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# Adaptive Techniques for Mechanical Systems

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## **Comments**

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# Adaptive Techniques for Mechanical Systems

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## Abstract

Strict Lyapunov functions are constructed for a class of nonlinear feedback compensated mechanical systems, requiring no a priori information concerning the initial conditions of the closed loop system. These Lyapunov functions may be used to design a stable adaptive version of the "computed torque" algorithm for tracking a reference trajectory. A particular Lyapunov function is then generalized to permit an adaptive version of a control scheme forced by reference dynamics rather than a reference trajectory.

## 1 Introduction

Within the last year, independent work by Slotine and Li [15] and Horowitz and Sadegh [14] has resulted in a solution to the challenging problem of adaptive nonlinear "computed torque" control of a mechanical system,

$$M[q]\ddot{q} + c[q, \dot{q}] = \tau, \quad (1)$$

Interestingly enough, their approach departs from the traditional paradigm of linear time invariant adaptive technique in that the "Lyapunov function" used to demonstrate stability is not positive definite. <sup>1</sup> Since the consequent lacunae in the standard arguments are successfully replaced with correct alternative reasoning, their results represent a very important first step in the theory of adaptive control for general mechanical systems (1). Extensions of this analysis to more complicated problems, however — e.g. situations where the higher order unmodeled dynamics must be taken into account; where not all degrees of mechanical freedom have actuators; or situations where the task is not intrinsically defined by means of reference trajectory which is known, a priori, for all future time — may be quite difficult, absent a suitable Lyapunov function.

A new class of strict Lyapunov functions for the mechanical systems has been developed independently by Arimoto [1], Wen and Bayard [17], and this author [8,10]. Perhaps most notably, this analysis demonstrates that any mechanical system, (1), may be made exponentially stable by the application of linear proportional and derivative feedback. However, a significant weakness of this work lies in the dependence of the Lyapunov

functions upon the initial state of the system to be analyzed. In consequence, such functions are not suitable for use in defining adaptive control laws.

This paper presents some work in progress toward the "regularization" of stability arguments for mechanical systems. In Section 2, we start with a Lyapunov function which combines some features of several previous constructions [8,10,17]. It is shown that strict Lyapunov functions may be constructed free of all a priori information about initial conditions, if certain more (Section 2.2) or less (Section 2.3) restrictive nonlinear feedback controllers are allowed to replace the simpler and more general linear proportional and derivative gains. That insight results in a globally stable adaptive controller for "computed torque" applications whose qualitative behavior is established by means of a strict Lyapunov function with no high gain requirement (Section 2.4).

Attention shifts in Section 3 to a task description and control paradigm of more central importance in the author's program of robotics research which replaces substitutes dynamical reference systems in place of fixed trajectories [5,7,6]. The new Lyapunov function of Section 2.3 is generalized and used to expand the class of reference systems whose limit properties may be "embedded" in a mechanical system. Since the more general embedding requires full information about the plant parameters, it is incumbent to provide an adaptive version of this algorithm, and this is accomplished in Section 3.4

### 1.1 Notation and Definitions

#### Operator Bounds:

It will be important to obtain bounds on the operator norm of matrix valued maps. If  $A : J \rightarrow \mathbb{R}^{n \times n}$  is a map taking matrix values then define

$$\mu_\eta(A) \triangleq \sup_{\|q\| \leq \eta} \sup_{\|x\|=1} |x^T A(q)x| \quad \nu_\eta(A) \triangleq \inf_{\|q\| \leq \eta} \inf_{\|x\|=1} |x^T A(q)x|$$

and

$$\mu(A) \triangleq \lim_{\eta \rightarrow \infty} \mu_\eta(A) \quad \nu(A) \triangleq \lim_{\eta \rightarrow \infty} \nu_\eta(A).$$

These bounds extend to bilinear operator valued maps in a straightforward fashion.

#### Lyapunov Functions:

We will use standard notions of Lyapunov theory. Given a smooth (possibly time varying) vector field,  $f$ , on  $\mathcal{P}$ , we shall say that,  $v$ , a positive definite map at  $p_d \in S$ , constitutes a *Lyapunov function for  $f$  at  $p_d$*  if the time derivative along any motion of the vector field is non-positive,

$$\dot{v} = [D_p v]f + D_t v \leq 0,$$

<sup>1</sup>We will call such functions "LaSalle functions", below.

