on E^n as $T_d(q) = q$, and let

$$\sigma_d(q,\lambda) \triangleq 1 - \sum_{i=0}^{M} \sigma_i$$

denote the "destination switch." A "linear combination of translated scalings," is the one-parameter family of transformations defined by

$$h_\lambda(q) \triangleq \sigma_d(q,\lambda) T_d(q) + \sum_{i=0}^{M} \sigma_i(q,\lambda) T_i(q). \tag{6}$$

The "switches," make h look like the jth deforming factor in the vicinity of the jth obstacle, and like the identity map away from all the obstacle boundaries. With some further geometric computation we are able to prove the following.

Theorem 3 ([20]). *For any valid star world, $\mathfrak{F}$, there exists a suitable model sphere world $\mathfrak{M}$, and a positive constant Λ, such that if $\lambda \geq \Lambda$, then*

$$h_\lambda : \mathfrak{F} \to \mathfrak{M},$$

in (6) is an analytic diffeomorphism.

Thus, if φ is a navigation function on $\mathfrak{M}$, the construction of h_λ automatically induces a navigation function on $\mathfrak{F}$ via composition, $\tilde{\varphi} \triangleq \varphi \circ h_\lambda$, according to Propostion 2.

This family of transformations, mapping any star world onto the corresponding sphere world, induces navigation functions on a much larger class than the original sphere worlds, thus advancing our program of research toward the goal of developing "geometric expressiveness" rich enough for navigation amidst real world obstacles. Progress in this area has advanced to the point that we can now construct navigation functions for arbitrarily close approximations of any topological sphere world [19].

3.3 A Navigation Function Induced by the Kinematic Transformation. The previous examples have been concerned with encoding tasks via navigation functions over the configuration space. It frequently occurs in robotics that we have a task expressed in terms of the workspace of the gripper, even though the robot must ultimately be controlled via the generation of torques acting on the jointspace. The workspace gripper frame may be expressed as the image of the robot's joints under the forward kinematics, $g: \mathcal{Q} \to \mathcal{W}$. Consider now the possibility of inducing a navigation function on $\mathcal{Q}$ through an expression of the task in terms of a navigation function on the workspace, $\mathcal{W}$[6].

Unfortunately, the forward kinematic map is almost never a diffeomorphism: there are often more degrees of freedom in the joint space than in the workspace; even if the number of degrees of freedom is the same, there are almost always kinematic singularities. Thus, Proposition 2 cannot be invoked to guarantee that the navigation properties φ are preserved. As a concrete example consider a planar revolute-revolute kinematic chain. For convenience, suppose that there are no joint limits so that $\mathcal{Q}$ is the torus with empty boundary.[7] Similarly, for ease of exposition, suppose that all the mass, m_1, of the first link is concentrated at position

$$k_1(q) \triangleq \begin{bmatrix} l_1 \cos(q_1) \\ l_1 \sin(q_1) \end{bmatrix},$$

and all the mass, m_2, of the second, at position

$$k_2(q) \triangleq k_1(q) + \begin{bmatrix} l_2 \cos(q_1 + q_2) \\ l_2 \sin(q_1 + q_2) \end{bmatrix},$$

which we take, as well, to be the origin of the gripper frame of reference. Ignoring the orientation of the gripper, we consider the workspace, $\mathcal{W}$ to be the entire plane. Thus, the forward kinematic map, $g: \mathcal{Q} \to \mathcal{W}$, is exactly $g \equiv k_2$.

Now suppose that an end-point regulation task has been specified as reaching a desired cartesian point, w_d within the annular region of the plane comprising the reachable portion of workspace. The Hook's Law potential

$$\varphi_H \triangleq \frac{1}{2} \|w - w_d\|^2$$

is an obvious navigation function (up to a scaling constant to adjust the height) on a bounded disk in $\mathbb{R}^2$ which encodes the task at hand. Its composition with g,

$$\hat{\varphi}_H \triangleq \varphi_H(g(q)) \triangleq \frac{1}{2} \|g(q) - w_d\|^2,$$

however, might not be, since one does not yet know how many minima $\hat{\varphi}_H$ may have.

The set of extrema of $\hat{\varphi}_H$ in $\mathcal{Q}$ consists of those points in jointspace at which the gradient vector of φ_H is in the null space of $G^T \triangleq [Dg]^T$—the transposed jacobian matrix of the kinematic map. Note that the jacobian is given by

$$G = Dg = [Jk_2, J(k_2 - k_1)] = J[k_2, k_2 - k_1]$$

[6] While inverse kinematic problems are often solved numerically, the results here seem to represent the first unambiguous demonstration that the intrinsic motion of the robot's links can solve such problems with probability one.

