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Aspects of Galileons

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Abstract

Galileons are a class of scalar field theories which have been found to arise in a disparate variety of contexts and exhibit a host of interesting properties by themselves, both classical and quantum. They obey non-trivial shift symmetries which restrict their self-interactions to be of higher derivative form, yet their equations of motion remain second order so that they are free of ghost instabilities. Further, when used as a force mediator between massive objects, galileons provide a natural realization of the Vainshtein screening mechanism which shuts off the fifth force at distances close to massive sources. As such, they are well suited for cosmology and are naturally incorporated into theories of modified gravity such as the Dvali-Gabadadze-Porrati braneworld model and the de Rham-Gabadadze-Tolley theory of massive gravity. Treated as a quantum field theory, galileons obey a non-trivial non-renormalization theorem which proves that they are not renormalized to any numbers of loops. In this thesis, we explore the properties of galileon theories and their generalizations through a combination of geometric and algebraic means. On the geometry side, we demonstrate that generic galileon theories are naturally thought of as the description of branes moving in higher dimensional spacetimes. On the algebraic side, we show that there exists a precise interpretation in which galileons can be thought of as Goldstone modes which arise when spacetime symmetries are spontaneously broken. In particular, when viewed in this light the galileons are the analogue of the Wess-Zumino-Witten term of the chiral lagrangian and thus represent interactions which are technically special. These methods provide both new technical tools for analyzing galileon-like theories and offer conceptual changes for how these theories can be viewed.

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ASPECTS OF GALILEONS

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ABSTRACT

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Mark Trodden

Galileons are a class of scalar field theories which have been found to arise in a disparate variety of contexts and exhibit a host of interesting properties by themselves, both classical and quantum. They obey non-trivial shift symmetries which restrict their self-interactions to be of higher derivative form, yet their equations of motion remain second order so that they are free of ghost instabilities. Further, when used as a force mediator between massive objects, galileons provide a natural realization of the Vainshtein screening mechanism which shuts off the fifth force at distances close to massive sources. As such, they are well suited for cosmology and are naturally incorporated into theories of modified gravity such as the Dvali-Gabadadze-Porrati braneworld model and the de Rham-Gabadadze-Tolley theory of massive gravity. Treated as a quantum field theory, galileons obey a non-trivial non-renormalization theorem which proves that they are not renormalized to any numbers of loops. In this thesis, we explore the properties of galileon theories and their generalizations through a combination of geometric and algebraic means. On the geometry side, we demonstrate that generic galileon theories are naturally thought of as the description of branes moving in higher dimensional spacetimes. On the algebraic side, we show that there exists a precise interpretation in which galileons can be thought of as Goldstone modes which arise when spacetime symmetries are spontaneously broken. In particular, when viewed in this light the galileons are the analogue of the Wess-Zumino-Witten term of the chiral lagrangian and

thus represent interactions which are technically special. These methods provide both new technical tools for analyzing galileon-like theories and offer conceptual changes for how these theories can be viewed.

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Introduction

1.1 Motivations and Overview

It has been known for more than fifteen years that the universe is accelerating [100], yet the source of the phenomenon remains elusive. Commonly proposed explanations consist of positing yet undetected forms of matter, altering the dynamics of gravity itself or admixtures of the two ideas. There are many proposals for modifying gravitational dynamics [30] and they all fall under the broad title of "modified gravity." Generically these proposals require the introduction of additional degrees of freedom. In this thesis we present work pertaining to particular classes of scalar field theory, collectively called Galileon models, which appear in a variety of modifications of gravity, and we analyze them from a variety of approaches.

Galileons originate from higher dimensional models. It is an old idea that the universe may contain more spatial dimensions beyond the three we commonly perceive and it has also proven to be a fruitful one, as many contemporary lines of research still draw their inspiration from these methods. The precise manner in which the dynamics of the higher-dimensional space manifests itself in the four dimensional world depends on the geometry and topology of the extra-dimensional manifold, and the matter content and action chosen. At low enough energies, the important physics is then captured by a four-dimensional effective field theory with properties inherited from the specific higher-dimensional model under consideration. The most relevant model at hand is the Dvali-Gabadadze-Porrati (DGP) model [51] in which the 3-brane we inhabit is floating in an ambient, five dimensional space. The gravitational action simply consists of two separate Einstein Hilbert terms – one in 5D, and the other only on the brane, constructed from the induced metric. At short distances, gravity behaves just as it would in everyday General Relativity (GR), but becomes modified at large distances. In an appropriate limit, the physics is described by a four-dimensional effective field theory which describes gravity plus a scalar degree of freedom which captures the brane bending mode [82, 90]. The scalar represents a modification of gravity and comes equipped with a screening mechanism which ensures that its fifth force turns off at short distances where the theory is expected to be equivalent to GR. The specific type of scalar interactions, whose explicit form we will describe in detail soon, was named the *galileon* due to a *Galilean* shift symmetry they enjoy [91].

While galileons originated in higher dimensional theories, they were later found to naturally arise in other, non-braneworld modifications of gravity. Namely, galileon interactions appear to be essential in the construction of four-dimensional massive gravity. Fierz and Pauli led the first attempt to give the graviton a mass and were the first to derive the appropriate form of the mass term [58], which was later shown to be unique [111]. Things are not as simple as one may have thought, however, and the basic, linear theory already displays fundamental problems. Primarily, when coupling the massive graviton to matter, one finds that the physics of the $m \to 0$ limit does not coincide with that of linearized GR [110, 121], a problem known as the van Dam-Veltman-Zakharov (vDVZ) discontinuity. In particular, the deflection of light by heavy sources is minimized by a factor of $\frac{3}{4}$ and observations of light bent by the sun are enough to rule out this simple massive theory on phenomenological grounds. A natural solution is to posit the existence of non-linear interactions which become important in the regime of the discrepancy and render the conclusions of the simple linear theory invalid [108]. Again, this line of research turned out to be highly non-trivial as generic interactions were found to develop the so-called Boulware-Deser (BD) ghost instability [13] which manifests itself as an extra, sixth, unstable degree of freedom for the massive graviton. Only in the last handful of years has it been discovered [36, 37] how to construct the appropriate non-linear interactions which sidestep the BD ghost. The very specific interactions of this de Rham-Gabadadze-Tolley (dRGT) massive gravity describe the appropriate five degrees of freedom and it was found that galileon interactions arise in the description of the longitudinal mode.

Therefore, a proper understanding of galileon interactions in their own right can shed light on multiple well motivated modifications of gravity. In particular, any potential pathologies or interesting phenomena displayed in galileon theories are likely to also be manifested in the modified theories of gravity in which they appear. These motivations provided the bulk of the impetus for the work performed in this thesis.

Further inspiration comes from the study of pure galileon theories themselves, as they were found to be quite non-trivial and interesting in their own right. At face value, they are simply theories of scalar fields which employ derivative interactions, but a deeper search reveals that they are connected to a diverse variety of concepts familiar throughout theoretical physics. For instance, a proper understanding of the properties of galileons requires considerations of strong coupling, intricacies of non-linear derivative interactions, ghost instabilities and the generic properties of UV completions. There are also a variety of viewpoints on the means through which galileons arise and thesis will primarily be focused on these varied interpretations. In particular, we will cover in detail how galileons appear in the study of probe branes inhabiting higher dimensional spaces and also their interpretation as the Goldstone modes resulting from spontaneous spacetime symmetry breaking. It is this eclectic mix of concepts and viewpoints which arise in the study of these theories that makes them so fascinating and the focus of this thesis is on the inherent properties of galileons and their connections to other aspects of physics.

In the remainder of this chapter we recall the history of galileons, their various properties and the relevant physical problems to which they have applications. After reviewing the canonical issues pertaining to cosmic acceleration and various relevant models and ideas in Sec. 1.2, we move on to the origins of galileons in the DGP model of modified gravity in Sec. 1.3. Abstracting the galileon theories away from their higher dimensional origins, we study them in their own right, define and categorize them in Sec. 1.4 and review the non-renormalization theorem they obey and study their use as a natural realization of the Vainshtein mechanism in Sec. 1.5. Later, we review how galileons arise in theories of modified gravity different from DGP in Sec. 1.6, in particular the manner in which they arise in the dRGT theory of massive gravity in Sec. 1.6.2. At the end of the introduction in Sec. 1.7 we briefly discuss common generalizations of the original galileon theory.

Part II of this thesis covers the geometry of galileons. In Chapter 2 we develop a geometrical interpretation of galileon theories as the description of a 3-brane living in an ambient 5D bulk. We review the construction of general brane actions and symmetries, and the ways in which these symmetries may be inherited by a four-dimensional effective field theory. In Sec. 2.3 we categorize the unique actions which lead to second order equations of motion. Then, in Chapter 3 we apply these methods to six separate examples, exhausting all the maximally symmetric possibilities: a 4D Minkowski brane embedded in a Minkowski bulk; a 4D Minkowski brane embedded in AdS_5; a 4D de Sitter brane embedded in a Minkowski bulk; a 4D de Sitter brane embedded in dS_5; a 4D de Sitter brane embedded in AdS_5; and a 4D Anti-de Sitter brane embedded in AdS_5. In each case, we describe the resulting 4D effective field theories and comment on their structure. In Sec. 3.4 we take small field limits to obtain galileon-like theories, discuss their stability and compare and contrast these theories with the special case of the original galileon. In Sec. 3.5 we study the phenomenology of one of these generalized classes of galileons, focusing on spherical symmetric solutions about a massive source. We demonstrate that the superluminalities displayed by the original galileons persist in the generalized model. Finally, in Chapter 4 we apply the brane construction to cosmological FRW spacetimes and construct the natural galileon theory inhabiting this spacetime.

Part III of this thesis covers the interpretation of galileons as the Goldstone modes resulting from the spontaneous breaking of spacetime symmetries. We re-derive the galileon interactions starting from the standard standard tools of spontaneous symmetry breaking: the coset methods of Callan, Coleman, Wess and Zumino [22, 31] and Volkov [112]. After reviewing the general coset construction in Sec. 6.2, we derive the algebra non-linearly realized by the galileons—the "galileon algebra." We show in Sec. 6.4, inspired by brane-world models, this is a contraction of a higher-dimensional Poincaré algebra only along particular auxiliary directions. That is, it can be thought of as the Poincaré algebra of a brane embedded in higher dimensions, where the speed of light in the directions transverse to the brane is sent to infinity, while the speed of light along the brane is kept constant. We show in Sec. 6.7 that, like the familiar Wess-Zumino-Witten term of the chiral Lagrangian [116, 118], the galileon terms in d-dimensions are not captured by the naive d-dimensional coset construction and a higher dimensional construction is required, making the galileons technically special. Additionally, we consider the conformal galileons in Sec. 6.11 and demonstrate that only one of the conformal galileons, the cubic term, appears as a Wess–Zumino term for spontaneously broken conformal symmetry. Finally, in Sec. 6.12 we demonstrate that, although the original galileons are Wess-Zumino terms for spontaneously broken space-time symmetries, this is not the case for the generalized relativistic DBI galileons [39] (covered in Sec. 3.3.1), which—aside from the tadpole term—are obtainable from the coset construction and hence are not Wess-Zumino terms. Finally, we explicitly construct the DBI galileons using the techniques of non-linear realizations.

1.2 Cosmic Acceleration and Modified Gravity

Before delving into the galileons, we review in greater detail the issues of cosmic acceleration, the cosmological constant problem and various proposals for explaining the source of the acceleration and for modifying gravitational dynamics

1.2.1 The Cosmological Constant

In this section we review the conundrums that arise when studying cosmic acceleration. There are many possible explanations for the acceleration and they are collectively called "dark energy." We will later briefly review dynamical models of dark energy, but since even these models demand a separate explanation for the smallness of the cosmological constant (CC), an extended discussion of the CC is appropriate. There exist many excellent, detailed reviews of the subject, see for example [24, 84, 99, 115], and we only cover the essentials here.

1.2.1.1 Basics: Dynamics and Forms of Matter

The simplest explanation for the acceleration of the universe is a cosmological constant, Λ, which enters the gravitational action as

$$S = \frac{M_{\rm pl}^2}{2}\int \mathrm{d}^4x\,\sqrt{-g}\,[R-2\Lambda]\ . \tag{1.1}$$

A cosmology which is only driven by positive Λ captures the qualitative properties of our current universe, namely it accelerates. The standard Einstein equations following from (1.1) are

$$R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}+\Lambda g_{\mu\nu}=0 \tag{1.2}$$

and writing the flat Friedmann-Robertson-Walker (FRW) metric in its standard form

$$\mathrm{d}s^2=-\mathrm{d}t^2+a(t)^2(\mathrm{d}x^2+\mathrm{d}y^2+\mathrm{d}z^3)\ , \tag{1.3}$$

where $a(t)$ is the scale factor, (1.1) reduces to

$$H^2=\Lambda/3\,,\qquad \frac{\ddot a}{a}=\frac{2}{3}\Lambda\ . \tag{1.4}$$

Above, $H \equiv \dot{a}/a$ is the Hubble parameter and (1.4) are nothing but the standard Friedmann equations applied to matter with equation of state $w \equiv p/\rho = -1$. As claimed, we see that positive Λ generates positive acceleration, $\ddot{a} > 0$, and the first Friedmann equation determines the scale factor to be $a(t) = e^{Ht}$ with $H = \sqrt{\Lambda/3}$.

While a positive cosmological constant certainly suits the task for generating an accelerating cosmology, it is a quite odd form of matter. If the universe were instead dominated by a more general form of matter with energy density ρ and pressure p, the cosmology would be governed by the Friedmann equations

$$H^2 = \frac{1}{3M_{\rm pl}^2}\rho\,, \quad \frac{\ddot{a}}{a} = -\frac{1}{6M_{\rm pl}^2}\left(\rho + 3p\right)\ . \tag{1.5}$$

Hence, the requirement for an accelerating universe is that equation of state obey $w < -1/3$. The everyday constituents of the world around us do not satisfy this requirement at all. For instance, pressure-less matter and radiation (that is, relativistic matter) have equations of state $w = 0$ and $w = \frac{1}{3}$, respectively and therefore no admixture of the matter most familiar to us will generate cosmic acceleration. This is simply a reflection of the fact that matter we're used to is gravitationally attractive and will not provide the repulsive force needed for an expanding universe.

1.2.1.2 Unnatural Λ: Classical and Quantum

Though unfamiliar, from an effective field theory viewpoint the cosmological constant is a natural object that can and should appear in the lagrangian as it obeys the full diffeomorphism symmetry of general relativity, and so we should not reject it out of hand simply because we haven't see it in our daily lives. The truly unnatural aspect of Λ, however, is its observed size. It is far smaller than one would a priori expect and the problem of explaining its apparent magnitude is known as the "old cosmological constant problem."

To be precise, the Planck mission determined the size of the current Hubble parameter to be 67.3 (km/s)/Mpc[2]. If the universe were entirely CC dominated[1], i.e. all other matter negligible, then (1.4) determines $\Lambda \sim 10^{-84}\text{GeV}^2$ corresponding to an energy scale $\sqrt{\Lambda} \sim 10^{-42}\text{GeV}$. Alternatively, the CC appears in the action (1.1) in the combination $\sim M_{\text{pl}}^2\Lambda$ which defines an energy density $\rho \sim M_{\text{pl}}^2\Lambda$ corresponding to an energy scale $(\rho)^{1/4} \sim 10^{-12}\text{GeV}$.

First, from a classical perspective, these energy scales are absurdly small. The zero point energy coming from potentials for all of the matter of the universe will behave as a cosmological constant and there is no particular reason for this to contribute at observed order of magnitude. That is, the matter fields $\{\psi\}$ have some given potential energy $V(\{\psi\})$ and the $\{\psi\}$ dynamics are unaffected by shifting $V(\{\psi\}) \to V(\{\psi\}) + V_0$, $V_0 =$ constant; only energy differences matter for $\{\psi\}$. On the other hand, gravity *is* sensitive to the overall size of $V(\{\psi\})$, as shifting $V(\{\psi\})$ corresponds to changing $\Lambda \to \Lambda + V_0/2M_{\text{pl}}^2$ in (1.1). A priori there is no compelling reason to choose one value of V_0 over another and there are in fact good reasons to expect that whatever its value is, it is unlikely that the contribution to the CC will end up near 10^{-11}GeV. The reason is that we expect that the universe went through a series of phase transitions [24] throughout its history which generically cause large jumps in the CC. For instance, if the Higgs potential were adjusted so that its potential energy was strictly zero at higher energies (i.e. when the Higgs vacuum expectation value vanishes), then we'd expect that after the electroweak phase transition the potential would become of the size $V \sim -(250\text{GeV})^4$ where 250GeV is the approximate electroweak scale. Combined with the contribution stemming from the QCD transition and other unknown, higher energy phase transitions, it appears horribly unnatural for these effects to nearly exactly cancel and give a result of the order required.

[1]The Planck mission actually determined that the effective cosmological constant accounts for 69% of the energy in the universe, but the approximations made here are adequate for illuminating the essence of the problem.

Next, the size of the required CC is unnatural from a quantum field theoretic viewpoint. In order to see this, we first review the typical effective field theory (EFT) argument that leads to the hierarchy problem, beginning with a simple statement of the general rules for EFT's and later justifying their origins. Then we apply the EFT reasoning to the gravitational action (1.1) and discuss the hierarchy problem from which it suffers.

The mantra of effective field theory is to start by specifying the field content, symmetries and high energy cutoff $\tilde{\Lambda}$ of the theory, below which the EFT is supposed to provide an accurate description of the physics. One then builds the action by including in the theory every operator compatible with the specified symmetries with $\mathcal{O}(1)$ coefficients and factors of $\tilde{\Lambda}$ to fix the proper dimensions [98]. Occasionally, operators can have coefficients that are much smaller than $\mathcal{O}(1)$. This is allowed if the relevant parameter is "technically natural" in the sense that the number of symmetries enjoyed by the system increases as the parameter is taken to zero [103].

The canonical example of such a natural parameter is the fermion mass m. A typical EFT of Dirac fermions with cutoff $\tilde{\Lambda}$ might take the form

$$\mathcal{L} = i\bar{\psi}\partial_\mu\gamma^\mu\psi - c\tilde{\Lambda}\bar{\psi}\psi + g^2\frac{\bar{\psi}\gamma^\mu\psi\bar{\psi}\gamma_\mu\psi}{\tilde{\Lambda}^2} + \ldots \tag{1.6}$$

with $g \sim \mathcal{O}(1)$ but $c \ll 1$ and hence $m \equiv c\tilde{\Lambda} \ll \tilde{\Lambda}$. This latter condition is technically natural because as $m \to 0$ the lagrangian (1.6) gains *chiral* symmetry, $\psi \to e^{i\theta\gamma_5}\psi$. Similarly, gauge boson masses are technically natural since we recover gauge symmetry as the mass is taken to zero.

However, the analogous scenario for scalar fields is generally unnatural. That is, given a generic EFT for ϕ, say

$$\mathcal{L} = -\frac{1}{2}(\partial\phi)^2 - c\tilde{\Lambda}^2\phi^2 - \lambda\phi^4 - g\frac{\phi^6}{\tilde{\Lambda}^2} + \ldots \ , \tag{1.7}$$

no symmetry is gained by taking $c \to 0$ and hence by the rules of EFT's we ought to take $c \sim \lambda \sim g \sim \mathcal{O}(1)$. This leads to a contradiction, however, since the theory (1.7) describes a scalar of mass $m_\phi \sim \tilde{\Lambda}$, but by assumption the EFT is only a good description of physics at energies much below the scale $\tilde{\Lambda}$ and hence should not accurately describe the physics of ϕ [98]. While we wish that our EFT could describe a light scalar of mass $m \ll \tilde{\lambda}$, this desire typically contradicts the general rules of EFT's. The unnaturalness of the gap $m \ll \tilde{\Lambda}$ is called the "Hierarchy Problem" and it is a well-known problem of the Higgs sector for the electroweak theory in which the Higgs mass $m \sim 125\text{GeV}$ is much smaller than the scale at which we know new physics must appear, $M_{\rm pl} \sim 10^{19}\text{GeV}$. As another example, a similar issue exists for QCD in which there are two possible interactions which are quadratic in the field strength tensor, i.e. $\mathcal{L} \sim a\text{Tr}F \wedge \star F + b\text{Tr}F \wedge F$. The a term is familiar kinetic term and is $a \sim \mathcal{O}(1)$. The b term on the other hand contributes to the electric dipole moment of the neutron and observations determine $b \ll 10^{-9}$ [48]. This discrepancy from $\mathcal{O}(1)$ known as the "strong CP problem."

The above rules for effective field theories originate from the Wilsonian viewpoint in which we generate EFT's by "integrating out" the heavy degrees of freedom. That is, the full microscopic theory may consist of heavy particles of mass M and light particles of mass $m \ll M$ and if we are only interested in physics at energy scales E with $m \ll E \ll M$, then there is not much sense in keeping detailed track of the heavy particles. In particular, we can capture the effects of the heavy fields on the physics by simply introducing effective interactions that only entail factors of the light fields. For instance, we can imagine that the fermionic EFT of (1.6) arises as a low energy description of fermions coupled to massive gauge fields, say A_μ with mass M. The full theory would have cubic interactions of the form $\sim g\bar{\psi}\gamma^\mu A_\mu \psi$ which can be combined to describe, say, $\bar{\psi}\psi \to \bar{\psi}\psi$ scattering which in the tree approximation would have amplitude $\mathcal{M} \sim g^2 \frac{i}{-p^2 - M^2}$. However, if we are only interested in scales much smaller than M, to a good approximation the amplitude is simply $\mathcal{M} \sim \frac{g^2}{M^2}$. Therefore, $\bar{\psi}\psi \to \bar{\psi}\psi$ scattering would be described perfectly well to the order we care about by instead considering an effective theory in which we remove the massive gauge bosons from the theory altogether and add in its place an interaction $\sim g^2 \bar{\psi}\gamma^\mu\psi\bar{\psi}\gamma_\mu\psi/M^2$.

This is precisely the case described in (1.6) with the identification $\tilde{\Lambda} \sim M$ and this is the generic expectation; the cutoff of the effective field theory is the energy scale where new physics (i.e. the emergence of heavier particles in this context) becomes important.

The naturalness of fermion masses and unnaturalness of scalar masses in EFT's then becomes apparent by considering the roles of the two fields in their full, microscopic theories. In the full theory, the fermion interacts with much heavier particles and direct calculations in such scenarios typically demonstrate that loop corrections to the bare fermion mass, m, are proportional to m themselves $\delta m \propto m$. As discussed, this must be the case since as $m \to 0$ the theory gains a chiral symmetry which is expected to be preserved by loop corrections. Therefore, the fermion mass is not very sensitive to the masses of heavy particles in the full theory and a small bare mass in the fundamental theory will generically translate into a small fermion mass in the effective theory since the final result is $\propto m$. On the other hand, repeating the same exercise for scalars typically results in corrections to the bare mass squared of a scalar, m^2, which are *not* proportional to m^2, but instead to the squared mass of the heavy particles, $\delta m^2 \sim M^2$. As no symmetry is gained by setting $m \to 0$, corrections need not be proportional to m. Hence, the mass squared of the scalar appearing in the effective theory is typically highly sensitive to the masses of heavy particles and is generally insensitive to the initial base mass. In particular, the effective mass will typically be pushed up to the scale $\sim M$. From the previous paragraph, this corresponds to pushing the mass of the scalar to energies approaching the cutoff of the EFT (i.e. we should be choosing the mass to be $\sim c\tilde{\Lambda}^2$ with $c \sim \mathcal{O}(1)$) and we find ourselves in the unnatural position of (1.7). While it is possible to precisely tune the scalar bare mass so that the resulting effective theory enjoys a wide hierarchy between the effective scalar mass and the cutoff, such a scenario is a highly optimistic assumption about the nature of high energy physics and is considered a contrived and unappealing possibility, hence the hierarchy problem.

The relevance of the hierarchy problem to general relativity is that the role played by the cosmological constant is analogous to the scalar mass in (1.7). Viewed as an effective field theory with cutoff $M_{\rm pl}$, gravitational physics should be described by a lagrangian of the schematic form

$$\mathcal{L} \sim \sqrt{-g}\left[aM_{\rm pl}^4 + bM_{\rm pl}^2 R + cR^2 + dR^4/M_{\rm pl}^2 + \ldots\right] \ . \tag{1.8}$$

No symmetry is gained in the limit $a \to 0$ and hence a is not a technically natural parameter and should be expected to be $\mathcal{O}(1)$. Indeed, direct calculation [84] demonstrates that a bare Λ receives loop corrections from heavy particles, say of mass M, in the form $\delta\Lambda \sim M^4/M_{\rm pl}^2$ and so the CC has the same naturalness issues as the scalar was found to have. The cosmological constant then has a natural size[2] $\Lambda_{\rm natural} \sim M_{\rm pl}^2 \sim (10^{19}{\rm GeV})^2$. This is hugely at odds with our observationally derived value $\Lambda_{\rm derived} \sim (10^{-42}{\rm GeV})^2$ and, understandably, is considered to be quite an embarrassing as there is no compelling explanation for the disparity.

1.2.2 The Coincidence Problem

A less dramatic conundrum is the observation that current energy density coming from the apparent cosmological constant is the same order of magnitude as the energy density arising from matter and only relatively recently (i.e. around redshift $z \sim 1$) has the universe begun to accelerate. In particular, the Planck mission [2] found that energy budget of the universe is 31% matter and 69% dark energy. Such a configuration is surprising in light of the differing rates at which the two energy components dilute. While dark energy is approximately constant, matter dilutes along with the expansion of the universe as $\propto a^{-3}$ and therefore in the far past the matter density hugely dominated over dark energy and it is only at relatively recent times that rough equivalence between the two was established. It is possible that there exists a dynamical origin for this fact. If dark energy were not simply a cosmological constant, but instead arose from some unknown degree of freedom, then it

[2] Other, more conservative estimates might put the natural size at the assumed SUSY breaking scale of, say $\mathcal{O}((1{\rm TeV})^4)$ [5, 49], but the discrepancy is still enormous.

could be that our observations are not simply a coincidence, but the generic expectation for the evolution of the universe.

1.2.3 Removing Λ: Adjustment Mechanisms

A natural first instinct when attempting to solve the cosmological constant problem is to play with the idea that there exists some set of fields $\{\phi_i\}$ whose dynamics will generically drive $\Lambda \to 0$. As Weinberg says, this idea has been "tried by virtually everyone" [115], but without much success.

There are quite general arguments that indicate that this route is a difficult one, in particular the "no-go" theorem of Weinberg himself [115], which we review here. The key idea is that a natural solution of the CC problem would require the fact $\Lambda \approx 0$ to follow entirely from the equations of motion for the fields $\{\phi_i\}$. For instance, an unnatural solution would be to simply assume that the potential for these additional fields $V(\{\phi_i\})$ simply takes on exactly the right value to cancel off the true value of Λ. Such a solution would be nothing but a reshuffling of where the fine tuning is taking place. Rather than small Λ originating from a fine tuning in the fundamental UV physics, it would originate from fine tuning within $V(\{\phi_i\})$, rendering the idea unpalatable.

Instead, it would be much more natural if the equations of motion for the fields $\{\phi_i\}$ were proportional to $T^\mu{}_\mu$. If this were the case, then the dynamics of $\{\phi_i\}$ drive $T^\mu{}_\mu \to 0$ and hence drive spacetime to be approximately flat, as Einsteins equations tell us $R \propto T^\mu{}_\mu$. Let us attempt to construct such a scenario in the restricted case where the fields $\{\phi_i\}$ and are constant throughout spacetime, i.e. they preserve Poincaré invariance, and we want the $\{\phi_i\}$ to drive us to a scenario in which the metric $g_{\mu\nu}$ is also spacetime independent. In terms of the lagrangian $\mathcal{L}$, we are in effect requiring the statement

$$\sum_i f_i \frac{\partial \mathcal{L}}{\partial \phi_i} \propto T^\mu{}_\mu \ , \tag{1.9}$$

where $\{f_i\}$ are generic functions of the $\{\phi_i\}$. If we can find a system which obeys a form of (1.9), then $T^\mu{}_\mu$ vanishes as the fields $\{\phi_i\}$ settle into their minima and we have accomplished our goal.

The condition (1.9) turns out to be a quite severe restriction, however. From the assumption that $g_{\mu\nu}$ is spacetime independent, we can write write $T^\mu{}_\mu = \frac{2}{\sqrt{-g}} g_{\mu\nu} \frac{\partial \mathcal{L}}{\partial g_{\mu\nu}}$ and so (1.9) becomes

$$\sum_i f_i \frac{\partial \mathcal{L}}{\partial \phi_i} = g_{\mu\nu} \frac{\partial \mathcal{L}}{\partial g_{\mu\nu}} \ . \tag{1.10}$$

This (1.10) is actually a statement of the symmetries of the fields; it expresses that the lagrangian must be symmetric under the combined transformations

$$\delta\phi_i = -\epsilon f_i \,, \quad \delta g_{\mu\nu} = \epsilon g_{\mu\nu} \ . \tag{1.11}$$

In order to see why this is problematic, it is useful to redefine our fields to a new diagonal basis, $\{\phi_i\} \to \{\varphi, \sigma_a\}$, which is chosen such that the symmetry in terms of $\{\varphi, \sigma_a\}$ is

$$\delta\varphi = -\epsilon \,, \quad \delta\sigma_a = 0 \,, \quad \delta g_{\mu\nu} = \epsilon g_{\mu\nu} \ . \tag{1.12}$$

Any non-derivative interaction which obeys this symmetry must then be built from the invariant effective metric $e^\varphi g_{\mu\nu}$ and general coordinate invariance further dictates that the interaction must be $\propto \sqrt{-\det e^\varphi g_{\mu\nu}} = e^{2\varphi}\sqrt{-g}$. The most general, symmetric, non-derivative interaction that will involve all the fields $\{g_{\mu\nu}, \varphi, \sigma_a\}$ is then $\mathcal{L} = \sqrt{-g}\, e^{2\phi} \mathcal{L}'(\{\sigma_a\})$ where $\mathcal{L}'(\{\sigma_a\})$ only involves the fields $\{\sigma_a\}$ and would generally include a cosmological constant piece $\mathcal{L}'(\{\sigma_a\}) \supset M_{\rm pl}^2 \Lambda$.

The problem we will find is that while we were explicitly looking for interactions which would naturally set $T^\mu{}_\mu \to 0$, the lagrangian $\mathcal{L} = \sqrt{-g}\, e^{2\phi}\mathcal{L}'(\{\sigma_a\})$ will only set $T^\mu{}_\mu \to 0$ for a precise tuning of the parameters in $\mathcal{L}'(\{\sigma_a\})$, which is exactly what we were trying to avoid. Specifically, the equations of motion for φ and σ_a are

$$\frac{\partial \mathcal{L}}{\partial \varphi} = 2\sqrt{-g}\, e^{2\phi} \mathcal{L}'(\{\sigma_a\}) \,, \quad \frac{\partial \mathcal{L}}{\partial \sigma_a} = \sqrt{-g}\, e^{2\phi} \frac{\partial \mathcal{L}'(\{\sigma_a\})}{\partial \sigma_a} \ . \tag{1.13}$$

The σ_a equations simply define the equilibrium values of $\{\sigma_a\}$ through the conditions $\frac{\partial \mathcal{L}'(\{\sigma_a\})}{\partial \sigma_b} = 0$. On the other hand, it is the φ equation which sets the trace of the energy momentum tensor,

$$\frac{\partial \mathcal{L}}{\partial \varphi} = 2\sqrt{-g}\, e^{2\varphi} \mathcal{L}'(\{\sigma_a\}) = g_{\mu\nu} \frac{\partial \mathcal{L}}{\partial g_{\mu\nu}} = \frac{1}{2}\sqrt{-g} T^{\mu}{}_{\mu} \ . \tag{1.14}$$

The only way to get a stationary configuration for φ, and hence set $T^{\mu}{}_{\mu} \to 0$, is to then enforce that the $\{\sigma_a\}$ equilibrium additionally sets $\mathcal{L}'(\{\sigma_a\}) = 0$, a condition which can only arise through fine tuning. That is, to satisfy this condition we must balance all couplings in $\mathcal{L}'(\{\sigma_a\})$ against one another, and against the cosmological constant that might appear in $\mathcal{L}'(\{\sigma_a\})$, so that they precisely cancel.

Therefore, the simplest attempts at adjustment mechanisms will generically suffer from fine tuning issues themselves and do not represent improvements of the problem. It is possible to evade this conclusion by bypassing the assumptions made above, for example by exploring scenarios in which the $\{\phi_i\}$ are not spacetime independent. These are active lines of research [26, 27], but we will not discuss them in detail here.

1.2.4 Shielding Λ: Degravitation

An entirely different approach supposes that Λ really does exist and is in fact enormous, but it just doesn't curve the universe as much as we assume it will. We derived that in our universe Λ must be quite small through the use of the Einstein equations, but if at long distances Einstein's equations are not an accurate description of the physics then our conclusion would be rendered faulty. That is, a modification of the dynamics of gravity at large distance would modify our interpretation of the size of Λ. In particular, we review here concepts which fall under the umbrella of "degravitation" and, as stated in [5], such approaches do not try to explain the size of Λ, but rather answer the question "Why does the vacuum energy gravitate so little?"

Quite generally, models of degravitation consist of regarding gravity as a high pass filter. Essentially they represent modifications of Einstein's equation of the form

$$M_{\rm pl}^2 G_{\mu\nu} = T_{\mu\nu} \to M_{\rm pl}^2(L^2\Box)G_{\mu\nu} = T_{\mu\nu}\ , \tag{1.15}$$

that is they promote the Planck mass to be a scale dependent function and introduce an additional length scale L. In Fourier space, the functional dependence should be chosen so that at short distance scales, $kL \gg 1$, $M_{\rm pl}(L^2\Box)$ should take on its familiar value $\sim 10^{19}$GeV, but at long scales, $kL \ll 1$, $M_{\rm pl}(L^2\Box)$ diverges. Since curvature is sourced by $T_{\mu\nu}/M_{\rm pl}^2(L^2\Box)$ gravity would ostensibly be unchanged at shorter distances, but would become entirely insensitive to long wavelength sources such as Λ.

Though the introduced scale dependence of $M_{\rm pl}$ may appear contrived, it can arise quite naturally in reasonable modifications of gravity. In particular, the physics of a massive graviton should be expected to resemble the degravitation mechanism. The reasoning is that since a boson of mass m only have characteristic range m^{-1} it will never know about sources which are homogeneous on scales larger than m^{-1} and is therefore insensitive to them.

The analysis for massive gravitons is long and detailed, see [50], and to demonstrate the main concepts it is sufficient for our purposes to consider the case of an analogue model of a massive $U(1)$ gauge bosons. We follow [50] and begin with the standard massive gauge boson lagrangian coupled to an external source which is assumed to be conserved, $\partial_\mu J^\mu = 0$,

$$\mathcal{L} = -\frac{1}{4}F[A]^2 - \frac{1}{2}m^2 A_\mu^2 + J_\mu A^\mu\ . \tag{1.16}$$

A massive gauge boson in $d = 4$ carries three independent degrees of freedom, one zero helicity longitudinal mode and two transverse modes of helicity $h = \pm 1$. A particularly clear way of separating different degrees of freedom in (1.16) is to rewrite the lagrangian by using the Stückelberg trick by making the replacement $A_\mu \to \tilde{A}_\mu + \frac{1}{m}\partial_\mu\phi$ everywhere, under which (1.16) becomes

$$\mathcal{L} = -\frac{1}{2}F^2[\tilde{A}] - \frac{1}{2}m^2\tilde{A}_\mu^2 - \frac{1}{2}(\partial\phi)^2 - m\tilde{A}^\mu\partial_\mu\phi + J_\mu\tilde{A}^\mu\ . \tag{1.17}$$

While the mass term in (1.16) ruins gauge invariance and leads to the three independent degrees of freedom in A_μ, the lagrangian (1.17) enjoys the gauge invariance

$$\delta\tilde{A}_\mu = \partial_\mu\Lambda(x)\,, \quad \delta\phi = -\Lambda(x) \tag{1.18}$$

which implies that $\tilde{A}_\mu$ only carries the two helicity ± 1 degrees of freedom. The two lagrangians (1.16) and (1.17) are entirely equivalent. This can be seen explicitly by going to "unitary gauge" in (1.17) where the gauge invariance (1.18) is used to set $\tilde{\phi} = 0$ so that the two lagrangians coincide.

We can then demonstrate explicitly that the effect of the mass is to generate a de-electrifying effect (the analogue of degravitation) for the helicity ± 1 modes of $\tilde{A}_\mu$. This simply follows from integrating out ϕ via its equations of motion. Explicitly, they read

$$\Box\phi = -m\partial^\mu\tilde{A}_\mu \tag{1.19}$$

and we simply replace $\phi \to -\frac{m}{\Box}\partial_\mu\tilde{A}^\mu$ everywhere in (1.17) to generate the effective action for $\tilde{A}_\mu$. The result is

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4}F[\tilde{A}]^2 - \frac{1}{2}m^2\tilde{A}_\mu^2 + \frac{m^2}{2}\partial_\mu\tilde{A}^\mu\frac{1}{\Box}\partial_\nu\tilde{A}^\nu + m^2\tilde{A}^\mu\frac{\Box}{\Box}\tilde{A}_\mu + J_\mu\tilde{A}^\mu \\ &= -\frac{1}{4}F[\tilde{A}]^2 - \frac{m^2}{2}\left[\partial_\mu\tilde{A}^\nu\frac{1}{\Box}\partial^\mu\tilde{A}_\nu - \partial_\mu\tilde{A}^\mu\frac{1}{\Box}\partial_\nu\tilde{A}^\nu\right] + J_\mu\tilde{A}^\mu\ .\end{aligned} \tag{1.20}$$

The two transverse modes then obey the effective equation of motion

$$\left(1 - \frac{m^2}{\Box}\right)\partial_\mu F^{\mu\nu} + J^\nu = 0 \tag{1.21}$$

which is precisely what is needed for de-electrification. As the $m^2/\Box$ term becomes large at long wavelengths and $\tilde{A}_\mu$ decouples from long wavelength sources. The analogous story hold for Fierz-Pauli massive gravity [50].

Therefore, it's possible that long wavelength modifications of gravity can alleviate the old cosmological constant problem. The simplest modification of gravity that can behave as a high pass filter is the addition of a hard mass, but it is not the only way. In particular, braneworld models of gravity where matter is confined to live on a 4D brane, but where gravity can propagate through a large 5D bulk space in which the brane is embedded, will also lead to equations of the form (1.15) [5, 49]. For instance, the momentum dependence of the graviton in the brane-world Dvali-Gabadadze-Porrati model (see Sec. 1.3) is $D(p) \sim \frac{-i}{p^2+2r_c^{-1}\sqrt{p^2}}$ for some length scale r_c, corresponding to a resonance graviton, and hence the DGP model employs a high pass filter $\sim \left(1 - \frac{1}{r_c\sqrt{\Box}}\right)$. In practice, it is often difficult for specific models to degravitate efficiently enough to reproduce our own universe without fine tuning and while leaving other gravitational physics unmodified, but research is still progressing along these directions.

1.2.5 Alternatives to Λ: Other Sources of Dark Energy

A final field of study we briefly review are attempts to explain cosmic acceleration via other means than a cosmological constant. Such approaches assume the existence of some symmetry or unknown mechanism which separately solves the old cosmological constant problem and sets $\Lambda \to 0$ and focus instead on types of dynamical matter or modifications of gravity that generate the late-time acceleration. There are an enormous variety of such proposals [30] and we only cover two here, briefly.

On the dynamical matter front, the use of scalar fields as a candidate source of dark energy is the most common tactic and these are known as "quintessence models" [19]. They typically consist of a standard kinetic term[3] and potential $V(\phi)$

$$\mathcal{L} = -\frac{1}{2}(\partial\phi)^2 - V(\phi) \ . \tag{1.22}$$

Assuming negligible spatial gradients, the pressure, density and equation of state are given

[3]Other models such as k-essence [7] employ derivative interactions.

by

$$p = \frac{1}{2}\dot{\phi}^2 + V(\phi)\,, \quad \rho = \frac{1}{2}\dot{\phi}^2 - V(\phi)\,, \quad w_\phi = \frac{\frac{1}{2}\dot{\phi}^2 + V(\phi)}{\frac{1}{2}\dot{\phi}^2 - V(\phi)} \tag{1.23}$$

and hence if the potential is such that the scalar is slowly rolling, $\dot{\phi}^2 \ll V(\phi)$, the quintessence field describes a fluid with equation of state similar to that of a cosmological constant, $w \approx -1$.

Typically, quintessence models fall into one of two categories: thawing or freezing [20]. The equation of state obeyed by the (assumed to be homogeneous) background ϕ is

$$\ddot{\phi} + 3H\dot{\phi} + \partial_\phi V(\phi) = 0 \tag{1.24}$$

and in the thawing models, the potential is such that at early times (large H) the Hubble parameter dominates and the field is frozen by Hubble friction and thus acts identically to a CC. At late times, when H has diminished substantially, the scalar unfreezes and the fluid's equation of state deviates from $w_\phi = -1$, leaving cosmological signatures. In freezing models, the quintessence field starts out rolling down the potential. The potential eventually flattens out and as the field slows its roll it approaches equation of state $w = -1$. In particular, there are "tracker solutions" in which the equation of state tracks the dominant form of matter in the universe. In such scenarios, the quintessence field begins by tracking radiation at early times with $w_\phi < 1/3$, then begins to track matter during the matter domination era with $w_\phi < 0$ and eventually as the matter density dilutes away the quintessence field comes to dominate and w_ϕ approaches -1, generating the observed acceleration. The quintessence field needs to roll over a large distance and hence these models often suffer from the standard fine tuning and naturalness issues which afflict typical scalar field theories. For reviews on quintessence fields with varying levels of detail, consult [83, 102, 107].

A different tack is to invoke new dynamics as the root of the acceleration. In particular, by positing that the gravitational action is not simply $\sim \sqrt{-g}R[g]$ but instead a complicated function of the Ricci scalar (as may be expected from an effective field theory viewpoint), it's possible to produce dynamics by which the universe begins to accelerate at late times, even in the absence of a cosmological constant or other matter. Such models take on the form

$$S = \int \mathrm{d}^4x \sqrt{-g} f(R) \tag{1.25}$$

and are hence known as $f(R)$ theories [30]. Among other possible uses, judicious choices of $f(R)$ will lead to cosmologies which self-accelerate at late times, again even in the absence of a cosmological constant or other matter [4, 25, 74, 101].

1.3 History: DGP and the Galileon

In this section we review the historical discovery and development of galileon theories in the context of the Dvali-Gabadadze-Porrati (DGP) model [51]. DGP begins by assuming that the 3-brane which we inhabit is embedded in a flat, five dimensional space where the extra dimension is infinitely large. This is a departure from more common theories in which extra dimensions are posited to be quite small, such as Kaluza-Klein models, because the motivations are different. Models with small extra dimensions modify the high energy, short distance physics and since we are interested in modifying the low energy, long distance behavior of gravity the large extra dimension scenario is the appropriate one. Both the higher dimensional space and the brane are given Einstein-Hilbert terms, so that gravity propagates in both spaces, while matter is confined to the brane.

We let $\mathcal{M}$ be our 3-brane with coordinates x^μ, $\mu \in \{0, 1, 2, 3\}$ and $\mathcal{N}$ be the five dimensional space with coordinates X^A, $A \in \{0, 1, 2, 3, 5\}$. The brane's position is given by embedding functions $X^A(x^\mu)$ and the bulk metric, G_{AB}, induces a metric on the brane,

$g_{\mu\nu} \equiv \frac{\partial X^A}{\partial x^\mu}\frac{\partial X^B}{\partial x^\nu} G_{AB}$. The DGP action is defined to be

$$S_{\rm DGP} = \frac{1}{2}M_5^3 \int_{\mathcal{N}} \mathrm{d}^5 X \sqrt{-G} R^{(5)}[G] + \int_{\mathcal{M}} \mathrm{d}^4 x \sqrt{-g}\left[-M_5^3 K[g] + \frac{1}{2}M_{\rm pl}^2 R^{(4)}[g] + \mathcal{L}_M(\psi)\right] . \tag{1.26}$$

where $R^{(5)}[G]$ is the 5D Ricci scalar generated from G_{AB}, $R^{(4)}[g]$ is the 4D Ricci scalar generated from $g_{\mu\nu}$, $K(g)$ is the extrinsic curvature of the 3-brane and $\mathcal{L}_M(\psi)$ is the matter action. The masses M_5 and $M_{\rm pl}$ are the five and four dimensional Planck masses, respectively. The dynamical variables of the theory are the bulk metric (G_{AB}), coordinates of the brane in the bulk ($X^A(x^\mu)$) and matter fields (ψ); the induced metric does not contain independent degrees of freedom as it is derived from G_{AB} and the X^A's.

