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Exact robot navigation using cost functions: the case of distinct spherical boundaries in E^n

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Abstract
The utility of artificial potential functions is explored as a means of translating automatically a robot task description into a feedback control law to drive the robot actuators. A class of functions is sought which will guide a point robot amid any finite number of spherically bounded obstacles in Euclidean n-space toward an arbitrary destination point. By introducing a set of additional constraints, the subclass of navigation functions is defined. This class is dynamically sound in the sense that the actual mechanical system will inherit the essential aspects of the qualitative behavior of the gradient lines of the cost function. An existence proof is given by constructing a one parameter family of such functions; the parameter is used to guarantee the absence of local minima.

Exact Robot Navigation using Cost Functions: The Case of Distinct Spherical Boundaries in E^n

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Abstrarct
The utility of artificial potential functions is explored, as a mean of translating automatically a robot task description into a feedback control law to drive the robot actuators. We seek a class of cost functions which will guide a point robot amidst any finite number of spherically bounded obstacles in Euclidean n-space toward an arbitrary destination point. By introducing a set of additional constraints, the subclass of *navigation functions* is defined. This class is "dynamically sound" in the sense that the actual mechanical system will inherit the essential aspects of the qualitative behavior of the gradient lines of the cost function. An existence proof is given, by the construction of a one-parameter family of such functions; the parameter is used to guarantee the absence of local minima.

1 Introduction

This paper, which is essentially a continuation and extension of the questions explored in [3], addresses the following problem. We are given perfect information concerning the location of a point-robot in Euclidean n-space, E^n; a workspace bounded by an $(n-1)$-sphere of finite radius; a finite number of obstacles bounded by smaller, non-intersecting $(n-1)$-spheres inside this workspace; and a destination point, q_d, in the resulting free space. We seek a real valued cost-function such that regardless of the starting point, its gradient vector field, if integrated, would guide the robot to q_d, while avoiding the obstacles. The gradient vector field must satisfy an additional set of requirements which guarantee that, with no further computation, it may be directly applied as a feedback control law to the robot's actuators. It can be shown that the resulting trajectory of the closed loop mechanical system will also arrive at q_d while avoiding the obstacles [2,3,5].

The idea of using "potential functions" for the specification of robot tasks was pioneered by Khatib [6] in the context of obstacle avoidance, and further advanced by fundamental work of Hogan [11] in the context of force control. The methodology was developed independently by Arimoto in Japan [13], and by Soviet investigators as well [14]. This paper introduces the notion of a *navigation function* (to be defined precisely in Section 2.1), which encompasses the more stringent requirements described above. A lengthier discussion of the motivation for this investigation is provided in the earlier paper [3], which explores the possibility of à priori topological obstructions to the project. Here, we present a construction and proof that it correctly solves the problem stated above.

Of course, the much more general problem of constructing a path between two points in a space obstructed by sets with arbitrary polynomial boundary (given perfect information) has already been completely solved [7]. Moreover, a near optimally efficient solution has recently been offered for this class of problems as well [15]. The motivation for the present direction of inquiry (beyond its academic interest) is the perspective that robot navigation, rather than being viewed as a challenging problem in constructive topology, might be fruitfully approached as a problem in the construction of control algorithms for a well characterized class of dynamical systems in the presence of well characterized constraints. We regard the present work as an interesting beginning in the approach to unified problems of robot task description and control in a geometrically characterized environment.

The paper is organized as follows. Technical definitions and a precise statement of the problem are followed by the presentation of a candidate solution in Section 2 . The proof of correctness is presented in Section 3 . Explicit computations required to implement the algorithm are presented in the Appendix of [1].

2 The Construction of a Navigation - Function

2.1 Definitions and Notation

If A is a matrix, then let $(A)_s$ denote its symmetric part — $\frac{1}{2}(A+A^T)$. Denote the boundary of a set, $\mathcal{X} \subseteq E^n$, by $\partial\mathcal{X}$. Suppose that $\mathcal{X},\mathcal{Y} \subseteq E^n$, and let $h : \mathcal{X} \to \mathcal{Y}$ be a smooth map. Denote its Jacobian by Dh. A point in the domain, $x \in \mathcal{X}$, is a *critical point of* h if $Dh(x)$ fails to be surjective. In the special case $\varphi : \mathcal{X} \to \mathbb{R}$, denote the transposed Jacobian (i.e. the gradient) by $\nabla\varphi \triangleq (D\varphi)^T$ (a column vector), and the Hessian by $\nabla^2\varphi \triangleq D(D\varphi)^T$. The set of critical points of φ will be denoted by $\mathcal{C}_\varphi$.

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In this section, we will define what is meant by workspace and obstacle, and introduce the notion of a navigation function.