in some neighborhood of  $p_d$ , for all  $t$ , and that it constitutes a *strict Lyapunov function* for  $f$  if the inequality is strict [4,12]. A strict Lyapunov function will be called a *quadratic Lyapunov function* for  $f$  on the domain,  $S$  if it is analytic and there exist three positive constants,  $\alpha_1, \alpha_2, \alpha_3$ , with the properties,

$$\alpha_1 \|p - p_d\|^2 \leq v(p, t) \leq \alpha_2 \|p - p_d\|^2 \text{ and } \dot{v}(p, t) \leq -\alpha_3 \|p - p_d\|^2 \text{ for all } t \in \mathbb{R}^+ \text{ and } p \in \mathcal{D}. \quad (2)$$

**Theorem 1 ([10,3])** *If  $v$  is a quadratic Lyapunov function for  $f$  on some domain,  $S$ , containing the origin, then that equilibrium state is exponentially stable and attracts all trajectories originating in  $S$ .*

A related, but more general notion of a Lyapunov function involves a definition used by LaSalle [12]. Let  $f$  be a smooth, *time invariant* vector field on  $\mathcal{P}$ . A smooth scalar valued function,  $v$ , on some domain,  $S \subset \mathcal{P}$ , will be called a *LaSalle function* for  $f$  on  $S$  if its derivative along the flow induced by  $f$  is negative semi-definite on  $S$ . A subset of  $\mathcal{P}$  is said to be *positive invariant* with respect to the vector field,  $f$ , if any trajectory originating there stays there for all future time.

**Theorem 2 (LaSalle's Invariance Principle [12])** *Suppose  $f$  is time invariant vector field defined on the entirety of  $\mathcal{P}$  which has no finite escape trajectories. Let  $v$  be a LaSalle function for  $f$  on  $\mathcal{P}$ , with*

$$\mathcal{E} \triangleq \{p \in \mathcal{P} : \dot{v}(p) = 0\}.$$

*Given any initial condition,  $p_0 \in \mathcal{P}$ , there is some real value,  $\eta$ , such that the positive limit set of the solution through  $p_0$  is contained in the largest positive invariant set within  $\mathcal{E} \cap v^{-1}[\eta]$ .*

## 1.2 The Dynamical Structure of Mechanical Systems

Define a *mechanical system* to be any lagrangian dynamical system whose kinetic energy,  $\kappa$ , may be expressed locally as a quadratic form in generalized velocity and which is analytic in generalized position.

It seems simplest to use the "stack-kronecker notation" (refer to [2,13,11,10]) in computing the equations of motion. Let  $\dot{M}_q \in \mathcal{L}(T_q J, \mathcal{L}(T_q J, T_q J))$  denote the linear map

$$\dot{M}_q : x \mapsto [x \otimes I]^T D_q M^{\otimes},$$

define a skew-symmetric valued operator,

$$J_q(x) \triangleq \dot{M}_q(x) - \dot{M}_q^T(x). \quad (3)$$

and a linear map,  $L_q \in \mathcal{L}(T_q J, \mathcal{L}(T_q J, T_q J))$ , whose action on any tangent vector,  $y \in T_q J$  is given by

$$L_q(x)y = \dot{M}_q(y)x - \dot{M}_q(x)y \quad (4)$$

**Lemma 1** *For any curve,  $q : \mathbb{R} \rightarrow J$ , and any vector,  $x \in T_{q(t_0)} J$ ,*

$$\dot{M}(q) |_{t_0} x = \dot{M}_{q(t_0)}(x)\dot{q} |_{t_0}.$$

**Lemma 2** *Given a lagrangian with kinetic energy,  $\kappa$ , with no potential forces present, and with an external torque or force acting at every degree of freedom as specified by the vector,  $\tau$ , the equations of motion may be written in the form (1),*

$$M(q)\ddot{q} + c(q, \dot{q}) = \tau,$$

where

$$c(q, x) = A(q, x)x = B(q, x)x = C(q, x)x, \quad (5)$$

and

$$\begin{aligned} A(q, x) &\triangleq \left( \dot{M}(q_x) - \frac{1}{2} (D_q M x)^T \right) x \\ B(q, x) &\triangleq \frac{1}{2} \dot{M}_q(x) - \frac{1}{2} J_q(x) \\ C(q, x) &\triangleq \frac{1}{2} \dot{M}(q_x) - \frac{1}{2} J_q(x). \end{aligned} \quad (6)$$

The proof of these Lemmas follows by direct computation.

The representation of coriolis and centripetal forces in terms of the bilinear operator valued maps,  $A, B, C$ , is merely a convenience of exposition, and it is worth including the reminder at this juncture that, while they coincide at  $\dot{q}$  with the quadratic expression,  $c(q, \dot{q})$ , they are not, in general, the same. In particular, note that they differ by the "defect",  $L_q$ , i.e.

$$A(q, x) - L_q(x) = B(q, x) + L_q(x) = C(q, x),$$

and the identity, (5), may be interpreted as a consequence of the fact that  $x \in \text{Ker } L_q(x)$ . In this paper, we will find the representation in terms of  $C$  the most useful.