DGP is a rich model with robust phenomenology and features and a detailed analysis is outside of the realm of this thesis. Instead, we concentrate on the basic features of the model and a sketch of how the galileon interactions arise. Of primary importance is the transition in the behavior of gravity as we progress to larger and larger distances. This transition is reflected in the graviton propagator derived from (1.26) whose momentum dependence goes as [71]

$$D(p) \sim \frac{-i}{p^2 + 2\frac{M_5^3}{M_4^2}\sqrt{p^2}} \ . \tag{1.27}$$

The propagator defines a "crossover distance" $r_c \equiv \frac{M_{\rm pl}^2}{M_5^3} \equiv m^{-1}$ below which gravity appears four dimensional and above which gravity appear five dimensional, that is

$$D(p) \sim \begin{cases} \frac{-i}{p^2} & p \ll r_c^{-1} \\ \frac{-i}{|p|} & p \gg r_c^{-1} \end{cases} . \tag{1.28}$$

In order to set r_c to be of order the Hubble radius, we must have $M_5 \sim$ 10MeV.

In addition to the tensor mode of the graviton, there is also a vector and a scalar mode which descend from $G_{\mu 5}$ and h_{55} respectively. A long and detailed analysis [82] demonstrates that the scalar sector of the effective 4D lagrangian is a derivatively self-coupled

$$\mathcal{L}_{\rm eff} \supset 3M_{\rm pl}^2 m^2 \pi \Box \pi - M_{\rm pl}^2 m (\partial\pi)^2 \Box \pi + \frac{1}{2} m\pi T \ . \tag{1.29}$$

The full action depends on tensor, vector and scalar fields $\{h'_{\mu\nu}, N'_\mu, \pi\}$, which are intricate field redefinitions of the bulk metric perturbations about flat space, $\{H_{\mu\nu}, H_{\mu 5}, H_{55}\}$, needed to diagonalize the action, but in the *decoupling limit* we can focus our attention on the scalar lagrangian (1.29).

In order to justify this focus, we compare the strong coupling scale of scalar interactions and demonstrate that is is parametrically lower than all other interaction scales appearing in the effective DGP lagrangian. More precisely, when we canonically normalize by using the field $\hat{\pi} \sim M_{\rm pl} m \pi$, we see that the scale suppressing the cubic derivative term in (1.29) is $\Lambda_3 \equiv M_5^2/M_{\rm pl} = \left(m^2 M_{\rm pl}\right)^{1/3}$ and $\hat{\pi}$ couples to matter with gravitational strength, $\sim \frac{\hat{\pi} T}{M_{\rm pl}}$. One can deduce that a typical interaction appearing in $\mathcal{L}_{\rm eff}$ is of the form [90]

$$\sim m M_{\rm pl}^2 \partial \left(\frac{\hat{N}_\mu}{m^{1/2} M_{\rm pl}}\right)^p \left(\frac{\partial \hat{\pi}}{m M_{\rm pl}}\right)^q (h'_{\mu\nu})^s \tag{1.30}$$

with $p + q + s \geq 3$ and where $\hat{h}_{\mu\nu}$ and $\hat{N}_\mu$ are the canonically normalized[4] tensor and vector fields. Then if we want to study modifications of gravity well outside of a source's Schwarzschild radius, i.e. where we can take $\hat{h}_{\mu\nu} \to 0$, we need only concern ourselves with interaction of the form (1.30) with $q = 0$. Inspection of (1.30) shows that the interaction of the type (p, q) is suppressed by the scale

$$\Lambda^{(p,q)} \sim \left(m^{q+p/2-1} M_{\rm pl}^{p+q-2}\right)^{\frac{1}{3p/2+2q-3}} = \left(\Lambda_3^q M_5^{3/2p+q-3}\right)^{\frac{1}{3p/2+2q-3}} \tag{1.31}$$

and since $m \ll M_{\rm pl}$ we see, as claimed, that the smallest this scale can ever be is Λ_3, corresponding to $p = 0$, $q = 3$. Thus, the π self interactions are the least suppressed and

[4] As indicated in (1.30), the normalization for N_μ is *not* the standard one for vector fields. Rather, the kinetic term for N_μ arises from $\sim N_\mu \triangle N^\mu$ where $\triangle = \sqrt{-\Box}$, and so the canonical normalization sets the dimension of N_μ to be $[N_\mu] = E^{3/2}$.

most important. A formal limit makes this statement more precise. By taking $M_5, M_{\rm pl}, T \to \infty$ with Λ_3 and $T/M_{\rm pl}$ held constant, it is clear from (1.31) that only the cubic scalar interaction survives and in this so-called decoupling limit the lagrangian becomes

$$\mathcal{L}_{\hat{\pi}} = \frac{1}{2}\hat{\pi}\Box\hat{\pi} - \frac{1}{6^{3/2}\Lambda_3^3}(\partial\hat{\pi})^2\Box\hat{\pi} + \frac{1}{2\sqrt{6}M_{\rm pl}}\hat{\pi}T \ . \tag{1.32}$$

This scalar degree of freedom is known as the "galileon" and we continue its history in the next section.

1.4 The Original Galileons

As we've seen, in the decoupling limit the physics of the DGP model [51] is well described by a scalar field π which couples to the trace of the energy momentum tensor $\sim \pi T$ and has derivative self-interactions $\sim \Box\pi(\partial\pi)^2$ [82]. In this section we review the properties of this theory and its generalizations.

Though the self-interaction is higher-derivative, it nevertheless has second order equations of motion, $\frac{\delta}{\delta\pi}\Box\pi(\partial\pi)^2 = 2\partial_\mu\partial_\nu\pi\partial^\mu\partial^\nu\pi - 2(\Box\pi)^2$. This guarantees that the theory does not propagate a ghost, which is the usual pathology associated with many higher-derivative scalar theories. For instance, consider a scalar field ϕ whose quadratic lagrangian is higher derivative

$$\mathcal{L} = \frac{1}{2}\phi\Box\phi + \frac{\lambda}{\tilde{\Lambda}^2}(\Box\phi)^2 \ . \tag{1.33}$$

The equations of motion are fourth order and hence more initial data is required to solve the resulting system. Generically, this scenario will lead to instabilities [119] due to the fact that the associated Hamiltonian will be unbounded from below. Such instabilities are known as Ostrogradski ghosts and we can explicitly demonstrate the connection between the higher derivatives stemming from (1.33) and the familiar notion of ghosts as degrees of freedom with wrong sign kinetic terms. Inserting an auxiliary variable ψ, the lagrangian (1.33) can be rewritten as

$$\mathcal{L} = \frac{1}{2}\phi\Box\phi + \frac{\lambda}{\tilde{\Lambda}^2}\left(2\psi\Box\phi - \psi^2\right) \ . \tag{1.34}$$

A field redefinition $\phi = \phi' - \frac{2\lambda}{\tilde{\Lambda}^2}\psi$ turns this into

$$\mathcal{L} = \frac{1}{2}\phi'\Box\phi' - 2\frac{\lambda^2}{\tilde{\Lambda}^4}\psi\Box\psi - \frac{\lambda}{\tilde{\Lambda}^2}\psi^2 \tag{1.35}$$

and canonical normalization demonstrates that (1.33) describes a healthy mode and a ghost of mass $\tilde{\Lambda}/\sqrt{\lambda}$ (which may also be tachyonic, depending on the sign of λ). Typically, higher derivative theories either display this pathology in the fundamental action or these instabilities can arise for perturbations about non-trivial background solutions and we wish to avoid these scenarios.

From the higher-dimensional viewpoint of DGP, the π field is the brane-bending mode—the Goldstone field associated with spontaneously broken five-dimensional Poincaré invariance, as we will see later. The broken symmetries manifest themselves as non-linearly realized symmetries for π, a type of "galilean" shift

$$\pi(x) \longrightarrow \pi(x) + c + b_\mu x^\mu \,, \quad c,\, b_\mu = \text{ constant} \tag{1.36}$$

under which the cubic galileon and kinetic term shift by total derivatives.

One can abstract these two properties, that the theory be symmetric under (1.36) and have at most second order equations of motion, and search for all terms obeying these requirements. Such theories describe the "galileon" degree of freedom $\pi(x)$ and it was in [91] where this line of research was first carried out (for a review of later developments, see [106]). This class of models, and its generalizations, was found to be quite interesting in its own right, as we shall see.

1.4.1 Definition and Fundamentals

A fundamental characteristic of galileon theories is that there are only a finite number of interactions which are both shift symmetric and have second order equations of motion. The issue is the second requirement; the first is easily satisfied by considering terms with many derivatives per π field.

The number of desired interactions is highly dimension dependent and in d-dimensions there exist $d+1$ satisfactory terms. There exists a concise method of writing down the relevant terms in terms of Levi-Civita symbols. Namely, in d-dimensions the interaction with $n+1$, $0 \leq n \leq d$, factors of π takes on the following form

$$\begin{aligned}\mathcal{L}_{n+1}^{(d)} &= \epsilon^{\mu_1 \ldots \mu_n \alpha_{n+1} \ldots \alpha_d} \epsilon_{\nu_1 \ldots \nu_n \alpha_{n+1} \ldots \alpha_d} \partial_{\mu_1} \pi \partial_{\nu_1} \pi \partial_{\mu_2} \partial_{\nu_2} \pi \ldots \partial_{\mu_n} \partial_{\nu_n} \pi \\ &\cong -\epsilon^{\mu_1 \ldots \mu_n \alpha_{n+1} \ldots \alpha_d} \epsilon_{\nu_1 \ldots \nu_n \alpha_{n+1} \ldots \alpha_d} \pi \partial_{\mu_1} \partial_{\nu_1} \pi \partial_{\mu_2} \partial_{\nu_2} \pi \ldots \partial_{\mu_n} \partial_{\nu_n} \pi \end{aligned} \tag{1.37}$$

where, here and elsewhere, $\cong$ signifies equivalence up to total derivatives. For instance the cubic term in $d=4$ arises from

$$\begin{aligned}\mathcal{L}_3^{(4)} &= \epsilon^{\mu_1 \mu_2 \alpha \beta} \epsilon^{\nu_1 \nu_2}{}_{\alpha \beta} \partial_{\mu_1} \pi \partial_{\nu_1} \pi \partial_{\mu_1} \partial_{\nu_2} \pi \\ &= -2 \left(\eta^{\mu_1 \nu_1} \eta^{\mu_2 \nu_2} - \eta^{\mu_1 \nu_2} \eta^{\mu_2 \nu_1} \right) \partial_{\mu_1} \pi \partial_{\nu_1} \pi \partial_{\mu_1} \partial_{\nu_2} \pi \\ &\cong -\frac{3}{2} \Box \pi (\partial \pi)^2 \ . \end{aligned} \tag{1.38}$$

The use of the Levi-Civita symbol helps to illuminate many of the interesting properties of galileons. The key is to note that whenever two μ_i or two ν_i derivatives act on a π the term vanishes due to the $\mu_i \leftrightarrow \mu_j$ and $\nu_i \leftrightarrow \nu_j$ anti-symmetry of the ϵ tensors.

First, their invariance under (1.36) is easily seen. Under the shift we have $\delta \partial_\mu \pi = b_\mu$ and $\delta \partial_\mu \partial_\nu \pi$ and therefore $\mathcal{L}_{n+1}^{(d)}$ changes by a total derivative

$$\begin{aligned}\delta \mathcal{L}_{n+1}^{(d)} &= 2 \epsilon^{\mu_1 \ldots \mu_n \alpha_{n+1} \ldots \alpha_d} \epsilon_{\nu_1 \ldots \nu_n \alpha_{n+1} \ldots \alpha_d} b_{\mu_1} \partial_{\nu_1} \pi \partial_{\mu_2} \partial_{\nu_2} \pi \ldots \partial_{\mu_n} \partial_{\nu_n} \pi \\ &= 2 \partial_{\nu_1} \left[\epsilon^{\mu_1 \ldots \mu_n \alpha_{n+1} \ldots \alpha_d} \epsilon_{\nu_1 \ldots \nu_n \alpha_{n+1} \ldots \alpha_d} b_{\mu_1} \pi \partial_{\mu_2} \partial_{\nu_2} \pi \ldots \partial_{\mu_n} \partial_{\nu_n} \pi \right] \ . \end{aligned} \tag{1.39}$$

The property that the equations of motion stemming from (1.37) are second order also follows rather simply. We get that $\mathcal{E}_{n+1}^{(d)} \equiv \frac{\delta \mathcal{L}_{n+1}^{(d)}}{\delta \pi}$ is

$$\begin{aligned}
\mathcal{E}_{n+1}^{(d)} &= -(n+1)\epsilon^{\mu_1\ldots\mu_n\alpha_{n+1}\ldots\alpha_d}\epsilon_{\nu_1\ldots\nu_n\alpha_{n+1}\ldots\alpha_d}\partial_{\mu_1}\partial_{\nu_1}\pi\partial_{\mu_2}\partial_{\nu_2}\pi\ldots\partial_{\mu_n}\partial_{\nu_n}\pi \\
&= \partial_\mu\left[-(n+1)\epsilon^{\mu\ldots\mu_n\alpha_{n+1}\ldots\alpha_d}\epsilon_{\nu_1\ldots\nu_n\alpha_{n+1}\ldots\alpha_d}\partial_{\nu_1}\pi\partial_{\mu_2}\partial_{\nu_2}\pi\ldots\partial_{\mu_n}\partial_{\nu_n}\pi\right] \\
&\equiv \partial_\mu \mathcal{J}_{n+1}^{\mu(d)} \,. \qquad (1.40)
\end{aligned}$$

Therefore we see that the equations are second order, as claimed, and further that the equations of motion take the form of the divergence of a current, a fact that will be useful later.

We also note that the construction in (1.37) demonstrates the finite number of terms in the action which fit out criteria. Namely, in d-dimensions once we try to create an interaction with $d+2$ π's we find that we've run out of indices in the Levi-Civita symbol with which to contract and so there can only be $d+1$ interactions of the form (1.37). All other interactions which obey (1.36) will be built from polynomials of $\partial^{2+k}\pi$, $k \geq 0$ and will inevitably lead to higher derivative equations of motion. The fact that an interaction of the class (1.37) with n π's will only have $2n-2$ derivatives while all other symmetric interactions with n π's will have at least $2n$ derivatives (think of a term like $\sim \partial^m(\partial^2\pi)^k$) is of crucial importance and lies at the core of what makes the galileons interesting.

Finally, as we are primarily interested in $d=4$ we provide here explicit expressions for the five relevant lagrangians. After many integrations by parts and an overall rescaling of the lagrangians, they can be brought into the form [71]

$$\begin{aligned}
\mathcal{L}_1 &= -\frac{1}{2}\pi \\
\mathcal{L}_2 &= -\frac{1}{2}(\partial\pi)^2 \\
\mathcal{L}_3 &= -\frac{1}{2}(\partial\pi)^2\,[\Pi] \\
\mathcal{L}_4 &= -\frac{1}{2}(\partial\pi)^2\left([\Pi]^2 - [\Pi^2]\right)
\end{aligned}$$

$$\mathcal{L}_5 = -\frac{1}{2}(\partial\pi)^2 \left([\Pi]^2 - 3\,[\Pi]\left[\Pi^2\right] + 2\left[\Pi^3\right]\right) \tag{1.41}$$

where the bracketed trace notation is defined in Appendix A.

1.5 Galileon Field Theory

Much of the interest in galileons is due to their attractive field-theoretic properties. The fact that they have fewer derivatives than other terms invariant under the shift symmetry makes it possible to find regimes in which the galileons can be consistently treated as the only important interactions [90]. Furthermore, around sources, galileon theories exhibit the Vainshtein screening mechanism [44, 108] at short distances from the source, potentially allowing them to evade fifth force constraints, such as those provided by measurements within the solar system. Finally, the galileon terms are not renormalized to any loop order in perturbation theory [73, 82], which allows them to be treated classically for certain purposes. We cover these properties in the following sections.

1.5.1 The Non-Renormalization Theorem

Here we review the well-known non-renormalization theorem obeyed by the galileons. As shown in [73], there exists a beautiful diagrammatic proof of the fact that loops generated from interactions invariant under (1.36) will only generate interactions with at least two derivatives per external leg and which are therefore not of the form (1.37). In other words, loops do not renormalize the interactions (1.37). We provide a sketch of the argument now.

Consider a generic 1PI diagram with n external legs which would contribute to quantum effective action. In particular, focus on a vertex within the diagram at which m legs meet, at least one of which is an external line. Every power of external momenta generated by the diagram corresponds to a derivative operator in the term we are generating. If we can demonstrate that to each external line at this generic vertex the Feynman rules associate at least two powers of external momentum, then we have demonstrated that the term generated by the Feynman diagram is not of the type (1.37), as these have *fewer* than two derivatives per π.

We can trivially cover the case where the vertex is generated by terms symmetric under (1.36) but not of the form (1.37) since, as discussed, these terms will always have at least two derivatives per π. For such vertices, the π associated to the external leg under consideration will always have at least two derivatives acting on it and will lead to at least two powers of the momenta along this external leg.

When the interaction is of the type (1.37), schematically denoted here by $\epsilon\epsilon\pi(\partial^2\pi)^{m-1}$ we need to examine the situation in greater detail. Let $m_{\rm ext}$ of the legs at the vertex under consideration be external legs and the remaining $m_{\rm int} = m - m_{\rm ext}$ be internal. The Feynman rule for this vertex comes from assigning the π's in $\epsilon\epsilon\pi(\partial^2\pi)^{m-1}$ to the $m_{\rm ext}$ and $m_{\rm int}$ legs in all possible combinations. In some of these combinations, all of the π's with two derivatives will be assigned to external legs, which we denote by $\epsilon\epsilon\pi_{\rm int}(\partial^2\pi_{\rm int})^{m_{\rm int}-1}(\partial^2\pi_{\rm ext})^{m_{\rm ext}}$, and therefore this contraction associates two powers of external momentum to each external line.

The slightly more subtle case is when a bare π in the interaction is assigned to the external leg, denoted as $\epsilon\epsilon\pi_{\rm ext}(\partial^2\pi_{\rm ext})^{m_{\rm ext}-1}(\partial^2\pi_{\rm int})^{m_{\rm int}}$. At first glance, it might appear that there are then only $2(m_{\rm ext}-1)$ powers of external momenta generated in this case, corresponding to the $(\partial^2\pi_{\rm ext})^{m_{\rm ext}-1}$ factor, but this is not the case. Due to the $\epsilon\epsilon$ tensor structure of (1.37), the final $(\partial^2\pi_{\rm int})^{m_{\rm int}}$ can be written as a double total derivative, i.e. $\epsilon\epsilon\pi_{\rm ext}(\partial^2\pi_{\rm ext})^{m_{\rm ext}-1}\left[\partial^2\left(\pi_{\rm int}(\partial^2\pi_{\rm int})^{m_{\rm int}-2}\right)\right]$. Therefore, each of the two outside derivatives in $\partial^2\left(\pi_{\rm int}(\partial^2\pi_{\rm int})^{m_{\rm int}-2}\right)$ will hit every factor of $\pi_{\rm int}$ once and will contribute a factor of the sum of all momenta associated to internal lines, schematically $\partial_\mu \to \sum_{\rm i=1}^{m_{\rm int}} p^i_\mu$. Since momenta is conserved at each vertex we can trade the sum of all internal momenta for the sum of all external momenta and this brings the total up to $2m_{\rm ext}$ powers of the external momenta for this contraction. Therefore, we see that only higher order interactions of the schematic form $\sim \partial^k(\partial^2\pi)^j$ are generated and no terms of the form (1.37), proving the theorem.

1.5.2 Coupling to Massive Sources

1.5.2.1 The Vainshtein Mechanism: Generalities

One of the most interesting facets of galileon theories is that they provide an explicit, natural realization of the Vainshtein screening mechanism [108] which allows the galileon field to mediate a force between a massive object and a far away target, but not a nearby one. The effect was first proposed as a possible solution to the vDVZ discontinuity of massive gravity [110, 121] in which GR is not recovered as the limit $m \to 0$ is taken in Fierz-Pauli massive gravity. The assumption is that the continuity could be restored if there existed a host of non-linear interactions which also became important in this $m \to 0$ limit and which would restore the expected behavior. The effect fundamentally follows from the non-linear interactions and the general idea is well captured by the following sketch.

Consider a canonical scalar field ϕ which couples to the trace of the stress tensor and has some set of complicated, derivative self-interactions, $V(\phi, \partial\phi)$,

$$\mathcal{L} = -\frac{1}{2}(\partial\phi)^2 + V(\phi, \partial\phi) + \frac{\phi T}{M} \ . \tag{1.42}$$

Taking the source of T to be a point mass of mass m, at far distances we assume that $V(\phi, \partial\phi)$ can be neglected and ϕ acquires the familiar profile $\phi_0 \sim \frac{m}{M}\frac{1}{r}$, leading to a $1/r$ potential for far away test particles. The Vainshtein screening mechanism relies on the assumption that there exists a distance $r_{\rm nl}$ below which $V(\phi, \partial\phi)$ becomes large and can no longer be ignored. If we expand about the background profile for ϕ in this regime, say $\phi = \phi_0 + \delta\phi$, then the quadratic lagrangian for fluctuations takes on the form

$$\mathcal{L} = \frac{1}{2}(\partial\delta\phi)^2 - \frac{1}{2}Z_{\mu\nu}(\phi_0, \partial\phi_0)\partial_\mu\delta\phi\partial_\nu\delta\phi + \frac{\delta\phi T}{M} \ , \tag{1.43}$$

for some function $Z_{\mu\nu}(\phi_0, \partial\phi_0)$ which depends on the background profile. Making the simplifying assumption that $Z_{\mu\nu}$ takes on the form $Z_{\mu\nu} = Z\eta_{\mu\nu}$, we find that if the interactions are such that $Z \gg 1$ then in the region at hand the coupling of $\delta\phi$ to matter is highly suppressed. In order to see this explicitly, we canonically normalize the quadratic lagrangian by letting $\delta\phi \equiv \delta\hat{\phi}/\sqrt{1+Z} \approx \delta\hat{\phi}/\sqrt{Z}$ we get

$$\mathcal{L} \approx -\frac{1}{2}(\partial\delta\hat{\phi})^2 + \frac{\delta\hat{\phi}T}{\sqrt{Z}M} \ . \tag{1.44}$$

Therefore, in the region $r \gg r_{\rm nl}$ the coupling between fluctuations of the field and matter is $\frac{1}{M}$, but in the non-linear region $r \ll r_{\rm nl}$ they are coupled with strength $\frac{1}{M\sqrt{Z}} \ll \frac{1}{M}$, by assumption, and so the force mediated by ϕ is highly screened for $r \ll r_{\rm nl}$.

The Vainshtein mechanism is a lovely concept, but it turns out to be quite hard to realize the idea in practice without running into major obstacles. First, it is difficult to generate a model which displays the non-linear effect in a regime where one can consistently neglect quantum corrections. Typically, quantum effects become important at the same distance scale (or at even larger distances) than classical non-linear effects do and hence one cannot calculate with control in the Vainshtein region. Next, even if quantum effects are ignored, the non-linear nature of the mechanism makes explicit, analytic calculations difficult. Almost the only non-trivial case that can be solved in an analytic manner is that of a single, pointlike mass source. Finally, since we are relying on non-linear, derivative operators there is also the constant worry that such terms will lead to ghost instability or superluminal propagation.

1.5.2.2 The Vainshtein Mechanisms: Specifics

In this section we will explore in greater detail various attempts to realize the Vainshtein mechanism by using scalar fields. We will see the various difficulties which are inherent in the construction of such theories. These failures will provide a good illustration of why galileon theories are special and ideally suited for realizations of Vainshtein screening. Many of the concepts and ideas in what follow draw from the arguments of [6, 71].

Ideal Non-linearities: General Relativity

Before analyzing scalar models, it is useful to first consider general relativity (with zero cosmological constant) as it shares many features which are analogous to the ones we wish to capture in our own model. Namely, at long distances from a source GR is well approximated by a linear model which takes on the schematic form

$$\mathcal{L} \sim (\partial h)^2 + \frac{hT}{M_{\rm pl}} \ . \tag{1.45}$$

The field h stands for the metric perturbation about flat space and we ignore all indices here as we will only require general ideas. Far away from the source, taken to be pointlike and of mass $M \gg M_{\rm pl}$, the gravitational potential takes on the familiar Newtonian form which we approximate as $h \sim \frac{M}{M_{\rm pl}}\frac{1}{r}$. However, as we approach the Schwarzschild radius of the source, non-linear GR effects become important, of course. Due to diffeomorphism symmetry, the gravitational self-interactions are packaged into the Ricci scalar $R[g]$ and the full non-linear lagrangian takes on the rough form

$$\mathcal{L} \sim (\partial h)^2 + \sum_i c_i \left(\frac{h}{M_{\rm pl}}\right)^i (\partial h)^2 + \frac{hT}{M_{\rm pl}} \ , \tag{1.46}$$

with $c_i \sim \mathcal{O}(1)$. We can estimate where the non-linear terms become important by evaluating the higher order terms in (1.46) on the background $h \sim \frac{M}{M_{\rm pl}}\frac{1}{r}$ and determining when they become the same size as the quadratic term. Inspection shows that this occurs when $\frac{h}{M_{\rm pl}} \sim \mathcal{O}(1)$ and hence the problem is fully non-linear around $r_{\rm nl} \sim \frac{M}{M_{\rm pl}^2}$ which is nothing but the usual Schwarzschild radius. Therefore, this simple estimate accurately determines when the non-linear corrections need to be taken into account.

One may worry that quantum corrections to (1.46) could affect the conclusion, but it is simple enough to demonstrate that this is not the case. In effect, the potential problem is that in writing down (1.46) we are ignoring a host of unknown operators which are expected to arise from quantum effects and a priori these operators could dominate the physics of the regime under consideration. We need to justify our implicit assumption that these interactions are negligible.

Quantum generated terms are still diffeomorphism invariant and hence also come packaged in the schematic form $\sim \nabla^m R^n$. In terms of h, this means that a typical correction to (1.46) will be of the form

$$\sim \partial^m h^l (\partial h)^{2n} / M_{\rm pl}^{l+m+4n-4}\,, n > 0\ .$$

It is of crucial importance that this term has more derivatives than the Ricci scalar does. Comparing this to the quadratic term $(\partial h)^2$, an analysis similar to that of the previous paragraph demonstrates that this non-linear term becomes important at the distance scale

$$r_{\rm q} \sim M_{\rm pl}^{-1} \left(\frac{M}{M_{\rm pl}} \right)^{\frac{2n+m-2}{4n+l+m-4}} . \tag{1.47}$$

This should be concerning since in the $m \to \infty$ limit $r_{\rm q}$ approaches its maximum which is the size of the Schwarzschild radius, $\lim_{m\to\infty} r_q \sim \frac{M}{M_{\rm pl}^2}$ and so it naively would appear that this is the distance scale where quantum effects are becoming important. If indeed the quantum generated terms turned out to be just as important as the Ricci scalar term at this distance, then we could not trust the usual Schwarzschild metric solution as we would be dropping terms which should not be neglected. Of course, this does not turn out to be the case.

Since both the non-linear Ricci interactions and the quantum corrections are naively becoming important at the same scale, the next level of analysis is to compare the two directly. The expansion of Ricci entails an infinite series in h with only two derivatives appearing in each term, i.e. $\sim \partial^2 h^m / M_{\rm pl}^{m-2}$, while the quantum corrections also have arbitrary powers of h but involve strictly more factors of derivatives, i.e. $\sim \partial^{2+k} h^m / M_{\rm pl}^{m+k-2}$. Therefore the quantum corrections are suppressed with respect to the non-linear Ricci term by factors of $\frac{\partial}{M_{\rm pl}}$ and hence we can ignore them until the distance scale $r_{\rm pl} \sim M_{\rm pl}^{-1}$. Said another way, gravitational quantum corrections only becomes important at the Planck scale. In summary, both the non-linear classical interaction and the quantum correction start to dominate over the quadratic term around $r_{\rm nl}$, but the non-linear classical terms dominate the quantum corrections until $r_{\rm pl} \ll r_{\rm nl}$ and thus we are justified in ignoring the infinite number of quantum generated operators as long as we are only interested in distance scales larger than the Planck length.

Scalar Vainshtein Screening: Initial Failures

Now we attempt to create a realization of Vainshtein screening using scalar fields. The goals are similar to those we had for GR. That is, we wish to construct a theory of ϕ in which the physics is linear at distances larger than a non-linear distance scale $r_{\rm nl}$ ($r_{\rm nl}$ is alternatively known as the Vainshtein scale), dominated by non-linear classical effects in a regime $r_q \ll r \ll r_{\rm nl}$ and only dominated by quantum effects for $r \ll r_q$, with a wide separation between r_q and $r_{\rm nl}$.

Start with the simplest possible attempt, a scalar field ϕ with cubic interaction $\phi(\partial\phi)^2/\Lambda$ which couples to the trace of matter with gravitational strength,

$$\mathcal{L} = -\frac{1}{2}(\partial\phi)^2 + \frac{\lambda}{\Lambda}\phi(\partial\phi)^2 + \frac{\phi T}{M_{\rm pl}} \ , \tag{1.48}$$

where λ is an $\mathcal{O}(1)$ parameter. We follow our analysis of GR and find that when we evaluate the cubic term on the long distance background $\phi \sim \frac{M}{M_{\rm pl}}\frac{1}{r}$ this interaction becomes comparable to the quadratic term at a distance $r_{\rm nl} \sim \Lambda^{-1}\frac{M}{M_{\rm pl}}$. In the asymptotic regimes, the field behaves as

$$\phi \sim \begin{cases} \frac{M}{M_{\rm pl}}\frac{1}{r} & r \gg r_{\rm nl} \\ \frac{M}{M_{\rm pl}}\frac{1}{r}\sqrt{\frac{r}{r_{\rm nl}}} & r \ll r_{\rm nl} \end{cases} \tag{1.49}$$

and so if this result holds, then the potential felt by a test particle in the Vainshtein region $r \ll r_{\rm nl}$ is suppressed relative to the usual r^{-1} potential by a factor of roughly $\sqrt{r/r_{\rm nl}} \ll 1$.

However, the conclusion is found to be inaccurate once quantum corrections are taken into account. The scalar sector of (1.48) has no symmetries and the action should be supplemented by an infinite tower of unknown quantum corrections whose form is unrestricted. A generated interaction of the form $\sim \partial^m\phi^n/\Lambda^{m+n-4}$, $n>2$ becomes comparable to the quadratic term at the scale $r_{\rm q}^{(n,m)} \sim \Lambda^{-1}\left(\frac{M}{M{\rm pl}}\right)^{\frac{n-2}{m+n-2}}$ and therefore terms with $m=0$ also become important at the distance $r_{\rm q}^{(n,0)} \sim \Lambda^{-1}\frac{M}{M_{\rm pl}} \sim r_{\rm nl}$. In particular, one expects a generated term of the form $\sim \Lambda\phi^3$ and comparing this to the cubic term in (1.48) we see that the quantum effect dominates by a factor of $\Lambda^2/\partial^2 \gg 1$ (for $r \gg \lambda^{-1}$). Therefore, in the non-linear region $r \ll r_{\rm nl}$, unknown quantum corrections dominate the physics, rendering the classical conclusion (1.49) invalid.

Since the main problem of the theory (1.48) was that quantum corrections produced interactions with fewer derivatives per ϕ, relative to the original cubic interaction, a natural next step is to consider a theory whose classical interactions are solely built from $\partial\phi$. The scalar sector of this theory will respect a $\phi \to \phi + c$, $c =$ constant symmetry which ensures that quantum corrections are also symmetric under this shift and therefore built from $\partial^n\phi$, $n>0$ and are thus not expected to dominate with the same Λ/∂ enhancement that was found in (1.48).

In order to be specific, consider the theory

$$\mathcal{L} = -\frac{1}{2}(\partial\phi)^2 + \frac{\lambda}{\Lambda^4}(\partial\phi)^4 + \frac{\phi T}{M_{\rm pl}} \ . \tag{1.50}$$

A repetition of our previous analyses demonstrates that the theory becomes non-linear at $r_{\rm nl} \sim \Lambda^{-1}\sqrt{\frac{M}{M_{\rm pl}}}$ and a generic quantum generated interaction of the form $\partial^m(\partial\phi)^n/\Lambda^{2n+m-4}$, $n>2$ becomes comparable to the quadratic term at the distance scale

$$r_q^{(m,n)} \sim \Lambda^{-1} \times \left(\frac{M}{M_{\rm pl}}\right)^{\frac{n-2}{m+2n-4}} \ . \tag{1.51}$$

As $n\to\infty$, the quantum scale asymptotes to the classical scale, $r_q^{(m,n)} \sim r_{\rm nl}$ and we see that an infinite number of operators are becoming important at $r_{\rm nl}$. We therefore need to compare the size of the classical and quantum interactions in this limit. In particular, examining quantum corrections of the form $(\partial\phi)^n/\Lambda^{2n-4}$, we see that there are many important quantum operators which will not generically be enhanced or suppressed relative to the classical quartic term in (1.50) and thus must be taken into account, nullifying any conclusions reached with the purely classical lagrangian (1.50).

Ideally, one could add interactions to (1.50) which have fewer than one derivative per field and still obey the shift symmetry $\phi \to \phi + c$. Such terms would be the analogue of the Ricci scalar in GR and if this were possible, then these classical terms would dominate over quantum corrections by factors of Λ/∂ and improve the validity of their use. Unfortunately, such terms do not exist; there are no interactions with fewer derivatives per field which are also symmetric under the shift symmetry. Adding any terms with fewer derivatives will generically lead us back to the scenario of (1.48) and the problems found within. However, as we will see in the next section, the desired story will hold true when we consider a theory with even more derivatives and the galileons will play the special role of terms which respect all symmetries and have fewer derivatives per field than generic interactions do.

Scalar Vainshtein Screening: A Galileon Improvement

We will find that things are quite different when we consider theories which interact via terms with more than one derivative per field. We motivate such theories by positing a shift symmetry $\phi \to \phi + c + b_\mu x^\mu$, with c, b_μ constants. Naively, the only interactions which obey the symmetry will be of the form $\sim \partial^m(\partial^2\phi)^n/\Lambda^{3n+m-4}$, but a theory constructed from such interactions will suffer from similar problems to (1.48) and (1.50). To be precise, if we consider the theory[5]

$$\mathcal{L} = -\frac{1}{2}(\partial\phi)^2 + \frac{\lambda}{\Lambda^5}(\partial^2\phi)^3 + \frac{\phi T}{M_{\rm pl}} \ . \tag{1.52}$$

[5]This theory naturally arises in the Stückelberg analysis of Fierz-Pauli massive gravity, see Sec. 1.6.2, as was masterfully demonstrated in [6]. As mentioned, much of our analysis follows their lead.

The derived Vainshtein scale is $r_{\rm nl} \sim \Lambda^{-1}\left(\frac{M}{M_{\rm pl}}\right)^{1/5}$ and a quantum generated interaction of the form $\sim \partial^m(\partial^2\phi)^n/\Lambda^{3n+m-4}$ will be become important at the scale $r_q^{(m,n)} \sim \Lambda^{-1}\left(\frac{M}{M_{\rm pl}}\right)^{\frac{n-2}{3n+m-4}}$. Here we see that the problem is even worse than it was in (1.48) or (1.50). As $n \to \infty$ we find that $r_q \to\sim \Lambda^{-1}\left(\frac{M}{M_{\rm p}}\right)^{1/3} \gg r_{\rm nl}$ and hence there exist quantum corrections to (1.52) which dominate the physics long *before* the classical cubic interaction becomes relevant.

This problem was already presaged by the higher derivative interactions [71]. In particular, if we were to expand about the classical background $\phi_0 \sim \frac{M}{M_{\rm pl}}\frac{1}{r}$ then the interaction term in (1.52) would lead to a higher derivative kinetic term for the fluctuation $\delta\phi = \phi - \phi_0$ of the form $\sim \frac{\partial^2\phi_0}{\Lambda^5}(\partial^2\delta\phi)^2$, corresponding to a ghost of mass $m^2_{\rm ghost} \sim \frac{\Lambda^5}{\partial^2\phi_0}$. Treating (1.52) as an effective field theory with cutoff Λ, the theory is only then valid when $m_{\rm ghost} > \Lambda$. One then finds that at distances shorter than $r_{\rm ghost} \sim \Lambda^{-1}\left(\frac{M}{M_{\rm pl}}\right)^{1/3}$ the ghost mass drops below the cutoff and the theory can no longer be trusted.

Galileon interactions now come to the rescue. Rather than solely building interactions from $\partial^2\phi$ we have the five galileon galileon interactions (1.41) at our disposal. In particular, consider the cubic theory

$$\mathcal{L} = -\frac{1}{2}(\partial\phi)^2 + \frac{\lambda}{\Lambda^3}\Box\phi(\partial\phi)^2 + \frac{\phi T}{M_{\rm pl}}\ . \tag{1.53}$$

Now the classical non-linear scale is found to be $r_{\rm nl} \sim \Lambda^{-1}\left(\frac{M}{M_{\rm pl}}\right)^{1/3}$. Due to the non-renormalization theorem in Sec.1.5.1, we know that generic quantum corrections are still of the form generated from (1.52), i.e. $\sim \partial^m(\partial^2\phi)^n/\Lambda^{3n+m-4}$, and so the quantum distance scale is still at most $r_q \sim \Lambda^{-1}\left(\frac{M}{M_{\rm pl}}\right)^{1/3}$. However the crucial point is that the cubic galileon has fewer derivatives per field and hence dominates over a generic quantum correction due to the factor of Λ/∂. Much like general relativity then, a classical theory of galileons (1.52) is linear above the scale $r_{\rm nl} \sim \Lambda^{-1}\left(\frac{M}{M_{\rm pl}}\right)^{1/3}$, dominated by classical non-linearities in the distance range $\Lambda^{-1} \lesssim r \lesssim \Lambda^{-1}\left(\frac{M}{M_{\rm pl}}\right)^{1/3}$ and swamped by quantum corrections only below $r \lesssim \Lambda^{-1}$. Qualitatively similar results hold when any of the other galileon interactions are included in the classical action. Therefore, we see that galileons are special in that they provide the simplest example of a classical Vainshtein mechanism whose classical interactions will also be generically enhanced in size relative to quantum corrections.

There is no need to consider extending this program to even higher derivative mechanisms. That is, the next natural step would be to consider scalar field theories symmetric under, $\phi \to \phi + c + b_\mu x^\mu + a_{\mu\nu} x^\mu x^\nu$, but such theories will built from even higher derivative order building blocks and will inevitably be ghostly. Therefore, the galileons represent the unique, natural way to implement the Vainshtein mechanism using scalars in a Lorentz invariant, ghost-free theory.

While in the case of galileons we have found a similar conclusion to the one we saw in GR, there is one important difference. Because the classical GR action contained an infinite series of terms $\sim h^n(\partial h)^2/M_{\rm pl}^n$, we were always able to compare a quantum generated interaction with, say, m powers of h to a classical interaction which also employs m powers of h. Then for every quantum interaction, there was a corresponding classical interaction which dominated by factors of Λ/∂. Because there are only a finite number of galileon interactions, $d+1$ in d-dimensions, the analogue does not hold. Instead, in say the example of (1.52), one will have to compare a classical interaction with 3 powers of the field to quantum terms with n powers of ϕ. As n increases, the domination of the cubic term diminishes and to retain theoretic control optimistic assumptions regarding the UV completion of the theory [90] must be made. Nevertheless, the galileon represent a concrete improvement over generic attempts to construct a viable Vainshtein screening mechanism.

1.5.2.3 The Cubic Galileon: Cutoffs, Scales and Superluminality

In this section, we further explore the details of the cubic galileon model (1.52). Though this is a truncation of the most general galileon model it will already capture the relevant effects we wish to demonstrate. Restoring the proper form of all index contractions and using the canonical galileon field $\pi(x)$, we study the lagrangian

$$\mathcal{L} = -\frac{1}{2}(\partial\pi)^2 - \frac{\lambda}{\Lambda^3}\Box\pi(\partial\pi)^2 + \frac{\pi T}{M_{\rm pl}} \tag{1.54}$$

with $T = -M\delta^3(\vec{r}) = -M\vec{\nabla}\cdot\frac{\hat{r}}{4\pi r^2}$. Because the galileon equations of motion take the form of a divergence equation (1.40) the system defined by (1.54) admits a first integral and becomes

$$\frac{\pi'}{r} + \frac{4\lambda}{\Lambda^3}\left(\frac{\pi'}{r}\right)^2 = \frac{M}{M_{\rm pl}}\frac{1}{4\pi r^3}\ . \tag{1.55}$$

Because the right hand side is strictly positive, the left hand must be, too, which leads to the requirement $\lambda > 0$. An examination of (1.54) shows that this is clearly the appropriate sign for obtaining a Vainshtein mechanism as this choice is needed for the cubic term to combine correctly with the quadratic kinetic term when expanded about an r-dependent solution. In the asymptotic regimes, the π profile becomes

$$\pi(r) = \begin{cases} -\frac{M}{M_{\rm pl}}\frac{1}{4\pi r} & r \gg r_{\rm nl} \\ -\frac{1}{\sqrt{64\pi\lambda}}\frac{M}{M_{\rm pl}}\frac{1}{r}\times\left(\frac{r}{r_{\rm nl}}\right)^{3/2} & r \ll r_{\rm nl} \end{cases} \tag{1.56}$$

where $r_{\rm nl} \equiv \Lambda^{-1}\left(M/M_{\rm pl}\right)^{1/3}$ and we once again see the characteristic $r/r_{\rm nl}$ suppression in the Vainshtein regime.

Exploring perturbations about the non-trivial $\pi(r)$ configuration demonstrates that with $\lambda > 0$ the system (1.54) is free from ghost and gradient instabilities, but also reveals a problem with radial modes: they propagate superluminally. Specifically, the speed of sound at large distances from the source behaves as

$$c_r^2 \approx 1 + \frac{8\lambda}{\Lambda^3}\frac{\pi'}{r} + \mathcal{O}\left(\left(\frac{\pi'}{r}\right)^2\right) \tag{1.57}$$

which is superluminal given our requirement $\lambda > 0$ and observation $\pi'/r > 0$.

The presence of superluminalities is generic. As demonstrated in [91], a general galileon theory coupled to a pointlike mass will exhibit the same type of superluminal perturbations. More precisely, requiring the existence of stable, spherical solutions one finds that angular excitations remain subluminal, but far from the source these two requirements force radial perturbations to become superluminal. Another demonstration of this general fact will be seen in Sec.3.5 for the case of a closely related model. Whether the presence of superluminality is truly a physical inconsistency of galileon theories is still a matter of active research [18, 32, 38].

Finally, in order to get a sense of the scales involved and levels of Vainshtein suppression, we consider the effect of the cubic theory applied to the solar system. The typical natural scale for Λ is $\Lambda^{-1} \sim 1000\text{km}$, or equivalently $\Lambda \sim 10^{-13}\text{eV}$, as derived from requiring that the DGP model only modify gravity on distances comparable to the Hubble length, i.e. $r_c = \frac{M_{\rm pl}^2}{M_5^3} \sim H^{-1}$. Then considering the sun to be the source of the galileon field, one finds that the corresponding Vainshtein radius below which the galileon field is screened is $r_{\rm nl}^{\odot} \sim 1000\text{km} \times \left(\frac{M_\odot}{M_{\rm pl}}\right) \sim 200\text{pc}$ while the size of the solar system is only $r_{\text{Solar System}} \sim 10^{-4}\text{pc}$ and hence the Earth lies well within the screened region. As seen from (1.56), within the Vainshtein regime the fifth force potential is suppressed by a factor of $\sim (r/r_{\rm nl})^{3/2}$ which comes out to $\sim 10^{-12}$ evaluated at the Earth's orbital radius and hence the galileon force on the Earth due to the Sun would be hugely suppressed.

1.6 Galileons And Other Modifications of Gravity

Galileons appear in other, non-braneworld modifications of gravity and we cover two such scenarios here.

1.6.1 Kinetically Mixed Galileons

In the original work which defined, classified and generalized galileons [91], the authors divorced the scalar from its higher dimensional origins and instead considered a scenario where π is kinetically mixed with gravity via a πR coupling. We assume that in this frame π has self interactions, but no direct coupling to the matter lagrangian, $\mathcal{L}_M$. We can return to Einstein frame by performing a field redefinition (more specifically a Weyl transformation) on the metric perturbation $h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$ via $h_{\mu\nu} \equiv \hat{h}_{\mu\nu} + 2\pi\eta_{\mu\nu}$. To quadratic order in $\hat{h}$, the redefined action takes on the form [91]

$$\mathcal{L} = \frac{M_{\rm pl}^2}{2}\sqrt{-\hat{g}}\hat{R} + \frac{1}{2}\hat{h}_{\mu\nu}T^{\mu\nu} + \mathcal{L}_\pi + \pi T^\mu{}_\mu \ , \tag{1.58}$$

where $\mathcal{L}_\pi$ contains the $d = 4$ galileon interactions of (1.41) and $T_{\mu\nu}$ is the stress tensor derived only from $\mathcal{L}_M$.