Definition 1 *The robot* workspace *is the closed n-ball of radius ρ_0 centered at the origin of E^n,*

$$\mathcal{W} \triangleq \{q \in E^n : \|q\|^2 \leq \rho_0^2\}.$$

An obstacle *is an open ball of radius ρ_j centered at $q_j \in \mathcal{W}$,*

$$\mathcal{O}_j \triangleq \{q \in E^n : \|q - q_j\|^2 < \rho_j^2\} \;\; j = 1 \ldots M.$$

The free space *is*

$$\mathcal{F} \triangleq \mathcal{W} - \bigcup_{j=1}^{M} \mathcal{O}_j.$$

If φ is a real valued map on $\mathcal{X}$ then a critical point, $q_c \in \mathcal{C}_\varphi$, is said to be *non-degenerate* if it is not a critical point of $\nabla\varphi$ — i.e. if the Hessian of φ at q_c has a zero kernel. Such a map is said to be a *Morse function on $\mathcal{X}$* if all its critical points in that set are *non-degenerate.* The *Morse index* of a critical point, denoted by $\mathrm{index}(\varphi)|_{q_c}$,is the number of negative eigenvalues of $(\nabla^2\varphi)(q_c)$ [12]. It is said to be *polar on $\mathcal{X}$ at q_d* if it has exactly one minimum, at q_d [10]. Finally, φ is *admissible on $\mathcal{X}$* if $\varphi(\partial\mathcal{X}) = 1$, and at any other point in the interior of $\mathcal{X}$ $0 \leq \varphi < 1$ [12].

Definition 2 *A map $\varphi : E^n \rightarrow [0,1]$, is a* navigation function *on a compact connected smooth manifold $\mathcal{F} \subseteq \mathcal{W} \subseteq E^n$ if it is:*
1. Analytic on some open set containing $\mathcal{F}$.
2. Polar at q_d, where $q_d \in \mathcal{F}$.
3. Morse on $\mathcal{F}$.
4. Admissible on $\mathcal{F}$.

The intuitive motivation for this definition is most simply provided by reference to the following fact which obtains from elementary properties of gradient vector fields, for example, as discussed in [9].

Proposition 2.1 (**[4]**) *Let $\mathcal{F}$ be a smooth manifold embedded in E^n, and let $\varphi : \mathcal{F} \rightarrow \mathbb{R}$ be smooth, with* isolated *critical points, whose gradient is directed* away from *$\mathcal{F}$ at any boundary point. Then $\mathcal{F}$ is* positive invariant *with respect to the flow induced by the gradient system,*

$$\dot{q} = -(\nabla\varphi)(q),$$

and, apart from a set of measure zero, all solutions originating in the interior of $\mathcal{F}$ approach a local minimum of φ.

First consider the implications of property 2 in Definition 2 . If $\mathcal{F}$ is disconnected it is clearly impossible to construct a continuous function which is polar. Supposing, however, that the freespace is connected, φ has a unique minimum at q_d, "respects the boundary" of $\mathcal{F}$, and that $\mathcal{C}_\varphi$ is a totally isolated set, it follows from the proposition that all initial conditions away from a set of measure zero are successfully brought to q_d without hitting any obstacles or running into the workspace boundary[1]. It has been argued in [3] that one cannot do better than this with smooth vector fields: topological obstructions prohibit the existence of vector fields which take *every* point in $\mathcal{F}$ to q_d.

Property 3 in Definition 2 is sufficient to guarantee that $\mathcal{C}_\varphi$ is, indeed, totally isolated as required by the Proposition. It is, in fact, a much more restrictive requirement which is imposed to permit a straightforward proof that the desirable limiting behavior of the gradient flow is "inherited" by the ultimate closed loop mechanical system formed by using $\nabla\varphi$ directly as a feedback control law for the robot's actuators. Similarly, property 4 in Definition 2, while sufficient to guarantee that $\nabla\varphi$ is transverse to the boundary of $\mathcal{F}$ (as additionally required by the proposition), is a much stronger condition imposed to insure that the transients of the resulting closed loop mechanical system "inherit" the desirable properties of the gradient flow which prevent collisions with the boundary. Obviously, a careful discussion of the control theoretic aspects of this work is beyond the scope of the present paper, and the reader is referred to [2,3,5] for details.

Finally, it might be said that property 1 in Definition 2 reflects the authors' "ideological" perspective that closed form mathematical expressions are a preferable encoding of actuator commands than algorithms which include logical decisions. Functions which are merely smooth ($C^{(\infty)}$) may be defined by "patching together" different closed form expressions on different portions of the space leading to the kind of branching and looping in the ultimate control algorithm that we would like to avoid as much as possible. Analytic navigation functions will be harder to construct, but once defined, yield a control algorithm directly by "parsing" the symbolic expression into its gradient[2]. Unquestionably, real world scenes will often not admit even a smooth, much less an analytic representation, and it may well turn out (the theoretical recourse to ever more accurate analytic approximations notwithstanding) that any serious attempt to extend this work beyond the class of ball obstacles requires a relaxation of property 1. Until such a time, we prefer to remain within the category of analytic maps on analytic manifolds.