**Corollary 3** *For any motion  $q : \mathbb{R} \rightarrow J$ , and any tangent vector,  $x \in T_{Jq}(t)$ ,*

$$x^T \left[ \frac{1}{2} \dot{M}(q) - C(q, \dot{q}) \right] x \equiv 0.$$

**Proof:** From the previous lemma,

$$x^T \left[ \frac{1}{2} \dot{M}(q) - C(q, \dot{q}) \right] x = -\frac{1}{2} x^T J_q(\dot{q}) x = 0.$$

□

## 2 Strict Lyapunov Functions for Compensated Mechanical Systems

In this Section we explore some of the tradeoffs between feedback compensation schemes for mechanical systems, (1), and the information required to construct a strict Lyapunov function proving the exponential stability of the resulting closed loop. To simplify the arguments in the sequel, it is convenient (but not necessary [10]) to assume that  $M(q)$  takes positive definite values and is bounded above over the entirety of  $J$ . Moreover, assume that the matrix  $M$ , and (hence)  $C$  is linear in a finite set of "dynamical parameters",  $\pi$

$$[M(q), C(q, \dot{q})] = H(q)\pi.$$

In the case of robotic manipulators, these assumptions actually hold true.

### 2.1 A Quadratic Lyapunov Function for Linear "PD" Compensated Mechanical Systems

A notational definition will prove helpful in the sequel. Given two positive definite symmetric arrays,  $K_1, K_2$ , define

$$Q_0 \triangleq \begin{bmatrix} \frac{1}{2}(K_1 K_2 + K_2 K_1) & \frac{1}{2} K_2^2 \\ \frac{1}{2} K_2^2 & K_2 \end{bmatrix}, \quad (7)$$

and note that  $Q_0$  is easily made positive definite by the appropriate adjustment of  $K_1, K_2$ . For example, it is sufficient that the symmetric part of  $K_1 K_2$  be positive definite, and that [10]

$$\nu(K_1) > \frac{1}{2} \mu(K_2)^4 / \nu(K_2)^2.$$

Now consider the nonlinear vector field defined by

$$A(q(t), \dot{q}(t)) \triangleq \begin{bmatrix} 0 \\ -M^{-1}K_1 & -M^{-1}(C + K_2) \end{bmatrix}. \quad (8)$$

This characterizes the closed loop dynamics of an unforced mechanical system, (1) subject to linear PD compensation,

$$\tau = -K_1\dot{q} - K_2q, \quad (9)$$

as well as the closed loop error equations resulting from the "computed torque" algorithm,

$$\tau_{gen} = -K_1[q - r(t)] - K_2[\dot{q} - \dot{r}] + M(q)\ddot{r} + C(q, \dot{q})\dot{r} \quad (10)$$

with respect to a specified reference trajectory,  $r(t)$ .

**Proposition 4** Let  $K_1, K_2$  be positive definite symmetric matrices which further satisfy the condition that  $Q_0$  in (7) be positive definite as well.

Then there exist two positive functions,

$$\gamma_0(M) \triangleq \max \left\{ 1 + \mu(M) \frac{\mu(K_2)}{\nu(K_2)}, \mu(K_2) \sqrt{\frac{\nu(M)}{4\nu(K_1)}} \right\},$$

and

$$\gamma_1[e(0)] \triangleq \|e(0)\| \mu(J_q - \frac{1}{2}K_2^{-1}\dot{M}K_2) \frac{\mu(K_2)}{\nu(K_2)} \sqrt{\frac{\mu(P)}{\nu(P)}},$$

such that

$$v_{pd}(e) \triangleq \frac{1}{2}e^T P_{pd} e = \frac{1}{2}e^T \begin{bmatrix} \gamma K_1 & K_2 M \\ M K_2 & \gamma M \end{bmatrix} e$$

is a quadratic Lyapunov Function for the system defined by (8) on the ball of radius  $\|e(0)\|$ ,

$$B \triangleq \{e \in P : \|e\| \leq \|e(0)\|\}$$

as long as

$$\gamma > \gamma_0 + \gamma_1.$$

**Proof:** The proof proceeds very much as in [8,10], and the highlights are merely suggested here.

$$P_{pd} = \begin{bmatrix} \gamma K_1 - K_2 M K_2 & 0 \\ 0 & (\gamma - 1)M \end{bmatrix} + \begin{bmatrix} K_2 \\ I \end{bmatrix} M [K_2, I]$$

is positive definite as long as  $\gamma > \gamma_0$ . We have

$$\dot{v} = -e^T Q_0 e - e_2^T [(\gamma - 1)K_2 - M K_2] e_2$$

$$-e_1^T K_2 [J_q - \frac{1}{2}K_2^{-1}\dot{M}K_2] e_2$$

The first term is negative definite by hypothesis. The second term is negative as long as  $\gamma > \gamma_0$ , as well.