[7] Otherwise, the techniques for handling holonomic constraints developed above could be added here.

where J is the unit skew symmetric matrix on $\mathbb{R}^2$. It follows that the critical values of g—the workspace image of the kinematic singularities—occur when the two masses lie on the same ray through the origin of $\mathcal{W}^8$, and the gripper lies on either the outer or inner boundary of the reachable workspace. In such a situation, it is clear that the nullspace of G^T must lie along the same ray as well. Thus, in addition to the extremum at $g^{-1}(w_d)$, it may also occur that G^T grad $\varphi_H = 0$ when grad $\varphi_H \neq 0$. Expressed intuitively, the cartesian force vector corresponding to an instantaneous error might resolve into a joint space generalized torque vector which points along a direction of "lost freedom."

Proposition 3. Let q^* be an extremum of $\hat{\varphi}_H$,

$$[D\hat{\varphi}_H](q^*)=0.$$

If q^* is a critical point of g,

$$|G^*|=0; \quad G^* \triangleq [[Dg](q^*)]^T,$$

and $g(q^*)$ is not an extremum of φ_H,

$$D\varphi_H(g^*(q^*))\neq 0$$

then q^* is not a local minimum of $\hat{\varphi}_H$.

Proof:

The differential of $\hat{\varphi}_H \triangleq [g(q)-w_d]^T [g(q)-w_d]$ is $D_q\hat{\varphi}_H = v^T G$, where v denotes the gradient vector field. It suffices to show that the Hessian,

$$D^2\hat{\varphi}_H = G^T D_v + (I\otimes v)DG^S,$$

could never be a positive definite matrix when evaluated at q^*.[9] Differentiating, we have $Dv = G^T$ and

$$DG^S = D\begin{bmatrix} Jk_2 \\ J(k_2-k_1)\end{bmatrix}$$

$$= \begin{bmatrix} -k_2 & k_1-k_2 \\ k_1-k_2 & k_1-k_2\end{bmatrix},$$

hence

$$D^2\hat{\varphi}_H = G^T G - \begin{bmatrix} (k_2-w_d)^T & 0 \\ 0 & (k_2-w_d)^T\end{bmatrix}$$

$$= \begin{bmatrix} -k_2 & k_1-k_2 \\ k_1-k_2 & k_1-k_2\end{bmatrix}$$

$$= \begin{bmatrix} w_d^T k_2 & w_d^T(k_2-k_1) \\ w_d^T(k_2-k_1) & (w_d-k_1)^T(k_2-k_1)\end{bmatrix}.$$

Since $|G^*| = 0$ we know that $k_2(q^*)$ and $k_1(q^*)$ are linearly dependent: since $k_1 = l_1 u$ for some unit vector, u, it follows that $k_2 = \kappa_2 u$ where $\kappa_2 = l_1 \pm l_2$ (recall that l_i isthe length of link i). Moreover, since the kernel of G^T is always along this same unit vector, and $v = g - w_d$ is in this kernel by assumption, we have $w_d = \omega u$ as well, where $|l_1 - l_2| \leq |\omega| \leq l_1 + l_2$. We may now write

$$[D^2\hat{\varphi}_H](q^*) = \begin{bmatrix} \omega\kappa_2 & \omega(\kappa_2-l_1) \\ \omega(\kappa_2-l_1) & [(\omega-l_1)(\kappa_2-l_1)\end{bmatrix},$$

and it will suffice to show that the leading principal minors of this matrix cannot both be positive.

The first principal minor is $\omega\kappa_2$. The second its given by the determinant,

$$|[D^2\hat{\varphi}_H](q^*)| = \omega(\kappa_2-l_1)(\kappa_2(\omega-l_1)-\omega(\kappa_2-l_1))$$

$$= -l_1\omega(\kappa_2-l_1)(\kappa_2-\omega)$$

Assume first that ω is positive. We must have $\kappa_2>0$ as well for the first principal minor to be positive. This implies that either $\kappa_2 = l_1 + l_2$ or that $l_1>l_2$ and $\kappa_2 = l_1 - l_2$. In both cases, $(\kappa_2 - l_1)(\kappa_2 - \omega)$ is positive, so that the second principal minor is negative. Alternatively, assume that ω is negative (if it is zero then the matrix cannot be positive definite). We must have $\kappa_2<0$ and this implies that both $l_1<l_2$ and $\kappa_2 = l_1 - l_2$, so that $(\omega)(\kappa_2 - l_1)$ is positive. Thus, the second minor cannot be positive unless $-\kappa_2> -\omega$ in contradiction to the assumption that $|\omega|>|l_1-l_2|$.