We assume perfect information, i.e. (q_j, ρ_j) $j = 0 \ldots M$ are given. It is also assumed that the given workspace arrangement has the following properties.

Definition 3 *A* valid workspace *is one in which:*
1. q_d is in the interior of $\mathcal{F}$;
2. all the obstacles are contained in the interior of $\mathcal{W}$;
3. the obstacles do not intersect,

where the term obstacle stands for the closure of the obstacle.

[1] In particular, according to this Proposition, it is impossible for any zero measure submanifold of points not attracted to q_d to disconnect $\mathcal{F}$ and "block" the flow toward q_d. For, this would imply that some maximum or saddle has an attracting domain which includes an open set — contradicting the fact that a non-degenerate unstable equilibrium state has a stable manifold of dimension less than n.

[2] We presume, as well, that such considerations will play an important role with respect to verifiability of implementations in complicated environments.

2.2 Deformations

A scalar valued function can be composed "from the right" by a map into its domain or "from the left" by a scalar function. In the present section we explore the utility of each "side" in advancing this program of research, when the composing function serves as a "deformation". A bijective map, $h : \mathcal{X} \to \mathcal{Y}$, between two topological spaces is said to be a *homeomorphism* if both h and h^{-1} are continuous. It is said to be a *diffeomorphism of class* $C^{(r)}$ if, in addition, both h and h^{-1} are continuously differentiable r-times. Given our stated intention to remain within the category of analytic maps on analytic manifolds, we will refer to any diffeomorphism of class $C^{(\omega)}$ (or "analytic diffeomorphism") as a *deformation* .

We first show that deformations from the right lead to a presently impracticable but theoretically unequivocal procedure for extending the results of this paper over a wide domain of workspaces and obstacles. In this context, we regard the particular free space of Definition 1 as a simplified "model", $\mathcal{F}_M$, of much larger family of spaces which are "deformable" into it. The following statement, suggested by Prof. M. Hirsch in the course of a personal conversation, constitutes a formal guarantee of the existence of navigation functions over every space in the deformation class of a given model.

Proposition 2.2 *Let* $\varphi : \mathcal{F}_M \to [0,1]$ *be a navigation function on* $\mathcal{F}_M$*, and* $h : \mathcal{F} \to \mathcal{F}_M$ *be an analytic diffeomorphism. Then*

$$\tilde{\varphi} \triangleq \varphi \circ h,$$

is a navigation function on $\mathcal{F}$.

A proof can be found in [1]. The second deformation scheme will be used explicitly in this paper. It will serve to deform a given cost function on $\mathcal{F}$, to a navigation function. Specifically, here it is used to make a cost function $\hat{\varphi}$ admissible on $\mathcal{F}$, and to change q_d to a non-degenerate critical point.

Proposition 2.3 *Let* $I_1, I_2 \subseteq \mathbb{R}$ *be intervals,* $\hat{\varphi} : \mathcal{F} \to I_1$ *and* $\sigma : I_1 \to I_2$ *be analytic. Define the composition* $\varphi : \mathcal{F} \to I_2$*, to be*

$$\varphi \triangleq \sigma \circ \hat{\varphi}.$$

If σ *is monotonically increasing on* I_1*, then the set of critical points of* $\hat{\varphi}$ *and* φ *coincide,*

$$\mathcal{C}_{\varphi} = \mathcal{C}_{\hat{\varphi}},$$

and the index of each point is identical,

$$index(\varphi)|_{\mathcal{C}_{\varphi}} = index(\hat{\varphi})\,|_{\mathcal{C}_{\hat{\varphi}}}.$$

In other words, the composition with σ neither changes the set of critical points, nor their type (minimum, maximum or a saddle) or degeneracy. The proof is given in [1].

2.3 A Navigation Function for a Ball with M Balls Removed

The proposed navigation function, $\varphi : \mathcal{F} \to [0,1]$, is a composition of three functions:

$$\varphi \triangleq \sigma_d \circ \sigma \circ \hat{\varphi}.$$

The function $\hat{\varphi}$ is polar, almost everywhere Morse, and analytic: it attains a uniform height on $\partial\mathcal{F}$ by blowing up there. It is "squashed" by the deformation, σ, of $[0,\infty]$ into $[0,1]$, where

$$\sigma(x) \triangleq \frac{x}{1+x},$$

resulting in a polar, admissible, and analytic function which is non-degenerate on $\mathcal{F}$ except at one point — the destination. This last flaw is repaired by σ_d.