A key feature of this model is that it can lead to an accelerating universe even if $\mathcal{L}_M = 0$. The idea is that even no matter the π self-interactions can support non-trivial $\pi(x)$ profiles and these generate a non-trivial $h_{\mu\nu}$ since $h_{\mu\nu} = \hat{h}_{\mu\nu} + 2\pi(x)\eta_{\mu\nu}$. In particular, one can write the 4D de Sitter metric as [91]

$$\mathrm{d}s^2 = \left(1 - H^2 x_\mu x^\mu\right)\eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu \tag{1.59}$$

and hence if $\mathcal{L}_\pi$ can generate a $\pi(x) \propto x_\mu x^\mu$ configuration then this modification of gravity can lead to self-acceleration. Indeed this is the case for appropriate choices of $\mathcal{L}_\pi$. Further, this profile is Lorentz invariant and it can be shown that the equations of motion for perturbations about the profile also obey galileon equations of motion and hence perturbations on top of the self-accelerating solution enjoy all the benefits and suffer the same pathologies of galileons that we encountered in Sec.1.5.2.

1.6.2 Massive Gravity

A perhaps more natural setting in which galileons play a prominent role is in dRGT theory of massive gravity [36, 37]. First, we briefly describe the quadratic Fierz-Pauli theory and its failings before covering how dRGT heals the issues of Fierz-Pauli and the manner in which galileon interactions arise. Both [6] and [71] do a masterful job of illuminating the physics behind massive gravity and we follow both references extensively.

1.6.2.1 Fierz-Pauli Massive Gravity

Consider building a simple quadratic theory of massive gravity. In terms of the metric perturbation $h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$, quadratic general relativity takes on the form

$$S = \int \mathrm{d}^4x \, \frac{1}{2} h_{\mu\nu} \mathcal{E}^{\mu\nu,\alpha\beta} h_{\alpha\beta} + h_{\mu\nu} T^{\mu\nu} \equiv \int \mathrm{d}^4x \, \mathcal{L}_{\mathrm{GR}}(h_{\mu\nu}) + h_{\mu\nu} T^{\mu\nu} \tag{1.60}$$

where $\mathcal{E}^{\mu\nu,\alpha\beta}$ is the quadratic differential operator which stems from the expanding out the Ricci scalar whose exact form we will not need. If we were to attempt to add a mass term to (1.60), we would add it in some combination $ah^2 + bh_{\mu\nu}h^{\mu\nu}$ where $h = h^{\mu}{}_{\mu}$ and all indices are raised and lowered with $\eta_{\mu\nu}$. However, a priori there is no obvious reasoning for choosing particular values of a or b.

Fierz and Pauli first demonstrated that the only stable combination combination is $\propto h^2 - h_{\mu\nu}^2$ [58] and [6] presented a wonderfully clear explanation for why this is the proper choice and we reproduce the argument here. The GR lagrangian (1.60) enjoys linearized diffeomorphism symmetry, $h_{\mu\nu} \to h_{\mu\nu} + \partial_{(\mu}\xi_{\nu)}$, but any mass term $\mathcal{L}_m(h_{\mu\nu})$ we add will generically ruin this. A useful trick is to reintroduce diffeomorphism invariance to the massive theory by introducing Stückelbeg fields which restore gauge invariance at the cost of introducing more fields into the theory. Though more fields are added, the total number of degrees of freedom are unchanged as the restored gauge invariance simultaneously removes degrees of freedom.

One patterns the introduction of Stückelberg fields after the gauge transformation. That is, everywhere in $\mathcal{L}_{\rm GR}(h_{\mu\nu})+\mathcal{L}_m(h_{\mu\nu})$ we make the replacement $h_{\mu\nu}\to h_{\mu\nu}+\partial_{(\mu}A_{\nu)}$. Since $\mathcal{L}_{\rm GR}$ descends from a curvature invariant, the factors of A_μ in $\mathcal{L}_{\rm GR}(h_{\mu\nu}+\partial_{(\mu}A_{\nu)})$ will disappear, but $\mathcal{L}_m(h_{\mu\nu}+\partial_{(\mu}A_{\nu)})$ will depend on A_μ. The theory will now contain two different fields $h_{\mu\nu}$ and A_μ, but also gains the gauge symmetry

$$h_{\mu\nu}\to h_{\mu\nu}+\partial_{(\mu}\xi_{\nu)}\,,\quad A_\mu\to A_\mu-\xi_\mu\ . \tag{1.61}$$

Similarly, it is fruitful to make the replacement $A_\mu\to A_\mu+\partial_\mu\pi$. A new field $\pi(x)$ is added, but again we gain a $U(1)$ gauge invariance

$$h_{\mu\nu}\to h_{\mu\nu}+\partial_\mu\partial_\nu\phi\,,\quad A_\mu\to A_\mu-\partial_\mu\phi\,,\quad \pi\to\pi+\phi\ . \tag{1.62}$$

We now explore the physics contained within $\mathcal{L}_m(h_{\mu\nu}+\partial_{(\mu}A_{\nu)}+\partial_\mu\partial_\nu\pi)$. The physics in the Stückelberg language is equivalent to the physics we started with and so any pathologies of the Stückelberged theory also afflict the original theory $\mathcal{L}_{\rm GR}(h_{\mu\nu})+\mathcal{L}_m(h_{\mu\nu})$.

In particular, consider our generic mass term from before, $\mathcal{L}_m\equiv ah^2+bh_{\mu\nu}^2$. Making the Stückelberg replacement, we find that $\mathcal{L}_m$ contains

$$\mathcal{L}_m\supset ah\Box\pi+bh_{\mu\nu}\partial^\mu\partial^\nu\pi+a(\Box\pi)^2+b\partial_\mu\partial_\nu\pi\partial^\mu\partial^\nu\pi\ . \tag{1.63}$$

Integrating by parts, we find that $\mathcal{L}_m\supset(a+b)(\Box\pi)^2$ which corresponds to a ghost in the theory of mass $m^2_{\rm ghost}\sim\frac{1}{(a+b)^2}$. Removing the ghost entirely corresponds to taking $a=-b$ and we see that this enforces the mass term to be of the Fierz-Pauli form $\mathcal{L}_m\propto h^2-h_{\mu\nu}^2$.

In order to determine the overall sign, we need to look at the vectors. Focusing on the quadratic vector terms that arise from the Stückelberg replacement in $\mathcal{L}_m=a(h^2-h_{\mu\nu}^2)$, we get

$$\mathcal{L}_m\supset a\left((\partial_\mu A^\mu)^2-(\partial_\mu A_\nu)^2\right)=-\frac{1}{2}aF_{\mu\nu}^2 \tag{1.64}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the usual field strength tensor. In order to avoid ghost instabilities for the vectors we then must have $a > 0$. Finally, the dispersion relation for $h_{\mu\nu}$ determines that we must normalize to $a = \frac{1}{2}M_{\rm pl}^2 m^2$ if the graviton is to have mass m. Therefore, the basic requirement that our theory be free from ghosts determines that the mass term take on the Fierz-Pauli form $\mathcal{L}_m = \frac{1}{2}M_{\rm pl}^2 m^2(h^2 - h_{\mu\nu})^2$.

We've figured out the correct mass term, but additionally (1.63) exhibits kinetic mixing, which ought be removed. Performing integrations by parts we have that $\mathcal{L}_m \supset \frac{M_{\rm pl}^2 m^2}{2}\left(\Box h - \partial_\mu\partial_\nu h^{\mu\nu}\right)\pi$ and we can diagonalize by defining $h_{\mu\nu} = h'_{\mu\nu} + \frac{m^2}{4}\pi\eta_{\mu\nu}$ (a Weyl rescaling). In particular, we have the relation [71]

$$\mathcal{L}_{\rm GR}(h_{\mu\nu}) \cong \mathcal{L}_{\rm GR}(h'_{\mu\nu}) - \frac{M_{\rm pl}^2 m^2}{2}\pi\left(\Box h' - \partial_\mu\partial_\nu h'^{\mu\nu}\right) + \frac{3}{16}M_{\rm pl}^2 m^4(\partial\pi)^2 \ . \tag{1.65}$$

and so in terms of $h'_{\mu\nu}$ and π we end up with

$$\mathcal{L} \supset \mathcal{L}_{\rm GR}(h'_{\mu\nu}) - \frac{3}{16}M_{\rm pl}^2 m^4(\partial\pi)^2 + \frac{m^2}{4}\pi T^\mu{}_\mu \ . \tag{1.66}$$

Importantly, the field redefinition has generated a coupling between the scalar and the trace of matter. The relative powers of m appearing in the kinetic term for π and in π's coupling to matter are what underly the vDVZ discontinuity in which the limit $m \to 0$ fails to recover general relativity. To be precise, when we canonically normalize by sending $\pi \sim \hat{\pi}/M_{\rm pl}m^2$ we see that m drops out entirely, i.e. schematically $\mathcal{L} \sim -(\partial\hat{\pi})^2 + \hat{\pi}T/M_{\rm pl}$ and so the coupling of the scalar to matter persists even as $m \to 0$, which is the root of the problem.

Finally we can consider higher order interactions that arise from the Fierz-Pauli mass term. Before we were working with linearized diffeomorphisms to make the Stückelberg replacement, but in order to find the interesting interactions arising from the mass term we need to use the full diffeomorphisms. That is, we insert the vector Stückelbergs via $g_{\mu\nu} \to \partial_\mu(x^\alpha + A^\alpha/2)\partial_\nu(x^\beta + A^\beta/2)g_{\mu\nu}$ and then insert the scalar via $A_\mu \to A_\mu + \partial_\mu\pi$, as before. For our purposes we are primarily interested in the scalar sector and focusing on these, the procedure amounts to making the replacement

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \partial_\nu \pi + \frac{1}{4} \partial_\mu \partial_\alpha \pi \partial^\alpha \partial_\nu \pi \tag{1.67}$$

everywhere in the Fierz-Pauli mass term.

Given the discussions in Sec. 1.5.2, the problems now become clear. Schematically, the Fierz-Pauli term gives rise to π self couplings of the form

$$\mathcal{L}_m \sim M_{\rm pl}^2 m^2 \left[(\partial^2 \pi)^3 + (\partial^2 \pi)^4 \right] \tag{1.68}$$

which when canonically normalized as $\hat{\pi} \sim M_{\rm pl} m^2 \pi$ read

$$\mathcal{L}_m \sim \frac{1}{\Lambda_5^5} (\partial^2 \pi)^3 + \frac{1}{\Lambda_4^8} (\partial^2 \pi)^4 \tag{1.69}$$

where we define the set of scales $\Lambda_n \equiv \left(M_{\rm pl} m^{n-1} \right)^{1/n}$ which are monotonically decreasing with n. For applications to our current universe, the graviton mass should be of order the Hubble scale, $m \sim H \sim 10^{-42}$GeV, which would make $\Lambda_5 \sim 10^{-30}$Gev and $\Lambda_3 \sim 10^{-22}$GeV. Note that Λ_3 is the same order as the strong coupling scale that arises in DGP (1.32).

We then recognize that we are back to the situation found in Sec. 1.5.2 where we saw that classically interactions of the form (1.69) appear to provide a realization of the Vainshtein mechanism, which in this context would provide a potential solution for the vDVZ discontinuity, but in reality the calculation is swamped by unknown quantum corrections and we cannot actually calculate within the non-linear regime with any control.

1.6.2.2 Galileons and dRGT

Since the fatal flaw we found in the Fierz-Pauli theory was the presence of scalar self-interactions built from $\partial^2 \pi$, we can attempt to do the natural thing and cure the theory by removing these terms. Specifically, we can add higher order interactions of $h_{\mu\nu}$ in specific combinations such that when we Stückelberg, all of the pure scalar self-interactions become total derivatives and can be dropped.

de Rham, Gabadadze and Tolley were the first to successfully carry out this program and demonstrated that the infinite number of $h_{\mu\nu}$ interactions required can further be nicely resummed into a closed form [36, 37]. The construction itself is quite interesting, but for the purposes of this brief survey of the uses of galileons we only need the schematic picture of what the resulting lagrangian looks like. In terms of canonically normalized fields $\hat{h}_{\mu\nu}$ and $\hat{\pi}$, the surviving relevant, lowest order terms in the dRGT lagrangian are schematically [71]

$$\mathcal{L} \supset \frac{1}{2}\hat{h}_{\mu\nu}\mathcal{E}^{\mu\nu,\alpha\beta}\hat{h}_{\alpha\beta} + \hat{h}^{\mu\nu}\left[X^{(1)}_{\mu\nu}(\hat{\pi}) + \frac{1}{\Lambda_3^3}X^{(2)}_{\mu\nu}(\hat{\pi}) + \frac{1}{\Lambda_3^6}X^{(3)}_{\mu\nu}(\hat{\pi})\right] + h_{\mu\nu}T^{\mu\nu} \tag{1.70}$$

where the $X^{(i)}_{\mu\nu}(\hat{\pi})$ are of the form

$$\begin{aligned} X^{(1)}_{\mu\nu}(\hat{\pi}) &\sim \epsilon_{\mu}{}^{\alpha_1\rho\sigma}\epsilon_{\nu}{}^{\beta_1}{}_{\rho\sigma}\partial_{\alpha_1}\partial_{\beta_1}\hat{\pi} \\ X^{(2)}_{\mu\nu}(\hat{\pi}) &\sim \epsilon_{\mu}{}^{\alpha_1\alpha_2\rho}\epsilon_{\nu}{}^{\beta_1\beta_2}{}_{\rho}\partial_{\alpha_1}\partial_{\beta_1}\hat{\pi}\partial_{\alpha_2}\partial_{\beta_2}\hat{\pi} \\ X^{(3)}_{\mu\nu}(\hat{\pi}) &\sim \epsilon_{\mu}{}^{\alpha_1\alpha_2\alpha_3}\epsilon_{\nu}{}^{\beta_1\beta_2\beta_3}\partial_{\alpha_1}\partial_{\beta_1}\hat{\pi}\partial_{\alpha_2}\partial_{\beta_2}\hat{\pi}\partial_{\alpha_3}\partial_{\beta_3}\hat{\pi}\ . \end{aligned} \tag{1.71}$$

Note that the smallest strong coupling scale is now Λ_3 instead of Λ_5; the dRGT construction removes interactions which have strong coupling scale Λ_n with $n > 3$, thereby raising the cutoff of the theory. The $X^{(i)}_{\mu\nu}(\hat{\pi})$ are conserved in each index, i.e. $\partial^\mu X^{(i)}_{\mu\nu}(\hat{\pi}) = 0$ and $\partial^\nu X^{(i)}_{\mu\nu}(\hat{\pi}) = 0$, and more importantly they are related to the galileon lagrangians (1.37) $\mathcal{L}^{(4)}_{i+1}(\hat{\pi})$ by $\mathcal{L}^{(4)}_{i+1}(\hat{\pi}) \propto \hat{\pi}\eta^{\mu\nu}X^{(i)}_{\mu\nu}(\hat{\pi})$.

The lagrangian (1.70) has not yet had its kinetic mixing removed and hence when we send $\hat{h}_{\mu\nu} \to h'_{\mu\nu} + \hat{\pi}\eta_{\mu\nu}$ the dRGT action will contain, among other interactions, a $\hat{\pi}T$ coupling and galileon self-interactions. As discussed in Sec. 1.5.2, the galileon interactions represent the terms necessary to have regimes in which classical non-linearities which are important and also not swamped by quantum corrections and hence represent the natural candidate for solving the problem of the vDVZ discontinuity. The presence of galileon interactions in massive gravity could potentially have been expected. A massive graviton has five degrees of freedom, one of which corresponds to the longitudinal, scalar mode whose associated force must be screened in the regime close to massive sources and for this screening mechanism to be healthy and reliable, it must use the galileons, as seen in Sec. 1.5.2.

1.7 Generalized Galileons

The development of the original galileons inspired a myriad of generalizations. Each generalization is interesting in its own right and we briefly cover some of the models here.

1.7.1 Multi-Galileon Theories

A natural extension to the galileons is the inclusion of multiple galileon fields. In particular, it's possible to construct SO(N) symmetric multi-galileon theories, where the fields π^I each have the shift symmetry (1.36) and also rotate in the fundamental representation of an internal SO(N) [73, 96]. In this case, in d dimensions there are $d/2$ possible galileon terms if d is even, and $(d+1)/2$ if d is odd and the terms only contain an even number of π^I's (thus, there is no tadpole). These are obtained by simply contracting indices with δ_{IJ},

$$\mathcal{L}_n \sim \delta_{I_1 J_1}\delta_{I_2 J_2}\cdots\delta_{I_{n/2}J_{n/2}}\eta^{\mu_1\nu_1\mu_2\nu_2\cdots\mu_{n-1}\nu_{n-1}} \tag{1.72}$$
$$\times\left(\pi^{I_1}\partial_{\mu_1}\partial_{\nu_1}\pi^{J_1}\partial_{\mu_2}\partial_{\nu_2}\pi^{I_2}\partial_{\mu_3}\partial_{\nu_3}\pi^{J_2}\cdots\partial_{\mu_{n-2}}\partial_{\nu_{n-2}}\pi^{I_{n/2}}\partial_{\mu_{n-1}}\partial_{\nu_{n-1}}\pi^{J_{n/2}}\right). \tag{1.73}$$

In Sec.6.8 we will explore these theories from an algebraic perspective and demonstrate the uniqueness of (1.73).

1.7.2 Conformal Galileons

The original galileons non-linearly realize a version of Galilean symmetry and a natural extension is to develop theories which non-linearly realize other symmetry groups. One such possibility is to non-linearly realize the conformal group, $SO(3,2)$. The relevant theory, known as the conformal galileons [91], can be built from the effective metric $e^{2\pi}\eta_{\mu\nu}$ and its derived curvature invariants, for instance $\sqrt{-g} = e^{4\pi}$ or $\sqrt{-g}R \cong 6e^{2\pi}(\partial\pi)^2$. Any theory thus constructed will enjoy the non-linear symmetries [91]

$$\pi(x^\mu) \to \pi(\lambda x^\mu) + \ln\lambda\,,\ \pi(x^\mu) \to \pi\left(x^\mu + b^\mu x^2 - 2b_\nu x^\nu x^\mu\right) - 2b_\nu x^\nu\ . \tag{1.74}$$

An interesting facet of $d = 4$ is that there is only a single independent interaction stemming from terms quadratic in the Riemann curvature. That is after, integrating by parts, the interactions generated by $\sqrt{-g}R^2$, $\sqrt{-g}R^2_{\mu\nu}$ and $\sqrt{-g}R^2_{\mu\nu\rho\sigma}$ with $g = e^{2\pi}\eta$ are all proportional to one another. In generic dimensions, two linearly independent combinations can be formed from the quadratic curvatures, but the presence of the $d = 4$ Gauss-Bonnet term forces one of these to be a total derivative only for this precise dimension. Working in d dimensions and taking a $d \to 4$ limit at the end, [91] demonstrated that it is possible to recover a form of the lost, independent interaction $\sim (\partial\pi)^4 + 2(\partial\pi)^2\Box\pi$ which transforms by a total derivative under (1.74). In Sec.6.11 we will provide an algebraic analysis of the conformal galileons and discuss in what sense this additional $d = 4$ interaction is special.

1.7.3 DBI Galileons

A different symmetry pattern that can be non-linearly realized by scalar fields is simply the 5D Poincaré group, $ISO(4,1)$. As the galileons non-linearly realize a Galilean group, the generalization to $ISO(4,1)$ can be viewed a relativistic extension. First derived in [39], the "DBI galileon" $\pi(x)$ non-linearly realizes the symmetries

$$\pi(x) \to \pi(x) + v_\nu x^\nu + \pi(x)v^\nu\partial_\nu\pi(x) \tag{1.75}$$

and, similar to the case of the conformal galileon, its interactions are derived from the effective metric $\eta_{\mu\nu} + \partial_\mu\pi(x)\partial_\nu\pi(x) \equiv g_{\mu\nu}$ and derived volume elements and curvatures, for instance $\sqrt{-g} = \sqrt{1 + (\partial\pi)^2}$.

However, different from the case of the conformal galileon, $g_{\mu\nu}$ has a natural interpretation as the metric induced on a Minkowski 3-brane living in an ambient $5D$ Minkowski spacetime. Given this interpretation, one can additionally use the extrinsic brane curvature $K_{\mu\nu}$ to generate actions. All actions built from $\{g_{\mu\nu}, K_{\mu\nu}, R_{\mu\nu\rho\sigma}\}$ which lead to second order equations of motion for $\pi(x)$ were classified in [39] and were found to be in one-to-one correspondence with the Lovelock terms and the corresponding Gibbons-Hawking-York boundary terms. We will cover this geometric perspective in detail in Sec.3.3.1 and develop the more general framework which encompasses all probe brane generalizations of galileons throughout Chapter 2.

1.7.4 Covariant Galileons

A final generalization concerns the coupling of galileons to gravity. As shown in [42], the minimal coupling of galileons via $\partial \to \nabla$ introduces the unwelcome feature of ghosts. Specifically, in $d = 4$ the equations of motion for the fourth and fifth order galileon interactions leads to derivatives of the Riemann tensor and hence generate exactly the type of higher order equations of motion that galileons were constructed to avoid. Non-minimal couplings can be included which precisely cancel off the offending higher order pieces, see [42] for the full expressions, but once this is done all traces of the galileon symmetry are gone. In Sec.3.2.2, we demonstrate how these non-minimal couplings naturally arise from geometric constructions of galileon theories.

Part II

Galileons and Geometry

Chapter 2

General Construction of Probe Brane Galileons[6]

2.1 Overview

In this chapter, we present the general procedure for constructing four-dimensional effective field theories generated through the description of a 3-brane probing a higher dimensional bulk. As we will see, the galileons and their generalizations are special cases of this construction. This extends the construction of [39] to its most general form. We observe that the symmetries inherited by scalar fields in the 4D theory descend from isometries of the bulk metric. The precise manner in which the symmetries are realized is determined by the choice of gauge, or foliation, against which brane fluctuations are measured. Actions are built from curvature invariants, but only judicious choices will avoid higher derivative equations of motion and hence ghost instabilities. We discuss the limited set of terms which avoid this potential catastrophe. The result of the chapter is a highly general prescription for deriving 4D actions for scalar fields which enjoy many non-linearly realized symmetries

[6]The work in this chapter was performed in collaboration with Kurt Hinterbichler and Mark Trodden.

and utilize intricate derivative interactions while carefully avoiding ghost instabilities. In subsequent chapters, we apply the prescription first to the scenario in which the bulk space and brane ground state are both maximally symmetric and later to the case where the bulk is flat and brane state is a Friedmann-Robertson-Walker (FRW) spacetime.

2.2 General Brane Actions and Symmetries

We begin with a completely general case - the theory of a dynamical 3-brane moving in a fixed but arbitrary (4+1)-dimensional background. The dynamical variables are the brane embedding $X^A(x)$, five functions of the world-volume coordinates x^μ.

The bulk has a fixed background metric $G_{AB}(X)$. From this and the X^A, we may construct the induced metric $\bar{g}_{\mu\nu}(x)$ and the extrinsic curvature $K_{\mu\nu}(x)$, via

$$\bar{g}_{\mu\nu} = e^A_{\ \mu} e^B_{\ \nu} G_{AB}(X), \tag{2.1}$$

$$K_{\mu\nu} = e^A_{\ \mu} e^B_{\ \nu} \nabla_A n_B \ . \tag{2.2}$$

Here $e^A_{\ \mu} = \frac{\partial X^A}{\partial x^\mu}$ are the tangent vectors to the brane, and n^A is the normal vector, defined uniquely (up to a sign) by the properties that it is orthogonal to the tangent vectors $e^A_{\ \mu} n^B G_{AB} = 0$, and normalized to unity $n^A n^B G_{AB} = 1$. (Note that the extrinsic curvature can be written $K_{\mu\nu} = e^B_{\ \nu}\partial_\mu n_B - e^A_{\ \mu} e^B_{\ \nu}\Gamma^C_{AB} n_C$, demonstrating that it depends only on quantities defined directly on the brane and their tangential derivatives.)

We require the world-volume action to be gauge invariant under reparametrizations of the brane,

$$\delta_g X^A = \xi^\mu \partial_\mu X^A \ , \tag{2.3}$$

where $\xi^\mu(x)$ is the gauge parameter. This requires that the action be written as a diffeomorphism scalar, F, of $\bar{g}_{\mu\nu}$ and $K_{\mu\nu}$ as well as the covariant derivative $\bar{\nabla}_\mu$ and curvature

$\bar{R}^{\alpha}{}_{\beta\mu\nu}$ constructed from $\bar{g}_{\mu\nu}$,

$$S = \int \mathrm{d}^4x\ \sqrt{-\bar{g}}\, F\left(\bar{g}_{\mu\nu}, \bar{\nabla}_{\mu}, \bar{R}^{\alpha}{}_{\beta\mu\nu}, K_{\mu\nu}\right)\ . \tag{2.4}$$

This action will have global symmetries only if the bulk metric has Killing symmetries. If the bulk metric has a Killing vector $K^A(X)$, i.e. a vector satisfying the Killing equation

$$K^C\partial_C G_{AB} + \partial_A K^C G_{CB} + \partial_B K^C G_{AC} = 0\ , \tag{2.5}$$

then the action will have the following global symmetry under which the X^A shift,

$$\delta_K X^A = K^A(X)\ . \tag{2.6}$$

It is straightforward to see that the induced metric and extrinsic curvature, and hence the action (2.4), are invariant under (2.6).

We are interested in creating non-gauge theories with global symmetries from the transverse fluctuations of the brane, so we now fix all the gauge symmetry of the action. We accomplish this by first choosing a foliation of the bulk by time-like slices. We then choose bulk coordinates such that the foliation is given by the surfaces $X^5 =$ constant. The remaining coordinates X^μ can be chosen arbitrarily and parametrize the leaves of the foliation. The gauge we choose is

$$X^\mu(x) = x^\mu, \quad X^5(x) \equiv \pi(x)\ . \tag{2.7}$$

In this gauge, the world-volume coordinates of the brane are fixed to the bulk coordinates of the foliation. We call the remaining unfixed coordinate $\pi(x)$, which measures the transverse position of the brane relative to the foliation (see Figure 2.1). This completely fixes the gauge freedom. The resulting gauge fixed action is then an action solely for π,

$$S = \int \mathrm{d}^4x\ \ \sqrt{-\bar{g}}\, F\left(\bar{g}_{\mu\nu}, \bar{\nabla}_{\mu}, \bar{R}^{\alpha}{}_{\beta\mu\nu}, K_{\mu\nu}\right)\Big|_{X^\mu = x^\mu,\ X^5 = \pi}\ . \tag{2.8}$$

Note that the only invariant data that go into constructing a brane theory are the background metric and the action. Theories with the same background metric and the same action are isomorphic, regardless of the choice of foliation (which is merely a choice of gauge).

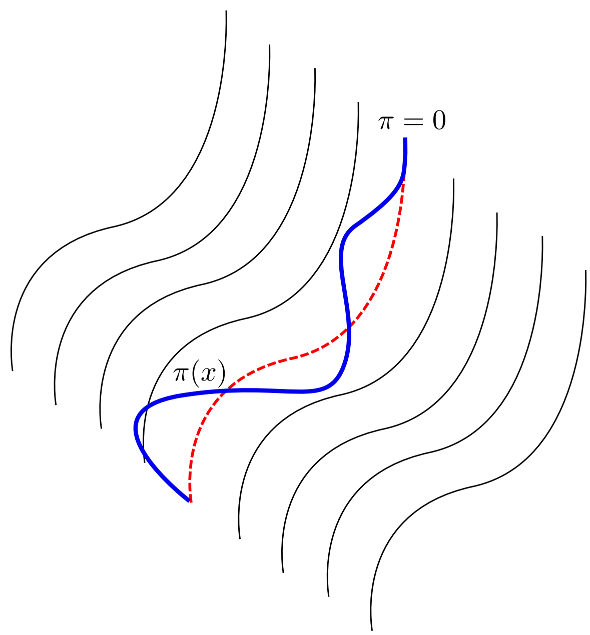

Figure 2.1: The field π measures the brane position with respect to some chosen foliation. Thin, black lines represent leaves of the foliation. The red, dotted line represents the ground state in which $\pi(x) = 0$, i.e. the leaf is unperturbed. The solid blue line represents a possible generic configuration, $\pi(x)$.

Global symmetries are physical symmetries that cannot be altered by the unphysical act of gauge fixing. Thus, if the original action (2.4) possesses a global symmetry (2.6), generated by a Killing vector K^A, then the gauge fixed action (2.8) must also have this symmetry. However, the form of the symmetry will be different because the gauge choice will not generally be preserved by the global symmetry. The change induced by K^A is

$$\delta_K x^\mu = K^\mu(x,\pi), \quad \delta_K \pi = K^5(x,\pi) \ . \tag{2.9}$$

To re-fix the gauge to (2.7), it is necessary to simultaneously perform a compensating gauge transformation with gauge parameter

$$\xi^\mu_{\rm comp} = -K^\mu(x,\pi) \ . \tag{2.10}$$

The combined symmetry acting on π,

$$(\delta_K + \delta_{g,\rm comp})\pi = -K^\mu(x,\pi)\partial_\mu \pi + K^5(x,\pi) \ , \tag{2.11}$$

is then a symmetry of the gauge fixed action (2.8).

2.2.1 Induced Metrics and Extrinsic Curvatures

We now specialize to the cases which we will need to consider for later chapter. First, we simply derive the induced metric and extrinsic curvature for the case where the bulk metric is written in Gaussian normal form,

$$G_{AB}\mathrm{d}x^A dX^B = f_{\mu\nu}(x,\rho)\mathrm{d}x^\mu \mathrm{d}x^\nu + \mathrm{d}\rho^2 \ . \tag{2.12}$$

Here $X^5 = \rho$ denotes the Gaussian normal transverse coordinate and the leaves of the foliation are defined by $\rho =$ constant. Recall that in the physical gauge (2.7), the transverse coordinate of the brane is set equal to the scalar field, $\rho(x) = \pi(x)$. After this, we further specialize to the case where the bulk metric is Gaussian normal and the extrinsic curvature of each leaf of the foliation is proportional to the induced metric on that leaf,

$$G_{AB}\mathrm{d}X^A\mathrm{d}X^B = \mathrm{d}\rho^2 + f(\rho)^2 g_{\mu\nu}(x)\mathrm{d}x^\mu\mathrm{d}x^\nu \ . \tag{2.13}$$

The former case will be required for the galileon theory derived on an FRW space, while the latter can cover every maximally symmetric space.

2.2.1.1 Gaussian Normal

We now proceed with the calculation of the induced metric and extrinsic curvature for the Gaussian normal metric (2.12). The induced metric is

$$\bar{g}_{\mu\nu} = f_{\mu\nu} + \partial_\mu \pi \partial_\nu \pi, \tag{2.14}$$

and its inverse is

$$\bar{g}^{\mu\nu} = f^{\mu\nu} - \tilde{\gamma}^2 \partial^\mu \pi \partial^\nu \pi \ , \tag{2.15}$$

where

$$\tilde{\gamma} \equiv 1/\sqrt{1 + (\partial \pi)^2} \ , \tag{2.16}$$

and the indices on the derivatives are raised with $f^{\mu\nu}$, the inverse of $f_{\mu\nu}$.

To calculate the extrinsic curvature we need to find the normal vector n^A, which satisfies

$$\begin{aligned} n^A e^B_\nu G_{AB} &= 0 \ , \\ n^A n^B G_{AB} &= 1 \ , \end{aligned} \tag{2.17}$$

where $e^B_\nu = \frac{\partial X^B}{\partial x^\nu}$ are the tangent vectors to the brane. Solving these equations in the gauge (2.7) where

$$e^A_{\ \mu} = \frac{\partial X^A}{\partial x^\mu} = \begin{cases} \delta^\nu_\mu & A = \nu \\ \nabla_\mu \pi & A = 5 \end{cases} \ . \tag{2.18}$$

yields

$$n_A = \tilde{\gamma}(-\partial_\mu \pi, 1). \tag{2.19}$$

The extrinsic curvature is given by

$$K_{\mu\nu} = e^A_\mu e^B_\nu \nabla_A n_B \ , \tag{2.20}$$

which can be written as $K_{\mu\nu} = e^B_\nu \partial_\mu n_B - e^A_\mu e^B_\nu \Gamma^C_{AB} n_C$.

The ∇_A is a covariant derivative of the bulk metric and so the Christoffel Γ^C_{AB} must be calculated with $X^5 = w$. The replacement $w \to \pi(x)$ is then made at the end of the calculation. Using the bulk coordinates in the form (4.4), the non-zero 5D Christoffels, Γ^A_{BC}, are

$$\begin{aligned}
\Gamma^\lambda_{\mu\nu} &= \Gamma^\lambda_{\mu\nu}(f), \\
\Gamma^5_{\mu\nu} &= -\frac{1}{2} f'_{\mu\nu}, \\
\Gamma^\mu_{5\nu} &= \frac{1}{2} f^{\mu\lambda} f'_{\lambda\nu} \ ,
\end{aligned} \tag{2.21}$$

where primes denote derivatives with respect to π. Note that on the right-hand side of the first line, the Christoffels of $f_{\mu\nu}$ are to be calculated with the π dependence held fixed. The extrinsic curvature then reads

$$K_{\mu\nu} = -\tilde{\gamma} \nabla_\mu \nabla_\nu \pi + \frac{1}{2} \tilde{\gamma} f'_{\mu\nu} + \tilde{\gamma} \partial^\lambda \pi \partial_{(\mu} \pi f'_{\nu)\lambda} \ , \tag{2.22}$$

where ∇_μ is the covariant derivative calculated from $f_{\mu\nu}$ at fixed π.

2.2.1.2 Gaussian Normal and $K_{\mu\nu} \propto \bar{g}_{\mu\nu}$

We now specialize further to the case of the metric (2.13), where the foliation is Gaussian normal with respect to the metric G_{AB}, and the extrinsic curvature on each of the leaves of the foliation is proportional to the induced metric. While we could read off these results from the previous section, they would not be in the optimal form for later use.

Working in the gauge (2.7), the induced metric is

$$\bar{g}_{\mu\nu} = f(\pi)^2 g_{\mu\nu} + \nabla_\mu \pi \nabla_\nu \pi \ . \tag{2.23}$$

It is useful to define the quantity

$$\gamma = \frac{1}{\sqrt{1 + \frac{1}{f^2} (\nabla \pi)^2}} \tag{2.24}$$

where indices are raised and lowered with $g_{\mu\nu}$ and $g^{\mu\nu}$. the square root of the determinant and the inverse metric may then be expressed as

$$\sqrt{-\bar{g}} = \sqrt{-g} f^4 \sqrt{1 + \frac{1}{f^2}(\nabla\pi)^2} = \sqrt{-g} f^4 \frac{1}{\gamma}, \tag{2.25}$$

and

$$\bar{g}^{\mu\nu} = \frac{1}{f^2}\left(g^{\mu\nu} - \gamma^2 \frac{\nabla^\mu\pi\nabla^\nu\pi}{f^2}\right) . \tag{2.26}$$

Again, in our gauge (2.7) the tangent vectors are

$$e^A{}_\mu = \frac{\partial X^A}{\partial x^\mu} = \begin{cases} \delta^\nu_\mu & A = \nu \\ \nabla_\mu\pi & A = 5 \end{cases} \tag{2.27}$$

and to find the normal vector n^A we solve the two equations

$$0 = e^A{}_\mu n^B G_{AB} = f^2 n^\nu g_{\mu\nu} + n^5 \partial_\mu \pi, \tag{2.28}$$

$$1 = n^A n^B G_{AB} = \frac{1}{f^2} g^{\mu\nu}\partial_\mu\pi\partial_\nu\pi (n^5)^2 + (n^5)^2 \ , \tag{2.29}$$

to obtain

$$n^A = \begin{cases} -\frac{1}{f^2}\gamma\nabla^\mu\pi & A = \mu \\ \gamma & A = 5 \end{cases}, \qquad n_A = \begin{cases} -\gamma\nabla_\mu\pi & A = \mu \\ \gamma & A = 5 \end{cases} . \tag{2.30}$$

Using the non-vanishing Christoffel symbols $\Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu}(g)$, $\Gamma^5_{\mu\nu} = -ff' g_{\mu\nu}$, $\Gamma^\mu_{\nu 5} = \delta^\mu_\nu \frac{f'}{f}$, the extrinsic curvature is then

$$K_{\mu\nu} = \gamma\left(-\nabla_\mu\nabla_\nu\pi + ff' g_{\mu\nu} + 2\frac{f'}{f}\nabla_\mu\pi\nabla_\nu\pi\right) . \tag{2.31}$$

Note that when the 4D coordinates have dimensions of length, π has mass dimension -1 and f is dimensionless.

2.3 Actions with second order equations of motion

Up until now we have discussed the degrees of freedom and their symmetries, but it is the choice of action that defines the dynamics. A general choice for the function F in (2.8) will lead to scalar field equations for π which are higher than second order in derivatives. When this is the case, the scalar will generally propagate extra degrees of freedom which are ghost-like [40, 95]. The presence of such ghosts signifies that either the theory is unstable, or the cutoff must be lowered so as to exclude the ghosts. Neither of these options is particularly attractive, and so it is desirable to avoid ghosts altogether. It is the Galileon terms which are special because they lead to equations of at most second order. Furthermore, as mentioned in the introduction, there can exist regimes in which the Galileon terms dominate over all others, so we will be interested only in these terms.

A key insight of de Rham and Tolley [39] is that there are a finite number of actions of the type (2.8), the Lovelock terms and their boundary terms, that do in fact lead to second order equations for π and become the Galileon terms (Galileon-like terms can also be obtained from Lovelock terms via a Kaluza-Klein dimensional reduction rather than a brane embedding [109]). The possible extensions of Einstein gravity which remain second order are given by Lovelock terms [80]. These terms are specific combinations of powers of the Riemann tensor which are topological (i.e. total derivatives) in some specific home dimension, but in lower dimensions have the property that equations of motions derived from them are second order. (For a short summary of some properties of these terms, see Appendix B of [73].) The Lovelock terms come with boundary terms. It is well known that, when a brane is present, bulk gravity described by the Einstein-Hilbert Lagrangian should be supplemented by the Gibbons-Hawking-York boundary term [63, 120]

$$S = \int_M \mathrm{d}^5 X \sqrt{-G} R[G] \ + 2 \int \mathrm{d}^4 x \sqrt{-\bar{g}} K \ . \tag{2.32}$$

Similarly, Lovelock gravity in the bulk must be supplemented by brane terms which depend on the intrinsic and extrinsic curvature of the brane (the so-called Myers terms [86, 87]), which are needed in order to make the variational problem for the brane/bulk system well posed [52]. Of course we are not considering bulk gravity to be dynamical, but the point here is that these boundary terms also yield second order equations of motion for π in the construction leading to (2.8).

The prescription of [39] is then as follows: on the 4-dimensional brane, we may add the first two Lovelock terms, namely the cosmological constant term $\sim \sqrt{-\bar{g}}$ and the Einstein-Hilbert term $\sim \sqrt{-\bar{g}}R[\bar{g}]$. (The higher Lovelock terms are total derivatives in 4-dimensions.) We may also add the boundary term corresponding to a bulk Einstein-Hilbert term, $\sqrt{-\bar{g}}K$, and the boundary term $\mathcal{K}_{\rm GB}$ corresponding to the Gauss-Bonnet Lovelock invariant $R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$ in the bulk,

$$\mathcal{K}_{\rm GB} = -\frac{1}{3}K^3 + K^2_{\mu\nu}K - \frac{2}{3}K^3_{\mu\nu} - 2\left(\bar{R}_{\mu\nu} - \frac{1}{2}\bar{R}\bar{g}_{\mu\nu}\right)K^{\mu\nu} \, . \tag{2.33}$$

The zero order cosmological constant Lovelock term in the bulk has no boundary term (although as we will see, we may construct a fifth term, the tadpole term, from it) and the higher order bulk Lovelock terms vanish identically.

While the above discussion captures all of the desired terms with derivative interactions, there is one term that contains no derivatives of π and is not of the form (2.8). This Lagrangian is called the tadpole term, denoted by $\mathcal{A}(\pi)$. The value of the tadpole action is the proper 5-volume between some $\rho =$ constant surface and the position of the brane,

$$S_1 = \int {\rm d}^4x \int^{\pi} {\rm d}\pi' \sqrt{-G(x^\mu, \pi')} \, , \tag{2.34}$$

where $G(x^\mu, \pi') = \det G_{AB}(x^\mu, \pi')$. This tadpole term (2.34) obeys all of the non-linear symmetries of the theory. Under the π symmetry (2.11), the shift of the tadpole term is

$$\delta S_1 = \int {\rm d}^4x \, \sqrt{-G(x,\pi)} \left[K^5(x,\pi) - K^\mu(x,\pi)\partial_\mu\pi\right] , \tag{2.35}$$

where $G(x,\pi) \equiv \det G_{AB}(x,\pi)$. We will show that the integrand of (2.35) is a total derivative by showing that its Euler-Lagrange variation vanishes. Taking a general variation of the right hand side gives

$$\begin{aligned}
&\int \mathrm{d}^4x \quad \sqrt{-G(\pi,x)}\Big\{\frac{1}{2}\,G^{AB}\partial_\pi G_{AB}\,\delta\pi\,\left[K^5(x,\pi)-K^\mu(x,\pi)\partial_\mu\pi\right] \\
&\quad+\left[\partial_\pi K^5(x,\pi)\delta\pi-\partial_\pi K^\mu(x,\pi)\delta\pi\partial_\mu\pi-K^\mu(x,\pi)\partial_\mu\delta\pi\right]\Big\} \\
&=\int \mathrm{d}^4x\,\sqrt{-G(\pi,x)}\Big\{\frac{1}{2}\,G^{AB}\partial_\pi G_{AB}\left[K^5(x,\pi)-K^\mu(x,\pi)\partial_\mu\pi\right] \\
&\quad+\partial_\pi K^5(x,\pi)-\partial_\pi K^\mu(x,\pi)\partial_\mu\pi+\partial_\mu K^\mu(x,\pi) \\
&\quad+\partial_\pi K^\mu(x,\pi)\partial_\mu\pi+\frac{1}{2}K^\mu G^{AB}\partial_\mu G_{AB}+\frac{1}{2}K^\mu G^{AB}\partial_\pi G_{AB}\partial_\mu\pi\Big\}\delta\pi \\
&=\int \mathrm{d}^4x\,\frac{1}{2}\sqrt{-G(\pi,x)}\,\Big\{G^{AB}\partial_\pi G_{AB}K^5(x,\pi)+K^\mu G^{AB}\partial_\mu G_{AB} \\
&\quad+2\partial_\pi K^5(x,\pi)+2\partial_\mu K^\mu(x,\pi)\Big\}\delta\pi\ .
\end{aligned} \tag{2.36}$$

Contracting the Killing equation (2.5) with G^{AB} gives

$$G^{AB}\partial_5 G_{AB}K^5(x,\pi)+K^\mu G^{AB}\partial_\mu G_{AB}+2\partial_5 K^5(x,\pi)+2\partial_\mu K^\mu(x,\pi)=0\ , \tag{2.37}$$

and so (2.36) vanishes, indicating that (2.35) is a total derivative and that, as claimed, the tadpole obeys all of the required symmetries.

In summary, the five ghost free lagrangians which we will use to define our generalized galileon actions are

$$\begin{aligned}
\mathcal{L}_1 &= \int^\pi \mathrm{d}\pi'\sqrt{-G(x^\mu,\pi')} \\
\mathcal{L}_2 &= -\sqrt{-\bar{g}}\ , \\
\mathcal{L}_3 &= \sqrt{-\bar{g}}K\ , \\
\mathcal{L}_4 &= -\sqrt{-\bar{g}}\bar{R}\ , \\
\mathcal{L}_5 &= \frac{3}{2}\sqrt{-\bar{g}}\mathcal{K}_{\mathrm{GB}}\ .
\end{aligned} \tag{2.38}$$

Chapter 3

Maximally Symmetric Cases[7]

3.1 Overview

In this chapter we will apply our methods of the previous chapter to the case in which the 5D background metric has 15 global symmetries, i.e. the maximal number. Thus, the bulk is either 5D anti-de Sitter space AdS_5 with isometry algebra $SO(4,2)$, 5D de-Sitter space dS_5 with isometry algebra $SO(5,1)$, or flat 5D Minkowski space M_5 with isometry algebra the five dimensional Poincare algebra $ISO(4,1)$. In addition, we focus on the case where the brane metric $g_{\mu\nu}$, and hence the extrinsic curvature, are maximally symmetric, so that the unbroken subalgebra has the maximal number of generators, 10. This means that the leaves of the foliation are either 4D anti-de Sitter space AdS_4 with isometry algebra $SO(3,2)$, 4D de-Sitter space dS_4 with isometry algebra $SO(4,1)$, or flat 4D Minkowski space M_4 with isometry algebra the four dimensional Poincare algebra $ISO(3,1)$. In fact, there are only 6 such possible foliations of 5D maximally symmetric spaces by 4D maximally symmetric time-like slices, such that the metric takes the form (2.13). Flat M_5 can be foliated by flat M_4 slices or by dS_4 slices; dS_5 can be foliated by flat M_4 slices, dS_4 slices, or AdS_4

[7]The work in this chapter was performed in collaboration with Kurt Hinterbichler and Mark Trodden.

slices; and AdS_5 can only be foliated by AdS_4 slices. Each of these 6 foliations, through the construction leading to (2.8), will generate a class of theories living on an AdS_4, M_4 or dS_4 background and having 15 global symmetries broken to the 10 isometries of the brane. These possibilities are summarized in Figure 3.1.