Now suppose  $\gamma > \gamma_0 + \gamma_1$  so that <sup>2</sup>

$$\gamma - \gamma_0 > \|e(0)\| \mu(J_q - \frac{1}{2}K_2^{-1}\dot{M}K_2) \frac{\mu(K_2)}{\nu(K_2)} \sqrt{\frac{\mu(P_{pd})}{\nu(P_{pd})}}.$$

<sup>2</sup>While the ratio of eigenvalues of  $P_{pd}$  in the following inequality is function of  $\gamma$ , it is easily verified that appears in the same power in both numerator and denominator: i.e. the ratio is globally bounded as a function of  $\gamma$ , hence, we may choose a large enough value of that constant to make the inequality correct.

This implies

$$(\gamma - \gamma_0)e_2(0)^T K_2 e_2(0) > |e_1(0)^T K_2 J_{q(0)} [e_2(0)] e_2(0)|, \quad (11)$$

demonstrating that  $\dot{v}(0) < 0$ .

If  $\dot{v}(0) < 0$ , then there is some open interval of time such that  $v(t) < v(0)$  and over this interval we have

$$\sqrt{\frac{v(0)}{\nu(P_{pd})}} \mu(J_q) > \sqrt{\frac{v(t)}{\nu(P_{pd})}} \mu(J_q)$$

so that the inequality (15) continues to hold. This implies that  $\dot{v}(t)$  is strictly negative over the interval, which must, in consequence, be unbounded to the right.

□

## 2.2 A Quadratic Lyapunov Function for a Specially Compensated Mechanical System

By choosing a very specialized nonlinear controller, it is possible to prove global exponential stability by means of a Lyapunov function whose definition involves no information regarding either the initial states or the inertial matrix,  $M$ . Specifically, consider the feedback law,

$$\tau = -[K_1 + C(q, \dot{q})K_2 + K_2 M K_2]q - [K_2 + 2M(q)K_2]\dot{q}, \quad (12)$$

and its associated inverse dynamics precompensator,

$$\tau_{spec} = -[K_1 + C(q, \dot{q})K_2 + K_2 M K_2][q - r(t)] - [K_2 + 2M(q)K_2][\dot{q} - \dot{r}] + M(q)\ddot{r} + C(q, \dot{q})\dot{r} \quad (13)$$

for tracking  $r(t)$ . These result in a vector field defined by

$$A_s \triangleq \begin{bmatrix} 0 & I \\ -M^{-1}(K_1 + C K_2 + K_2 M K_2) & -M^{-1}(C + K_2 + 2M K_2) \end{bmatrix}. \quad (14)$$

We will find it necessary to further restrict this class of controllers by the requirement that the symmetric part of  $K_2 M$  be positive definite: barring the case that  $K_2$  is chosen to be a scalar multiple of  $M$  (which would, in turn, necessitate a similar condition on  $K_1$ ), this amounts, in practice, to the requirement that  $K_2 \triangleq \kappa_2 I$  be a multiple of the identity matrix.

**Proposition 5** Let  $K_1, K_2, Q_0$  all be positive definite symmetric matrices subject to the further condition that the symmetric part of  $K_2 M$  is positive definite.

Then

$$v_s(e) \triangleq \frac{1}{2}e^T P_s(q)e = \frac{1}{2}e^T \begin{bmatrix} K_1 + K_2 M K_2 & K_2 M \\ M K_2 & M \end{bmatrix} e$$

is a strict Lyapunov Function for the vector field defined by  $A_s$  in (14).

**Proof:** Noting that

$$P_s = \begin{bmatrix} K_1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} K_2 \\ I \end{bmatrix} M [K_2, I],$$

it is clear that  $v_s$  is always positive definite.

Evaluating  $v_s$  along the motion of the system defined by

$A_s$ , we have

$$\dot{v} = -e^T \left( Q_0 + \begin{bmatrix} K_2 \\ I \end{bmatrix} K_2 M [K_2, I] + \begin{bmatrix} K_2 \\ I \end{bmatrix} J_q [K_2, I] \right) e$$

The last term is identically zero, the second term is a positive semi-definite matrix, in consequence of the assumption in the hypothesis that  $K_2 M$  has a positive definite symmetric part, and the first term is a positive definite matrix, by hypothesis as well. The result follows.