We distinguish between "good" and "bad" subsets of $\mathcal{F}$. When a point belongs to the "good" set, we expect the gradient lines to lead to it (here it is just $\{q_d\}$). The "bad" subset includes all the boundary points of the free space, and we expect the cost at such a point to be high.
Let γ and β denote analytic real valued maps whose zero-levels, i.e. $\gamma^{-1}(0), \beta^{-1}(0)$, are respectively, the "good" and "bad" sets. We define $\hat{\varphi}$ to be,

$$\hat{\varphi} \triangleq \frac{\gamma}{\beta},$$

where $\gamma : \mathcal{F} \to [0,\infty)$ is

$$\gamma \triangleq {\gamma_d}^k \quad k \in \mathbb{N}; \qquad \gamma_d \triangleq \|q - q_d\|^2,$$

and $\beta : \mathcal{F} \to [0,\infty)$ is,

$$\beta \triangleq \textstyle\prod_{j=0}^{M} \beta_j,$$

where

$$\beta_0 \triangleq \rho_0^2 - \|q\|^2 \quad ; \quad \beta_j \triangleq \|q - q_j\|^2 - \rho_j^2 \;\; j = 1 \ldots M.$$

In the sequel we will denote the "omitted product" by the symbol

$$\bar{\beta}_i \triangleq \prod_{j=0, j\neq i}^{M} \beta_j.$$

Due to the parameter k in $\hat{\varphi}$, the destination point is a *degenerate* critical point. To counteract this effect, the "distortion" $\sigma_d : [0,1] \to [0,1]$,

$$\sigma_d(x) \triangleq (x)^{\frac{1}{k}} \quad k \in \mathbb{N},$$

is introduced, to change q_d to a non-degenerate critical point.

To make the notion of a valid workspace more precise, we give here an algebraic interpretation for Definition 3 . A *valid workspace* is one in which:

1. q_d is in the interior of $\mathcal{F}$,

$$\beta_i(q_d) > 0 \quad i \in \{0, \ldots M\}.$$

2. All the obstacles are contained in the interior of $\mathcal{W}$,

$$\beta_0(q_i) > 0 \text{ and } \|q_i\| + \rho_i < \rho_0 \quad i \in \{1, \ldots M\}.$$

3. The obstacles do not intersect,

$$\|q_i - q_j\| > \rho_i + \rho_j \quad i, j \in \{1, \ldots M\}.$$

The following theorem is the main contribution of this paper.

Theorem 1 *If the workspace is valid (in the sense of Definition 8), then there exists a positive integer $N \in \mathbb{N}$ such that for every $k \geq N$*

$$\varphi = \sigma_d \circ \sigma \circ \hat{\varphi} = \left(\frac{\gamma_d{}^k}{\gamma_d{}^k + \beta} \right)^{\frac{1}{k}} \tag{1}$$

is a navigation function *on $\mathcal{F}$.*

Remark: While $\hat{\varphi}$ is analytic only on the interior of $\mathcal{F}$, σ is analytic on $[0, \infty)$, and σ_d is analytic only on $(0, \infty)$, their composition, φ, is analytic on the entirety of of some open neighborhood containing $\mathcal{F}$.

3 Proof of Correctness

Let $\epsilon > 0$, define $\mathcal{B}_i(\epsilon) \stackrel{\Delta}{=} \{q \in E^n : 0 < \beta_i < \epsilon\}$ (an open annulus — i.e. an n-ball "without a core"). In the proof that follows, the free space is partitioned into five subsets:

1. the destination point,
$$\{q_d\};$$
2. the free space boundary,
$$\partial \mathcal{F} \stackrel{\Delta}{=} \beta^{-1}(0);$$
3. the set "near the obstacles" ,
$$\mathcal{F}_0(\epsilon) \stackrel{\Delta}{=} \bigcup_{i=1}^{M} \mathcal{B}_i(\epsilon) - \{q_d\};$$
4. the set "near the workspace boundary",
$$\mathcal{F}_1(\epsilon) \stackrel{\Delta}{=} \mathcal{B}_0(\epsilon) - (\{q_d\} \bigcup \mathcal{F}_0(\epsilon));$$
5. the set "away from the obstacles" ,
$$\mathcal{F}_2(\epsilon) \stackrel{\Delta}{=} \mathcal{F} - (\{q_d\} \bigcup \partial \mathcal{F} \bigcup \mathcal{F}_0(\epsilon) \bigcup \mathcal{F}_1(\epsilon)).$$

We assume to begin with that ϵ is sufficiently small to guarantee

$$\mathcal{F}_0(\epsilon) \subseteq \mathcal{F}.$$

This assumption is interpreted algebraically as

$$\epsilon < (\|q_i - q_j\| - \rho_j)^2 - \rho_i^2 \quad i, j \in \{1, \ldots M\}, i \neq j, \tag{2}$$

and

$$\epsilon < (\rho_0 - \|q_i\|)^2 - \rho_i^2 \quad i \in \{1, \ldots M\}. \tag{3}$$

Note that in practicality, ϵ is expected to be small enough so that the exclusion of $\{q_d\}$ from $\mathcal{F}_0(\epsilon)$ and $\mathcal{F}_1(\epsilon)$ is reduntent.