Brane metric

Bulk metric

	AdS_4	M_4	dS_4
AdS_5	AdS DBI galileons $SO(4,2) \to SO(3,2)$ $f(\pi) = \mathcal{R}\cosh(\pi/\mathcal{R})$	Conformal DBI galileons $SO(4,2) \to ISO(3,1)$ $f(\pi) = e^{-\pi/\mathcal{R}}$	type III dS DBI galileons $SO(4,2) \to SO(4,1)$ $f(\pi) = \mathcal{R}\sinh(\pi/\mathcal{R})$
M_5		DBI galileons $ISO(4,1) \to ISO(3,1)$ $f(\pi) = 1$	Type II dS DBI galileons $ISO(4,1) \to SO(4,1)$ $f(\pi) = \pi$
dS_5			Type I DBI galileons $SO(5,1) \to SO(4,1)$ $f(\pi) = \mathcal{R}\sin(\pi/\mathcal{R})$

Figure 3.1: Types of maximally symmetric embedded brane effective field theories, their symmetry breaking patterns, and functions $f(\pi)$. The relationships to the Galileon and DBI theories are also noted.

It should be noted that the missing squares in Figure 3.1 may be filled in if we are willing to consider a bulk which has more than one time direction[8]. For example, it is possible to embed AdS_4 into a five-dimensional Minkowski space with two times (indeed, this is the

[8]We thank Sergei Dubovsky for pointing this out.

standard way of constructing AdS spaces). From the point of view that the bulk is physical, and hence should be thought of as dynamical, these possibilities may be unacceptable on physical grounds. However, if one thinks of the bulk as merely a mathematical device for constructing novel four-dimensional effective theories, then there is nothing a priori to rule out these possibilities. In this paper, we focus on those cases in which the bulk has only one time dimension. The construction in the other cases will, however, follow the same pattern.

We first exhaust all the maximally symmetric possibilities in Figure 3.1, providing explicit expressions for ghost free actions of Sec.2.3 and for the non-linear π symmetries. As we will see, these generalized galileon theories have their own unique properties. For example, in curved space the field acquires a potential which is fixed by the symmetries – something that is not allowed for the flat space galileons. In particular, the scalars acquire a mass of order the inverse radius of the background, and the value of the mass is fixed by the nonlinear symmetries. Although not addressed in detail here, allowing for de Sitter solutions on the brane opens up the possibility of adapting these new effective field theories to cosmological applications such as inflation or late time cosmic acceleration in such a way that their symmetries ensure technical naturalness.

In Sec.3.4 we take the small field limits to obtain Galileon-like theories, discuss their stability, and compare and contrast these theories with the special case of the original galileon. Finally, in Sec.3.5 we explore whether or not the DBI galileon theory, arising from a Minkowski probe brane in a Minkowski bulk, are free of the issues that plague the original galileons, namely the superluminal propagation of perturbations when the field is sourced by a heavy object. For conventions, refer to Appendix A.

3.2 General Construction

We now proceed to construct explicitly the maximally symmetric examples catalogued in Section 3 and Figure 3.1. Since the bulk metric can be put in the same form (2.13) in every case considered here, it is efficient derive the form of the π symmetries and lagrangians for the generic metric (2.13) and then simply plug in to get the explicit result for each of the cases in Figure 3.1.

3.2.1 Symmetries

The algebra of Killing vectors of G_{AB} contains a natural subalgebra consisting of the Killing vectors for which $K^5 = 0$. This is the subalgebra of Killing vectors that are parallel to the foliation of constant ρ surfaces, and it generates the subgroup of isometries which preserve the foliation. We choose a basis of this subalgebra and index the basis elements by i,

$$K_i^A(X) = \begin{cases} K_i^\mu(x) & A = \mu \\ 0 & A = 5 \end{cases} , \tag{3.1}$$

where we have written $K_i^\mu(x)$ for the $A = \mu$ components, indicating that these components are independent of ρ. To see that this is the case, note that, for those vectors with $K^5 = 0$, the $\mu 5$ Killing equations (2.5) tell us that $K_i^\mu(x)$ is independent of ρ. Furthermore, the $\mu\nu$ Killing equations tell us that $K_i^\mu(x)$ is a Killing vector of $g_{\mu\nu}$.

We now extend our basis of this subalgebra to a basis of the algebra of all Killing vectors by appending a suitably chosen set of linearly independent Killing vectors with non-vanishing K^5. We index these with I, so that (K_i, K_I) is a basis of the full algebra of Killing vectors. From the 55 component of Killing's equation, we see that K^5 must be independent of ρ, so we may write $K^5(x)$.

A general global symmetry transformation thus reads

$$\delta_K X^A = a^i K_i^A(X) + a^I K_I^A(X) , \tag{3.2}$$

where a^i and a^I are arbitrary constant coefficients of the transformation. In the gauge (2.7), the transformations become, from (2.11),

$$(\delta_K + \delta_{g,\text{comp}})\pi = -a^i K_i^\mu(x)\partial_\mu \pi + a^I K_I^5(x) - a^I K_I^\mu(x,\pi)\partial_\mu \pi . \tag{3.3}$$

From this, we see that the K_i symmetries are linearly realized, whereas the K_I are realized nonlinearly. Thus, the algebra of all Killing vectors is spontaneously broken to the subalgebra of Killing vectors preserving the foliation.

3.2.2 Explicit expressions for the terms

From (2.38), the five terms that lead to second order equations for π are

$$\begin{aligned}
\mathcal{L}_1 &= \sqrt{-g}\int^{\pi} \mathrm{d}\pi' f(\pi')^4, \\
\mathcal{L}_2 &= -\sqrt{-\bar{g}}\ , \\
\mathcal{L}_3 &= \sqrt{-\bar{g}}K\ , \\
\mathcal{L}_4 &= -\sqrt{-\bar{g}}\bar{R}\ , \\
\mathcal{L}_5 &= \frac{3}{2}\sqrt{-\bar{g}}\mathcal{K}_{\mathrm{GB}}\ ,
\end{aligned} \tag{3.4}$$

where the expression for the tadpole term follows from the form of the metric (2.13) and the explicit form of the Gauss-Bonnet boundary term is given in (2.33).

En route to presenting specific examples of our new theories, we now evaluate these terms on the special case metric (2.13)

$$G_{AB}\mathrm{d}X^A\mathrm{d}X^B = \mathrm{d}\rho^2 + f(\rho)^2 g_{\mu\nu}(x)\mathrm{d}x^\mu\mathrm{d}x^\nu\ . \tag{3.5}$$

We make use of formulae catalogued in Appendix B. Our strategy is to collect coefficients of f'', f', f'^2 and f'^3, eliminate everywhere $(\partial\pi)^2$ in favor of $\gamma = \frac{1}{\sqrt{1+\frac{1}{f^2}(\partial\pi)^2}}$, and then to group like terms by powers of γ. A lengthy calculation yields

$$\begin{aligned}
\mathcal{L}_1 &= \sqrt{-g}\int^{\pi} \mathrm{d}\pi' f(\pi')^4, \\
\mathcal{L}_2 &= -\sqrt{-g}f^4\sqrt{1+\frac{1}{f^2}(\partial\pi)^2}, \\
\mathcal{L}_3 &= \sqrt{-g}\left[f^3 f'(5-\gamma^2) - f^2[\Pi] + \gamma^2[\pi^3]\right],
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}_4 = &-\sqrt{-g}\left\{\frac{1}{\gamma}f^2R - 2\gamma R_{\mu\nu}\nabla^\mu\pi\nabla^\nu\pi\right.\\
&+\gamma\left[[\Pi]^2 - [\Pi^2] + 2\frac{\gamma^2}{f^2}\left(-[\Pi][\pi^3] + [\pi^4]\right)\right] + 6\frac{f^3f''}{\gamma}\left(-1+\gamma^2\right)\\
&\left.+2\gamma ff'\left[-4[\Pi] + \frac{\gamma^2}{f^2}\left(f^2[\Pi] + 4[\pi^3]\right)\right] - 6\frac{f^2f'^2}{\gamma}\left(1-2\gamma^2+\gamma^4\right)\right\},\\
\mathcal{L}_5 = &\frac{3}{2}\sqrt{-g}\left\{R\left[3ff' - [\Pi] + \frac{\gamma^2}{f^2}\left(-f^3f' + [\pi^3]\right)\right] - 2\frac{\gamma^2}{f^2}R^{\mu\nu\alpha\beta}\nabla_\mu\pi\nabla_\alpha\pi\Pi_{\nu\beta}\right.\\
&+2R^{\mu\nu}\left[\Pi_{\mu\nu} + \frac{\gamma^2}{f^2}\left((-3ff' + [\Pi])\nabla_\mu\pi\nabla_\nu\pi - 2\Pi_{\alpha(\mu}\nabla_{\nu)}\pi\nabla^\alpha\pi\right)\right]\\
&-\frac{\gamma^2}{f^2}\left[\frac{2}{3}\left([\Pi]^3 - 3[\Pi][\Pi^2] + 2[\Pi^3]\right) + 2\frac{\gamma^2}{f^2}\left(-[\pi^3]([\Pi]^2 - [\Pi^2]) + 2[\Pi][\pi^4] - 2[\pi^5]\right)\right]\\
&+4ff''\left[-3ff' + [\Pi] + \frac{\gamma^2}{f^2}\left(3f^3f' - f^2[\Pi] - [\pi^3]\right)\right] - 2ff'^3\left(9 - 11\gamma^2 + 6\gamma^4\right)\\
&+2f'^2\left[[\Pi] - \frac{\gamma^2}{f^2}\left(8f^2[\Pi] + [\pi^3]\right) + 2\frac{\gamma^4}{f^2}\left(2f^2[\Pi] + 5[\pi^3]\right)\right]\\
&\left.+2\gamma^2\frac{f'}{f}\left[3\left([\Pi]^2 - [\Pi^2]\right) - \frac{\gamma^2}{f^2}\left(f^2([\Pi]^2 - [\Pi^2]) + 6([\Pi][\pi^3] - [\pi^4])\right)\right]\right\}\ .
\end{aligned} \tag{3.6}$$

The quantities $[\Pi^n]$ and $[\pi^n]$ are various contractions of derivatives of the π field as is explained in the conventions Appendix A. In these expressions, all curvatures are those of the metric $g_{\mu\nu}$, and all derivatives are covariant derivatives with respect to $g_{\mu\nu}$. We point out that no integrations by parts have been performed in obtaining these expressions.

The equations of motion derived from any of these five terms will contain no more than two derivatives on each field, ensuring that no extra degrees of freedom propagate around any background. After suitable integrations by parts, these actions should therefore conform to the general structure presented in [43] for actions of a single scalar with second order equations (see also the Euler hierarchy constructions [54, 55, 56, 57]). In the above construction, however, we can immediately identify the nonlinear symmetries by reading them off from the isometries of the bulk.

Finally, we note that by keeping the metric $g_{\mu\nu}$ in (2.13) arbitrary rather than fixing it to the foliation, we can automatically obtain the covariantizaton of these various Galileon actions, including the non-minimal curvature terms required to keep the equations of motion second order, the same terms obtained by purely 4-d methods in [41, 42, 43]. Of course, this in general ruins the symmetries we are interested in considering. But from this point of view, we can see exactly when such symmetries will be present. The symmetries will only be present if the $g_{\mu\nu}$ which is used to covariantly couple is such that the full metric (2.13) has isometries.

3.3 All Examples of Maximal Symmetry

We now apply the work of Sec.3.2.1 and Sec.3.2.2 to each of the cases in Figure 3.1. The construction starts by finding coordinates which are adapted to the desired foliation, so that the metric in the bulk takes the form (2.13), allowing us to read off the function $f(\pi)$. Plugging into (3.6) then gives us the explicit Lagrangians. To find the form of the global symmetries, we must write the explicit Killing vectors in the bulk, and identify those which are parallel and not parallel to the foliation. We may then read off the symmetries from (3.3).

The construction for each case is similar, and some of the results are related by analytic continuation, but there are enough differences in the forms of the embeddings and the Killing vectors that we thought it worthwhile to display each case explicitly. The reader interested only in a given case may skip directly to it.

3.3.1 A Minkowski brane in a Minkowski bulk: M_4 in M_5 – DBI Galileons

Choosing cartesian coordinates (x^μ, ρ) on M_5, the foliation of M_5 by M_4 is simply given by $\rho =$ constant slices, and the metric takes the form

$$\mathrm{d}s^2 = (\mathrm{d}\rho)^2 + \eta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu \ . \tag{3.7}$$

Comparing this to (2.13), we obtain

$$f(\pi) = 1, \quad g_{\mu\nu} = \eta_{\mu\nu}, \tag{3.8}$$

and the terms (3.6) become (again, without integration by parts)

$$\begin{aligned}
\mathcal{L}_1 &= \pi, \\
\mathcal{L}_2 &= -\sqrt{1+(\partial\pi)^2}\ , \\
\mathcal{L}_3 &= -[\Pi] + \gamma^2 \left[\pi^3\right]\ , \\
\mathcal{L}_4 &= -\gamma\left([\Pi]^2 - \left[\Pi^2\right]\right) - 2\gamma^3\left(\left[\pi^4\right] - [\Pi]\left[\pi^3\right]\right)\ , \\
\mathcal{L}_5 &= -\gamma^2\left([\Pi]^3 + 2\left[\Pi^3\right] - 3[\Pi]\left[\Pi^2\right]\right) - \gamma^4\left(6[\Pi]\left[\pi^4\right] - 6\left[\pi^5\right] - 3\left([\Pi]^2 - \left[\Pi^2\right]\right)\left[\pi^3\right]\right)\ ,
\end{aligned} \tag{3.9}$$

where $\gamma = \frac{1}{\sqrt{1+(\partial\pi)^2}}$. These are the DBI Galileon terms, first written down in [39] and further studied in [67].

3.3.1.1 Killing vectors and symmetries

The Killing vectors of 5D Minkowski space are the 10 boosts $L_{AB} = X_A\partial_B - X_B\partial_A$, and the 5 translations $P_A = -\partial_A$. The 6 boosts $J_{\mu\nu}$ and the 4 translations P_μ are parallel to the foliation and form the unbroken $ISO(3,1)$ symmetries of M_4. The 5 broken generators are

$$K \equiv -P_5 = \partial_\rho, \tag{3.10}$$

$$K_\mu \equiv L_{\mu 5} = x_\mu\partial_\rho - \rho\partial_\mu\ . \tag{3.11}$$

Using the relation $\delta_K\pi = K^5(x) - K^\mu(x,\pi)\partial_\mu\pi$ from (3.3), we obtain the transformation rules

$$\delta\pi = 1,$$

$$\delta_\mu \pi = x_\mu + \pi \partial_\mu \pi \ , \tag{3.12}$$

under which the terms (3.9) are each invariant up to a total derivative. The symmetry breaking pattern is

$$ISO(4,1) \to ISO(3,1) \ . \tag{3.13}$$

3.3.2 A Minkowski brane in an anti-de Sitter bulk: M_4 in AdS_5 – Conformal Galileons

In this section, indices $\mathcal{A}, \mathcal{B}, \cdots$ run over six values $0, 1, 2, 3, 4, 5$ and $Y^{\mathcal{A}}$ are cartesian coordinates in an ambient 6d two-time Minkowski space with metric $\eta_{\mathcal{A}\mathcal{B}} = \text{diag}(-1, -1, 1, 1, 1, 1)$, which we call $M_{4,2}$.

Five dimensional anti-de Sitter space AdS_5 (more precisely, a quotient thereof) can be described as the subset of points $(Y^0, Y^1, Y^2 \ldots, Y^5) \in M_{4,2}$ in the hyperbola of one sheet satisfying

$$\eta_{\mathcal{A}\mathcal{B}} Y^{\mathcal{A}} Y^{\mathcal{B}} = -(Y^0)^2 - (Y^1)^2 + (Y^2)^2 + \cdots + (Y^5)^2 = -\mathcal{R}^2 \ , \tag{3.14}$$

with $\mathcal{R} > 0$ the radius of curvature of AdS_5, and where the metric is induced from the flat metric on $M_{4,2}$. This space is not simply connected, but its universal cover is AdS_5. The scalar curvature R and cosmological constant Λ are given by $R = -\frac{20}{\mathcal{R}^2}$, $\Lambda = -\frac{6}{\mathcal{R}^2}$.

We use Poincare coordinates (ρ, x^μ) on AdS_5 which cover the region $Y^0 + Y^2 > 0$,

$$\begin{aligned}
Y^0 &= \mathcal{R} \cosh\left(\frac{\rho}{\mathcal{R}}\right) + \frac{1}{2\mathcal{R}} e^{-\rho/\mathcal{R}} x^2 \ , \\
Y^1 &= e^{-\rho/\mathcal{R}} x^0 \ , \\
Y^2 &= -\mathcal{R} \sinh\left(\frac{\rho}{\mathcal{R}}\right) - \frac{1}{2\mathcal{R}} e^{-\rho/\mathcal{R}} x^2 \ , \qquad (3.15) \\
Y^{i+2} &= e^{-\rho/\mathcal{R}} x^i \ , \quad i = 1, 2, 3 \ , \qquad (3.16)
\end{aligned}$$

where $x^2 \equiv \eta_{\mu\nu} x^\mu x^\nu$, and $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$ is the Minkowski 4-metric. The coordinates u and x^μ all take the range $(-\infty, \infty)$. Lines of constant ρ foliate the Poincare patch of AdS_5

with Minkowski M_4 time-like slices, given by intersecting the planes $Y^0 + Y^2 =$ constant with the hyperbola.

The induced metric is

$$\mathrm{d}s^2 = \mathrm{d}\rho^2 + e^{-2\rho/\mathcal{R}}\eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu \ . \tag{3.17}$$

Comparing this with (2.13) we obtain

$$f(\pi) = e^{-\pi/\mathcal{R}}, \qquad g_{\mu\nu} = \eta_{\mu\nu} \ , \tag{3.18}$$

and the terms (3.6) become (without integration by parts)

$$\begin{aligned}
\mathcal{L}_1 &= -\frac{\mathcal{R}}{4}e^{-4\pi/\mathcal{R}} \ , \\
\mathcal{L}_2 &= -e^{-4\pi/\mathcal{R}}\sqrt{1+e^{2\pi/\mathcal{R}}(\partial\pi)^2} \ , \\
\mathcal{L}_3 &= \gamma^2[\pi^3] - e^{-2\pi/\mathcal{R}}[\Pi] + \frac{1}{\mathcal{R}}e^{-4\pi/\mathcal{R}}(\gamma^2-5) \ , \\
\mathcal{L}_4 &= -\gamma([\Pi]^2-[\Pi^2]) - 2\gamma^3 e^{2\pi/\mathcal{R}}([\pi^4]-[\Pi][\pi^3]) \\
&\quad + \frac{6}{\mathcal{R}^2}e^{-4\pi/\mathcal{R}}\frac{1}{\gamma}\left(2-3\gamma^2+\gamma^4\right) + \frac{8}{\mathcal{R}}\gamma^3[\pi^3] - \frac{2}{\mathcal{R}}e^{-2\pi/\mathcal{R}}\gamma\left(4-\gamma^2\right)[\Pi] \, , \\
\mathcal{L}_5 &= -\gamma^2 e^{2\pi/\mathcal{R}}\left([\Pi]^3 - 3[\Pi][\Pi^2] + 2[\Pi^3]\right) \\
&- 3\gamma^4 e^{4\pi/\mathcal{R}}\left[2([\Pi][\pi^4]-[\pi^5]) - ([\Pi]^2-[\Pi^2])[\pi^3]\right] \\
&+ \frac{18}{\mathcal{R}}e^{2\pi/\mathcal{R}}\gamma^4([\Pi][\pi^3]-[\pi^4]) - \frac{3}{\mathcal{R}}\gamma^2(3-\gamma^2)([\Pi]^2-[\Pi^2]) \\
&- \frac{3}{\mathcal{R}^2}\gamma^2(3-10\gamma^2)[\pi^3] - \frac{3}{\mathcal{R}^2}e^{-2\pi/\mathcal{R}}(-3+10\gamma^2-4\gamma^4)[\Pi] \\
&\quad + \frac{3}{\mathcal{R}^3}e^{-4\pi/\mathcal{R}}(15-17\gamma^2+6\gamma^4) \ ,
\end{aligned} \tag{3.19}$$

where

$$\gamma = \frac{1}{\sqrt{1+e^{2\pi/\mathcal{R}}(\partial\pi)^2}} \ . \tag{3.20}$$

These are the conformal DBI Galileons, first written down in [39].

3.3.2.1 Killing vectors and symmetries

The 15 Lorentz generators of $M_{4,2}$; $M_{\mathcal{AB}} = Y_{\mathcal{A}}\bar{\partial}_{\mathcal{B}} - Y_{\mathcal{B}}\bar{\partial}_{\mathcal{A}}$ (here $\bar{\partial}_{\mathcal{A}}$ are the coordinate basis vectors in the ambient space $M_{4,2}$, and indices are lowered with the $M_{4,2}$ flat metric $\eta_{\mathcal{AB}}$) are all tangent to the AdS_5 hyperboloid, and become the 15 isometries of the $SO(4,2)$ isometry algebra of AdS_5. Of these, 10 have no ∂_ρ components and are parallel to the M_4 foliation. These form the unbroken $ISO(3,1)$ isometry algebra of the M_4 slices.

First we have

$$Y^{i+2}\bar{\partial}_1 + Y^1\bar{\partial}_{i+2} \to x^i\partial_0 + x^0\partial_i, \quad i = 1,2,3, \tag{3.21}$$

$$Y^{i+2}\bar{\partial}_{j+2} - Y^{j+2}\bar{\partial}_{i+2} \to x_i\partial_j - x_j\partial_i, \quad i,j = 1,2,3, \tag{3.22}$$

which taken together are the 6 Lorentz transformations $L_{\mu\nu} = x_\mu\partial_\nu - x_\nu\partial_\mu$ of the x^μ.

For the remaining 4, we focus on

$$\begin{aligned}
-Y^1\bar{\partial}_0 + Y^0\bar{\partial}_1 &\to x^0\partial_\rho + \left[\frac{\mathcal{R}}{2}\left(1+e^{\frac{2\rho}{\mathcal{R}}}\right) + \frac{1}{2\mathcal{R}}x^2\right]\partial_0 + \frac{1}{\mathcal{R}}x^0x^\mu\partial_\mu \ , \\
-Y^{i+2}\bar{\partial}_0 - Y^0\bar{\partial}_{i+2} &\to x^i\partial_\rho + \left[-\frac{\mathcal{R}}{2}\left(1+e^{\frac{2\rho}{\mathcal{R}}}\right) - \frac{1}{2\mathcal{R}}x^2\right]\partial_i + \frac{1}{\mathcal{R}}x^ix^\mu\partial_\mu \ , \qquad i = 1,2,3 \ , \\
-Y^2\bar{\partial}_1 - Y^1\bar{\partial}_2 &\to x^0\partial_\rho + \left[-\frac{\mathcal{R}}{2}\left(1-e^{\frac{2\rho}{\mathcal{R}}}\right) + \frac{1}{2\mathcal{R}}x^2\right]\partial_0 + \frac{1}{\mathcal{R}}x^0x^\mu\partial_\mu \ , \\
-Y^{i+2}\bar{\partial}_2 + Y^2\bar{\partial}_{i+2} &\to x^i\partial_\rho + \left[\frac{\mathcal{R}}{2}\left(1-e^{\frac{2\rho}{\mathcal{R}}}\right) - \frac{1}{2\mathcal{R}}x^2\right]\partial_i + \frac{1}{\mathcal{R}}x^ix^\mu\partial_\mu \ , \qquad i = 1,2,3 \ ,
\end{aligned} \tag{3.23}$$

which may be grouped as

$$\begin{aligned}
V_\mu &= x_\mu\partial_\rho + \left[-\frac{\mathcal{R}}{2}\left(1+e^{\frac{2\rho}{\mathcal{R}}}\right) - \frac{1}{2\mathcal{R}}x^2\right]\partial_\mu + \frac{1}{\mathcal{R}}x_\mu x^\nu\partial_\nu \ , \qquad \mu = 0,1,2,3 \\
V'_\mu &= x_\mu\partial_\rho + \left[\frac{\mathcal{R}}{2}\left(1-e^{\frac{2\rho}{\mathcal{R}}}\right) - \frac{1}{2\mathcal{R}}x^2\right]\partial_\mu + \frac{1}{\mathcal{R}}x_\mu x^\nu\partial_\nu, \qquad \mu = 0,1,2,3 \ .
\end{aligned} \tag{3.24}$$

If we now take the following linear combinations,

$$P_\mu = \frac{1}{\mathcal{R}}(V_\mu - V'_\mu) = -\partial_\mu \ , \tag{3.25}$$

$$K_\mu = (V_\mu + V'_\mu) = 2x_\mu \partial_\rho - \left[\mathcal{R} e^{\frac{2\rho}{\mathcal{R}}} + \frac{1}{\mathcal{R}} x^2\right] \partial_\mu + \frac{2}{\mathcal{R}} x_\mu x^\nu \partial_\nu \ , \tag{3.26}$$

the P_μ are the translations on the x^μ, the remaining 4 unbroken vectors.

The K_μ are broken generators and, in addition, there is one more broken vector,

$$-Y^2 \bar{\partial}_0 - Y^0 \bar{\partial}_2 = \mathcal{R} \partial_\rho + x^\mu \partial_\mu \ . \tag{3.27}$$

Using the relation $\delta_K \pi = K^5(x) - K^\mu(x, \pi)\partial_\mu \pi$ from (3.3), we obtain the transformation rules for the π field from this and from the K_μ as

$$\begin{aligned} \delta \pi &= \mathcal{R} - x^\mu \partial_\mu \pi, \\ \delta_\mu \pi &= 2x_\mu + \left[\mathcal{R} e^{\frac{2\pi}{\mathcal{R}}} + \frac{1}{\mathcal{R}} x^2\right] \partial_\mu \pi - \frac{2}{\mathcal{R}} x_\mu x^\nu \partial_\nu \pi \ . \end{aligned} \tag{3.28}$$

The terms (3.19) are each invariant up to a total derivative under these transformations, and the symmetry breaking pattern is

$$SO(4, 2) \to ISO(3, 1) \ . \tag{3.29}$$

3.3.3 A de Sitter brane in a Minkowski bulk: dS_4 in M_5

We describe the Minkowski bulk with the usual metric in cartesian coordinates

$$\mathrm{d}s^2 = \eta_{AB} \mathrm{d}X^A \mathrm{d}X^B = -(\mathrm{d}X^0)^2 + (\mathrm{d}X^1)^2 + (\mathrm{d}X^2)^2 + (\mathrm{d}X^3)^2 + (\mathrm{d}X^4)^2 \ . \tag{3.30}$$

The region $\eta_{AB} X^A X^B > 0$ (i.e. outside the lightcone) can be foliated by de Sitter slices. To see this, we use Rindler coordinates which cover this region,

$$\begin{aligned} X^0 &= r \sinh \tau, \\ X^1 &= \rho \cosh \tau \ \cos \theta_1 \ , \\ X^2 &= \rho \cosh \tau \ \sin \theta_1 \cos \theta_2 \ , \\ X^3 &= \rho \cosh \tau \ \sin \theta_1 \sin \theta_2 \cos \theta_3 \ , \end{aligned} \tag{3.31}$$

$$X^4 = \rho \cosh \tau \ \sin \theta_1 \sin \theta_2 \sin \theta_3 \ , \tag{3.32}$$

where $\rho \in (0, \infty)$, $\tau \in (-\infty, \infty)$, and the θ_i $(i = 1, 2, 3)$ parametrize a 3 sphere. The metric in Rindler coordinates is then

$$\mathrm{d}s^2 = \mathrm{d}\rho^2 + \rho^2 \left[-\mathrm{d}\tau^2 + \cosh^2 \tau \ \mathrm{d}\Omega^2_{(3)} \right] \ . \tag{3.33}$$

This metric is $\mathrm{d}s^2 = \mathrm{d}\rho^2 + \rho^2 \mathrm{d}s^2_{dS_4}$, where $\mathrm{d}s^2_{dS_4}$ is the global metric on a unit radius 4D de Sitter space. The foliation by $\mathrm{d}s^4$ thus corresponds to ρ = constant surfaces (or to $-(X^0)^2 + (X^i)^2 = \text{constant} > 0$ in cartesian coordinates).

Comparing this with (2.13), we obtain

$$f(\pi) = \pi, \quad g_{\mu\nu} = g^{(dS_4)}_{\mu\nu} \ , \tag{3.34}$$

and the terms (3.6) become (without any integrations by parts)

$$\begin{aligned}
\mathcal{L}_1 &= \frac{1}{5}\sqrt{-g}\pi^5 \ , \\
\mathcal{L}_2 &= -\sqrt{-g}\pi^4 \sqrt{1 + \frac{1}{\pi^2}(\partial\pi)^2} \ , \\
\mathcal{L}_3 &= \sqrt{-g}\left[\pi^3(5 - \gamma^2) - \pi^2[\Pi] + \gamma^2[\pi^3]\right] \ , \\
\mathcal{L}_4 &= \sqrt{-g}\ \gamma \left[-[\Pi]^2 + [\Pi^2] + 8\pi[\Pi] - 18\pi^2 - 2\frac{\gamma^2}{\pi^2}\left([\pi^4] + 4\pi[\pi^3] - 3\pi^4 - [\Pi][\pi^3] + \pi^3[\Pi]\right)\right] \ , \\
\mathcal{L}_5 &= \sqrt{-g}\ \frac{\gamma^2}{\pi^2}\Bigg[- [\Pi]^3 + 3[\Pi][\Pi^2] - 2[\Pi^3] + 9\pi([\Pi]^2 - [\Pi^2]) + 42\pi^3 - 30\pi^2[\Pi] \\
&\quad + 3\frac{\gamma^2}{\pi^2}\Big(([\Pi]^2 - [\Pi^2])[\pi^3] + 2[\pi^5] + 6\pi[\pi^4] + 10\pi^2[\pi^3] - \pi^3([\Pi]^2 - [\Pi^2]) \\
&\quad - 6\pi^5 - 2[\Pi]([\pi^4] + 3\pi[\pi^3] - 2\pi^4)]\Big)\Bigg] \ ,
\end{aligned} \tag{3.35}$$

where the background metric and covariant derivatives are those of unit-radius 4D de Sitter space, and

$$\gamma = \frac{1}{\sqrt{1 + \frac{1}{\pi^2}(\partial\pi)^2}} \ . \tag{3.36}$$

Note that, since we have chosen the 4D space to be a unit-radius dS_4 with dimensionless coordinates, π and f have mass dimension -1. In evaluating (3.35), we have used that the scalar curvature and cosmological constant of this space are $R = 12$ and $\Lambda = 3$ respectively, and used the relations $R_{\mu\nu\alpha\beta} = \frac{R}{12}\left(g_{\mu\alpha}g_{\nu\beta} - g_{\mu\beta}g_{\nu\alpha}\right)$, and $R_{\mu\nu} = \frac{R}{4}g_{\mu\nu}$, valid for a maximally symmetric space. It is possible, of course, to rescale the coordinates, canonically normalize the field, and/or rescale f to bring these quantities to their usual dimensions. Given a suitable combinations of these Lagrangians so that a constant field $\pi(x) = \pi_0 =$ constant is a solution to the equations of motion, π_0 sets the radius of the de Sitter brane in its ground state.

We call these Type II de Sitter DBI Galileons (see Figure 3.1), and they are our first example of a Galileon that lives on curved space yet still retains the same number of shift-like symmetries as their flat space counterparts.

3.3.3.1 Killing vectors and symmetries

The 10 Lorentz transformations of M_5 are parallel to the de Sitter slices and become the unbroken $SO(4,1)$ isometries of dS_4. The 5 translations are not parallel and will be nonlinearly realized.

With a future application to cosmology in mind, we will calculate the transformation laws explicitly using conformal inflationary coordinates (u, y^i) on the de Sitter slices, even though these coordinates only cover half of each de Sitter slice. The embedding becomes

$$
\begin{aligned}
X^0 &= \frac{\rho}{2u}\left(1 - u^2 + y^2\right) \ , \\
X^1 &= \frac{\rho}{2u}\left(1 + u^2 - y^2\right) \ , \\
X^{i+1} &= \frac{\rho y^i}{u}, \quad i = 1, 2, 3 \ ,
\end{aligned}
\tag{3.37}
$$

where $y^2 \equiv \delta_{ij}y^i y^j$, and the coordinate ranges are $\rho \in (0, \infty)$, $u \in (0, \infty)$, $y^i \in (-\infty, \infty)$.

The metric takes the form

$$ds^2 = d\rho^2 + \rho^2 \left[\frac{1}{u^2}\left(-du^2 + dy^2\right)\right] \,, \tag{3.38}$$

so that the dS_4 slices have conformal inflationary coordinates, with u the conformal time.

We are interested in the form of the nonlinear symmetries stemming from the broken translation generators of M_5. In the coordinates (3.37), the broken Killing vectors $\bar{\partial}_A$ are

$$\bar{\partial}_0 = \frac{1}{2u}\left(-1 + u^2 - y^2\right)\partial_\rho - \frac{1}{2\rho}\left(1 + u^2 + y^2\right)\partial_u - \frac{u}{\rho}y^i\partial_i \,, \tag{3.39}$$

$$\bar{\partial}_1 = \frac{1}{2u}\left(1 + u^2 - y^2\right)\partial_\rho - \frac{1}{2\rho}\left(-1 + u^2 + y^2\right)\partial_u - \frac{u}{\rho}y^i\partial_i \,, \tag{3.40}$$

$$\bar{\partial}_i = \frac{y_i}{u}\partial_\rho + \frac{y_i}{\rho}\partial_u + \frac{u}{r}\partial_i, \quad i = 1, 2, 3 \,. \tag{3.41}$$

Taking the following linear combinations

$$K_+ = \bar{\partial}_0 + \bar{\partial}_1 = \frac{1}{u}\left(u^2 - y^2\right)\partial_\rho - \frac{1}{\rho}\left(u^2 + y^2\right)\partial_u - \frac{2u}{\rho}y^i\partial_i \,, \tag{3.42}$$

$$K_- = \bar{\partial}_0 - \bar{\partial}_1 = -\frac{1}{u}\partial_\rho - \frac{1}{\rho}\partial_u \,, \tag{3.43}$$

$$K_i = \bar{\partial}_i = \frac{y_i}{u}\partial_\rho + \frac{y_i}{\rho}\partial_u + \frac{u}{\rho}\partial_i \,, \tag{3.44}$$

and using the relation $\delta_K\pi = K^5(x) - K^\mu(x,\pi)\partial_\mu\pi$ from (3.3), we then obtain the transformation rules

$$\begin{aligned}
\delta_+\pi &= \frac{1}{u}\left(u^2 - y^2\right) + \frac{1}{\pi}\left(u^2 + y^2\right)\pi' + \frac{2u}{\pi}y^i\partial_i\pi \,, \\
\delta_-\pi &= -\frac{1}{u} + \frac{1}{\pi}\pi' \,, \\
\delta_i\pi &= \frac{y_i}{u} - \frac{y_i}{\pi}\pi' - \frac{u}{\pi}\partial_i\pi \,,
\end{aligned} \tag{3.45}$$

where $\pi' \equiv \partial_u\pi$.

The terms (3.35) are each invariant up to a total derivative under these transformations, and the symmetry breaking pattern is

$$ISO(4,1) \to SO(4,1) \,. \tag{3.46}$$

3.3.4 A de Sitter brane in a de Sitter bulk: dS_4 in dS_5

In this section, indices $\mathcal{A}, \mathcal{B}, \cdots$ run over six values $0,1,2,3,4,5$ and $Y^{\mathcal{A}}$ are coordinates in an ambient 6d Minkowski space with metric $\eta_{\mathcal{AB}} = \mathrm{diag}(-1,1,1,1,1,1)$, which we call M_6.

Five-dimensional de Sitter space dS_5 can be described as the subset of points $(Y^0, Y^1, Y^2 \ldots, Y^5) \in M_6$ in the hyperbola of one sheet satisfying

$$\eta_{\mathcal{AB}} Y^{\mathcal{A}} Y^{\mathcal{B}} = -(Y^0)^2 + (Y^1)^2 + (Y^2)^2 + \cdots + (Y^5)^2 = \mathcal{R}^2 \ , \tag{3.47}$$

with the metric induced from the metric on M_6, for some constant $\mathcal{R} > 0$, the radius of curvature of the dS_5. The scalar curvature R and cosmological constant Λ are given by $R = 20/\mathcal{R}^2$ and $\Lambda = 6/\mathcal{R}^2$, respectively.

We use coordinates in which the constant ρ surfaces are the intersections of the planes $Y^1 =$ constant with the hyperbola, and are themselves four-dimensional de Sitter spaces $\mathrm{d}s^4$,

$$Y^0 = \mathcal{R} \sin\rho \sinh\tau \ , \tag{3.48}$$
$$Y^1 = \mathcal{R} \cos\rho \ , \tag{3.49}$$
$$Y^2 = \mathcal{R} \cosh\tau \sin\rho \ \cos\theta_1 \ , \tag{3.50}$$
$$Y^3 = \mathcal{R} \cosh\tau \sin\rho \ \sin\theta_1 \cos\theta_2 \ , \tag{3.51}$$
$$Y^4 = \mathcal{R} \cosh\tau \sin\rho \ \sin\theta_1 \sin\theta_2 \cos\theta_3 \ , \tag{3.52}$$
$$Y^5 = \mathcal{R} \cosh\tau \sin\rho \ \sin\theta_1 \sin\theta_2 \sin\theta_3 \ . \tag{3.53}$$
$$\tag{3.54}$$

Here $\tau \in (-\infty, \infty)$, $\rho \in (0, \pi)$ and θ_i, $i = 1,2,3$ parametrize a 3-sphere. These coordinates cover the region $0 < Y^1 < \mathcal{R}$, $0 < Y^2 < \mathcal{R}$.

The metric is

$$\mathrm{d}s^2 = \mathcal{R}^2 \left[\mathrm{d}\rho^2 + \sin^2\rho \left(-\mathrm{d}\tau^2 + \cosh^2\tau \ \mathrm{d}\Omega_{(3)} \right) \right] \ . \tag{3.55}$$

Scaling ρ so that it lies in the range $(0, \pi\mathcal{R})$, the metric becomes $\mathrm{d}s^2 = \mathrm{d}\rho^2 + \mathcal{R}^2 \sin^2\left(\frac{\rho}{\mathcal{R}}\right) \mathrm{d}s^2_{dS_4}$, where $\mathrm{d}s^2_{dS_4}$ is the global metric on a four-dimensional de Sitter space dS_4 of unit radius. The foliation by $\mathrm{d}s^4$ thus corresponds to $\rho =$ constant surfaces. These slices are given by intersecting the planes $Y^1 =$ constant with the hyperbola, for values $0 < Y^1 < \mathcal{R}$. (By taking $\rho < 0$ we cover instead $-\mathcal{R} < Y^2 < 0$. This is the maximum extent to which we may extend the foliation.)

Comparing this with (2.13), we obtain

$$f(\pi) = \mathcal{R} \sin(\pi/\mathcal{R}), \quad g_{\mu\nu} = g^{(dS_4)}_{\mu\nu} \ , \tag{3.56}$$

and the terms (3.6) become (using no integrations by parts)

$$\mathcal{L}_1 = \sqrt{-g}\frac{\mathcal{R}^4}{32}\left(12\ \pi - 8\mathcal{R}\sin\left(\frac{2\pi}{\mathcal{R}}\right) + \mathcal{R}\sin\left(\frac{4\pi}{\mathcal{R}}\right)\right) \ , \tag{3.57}$$

$$\mathcal{L}_2 = -\sqrt{-g}\frac{\mathcal{R}^4}{\gamma}\sin^4\left(\frac{\pi}{\mathcal{R}}\right) \ , \tag{3.58}$$

$$\mathcal{L}_3 = \sqrt{-g}\left[\gamma^2[\pi^3] - \mathcal{R}^2[\Pi]\sin^2\left(\frac{\pi}{\mathcal{R}}\right) + \mathcal{R}^3(5-\gamma^2)\sin^3\left(\frac{\pi}{\mathcal{R}}\right)\cos\left(\frac{\pi}{\mathcal{R}}\right)\right] \ , \tag{3.59}$$

$$\begin{aligned}\mathcal{L}_4 = \sqrt{-g}\Bigg[&\frac{2\gamma^3}{\mathcal{R}^2}\left([\Pi][\pi^3] - [\pi^4]\right)\csc^2\left(\frac{\pi}{\mathcal{R}}\right) - \gamma\left([\Pi]^2 - [\Pi^2] + \frac{8\gamma^2}{\mathcal{R}}[\pi^3]\cot\left(\frac{\pi}{\mathcal{R}}\right)\right)\\ &+ \mathcal{R}\gamma(4-\gamma^2)[\Pi]\sin\left(\frac{2\pi}{\mathcal{R}}\right)\\ &+ \frac{3\mathcal{R}^2}{\gamma}\sin^2\left(\frac{\pi}{\mathcal{R}}\right)\left(-2-3\gamma^2+\gamma^4+(2-3\gamma^2+\gamma^4)\cos\left(\frac{2\pi}{\mathcal{R}}\right)\right)\Bigg] \ ,\end{aligned} \tag{3.60}$$

$$\begin{aligned}\mathcal{L}_5 = \sqrt{-g}\Bigg[&\frac{3\gamma^4}{\mathcal{R}^4}\left(2([\pi^5] - [\Pi][\pi^4]) + [\pi^3]([\Pi]^2 - [\Pi^2])\right)\csc^4\left(\frac{\pi}{\mathcal{R}}\right)\\ &- \frac{18\gamma^4}{\mathcal{R}^3}\left([\Pi][\pi^3] - [\pi^4]\right)\csc^2\left(\frac{\pi}{\mathcal{R}}\right)\cot\left(\frac{\pi}{\mathcal{R}}\right)\\ &- \frac{\gamma^2}{\mathcal{R}^2}\csc^2\left(\frac{\pi}{\mathcal{R}}\right)\Bigg([\Pi]^3 - 3[\Pi][\Pi^2] + 2[\Pi^3] - \frac{3}{2}(3+10\gamma^2)[\pi^3]\\ &+ \frac{3}{2}(3-10\gamma^2)[\pi^3]\cos\left(\frac{2\pi}{\mathcal{R}}\right)\Bigg) + \frac{3\gamma^2}{\mathcal{R}}(3-\gamma^2)([\Pi]^2 - [\Pi^2])\cot\left(\frac{\pi}{\mathcal{R}}\right)\end{aligned} \tag{3.61}$$

$$+\frac{3}{2}[\Pi]\left(-3-10\gamma^2+4\gamma^4+(3-10\gamma^2+4\gamma^4)\cos\left(\frac{2\pi}{\mathcal{R}}\right)\right)$$
$$-\frac{3\mathcal{R}}{4}\left(-15-11\gamma^2+6\gamma^4+(15-17\gamma^2+6\gamma^4)\cos\left(\frac{2\pi}{\mathcal{R}}\right)\right)\sin\left(\frac{2\pi}{\mathcal{R}}\right)\Bigg], \tag{3.62}$$

where the background metric and covariant derivatives are those of the unit-radius 4D de Sitter space, and

$$\gamma = \frac{1}{\sqrt{1+\frac{(\partial\pi)^2}{\mathcal{R}^2\sin^2\left(\frac{\pi}{\mathcal{R}}\right)}}} . \tag{3.63}$$

Since we have chosen the 4D space to have unit radius in dimensionless coordinates, π and f have mass dimension -1. In evaluating (3.35), we have used that fact that the scalar curvature and cosmological constant of this space are $R = 12$ and $\Lambda = 3$ respectively, and the relations $R_{\mu\nu\alpha\beta} = \frac{R}{12}\left(g_{\mu\alpha}g_{\nu\beta} - g_{\mu\beta}g_{\nu\alpha}\right)$ and $R_{\mu\nu} = \frac{R}{4}g_{\mu\nu}$ valid for a maximally symmetric space. Given a suitable combination of these Lagrangians so that a constant field $\pi(x) = \pi_0 = const.$ is a solution to the equations of motion, $f(\pi_0) = \mathcal{R}\sin\left(\frac{\pi_0}{\mathcal{R}}\right)$ sets the radius of the de Sitter brane. We call these Type I de Sitter DBI Galileons (see Figure 3.1).