□

### 2.3 A Quadratic Lyapunov Function for a Larger Class of Nonlinear Compensated Mechanical Systems

While the control algorithm, (12), has the advantage of admitting a quadratic Lyapunov function which involves no a priori information regarding the initial conditions of the system, and the magnitudes of its parameters, it has been remarked that the restriction to a decoupled damping term,

$$K_2 \triangleq \kappa_2 I,$$

is unfortunate. Consider, instead, the specialized (in comparison to (9)) but less restricted (in comparison to (12)) nonlinear feedback law,

$$\tau = -[K_1 + C(q, \dot{q})K_2]q - [K_2 + MK_2]\dot{q} \quad (15)$$

and its associated "inverse dynamics" pre-compensator version,

$$\tau_{nl} = -[K_1 + C(q, \dot{q})K_2][q - r(t)] - [K_2 + MK_2][\dot{q} - \dot{r}] + M(q)\ddot{r} + C(q, \dot{q})\dot{r} \quad (16)$$

used for tracking the reference trajectory,  $r(t)$ . These algorithms, when applied to a mechanical system, (1), result in the closed loop vector field defined by

$$A_{nl} \triangleq \begin{bmatrix} 0 & I \\ -M^{-1}(K_1 + CK_2) & -M^{-1}(C + K_2 + MK_2) \end{bmatrix}. \quad (17)$$

**Proposition 6** Let  $K_1, K_2, Q_0$  all be positive definite symmetric matrices. Then the function,  $v_{pd}$ , defined in Proposition 5 is a quadratic Lyapunov function for the system defined by (16).

**Proof:** Recall from the proof of Proposition 5, that  $P_s$  is guaranteed to be positive definite. Evaluating the time derivative of  $v_s$  along the motion of the system defined by  $A_{nl}$ , we have

$$\dot{v}_s = -e^T Q_0 e - e^T \begin{bmatrix} K_2 \\ I \end{bmatrix} J_q [K_2, I] e$$

The last term is identically zero according to Corollary 3. The first term is strictly negative by hypothesis.

□

**Remark:** It is interesting to note that by adopting a particular choice of gains, parametrized by two positive definite symmetric matrices,  $\Lambda, K_D$ , as follows,

$$K_2 \triangleq \Lambda, \quad K_1 \triangleq K_D \Lambda,$$

and rearranging the terms in (15),

$$\begin{aligned} \tau_{nl} &= -[K_D \Lambda + C \Lambda](q - r) - [\Lambda + M \Lambda](\dot{q} - \dot{r}) + C \dot{r} + M \ddot{r} \\ &= -K_D [\Lambda, I] \begin{bmatrix} q - r \\ \dot{q} - \dot{r} \end{bmatrix} + C \left( \dot{q} - [\Lambda, I] \begin{bmatrix} q - r \\ \dot{q} - \dot{r} \end{bmatrix} \right) \\ &\quad + M \left( \ddot{q} - [\Lambda, I] \begin{bmatrix} \dot{q} - \dot{r} \\ \ddot{q} - \ddot{r} \end{bmatrix} \right) \end{aligned}$$

one obtains exactly the control law proposed by Slotine and Li [15]!

### 2.4 Applications to Adaptive Control of Mechanical Systems

Consider the nonlinear plant

$$\dot{x}_p = \begin{bmatrix} 0 & I \\ 0 & M^{-1}C \end{bmatrix} x_p + \begin{bmatrix} 0 \\ M^{-1} \end{bmatrix} \tau,$$

and the "special" nonlinear model,

$$\dot{x}_m = A_s x_m + \begin{bmatrix} 0 \\ M^{-1} \end{bmatrix} \tau,$$

which would result from the application to the plant of the special feedback law, (12), of Section 2.2.

It is natural to apply the adaptive control input to the plant,

$$\begin{aligned} \hat{\tau}_{spec} &= -[K_1 + \hat{C}(q_p, \dot{q}_p)K_2 + K_2 \hat{M}K_2][q_p - r(t)] \\ &\quad - [K_2 + 2\hat{M}(q_p)K_2 + \hat{C}(q_p, \dot{q}_p)][\dot{q}_p - \dot{r}] + \hat{M}(q_p)\ddot{r} \\ &= [K_1 + C(q_p, \dot{q}_p)K_2 + K_2 M K_2][q_p - r(t)] \\ &\quad - [K_2 + 2M(q_p)K_2 + C(q_p, \dot{q}_p)][\dot{q}_p - \dot{r}] + M(q_p)\ddot{r} \\ &\quad + H[q_m, \dot{q}_m, r(t), \dot{r}(t)]\phi(t), \end{aligned}$$

where  $\phi \triangleq \pi - \hat{\pi}$ , and the last line is a consequence of the original assumption that  $M$  is linear in the dynamical parameters. Let  $x \triangleq x_p - x_m$ , and define an adaptive law as

$$\begin{aligned} \dot{\phi} &= -H^T [0, M^{-1}] P_s x \\ &= -H^T [K_2, I] x. \end{aligned}$$