We will begin by showing that q_d is a non-degenerate local minimum and that φ has no critical points on $\partial \mathcal{F}$, using the navigation function itself. Then, since Proposition 2.3 applies to $\mathcal{F} - \partial \mathcal{F} - \{q_d\}$, it will suffice to assert the theorem in consideration of $\mathcal{F}_0(\epsilon)$, $\mathcal{F}_1(\epsilon)$, and $\mathcal{F}_2(\epsilon)$, using $\hat{\varphi}$, which is much simpler to deal with.

The following technical lemma gives formulas for the Gradient and Hessian of a rational function at a critical point, to which we will continually refer in the sequel.

Lemma 3.1 *Let $\nu, \delta \in C^{(2)}[E^n, \mathbb{R}]$, and define,*

$$\rho \stackrel{\Delta}{=} \frac{\nu}{\delta}.$$

Then

$$\nabla^2 \rho|_{\mathcal{C}_\rho} = \frac{1}{\delta^2}[\delta \nabla^2 \nu - \nu \nabla^2 \delta]. \tag{4}$$

Proof: Since

$$\nabla \rho = \frac{1}{\delta^2}(\delta \nabla \nu - \nu \nabla \delta), \tag{5}$$

we have,

$$\nabla^2 \rho = \frac{1}{\delta^2} \left[\delta \nabla^2 \nu + \nabla \nu \nabla \delta^T - \nabla \delta \nabla \nu^T - \nu \nabla^2 \delta \right] + \delta^2 \nabla \rho (\nabla \tfrac{1}{\delta^2})^T.$$

But at a critical point $\nabla \rho = 0$ and $\nabla \nu = \rho \nabla \delta$, hence

$$\nabla^2 \rho|_{\mathcal{C}_\rho} = \frac{1}{\delta^2}[\, \delta \nabla^2 \nu - \nu \nabla^2 \delta]. \qquad \square$$

3.1 The Destination and the Boundary of $\mathcal{F}$

Proposition 3.2 *For a valid workspace, the destination point, q_d, is a non-degenerate local minimum of φ.*

This can be easily checked by evaluating $\nabla \varphi$ and $\nabla^2 \varphi$ at q_d.

Proposition 3.3 *For a valid workspace, all the critical points of $\varphi|_{\mathcal{F}}$ are in the interior of the free space.*

Proof: Let $q_0 \in \partial \mathcal{F}$, and suppose that $\beta_i(q_0) = 0$ for some $i \in \{0, \ldots M\}$. Applying equation 5 on φ,

$$\begin{aligned}(\nabla \varphi)(q_0) &= \frac{1}{(\gamma_d{}^k + \beta)^{\frac{2}{k}}} \left((\gamma_d{}^k + \beta)^{\frac{1}{k}} \nabla \gamma_d - \gamma_d \nabla (\gamma_d{}^k + \beta)^{\frac{1}{k}} \right)\Big|_{q_0} \\ &= -\tfrac{1}{k} \gamma_d{}^{-k} (\textstyle\prod_{j=0, j \neq i}^{M} \beta_j) \nabla \beta_i \neq 0. \end{aligned} \qquad \square$$

3.2 The Absence of Minima in the Interior of $\mathcal{F}$

From now on, we will assert the theorem using $\hat{\varphi}$. The trick is to use k in $\hat{\varphi}$ as a tuning parameter. Intuitively, $\nabla \hat{\varphi}$ (see equation 5) consists of the terms $\nabla \gamma$ and $\nabla \beta$. By increasing k, the first term dominates, forcing $-(\nabla \hat{\varphi})(q)$ to be directed toward q_d and have a larger magnitude. The overall effect will be to shift the critical points of $\hat{\varphi}$ toward the obstacle boundaries. But we may as well expect that when k is high enough, each critical point is *not* a local minimum, since the overall behavior of $\hat{\varphi}$ tends to that of γ. In such a case any test direction which is parallel to the near obstacle boundary should prove that this critical point is not a local minimum.

The proof that follows has two steps: first we show that all the critical points can be shifted arbitrarily close to the boundary of the free space. Then we find a test direction along which $\hat{\varphi}$ does not have a local minimum. As a result, q_d is the unique minimum of $\hat{\varphi}$.

The following Proposition shows that $\mathcal{F}_2(\epsilon)$, the set "away from the obstacles", can be "cleaned" of critical points.