3.3.4.1 Killing vectors and symmetries

Once again, we calculate the transformation laws using conformal inflationary coordinates (u, y^i) on the de Sitter slices, even though they only cover half of each de Sitter slice. The embedding becomes

$$Y^0 = \mathcal{R}\sin\left(\frac{\rho}{\mathcal{R}}\right)\frac{1}{2u}\left(1-u^2+y^2\right) , \tag{3.64}$$
$$Y^1 = \mathcal{R}\cos\left(\frac{\rho}{\mathcal{R}}\right) , \tag{3.65}$$
$$Y^2 = \mathcal{R}\sin\left(\frac{\rho}{\mathcal{R}}\right)\frac{1}{2u}\left(1+u^2-y^2\right) , \tag{3.66}$$
$$Y^{i+2} = \mathcal{R}\sin\left(\frac{\rho}{\mathcal{R}}\right)\frac{y^i}{u}, \quad i = 1,2,3 . \tag{3.67}$$

The coordinate ranges are $\rho \in (0, \pi\mathcal{R})$, $u \in (0, \infty)$ and $y^i \in (-\infty, \infty)$, and the induced metric then becomes

$$\mathrm{d}s^2 = \mathrm{d}\rho^2 + \mathcal{R}^2 \sin^2\left(\frac{\rho}{\mathcal{R}}\right)\left[\frac{1}{u^2}\left(-\mathrm{d}u^2 + \mathrm{d}y^2\right)\right] . \tag{3.68}$$

The 15 Lorentz generators of M_6 are all tangent to the dS_5 hyperboloid, and become the 15 isometries of its $SO(5,1)$ isometry algebra. Of these, 10 have no ∂_ρ components and are parallel to the dS_4 foliation: these form the $SO(4,1)$ isometry algebra of the dS_4 slices,

$$-Y^2\bar{\partial}_0 - Y^0\bar{\partial}_2 \to d = u\partial_u + y^i\partial_i \ , \tag{3.69}$$

$$-Y^{i+2}\bar{\partial}_0 - Y^0\bar{\partial}_{i+2} \to j_i^+ = uy_i\partial_u + \frac{1}{2}\left(-1 + u^2 - y^2\right)\partial_i + y_i y^j\partial_j, \quad i = 1,2,3, \tag{3.70}$$

$$-Y^{i+2}\bar{\partial}_2 + Y^2\bar{\partial}_{i+2} \to j_i^- = uy_i\partial_u + \frac{1}{2}\left(1 + u^2 - y^2\right)\partial_i + y_i y^j\partial_j, \quad i = 1,2,3, \tag{3.71}$$

$$Y^{i+2}\bar{\partial}_{j+2} - Y^{j+2}\bar{\partial}_{i+2} \to j_{ij} = y_i\partial_j - y_j\partial_i, \quad i,j = 1,2,3. \tag{3.72}$$

Taking the combinations

$$p_i = j_i^+ - j_i^- = -\partial_i \ , \tag{3.73}$$

$$k_i = j_i^+ + j_i^- = 2uy_i\partial_u + (u^2 - y^2)\partial_i + 2y_i y^j\partial_j \ , \tag{3.74}$$

we then recognize p_i and j_{ij} as translations and rotations on the y plane, while d and k_i fill out the $SO(4,1)$ algebra.

The remaining 5 Killing vectors do have a ∂_ρ component,

$$\begin{aligned}
-Y^1\bar{\partial}_0 - Y^0\bar{\partial}_1 &\to K = \frac{\mathcal{R}}{2u}\left(1-u^2+y^2\right)\partial_\rho + \frac{1}{2}\left(1+u^2+y^2\right)\cot\left(\frac{\rho}{\mathcal{R}}\right)\partial_u + u\cot\left(\frac{\rho}{\mathcal{R}}\right)y^i\partial_i \ , \\
-Y^2\bar{\partial}_1 + Y^1\bar{\partial}_2 &\to K' = \frac{\mathcal{R}}{2u}\left(1+u^2-y^2\right)\partial_\rho + \frac{1}{2}\left(1-u^2-y^2\right)\cot\left(\frac{\rho}{\mathcal{R}}\right)\partial_u - u\cot\left(\frac{\rho}{\mathcal{R}}\right)y^i\partial_i \ , \\
-Y^{i+2}\bar{\partial}_1 + Y^1\bar{\partial}_{i+2} &\to K_i = \frac{\mathcal{R}}{u}y_i\partial_\rho + y_i\cot\left(\frac{\rho}{\mathcal{R}}\right)\partial_u + u\cot\left(\frac{\rho}{\mathcal{R}}\right)\partial_i, \quad i = 1,2,3.
\end{aligned} \tag{3.75}$$

Defining the following linear combinations,

$$K_+ = K + K' = \frac{\mathcal{R}}{u}\partial_\rho + \cot\left(\frac{\rho}{\mathcal{R}}\right)\partial_u \ ,$$

$$\begin{aligned}
K_- = K - K' &= \frac{\mathcal{R}}{u}\left(-u^2+y^2\right)\partial_\rho + \left(u^2+y^2\right)\cot\left(\frac{\rho}{\mathcal{R}}\right)\partial_u + 2u\cot\left(\frac{\rho}{\mathcal{R}}\right)y^i\partial_i \ , \\
K_i &= \frac{\mathcal{R}}{u}y_i\partial_\rho + y_i\cot\left(\frac{\rho}{\mathcal{R}}\right)\partial_u + u\cot\left(\frac{\rho}{\mathcal{R}}\right)\partial_i \ , \qquad (3.76)
\end{aligned}$$

and using the relation $\delta_K\pi = K^5(x) - K^\mu(x,\pi)\partial_\mu\pi$ from (3.3), we obtain the transformation rules

$$\begin{aligned}
\delta_+\pi &= \frac{\mathcal{R}}{u} - \cot\left(\frac{\pi}{\mathcal{R}}\right)\pi' \ , \\
\delta_-\pi &= \frac{\mathcal{R}}{u}\left(-u^2+y^2\right) - \left(u^2+y^2\right)\cot\left(\frac{\pi}{\mathcal{R}}\right)\pi' - 2u\cot\left(\frac{\pi}{\mathcal{R}}\right)y^i\partial_i\pi \ , \\
\delta_i\pi &= \frac{\mathcal{R}}{u}y_i - y_i\cot\left(\frac{\pi}{\mathcal{R}}\right)\pi' - u\cot\left(\frac{\pi}{\mathcal{R}}\right)\partial_i\pi \ , \qquad (3.77)
\end{aligned}$$

where $\pi' \equiv \partial_u\pi$. The terms (3.62) are each invariant up to a total derivative under these transformations, and the symmetry breaking pattern is

$$SO(5,1) \to SO(4,1) \ . \qquad (3.78)$$

3.3.5 A de Sitter brane in an anti-de Sitter bulk: dS_4 in AdS_5

Using the description and notation for the AdS_5 embedding in section 3.3.2, the following coordinates cover the intersection of the AdS_5 hyperbola with the region $Y^0 > \mathcal{R}$,

$$\begin{aligned}
Y^0 &= \mathcal{R}\cosh\rho \ , \\
Y^1 &= \mathcal{R}\sinh\rho\sinh\tau \ , \\
Y^2 &= \mathcal{R}\sinh\rho\cosh\tau\cos\theta_1 \ , \\
Y^3 &= \mathcal{R}\sinh\rho\cosh\tau\sin\theta_1\cos\theta_2 \ , \qquad (3.79) \\
Y^4 &= \mathcal{R}\sinh\rho\cosh\tau\sin\theta_1\sin\theta_2\cos\theta_3 \ , \qquad (3.80) \\
Y^5 &= \mathcal{R}\sinh\rho\cosh\tau\sin\theta_1\sin\theta_2\sin\theta_3 \ , \qquad (3.81) \\
& \qquad (3.82)
\end{aligned}$$

where $\tau \in (-\infty, \infty)$, $\rho \in (0, \infty)$, and θ_i, $i = 1, 2, 3$ parametrize a 3-sphere.

The metric reads

$$\mathrm{d}s^2 = \mathcal{R}^2 \left[\mathrm{d}\rho^2 + \sinh^2\rho \left(-\mathrm{d}\tau^2 + \cosh^2\tau \; \mathrm{d}\Omega^2_{(3)}\right)\right] \; . \tag{3.83}$$

Scaling ρ, the metric becomes $\mathrm{d}s^2 = \mathrm{d}\rho^2 + \mathcal{R}^2 \sinh^2\left(\frac{\rho}{\mathcal{R}}\right) \mathrm{d}s^2_{dS_4}$, where $\mathrm{d}s^2_{dS_4}$ is the global metric on a four-dimensional de Sitter space dS_4 of unit radius. The foliation by dS_4 thus corresponds to $\rho =$ constant surfaces. These slices are given by intersecting the planes $Y^0 =$ constant with the hyperbola in the region $Y^0 > \mathcal{R}$. (If we map $Y^0 \to -Y^0$ then the coordinates cover the region $Y^0 < -\mathcal{R}$, and the metric remains identical to (3.83), and this is the maximum extent to which we can extend the foliation.)

Comparing this with (2.13), we obtain

$$f(\pi) = \mathcal{R}\sinh(\pi/\mathcal{R}), \quad g_{\mu\nu} = g^{(dS_4)}_{\mu\nu} \; , \tag{3.84}$$

and the terms (3.6) become (without integration by parts)

$$\mathcal{L}_1 = \sqrt{-g}\frac{\mathcal{R}^4}{32}\left(12\;\pi - 8\mathcal{R}\sinh\left(\frac{2\pi}{\mathcal{R}}\right) + \mathcal{R}\sinh\left(\frac{4\pi}{\mathcal{R}}\right)\right) \; , \tag{3.85}$$

$$\mathcal{L}_2 = -\sqrt{-g}\frac{\mathcal{R}^4}{\gamma}\sinh^4\left(\frac{\pi}{\mathcal{R}}\right) \; , \tag{3.86}$$

$$\mathcal{L}_3 = \sqrt{-g}\left[\gamma^2[\pi^3] - \mathcal{R}^2[\Pi]\sinh^2\left(\frac{\pi}{\mathcal{R}}\right) + \mathcal{R}^3(5-\gamma^2)\sinh^3\left(\frac{\pi}{\mathcal{R}}\right)\cosh\left(\frac{\pi}{\mathcal{R}}\right)\right] \; , \tag{3.87}$$

$$\begin{aligned}\mathcal{L}_4 = \sqrt{-g}\Bigg[&\frac{2\gamma^3}{\mathcal{R}^2}\left([\Pi][\pi^3] - [\pi^4]\right)\operatorname{csch}^2\left(\frac{\pi}{\mathcal{R}}\right) - \gamma\left([\Pi]^2 - [\Pi^2] + \frac{8\gamma^2}{\mathcal{R}}[\pi^3]\coth\left(\frac{\pi}{\mathcal{R}}\right)\right) \\ &+ \mathcal{R}\gamma(4-\gamma^2)[\Pi]\sinh\left(\frac{2\pi}{\mathcal{R}}\right) \\ &+ \frac{3\mathcal{R}^2}{\gamma}\sinh^2\left(\frac{\pi}{\mathcal{R}}\right)\left(-2 - 3\gamma^2 + \gamma^4 + (2 - 3\gamma^2 + \gamma^4)\cosh\left(\frac{2\pi}{\mathcal{R}}\right)\right)\Bigg] \; ,\end{aligned} \tag{3.88}$$

$$\mathcal{L}_5 = \sqrt{-g}\Bigg[\frac{3\gamma^4}{\mathcal{R}^4}\left(2([\pi^5] - [\Pi][\pi^4]) + [\pi^3]([\Pi]^2 - [\Pi^2])\right)\operatorname{csch}^4\left(\frac{\pi}{\mathcal{R}}\right)$$

$$
\begin{aligned}
&- \frac{18\gamma^4}{\mathcal{R}^3}\left([\Pi][\pi^3]-[\pi^4]\right)\operatorname{csch}^2\left(\frac{\pi}{\mathcal{R}}\right)\coth\left(\frac{\pi}{\mathcal{R}}\right) \\
&- \frac{\gamma^2}{\mathcal{R}^2}\operatorname{csch}^2\left(\frac{\pi}{\mathcal{R}}\right)\left([\Pi]^3-3[\Pi][\Pi^2]+2[\Pi^3]-\frac{3}{2}(3+10\gamma^2)[\pi^3]\right. \\
&\left.+\frac{3}{2}(3-10\gamma^2)[\pi^3]\cosh\left(\frac{2\pi}{\mathcal{R}}\right)\right)+\frac{3\gamma^2}{\mathcal{R}}(3-\gamma^2)([\Pi]^2-[\Pi^2])\coth\left(\frac{\pi}{\mathcal{R}}\right) \\
&+\frac{3}{2}[\Pi]\left(-3-10\gamma^2+4\gamma^4+(3-10\gamma^2+4\gamma^4)\cosh\left(\frac{2\pi}{\mathcal{R}}\right)\right) \\
&\left.-\frac{3\mathcal{R}}{4}\left(-15-11\gamma^2+6\gamma^4+(15-17\gamma^2+6\gamma^4)\cosh\left(\frac{2\pi}{\mathcal{R}}\right)\right)\sinh\left(\frac{2\pi}{\mathcal{R}}\right)\right] ,
\end{aligned} \tag{3.89}
$$

where the background metric and covariant derivatives are those of the unit-radius 4D de Sitter space, and

$$
\gamma = \frac{1}{\sqrt{1+\frac{(\partial\pi)^2}{\mathcal{R}^2\sinh^2\left(\frac{\pi}{\mathcal{R}}\right)}}} . \tag{3.90}
$$

Given suitable combinations of these Lagrangians so that a constant field $\pi(x) = \pi_0 =$ constant is a solution to the equations of motion, $f(\pi_0) = \mathcal{R}\sinh\left(\frac{\pi_0}{\mathcal{R}}\right)$ sets the radius of the de Sitter brane. We call these Type III de Sitter DBI Galileons (see Figure 3.1).

3.3.5.1 Killing vectors and symmetries

Once again we use conformal inflationary coordinates on the dS_4 slices. The embedding becomes,

$$
Y^0 = \mathcal{R}\cosh\left(\frac{\rho}{\mathcal{R}}\right) , \tag{3.91}
$$

$$
Y^1 = \mathcal{R}\sinh\left(\frac{\rho}{\mathcal{R}}\right)\frac{1}{2u}\left(1-u^2+y^2\right) , \tag{3.92}
$$

$$
Y^2 = \mathcal{R}\sinh\left(\frac{\rho}{\mathcal{R}}\right)\frac{1}{2u}\left(1+u^2-y^2\right) , \tag{3.93}
$$

$$
Y^{i+2} = \mathcal{R}\sinh\left(\frac{\rho}{\mathcal{R}}\right)\frac{y^i}{u}, \quad i=1,2,3 , \tag{3.94}
$$

where $\rho \in (0,\infty)$ and $u \in (0,\infty)$. The coordinate ranges are $\rho \in (0,\infty)$, $u \in (0,\infty)$, $y^i \in (-\infty,\infty)$, and the induced metric is

$$ds^2 = d\rho^2 + \mathcal{R}^2 \sinh^2\left(\frac{\rho}{\mathcal{R}}\right)\left[\frac{1}{u^2}\left(-du^2 + dy^2\right)\right] . \tag{3.95}$$

The 15 Lorentz generators of $M_{4,2}$, $M_{AB} = Y_A\bar{\partial}_B - Y_B\bar{\partial}_A$, are all tangent to the AdS_5 hyperboloid, and become the 15 isometries of the $SO(4,2)$ isometry algebra of AdS_5. Of these, 10 have no ∂_ρ components and are parallel to the dS_4 foliation. These form the $SO(4,1)$ isometry algebra of the dS_4 slices

$$-Y^2\bar{\partial}_1 - Y^1\bar{\partial}_2 \to d = u\partial_u + y^i\partial_i , \tag{3.96}$$

$$-Y^{i+2}\bar{\partial}_1 - Y^1\bar{\partial}_{i+2} \to j_i^+ = uy_i\partial_u + \frac{1}{2}\left(-1 + u^2 - y^2\right)\partial_i + y_iy^j\partial_j, \quad i = 1,2,3, \tag{3.97}$$

$$-Y^{i+2}\bar{\partial}_2 + Y^2\bar{\partial}_{i+2} \to j_i^- = uy_i\partial_u + \frac{1}{2}\left(1 + u^2 - y^2\right)\partial_i + y_iy^j\partial_j, \quad i = 1,2,3, \tag{3.98}$$

$$Y^{i+2}\bar{\partial}_{j+2} - Y^{j+2}\bar{\partial}_{i+2} \to j_{ij} = y_i\partial_j - y_j\partial_i, \quad i,j = 1,2,3. \tag{3.99}$$

Taking the combinations

$$p_i = j_i^+ - j_i^- = -\partial_i , \tag{3.100}$$

$$k_i = j_i^+ + j_i^- = 2uy_i\partial_u + (u^2 - y^2)\partial_i + 2y_iy^j\partial_j , \tag{3.101}$$

we recognize p_i and j_{ij} as translations and rotations on the y plane, with d and k_i filling out the rest of the $SO(4,1)$ algebra.

The remaining 5 Killing vectors do have a ∂_ρ component,

$$\begin{aligned} Y^1\bar{\partial}_0 - Y^0\bar{\partial}_1 \to K &= \frac{\mathcal{R}}{2u}\left(1 - u^2 + y^2\right)\partial_\rho + \frac{1}{2}\left(1 + u^2 + y^2\right)\coth\left(\frac{\rho}{\mathcal{R}}\right)\partial_u \\ &\quad + u\coth\left(\frac{\rho}{\mathcal{R}}\right)y^i\partial_i , \\ Y^2\bar{\partial}_0 + Y^0\bar{\partial}_2 \to K' &= \frac{\mathcal{R}}{2u}\left(1 + u^2 - y^2\right)\partial_\rho + \frac{1}{2}\left(1 - u^2 - y^2\right)\coth\left(\frac{\rho}{\mathcal{R}}\right)\partial_u \\ &\quad - u\coth\left(\frac{\rho}{\mathcal{R}}\right)y^i\partial_i , \end{aligned}$$

$$Y^{i+2}\bar{\partial}_0 + Y^0\bar{\partial}_{i+2} \to K_i = \frac{\mathcal{R}}{u}y_i\partial_\rho + y_i \coth\left(\frac{\rho}{\mathcal{R}}\right)\partial_u + u\coth\left(\frac{\rho}{\mathcal{R}}\right)\partial_i, \quad i = 1,2,3. \tag{3.102}$$

Taking the following linear combinations

$$\begin{aligned}
K_+ &= K + K' = \frac{\mathcal{R}}{u}\partial_\rho + \coth\left(\frac{\rho}{\mathcal{R}}\right)\partial_u \ , \\
K_- &= K - K' = \frac{\mathcal{R}}{u}\left(-u^2+y^2\right)\partial_\rho + \left(u^2+y^2\right)\coth\left(\frac{\rho}{\mathcal{R}}\right)\partial_u + 2u\coth\left(\frac{\rho}{\mathcal{R}}\right)y^i\partial_i \ , \\
K_i &= \frac{\mathcal{R}}{u}y_i\partial_\rho + y_i\coth\left(\frac{\rho}{\mathcal{R}}\right)\partial_u + u\coth\left(\frac{\rho}{\mathcal{R}}\right)\partial_i \ ,
\end{aligned} \tag{3.103}$$

and using the relation $\delta_K\pi = K^5(x) - K^\mu(x,\pi)\partial_\mu\pi$ from (3.3), we obtain the transformation rules

$$\begin{aligned}
\delta_+\pi &= \frac{\mathcal{R}}{u} - \coth\left(\frac{\pi}{\mathcal{R}}\right)\pi' \ , \\
\delta_-\pi &= \frac{\mathcal{R}}{u}\left(-u^2+y^2\right) - \left(u^2+y^2\right)\coth\left(\frac{\pi}{\mathcal{R}}\right)\pi' - 2u\coth\left(\frac{\pi}{\mathcal{R}}\right)y^i\partial_i\pi \ , \\
\delta_i\pi &= \frac{\mathcal{R}}{u}y_i - y_i\coth\left(\frac{\pi}{\mathcal{R}}\right)\pi' - u\coth\left(\frac{\pi}{\mathcal{R}}\right)\partial_i\pi \ ,
\end{aligned} \tag{3.104}$$

where $\pi' \equiv \partial_u\pi$.

The terms (3.89) are each invariant up to a total derivative under these transformations, and the symmetry breaking pattern is

$$SO(4,2) \to SO(4,1) \ . \tag{3.105}$$

3.3.6 An anti-de Sitter brane in an anti-de Sitter bulk: AdS_4 in AdS_5

Using the description and notation for the AdS_5 embedding from section 3.3.2, hyperbolic coordinates on AdS^5 are

$$Y^0 = \mathcal{R} \cos\tau \cosh\rho \cosh\psi \ ,$$
$$Y^1 = \mathcal{R} \sin\tau \cosh\rho \cosh\psi \ ,$$
$$Y^2 = \mathcal{R} \sinh\rho \ ,$$
$$Y^3 = \mathcal{R} \cosh\rho \sinh\psi \cos\theta_1 \ ,$$
$$Y^4 = \mathcal{R} \cosh\rho \sinh\psi \sin\theta_1 \cos\theta_2 \ , \tag{3.106}$$
$$Y^5 = \mathcal{R} \cosh\rho \sinh\psi \sin\theta_1 \sin\theta_2 \ , \tag{3.107}$$

where $\tau \in (-\pi, \pi)$ (the universal cover is obtained by extending this to $\tau \in (-\infty, \infty)$), $\rho \in (-\infty, \infty)$, $\psi \in (0, \infty)$, and θ_1, θ_2 parametrize a 2-sphere. These coordinates cover the entire AdS_5 hyperbola, and after extending τ, the whole of AdS_5.

The metric reads

$$\mathrm{d}s^2 = \mathcal{R}^2 \left[\mathrm{d}\rho^2 + \cosh^2\rho \left(-\cosh^2\psi \ \mathrm{d}\tau^2 + \mathrm{d}\psi^2 + \sinh^2\psi \ \mathrm{d}\Omega^2_{(2)} \right) \right] \ , \tag{3.108}$$

and after scaling ρ, this becomes $\mathrm{d}s^2 = \mathrm{d}\rho^2 + \mathcal{R}^2 \cosh^2\left(\frac{\rho}{\mathcal{R}}\right) \mathrm{d}s^2_{AdS_4}$, where $\mathrm{d}s^2_{AdS_4}$ is the global metric on an anti-de Sitter space AdS_4 of unit radius. The foliation by AdS_4 thus corresponds to $\rho =$ constant surfaces, and these slices are given by intersecting the planes $Y^2 =$ constant with the hyperbola. This foliation covers the entire AdS_5 space.

Comparing this with (2.13), we obtain

$$f(\pi) = \mathcal{R} \cosh(\pi/\mathcal{R}), \quad g_{\mu\nu} = g^{(AdS_4)}_{\mu\nu} \ , \tag{3.109}$$

and the terms (3.6) become (without any integrations by parts)

$$\mathcal{L}_1 = \sqrt{-g} \frac{\mathcal{R}^4}{32} \left(12\ \pi + 8\mathcal{R} \sinh\left(\frac{2\pi}{\mathcal{R}}\right) + \mathcal{R} \sinh\left(\frac{4\pi}{\mathcal{R}}\right) \right) \ , \tag{3.110}$$
$$\mathcal{L}_2 = -\sqrt{-g} \frac{\mathcal{R}^4}{\gamma} \cosh^4\left(\frac{\pi}{\mathcal{R}}\right) \ , \tag{3.111}$$
$$\mathcal{L}_3 = \sqrt{-g} \left[\gamma^2 [\pi^3] - \mathcal{R}^2 [\Pi] \cosh^2\left(\frac{\pi}{\mathcal{R}}\right) + \mathcal{R}^3 (5 - \gamma^2) \cosh^3\left(\frac{\pi}{\mathcal{R}}\right) \sinh\left(\frac{\pi}{\mathcal{R}}\right) \right] \ , \tag{3.112}$$

$$\begin{aligned}\mathcal{L}_4 = \sqrt{-g}\Bigg[&\frac{2\gamma^3}{\mathcal{R}^2}\left([\Pi][\pi^3]-[\pi^4]\right)\operatorname{sech}^2\left(\frac{\pi}{\mathcal{R}}\right)-\gamma\left([\Pi]^2-[\Pi^2]+\frac{8\gamma^2}{\mathcal{R}}[\pi^3]\tanh\left(\frac{\pi}{\mathcal{R}}\right)\right)\\ &+\mathcal{R}\gamma(4-\gamma^2)[\Pi]\sinh\left(\frac{2\pi}{\mathcal{R}}\right)\\ &+\frac{3\mathcal{R}^2}{\gamma}\cosh^2\left(\frac{\pi}{\mathcal{R}}\right)\left(2+3\gamma^2-\gamma^4+(2-3\gamma^2+\gamma^4)\cosh\left(\frac{2\pi}{\mathcal{R}}\right)\right)\Bigg]\,, \end{aligned}\tag{3.113}$$

$$\begin{aligned}\mathcal{L}_5 = \sqrt{-g}\Bigg[&\frac{3\gamma^4}{\mathcal{R}^4}\left(2([\pi^5]-[\Pi][\pi^4])+[\pi^3]([\Pi]^2-[\Pi^2])\right)\operatorname{sech}^4\left(\frac{\pi}{\mathcal{R}}\right)\\ &-\frac{18\gamma^4}{\mathcal{R}^3}\left([\Pi][\pi^3]-[\pi^4]\right)\operatorname{sech}^2\left(\frac{\pi}{\mathcal{R}}\right)\tanh\left(\frac{\pi}{\mathcal{R}}\right)\\ &-\frac{\gamma^2}{\mathcal{R}^2}\operatorname{sech}^2\left(\frac{\pi}{\mathcal{R}}\right)\left([\Pi]^3-3[\Pi][\Pi^2]+2[\Pi^3]+\frac{3}{2}(3+10\gamma^2)[\pi^3]\right.\\ &\left.+\frac{3}{2}(3-10\gamma^2)[\pi^3]\cosh\left(\frac{2\pi}{\mathcal{R}}\right)\right)+\frac{3\gamma^2}{\mathcal{R}}(3-\gamma^2)([\Pi]^2-[\Pi^2])\tanh\left(\frac{\pi}{\mathcal{R}}\right)\\ &+\frac{3}{2}[\Pi]\left(3+10\gamma^2-4\gamma^4+(3-10\gamma^2+4\gamma^4)\cosh\left(\frac{2\pi}{\mathcal{R}}\right)\right)\\ -&\frac{3\mathcal{R}}{4}\left(15+11\gamma^2-6\gamma^4+(15-17\gamma^2+6\gamma^4)\cosh\left(\frac{2\pi}{\mathcal{R}}\right)\right)\sinh\left(\frac{2\pi}{\mathcal{R}}\right)\Bigg]\,,\end{aligned}\tag{3.114}$$

where the background metric and covariant derivatives are those of a unit-radius AdS_4, and

$$\gamma=\frac{1}{\sqrt{1+\frac{(\partial\pi)^2}{\mathcal{R}^2\cosh^2\left(\frac{\pi}{\mathcal{R}}\right)}}}\,.\tag{3.115}$$

In evaluating (3.35), we have used that fact that the scalar curvature and cosmological constant of the unit-radius AdS_4 are $R=-12$ and $\Lambda=-3$ respectively, as well as the relations $R_{\mu\nu\alpha\beta}=\frac{R}{12}\left(g_{\mu\alpha}g_{\nu\beta}-g_{\mu\beta}g_{\nu\alpha}\right)$, $R_{\mu\nu}=\frac{R}{4}g_{\mu\nu}$ valid for a maximally symmetric space. Given suitable combinations of these Lagrangians so that a constant field $\pi(x)=\pi_0=\text{constant}$ is a solution to the equations of motion, $f(\pi_0)=\mathcal{R}\cosh\left(\frac{\pi_0}{\mathcal{R}}\right)$ sets the radius of the anti-de Sitter brane. We call these anti-de Sitter DBI Galileons (see Figure 3.1).

3.3.6.1 Killing vectors and symmetries

We use Poincare coordinates (u, x^0, x^1, x^2) on the AdS_4 slices. The embedding becomes,

$$Y^0 = \mathcal{R} \cosh\left(\frac{\rho}{\mathcal{R}}\right) \frac{1}{2u} \left(1 + u^2 + x^2\right) \ , \tag{3.116}$$

$$Y^1 = \mathcal{R} \cosh\left(\frac{\rho}{\mathcal{R}}\right) \frac{x^0}{u} \ , \tag{3.117}$$

$$Y^2 = \mathcal{R} \sinh\left(\frac{\rho}{\mathcal{R}}\right) \ , \tag{3.118}$$

$$Y^3 = \mathcal{R} \cosh\left(\frac{\rho}{\mathcal{R}}\right) \frac{1}{2u} \left(1 - u^2 - x^2\right) \ , \tag{3.119}$$

$$Y^{i+3} = \mathcal{R} \cosh\left(\frac{\rho}{\mathcal{R}}\right) \frac{x^i}{u}, \quad i = 1, 2 \ . \tag{3.120}$$

Here $x^2 \equiv \eta_{ij} x^i x^j$, where $\eta_{ij} = \mathrm{diag}(-1, 1, 1)$ is the Minkowski 3-metric. The coordinate ranges are $\rho \in (0, \infty)$, $u \in (0, \infty)$ and $x^i \in (-\infty, \infty)$, and the induced metric is

$$\mathrm{d}s^2 = \mathrm{d}\rho^2 + \mathcal{R}^2 \cosh^2\left(\frac{\rho}{\mathcal{R}}\right) \left[\frac{1}{u^2} \left(\mathrm{d}u^2 + \eta_{ij} \mathrm{d}x^i \mathrm{d}x^j\right)\right] \ . \tag{3.121}$$

The 15 Lorentz generators of $M_{4,2}$ are all tangent to the AdS_5 hyperboloid, and become the 15 isometries of the $SO(4,2)$ isometry algebra of AdS_5. Of these, 10 have no ∂_ρ components and are parallel to the AdS_4 foliation - these form the $SO(3,2)$ isometry algebra of the AdS_4 slices,

$$\begin{aligned}
-Y^3 \bar{\partial}_0 - Y^0 \bar{\partial}_3 &\to u \partial_u + x^i \partial_i \ , \\
-Y^1 \bar{\partial}_0 + Y^0 \bar{\partial}_1 &\to u x^0 \partial_u + \frac{1}{2} \left(1 + u^2 + x^2\right) \partial_0 + x^0 x^j \partial_j \ , \\
-Y^{i+3} \bar{\partial}_0 - Y^0 \bar{\partial}_{i+3} &\to u x_i \partial_u - \frac{1}{2} \left(1 + u^2 + x^2\right) \partial_i + x_i x^j \partial_j, \quad i = 1, 2 \\
-Y^3 \bar{\partial}_1 - Y^1 \bar{\partial}_3 &\to u x^0 \partial_u + \frac{1}{2} \left(-1 + u^2 + x^2\right) \partial_0 + x^0 x^j \partial_j \ , \\
-Y^{i+3} \bar{\partial}_3 + Y^3 \bar{\partial}_{i+3} &\to u x_i \partial_u - \frac{1}{2} \left(-1 + u^2 + x^2\right) \partial_i + x_i x^j \partial_j, \quad i = 1, 2 \\
Y^{i+3} \bar{\partial}_1 + Y^1 \bar{\partial}_{i+3} &\to x^i \partial_0 + x^0 \partial_i, \quad i = 1, 2
\end{aligned}$$

$$Y^5\bar{\partial}_4 + Y^4\bar{\partial}_5 \to x^2\partial_1 - x^1\partial_2 \ , \tag{3.122}$$

where the sums are over $j = 0, 1, 2$, and indices are raised and lowered with η_{ij}. These may be grouped as

$$d = u\partial_u + x^i\partial_i \ , \tag{3.123}$$
$$j_i^+ = ux_i\partial_u - \frac{1}{2}\left(1 + u^2 + x^2\right)\partial_i + x_ix^j\partial_j, \quad i = 0, 1, 2 \tag{3.124}$$
$$j_i^- = ux_i\partial_u - \frac{1}{2}\left(-1 + u^2 + x^2\right)\partial_i + x_ix^j\partial_j, \quad i = 0, 1, 2 \tag{3.125}$$
$$j_{ij} = x_i\partial_j - x_j\partial_i, \quad i, j = 0, 1, 2 \ , \tag{3.126}$$

and by taking the combinations

$$p_i = j_i^+ - j_i^- = -\partial_i \ , \tag{3.127}$$
$$k_i = j_i^+ + j_i^- = 2ux_i\partial_u - (u^2 + x^2)\partial_i + 2x_ix^j\partial_j \ , \tag{3.128}$$

we recognize p_i and j_{ij} as translations and rotations on the x-space, with d and k_i filling out the rest of the $SO(3,2)$ algebra.

The remaining 5 Killing vectors do have a ∂_ρ component,

$$\begin{aligned}
Y^2\bar{\partial}_0 + Y^0\bar{\partial}_2 \to K &= \frac{\mathcal{R}}{2u}\left(1 + u^2 + x^2\right)\partial_\rho + \frac{1}{2}\left(1 - u^2 + x^2\right)\tanh\left(\frac{\rho}{\mathcal{R}}\right)\partial_u \\
&\quad - u\tanh\left(\frac{\rho}{\mathcal{R}}\right)x^i\partial_i \ , \\
Y^3\bar{\partial}_2 - Y^2\bar{\partial}_3 \to K' &= \frac{\mathcal{R}}{2u}\left(1 - u^2 - x^2\right)\partial_\rho + \frac{1}{2}\left(1 + u^2 - x^2\right)\tanh\left(\frac{\rho}{\mathcal{R}}\right)\partial_u \\
&\quad + u\tanh\left(\frac{\rho}{\mathcal{R}}\right)x^i\partial_i \ , \\
Y^2\bar{\partial}_1 + Y^1\bar{\partial}_2 &\to \frac{\mathcal{R}}{u}x^0\partial_\rho + x^0\tanh\left(\frac{\rho}{\mathcal{R}}\right)\partial_u + u\tanh\left(\frac{\rho}{\mathcal{R}}\right)\partial_0 \ , \\
Y^{i+3}\bar{\partial}_2 - Y^2\bar{\partial}_{i+3} &\to \frac{\mathcal{R}}{u}x^i\partial_\rho + x^i\tanh\left(\frac{\rho}{\mathcal{R}}\right)\partial_u - u\tanh\left(\frac{\rho}{\mathcal{R}}\right)\partial_i, \quad i = 1, 2 \ ,
\end{aligned} \tag{3.129}$$

which may be combined to form

$$
\begin{aligned}
K_+ &= K + K' = \frac{\mathcal{R}}{u}\partial_\rho + \tanh\left(\frac{\rho}{\mathcal{R}}\right)\partial_u \ , \\
K_- &= K - K' = \frac{\mathcal{R}}{u}\left(u^2 + x^2\right)\partial_\rho + \left(-u^2 + x^2\right)\tanh\left(\frac{\rho}{\mathcal{R}}\right)\partial_u - 2u\tanh\left(\frac{\rho}{\mathcal{R}}\right)x^i\partial_i \ , \\
K_i &= \frac{\mathcal{R}}{u}x_i\partial_\rho + x_i\tanh\left(\frac{\rho}{\mathcal{R}}\right)\partial_u - u\tanh\left(\frac{\rho}{\mathcal{R}}\right)\partial_i, \quad i = 0, 1, 2.
\end{aligned} \tag{3.130}
$$

Using the relation $\delta_K\pi = K^5(x) - K^\mu(x,\pi)\partial_\mu\pi$ from (3.3), we obtain the transformation rules

$$
\delta_+\pi = \frac{\mathcal{R}}{u} - \tanh\left(\frac{\pi}{\mathcal{R}}\right)\pi' \ , \tag{3.131}
$$

$$
\delta_-\pi = \frac{\mathcal{R}}{u}\left(u^2 + x^2\right) - \left(-u^2 + x^2\right)\tanh\left(\frac{\pi}{\mathcal{R}}\right)\pi' + 2u\tanh\left(\frac{\pi}{\mathcal{R}}\right)x^i\partial_i\pi \ , \tag{3.132}
$$

$$
\delta_i\pi = \frac{\mathcal{R}}{u}x_i - x_i\tanh\left(\frac{\pi}{\mathcal{R}}\right)\pi' + u\tanh\left(\frac{\pi}{\mathcal{R}}\right)\partial_i\pi, \quad i = 0, 1, 2 \ , \tag{3.133}
$$

where $\pi' \equiv \partial_u\pi$.

The terms (3.114) are each invariant up to a total derivative under these transformations, and the symmetry breaking pattern is

$$
SO(4,2) \to SO(3,2) \ . \tag{3.134}
$$

3.4 Small field limits: the analogues of Galileons

The Lagrangians we have uncovered have a fairly complicated, non-polynomial form. We know in the Minkowski case that the special case of the Galileon symmetry arises in a particular limit [39], and that this limit greatly simplifies the actions. In this section, we consider similar limits for the general theories we have constructed.

Consider a Lagrangian $\mathcal{L}$ that may be expanded in some formal series in a parameter λ as

$$
\mathcal{L} = \lambda^n\left(\mathcal{L}_{(0)} + \lambda\mathcal{L}_{(1)} + \lambda^2\mathcal{L}_{(2)} + \cdots\right) \ , \tag{3.135}
$$

where n is an integer, indicating that the series need not start at order λ^0. Suppose $\mathcal{L}$ possesses a symmetry that may also be expanded in such a series

$$\delta\pi = \lambda^m \left(\delta_{(0)}\pi + \lambda\delta_{(1)}\pi + \lambda^2\delta_{(2)}\pi + \cdots\right) \ , \tag{3.136}$$

where m is another integer, again indicating that this series also need not start at order λ^0. The statement that $\delta\pi$ is a symmetry of $\mathcal{L}$ is

$$\frac{\delta^{EL}\mathcal{L}}{\delta\pi}\delta\pi \simeq 0 \ , \tag{3.137}$$

where $\frac{\delta^{EL}\mathcal{L}}{\delta\pi}$ is the Euler-Lagrange derivative and $\simeq$ indicates equality up to a total derivative.

Expanding (3.137) in powers of λ yields a series of equations

$$\frac{\delta^{EL}\mathcal{L}_{(0)}}{\delta\pi}\delta_{(0)}\pi \simeq 0 \ , \tag{3.138}$$
$$\begin{aligned} &\frac{\delta^{EL}\mathcal{L}_{(1)}}{\delta\pi}\delta_{(0)}\pi + \frac{\delta^{EL}\mathcal{L}_{(0)}}{\delta\pi}\delta_{(1)}\pi \simeq 0 \ , \\ &\vdots \end{aligned} \tag{3.139}$$

with the first of these indicating that $\delta_{(0)}$ is a symmetry of $\mathcal{L}_{(0)}$. Our goal in this section is to seek expansions of this form for the various examples we have constructed, in order to find simpler, but still non-trivial, theories with the same number of symmetries.

The expansion we choose is one in powers of the field π around some background. We expand π around a constant background value π_0 and let λ count powers of the deviation from this background; i.e. we make the replacement

$$\pi \to \pi_0 + \lambda\pi \ , \tag{3.140}$$

and then expand the Lagrangians and symmetries in powers of λ.

Applying this small field limit to the DBI Galileons (3.9) gives rise to the original Galileons first studied in [91]. These are, up to total derivatives,

$$
\begin{aligned}
\mathcal{L}_2 &= \pi \ , \\
\mathcal{L}_2 &= -\frac{1}{2}(\partial\pi)^2 \ , \\
\mathcal{L}_3 &= -\frac{1}{2}(\partial\pi)^2[\Pi] \ , \\
\mathcal{L}_4 &= -\frac{1}{2}(\partial\pi)^2\left([\Pi]^2 - [\Pi^2]\right) \ , \\
\mathcal{L}_5 &= -\frac{1}{2}(\partial\pi)^2\left([\Pi]^3 - 3[\Pi][\Pi^2] + 2[\Pi^3]\right) \ .
\end{aligned}
\tag{3.141}
$$

Note that lower order terms in the expansion are total derivatives. For example, in the expansion of $\mathcal{L}_4$ there exists an $\mathcal{O}\left(\pi^2\right)$ piece, but this is a total derivative in Minkowski space, and the first non-trivial term is the $\mathcal{O}\left(\pi^4\right)$ piece shown above.

Applying the small field limit to the transformation laws (3.12) yields

$$
\begin{aligned}
\delta\pi &= 1 \ , \\
\delta_\mu\pi &= x_\mu \ ,
\end{aligned}
\tag{3.142}
$$

under which the terms (3.141) are invariant. This is the original Galilean symmetry considered in [91]. The small field limit can also be applied to the case of a flat brane embedded in an AdS_5 bulk (3.19), but the resulting actions and transformation laws are identical to those of (3.141), (3.142).

Applying this technique to a de Sitter brane embedded in a flat bulk, we expand (3.35) around some constant background. The following linear combinations allow us to successively cancel the lowest order terms in λ up to total derivatives on dS_4, yielding terms which start at order λ, λ^2, etc.

$$
\bar{\mathcal{L}}_1 = \frac{1}{\pi_0^4}\mathcal{L}_1 = \sqrt{-g}\pi \ ,
$$

$$\bar{\mathcal{L}}_2 = \frac{1}{\pi_0^2}\left(\mathcal{L}_2 + \frac{4}{\pi_0}\mathcal{L}_1\right) = -\frac{1}{2}\sqrt{-g}\left((\partial\pi)^2 - 4\pi^2\right) ,$$
$$\bar{\mathcal{L}}_3 = \mathcal{L}_3 + \frac{6}{\pi_0}\mathcal{L}_2 + \frac{12}{\pi_0^2}\mathcal{L}_1 = \sqrt{-g}\left(-\frac{1}{2}(\partial\pi)^2[\Pi] - 3(\partial\pi)^2\pi + 4\pi^3\right) ,$$
$$\bar{\mathcal{L}}_4 = \pi_0^2\left(\mathcal{L}_4 + \frac{6}{\pi_0}\mathcal{L}_3 + \frac{18}{\pi_0^2}\mathcal{L}_2 + \frac{24}{\pi_0^3}\mathcal{L}_1\right)$$
$$= \sqrt{-g}\left[-\frac{1}{2}(\partial\pi)^2\left([\Pi]^2 - [\Pi^2] + \frac{1}{2}(\partial\pi)^2 + 6\pi[\Pi] + 18\pi^2\right) + 6\pi^4\right] ,$$
$$\bar{\mathcal{L}}_5 = \pi_0^4\left(\mathcal{L}_5 + \frac{4}{\pi_0}\mathcal{L}_4 + \frac{12}{\pi_0^2}\mathcal{L}_3 + \frac{24}{\pi_0^3}\mathcal{L}_2 + \frac{24}{\pi_0^4}\mathcal{L}_1\right)$$
$$= \sqrt{-g}\left[-\frac{1}{2}\left((\partial\pi)^2 + \frac{1}{5}\pi^2\right)\left([\Pi]^3 - 3[\Pi][\Pi^2] + 2[\Pi^3]\right)\right.$$
$$\left. -\frac{12}{5}\pi(\partial\pi)^2\left([\Pi]^2 - [\Pi^2] + \frac{27}{12}[\Pi]\pi + 5\pi^2\right) + \frac{24}{5}\pi^5\right] . \tag{3.143}$$

Scaling the coordinates to $(\hat{u}, \hat{y}^i) \equiv (Lu, Ly^i)$, carrying dimensions of length, the dS_4 curvature becomes $R = \frac{12}{L^2}$, and canonically normalizing the field to $\hat{\pi} = \frac{1}{L^2}\pi$, we then obtain

$$\hat{\mathcal{L}}_1 = \sqrt{-g}\hat{\pi} ,$$
$$\hat{\mathcal{L}}_2 = -\frac{1}{2}\sqrt{-g}\left((\partial\hat{\pi})^2 - \frac{4}{L^2}\hat{\pi}^2\right) ,$$
$$\hat{\mathcal{L}}_3 = \sqrt{-g}\left(-\frac{1}{2}(\partial\hat{\pi})^2[\hat{\Pi}] - \frac{3}{L^2}(\partial\hat{\pi})^2\hat{\pi} + \frac{4}{L^4}\hat{\pi}^3\right) ,$$
$$\hat{\mathcal{L}}_4 = \sqrt{-g}\left[-\frac{1}{2}(\partial\hat{\pi})^2\left([\hat{\Pi}]^2 - [\hat{\Pi}^2] + \frac{1}{2L^2}(\partial\hat{\pi})^2 + \frac{6}{L^2}\hat{\pi}[\hat{\Pi}] + \frac{18}{L^4}\hat{\pi}^2\right) + \frac{6}{L^6}\hat{\pi}^4\right] ,$$
$$\hat{\mathcal{L}}_5 = \sqrt{-g}\left[-\frac{1}{2}\left((\partial\hat{\pi})^2 + \frac{1}{5L^2}\hat{\pi}^2\right)\left([\hat{\Pi}]^3 - 3[\hat{\Pi}][\hat{\Pi}^2] + 2[\hat{\Pi}^3]\right)\right.$$
$$\left. -\frac{12}{5L^2}\hat{\pi}(\partial\hat{\pi})^2\left([\hat{\Pi}]^2 - [\hat{\Pi}^2] + \frac{27}{12L^2}[\hat{\Pi}]\hat{\pi} + \frac{5}{L^4}\hat{\pi}^2\right) + \frac{24}{5L^8}\hat{\pi}^5\right] , \tag{3.144}$$

where $\hat{\mathcal{L}}_n = \frac{1}{L^{4n+2}}\bar{\mathcal{L}}_n$.