Since φ_H is a navigation function on $\mathcal{W}$ it now follows that $\hat{\varphi}_H$ is a navigation function on $\mathcal{Q}$. Thus, the feedback law [13, Eq. (5)] may be used to achieve end-point tasks for a SCARA arm with no computation of inverse kinematics.

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References

1 Akishita, Sadao, Kawamura, Sadao, and Kei ichi Hayashi, New Navigation Function Utilizing Hydrodynamic Potential for Mobile Robot, in *Proceedings 1990 IEEE Motion Control Conference*, 1990.

2 Bellman, R., *Stability of Differential Equations*, Dover Publications, New York, 1953.

3 Bhatia, N. P., and Szegö, G. P., *Dynamical Systems: Stability Theory and Applications*, Springer-Verlag, Berlin, 1967.

4 Chillingworth, D. R. J., Marsden, J. E., and Wan, Y. H., "Symmetry and Bifurcation in Three Dimensional Elasticity, Part I." *Arch. Rat. Mech. Anal.*, Vol. 80(4), 1982, pp. 295-331.

5 Crouch, Peter, E., "Spacecraft Attitude Control and Stabilization: Applications of Geometric Control Theory to Rigid Body Models," *IEEE Transactions on Automatic Control*, Vol. 29(4), 1984, pp. 321-331.

6 Hogan, Neville, "Impedance Control: An Approach to Manipulation," ASME *Journal of Dynamics Systems, Measurement, and Control*, Vol. 107, Mar. 1865, pp. 1-7.

7 Hunt, L. R., Su, R., and Meyer, G., "Design for Multi-Input Nonlinear Systems," R. W. Brockett, R. S. Millman, and H. J. Sussman, eds; *Differential Geometric Control Theory*, Birkauser, Boston, MA, 1983, pp. 268-297.

8 Dwyer III, T. A. W., "Exact Nonlinear Control of Large Angle Rotational Maneuvers," *IEEE Transactions on Automatic Control*, Vol. AC-29(9); 1984, pp. 769-774.

9 Khatib, O., and Le Maitre, J.-F., "Dynamic Control of Manipulators Operating in a Complex Environment, *Proceedings Third International CISM-IFToMM Symposium*, Udine, Italy, Sept. 1978, pp. 267-282.

10 Jin-Oh, Kim, and Khosla, K. Pradeep, "Real-time Obstacle Avoidance Using Harmonic Potential Functions," Technical report, Carnegie Mellon University, 1990.

11 Koditschek, D. E., "Application of a New Lyapunov Function to Global Adaptive Attitude Tracking," *Proc. 27th IEEE Conference on Decision and Control*, Austin, TX, Dec., 1988, pp. 63-68.

12 Koditschek, Daniel, E., "Robot Control Systems," Stuart Shapiro, editor, *Encyclopedia of Artificial Intelligence*, J. Wiley, Inc., 1987, pp. 902-923.

13 Koditschek, Daniel, E., "The Control of Natural Motion in Mechanical Systems," ASME JOURNAL OF DYNAMICS SYSTEMS AND MEASUREMENT, Vol. 111, No. 4, pp. 547-551.

14 Koditschek, Daniel, E., and Rimon, Elon, "Robot Navigation Functions on Manifolds with Boundary, *Advances in Applied Mathematics*, Vol. 11, 1990, pp. 412-442.

15 Meyer, George, "Design and Global Analysis of Spacecraft Attitude Control Systems," NASA Technical Report TR R-361, Ames Research Center, Moffett Field, CA, Mar. 1971.

16 Miyazaki, Fumio, and Arimoto, S., "Sensory Feedback Based on the Artificial Potential for Robots," *Proceedings 9th IFAC*, Budapest, Hungary, 1984.

17 Newman, W. S., and Hogan, N., "High Speed Robot Control and Obstacle Avoidance Using Dynamic Potential Functions." *Proc. IEEE International Conference on Robotics and Automation*, IEEE, Raleigh, NC, 1987, pp. 14-24.