Proposition 3.4 *For every $\epsilon > 0$ there exists a positive integer $N(\epsilon) \in I\!N$ such that if $k \geq N(\epsilon)$ then there are no critical points of $\hat{\varphi}$ in $\mathcal{F}_2(\epsilon)$.*

Proof: At a critical point, $q_c \in \mathcal{C}_{\hat{\varphi}} \bigcap \mathcal{F}_2(\epsilon)$, according to equation 5 we have,

$$k\beta\nabla\gamma_d = \gamma_d\nabla\beta.$$

Taking the magnitude of both sides yields

$$2k\beta = \sqrt{\gamma_d}\|\nabla\beta\|.$$

Since $\|\nabla\gamma_d\| = 2\sqrt{\gamma_d}$. A sufficient condition for the above equality not to hold is given by

$$\frac{1}{2}\frac{\sqrt{\gamma_d}\|\nabla\beta\|}{\beta} < k \quad \text{for all } q \in \mathcal{F}_2(\epsilon).$$

An upper bound on the left side is given by

$$\frac{1}{2}\frac{\sqrt{\gamma_d}\|\nabla\beta\|}{\beta} \leq \frac{1}{2}\sqrt{\gamma_d}\sum_{i=0}^{M}\frac{\bar{\beta}_i}{\beta}\|\nabla\beta_i\|$$

$$< \frac{1}{2}\frac{1}{\epsilon}\max_{\mathcal{W}}\{\sqrt{\gamma_d}\}\sum_{i=0}^{M}\max_{\mathcal{W}}\{\|\nabla\beta_i\|\} \triangleq N(\epsilon). \tag{6}$$

Since $\beta_j \geq \epsilon$, $j \in \{0, \ldots M\}$. □

In the proof of Proposition 3.6 , it will prove important to have an upper bound for

$$\nu_i \triangleq \frac{1}{4}\nabla\beta_i \cdot \nabla\gamma_d - \gamma_d$$

over the closure of $\mathcal{B}_i(\epsilon)$, the set $\overline{\mathcal{B}_i(\epsilon)} \triangleq \{q : 0 \leq \beta_i(q) \leq \epsilon\}$. This is readily obtained using Lagrange multipliers.

Lemma 3.5

$$\max_{\overline{\mathcal{B}_i(\epsilon)}}\{\nu_i\} = \|q_d - q_i\|(\sqrt{\epsilon + \rho_i^2} - \|q_d - q_i\|).$$

In the proof, detailed in [1], it can be observed that $\max_{\overline{\mathcal{B}_i(\epsilon)}}\{\nu_i\}$ is negative for ϵ small enough, in consequence of the assumption of Definition 3 — namely that q_d is not inside the obstacle $\mathcal{O}_i$. The following proposition shows that for ϵ small enough, the set "near the obstacles", $\mathcal{F}_0(\epsilon)$, is free of local minima.

Proposition 3.6 *For any valid workspace, there exists an $\epsilon_0 > 0$ such that $\hat{\varphi}$ has* no local minimum *in $\mathcal{F}_0(\epsilon)$, as long as $\epsilon < \epsilon_0$.*

Proof: If $q_c \in \mathcal{F}_0(\epsilon) \bigcap \mathcal{C}_{\hat{\varphi}}$, then $q_c \in \mathcal{B}_i(\epsilon)$ for at least one $i \in \{1, \ldots M\}$ — i.e. q_c is very close to some obstacle boundary. We will use a unit vector orthogonal to $\nabla\beta_i$ at q_c as a test direction to demonstrate that $(\nabla^2\hat{\varphi})(q_c)$ has at least one negative eigenvalue. Using equation 4,

$$(\nabla^2\hat{\varphi})(q_c) = \frac{1}{\beta^2}(\beta\nabla^2\gamma_d{}^k - \gamma_d{}^k\nabla^2\beta)$$
$$= \frac{\gamma_d{}^{k-2}}{\beta^2}\left(k\beta\left[\gamma_d\nabla^2\gamma_d + (k-1)\nabla\gamma_d\nabla\gamma_d{}^T\right] - \gamma_d{}^2\nabla^2\beta\right). \tag{7}$$

At a critical point, $k\beta\nabla\gamma_d = \gamma_d\nabla\beta$, according to equation 5 . Hence, taking the outer-product of both sides,

$$(k\beta)^2\nabla\gamma_d\nabla\gamma_d{}^T = \gamma_d{}^2\nabla\beta\nabla\beta^T.$$

Substituting for $k(k-1)\beta\nabla\gamma_d\nabla\gamma_d{}^T$ ($q \neq q_d$) in equation 7 yields

$$(\nabla^2\hat{\varphi})(q_c) = \frac{\gamma_d{}^{k-1}}{\beta^2}\left(k\beta\nabla^2\gamma_d + (1-\frac{1}{k})\frac{\gamma_d}{\beta}\nabla\beta\nabla\beta^T - \gamma_d\nabla^2\beta\right). \tag{8}$$