These expressions are invariant under the lowest order symmetry transformations obtained by taking the small field limit of (3.45),

$$\begin{aligned}\delta_+\hat{\pi} &= \frac{1}{u}\left(u^2 - y^2\right) \ , \\ \delta_-\hat{\pi} &= -\frac{1}{u} \ , \\ \delta_i\hat{\pi} &= \frac{y_i}{u} \ . \end{aligned} \tag{3.145}$$

The terms (3.144) are Galileons which naturally live in de Sitter space, and become the original Galileons in the limit where the dS_4 radius goes to infinity. They have the same number of nonlinear shift-like symmetries as the original flat space Galileons, despite the fact that they live on a curved space. As such, we anticipate them being naturally suited to models of inflation and dark energy.

Another fascinating new feature that is not shared by the original Galileons is the existence of a potential. In particular, the quadratic term $\hat{\mathcal{L}}_2$ comes with a mass term of order the 4D de Sitter radius. The symmetries (3.145) fix the value of the mass (in fact, each of the symmetries in (3.145) is alone sufficient to fix the mass). If the coefficient of $\hat{\mathcal{L}}_2$ is chosen to be positive, so that the scalar field is not a ghost, then this mass is tachyonic. However, this instability is not necessarily worrisome because its timescale is of order the de Sitter time. Furthermore, this small mass should not be renormalized, because its value is protected by symmetry. The higher terms also come with cubic, quartic, and quintic terms in the potential, with values tied to the kinetic structure by the symmetries.

The small field limit may also be applied to the examples of a de Sitter brane embedded in either a de Sitter (3.62) or anti-de Sitter (3.89) bulk. The resulting actions and transformation laws are identical to those of (3.144) and (3.145).

Finally, we apply the small field expansion to the case of an anti-de Sitter brane embedded in an anti-de Sitter bulk, by expanding the terms (3.114) around a constant background π_0. In a similar manner to the previous case, the following linear combinations yield terms which start at order λ, λ^2, etc. up to total derivatives.

$$
\begin{aligned}
\bar{\mathcal{L}}_1 &= \frac{1}{L^4}\mathcal{L}_1 = \sqrt{-g}\pi \ , \\
\bar{\mathcal{L}}_2 &= \frac{1}{L^2}\left[\mathcal{L}_2 + \frac{4}{\mathcal{R}}\tanh\left(\frac{\pi_0}{\mathcal{R}}\right)\mathcal{L}_1\right] = -\frac{1}{2}\sqrt{-g}\left((\partial\pi)^2 + 4\pi^2\right) \ , \\
\bar{\mathcal{L}}_3 &= \mathcal{L}_3 + \frac{6}{\mathcal{R}}\tanh\left(\frac{\pi_0}{\mathcal{R}}\right)\mathcal{L}_2 + \frac{4}{\mathcal{R}^2}\left(2 - 3\ \mathrm{sech}^2\left(\frac{\pi_0}{\mathcal{R}}\right)\right)\mathcal{L}_1 \\
&= \sqrt{-g}\left(-\frac{1}{2}(\partial\pi)^2[\Pi] + 3(\partial\pi)^2\pi + 4\pi^3\right) \ , \\
\bar{\mathcal{L}}_4 &= L^2\left[\mathcal{L}_4 + \frac{6}{\mathcal{R}}\tanh\left(\frac{\pi_0}{\mathcal{R}}\right)\mathcal{L}_3 + \frac{6}{\mathcal{R}^2}\left(4 - 3\ \mathrm{sech}^2\left(\frac{\pi_0}{\mathcal{R}}\right)\right)\mathcal{L}_2 - \frac{24}{\mathcal{R}^3}\mathrm{sech}^2\left(\frac{\pi_0}{\mathcal{R}}\right)\tanh\left(\frac{\pi_0}{\mathcal{R}}\right)\mathcal{L}_1\right] \\
&= \sqrt{-g}\left[-\frac{1}{2}(\partial\pi)^2\left([\Pi]^2 - [\Pi^2] - \frac{1}{2}(\partial\pi)^2 - 6\pi[\Pi] + 18\pi^2\right) - 6\pi^4\right] \ , \\
\bar{\mathcal{L}}_5 &= L^4\left[\mathcal{L}_5 + \frac{4}{\mathcal{R}}\tanh\left(\frac{\pi_0}{\mathcal{R}}\right)\mathcal{L}_4 + \frac{3}{\mathcal{R}^2}\left(5 - 4\ \mathrm{sech}^2\left(\frac{\pi_0}{\mathcal{R}}\right)\right)\mathcal{L}_3\right. \\
&\quad\left. + \frac{12}{\mathcal{R}^3}\mathrm{sech}^3\left(\frac{\pi_0}{\mathcal{R}}\right)\left(\sinh\left(\frac{3\pi_0}{\mathcal{R}}\right) - \sinh\left(\frac{\pi_0}{\mathcal{R}}\right)\right)\mathcal{L}_2 + \frac{24}{\mathcal{R}^4}\mathrm{sech}^4\left(\frac{\pi_0}{\mathcal{R}}\right)\mathcal{L}_1\right] \\
&= \sqrt{-g}\left[-\frac{1}{2}\left((\partial\pi)^2 - \frac{1}{5}\pi^2\right)\left([\Pi]^3 - 3[\Pi][\Pi^2] + 2[\Pi^3]\right)\right. \\
&\quad\left. + \frac{12}{5}\pi(\partial\pi)^2\left([\Pi]^2 - [\Pi^2] - \frac{27}{12}[\Pi]\pi + 5\pi^2\right) + \frac{24}{5}\pi^5\right] \ ,
\end{aligned}
\tag{3.146}
$$

where $L = \mathcal{R}\cosh^4\left(\frac{\pi_0}{\mathcal{R}}\right)$ is the $AdS_{3,1}$ radius.

Scaling the coordinates to $(\hat{u}, \hat{x}^i) \equiv (Lu, Ly^i)$ so that they carry dimensions of length, the AdS_4 curvature becomes $R = -\frac{12}{L^2}$, and canonically normalizing the field to $\hat{\pi} = \frac{1}{L^2}\pi$, we then obtain

$$
\begin{aligned}
\hat{\mathcal{L}}_1 &= \sqrt{-g}\hat{\pi} \ , \\
\hat{\mathcal{L}}_2 &= -\frac{1}{2}\sqrt{-g}\left((\partial\hat{\pi})^2 + \frac{4}{L^2}\hat{\pi}^2\right) \ , \\
\hat{\mathcal{L}}_3 &= \sqrt{-g}\left(-\frac{1}{2}(\partial\hat{\pi})^2[\hat{\Pi}] + \frac{3}{L^2}(\partial\hat{\pi})^2\hat{\pi} + \frac{4}{L^4}\hat{\pi}^3\right) \ ,
\end{aligned}
$$

$$\hat{\mathcal{L}}_4 = \sqrt{-g}\left[-\frac{1}{2}(\partial\hat{\pi})^2\left([\hat{\Pi}]^2 - [\hat{\Pi}^2] - \frac{1}{2L^2}(\partial\hat{\pi})^2 - \frac{6}{L^2}\hat{\pi}[\hat{\Pi}] + \frac{18}{L^4}\hat{\pi}^2\right) - \frac{6}{L^6}\hat{\pi}^4\right] ,$$
$$\hat{\mathcal{L}}_5 = \sqrt{-g}\left[-\frac{1}{2}\left((\partial\hat{\pi})^2 - \frac{1}{5L^2}\hat{\pi}^2\right)\left([\hat{\Pi}]^3 - 3[\hat{\Pi}][\hat{\Pi}^2] + 2[\hat{\Pi}^3]\right)\right.$$
$$\left. + \frac{12}{5L^2}\hat{\pi}(\partial\hat{\pi})^2\left([\hat{\Pi}]^2 - [\hat{\Pi}^2] - \frac{27}{12L^2}[\hat{\Pi}]\hat{\pi} + \frac{5}{L^4}\hat{\pi}^2\right) + \frac{24}{5L^8}\hat{\pi}^5\right] , \qquad (3.147)$$

where $\hat{\mathcal{L}}_n = \frac{1}{L^{4n+2}}\bar{\mathcal{L}}_n$.

These terms are invariant under the lowest order symmetry transformations obtained by taking the small field limit of (3.133)

$$\delta_{+(0)}\hat{\pi} = \frac{\mathcal{R}}{u} ,$$
$$\delta_{-(0)}\hat{\pi} = \frac{\mathcal{R}}{u}\left(u^2 + x^2\right) ,$$
$$\delta_{i(0)}\hat{\pi} = \frac{\mathcal{R}}{u}x_i, \quad i = 0, 1, 2 . \qquad (3.148)$$

These are Galileons that live on anti-de Sitter space. In this case, the quadratic term comes with a non-tachyonic mass of order the AdS_4 radius.

While we have focused on the construction of new effective field theories through the small field expansion of embedded brane models, it is important to note that there may well exist other expansions that lead to different theories in the limit. For the example of a flat brane embedded in an anti-de Sitter bulk (3.19), the theory admits an expansion in powers of derivatives. Up to total derivatives, the derivative expansion yields

$$\bar{\mathcal{L}}_1 = \frac{1}{\mathcal{R}}\mathcal{L}_1 = -\frac{1}{4}e^{-4\hat{\pi}} ,$$
$$\bar{\mathcal{L}}_2 = \frac{1}{\mathcal{R}^2}\left(\mathcal{L}_2 - \frac{4}{\mathcal{R}}\mathcal{L}_1\right) = -\frac{1}{2}e^{-2\hat{\pi}}(\partial\hat{\pi})^2 ,$$
$$\bar{\mathcal{L}}_3 = \frac{1}{\mathcal{R}^3}\left(\mathcal{L}_3 - \frac{6}{\mathcal{R}}\mathcal{L}_2 + \frac{8}{\mathcal{R}^2}\mathcal{L}_1\right) = -\frac{1}{2}(\partial\hat{\pi})^2\Box\hat{\pi} + \frac{1}{4}(\partial\hat{\pi})^4 ,$$

$$
\begin{aligned}
\bar{\mathcal{L}}_4 &= \frac{1}{\mathcal{R}^4}\left(\mathcal{L}_4 - \frac{6}{\mathcal{R}}\mathcal{L}_3 + \frac{24}{\mathcal{R}^2}\mathcal{L}_2\right) \\
&= -\frac{1}{2}e^{2\hat{\pi}}(\partial\hat{\pi})^2\left([\hat{\Pi}]^2 - [\hat{\Pi}^2] + \frac{2}{5}((\partial\hat{\pi})^2\Box\hat{\pi} - [\hat{\pi}^3]) + \frac{3}{10}(\partial\hat{\pi})^4\right)\ , \\
\bar{\mathcal{L}}_5 &= \frac{1}{\mathcal{R}^5}\left(\mathcal{L}_5 - \frac{4}{\mathcal{R}}\mathcal{L}_4 + \frac{15}{\mathcal{R}^2}\mathcal{L}_3 - \frac{48}{\mathcal{R}^3}\mathcal{L}_2\right) \\
&= -\frac{1}{2}e^{4\hat{\pi}}(\partial\hat{\pi})^2\Big[[\hat{\Pi}]^3 - 3[\hat{\Pi}][\hat{\Pi}^2] + 2[\hat{\Pi}^3] + 3(\partial\hat{\pi})^2([\hat{\Pi}]^2 - [\hat{\Pi}^2]) \\
&\quad + \frac{30}{7}(\partial\hat{\pi})^2((\partial\hat{\pi})^2[\hat{\Pi}] - [\hat{\pi}^3]) - \frac{3}{28}(\partial\hat{\pi})^6\Big]\ ,
\end{aligned} \tag{3.149}
$$

where $\hat{\pi} \equiv \pi/\mathcal{R}$. These are the conformal Galileons [39, 77, 91]. Their transformation laws come from applying the derivative expansion to the transformation laws (3.28),

$$
\begin{aligned}
\delta\hat{\pi} &= 1 - x^\mu\partial_\mu\hat{\pi}, \\
\delta_\mu\hat{\pi} &= 2x_\mu + x^2\partial_\mu\hat{\pi} - 2x_\mu x^\nu\partial_\nu\hat{\pi}\ .
\end{aligned} \tag{3.150}
$$

In taking the limit in powers of derivatives, we must remember that the explicit factors of the coordinates in the transformation laws are assigned a power of inverse derivatives. The terms (3.149) are each invariant up to a total derivative under (3.150). As mentioned in [39], it is remarkable that this limit does not alter the commutation relations of the symmetries, so that the algebra remains $SO(4,2)$.

The derivative expansion can also be applied to the DBI Galileons (3.9). The result is identical to the small field limit, since the powers of π and powers of ∂ within each limiting Lagrangian are identical.

A derivative expansion does not, however, seem applicable in general. To see the problem, attempt to construct an order four derivative term from the general Lagrangians in (3.6). It is necessary to find a constant A such that the two derivative part in the expression $\mathcal{L}_3 + A\mathcal{L}_2$ is a total derivative. The two derivative part reads $\sqrt{-g}\left(3ff' - \frac{A}{2}f^2\right)(\partial\pi)^2$, up to a total derivative, and for this to vanish we must have $f \propto e^{A\pi/6}$. The only cases of ours that conform to this are the conformal DBI Galileons ($A \neq 0$) and the ordinary DBI Galileons ($A = 0$).

3.4.1 Symmetry breaking and ghosts

By writing the actions of the previous section in terms of the scalar curvature, $R = \frac{12}{L^2}$ for dS_4, $R = -\frac{12}{L^2}$ for AdS_4, and $R = 0$ for M_4, it is possible to combine the dS_4 Galileons (3.144), the AdS_4 Galileons (3.147) and the flat space Galileons (3.141) into the single set of expressions

$$\begin{aligned}
\hat{\mathcal{L}}_1 &= \sqrt{-g}\hat{\pi}\ , \\
\hat{\mathcal{L}}_2 &= -\frac{1}{2}\sqrt{-g}\left((\partial\hat{\pi})^2 - \frac{R}{3}\hat{\pi}^2\right)\ , \\
\hat{\mathcal{L}}_3 &= \sqrt{-g}\left(-\frac{1}{2}(\partial\hat{\pi})^2[\hat{\Pi}] - \frac{R}{4}(\partial\hat{\pi})^2\hat{\pi} + \frac{R^2}{36}\hat{\pi}^3\right)\ , \\
\hat{\mathcal{L}}_4 &= \sqrt{-g}\left[-\frac{1}{2}(\partial\hat{\pi})^2\left([\hat{\Pi}]^2 - [\hat{\Pi}^2] + \frac{R}{24}(\partial\hat{\pi})^2 + \frac{R}{2}\hat{\pi}[\hat{\Pi}] + \frac{R^2}{8}\hat{\pi}^2\right) + \frac{R^3}{288}\hat{\pi}^4\right]\ , \\
\hat{\mathcal{L}}_5 &= \sqrt{-g}\left[-\frac{1}{2}\left((\partial\hat{\pi})^2 + \frac{R}{60}\hat{\pi}^2\right)\left([\hat{\Pi}]^3 - 3[\hat{\Pi}][\hat{\Pi}^2] + 2[\hat{\Pi}^3]\right)\right. \\
&\quad \left. -\frac{R}{5}\hat{\pi}(\partial\hat{\pi})^2\left([\hat{\Pi}]^2 - [\hat{\Pi}^2] + \frac{3R}{16}[\hat{\Pi}]\hat{\pi} + \frac{5R^2}{144}\hat{\pi}^2\right) + \frac{R^4}{4320}\hat{\pi}^5\right]\ . \qquad (3.151)
\end{aligned}$$

Focusing on $\hat{\mathcal{L}}_2$, we note that the non-linear symmetries fix the sign of the mass term relative to that of the kinetic term. Therefore, in de Sitter space, where R is positive, the scalar is either a tachyon or a ghost, depending on the overall sign of $\hat{\mathcal{L}}_2$. In AdS on the other hand, where $R < 0$, the scalar can be stable and ghost free if the sign of $\hat{\mathcal{L}}_2$ is chosen to be positive[9].

The presence of a tachyon suggests spontaneous symmetry breaking, as there may be higher order terms in the potential which stabilize it. In this section, we explore the possibility of using the tachyon of the de Sitter Galileons to induce spontaneous symmetry

[9] A scalar in AdS can tolerate a slightly negative mass without instability. Any mass squared larger than the Breitenhloer Friedman bound $m^2 \geq -\frac{9}{4L^2} = \frac{3}{16}R$ is stable [14]. However, we cannot make use of this in any way, since the AdS scalar is ghostlike whenever its mass squared is negative.

breaking. More specifically, consider imposing a Z_2 symmetry $\pi \to -\pi$, which forbids the odd terms $\hat{\mathcal{L}}_3$ and $\hat{\mathcal{L}}_5$.[10] In the dS case and AdS case respectively, a symmetry breaking potential can be achieved by choosing

$$\hat{\mathcal{L}}_2 - a\hat{\mathcal{L}}_4, \quad dS \ , \tag{3.152}$$

$$-\hat{\mathcal{L}}_2 + a\hat{\mathcal{L}}_4, \quad AdS \ , \tag{3.153}$$

with coupling constant $a > 0$. In both cases, the potential is

$$V(\pi) = \frac{|R|}{288}\left(-48\pi^2 + aR^2\pi^4\right) \ . \tag{3.154}$$

This has a Z_2 preserving vacuum at $\pi = 0$ and Z_2 breaking vacua at $\pi = \pm\sqrt{\frac{24}{a}\frac{1}{|R|}}$.

None of these vacua alter any of the Galilean symmetries of these models. Thus, expanding around one of the minima (the positive one, say), we obtain a Lagrangian which is also a combination of the terms (3.151), with coefficients depending only on the original coefficient a,

$$-2\hat{\mathcal{L}}_2 - \sqrt{6a}\hat{\mathcal{L}}_3 - a\hat{\mathcal{L}}_4, \quad dS \ , \tag{3.155}$$

$$2\hat{\mathcal{L}}_2 - \sqrt{6a}\hat{\mathcal{L}}_3 + a\hat{\mathcal{L}}_4, \quad AdS \ . \tag{3.156}$$

In the dS case, the field has a normal sign kinetic term around the tachyonic $\pi = 0$ solution, and a ghostly kinetic term around the symmetry breaking vacuum. In the AdS case, the field is a ghost around the tachyonic $\pi = 0$ solution, and is ghost-free around the symmetry breaking vacuum. In this case we see a version of ghost condensation along with the usual tachyon condensation. See figure 3.2.

[10] This is interesting in its own right. Imposing this symmetry on the original Galileons gives an interacting scalar field theory which in suitable regimes has only one possible interaction term $\hat{\mathcal{L}}_4$, which furthermore is not renormalized. This is the co-dimension one version of introducing an internal $SO(N)$ symmetry in a theory with a multiplet of N Galileons, which also yields a single possible interaction term [73].

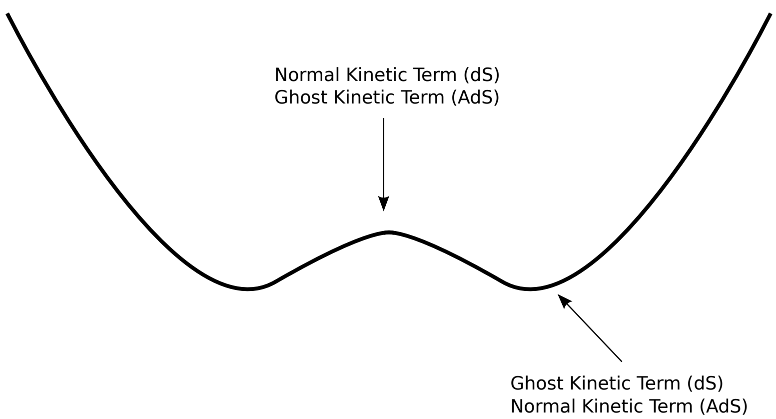

Figure 3.2: Z_2 symmetry breaking for the dS/AdS Galileons.

3.5 Stability and Subluminality for DBI Galileons

In this section we find static, spherically symmetric solutions of DBI galileon theory of Sec. 3.3.1, and explore their stability. Such an analysis was performed for the ordinary galileons in [91], and for multi-galileon theories in [3, 97]. For the original galileon model [91], it was found that for some choices of parameters, stable solutions exists but always contain superluminal signal propagation. We follow the same approach here, extending the results to the DBI galileons, and reach similar conclusions. The analysis is only valid in the $M_p \to \infty$ limit. As shown in [45], the stability of these theories depends on terms suppressed by the square of the Planck mass.

3.5.1 Equations of Motion

The first task is to derive the equations of motion for DBI galileons coupled to matter. From (3.9), the lagrangians under consideration are

$$\begin{aligned}
\mathcal{L}_2 &= -\sqrt{1+(\partial\pi)^2}\ , \\
\mathcal{L}_3 &= -[\Pi] + \gamma^2\left[\pi^3\right]\ , \\
\mathcal{L}_4 &= -\gamma\left([\Pi]^2 - \left[\Pi^2\right]\right) - 2\gamma^3\left(\left[\pi^4\right] - [\Pi]\left[\pi^3\right]\right)\ , \\
\mathcal{L}_5 &= -\gamma^2\left([\Pi]^3 + 2\left[\Pi^3\right] - 3[\Pi]\left[\Pi^2\right]\right) - \gamma^4\left(6[\Pi]\left[\pi^4\right] - 6\left[\pi^5\right] - 3\left([\Pi]^2 - \left[\Pi^2\right]\right)\left[\pi^3\right]\right)\ ,
\end{aligned} \tag{3.157}$$

where we have explicitly retained all total derivatives and we neglect the tadpole term.

The resulting equations of motion take the form $\mathcal{E}_n = 0$, with $n = 2, 3, 4, 5$, and

$$\mathcal{E}_2 = \gamma[\Pi] - \gamma^3\left[\pi^3\right]\ , \tag{3.158}$$

$$\mathcal{E}_3 = \gamma^2\left([\Pi]^2 - \left[\Pi^2\right]\right) + 2\gamma^4\left(\left[\pi^4\right] - [\Pi]\left[\pi^3\right]\right)\ , \tag{3.159}$$

$$\mathcal{E}_4 = \gamma^3\left([\Pi]^3 + 2\left[\Pi^3\right] - 3[\Pi]\left[\Pi^2\right]\right) + \gamma^5\left(6[\Pi]\left[\pi^4\right] - 6\left[\pi^5\right] - 3\left([\Pi]^2 - \left[\Pi^2\right]\right)\left[\pi^3\right]\right)\ , \tag{3.160}$$

$$\mathcal{E}_5 = \gamma^6 \left([\Pi]^4 - 6\left[\Pi^2\right][\Pi]^2 + 8[\Pi]\left[\Pi^3\right] + 3\left[\Pi^2\right]^2 - 6\left[\Pi^4\right] \right) \ . \tag{3.161}$$

These satisfy the following interesting recursion relation noticed in [39],

$$\frac{\delta}{\delta\pi}\left(\sqrt{-g}\,\right) = K \ , \tag{3.162}$$

$$\frac{\delta}{\delta\pi}\left(\sqrt{-g}\,K\right) = R \ , \tag{3.163}$$

$$\frac{\delta}{\delta\pi}\left(\sqrt{-g}\,R\right) = \frac{3}{2}\mathcal{K}_{GB} \ , \tag{3.164}$$

$$\frac{\delta}{\delta\pi}\left(\sqrt{-g}\,\mathcal{K}_{GB}\right) = \frac{2}{3}\mathcal{L}_{GB_4} \ , \tag{3.165}$$

where $\mathcal{L}_{GB_4} = R^2 - 4R^2_{\mu\nu} + R^2_{\mu\nu\alpha\beta}$ is the second order Lovelock invariant.

In this paper we consider a theory containing these terms with arbitrary coefficients d_n, and which is linearly coupled to the trace T of the energy momentum tensor of matter, so that the complete Lagrangian density is

$$\mathcal{L} = \sum_{n=2}^{5} d_n \mathcal{L}_n + \pi T \ , \tag{3.166}$$

with equation of motion $\mathcal{E} = 0$, where

$$\mathcal{E} \equiv \sum_{n=2}^{5} d_n \mathcal{E}_n + T \ . \tag{3.167}$$

The linear coupling is not invariant under the symmetry operation (3.12). Rather, it was chosen for simplicity and for comparison with the results of [91] where the same choice was made. It is also the coupling that arises if the scalar is considered as a modification to gravity that conformally mixes with the graviton, as happens in the DGP model. In like of the probe brane procedure, a perhaps more natural coupling between the galileon and matter which respects (3.12) could come from $\bar{g}_{\mu\nu}T^{\mu\nu} \supset \partial_\mu\pi\partial_\nu\pi T^{\mu\nu}$, but this gives no contribution to the equations of motion for static sources and is hence uninteresting for our purposes.

Our goal is to derive constraints on this model from the requirements of stability and subluminality of mode propagation around spherically symmetric backgrounds. We shall begin this analysis in the next section, but it is important to note that one constraint can be seen immediately;

$$d_2 > 0 \ , \tag{3.168}$$

since otherwise the kinetic term will yield a ghost (or will be absent, indicating strong coupling, if we set $d_2 = 0$).

3.5.2 Spherical solutions

We search for static spherically symmetric solutions to the equations of motion in spherical polar coordinates (r, θ, ϕ), in the presence of a positive mass delta function source at the origin

$$T = -M\delta^3(r), \quad M > 0 \ . \tag{3.169}$$

To evaluate the equations of motion we need find only the non-vanishing elements of $\Pi_{\mu\nu} = \partial_\mu \partial_\nu \pi - \Gamma^\alpha_{\mu\nu} \partial_\alpha \pi$. These are $\Pi_{rr} = \pi_{,rr}$, $\Pi_{\theta\theta} = r\pi_{,r}$, and $\Pi_{\phi\phi} = r \sin^2\theta \pi_{,r}$. Since the flat metric is diagonal we then have

$$[\Pi^n] = (\Pi_{rr}\eta^{rr})^n + \left(\Pi_{\theta\theta}\eta^{\theta\theta}\right)^n + \left(\Pi_{\phi\phi}\eta^{\phi\phi}\right)^n = \pi^n_{,rr} + \frac{2\pi^n_{,r}}{r^n} \ , \tag{3.170}$$

$$\left[\pi^{n+2}\right] = \pi^2_{,r} \left(\Pi_{rr}\right)^n \left(\eta^{rr}\right)^{n+1} = \pi^2_{,r}\pi^n_{,rr} \ . \tag{3.171}$$

Using these, the equations of motion (3.167) become

$$\mathcal{E}_2 = \frac{1}{r^2}\frac{d}{dr}\left[r^3 y\right] \ , \tag{3.172}$$

$$\mathcal{E}_3 = \frac{2}{r^2}\frac{d}{dr}\left[r^3 y^2\right] \ , \tag{3.173}$$

$$\mathcal{E}_4 = \frac{2}{r^2}\frac{d}{dr}\left[r^3 y^3\right] \ , \tag{3.174}$$

$$\mathcal{E}_5 = 0 \ , \tag{3.175}$$

where we have defined

$$y \equiv \frac{\gamma\pi'}{r} \ . \tag{3.176}$$

The fifth order term vanishes because our focus on static solutions causes the problem to effectively reduce to a three dimensional one, and the fifth order term is trivial in three dimensions. The remaining equations of motion can be written as a polynomial in y as

$$\frac{1}{r^2}\frac{d}{dr}\left[r^3 P(y)\right] = M\delta^3(r) \ , \tag{3.177}$$

with

$$P(y) \equiv d_2 y + 2d_3 y^2 + 2d_4 y^3 \ . \tag{3.178}$$

Note that the equations of motion are a total r-derivative. This is a consequence of the shift invariance $\pi \to \pi + c$ of the Lagrangian, which has an associated Noether current J^μ, in terms of which the equations of motion take the form $\partial_\mu(-J^\mu) = 0$. We may therefore integrate the equations of motion once to obtain

$$P(y) = \frac{M}{4\pi r^3} \ . \tag{3.179}$$

We now study the existence of spherically symmetric solutions, and the resulting constraints on the coefficients d_2, d_3, d_4. Our boundary condition is that π approaches a constant as $r \to \infty$. The other boundary condition is fixed by the delta function at the origin. Focusing on small r, (3.179) yields

$$\frac{\pi'^3}{(1+\pi'^2)^{3/2}} d_4 = \frac{M}{8\pi} \ . \tag{3.180}$$

This determines a finite value for π' at the origin, and therefore implies that π must also be finite there. Since the absolute value of the prefactor in front of d_4 on the left hand side is always less than unity, we then obtain the constraint

$$|d_4| > \frac{M}{8\pi} \ . \tag{3.181}$$

This constraint is unique to the DBI action - no such constraint arises in the usual galileon theories. The fourth-order term dominates at short distances, and its non-linearities render π finite at the origin. In particular therefore, note that there are no spherically symmetric static solutions in the pure DBI model, for which $d_3 = d_4 = 0$.

As we have demonstrated, $\pi'(r)$ ranges from some finite non-zero value at $r = 0$, to zero as $r \to \infty$ (since π itself goes to a constant). Thus, the variable $y = \gamma\pi'/r$ ranges from infinity to zero as r ranges from zero to infinity (we will see shortly that it does so monotonically).

As r varies from the origin to infinity, the right hand side of (3.179) ranges from zero to infinity, so the cubic polynomial on the left must do so as well. Looking at small y, along with the requirement $d_2 > 0$ for a healthy kinetic term, tells us that $P(y)$ intersects the origin and is monotonically increasing near the origin, and hence that y as a function of r is monotonically decreasing in the same region. As y gets larger (r smaller, $P(y)$ larger), the solution for $y(r)$ must continue to exist and be smooth, which means that $P(y)$ must not have any of its critical points in the region $y > 0$. Thus $P(y)$ monotonically increases for $y > 0$, and hence $y(r)$ is monotonically decreasing for $r > 0$. Looking at the form of y, this implies in turn that $\pi'(r)$ is monotonic, ranging from some finite value to zero as r goes from zero to infinity. Integrating, we see that $\pi(r)$ is monotonic as well.

The condition we have then is

$$P'(y) = d_2 + 4d_3 y + 6d_4 y^2 > 0, \quad \text{for } y > 0 \ . \tag{3.182}$$

Focusing on large y implies that $d_4 \geq 0$, so that we can now remove the absolute value sign in (3.181). We already know that $d_2 > 0$, from the requirement of a healthy kinetic term, but it is worth pointing out that a direct implication of (3.182), applied at small y, is that spherical solutions do not exist for a ghost-like theory with $d_2 < 0$. Furthermore, we are safe if the minimum of $P'(y)$ occurs above zero, which happens if

$$|d_3| < \sqrt{\frac{3}{2} d_2 d_4} \ . \tag{3.183}$$

Otherwise, the largest root of $P'(y)$ must occur for $y \le 0$, which happens if $d_3 \ge 0$.

In summary, the flat space theory is ghost-free and spherical solutions exist if and only if

$$d_2 > 0, \quad d_4 > \frac{M}{8\pi}, \quad d_3 > -\sqrt{\frac{3}{2} d_2 d_4} \ . \tag{3.184}$$

3.5.3 Stability

The existence of spherically symmetric solutions is, of course, not sufficient to guarantee viability of the theories in question. The next test is to examine the stability of these solutions. To do this, we expand the action in perturbations around the spherical solutions

$$\pi(x) = \pi_0(r) + \varphi(x) \ , \tag{3.185}$$

and isolate the terms quadratic in φ. These terms take the form

$$\mathcal{S}_\varphi = \frac{1}{2} \int dt \int d^2\Omega \int_0^\infty r^2 dr \ \left[K_t(r)\dot{\varphi}^2 - K_r(r)(\partial_r\varphi)^2 - K_\Omega(r)(\partial_\Omega\varphi)^2 \right] \ , \tag{3.186}$$

where overdots denote time derivatives, $(\partial_\Omega\varphi)^2 = (\partial_\theta\varphi)^2 + \frac{1}{\sin^2\theta}(\partial_\phi\varphi)^2$ is the angular part of $(\vec{\nabla}\varphi)^2$, and the kinetic coefficients K depend on r through the background radial solution $\pi_0(r)$ and its derivatives. Note that the quadratic action contains only second derivatives acting on the perturbations. This is because the field equations are second order, despite the fact that the lagrangian is higher derivative, as we mentioned earlier.

In order for the solution to be stable, each $K_i(r)$ $(i = t, \ r, \ \Omega)$ must be positive for all $r > 0$. If K_t is negative in some region, then localized excitations will be ghostlike and will carry negative energy. If either of K_r, K_Ω are negative in some region, then it is possible to find localized perturbations for which gradients lower the energy of the background solution. This kind of instability, associated with negative gradient energy for certain classes of fluctuations, is more troublesome than a tachyon-like instability associated with a negative mass squared term or upside down potential. A tachyon-like instability is, like the Jeans instability, dominated by modes with momenta of order the tachyonic mass scale, which can be parametrically smaller than the UV cutoff, and thus computable within the effective theory. By contrast, the gradient instability can be due to very short wavelength wave-packets with high momentum. Thus, this instability also plagues fluctuations right down to the UV cutoff of the theory, so that quantities such as decay rates are dominated by the shortest distances in the theory, and cannot be reliably computed within the effective theory.

To obtain explicit expressions for the functions $K_i(r)$, we expand the equations of motion to linear order in φ

$$\mathcal{E}[\pi_0 + \varphi] \to \frac{\delta \mathcal{S}_\varphi}{\delta \varphi} = -K_t(r)\,\ddot{\varphi} + \frac{1}{r^2}\partial_r\left(r^2 K_r(r)\,\partial_r\varphi\right) + K_\Omega(r)\,\partial_\Omega^2\varphi \ , \tag{3.187}$$

where $\partial_\Omega^2 = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$ is the angular part of the laplacian.

We begin with the radial perturbations, and find K_r simply by perturbing the radial equation (3.177), using a perturbation that depends only on r

$$\delta\mathcal{E} = \frac{1}{r^2}\frac{d}{dr}\left[r^3 P'(y)\delta y\right] = \frac{1}{r^2}\frac{d}{dr}\left[r^2 P'(y)\gamma^3\varphi'\right] \ . \tag{3.188}$$

From this we read off

$$K_r(r) = \gamma^3 P'(y) \ . \tag{3.189}$$

From (3.182), we then see that if the solution exists, then $K(r)$ is automatically positive, since $\gamma > 0$.

Now turn to the angular perturbations. To find K_Ω, we vary the full equations (3.167), allowing the perturbation to depend only on angular variables, and keeping in mind that the background depends only on r. Using the following useful expressions

$$\delta\left[\Pi^n\right] = \frac{n\pi'^{n-1}}{r^{n-1}}\partial_\Omega^2\varphi \ , \quad \delta\left[\pi^n\right] = 0 \ , \quad \delta\gamma = 0 \ , \tag{3.190}$$

it is simple to show that

$$K_\Omega(r) = \frac{\gamma}{2r}\frac{d}{dr}\left[r^2 P'(y)\right] \ . \tag{3.191}$$

Recall that the coefficient d_5 does not enter in either K_r or K_Ω, because we are still considering static configurations, for which the fifth DBI term vanishes.

Lastly, we consider the temporal perturbations. We find K_t by varying the full equations (3.167), this time allowing the perturbation to depend only on time. Once again, some useful expressions

$$\delta\left[\Pi\right] = -\ddot{\varphi} \ , \quad \delta\left[\Pi^n\right] = 0 \ \ (n > 1) \ , \quad \delta\left[\pi^n\right] = 0 \ , \quad \delta\gamma = 0 \ , \tag{3.192}$$

allow us to show that

$$K_t(r) = \frac{\gamma}{3r^2}\frac{d}{dr}\left[r^3\left(d_2 + 6d_3 y + 18 d_4 y^2 + 24 d_5 y^3\right)\right] \ . \tag{3.193}$$

We see that d_5 enters here for the first time, since we have deviated, at last, from static equations.

As we have written them, the functions $K_i(r)$ depend on γ, r, $\frac{dy}{dr}$ and y. However, we may eliminate $\frac{dy}{dr}$ in favor of y by using the implicit function theorem on the function $F(y,r) = P(y) - \frac{M}{4\pi r^3} = 0$. This yields

$$\frac{dy}{dr} = -\frac{\partial_r F}{\partial_y F} = -\frac{3}{r}\frac{P(y)}{P'(y)}. \tag{3.194}$$

Substituting this into our expressions for the $K_i(r)$ we obtain

$$
\begin{aligned}
K_r &= \gamma^3 \left[d_2 + 4d_3 y + 6d_4 y^2\right] \ , \\
K_\Omega &= \gamma \left[\frac{d_2^2 + 2d_2 d_3 y + \left(4d_3^2 - 6d_2 d_4\right) y^2}{d_2 + 4d_3 y + 6d_4 y^2}\right] \ , \\
K_t &= \gamma \left[\frac{d_2^2 + (4d_2 d_3)\, y + 12\left(d_3^2 - d_2 d_4\right) y^2 + 24\,(d_3 d_4 - 2d_5 d_2)\, y^3 + 12\left(3d_4^2 - 4d_3 d_5\right) y^4}{d_2 + 4d_3 y + 6d_4 y^2}\right] \ .
\end{aligned}
\tag{3.195}
$$

Note that the explicit r dependence has canceled out.

Since the solution spans all positive values of y as r varies from zero to infinity, we require K_t and K_Ω to be positive for all $y > 0$. The denominators in (3.195) are automatically positive, from (3.182). Given the constraints (3.184), The numerator in K_Ω is positive for $d_3 \geq \sqrt{\frac{3}{2}\, d_2\, d_4}$ which also ensures that the numerator in K_t is positive provided $d_5 \leq \frac{3}{4}\frac{d_4^2}{d_3}$.

The radial solution therefore exists and is stable if and only if

$$
d_2 > 0 \ , \quad d_4 > \frac{M}{8\pi} \ , \quad d_3 \geq \sqrt{\frac{3}{2} d_2 d_4} \ , \quad d_5 \leq \frac{3}{4}\frac{d_4^2}{d_3} \ . \tag{3.196}
$$

3.5.4 Propagation speed of fluctuations

As a final test of the viability of the DBI galileon theories, we consider the propagation speeds of small fluctuations around the stable spherical solutions. For radially propagating fluctuations this speed is

$$
c_r^2 = \frac{K_r}{K_t} \ . \tag{3.197}
$$

At large distances from the source (small y), this becomes

$$
c_r^2 = 1 + 4\frac{d_3}{d_2} y + \mathcal{O}(y^{4/3}) > 1 \ , \tag{3.198}
$$

where here and in what follows we express γ in terms of y via

$$\gamma = \sqrt{1 - r^2 y^2} = \sqrt{1 - \left(\frac{M}{4\pi P(y)}\right)^{2/3} y^2} \ . \tag{3.199}$$

Therefore, given the constraints implied by existence and stability of the solutions, this is always superluminal.

At smaller distances (larger y), the speed is

$$c_r^2 = \frac{3d_4^2}{3d_4^2 - 4d_3 d_5}\left[1 - \left(\frac{M}{8\pi d_4}\right)^{2/3}\right] + \mathcal{O}\left(\frac{1}{y}\right) , \tag{3.200}$$

so the propagation speed is subluminal in this region if

$$d_5 < \frac{3d_4^2}{4d_3}\left(\frac{M}{8\pi d_4}\right)^{2/3} \ . \tag{3.201}$$

The speed of angular excitations is

$$c_\Omega^2 = \frac{K_\Omega}{K_t} \ . \tag{3.202}$$

The difference between the numerator and the denominator is, apart from an overall positive factor,

$$K_\Omega - K_t \sim -2d_2 d_3 y - \left(8d_3^2 - 6d_2 d_4\right) y^2 - 24\left(d_3 d_4 - 2d_2 d_5\right) y^3 - 12\left(3d_4^2 - 4d_3 d_5\right) y^4 \ . \tag{3.203}$$

Given the constraints (3.196), this is always negative, so the speed of angular excitations is always subluminal. Also, the angular speed goes to zero as r goes to zero. The radial and angular speeds for a sample solution are shown in figure 3.3.

Certainly the existence of superluminally propagating modes raises questions about the viability of galileon DBI theories. Whether such a feature is really a problem that conclusively rules out a low-energy effective theory is still being debated [9, 16, 60], but it has been argued that, at the least, it may preclude the possibility of embedding the theory into a local, Lorentz invariant UV completion [1].

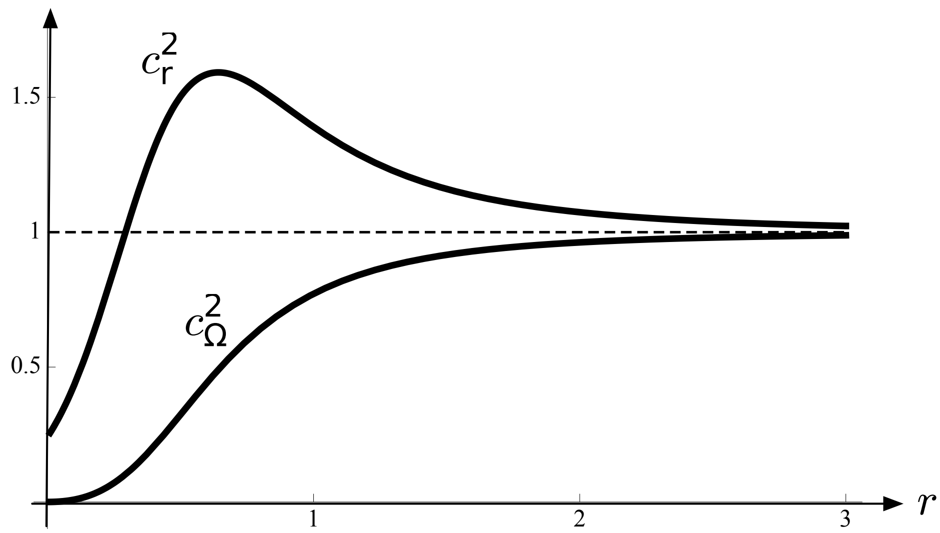

Figure 3.3: Speed of fluctuations c_r^2 and c_Ω^2, in the radial and angular directions respectively, for a sample solution satisfying the existence and stability constraints (3.196), as well as (3.201). The values chosen are $d_2 = 1$, $d_3 = 2$, $d_4 = 1$, $d_5 = -1$, $M = 1$.

3.5.5 Discussion

In this section we have studied spherically symmetric solutions to the DBI galileon models, demonstrating that there exists a range of parameters in which such solutions exist. We have also examined the stability of these solutions and computed the propagation speeds of perturbations around the solutions. While we have found that there exists a region of parameter space in which our solutions are stable, we have shown that these solutions always exhibit superluminal propagation. Such behavior is familiar from that of the ordinary galileon theories. Thus, although one might have thought that the γ factors appearing for DBI galileons could cure the superluminality issues, the results we find here indicate that they do not.

We have worked in dimensionless units, which corresponds to setting to unity a scale,

Λ, suppressing all the non-linearities in the Lagrangian. In addition, we have absorbed into the stress tensor a scale, M_p, representing the coupling strength. Restoring these scales, the condition (3.181) tells us $d_4 \gtrsim M/M_p$, so in gravitational applications, where M is the mass of the Sun and M_p the Planck mass, this tells us that d_4 must be huge, of order the solar mass in Planck units. One might worry that this necessitates strong coupling, but this is not the case because the coefficient d_2, which multiplies the kinetic term, may also be chosen to be very large, so that after canonical normalization the true couplings are still small.

To see the consequences of this, consider expanding the action with the scale Λ restored. The DBI term reads schematically $d_2\Lambda^4\sqrt{1+\frac{(\partial\pi)^2}{\Lambda^4}} \sim (\partial\hat{\pi})^2 + \frac{1}{d_2\Lambda^4}(\partial\hat{\pi})^4 + \cdots$, with the canonically normalized field $\hat{\pi} = d_2^{1/2}\pi$. The scale suppressing the non-linear terms here is $d_2^{1/4}\Lambda$. Similarly, the quartic galileon term is, schematically,

$$\begin{aligned} &\sim d_4\left[1+\frac{(\partial\pi)^2}{\Lambda^4}+\cdots\right]\frac{1}{\Lambda^6}(\partial^2\pi)^2(\partial\pi)^2 \\ &= \frac{d_4}{d_2\Lambda^2}\frac{1}{d_2\Lambda^4}(\partial^2\hat{\pi})^2(\partial\hat{\pi})^2 + \frac{d_4}{d_2\Lambda^2}\frac{1}{d_2^2\Lambda^8}(\partial^2\hat{\pi})^2(\partial\hat{\pi})^4+\cdots\ , \end{aligned} \tag{3.204}$$

which means that the strong coupling scales are $\left(\frac{d_2^2}{d_4}\right)^{1/6}\Lambda, \left(\frac{d_2^3}{d_4}\right)^{1/10}\Lambda, \cdots$. Since d_4 is so large, keeping the lowest strong coupling scale reasonably high requires choosing d_2 large, say $d_2^2 \sim d_4$, in which case all the higher order DBI scales are much higher (corresponding to small coupling), and the theory becomes very similar to the ordinary galileons, explaining why we find conclusions similar to the conclusions in that case. In addition, note that the coupling to the stress tensor, in terms of the canonically normalized field, is $\sim \frac{1}{d_2^{1/2}M_p}\hat{\pi}T$, so that the true Planck mass is actually $\sim d_2^{1/2}M_p$, and the necessary size of d_4 is actually larger than the solar mass in physical Planck units.