Fortunately, this is free of any dependence upon initial conditions, or parameters upper bounds. The resulting error equations may be written as

$$\begin{aligned} \dot{x} &= A_s x + \begin{bmatrix} 0 \\ M^{-1} \end{bmatrix} H \phi \\ \dot{\phi} &= -H^T [0, M^{-1}] P_s x. \end{aligned} \quad (18)$$

It is easy to see, using Proposition 5, that the Lyapunov function,  $\tilde{v} \triangleq v_s + \frac{1}{2}\phi^T \phi$ , has a negative semi-definite derivative along the motion of this system. Thus the overall system is Lagrange stable — i.e. all initial conditions give rise to bounded trajectories. Standard arguments from the linear adaptive control literature may now be used to show that the state errors,  $x$  approach zero asymptotically.

### 3 Adaptive Control of Natural Motion

A different paradigm of robot control [5,7,11] departs from the prevailing tradition of task specification by means of a reference trajectory,  $r(t)$ , in favor of specification by means of a "dynamical reference model". Namely, we assume that the problem of interest has been encoded in terms of the flow of a dynamical system,<sup>3</sup>

$$\dot{r} = f(r), \quad (19)$$

where  $f: J \rightarrow \mathcal{P}$  is a well behaved vector field on the configuration space,  $J$ .

#### 3.1 Embedding the Limit Behavior of Gradient Dynamics in Mechanical Systems

The use of potential functions to describe some desired geometric behavior is merely a special case of task encoding by means of a vector field, (19), which happens to be the gradient of some cost function

$$f \triangleq -(D\varphi)^T.$$

In fact, the restriction to the class of gradient vector fields has a most fortuitous consequence: namely, it is possible to duplicate the limiting behavior of the  $n^{\text{th}}$  order reference system, (19), within the  $2n^{\text{th}}$  order mechanical system, (1), using a feedback algorithm whose construction requires no explicit knowledge of the latter system. The following more precise statement of this fact amounts to a global extension of the century old discovery by Lord Kelvin [16] that conservative systems in the presence of dissipative forcing terms "decay" toward the local minima of their potential energy.

**Theorem 3 ([9])** *Let  $\varphi$  be a Morse function on  $J$  which is exterior directed on the boundary  $\partial J$ , surpasses the value  $\eta > 0$  on the boundary, and has local minima at the points  $S \triangleq \{q_i\}_{i=1}^n \subset J$ . Let  $K_2 > 0$  denote some positive definite symmetric matrix. Consider the set of "bounded total energy" states*

$$\mathcal{P}^\eta \triangleq \left\{ \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} \in \mathcal{P} : \varphi(p_1) + \frac{1}{2} p_2^T M p_2 \leq \eta \right\}.$$

*Under the feedback algorithm*

$$\tau \triangleq -K_2 p_2 - [D\varphi]^T(p_1),$$

*$\mathcal{P}^\eta$  is a positive invariant set of the closed loop dynamical system within which all initial conditions excluding a set of measure zero take  $S$  as their positive limit set.*

#### 3.2 Limitations in the "Expressive Power" of Gradient Dynamics

Consider the problem of tracking an implicitly specified curve,

$$\mathcal{C} \triangleq \{q \in J : c^{-1}(q) = 0\},$$

where it is known that 0 is a regular value of the smooth map,  $c: J \rightarrow \mathbb{R}^{n-1}$ , so that  $\mathcal{C}$  is guaranteed to be a one dimensional smooth manifold. If  $q_0$  is an explicitly known desired end-point along such an implicitly represented curve, then

$$\varphi_+ \triangleq c(q)^T c(q) + [q - q_0]^T [q - q_0],$$

<sup>3</sup>Here, and in the sequel, we assume that all vector fields under consideration are at least once continuously differentiable, hence, are guaranteed to generate a local flow.

$$\varphi_{(x,+)} \triangleq (c(q)^T c(q) + 1) [q - q_0]^T [q - q_0],$$

are two examples of the many different cost functions that might be constructed to represent the task of moving along  $\mathcal{C}$  until reaching the point  $q_0$ .<sup>4</sup> It is important to note, however, that in both cases, mimicking the transient response of the planning system, (19), by the closed loop mechanical system, is as important as mimicking the limiting behavior. Unfortunately, Theorem 3 provides guarantees only with regard to the latter criterion.