Recalling that $\bar{\beta}_i = \prod_{j=0,j\neq i}^{M}\beta_j$, note that

$$\begin{aligned}(\nabla^2\hat{\varphi})(q_c) \;&= \tfrac{\gamma_d{}^{k-1}}{\beta^2}\big(k\beta\nabla^2\gamma_d + (1-\tfrac{1}{k})\tfrac{\gamma_d}{\beta}[\beta_i{}^2\nabla\bar{\beta}_i\nabla\bar{\beta}_i{}^T \\ &\quad + 2\beta_i\bar{\beta}_i(\nabla\bar{\beta}_i\nabla\beta_i{}^T)_s + \bar{\beta}_i{}^2\nabla\beta_i\nabla\beta_i{}^T] \\ &\quad - \gamma_d[\beta_i\nabla^2\bar{\beta}_i + 2(\nabla\bar{\beta}_i{}^T\nabla\beta_i)_s + \bar{\beta}_i\nabla^2\beta_i]\big).\end{aligned}$$

Evaluating the quadratic form associated with $(\nabla^2\hat{\varphi})(q_c)$ at $\hat{v} \triangleq \widehat{(\nabla\beta_i)(q_c)}^{\perp}$ yields

$$\begin{aligned}\tfrac{\beta^2}{\gamma_d{}^{k-1}}\hat{v}^T(\nabla^2\hat{\varphi})(q_c)\hat{v} \;&= 2k\beta - 2\gamma_d\bar{\beta}_i \\ &\quad + \hat{v}^T\left[(1-\tfrac{1}{k})\tfrac{\gamma_d}{\beta}\beta_i{}^2\nabla\bar{\beta}_i\nabla\bar{\beta}_i{}^T - \gamma_d\beta_i\nabla^2\bar{\beta}_i\right]\hat{v},\end{aligned} \tag{9}$$

since $\nabla^2\gamma_d = \nabla^2\beta_i = 2I$. Now take the inner-product of both side of equation 5 with $\nabla\gamma_d$ to obtain

$$\begin{aligned}4k\beta \;&= \nabla\beta \cdot \nabla\gamma_d \\ &= \bar{\beta}_i\nabla\beta_i \cdot \nabla\gamma_d + \beta_i\nabla\bar{\beta}_i \cdot \nabla\gamma_d.\end{aligned}$$

Substituting this for $2k\beta$ in equation 9 and grouping the terms which are proportional to β_i, we have

$$\begin{aligned}\tfrac{\beta^2}{\gamma_d{}^{k-1}}\hat{v}^T(\nabla^2\hat{\varphi})(q_c)\hat{v} &= 2\bar{\beta}_i\overbrace{\left(\frac{1}{4}\nabla\beta_i \cdot \nabla\gamma_d - \gamma_d\right)}^{\nu_i(q)} \\ + \beta_i\Big(\tfrac{1}{4}\nabla\bar{\beta}_i \cdot \nabla\gamma_d &+ \gamma_d\hat{v}^T\left[(1-\tfrac{1}{k})\frac{1}{\bar{\beta}_i}\nabla\bar{\beta}_i\nabla\bar{\beta}_i{}^T - \nabla^2\bar{\beta}_i\right]\hat{v}\Big).\end{aligned} \tag{10}$$

The second term is proportional to β_i, and can be made arbitrarily small by a choice of ϵ, but it can still be positive, so the first term should be strictly negative. According to Lemma 3.5 , this is guaranteed by the condition

$$\epsilon < \|q_d - q_i\|^2 - \rho_i^2 \triangleq \epsilon_{0i}' \quad i \in \{1, \ldots M\}. \tag{11}$$

In order to assure the inequality $\widehat{\nabla\beta_i}^{\perp T}(\nabla^2\hat{\varphi})(q_c))\widehat{\nabla\beta_i}^{\perp} < 0$, it now follows from equation 10 that ϵ must be further constrained to satisfy,

$$\epsilon < \frac{2(-\nu_i)\bar{\beta}_i{}^2}{\frac{1}{4}\bar{\beta}_i\nabla\bar{\beta}_i \cdot \nabla\gamma_d + \gamma_d\widehat{\nabla\beta_i}^{\perp T}\left[(1-\frac{1}{k})\nabla\bar{\beta}_i\nabla\bar{\beta}_i{}^T - \bar{\beta}_i\nabla^2\bar{\beta}_i\right]\widehat{\nabla\beta_i}^{\perp}},$$

for which it will suffice that

$$\epsilon < \frac{\min_{\overline{\mathcal{B}_i(\epsilon)}}\{2|\nu(q)|\bar{\beta}_i{}^2\}}{\max_{\overline{\mathcal{B}_i(\epsilon)}}\{\frac{\bar{\beta}_i}{4}\nabla\bar{\beta}_i \cdot \nabla\gamma_d + \gamma_d\hat{v}^T\left[(1-\frac{1}{k})\nabla\bar{\beta}_i\nabla\bar{\beta}_i{}^T - \bar{\beta}_i\nabla^2\bar{\beta}_i\right]\hat{v}\}}.$$