[8] Here and in the sequel, $|\cdot|$ denotes the determinant of the enclosed array. The mathematical name for S^3 with antipodal points identified is $\mathbb{RP}^3$—projective three space. A suitable multiplication law on S^3 can be shown to preserve the composition properties of rotations: this representation of the rotations is then called the "quaternion" representation.

[9] The superscript, ^{S} denotes the vector formed by "stacking" stacking each of the columns of an array vertically; $\otimes$ denotes the Kronecker product [2]. See [12] for a more detailed exposition of the stack-Kronecker notation in matrix calculus.

18 Pavlov, V. V., and Voronin, A. N., "The Method of Potential Functions for Coding Constraints of the External Space in an Intelligent Mobile Robot," *Soviet Automatic Control*, Vol. (6), 1984.

19 Rimon, E., and Koditschek, D. E., "Exact Robot Navigation in Topolgically Simple but Geometrically Complicated Environments," *Proc. IEEE International Conference on Robotics and Automation*, Cincinnati, OH, May 1990, pp. 1937–1943.

20 Rimon, E., and Koditschek, D. E., "The Construction of Analytic diffeomorphisms for exact robot navigation on star worlds," *Transactions of the American Mathematical Society*, Vol. 327, No. 1, Sept. 1991 pp. 71–116.

21 Schwartz, Jacob, T., and Sharir, Micha, "On the "Piano Movers" Problem I. The Case of a Two-Dimensional Rigid Polygonal Body Moving Amidst Polygonal Barriers," Technical Report 39, N.Y.U. Courant Institute Department of Computer Science, New York, 1981.

22 Wen, John, T., and Kreutz, Kenneth, "Globally Stable Control Laws for the Attitude Maneuver Problem: Tracking Control and Adaptive Control, *Proc. 27th IEEE Conference on Decision and Control*, Austin, Texas, Dec. 1988.

APPENDIX

Rigid Body Kinematies

We will use the symbol $J(w)$ throughout the paper to denote the skew symmetric matrix corresponding to the vector, ω

$$J{:}\,\mathbb{R}^3 \to \mathbb{R}^{3\times 3}{:}\ \begin{bmatrix}\omega_1\\ \omega_2\\ \omega_2\end{bmatrix} \to \begin{bmatrix}0 & -\omega_3 & \omega_2\\ \omega_3 & 0 & -\omega_1\\ -\omega_2 & \omega_1 & 0\end{bmatrix},$$

We may identify the configuration space, $\mathfrak{I}$ = SO(3), with a subset of $\mathbb{R}^9$,

$$\mathrm{SO}(3) \triangleq \{R\in\mathbb{R}^{3\times 3}{:}\ R^TR = I \text{ and } |R| = 1\},$$

in analogy to [13, Ex. 2.1.2.]. This set may also be put into correspondence with a sphere—$\mathbb{S}^3$, the unit sphere of $\mathbb{R}^4$—but now there are two antipodal unit vectors, $\mathbf{u}, -\mathbf{u}\in\mathbb{S}^3$ associated with each rotation matrix. Specifically, letting v denote the first component, and u denote the three vector comprising the remaining components of $\mathbf{u} = (v,u)\in\mathbb{S}^3$, we have

$$R(\mathbf{u}) = uu^T + [vI - J(u)]^2.$$

Let R: $\mathcal{I} \to$ SO(3) be a parametrized curve. Since $R^TR \equiv I$ for all $t\in\mathcal{I}$ according to the definition of SO(3), we have

$$0 = \frac{d}{dt} R^TR$$

$$= \dot{R}^TR + R^T\dot{R},$$

from which it follows that $R^T\dot{R}\in$ skew(3), hence, for all t there exists an $\omega\in\mathbb{R}^3$ such that

$$\dot{R} = RJ(\omega).$$

The vector ω is called the "angular velocity." We take the phase space to be the set of pairs

$$\mathcal{P} = \mathrm{SO}(3)\times\mathbb{R}^3 = \{(R,r){:}\ R\in\mathrm{SO}(3) \text{ and } r\in\mathbb{R}^3\},$$

as justified above. Physically speaking, if M, a positive definite symmetric 3×3 matrix, is the moment of inertia of the rigid body then the kinetic energy at a phase, $p = (R,r)\in\mathcal{P}$ is

$$\kappa(p) = \frac{1}{2}\, tr[J(r)R][J(r)RM]^T = \frac{1}{2}\, r^TMr. \qquad (7)$$

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