In fact, it is relatively easy to adjust the transient response of a gradient system, (19) since the dynamical properties of that class are so constrained. For example, changing the velocity of a reference gradient flow without altering the locus of its phase curves may be done as follows.

**Lemma 7** *Let  $f_1, f_2$  be vector fields on  $J$  which differ by a scalar function,  $\alpha$ , i.e.*

$$f_1 = \alpha f_2.$$

*Then, on the intersection of their respective domains, the flow,  $F_1^t$ , generated by  $f_1$  has the relation to the flow,  $F_2^t$ , generated by  $f_2$ , as follows,*

$$F_1^t = F_2^{\alpha(t)},$$

*where*

$$\dot{s} = \alpha$$

**Proof:** This fact obtains from simple application of the chain rule:

$$\begin{aligned} \frac{d}{dt} F_2^{\alpha(t)} r_0 &= \frac{d}{ds} F_2^{\alpha(t)} r_0 \frac{ds}{dt} \\ &= f_2(F_2^{\alpha(t)} r_0) \alpha \\ &= f_1(F_2^{\alpha(t)} r_0) \\ &= \frac{d}{dt} F_1^t r_0 \end{aligned}$$

□

To apply this result in a specific setting, suppose that  $f_0 \triangleq -D\varphi_+$ , whose trajectories approach  $q_0$  along the desired curve,  $c^{-1}[0]$ , develops unacceptably high velocities along the way. According to Lemma 7, the vector field,

$$f_1 \triangleq \frac{[D\varphi_+]^T}{\|D\varphi_+\|}$$

has a phase portrait which is the duplicate of the original gradient system — i.e solutions originating at the same initial condition,  $r_0 \in J$ , are restricted to lie on the same curves in  $J$  for all future time — yet it is clear that every trajectory of  $f_1$  has unit velocity for all future time.<sup>5</sup> Suppose, further, that we would like to travel at a nominal velocity,  $v_{nom}$ , but slow down as the curvature of our path,

$$\kappa(r) \triangleq \frac{\dot{r}^T J \ddot{r}}{\|\dot{r}\|} = \frac{f_0^T J [Df_0] f_0}{\sqrt{f_0^T [Df_0]^T [Df_0] f_0}}$$

<sup>4</sup>Note that while  $c^T c$  is not a Morse function — i.e. it has a connected set of critical points along the curve  $c^{-1}[0]$  — a relatively mild assumption, e.g.  $c^{-1}[0]$  has non-vanishing curvature at every point, would suffice to guarantee that  $\varphi_+$  or  $\varphi_{(x,+)}$  are Morse. There would remain the important question of whether there are additional critical points besides  $q_0$ . We ignore here these technical, but crucial, caveats.

<sup>5</sup>Of course, this is not a useful reference system since the vector field is not defined at the equilibrium states of the original gradient system.



increases along the way. One obvious encoding of this task specification takes the form

$$\dot{r} = f_2(r) \triangleq \frac{v_{nom}}{1 + \kappa(r)^2} f_0(r).$$

Unfortunately, since Theorem 3 is concerned only with limit sets, it can in no way assist us in translating this reference behavior into a control algorithm of torques applied to the mechanical system, (1). Indeed, the situation is even more unsatisfactory since it is clear that  $f_1, f_2$ , useful though they may seem, are no longer contained within the class of gradient vector fields on  $J$ : Theorem 3 cannot even apply to their limiting behavior!

### 3.3 Embedding More General Dynamics in Mechanical Systems

It has seemed clear for some time that more general reference dynamics may be embedded in mechanical systems. For example, given a continuously differentiable vector field,  $f: J \rightarrow \mathcal{P}$ , define the map  $F: \mathcal{P} \rightarrow \mathbb{R}^n$

$$F(p) \triangleq p_2 - f(p_1),$$

consider the control algorithm

$$\tau = -K_2 F - C f(p_1) + M D f p_2, \quad (20)$$

which, when applied to the mechanical system, (1), yields a closed loop of the form,  $\dot{p} = h(p)$ ,

$$h(p) \triangleq \begin{bmatrix} D f p_2 + M^{-1} [K_2 F + C F], \\ p_2 \end{bmatrix} \quad (21)$$

**Proposition 8** *The "embedded reference dynamics",  $\mathcal{R} \triangleq F^{-1}[0]$ , is a globally attracting positive invariant submanifold of  $\mathcal{P}$  under the flow of the closed loop vector field,  $h$ , defined in (21), provided the flow is complete.*

**Proof:** The proof proceeds by noting that  $D F$  is always surjective, so that  $\mathcal{R}$  is a submanifold, and that  $h(p) |_{\mathcal{R}} \in \text{Ker} [D_p f, I]$  so that it is a positive invariant submanifold, as well. Finally, it is not hard to show that

$$v \triangleq \frac{1}{2} F^T M F$$

is a LaSalle function for  $h$ , from which the result follows.