Consider the right hand side of the above inequality to be a scalar valued function $\varsigma(\epsilon)$. If $\epsilon < \epsilon'$ then $\mathcal{B}_i(\epsilon) \subseteq \mathcal{B}_i(\epsilon')$, and it follows that $\varsigma(\epsilon) \geq \varsigma(\epsilon')$. Hence it will also suffice that

$$\epsilon < \frac{\min_{\overline{\mathcal{B}_i(\epsilon_{0i}')}}\{2|\nu(q)|\bar{\beta}_i{}^2\}}{\max_{\overline{\mathcal{B}_i(\epsilon_{0i}')}}\{\frac{1}{4}\bar{\beta}_i\nabla\bar{\beta}_i \cdot \nabla\gamma_d + \gamma_d\hat{v}^T\left[(1-\frac{1}{k})\nabla\bar{\beta}_i\nabla\bar{\beta}_i{}^T - \bar{\beta}_i\nabla^2\bar{\beta}_i\right]\hat{v}\}} \triangleq \epsilon_{0i}''.$$

By making $\epsilon_0 = \min\{\epsilon'_{0i}, \epsilon''_{0i}\}$ $i \in \{1, \ldots M\}$, the proof is completed. □

We now consider the set $\mathcal{F}_1(\epsilon)$. By adjusting ϵ, a point in this set can be made so close to the workspace boundary that $\nabla\beta_0$ dominates any obstacle gradient. We will show that such a point cannot be a critical point of $\hat{\varphi}$, provided that it is far enough from any obstacle.

Proposition 3.7 *If $k \geq N(\epsilon)$, then there exists an $\epsilon_1 > 0$ such that $\hat{\varphi}$ has no critical points on $\mathcal{F}_1(\epsilon)$, as long as $\epsilon < \epsilon_1$.*

The proof can be found in [1].

3.3 Non–Degeneracy of Critical Points in the Interior of $\mathcal{F}$

The proof that $\hat{\varphi}$ is polar was completed in the previous section. We now show that it is also Morse. The following Lemma, which will be used in Proposition 3.9 , asserts that the non–singularity of a linear operator follows from the fact that its associated quadratic form is sign definite on complementary subspaces of E^n.

Lemma 3.8 *Let $E^n = \mathcal{P} \oplus \mathcal{N}$, and let the symmetric matrix $Q \in \mathbb{R}^{n \times n}$ define a quadratic form on E^n*

$$\xi(v) \stackrel{\Delta}{=} v^T Q v.$$

If $\xi|_{\mathcal{P}}$ is positive definite and $\xi|_{\mathcal{N}}$ is negative definite, then Q is non–singular and

$$index(Q) = dim(\mathcal{N}).$$

The proof is detailed in [1]. Let $\xi_q(v)$ denote the quadratic form associated with the Hessian of $\hat{\varphi}$, $(\nabla^2\hat{\varphi})(q)$, on the tangent space to the set "near the obstacles" at $q \in \mathcal{F}_0(\epsilon)$, denoted as $T_q\mathcal{F}_0(\epsilon)$.

Proposition 3.9 *There exists an $\epsilon_2 > 0$ such that for every $\epsilon < \epsilon_2$ at each critical point of $\hat{\varphi}$ in $\mathcal{F}_0(\epsilon)$, $q_c \in \mathcal{C}_{\hat{\varphi}} \bigcap \mathcal{F}_0(\epsilon)$, there is a direct sum decomposition $T_{q_c}\mathcal{F}_0(\epsilon) = \mathcal{P}_{q_c} \oplus \mathcal{N}_{q_c}$, where $dim(\mathcal{P}_{q_c}) = 1$, for which $\xi_{q_c}|_{\mathcal{P}_{q_c}}$ is positive definite and $\xi_{q_c}|_{\mathcal{N}_{q_c}}$ is negative definite.*

According to Lemma 3.8 , this implies that all the critical points of $\hat{\varphi}$ are non–degenerate. The proof of this Proposition can be found in [1].

Finally, if we will choose $N(\epsilon) = N(\epsilon_{min})$ in equation 6 , where

$$\epsilon_{min} \stackrel{\Delta}{=} \frac{1}{2}\min\{\epsilon_0, \epsilon_1, \epsilon_2\},$$

the proof of theorem 1 is completed.

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