On the other hand, in some situations, it may be too much to demand that the spherical solutions exist for all r. For example, if π represents the fifth coordinate of a brane embedding, we should not expect that the brane configuration should be everywhere expressible as a single valued function of the four coordinates x^μ (the solutions of [21, 61, 62, 88] are examples of this). In this case, the restrictions on the coefficient d_4 may be relaxed.

DBI galileon theories therefore, like the ordinary galileons, face a challenge from the superluminal propagation of perturbations around simple spherically symmetric solutions. Whether these theories are viable depends on the development of an argument that this superluminality does not lead to the pathologies that are traditionally associated with this behavior, or whether a modification to the theory or its couplings to matter or gravity can eliminate this behavior. It should be mentioned that the coupling of galileons to gravity is non-trivial if one wishes to keep the equations of motion second order [41, 42], and the issue of superluminality should in principle be re-examined in the full covariant context, though the effects should be Planck suppressed.

Chapter 4

Cosmological Galileons [11]

4.1 Overview

In this chapter we use the brane construction methods of Sec.2.2 in order to construct a galileon-like theory on cosmological FRW spacetimes, and to identify the non-linear symmetries of the resulting theories (this possibility was commented on in [17]). The construction begins with an embedding of FRW in a flat 5D bulk, so that the symmetry group will be the 15-dimensional Poincare group of 5D flat space, of which the 6 symmetries of FRW (spatial translations and rotations) will be linearly realized. The resulting theory of "FRW galileons" turns out to be much more complicated and cumbersome than the scenarios of Sec.3 and after deriving the curvature terms necessary to generate the generic brane action (2.4) we restrict ourselves and only derive explicit lagrangians in the minisuperspace approximation as we mainly have in mind cosmological applications. Additionally, we discuss the small π limits and explore the existence and stability of simple solutions for π.

[11] The work in this chapter was performed in collaboration with Kurt Hinterbichler and Mark Trodden.

4.2 Embedding 4D FRW in 5D Minkowski

We consider the case of a spatially flat FRW 3-brane embedded in 5D Minkowski space. In order to begin the brane construction, we need to derive a convenient form of the bulk Minkowski metric in which we foliate the bulk by leaves which are themselves 4D FRW spacetimes. Starting from the bulk Minkowski metric with coordinates Y^A

$$\mathrm{d}s^2 = -\left(\mathrm{d}Y^0\right)^2 + \left(\mathrm{d}Y^1\right)^2 + \left(\mathrm{d}Y^2\right)^2 + \left(\mathrm{d}Y^3\right)^2 + \left(\mathrm{d}Y^5\right)^2 \ , \tag{4.1}$$

we make a change to coordinates to t, x^i, ρ, where $i = 1, 2, 3$ runs over the spatial indices on the brane[12],

$$\begin{aligned} Y^0 &= S(t,\rho)\left(\frac{x^2}{4} + 1 - \frac{1}{4H^2a^2}\right) - \frac{1}{2}\int dt\, \frac{\dot{H}}{H^3 a}, \\ Y^i &= S(t,\rho)x^i, \\ Y^5 &= S(t,\rho)\left(\frac{x^2}{4} - 1 - \frac{1}{4H^2a^2}\right) - \frac{1}{2}\int dt\, \frac{\dot{H}}{H^3 a} \ . \end{aligned} \tag{4.2}$$

Here, $a(t)$ is an arbitrary function of t which will become the scale factor of the 4D space, and overdots denote derivatives with respect to t. We have defined $x^2 \equiv x^i x^j \delta_{ij}$, $H \equiv \dot{a}/a$, and

$$S(t,\rho) \equiv a - \dot{a}\rho. \tag{4.3}$$

The lower limits on the integrals in (4.2) are arbitrary, and different choices merely shift the embedding. In the case of power law expansions $a(t) \sim t^\alpha$, $\alpha > 0$, taking the lower limit to be zero puts the big bang at the origin of the embedding space.

In these new coordinates, the Minkowski metric reads

$$\mathrm{d}s^2 = -n^2(t,\rho)\mathrm{d}t^2 + S^2(t,\rho)\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j + \mathrm{d}\rho^2 \ , \tag{4.4}$$

[12]This is the transformation used in [46], except that we have not imposed a Z_2 symmetry.

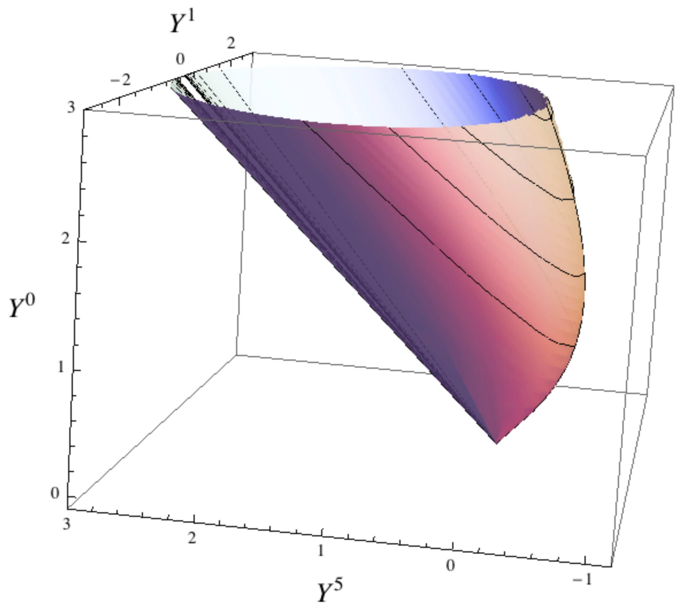

Figure 4.1: The embedding of an FRW brane in 5D Minkowski space for the case $a(t) = t^{1/2}$.

where

$$n(t, w) \equiv 1 - \frac{\ddot{a}}{\dot{a}}\rho \ . \tag{4.5}$$

On any $\rho = $ const. slice, the induced metric is

$$\mathrm{d}\tilde{s}^2 = -n^2(t,\rho)\mathrm{d}t^2 + S^2(t,\rho)\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j, \tag{4.6}$$

and so after a slice by slice time redefinition $n(t,\rho)dt = dt'$ we verify that we have indeed foliated M_5 with spatially flat FRW slices. Furthermore, the coordinates are Gaussian normal with respect to this foliation. A plot of the embedding in the case $a \sim t^{1/2}$ is shown in Fig.(4.1).

4.3 Symmetries

Before building lagrangians, we discuss and derive the symmetries that they will obey. The general analysis exactly mirrors that of Sec.3.2.1. The algebra of Killing vectors of G_{AB} contains a subalgebra consisting of those Killing vectors for which $K^5 = 0$ and we choose a basis of this subalgebra with elements indexed by $\mathcal{J}$,

$$K_{\mathcal{J}}^A(X) = \begin{cases} K_{\mathcal{J}}^\mu(x) & A = \mu \\ 0 & A = 5 \end{cases} . \tag{4.7}$$

We then extend this to a basis for the algebra of all Killing vectors by adding a suitably chosen set of linearly independent Killing vectors with non-vanishing K^5. We index these with I, so that $(K_{\mathcal{J}}, K_I)$ is a basis of the full algebra of Killing vectors. We work in our preferred gauge (2.7) and the gauge preserving symmetries take the form

$$(\delta_K + \delta_{g,\mathrm{comp}})\pi = -a^{\mathcal{J}} K_{\mathcal{J}}^\mu(x)\partial_\mu\pi + a^I K_I^5(x) - a^I K_I^\mu(x,\pi)\partial_\mu\pi \ , \tag{4.8}$$

where $\{a^{\mathcal{J}}, a^I\}$ are constants, demonstrating that the $K_{\mathcal{J}}$ symmetries are linearly realized, whereas the K_I symmetries are non-linearly realized. This pattern corresponds to the spontaneous breaking of the bulk symmetry algebra down to the subalgebra which preserves the leaves of the foliation.

In order to derive the basis of Killing vectors appropriate for the scenario under consideration, we start with be the cartesian coordinates used in (4.1), the Y^A's, whose associated basis vectors are $\bar{\partial}_A$. In these coordinates, the Killing vectors take the familiar form of the ten rotations and boosts, L_{AB}, and the five translations P_A,

$$L_{AB} = Y_A\bar{\partial}_B - Y_B\bar{\partial}_A, \qquad P_A = -\bar{\partial}_A. \tag{4.9}$$

After rewriting these Killing vectors in terms of the brane-adapted coordinates $\{t, x^i, \rho\}$ and the associated basis vectors $\{\partial_t, \partial_i, \partial_\rho\}$, we find the following combinations which contain no K^5 component,

$$L_{ij} = x^i\partial_j - x^j\partial_i, \qquad -\frac{1}{2}\left[L_{i0} + L_{i5}\right] = -\partial_i \ . \tag{4.10}$$

These generate the three rotations and three spatial translations of the FRW leaves and thus represent the K_J^A's.

The remaining vectors form the K_I^A's, which we take in the following combinations,

$$
\begin{aligned}
v_i &= \frac{1}{2}\left[L_{i0}-L_{i5}\right] = \frac{1}{2}x^i\dot{a}\left[\int dt\,\frac{\dot{H}}{H^3a}\right]\partial_\rho + \frac{x^i\big(a-\dot{a}\pi+\dot{a}^2\int dt\,\frac{\dot{H}}{H^3a}\big)}{2\dot{a}-2\pi\ddot{a}}\partial_t \\
&\quad - \left[\frac{x^ix^i\dot{a}^2+1}{4\dot{a}^2}+\frac{\int dt\,\frac{\dot{H}}{H^3a}}{2a-2\pi\dot{a}}\right]\partial_i + \sum_{j\neq i}\left[-\frac{x^ix^j}{2}\partial_j+\frac{x^jx^j}{4}\partial_i\right], \\
k_i &= -P_i = \frac{1}{a-\pi\dot{a}}\partial_i + x^i\dot{a}\big(\frac{\dot{a}}{\pi\ddot{a}-\dot{a}}\partial_t-\partial_\rho\big), \\
q &= -\frac{1}{2}\left[P_0+P_5\right] = \dot{a}\big(\partial_\rho+\frac{\dot{a}}{\dot{a}-\pi\ddot{a}}\partial_t\big), \\
u &= -\frac{1}{2}\left[P_0-P_5\right] = \frac{x^2\dot{a}^2-1}{4\dot{a}}\partial_\rho + \frac{x^2\dot{a}^2+1}{4\dot{a}-4\pi\ddot{a}}\partial_t - \frac{1}{2a-2\pi\dot{a}}\sum_i x^i\partial_i, \\
s &= L_{50} = \left[\frac{a-\pi\dot{a}+\dot{a}^2\int dt\,\frac{\dot{H}}{H^3a}}{\pi\ddot{a}-\dot{a}}\right]\partial_t - \dot{a}\left[\int dt\,\frac{\dot{H}}{H^3a}\right]\partial_\rho + \sum_i x^i\partial_i\ ,
\end{aligned}
\tag{4.11}
$$

where $H=\dot{a}/a$, $x^2=\delta_{ij}x^ix^j$, the summation convention has been suspended and we've replaced $\rho\to pi$. The lower limits on the integrals should be the same as those in (4.2).

The non-linear symmetries of the π field are then obtained from (4.8),

$$
\begin{aligned}
\delta_{v_i}\pi &= \frac{1}{2}x^i\dot{a}\int dt\,\frac{\dot{H}}{H^3a} - \frac{x^i\big(a-\dot{a}\pi+\dot{a}^2\int dt\,\frac{\dot{H}}{H^3a}\big)}{2\dot{a}-2\pi\ddot{a}}\dot{\pi} \\
&\quad + \left[\frac{x^ix^i\dot{a}^2+1}{4\dot{a}^2}+\frac{\int dt\,\frac{\dot{H}}{H^3a}}{2a-2\pi\dot{a}}\right]\partial_i\pi - \sum_{j\neq i}\left[-\frac{x^ix^j}{2}\partial_j\pi+\frac{x^jx^j}{4}\partial_i\pi\right], \\
\delta_{k_i}\pi &= x^i\dot{a}\left(\frac{\dot{a}\dot{\pi}}{\dot{a}-\pi\ddot{a}}-1\right) - \frac{\partial_i\pi}{a-\pi\dot{a}}, \\
\delta_q\pi &= \frac{\dot{\pi}\dot{a}^2}{\pi\ddot{a}-\dot{a}}+\dot{a}, \\
\delta_u\pi &= \frac{x^2\dot{a}^2-1}{4\dot{a}} - \frac{x^2\dot{a}^2+1}{4\dot{a}-4\pi\ddot{a}}\dot{\pi} + \frac{1}{2a-2\pi\dot{a}}\sum_i x^i\partial_i\pi,
\end{aligned}
$$

$$\delta_s \pi = -\dot{a} \int dt \, \frac{\dot{H}}{H^3 a} + \frac{\left(a - \dot{a}\pi + \dot{a}^2 \int dt \, \frac{\dot{H}}{H^3 a} \right) \dot{\pi}}{\dot{a} - \pi \ddot{a}} - \sum x^i \partial_i \pi, \tag{4.12}$$

where now, of course, $\pi = \pi(x)$ is the galileon-like field. These are clearly complicated and highly non-linear transformations, and without the brane formalism it would be nearly impossible to guess their form.

4.4 Lagrangians

Given our desired bulk metric (4.4) we can begin to derive the appropriate lagrangians and their symmetries and we do so in this section. The building blocks for the action are the induced metric (already derived in (4.6)), the extrinsic curvature and induced Riemann curvature. Since the bulk metric is Gaussian normal, we can use the results of Sec.2.2.1.2 to read off the extrinsic curvature. As for the induced Riemann curvature, we find that because the bulk metric is flat (and hence $R_{ABCD} = 0$) it can be expressed solely in terms of the extrinsic curvature tensor and induced metric via the Gauss-Codazzi equations,

$$R^{(5)}_{ABCD} e^A{}_\mu e^B{}_\nu e^C{}_\rho e^D{}_\sigma = 0 = \bar{R}_{\mu\nu\rho\sigma} - K_{\mu\rho} K_{\nu\sigma} + K_{\mu\sigma} K_{\nu\rho} \ . \tag{4.13}$$

In particular, this simplifies the expressions for $\mathcal{L}_4$ and $\mathcal{L}_5$ in (2.38) as they reduce to

$$\mathcal{L}_4 = -\sqrt{-\bar{g}} \left[K^2 - K^2_{\mu\nu} \right] , \tag{4.14}$$

$$\mathcal{L}_5 = \sqrt{-\bar{g}} \left[K^3 - 3K^2_{\mu\nu} K + 2K^3_{\mu\nu} \right] . \tag{4.15}$$

Therefore, only the knowledge of $\bar{g}_{\mu\nu}$ and $K_{\mu\nu}$ is necessary to construct the desired lagrangians.

Similar to the procedure of Sec.3.2.2, one can derive an explicit form of the $\mathcal{L}_i$'s for flat (i.e. $R_{ABCD} = 0$) bulk metrics written in Gaussian normal form. It is again a quite lengthy calculation and the result is given in Appendix C. Using the general formulas (C.0.4), the first two FRW galileon lagrangians are found to be

$$
\begin{aligned}
\mathcal{L}_1 &= a^3\pi - \frac{a^2\left(3\dot{a}^2 + a\ddot{a}\right)\pi^2}{2\dot{a}} + a\left(\dot{a}^2 + a\ddot{a}\right)\pi^3 - \frac{1}{4}\dot{a}\left(\dot{a}^2 + 3a\ddot{a}\right)\pi^4 + \frac{1}{5}\ddot{a}\dot{a}^2\pi^5, \\
\mathcal{L}_2 &= -(1 - \frac{\ddot{a}}{\dot{a}}\pi)(a - \dot{a}\pi)^3\sqrt{1 - \left(1 - \frac{\ddot{a}}{\dot{a}}\pi\right)^{-2}\dot{\pi}^2 + (a - \dot{a}\pi)^{-2}(\vec{\nabla}\pi)^2}.
\end{aligned}
\tag{4.16}
$$

where no integrations by parts have been made. Higher order lagrangians become unfortunately complicated. We relegate the expression for $\mathcal{L}_3$ to Appendix D, due to its complexity, and opt not to write out explicit expressions for $\mathcal{L}_4$ and $\mathcal{L}_5$ due to their even more unmanageable length.

4.4.1 Minisuperspace Lagrangians

In their full form, the higher order FRW galileon lagrangians are nearly impossible to use due to the sheer number of terms they carry. However, for cosmological applications where we are not considering fluctuations, we may be most interested in the limiting case in which spatial gradients are set to zero, so that $\pi = \pi(t)$. In this minisuperspace approximation, the lagrangians simplify significantly, and we display their full forms here. We present these lagrangians with their numerators ordered by increasing powers of π and then by patterns of derivatives on the π fields. No integrations by parts have been made.

$$
\begin{aligned}
\mathcal{L}_1 &= a^3\pi - \frac{a^2\left(3\dot{a}^2 + a\ddot{a}\right)\pi^2}{2\dot{a}} + a\left(\dot{a}^2 + a\ddot{a}\right)\pi^3 - \frac{1}{4}\dot{a}\left(\dot{a}^2 + 3a\ddot{a}\right)\pi^4 + \frac{1}{5}\ddot{a}\dot{a}^2\pi^5, \\
\mathcal{L}_2 &= -\left(a - \pi\dot{a}\right)^3\sqrt{\left(1 - \frac{\pi\ddot{a}}{\dot{a}}\right)^2 - \dot{\pi}^2}, \\
\mathcal{L}_3 &= \Big[3a^2\dot{a}^4 + a^3\ddot{a}\dot{a}^2 + \left(-6a\dot{a}^5 - 12a^2\ddot{a}\dot{a}^3 - 2a^3\ddot{a}^2\dot{a}\right)\pi - 3a^2\dot{a}^4\dot{\pi} - a^3\dot{a}^3\ddot{\pi} \\
&\quad + \left(3\dot{a}^6 + 21a\ddot{a}\dot{a}^4 + 15a^2\ddot{a}^2\dot{a}^2 + a^3\ddot{a}^3\right)\pi^2 + (6a\dot{a}^5 + 6a^2\ddot{a}\dot{a}^3 \\
&\quad - a^3\dddot{a}\dot{a}^2 + a^3\ddot{a}^2\dot{a})\pi\dot{\pi} + (-3a^2\dot{a}^4 - 2a^3\ddot{a}\dot{a}^2)\dot{\pi}^2 + (3a^2\dot{a}^4 + a^3\ddot{a}\dot{a}^2)\pi\ddot{\pi} \\
&\quad + (-10\ddot{a}\dot{a}^5 - 24a\ddot{a}^2\dot{a}^3 - 6a^2\ddot{a}^3\dot{a})\pi^3 + (-3\dot{a}^6 - 12a\ddot{a}\dot{a}^4 + 3a^2\dddot{a}\dot{a}^3
\end{aligned}
$$

$$
\begin{aligned}
&- 6a^2\ddot{a}^2\dot{a}^2)\pi^2\dot{\pi} + (6a\dot{a}^5 + 9a^2\ddot{a}\dot{a}^3)\pi\dot{\pi}^2 + (-3a\dot{a}^5 - 3a^2\ddot{a}\dot{a}^3)\pi^2\ddot{\pi} \\
&+ 3a^2\dot{a}^4\dot{\pi}^3 + (9a\dot{a}^2\ddot{a}^3 + 11\dot{a}^4\ddot{a}^2)\pi^4 + (6\ddot{a}\dot{a}^5 - 3a\dddot{a}\dot{a}^4 + 9a\ddot{a}^2\dot{a}^3)\pi^3\dot{\pi} \\
&+ (-3\dot{a}^6 - 12a\ddot{a}\dot{a}^4)\pi^2\dot{\pi}^2 + (\dot{a}^6 + 3a\ddot{a}\dot{a}^4)\pi^3\ddot{\pi} - 6a\dot{a}^5\pi\dot{\pi}^3 - 4\dot{a}^3\ddot{a}^3\pi^5 \\
&+ (\dot{a}^5\dddot{a} - 4\dot{a}^4\ddot{a}^2)\pi^4\dot{\pi} + 5\dot{a}^5\ddot{a}\pi^3\dot{\pi}^2 - \dot{a}^5\ddot{a}\ddot{\pi}\pi^4 \\
&+ 3\dot{a}^6\dot{\pi}^3\pi^2\Big]\Big/\Big[\dot{a}\left(\left(\dot{\pi}^2 - 1\right)\dot{a}^2 + 2\pi\ddot{a}\dot{a} - \pi^2\ddot{a}^2\right)\Big],
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{L}_4 = \Big[&- 6a\dot{a}^4 - 6a^2\ddot{a}\dot{a}^2 + (6\dot{a}^5 + 30a\ddot{a}\dot{a}^3 + 12a^2\ddot{a}^2\dot{a})\pi + 6a\dot{a}^4\dot{\pi} + 6a^2\dot{a}^3\ddot{\pi} \\
&+ (-24\ddot{a}\dot{a}^4 - 42a\ddot{a}^2\dot{a}^2 - 6a^2\ddot{a}^3)\pi^2 + (-6\dot{a}^5 - 12a\ddot{a}\dot{a}^3 + 6a^2\dddot{a}\dot{a}^2 \\
&- 6a^2\ddot{a}^2\dot{a})\pi\dot{\pi} + (6a\dot{a}^4 + 12a^2\ddot{a}\dot{a}^2)\dot{\pi}^2 + (-12a\dot{a}^4 - 6a^2\ddot{a}\dot{a}^2)\pi\ddot{\pi} \\
&+ (30\ddot{a}^2\dot{a}^3 + 18a\ddot{a}^3\dot{a})\pi^3 + (12\ddot{a}\dot{a}^4 - 12a\dddot{a}\dot{a}^3 + 18a\ddot{a}^2\dot{a}^2)\pi^2\dot{\pi} \\
&+ (-6\dot{a}^5 - 30a\ddot{a}\dot{a}^3)\pi\dot{\pi}^2 + (6\dot{a}^5 + 12a\ddot{a}\dot{a}^3)\pi^2\ddot{\pi} - 6a\dot{a}^4\dot{\pi}^3 - 12\dot{a}^2\ddot{a}^3\pi^4 \\
&+ (6\dot{a}^4\dddot{a} - 12\dot{a}^3\ddot{a}^2)\pi^3\dot{\pi} + 18\dot{a}^4\ddot{a}\pi^2\dot{\pi}^2 - 6\dot{a}^4\ddot{a}\ddot{\pi}\pi^3 \\
&+ 6\dot{a}^5\pi\dot{\pi}^3\Big]\Big/\Big[\dot{a}\left(\dot{a}\left(\dot{\pi} + 1\right) - \pi\ddot{a}\right)\sqrt{\left(1 - \frac{\pi\ddot{a}}{\dot{a}}\right)^2 - \dot{\pi}^2}\Big],
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{L}_5 = \Big[&- 6\dot{a}^5 - 18a\ddot{a}\dot{a}^3 + (36\ddot{a}\dot{a}^4 + 36a\ddot{a}^2\dot{a}^2)\pi + 6\dot{a}^5\dot{\pi} \\
&+ 18a\dot{a}^4\ddot{\pi} + (-54\ddot{a}^2\dot{a}^3 - 18a\ddot{a}^3\dot{a})\pi^2 + (-12\ddot{a}\dot{a}^4 + 18a\dddot{a}\dot{a}^3 \\
&- 18a\ddot{a}^2\dot{a}^2)\pi\dot{\pi} + (6\dot{a}^5 + 36a\ddot{a}\dot{a}^3)\dot{\pi}^2 + (-18\dot{a}^5 - 18a\ddot{a}\dot{a}^3)\pi\ddot{\pi} \\
&+ 24\dot{a}^2\ddot{a}^3\pi^3 + (24\dot{a}^3\ddot{a}^2 - 18\dot{a}^4\dddot{a})\pi^2\dot{\pi} + 18\dot{a}^4\ddot{a}\pi^2\ddot{\pi} \\
&- 42\dot{a}^4\ddot{a}\pi\dot{\pi}^2 - 6\dot{a}^5\dot{\pi}^3\Big]\Big/\Big[\dot{a}\left(\dot{\pi} + 1\right) - \pi\ddot{a}\Big]^2.
\end{aligned} \tag{4.17}
$$

The π equations of motion derived from these are second order in time derivatives. As before, the scale factor $a(t)$ describes the fixed background cosmological evolution, and does not represent a dynamical degree of freedom. Out of all the symmetries in (4.12), only $\delta_q\pi$,

$$\delta_q\pi = \frac{\dot{\pi}\dot{a}^2}{\pi\ddot{a} - \dot{a}} + \dot{a}. \tag{4.18}$$

is free of explicit dependence on the spatial coordinates and hence it is the only symmetry of the lagrangians (4.17),

4.5 Solutions, fluctuations, and small field limits

In this section, we explore the existence and stability of simple solutions for π. In particular, we focus on the properties of the possible $\pi = 0$ solutions.

4.5.1 Simple solutions and stability

Retaining all temporal and spatial derivatives, we expand the lagrangians to second order in π, and find, after much integration by parts,

$$\begin{aligned}
\mathcal{L}_1 &= a^3\pi - \frac{1}{2}\Big(\frac{\ddot{a}a^3}{\dot{a}} + 3\dot{a}a^2\Big)\pi^2 + \mathcal{O}\left(\pi^3\right), \\
\mathcal{L}_2 &= \left(3a^2\dot{a} + \frac{a^3\ddot{a}}{\dot{a}}\right)\pi + \frac{1}{2}a^3\dot{\pi}^2 - \frac{1}{2}a\left(\vec{\nabla}\pi\right)^2 - 3\left(\ddot{a}a^2 + \dot{a}^2 a\right)\pi^2 + \mathcal{O}\left(\pi^3\right), \\
\mathcal{L}_3 &= 6\left(a\dot{a}^2 + a^2\ddot{a}\right)\pi + 3\dot{a}a^2\dot{\pi}^2 - \left(2\dot{a} + \frac{a\ddot{a}}{\dot{a}}\right)\left(\vec{\nabla}\pi\right)^2 - 3\left(3\dot{a}\ddot{a}a + \dot{a}^3\right)\pi^2 + \mathcal{O}\left(\pi^3\right), \\
\mathcal{L}_4 &= 6\left(\dot{a}^3 + 3a\dot{a}\ddot{a}\right)\pi + 9\dot{a}^2 a\dot{\pi}^2 - 3\left(\frac{\dot{a}^2}{a} + 2\ddot{a}\right)\left(\vec{\nabla}\pi\right)^2 - 12\dot{a}^2\ddot{a}\pi^2 + \mathcal{O}\left(\pi^3\right), \\
\mathcal{L}_5 &= 24\dot{a}^2\ddot{a}\,\pi + 12\dot{a}^3\dot{\pi}^2 - 12\frac{\ddot{a}^2\dot{a}}{a}\left(\vec{\nabla}\pi\right)^2 + \mathcal{O}\left(\pi^3\right).
\end{aligned} \tag{4.19}$$

Note that at quadratic order all the higher derivative terms have canceled out up to total derivative, a consequence of the fact that the equations of motion are second order.

Consider a theory which is an arbitrary linear combination of the five lagrangians,

$$\mathcal{L} = \sum_{n=1}^{5} c_n \mathcal{L}_n, \tag{4.20}$$

where the c_n are (dimensionful) constants. If $\pi = 0$ is to be a solution to the full equations of motion, the linear terms in $\mathcal{L}$ must vanish, which gives the condition

$$c_1 a^3 + c_2 \left(3a^2\dot{a} + \frac{a^3\ddot{a}}{\dot{a}}\right) + 6c_3 \left(a\dot{a}^2 + a^2\ddot{a}\right) + 6c_4 \left(\dot{a}^3 + 3a\dot{a}\ddot{a}\right) + 24c_5\dot{a}^2\ddot{a} = 0. \tag{4.21}$$

For generic values of the c_n, this is a non-linear second order equation for $a(t)$ which can be solved to yield a background for which $\pi = 0$ is a solution. If we look for standard power-law solutions, $a(t) = (t/t_0)^\alpha$, the condition (4.21) becomes

$$\left[24c_5(\alpha-1)\alpha^3 + 6c_4(4\alpha-3)\alpha^2 t + 6c_3\alpha(2\alpha-1)t^2 + c_2(4\alpha-1)t^3 + c_1 t^4\right]\left(\frac{t}{t_0}\right)^{3\alpha} = 0\ . \tag{4.22}$$

Each power of t must vanish independently, so we see that the only non-trivial power-law solutions are for $\alpha = 1, 3/4, 1/2, 1/4$. For these solutions, the corresponding c_n must be non-zero and the others must be set to zero.

To test the stability around a given solution, we look at the quadratic part of the lagrangian, which has the following form,

$$\mathcal{L} = \frac{1}{2}A(a(t), c_n)\dot{\pi}^2 - \frac{1}{2}B(a(t), c_n)(\vec{\nabla}\pi)^2 - \frac{1}{2}C(a(t), c_n)\pi^2\ , \tag{4.23}$$

where

$$\begin{aligned}
A(a(t), c_n) &= c_2 a^3 + 6c_3\dot{a}a^2 + 18c_4\dot{a}^2 a + 24c_5\dot{a}^3, \\
B(a(t), c_n) &= c_2 a + 2c_3\left(2\dot{a} + \frac{a\ddot{a}}{\dot{a}}\right) + 6c_4\left(\frac{\dot{a}^2}{a} + 2\ddot{a}\right) + 24c_5\frac{\ddot{a}\dot{a}}{a}\ddot{a}, \\
C(a(t), c_n) &= c_1\left(\frac{\ddot{a}a^3}{\dot{a}} + 3\dot{a}a^2\right) + 6c_2\left(\ddot{a}a^2 + \dot{a}^2 a\right) + 6c_3\left(3\dot{a}\ddot{a}a + \dot{a}^3\right) + 24c_4\dot{a}^2\ddot{a}\ .
\end{aligned} \tag{4.24}$$

α	c_1	c_2	c_3	c_4	c_5	A	B	C	$H\tau$
1	0	0	0	0	c_5	$24\frac{c_5}{t_0^3}$	0	0	0
$\frac{3}{4}$	0	0	0	c_4	0	$\frac{81}{8t^2}c_4\left(t/t_0\right)^{9/4}$	$\frac{9}{8t^2}c_4\left(t/t_0\right)^{3/4}$	$-\frac{81}{32t^4}c_4\left(t/t_0\right)^{9/4}$	$3/2$
$\frac{1}{2}$	0	0	c_3	0	0	$\frac{3}{t}c_3\left(t/t_0\right)^{3/2}$	$\frac{1}{t}c_3\left(t/t_0\right)^{1/2}$	$-\frac{3}{2t^3}c_3\left(t/t_0\right)^{3/2}$	$\frac{1}{\sqrt{2}}$
$\frac{1}{4}$	0	c_2	0	0	0	$c_2\left(t/t_0\right)^{3/4}$	$c_2\left(t/t_0\right)^{1/4}$	$-\frac{3}{4t^2}c_2\left(t/t_0\right)^{3/4}$	$\frac{1}{2\sqrt{3}}$

Table 4.1: Lagrangian coefficients, stability coefficients, and time scale comparisons for fluctuations about $\pi = 0$ for all possible non-trivial power law solutions $a(t) = \left(t/t_0\right)^n$.

The stability of the theory against ghost and gradient instability, which is catastrophic at the shortest length scales, requires $A > 0$ and $B \geq 0$. Freedom from tachyon-like instabilities requires $C \geq 0$. However a tachyonic instability where $C < 0$ only affects the large-scale stability of the field, and may be tolerable as long as the time scale associated with the tachyonic mass is of the same order or larger than the Hubble time. The equations of motion take the form of a damped harmonic oscillator, $A\ddot{\pi} + \dot{A}\dot{\pi} - B\nabla^2\pi + C\pi = 0$. Thus, the time scale τ associated with a tachyonic mass term is given by $\tau = \sqrt{A/|C|}$ and the tachyonic instability is tolerable if $H\tau \gtrsim 1$.

In Table 4.1, we display the coefficients (4.24) for the four possible power-law solutions. For the solution $a(t) \sim t$, the choice $c_5 > 0$ leads to a stable solution, albeit marginally so, since there is no mass or gradient energy. For each of the other three cases, choosing the relevant coefficient to be positive ensures that $A > 0$, $B > 0$, at which point we necessarily have $C < 0$ and hence a tachyonic instability. The tachyon time scale is however $\tau H \sim 1$ (and happens to be independent of time). Therefore, each of the four power law solutions are stable to fluctuations over time scales shorter than the age of the universe.

Repeating the analysis in the case of a de-Sitter universe, the condition (4.21) for a $\pi = 0$ solution becomes

$$c_1 + 4Hc_2 + 12c_3H^2 + 24c_4H^3 + 24c_5H^4 = 0, \tag{4.25}$$

and the coefficients (4.24) of the quadratic part are

$$
\begin{aligned}
A(a(t), c_i) &= a_0^3 e^{3Ht}\left(c_2 + 6c_3 H + 18 c_4 H^2 + 24 c_5 H^3\right), \\
B(a(t), c_i) &= a_0 e^{Ht}\left(c_2 + 6c_3 H + 18 c_4 H^2 + 24 c_5 H^3\right), \\
C(a(t), c_i) &= -4a_0^3 e^{3Ht} H^2\left(c_2 + 6c_3 H + 18 c_4 H^2 + 24 c_5 H^3\right).
\end{aligned}
\tag{4.26}
$$

All the coefficients share a common factor, so the field is either a ghost or a tachyon, in agreement with the findings in Sec.3.4.1. Comparing the tachyon time scale against $1/H$ gives $H\tau = 1/2$, so the tachyon time scale is approximately the Hubble time. This would be disastrous for inflation, since the instability would manifest itself after one e-fold, but it may be tolerable for late-time cosmic acceleration.

4.5.2 Small π symmetries

The small π limits of the symmetries (4.12) expanded to lowest order in π, are

$$
\begin{aligned}
\delta_{v_i}\pi &= \frac{1}{2}x^i \int dt\, \frac{\dot{H}}{H^3 a}\dot{a}, \\
\delta_{k_i}\pi &= -x^i \dot{a}, \\
\delta_q \pi &= \dot{a}, \\
\delta_u \pi &= \frac{x^2\dot{a}^2 - 1}{4\dot{a}}, \\
\delta_s \pi &= -\dot{a}\int dt\, \frac{\dot{H}}{H^3 a}\ .
\end{aligned}
\tag{4.27}
$$

In the case where $\pi = 0$ is a solution, these are symmetries of the quadratic action for π. Otherwise, they are symmetries of the action linear in π.

4.5.3 Galileon-like limits

When we generate galileon theories by foliating a maximally symmetric bulk by maximally symmetric branes, as in [65], there exist small field limits which greatly simplify the lagrangians (C.0.5). To take these limits, we form linear combinations $\bar{\mathcal{L}}_n = \sum_{m=1}^{n} c_{n,m}\mathcal{L}_m$

of the original lagrangians, with constant coefficients $c_{n,m}$ chosen such that a perturbative expansion of $\mathcal{L}_n$ around a constant background $\pi \to \pi_0 + \delta\pi$ begins at $\mathcal{O}(\delta\pi^n)$. In particular, as first shown in [39], when applied to the case of a flat brane in a flat bulk, this procedure reproduces the flat space galileons of [91].

The ability to carry out such an expansion appears to be an artifact of maximal symmetry. The small π limit in the present case of a flat bulk and an FRW brane does not, for general $a(t)$, admit a choice of $c_{n,m}$ with the above mentioned properties.

One case which does work is $a(t) \sim e^{Ht}$, corresponding to a de Sitter brane, which has maximal symmetry. The induced metric on any $w = \text{const}$ hypersurface is

$$\begin{aligned} ds^2 &= (1 - Hw)^2 \left[-dt^2 + e^{2Ht} d\vec{x}^2 \right] \\ &= (1 - Hw)^2 \, g^{(dS)}_{\mu\nu} dx^\mu dx^\nu \ , \end{aligned} \tag{4.28}$$

where $g^{(dS)}_{\mu\nu}$ is the 4D de Sitter metric in inflationary coordinates, and so we are simply foliating 5D Minkowski by dS_4, returning to the setup of a maximally symmetric brane in a maximally symmetric bulk. In the gauge (2.7), the induced metric becomes

$$\bar{g}_{\mu\nu} = (-1 + H\pi)^2 \, g^{(dS)}_{\mu\nu} + \partial_\mu \pi \partial_\nu \pi \ . \tag{4.29}$$

If we then make the field redefinition $\tilde{\pi} = -1 + H\pi$ and switch to coordinates $\hat{x}^\mu = Hx^\mu$, the lagrangians calculated from the induced metric (4.29) and associated extrinsic curvature take the forms of those in Sec.3.3.3, from which small $\tilde{\pi}$ limits can be constructed.

Chapter 5

Summary of Part II

In Part II of this thesis, we have shown that the galileon theory is a special case of a class of effective field theories that may be identified as the description of a brane embedded in a bulk space. The theories obtained in this way may be interesting as examples of higher dimensional gravitating theories, or may merely provide new nontrivial examples of 4D effective field theories.

In Chapter 2, we explicitly laid out the general construction for galileon-like probe brane theories. In particular, we discussed the conditions necessary for generating well-behaved brane constructions and the methods used in deriving the symmetries of the resulting lagrangians.

In Chapter 3 we applied this construction to all possible special cases in which both the bulk and relaxed brane state are maximally symmetric spaces (with the bulk metric having only a single time direction). The results are new classes of effective field theories which share important properties of the galileons while exhibiting distinctive new features such as the existence of potentials with masses fixed by symmetries. These potentials open up the possibility of new, natural implementations of accelerating cosmological solutions in theories naturally having a de Sitter solution. Furthermore, in some cases the potentials allow both spontaneous symmetry breaking and ghost condensation at the same time. This may allow for other new consequences of these theories, including the possibility of novel topological defects in these theories.

One of the most interesting features of the original galileon model is that it provides a relatively simple realization of the Vainshtein mechanism [108] whereby a galileon field sourced by a heavy object exerts a fifth force at large distances from the object, but not at short distances. This screening mechanism is required if there's any hope for galileons to be in concert with solar system tests of gravity. However, the mechanism comes at a price: in the original galileon model, perturbations about the screening profile propagate superluminally. A natural hope would be that some version of the probe brane galileons derived in the previous sections could retain the Vainshtein mechanism yet avoid superluminally propagating perturbations. In Sec.3.5, we explored this possibility for the DBI galileons of Sec.3.3.1. While the screening mechanism was preserved, it was, unfortunately, found that superluminal propagation persisted, too.

In Chapter 4 we used the brane methods to construct a theory of a galileon-like scalar on FRW spacetime. We have derived the relevant operators allowed in the lagrangians, and identified the highly nontrivial symmetry transformations under which they are invariant. These general expressions are much longer for FRW spacetimes than they are for maximally symmetric ones. By specializing to the minisuperspace approximation, in which the galileons depend only on cosmic time, we are able to provide somewhat more compact versions suitable for understanding the effects of galileons on the background cosmology. However, more complicated questions, such as those involving spatially dependent galileon perturbations, will require the full expressions. It is possible that integrations by parts would greatly simplify the expressions, but we have not attempted these here. Though the higher order lagrangians are a bit impractical to use, their complexity also serves as an illustration of the power of brane methods as they provide a tool through which to generate lagrangians which can nearly be arbitrarily complicated, yet which obey still an enormous number of symmetries.

We then sought interesting small-field limits of the FRW lagrangians and their symmetry transformations, as was done for galileons propagating on maximally symmetric backgrounds. Due to the fewer isometries of FRW, the analogous expressions do not seem to exist, except in the special cases in which the FRW space coincides with de Sitter.

Finally, we studied the stability of simple solutions, namely $\pi = 0$ with $a(t) = (t/t_0)^n$, and found that given a correct sign for coefficients in the lagrangians, all four possible solutions are stable, at least on the time scales of the background. One of the four cases leads to a massless field without any gradient energy and the remaining three cases lead to scalar fields with tachyonic masses but the associated time scales are large enough to avoid the potential instability. For exponential scale factor growth, the $\pi = 0$ solution also leads to a tachyon whose time scale is again large enough to stabilize the theory for one e-fold.

Part III

Galileons and Spontaneous Symmetry Breaking

Chapter 6

Galileons As Wess-Zumino Terms[13]

6.1 Overview

In this chapter we present a different method of deriving the galileon terms—an algebraic method, treating them as Goldstone modes of spontaneously broken space-time symmetries. The inspiration for such a viewpoint comes from the general concepts outlined in Part II of this thesis. Indeed, this line of research was foreshadowed in various previous sections and, conversely, some of the language in the following sections will echo the ideas of the proceeding chapters. The main idea is that if we consider a bulk which enjoys some set of isometries, the placement of a probe brane within the bulk will generically spontaneously break some of these symmetries and a Goldstone mode will correspondingly arise.

The spontaneous nature of the breaking is perhaps most evident in the DBI galileon case of Sec.3.3.1 in which the probe brane's ground state is M_4 and the bulk is M_5. There are many inequivalent ways to embed M_4 in M_5, none of which is preferred. That is, given one acceptable placement of the brane there are many related setups which are just Lorentz rotations of the initial configuration, each of which would correspond to an equally

[13]The work in Part III was performed in collaboration with Kurt Hinterbichler, Austin Joyce and Mark Trodden.

acceptable ground state. Therefore, in placing the probe brane we are effectively choosing one out of many equivalent possible vacua which is the essential nature of spontaneous symmetry breaking (SSB). As evidence of this interpretation, we already saw in Part II that the bulk symmetries which are broken by the presence of the brane end up manifesting themselves as non-linear symmetries for the π field and it is well known that the non-linear realization of broken symmetries is one of the hallmarks of SSB.

There are standard tools in the literature for analyzing systems which display SSB, both in the case of internal and spacetime symmetries. Namely, there are the classic techniques of non-linear realizations developed by Callan, Coleman, Wess and Zumino [22, 31] and Volkov [112] and we adapt these "coset constructions" to the case of galileon theories. We show that, like the familiar Wess–Zumino–Witten term of the chiral Lagrangian [116, 118], the galileon terms in d dimensions are not captured by the naive d-dimensional coset construction. Instead, they require a higher dimensional construction and arise from invariant $(d+1)$-forms created via the coset construction which are then pulled back to our d-dimensional space-time in order to create galileon invariant actions. The relevant $(d+1)$-forms, and hence the galileons, are associated with non-trivial co-cycles in an appropriate Lie algebra cohomology [28, 33, 34], which is a cohomology theory on forms which are left-invariant under vector fields that generate the symmetry algebra.[14] This is related to the internal symmetry case, where it was shown in [47] that Wess–Zumino terms are counted by de Rham cohomology. Indeed, for compact groups, de Rham and Lie algebra cohomology are isomorphic [35].

After reviewing the general coset construction, we describe the algebra non-linearly realized by the galileons—the "galileon algebra." We show that, inspired by brane-world models of Part II, this is a contraction of a higher-dimensional Poincaré algebra only along particular auxiliary directions, that is, it can be thought of as the Poincaré algebra of a

[14] A similar viewpoint was conveyed in [15], where the low-energy effective actions for non-relativistic strings and branes were obtained as Wess–Zumino terms.

brane embedded in higher dimensions, where the speed of light in the directions transverse to the brane is sent to infinity, while the speed of light along the brane is kept constant. The most familiar example of a galileon theory is the non-relativistic free point particle, which can be thought of as a $(0+1)$-dimensional field theory invariant under the galilean group. We review the construction of the kinetic term for the free particle as a Wess–Zumino term before applying our arguments to the most physically relevant situation of galileons in four dimensions. As the galileons are Wess–Zumino terms, we argue that the number of such terms for both single and multi-galileon situations is bounded by the dimension of the appropriate Lie algebra cohomology groups.

Additionally, we consider the conformal galileons of (3.149). In this case, only one of the conformal galileons, the cubic term, appears as a Wess–Zumino term for spontaneously broken conformal symmetry. We construct this Wess–Zumino term explicitly and comment on its relation to the curvature invariant technique employed in [91] to construct the conformal galileons.

Finally, we demonstrate that, although the original galileons are Wess–Zumino terms for spontaneously broken space-time symmetries, this is not the case for the relativistic DBI galileons of Sec.3.3.1, first derived in [39], which—aside from the tadpole term—are obtainable from the coset construction and hence are not Wess–Zumino terms. We show how to construct the DBI galileons using the techniques of non-linear realizations.

6.2 Nonlinear realizations and the coset construction

Broken symmetries and effective field theory have historically been extremely profitable viewpoints from which to study the low-energy dynamics of physical systems. Motivated by the successes of phenomenological Lagrangians in describing low energy pion scattering [114], Callan, Coleman, Wess and Zumino [22, 31], as well as Volkov [112], developed a powerful formalism for constructing the most general effective action for a given symmetry breaking pattern. This is the now well-known technique of non-linear realizations, or coset construction, which we review briefly here. More comprehensive reviews are given in [94, 122].