□

Unfortunately, this result is much too weak to be useful. To begin with there is the assumption of completeness — i.e. that no finite escape trajectories result from the application of the control law. More fundamentally, the mere guarantee that  $\mathcal{R}$  is a globally attracting set does not afford the conclusion that the behavior of trajectories starting away from  $\mathcal{R}$  end up behaving at all like those which do originate in  $\mathcal{R}$ . In particular, if  $f$  is known to induce a positive limit set  $\Omega \subset \mathcal{R}$ , it is by no means clear that  $\Omega$  is the positive limit set of the flow on the entire space.

It is in this context that the new result, Proposition 6, becomes very important. In general, having synthesized the reference vector field,  $f$ , with some limiting behavior in mind, it is reasonable to assume that there is a known Lyapunov function,  $\varphi$  for  $f$  on  $J$ . For example, even though  $f_1, f_2$ , introduced in the previous section are not gradient vector fields, it is easy to

see that the original potential function,  $\varphi_+$ , is still a Lyapunov function for both.

Given a reference vector field,  $f$ , with Lyapunov function,  $\varphi$ , consider the control algorithm,

$$\tau = -K_2 F - D \varphi^T + M D f p_2 - C f, \quad (22)$$

which, when applied to the mechanical system, (1), yields a closed loop of the form,  $\dot{p} = h(p)$ ,

$$h(p) \triangleq \begin{bmatrix} D f - M^{-1} [K_2 F + C F + D \varphi^T], \\ p_2 \end{bmatrix} \quad (23)$$

**Theorem 4** *If  $\varphi$  is a strict Lyapunov function for  $f$  on  $J$ , then*

$$v \triangleq \varphi + \frac{1}{2} F^T M F$$

*is a strict Lyapunov function for  $h$  on  $\mathcal{P}$ .*

**Proof:** The derivative of  $v$  along the motion of  $h$  is given by

$$\begin{aligned} \dot{v} &= D \varphi p_2 + \frac{1}{2} F^T M \dot{F} - F^T (C + K_2) F - F^T D \varphi^T \\ &= -F^T K_2 F + D \varphi f. \end{aligned}$$

The first term is negative except on  $F^{-1}[0]$ , where it vanishes. According to the hypothesis, the second term is negative except on the subspace,  $p_1 = 0$ , where it is zero as well. The intersection of these two zero sets is the origin of  $\mathcal{P}$ .

□

Although there are many interesting consequences which obtain from weakening  $\varphi$ , e.g. assuming it is merely a LaSalle function for  $f$ , the primary focus of this discussion is the ultimate replacement of the control law (22) with its adaptive version, below.

### 3.4 Adaptive Embedding of General Transient Behavior

Suppose, once again, that the mechanical system is defined by parameters,  $\pi$ , which enter linearly in  $M$  (and, therefore,  $C$ ). Given a reference vector field,  $f$ , with Lyapunov function,  $\varphi$ , consider the control algorithm,

$$\begin{aligned} \tau &= -K_2 F - D \varphi^T + \hat{M} D f p_2 - \hat{C} f \\ &= -K_2 F - D \varphi^T + M D f p_2 - C f + H \phi \end{aligned} \quad (24)$$

where, once again,  $\phi \triangleq \pi - \hat{\pi}$  is the vector of unknown parameter errors, and  $H$  is a matrix of known nonlinear functions in the state variables. Set the adaptive law as

$$\dot{\phi} = -H^T F.$$

When applied to the mechanical system, (1), this yields a closed loop of the form,

$$\begin{aligned} \dot{p} &= h(p) + H \phi \\ \dot{\phi} &= -H^T F. \end{aligned} \quad (25)$$

Since  $v \triangleq \varphi + \frac{1}{2} F^T M F$ , is a strict Lyapunov function for  $h$ , according to Theorem 4, it is clear that  $\tilde{v} \triangleq v + \frac{1}{2} \phi^T \phi$  is a positive definite function with negative semi-definite derivative along



the motion of system (25), hence, both state and position errors remain bounded for all time. Since the overall error system, (25), is time invariant, LaSalle's Invariance Theorem is found to apply, and we immediately conclude that the states tend to zero,  $p \rightarrow 0$ .

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