6.2.1 Spontaneously broken internal symmetries

We begin by reviewing the problem of constructing a Lagrangian for Goldstone fields corresponding to the breaking of an internal (*i.e.*, commuting with the Poincaré group) symmetry group G down to a subgroup H; that is, we seek the most general Lagrangian which is invariant under G transformations, where the H transformations act linearly on the fields and those not in H act non-linearly. As is well known [22, 31], there will be $\dim(G/H)$ Goldstone bosons, which parametrize the space of (left) cosets G/H.

However, to start with, we use fields $V(x)$ that take values in the group G, $V(x) \in G$, so that there are $\dim(G)$ fields. We then count as equivalent fields that differ by an element of the subgroup, so $V(x) \sim V(x)h(x)$, where $h(x) \in H$. To implement this equivalence, we demand that the theory be gauge invariant under local $h(x)$ transformations $V(x) \to V(x)h(x)$. There are $\dim(H)$ gauge transformations, so the number of physical Goldstone bosons will be $\dim(G) - \dim(H) = \dim(G/H)$, the expected number.

The global G transformations act on the left as $V(x) \to gV(x)$, where $g \in G$. The theory should therefore be invariant under the symmetries

$$V(x) \longmapsto gV(x)h^{-1}(x), \tag{6.2.1}$$

where g is a global G transformation, and $h^{-1}(x)$ (written as an inverse for later convenience) is a local H transformation.

A Lie group, G, possesses a distinguished left-invariant Lie algebra-valued 1-form, the so-called Maurer–Cartan form, given by $V^{-1}\mathrm{d}V$. Since this is Lie algebra-valued we may expand over a basis $\{V_I, Z_a\}$ where $\{V_I\}$, $I = 1, \ldots, \dim(H)$ is a basis of the Lie algebra

$\mathfrak{h}$ of H, and $\{Z_a\}$, $a = 1, \ldots, \dim(G/H)$ is any completion to a basis of $\mathfrak{g}$. We expand the Maurer–Cartan form over this basis,

$$V^{-1}\mathrm{d}V = \omega_V^I V_I + \omega_Z^a Z_a \ , \tag{6.2.2}$$

where ω_V^I and ω_Z^a are the coefficients, which depend on the fields and their derivatives. The Maurer–Cartan form (6.2.2), and hence the coefficients in the expansion on the right hand side, are invariant under global G transformations.

Under the local $h(x)$ transformation, the pieces $\omega_V \equiv \omega_V^I V_I$ and $\omega_Z \equiv \omega_Z^I Z_I$ transform as

$$\begin{aligned} \omega_Z &\longmapsto h\,\omega_Z h^{-1}, \\ \omega_V &\longmapsto h\,\omega_V h^{-1} + h\,\mathrm{d}h^{-1} \ . \end{aligned} \tag{6.2.3}$$

We see that ω_Z transforms covariantly as the adjoint representation of the subgroup, and we use it as the basic ingredient to construct invariant Lagrangians [22, 31, 94, 112]. On the other had, ω_V transforms as a gauge connection.[15] If we have additional matter fields $\psi(x)$ which transform under some linear representation D of the local group H (and do not change under global G transformations),

$$\psi \longrightarrow D\,(h)\,\psi \ , \tag{6.2.4}$$

we may construct a covariant derivative using ω_V via

$$\mathcal{D}\psi \equiv \mathrm{d}\psi + D(\omega_V)\psi, \qquad \mathcal{D}\psi \to D\,(h)\,\mathcal{D}\psi \ . \tag{6.2.5}$$

Thus, the most general Lagrangian is any Lorentz and globally H-invariant scalar constructed from the components of ω_Z, ψ, and the covariant derivative,

$$\mathcal{L}\left(\omega_{Z\,\mu}^{\ I}, \psi, \mathcal{D}_\mu\right) \ . \tag{6.2.6}$$

[15] This is a reflection of the well-known fact that the pullback of the Maurer–Cartan form defines a natural H-connection on G/H [23, 34, 93].

To obtain a theory with global G symmetry, we fix the $h(x)$ gauge symmetry by imposing some canonical choice for $V(x)$, which we call $\tilde{V}(x)$. This canonical choice should smoothly pick out one representative element from each coset, so $\tilde{V}(x)$ contains $\dim(G/H)$ fields. In general, a global g transformation will not preserve this choice, so a compensating h transformation—depending on g and $\tilde{V}$—will have to be made at the same time to restore the gauge choice. The gauge fixed theory will then have the global symmetry

$$\tilde{V}(x) \longmapsto g\tilde{V}(x)h^{-1}(g, \tilde{V}(x)). \tag{6.2.7}$$

If we can choose the parametrization such that the transformation (6.2.7) is linear in the fields $\tilde{V}$ only when $g \in H$, then we will have realized the symmetry breaking pattern $G \to H$. When the commutation relations of the algebra are such that the commutator of a broken generator with a subgroup generator is again a subgroup generator $[V_I, Z] \sim Z$, (which is true if G is a compact group), one way to accomplish this is to choose the parametrization

$$\tilde{V}(x) = e^{\xi(x)\cdot Z} \quad . \tag{6.2.8}$$

Here the real scalar fields $\xi^a(x)$ are the $\dim(G/H) = \dim G - \dim H$ different Goldstone fields associated with the symmetry breaking pattern. Under left action by some $g \in G$, (6.2.7) gives the transformation law for the $\xi^a(x)$ as,

$$e^{\xi\cdot Z} \to e^{\xi'\cdot Z} = g e^{\xi\cdot Z} h^{-1}(g, \xi) \ , \tag{6.2.9}$$

As can be seen using the Baker–Campbell–Hausdorff formula and the commutation condition $[V_I, Z] \sim Z$, the action on ξ is linear when $g \in H$.

6.2.2 Spontaneously broken space-time symmetries

In the preceding subsection we reviewed the case of spontaneously broken internal symmetries. Galileons, however, arise as Goldstone modes of spontaneously broken space-time symmetries (the non-linear symmetry $\pi \to \pi + c + b_\mu x^\mu$ does not commute with the Poincaré generators). Consequently, we must extend the coset procedure to account for subtleties involved in non-linear realizations of symmetries which do not commute with the Poincaré group. This was worked out comprehensively by Volkov [112] and is reviewed nicely in [94]. While the construction is generally similar to the internal symmetry case, the main subtlety is that now we must explicitly keep track of the generators of space-time symmetries in the coset construction.

Following [94], we assume that our full symmetry group G contains the unbroken generators of space-time translations P_α, unbroken Lorentz rotations $J_{\alpha\beta}$, an unbroken symmetry subgroup H generated by V_I (which all together form a subgroup), and finally the broken generators denoted by Z_a. The broken generators may in general be a mix of internal and space-time symmetry generators. As before, we want to parameterize the coset G/H, but the parametrization now takes the form [81, 94, 112]

$$\tilde{V} = e^{x\cdot P} e^{\xi(x)\cdot Z} \ . \tag{6.2.10}$$

Note that we treat the unbroken translation generators on the same footing as the broken generators, with the coefficients simply the space-time coordinates.[16] As in the case of the internal symmetries, under left action by some $g \in$ G, (6.2.10) transforms non-linearly

$$e^{x\cdot P} e^{\xi(x)\cdot Z} \longmapsto e^{x'\cdot P} e^{\xi'(x')\cdot Z} = g\, e^{x\cdot P} e^{\xi(x)\cdot Z} h^{-1}(g, \xi(x)) \ , \tag{6.2.11}$$

where $h(g, \xi(x))$ belongs to the unbroken group spanned by V_I and $J_{\mu\nu}$, but has dependence on ξ.

As in the internal symmetry case, the object in which we are interested is the Maurer–Cartan form

$$\tilde{V}^{-1}\mathrm{d}\tilde{V} = \omega_P^\alpha P_\alpha + \omega_Z^a Z_a + \omega_V^I V_I + \frac{1}{2}\omega_J^{\alpha\beta} J_{\alpha\beta} \ , \tag{6.2.12}$$

[16] This is little more than bookkeeping, as the coordinates formally transform non-linearly under a translation $x^\mu \to x^\mu + \epsilon^\mu$. One intuitive way to understand this is to think of Minkowski space as the coset Poincaré/Lorentz, as is pointed out in [81].

where we have again expanded in the basis of the Lie algebra $\mathfrak{g}$. We may act with the transformation (6.2.11) to determine that the components, $\omega_P \equiv \omega_P^\alpha P_\alpha,\ \omega_Z \equiv \omega_Z^a Z_a,\ \omega_V \equiv \omega_V^I V_I,\ \omega_J \equiv \frac{1}{2}\omega_J^{\alpha\beta} J_{\alpha\beta}$ of the Maurer–Cartan 1-form transform as [94]

$$
\begin{aligned}
\omega_P &\to h\ \omega_P\ h^{-1}, \\
\omega_Z &\to h\ \omega_Z\ h^{-1}, \\
\omega_V + \omega_J &\to h\,(\omega_V + \omega_J)\,h^{-1} + h\,\mathrm{d}h^{-1}\ .
\end{aligned} \tag{6.2.13}
$$

The covariant transformation rule for ω_P and ω_Z tells us that these are the ingredients to use in constructing invariant Lagrangians [81, 94, 112]. The form ω_P, expanded in components is

$$
\omega_P = \mathrm{d}x^\nu (\omega_P)_\nu{}^\alpha P_\alpha, \tag{6.2.14}
$$

Here the components $(\omega_P)_\nu{}^\alpha$ should be thought of as an invariant vielbein, with α a Lorentz index, from which we can construct an invariant metric

$$
g_{\mu\nu} = (\omega_P)_\mu{}^\alpha (\omega_P)_\nu{}^\beta \eta_{\alpha\beta}, \tag{6.2.15}
$$

and an invariant measure

$$
-\frac{1}{4!}\epsilon_{\alpha\beta\gamma\delta}\omega_P^\alpha \wedge \omega_P^\beta \wedge \omega_P^\gamma \wedge \omega_P^\delta = \mathrm{d}^4x\sqrt{-g}\ . \tag{6.2.16}
$$

The form ω_Z, expanded in components

$$
\omega_Z = \mathrm{d}x^\mu (\omega_Z)_\mu{}^a Z_a, \tag{6.2.17}
$$

yields the basic ingredient $\mathcal{D}_\alpha \xi^a$, the covariant derivative of the Goldstones, through

$$
(\omega_Z)_\mu{}^a = (\omega_P)_\mu{}^\alpha \mathcal{D}_\alpha \xi^a. \tag{6.2.18}
$$

We can construct covariant derivatives $\mathcal{D}$ for matter fields ψ, transforming as some combined Lorentz and H representation, which we call D, by using $\omega_V + \omega_J$ as a connection,

$$\omega_P^\alpha \mathcal{D}_\alpha \psi = \mathrm{d}\psi + D(\omega_V)\psi + D(\omega_J)\psi \ . \tag{6.2.19}$$

This can also be used to take higher covariant derivatives of the Goldstones. From these pieces, $e_\mu{}^\alpha$, $\mathcal{D}_\alpha \xi^a$, ψ and $\mathcal{D}_\alpha$, we can build the most general invariant Lagrangian by combining them in a Lorentz and H invariant way, and then multiplying against the invariant measure (6.2.16).

6.2.3 Inverse Higgs constraint

There is another subtlety that arises in extending the coset construction to the case of space-time symmetries—there can be non-trivial relations between different Goldstone modes leading to fewer degrees of freedom than naive counting would suggest. This is the well-known statement that the counting of massless degrees of freedom in Goldstone's theorem fails in the case of broken space-time symmetries [11, 70, 76, 81, 85, 92, 112, 113]; that is, the number of Goldstone modes will not in general be equal to $\dim(G/H)$. This phenomenon is sometimes referred to as the *inverse Higgs effect* [76].

Accounting for this is simple—if the commutator of an unbroken translation generator with a broken symmetry generator, say Z_1, contains a component along some linearly independent broken generator, say Z_2,

$$[P, Z_1] \sim Z_2 + \cdots \ , \tag{6.2.20}$$

(where the dots represent a component along the broken directions), it is possible to eliminate the Goldstone field corresponding to the generator Z_1 [76, 81, 85]. The relation between the Goldstone modes is obtained by setting the coefficient of Z_2 in the Maurer–Cartan form to zero.

This is a covariant constraint; *i.e.*, it is invariant under G because the Maurer–Cartan form itself is invariant (often, the inverse Higgs constraint is imposed automatically in a constructed Lagrangian because it is equivalent to integrating out the redundant Goldstone field via its equation of motion [85]). We will need to use the inverse Higgs constraint in constructing the galileons.

6.3 Cohomology

As we shall see, the galileon terms are in fact not captured by the coset construction of the previous section. This is essentially due to the fact that the coset construction produces Lagrangians which are strictly invariant under the desired symmetries, but the galileon Lagrangians are not strictly invariant—they change by a total derivative (so the action is still invariant). As we shall also see, it will turn out that they can be thought of and categorized as non-trivial elements of Lie algebra cohomology.

In this section, we introduce the necessary concepts and definitions of Lie algebra cohomology and relative Lie algebra cohomology needed for classifying the galileons. For a more comprehensive introduction, including applications, see [33].

6.3.1 Lie algebra cohomology

Given a Lie algebra $\mathfrak{g}$, an n-co-chain, $n = 0, 1, 2, \ldots$, is a totally anti-symmetric multi-linear mapping $\omega_n : \bigwedge^n \mathfrak{g} \to \mathbb{R}$, taking values in the reals.[17] The space of n-co-chains is denoted $\Omega^n(\mathfrak{g})$. One then forms a co-boundary operator $\delta_n : \Omega^n(\mathfrak{g}) \to \Omega^{n+1}(\mathfrak{g})$ whose action is defined by [33]

$$\delta\omega(X_1, X_2, \ldots, X_{n+1}) = \sum_{\substack{j,k=1 \\ j<k}}^{n+1} (-1)^{j+k} \omega([X_j, X_k], X_1, \ldots, \hat{X}_j, \ldots, \hat{X}_k, \ldots, X_{n+1}), \quad (6.3.1)$$

[17] In general, one can consider the case in which the co-chains take values in an arbitrary vector space on which acts a non-trivial representation of $\mathfrak{g}$, but we do not need that here.

for $X_1, X_2, \ldots \in \mathfrak{g}$ and where $\hat{X}$ means the argument is omitted, and [,] is the Lie algebra commutator. The first few instances are

$$\begin{aligned}
&\delta\omega_0(X_1) = 0, \\
&\delta\omega_1(X_1, X_2) = -\omega_1([X_1, X_2]), \\
&\delta\omega_2(X_1, X_2, X_3) = -\omega_2([X_1, X_2], X_3) + \omega_2([X_1, X_3], X_2) - \omega_2([X_2, X_3], X_1), \\
&\quad\vdots
\end{aligned} \tag{6.3.2}$$

One can show, using the Jacobi identity $[X_1, [X_2, X_3]] + [X_2, [X_3, X_1]] + [X_3, [X_1, X_2]] = 0$, that the co-boundary operator is nilpotent

$$\delta^2 = 0 \ . \tag{6.3.3}$$

Thus we have $\mathrm{Im}_{\delta_{n-1}}\left(\Omega^{n-1}\right) \subset \mathrm{Ker}_{\delta_n}\left(\Omega^n\right)$, and we can define the cohomology spaces

$$H^n(\mathfrak{g}) = \frac{\mathrm{Ker}_{\delta_n}\left(\Omega^n(\mathfrak{g})\right)}{\mathrm{Im}_{\delta_{n-1}}\left(\Omega^{n-1}(\mathfrak{g})\right)} \ . \tag{6.3.4}$$

There is another way to represent the co-boundary operator that is often more convenient when we have an explicit basis. Let $\{e_i\}$, $i = 1, \cdots, \dim(\mathfrak{g})$, be a basis for the Lie algebra $\mathfrak{g}$. The structure constants ${c_{ij}}^k$ are given by

$$[e_i, e_j] = {c_{ij}}^k e_k \ . \tag{6.3.5}$$

They are anti-symmetric in their first indices, ${c_{ij}}^k = -{c_{ji}}^k$. The Jacobi identity becomes ${c_{il}}^m {c_{jk}}^l + {c_{jl}}^m {c_{ki}}^l + {c_{kl}}^m {c_{ij}}^l = 0$. Let $\{\omega^i\}$ be a basis of the dual space $\mathfrak{g}^*$, dual to the basis $\{e_i\}$, so that $\omega^i(e_j) = \delta^i_j$. Then we can write any n-co-chain ω_n as sums of wedge products of the ω^i,

$$\omega_n = \frac{1}{n!}\Omega_{i_1 i_2 \cdots i_n} \omega^{i_1} \wedge \omega^{i_2} \wedge \cdots \wedge \omega^{i_n} \ , \tag{6.3.6}$$

where $\Omega_{i_1 i_2 \cdots i_n}$ is the totally anti-symmetric tensor of coefficients. The action of the co-boundary operator on a single ω^i is given by

$$\delta\omega^i = -\frac{1}{2}c_{jk}{}^i\omega^j \wedge \omega^k \ , \tag{6.3.7}$$

and is extended to wedge products of multiple ω's by using linearity and the Leibniz product rule, where we are careful to include the addition of a minus sign every time δ has to pass through an ω.[18] For example, we have $\delta\left(\omega^i \wedge \omega^j\right) = -\frac{1}{2}c_{kl}{}^i\omega^k \wedge \omega^l \wedge \omega^j + \frac{1}{2}c_{kl}{}^j\omega^i \wedge \omega^k \wedge \omega^l$. In terms of components, we have

$$(\delta\Omega)_{i_1 \cdots i_{n+1}} = -\frac{n(n+1)}{2}c_{[i_1 i_2}{}^j\Omega_{|j|i_3 \cdots i_{n+1}]}. \tag{6.3.8}$$

Lie algebra cohomology also has a geometric interpretation.[19] Consider the simply connected Lie group G associated to the Lie algebra $\mathfrak{g}$. The space of p-forms on G which are invariant under the left action of G on itself can be identified with the co-chains of Lie algebra cohomology. In fact, there is one left invariant 1-form for each generator of the Lie algebra, and wedging them together in all ways generates all the invariant p-forms. The usual exterior derivative operator on G, $\mathrm{d}_p : \Omega^p(G) \to \Omega^{p+1}(G)$ satisfies $\mathrm{d}\omega^i = -\frac{1}{2}c_{jk}{}^i\omega^j \wedge \omega^k$, and can be identified with the operator δ of Lie algebra cohomology. Thus, Lie algebra cohomology counts the number of left-invariant forms on G which cannot be written as the exterior derivative of a form which is also left-invariant.

6.3.2 Relative Lie algebra cohomology

For characterizing symmetry breaking to a subalgebra, we will need a slightly more refined notion of Lie algebra cohomology, known as relative Lie algebra cohomology. Consider a subalgebra $\mathfrak{h} \subset \mathfrak{g}$. We define the space of relative co-chains $\Omega^n(\mathfrak{g}, \mathfrak{h})$, as the subspace of co-chains satisfying the following two conditions,

$$\Omega_n(V, X_2, \ldots, X_n) = 0 \ , \tag{6.3.9}$$

[18] The co-boundary operator, δ, is an anti-derivation on the algebra of co-chains.

[19] In this geometric context, Lie algebra cohomology is known as Chevalley–Eilenberg Cohomology [28].

$$\Omega_n([V, X_1], X_2, \ldots, X_n) + \Omega_n(X_1, [V, X_2], \ldots, X_n) + \cdots + \Omega_n(X_1, X_2, \ldots, [V, X_n]) = 0 \ ,$$
$$\text{for all} \quad V \in \mathfrak{h}, \ \text{and} \quad X_2, \cdots, X_n \in \mathfrak{g} \ . \tag{6.3.10}$$

The first requirement says that if any of the arguments lie completely in $\mathfrak{h}$, then we get zero. This means that the form is well defined on the quotient $\mathfrak{g}/\mathfrak{h}$. Equivalently, the n-co-chains are only constructed from wedging together one-forms which annihilate $\mathfrak{h}$. To see what this means in terms of components, choose a basis $\{h_I, f_a\}$ for $\mathfrak{g}$, where $\{h_I\}$, $I = 1, \ldots, \dim(\mathfrak{h})$ is a basis of $\mathfrak{h}$ and $\{f_a\}$, $a = 1, \ldots, \dim(\mathfrak{g}/\mathfrak{h})$ completes to a basis of $\mathfrak{g}$. Let the dual basis be $\{\eta^I, \omega^a\}$. To satisfy (6.3.9), forms are constructed by wedging together only the forms ω^a, so the components $\Omega_{i_1 \cdots i_n}$ of (6.3.6) are zero if any of the indices are in the $\mathfrak{h}$ directions.

The second condition, in terms of components (6.3.6), reads $c_{Ii_1}{}^j \Omega_{j i_2 \cdots i_n} + c_{Ii_2}{}^j \Omega_{i_1 j \cdots i_n} + \cdots + c_{Ii_n}{}^j \Omega_{i_1 i_2 \cdots j} = 0$. The combination of the two conditions (6.3.9) and (6.3.10) on the components, along with the fact that $c_{IJ}{}^a = 0$ since $\mathfrak{h}$ is a subgroup, gives our final conditions in terms of components for a co-chain to be a relative co-chain,

$$\Omega_{I i_2 \cdots i_n} = 0 \ , \tag{6.3.11}$$
$$c_{Ia_1}{}^b \Omega_{b a_2 \cdots a_n} + c_{Ia_2}{}^b \Omega_{a_1 b \cdots a_n} + \cdots + c_{Ia_n}{}^b \Omega_{a_1 a_2 \cdots b} = 0 \ . \tag{6.3.12}$$

Given our basis, the matrices

$$\phi(h_I)_a{}^b = -c_{Ia}{}^b \tag{6.3.13}$$

form a representation of the subalgebra $\mathfrak{h}$,

$$\phi(h_I)\phi(h_J) - \phi(h_J)\phi(h_I) = c_{IJ}{}^K \phi(h_K) \ , \tag{6.3.14}$$

as can be straightforwardly shown using the Jacobi identity, as well as the condition $c_{IJ}{}^a = 0$ which follows from the fact that $\mathfrak{h}$ is a subalgebra. Thus, the indices $a, b, \ldots$ of the space $\mathfrak{g}/\mathfrak{h}$ furnish a representation of the subgroup $\mathfrak{h}$, and the condition (6.3.12) says that the co-chain coefficients must be invariant tensors under the action of $\mathfrak{h}$ in this space.

The δ operator preserves the two conditions (6.3.9) and (6.3.10), so $\delta_n\left(\Omega^n(\mathfrak{g},\mathfrak{h})\right) \subset \Omega^{n+1}(\mathfrak{g},\mathfrak{h})$. Thus we may think of δ as acting on the spaces $\Omega^n(\mathfrak{g},\mathfrak{h})$. The cohomology classes of this action are denoted by $H^p(\mathfrak{g},\mathfrak{h})$ and the construction is known as *relative Lie algebra cohomology* [33],

$$H^n(\mathfrak{g},\mathfrak{h}) = \frac{\mathrm{Ker}_{\delta_n}\left(\Omega^n(\mathfrak{g},\mathfrak{h})\right)}{\mathrm{Im}_{\delta_{n-1}}\left(\Omega^{n-1}(\mathfrak{g},\mathfrak{h})\right)} \ . \tag{6.3.15}$$

Each non-trivial element of $H^{d+1}(\mathfrak{g},\mathfrak{h})$ corresponds to a Wess–Zumino term for a d-dimensional space-time [33, 34].

Relative Lie algebra cohomology also has a geometric interpretation. Consider the connected Lie group G and subgroup H, corresponding to the algebra $\mathfrak{g}$ and subalgebra $\mathfrak{h}$. We can think of the group G as a fiber bundle, consisting of spaces H fibered over the base space G/H. The group G acts naturally on G/H (which is a homogeneous space with isotropy subgroup H). The relative co-chains can be thought of as left invariant form on G which are projectable to G/H, *i.e.*, can be written as a pullback through the projection $G \to G/H$ of a unique form on G/H. Thus they can be identified with invariant forms on G/H. The operator δ can be identified with the usual exterior derivative d, so relative Lie algebra cohomology counts the number of left-invariant forms on G/H which cannot be written as the exterior derivative of a form which is also left-invariant.

6.4 The galileon algebra

Having briefly introduced the standard techniques of non-linear realizations and made our acquaintance with Lie algebra cohomology, we now move on to the problem of principal interest—the construction of galileons using this machinery. In order to do this, however, we must first describe the symmetry algebra which the galileons non-linearly realize. We will call this algebra the *galileon algebra*.

A theory of N galileons, π^I, $I = 1, \ldots, N$, in d space-time dimensions has the usual Poincaré invariance $\mathfrak{iso}(d-1,1)$, of a relativistic field theory, under which all the galileons are scalars,

$$\begin{aligned} \delta_{P_\mu}\pi^I &= -\partial_\mu \pi^I \ , \\ \delta_{J_{\mu\nu}}\pi^I &= (x_\mu\partial_\nu - x_\nu\partial_\mu)\pi^I \ . \end{aligned} \tag{6.4.1}$$

These satisfy the usual commutation relations

$$\begin{aligned} [P_\mu, P_\nu] &= 0 \ , \\ [J_{\mu\nu}, P_\sigma] &= \eta_{\mu\sigma}P_\nu - \eta_{\nu\sigma}P_\mu \ , \\ [J_{\mu\nu}, J_{\sigma\rho}] &= \eta_{\mu\sigma}J_{\nu\rho} - \eta_{\nu\sigma}J_{\mu\rho} + \eta_{\nu\rho}J_{\mu\sigma} - \eta_{\mu\rho}J_{\nu\sigma} \ . \end{aligned} \tag{6.4.2}$$

There is also a linearly realized internal $\mathfrak{so}(N)$ symmetry under which the π^I rotate in the fundamental representation,

$$\delta_{J_{IJ}}\pi^K = (\delta_I^K\delta_{JL} - \delta_J^K\delta_{IL})\pi^L \ , \tag{6.4.3}$$

satisfying

$$[J_{IJ}, J_{KL}] = \delta_{IK}J_{JL} - \delta_{JK}J_{IL} + \delta_{JL}J_{IK} - \delta_{IL}J_{JK} \ . \tag{6.4.4}$$

Finally, there are the non-linear shift symmetries[20],

$$\delta_{C^I}\pi^J = \delta^{IJ} \ , \qquad \delta_{B^I{}_\mu}\pi^J = x_\mu\delta^{IJ}. \tag{6.4.5}$$

These shift symmetries commute amongst themselves, but have the following non-trivial commutation relations with the linearly realized symmetries,

$$\begin{aligned} \left[P_\mu, B^I{}_\nu\right] &= \eta_{\mu\nu}C^I, & \left[J_{IJ}, C^K\right] &= \delta_I^K C_J - \delta_J^K C_I, \\ \left[J_{\mu\nu}, B^I{}_\sigma\right] &= \eta_{\mu\sigma}B^I{}_\nu - \eta_{\nu\sigma}B^I{}_\mu, & \left[J_{IJ}, B^K{}_\mu\right] &= \delta_I^K B_{J\mu} - \delta_J^K B_{I\mu} \ . \end{aligned} \tag{6.4.6}$$

[20] For an interpretation of the conserved charges associated with these symmetries, see [89].

We will call the algebra satisfying these commutation relations the *galileon algebra* in d space-time dimensions, co-dimension N, and denote it by

$$\mathfrak{Gal}((d-1)+1, N)\ , \tag{6.4.7}$$

where the first argument indicates that there are $d-1$ space dimensions, and 1 time dimension. Correspondingly, we will denote the galileon group by $\mathrm{Gal}((d-1)+1, N)$.

Consider first the special case when $d = 1$, *i.e.*, a $0+1$ dimensional space-time. The algebra $\mathfrak{Gal}(0+1, N)$ is the algebra of galilean transformations, the symmetries of a free non-relativistic point particle moving in N dimensions. The $0+1$ dimensional space-time is the particle world-line, and the N co-dimensions are the dimensions in which the particle moves. The case $N = 1$ gives the symmetries of the single field galileons (1.37), and the case $N \geq 2$ gives the symmetries of the $\mathfrak{so}(N)$ symmetric multi-field galileons (1.73).

6.4.1 Geometric interpretation of the galileon algebra

The galileon algebras can readily be given a geometric interpretation. Recall that the Poincaré transformations can be thought of as the algebra of infinitesimal transformations that preserve the metric tensor $\eta_{\mu\nu} = \mathrm{diag}\,(-1, 1, 1, \ldots, 1)$. The galileon algebra $\mathfrak{Gal}((d-1)+1, N)$ is the algebra of infinitesimal transformations of $\mathbb{R}^{d+N}$ that preserves *two* different tensors, one covariant and one contravariant,

$$f_{\mu\nu} = \mathrm{diag}(\underbrace{-1, 1, \ldots, 1}_{d \text{ slots}}, 0, \ldots 0), \tag{6.4.8}$$

$$\tilde{f}^{\mu\nu} = \mathrm{diag}(0, 0, \ldots, 0, \underbrace{1, \ldots 1}_{N \text{ slots}})\ . \tag{6.4.9}$$

The finite form of this transformation can be given most easily by grouping the coordinates (x^μ, y^I) of $\mathbb{R}^{d+N}$ into a column vector with the addition of a 1 in the last slot, and then giving the transformation in matrix form as

$$\begin{pmatrix} y^I \\ x^\mu \\ 1 \end{pmatrix} \longmapsto \begin{pmatrix} R^I{}_J & b^I{}_\nu & c^I \\ 0 & \Lambda^\mu{}_\nu & p^\mu \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y^J \\ x^\nu \\ 1 \end{pmatrix}. \tag{6.4.10}$$

Here $R^I{}_J$ is a rotation matrix, $\Lambda^\mu{}_\nu$ is a Lorentz transformation, and $b^I{}_\mu$, c^I and p^μ are any real numbers.

6.4.2 The galileon algebra as a contraction

Yet another way to think of the galileon algebras as a Wigner–İnönü contraction [75] of the $(d+N)$-dimensional Poincaré algebra along N of the spatial directions. Physically, we can think of the galileons as describing a co-dimension N brane, where the speed of light has been sent to infinity in the directions transverse to the brane, but remains finite in the directions along the brane.

To see this, begin with the $(d+N)$ dimensional Poincaré algebra $\mathfrak{iso}(d-1+N,1)$, with non-zero commutators

$$\begin{aligned} [J_{BC}, P_A] &= \eta_{BA} P_C - \eta_{CA} P_B, \\ [J_{AB}, J_{CD}] &= \eta_{AC} J_{BD} - \eta_{BC} J_{AD} + \eta_{BD} J_{AC} - \eta_{AD} J_{BC} \ , \end{aligned} \tag{6.4.11}$$

where $A, B \cdots = 0, 1, 2, \ldots, d+N$ and $\eta_{AB} = \mathrm{diag}(-1, 1, 1, \ldots, 1)$. Now break apart the indices, using Greek letters for the first d directions and Latin letters for the N co-dimension directions,

$$\begin{aligned} [J_{\nu\rho}, P_\mu] &= \eta_{\nu\mu} P_\rho - \eta_{\rho\mu} P_\nu, \\ [J_{\mu\nu}, J_{\rho\sigma}] &= \eta_{\mu\rho} J_{\nu\sigma} - \eta_{\nu\rho} J_{\mu\sigma} + \eta_{\nu\sigma} J_{\mu\rho} - \eta_{\mu\sigma} J_{\nu\rho}, \end{aligned}$$

$$\begin{aligned} [J_{IJ}, J_{KL}] &= \delta_{IK} J_{JL} - \delta_{JK} J_{IL} + \delta_{JL} J_{IK} - \delta_{IL} J_{JK}, \\ [J_{JK}, P_I] &= \delta_{JI} P_K - \delta_{KI} P_J, \end{aligned}$$

$$
\begin{aligned}
&[P_\mu, J_{I\nu}] = \eta_{\nu\mu} P_I, \\
&[P_I, J_{J\rho}] = -\delta_{IJ} P_\rho, \\
&[J_{I\nu}, J_{K\sigma}] = \delta_{IK} J_{\nu\sigma} + \eta_{\nu\sigma} J_{IK}, \\
&[J_{KL}, J_{I\nu}] = \delta_{IK} J_{L\nu} - \delta_{IL} J_{K\nu}, \\
&[J_{\rho\sigma}, J_{I\nu}] = \eta_{\rho\nu} J_{I\sigma} - \eta_{\sigma\nu} J_{I\rho} \ .
\end{aligned} \tag{6.4.12}
$$

The contraction is performed by introducing a parameter, v, which will be sent to infinity and which is inserted into the algebra by changing co-dimensional entries of η_{AB} to v, so that $\eta_{AB} \to \text{diag}\ (-1, 1, \ldots, 1, v, v, \ldots, v)$ and making the following re-scalings

$$
P_I \longrightarrow vC_I \ , \qquad J_{I\nu} \longrightarrow vB_{I\nu} \ , \qquad J_{IJ} \longrightarrow vJ_{IJ} \ . \tag{6.4.13}
$$

After sending $v \to \infty$, the surviving non-trivial commutation relations are

$$
\begin{aligned}
&[J_{\nu\rho}, P_\mu] = \eta_{\nu\mu} P_\rho - \eta_{\rho\mu} P_\nu \ , \\
&[J_{\mu\nu}, J_{\rho\sigma}] = \eta_{\mu\rho} J_{\nu\sigma} - \eta_{\nu\rho} J_{\mu\sigma} + \eta_{\nu\sigma} J_{\mu\rho} - \eta_{\mu\sigma} J_{\nu\rho} \ , \\
\\
&[J_{IJ}, J_{KL}] = \delta_{IK} J_{JL} - \delta_{JK} J_{IL} + \delta_{JL} J_{IK} - \delta_{IL} J_{JK} \ , \\
&[J_{JK}, C_I] = \delta_{JI} C_K - \delta_{KI} C_J \ , \\
\\
&[P_\mu, B_{I\nu}] = \eta_{\nu\mu} C_I \ , \\
&[J_{KL}, B_{I\nu}] = \delta_{IK} B_{L\nu} - \delta_{IL} B_{K\nu} \ , \\
&[J_{\rho\sigma}, B_{I\nu}] = \eta_{\rho\nu} B_{I\sigma} - \eta_{\sigma\nu} B_{I\rho} \ .
\end{aligned} \tag{6.4.14}
$$

These are exactly the commutations relations of $\mathfrak{Gal}((d-1)+1, N)$.

6.5 Non-relativistic point particle moving in one dimension

We now proceed with the coset construction, first considering the simplest case of a galileon: the one-dimensional non-relativistic free point particle. We can think of this as a $0+1$ dimensional brane probing a non-relativistic $1+1$ dimensional bulk. The Wess–Zumino nature of the kinetic term was pointed out in [59] and is elegantly treated using jet bundles in [35]. Here, instead, we will derive equivalent results from the coset perspective.

We denote the single degree of freedom as $q(t)$, where t is the one and only space-time coordinate. We want to construct Lagrangians which are invariant under the algebra $\mathfrak{Gal}(0+1,1)$, which is three dimensional and whose generators act on $q(t)$ as follows

$$\delta_C q = 1, \qquad \delta_B q = -t, \qquad \delta_P q = -\dot{q} \; . \tag{6.5.1}$$

Here δ_C is the shift symmetry on the field, δ_B is the analogue of the "galilean" shift symmetry (the galilean boost of the non-relativistic particle) and δ_P is time translation of the field. The algebra has only a single non-zero commutator[21]

$$[B,P] = C \; . \tag{6.5.2}$$

The only transformation among (6.5.1) which is linear is δ_P, the rigid translations of the line, so the breaking pattern is

$$\mathfrak{Gal}(0+1,1) \longrightarrow \mathfrak{iso}(1). \tag{6.5.3}$$

To construct the most general Lagrangian which realizes these symmetries (6.5.1), we employ the coset construction for space-time symmetries reviewed in Section (6.2.2). The parametrization of the coset (6.2.10) is given by

$$\tilde{V} = e^{tP} e^{qC+\xi B} \; , \tag{6.5.4}$$

[21] In relation to the d-dimensional algebra, we are defining $P \equiv P_0$, $B \equiv B_0$.

where q is the Goldstone field that will become the physical field associated with the shift symmetry, and ξ is the Goldstone field associated with the galilean boost symmetry. Since the momentum P is to be included in the coset, there is no subgroup H to be linearly realized. Thus the coset is the galilean group itself,

$$\mathrm{Gal}(0+1,1) \ . \tag{6.5.5}$$

Next we compute the Maurer–Cartan form (6.2.12),

$$\omega = \tilde{V}^{-1}\mathrm{d}\tilde{V} = \mathrm{d}tP + (\mathrm{d}q - \xi\mathrm{d}t)\, C + \mathrm{d}\xi B \ , \tag{6.5.6}$$

and the component 1-forms used to build Lagrangians can then be read off as

$$\omega_P = \mathrm{d}t \ , \qquad \omega_C = \mathrm{d}q - \xi\mathrm{d}t \ , \qquad \omega_B = \mathrm{d}\xi \ . \tag{6.5.7}$$

Now, it is important to note that there is an inverse Higgs constraint. Inspection of the only commutator of the algebra (6.5.2) shows that we can eliminate the ξ field in favor of q by setting $\omega_C = 0$, implying the relation

$$\xi = \dot{q} \ . \tag{6.5.8}$$

Substitution into (6.5.7) then provides simplified expressions for the basis 1-forms

$$\omega_P = \mathrm{d}t \ , \qquad \omega_B = \ddot{q}\, \mathrm{d}t \ . \tag{6.5.9}$$

Thus, all the ingredients available for constructing invariant Lagrangians involve at least two derivatives on each q. There is also the covariant derivative, but this turns out to be just $\frac{d}{dt}$, so taking higher covariant derivatives will only add more time derivatives. Lagrangians constructed in this way are all strictly invariant under the shift symmetries δ_B and δ_C.

This presents a puzzle, since we know that the free particle kinetic term, $\mathcal{L} = \frac{1}{2}\dot{q}^2$, is also galilean invariant. Although it is not invariant under δ_B, it is invariant up to a total derivative, so it represents a perfectly good Lagrangian which is missed by the coset construction since it contains fewer than two derivatives per q. Another missed example is the tadpole term $\mathcal{L} = q$, which changes up to a total derivative under both δ_B and δ_C. How do we construct these missing terms?

The answer is that these terms will appear as particular shift and boost invariant 2-forms which are themselves constructible from the Maurer–Cartan form (6.5.7). These terms will live on the coset space, that is, the space in which q and ξ are considered as new coordinates in addition to the t direction of space-time. These 2-forms will also be total derivatives in this higher dimensional space, writable as d of a 1-form. The Lagrangian will be obtained by integrating this 1-form on the 1 dimensional subspace where $q = q(t)$ and $\xi = \xi(t)$.

The symmetries on this space in our case are generated by the vector fields [35][22]

$$C = \partial_q \ , \qquad B = \partial_\xi + t\partial_q \ , \qquad P = \partial_t \ . \tag{6.5.10}$$

The components of the Maurer–Cartan form (6.5.7), where we treat q and ξ as independent coordinates, are the (left) invariant 1-forms on the coset space parametrized by $\{q, \xi, t\}$; that is we have $\pounds_X \omega = 0$ where X is any of the vector fields (6.5.10) and ω is any of the forms (6.5.7).

Consider the invariant 2-forms, which are all obtained by wedging together all combinations of the invariant one-forms (6.5.7). There are three of these, with the first being

$$\omega_1^{\mathrm{wz}} = \omega_B \wedge \omega_C = \mathrm{d}\xi \wedge (\mathrm{d}q - \xi \mathrm{d}t) \ . \tag{6.5.11}$$

We note that this can be written as the exterior derivative of a 1-form,

$$\omega_1^{\mathrm{wz}} = \mathrm{d}\beta_1^{\mathrm{wz}} \ , \qquad \beta_1^{\mathrm{wz}} = \xi \mathrm{d}q - \frac{1}{2}\xi^2 \mathrm{d}t \ . \tag{6.5.12}$$

This 1-form can be used to construct an invariant action by pulling back to the surface space-time manifold M, defined by $q = q(t)$, $\xi = \xi(t)$, and then integrating,

$$S_1^{\mathrm{wz}} = \int_M \beta_1^{\mathrm{wz}} = \int \mathrm{d}t \ \xi\dot{q} - \frac{1}{2}\xi^2 \ . \tag{6.5.13}$$

[22] Note that the Lie bracket of left-invariant vector fields is minus the commutator of the algebra.

Imposing the inverse Higgs constraint $\xi = \dot{q}$ (or, equivalently, integrating out ξ), we recover the well-known kinetic term for the non-relativistic free point particle which was missed in the coset construction,

$$S_1^{\mathrm{wz}} = \int_M \beta^{\mathrm{wz}} = \int \mathrm{d}t \ \frac{1}{2}\dot{q}^2 \ . \tag{6.5.14}$$

The tadpole term may be constructed similarly from the two form

$$\omega_2^{\mathrm{wz}} = \omega_C \wedge \omega_P = \mathrm{d}q \wedge \mathrm{d}t = \mathrm{d}\beta_2^{\mathrm{wz}} \ , \qquad \beta_2^{\mathrm{wz}} = q\mathrm{d}t \ . \tag{6.5.15}$$

$$S_2^{\mathrm{wz}} = \int_M \beta^{\mathrm{wz}} = \int \mathrm{d}t \ q \ . \tag{6.5.16}$$

The final possible invariant 2-form constructible from the invariant one forms (6.5.7) is $\omega_3^{\mathrm{wz}} = \omega_B \wedge \omega_P = \mathrm{d}\xi \wedge \mathrm{d}t = \mathrm{d}(\xi \mathrm{d}t)$. This leads to an action which is a total derivative once the Higgs constraint is imposed, and so nothing new results. (This illustrates that the dimension of the relevant cohomology groups may not in general count the number of galileons exactly, but will only put an upper bound on the possible number.)

In all cases, the 2-form ω^{wz} is closed since it can be written as d of a one form β^{wz} (so that we may use it to construct an action). Furthermore, the 2-form ω^{wz} is by construction (left) invariant under the vector fields that generate the symmetries we are interested in (6.5.1). However, the 1-form β^{wz} is not invariant—it shifts by a total d (as it must since ω^{wz} is invariant, $\omega^{\mathrm{wz}} = \mathrm{d}\beta^{\mathrm{wz}}$, and de Rham cohomology is trivial on all the spaces we're considering), but this still leaves the action invariant.

The interesting 2-forms are therefore those which are invariant under the action of the vector fields (6.5.10) but which *cannot* be written as the exterior derivative of a 1-form which is itself invariant [35] (since otherwise the corresponding 1-form on the boundary would be strictly invariant and would have already been captured by the coset construction). They can thus be identified with non-trivial elements of the Lie algebra cohomology

$$H^2\left(\mathfrak{Gal}(0+1,1)\right) . \tag{6.5.17}$$

Lagrangians constructed in this manner are what we call Wess–Zumino terms. For a d-dimensional space-time, they are terms that correspond to non-trivial $d+1$ co-cycles in the cohomology of d acting on invariant vector fields on the coset space (we will review this more carefully in the next section) [28].

6.6 Non-relativistic point particle moving in higher dimensions

Now that we have understood a familiar system as the simplest example of a galileon theory, we are ready to apply the same techniques to the next most complicated case. We consider the point particle in higher co-dimensions, where in addition to space-time transformations, the fields can also rotate into each other in field space. This describes a non-relativistic particle moving in the plane $\mathbb{R}^N$.

The fields q^I now have an extra index, $I = 1, \cdots, N$, the shift symmetries and time translation symmetry act as

$$\delta_{C_J} q^I = \delta^I_J \ , \qquad \delta_{B_J} q^I = -t\delta^I_J \ , \qquad \delta_P q^I = -\dot{q}^I \ , \tag{6.6.1}$$

and there is now an internal $\mathfrak{so}(N)$ symmetry,

$$\delta_{J_{IJ}} q^K = (\delta^K_I \delta_{JL} - \delta^K_J \delta_{IL}) q^L \ . \tag{6.6.2}$$

The non-trivial commutation relations are

$$\begin{aligned}
&[B_I, P] = C_I \ , \\
&[J_{JK}, C_I] = \delta_{JI} C_K - \delta_{KI} C_J \ , \\
&[J_{KL}, B_I] = \delta_{IK} B_L - \delta_{IL} B_K \ , \\
&[J_{IJ}, J_{KL}] = \delta_{IK} J_{JL} - \delta_{JK} J_{IL} + \delta_{JL} J_{IK} - \delta_{IL} J_{JK} \ ,
\end{aligned} \tag{6.6.3}$$

and the symmetry breaking pattern is

$$\mathfrak{Gal}(0+1,N) \longrightarrow \mathfrak{iso}(1) \oplus \mathfrak{so}(N) \ . \tag{6.6.4}$$

The coset we are interested in is then

$$\mathrm{Gal}(0+1,N)/\mathrm{SO}(N) \ , \tag{6.6.5}$$

which is parametrized by (6.2.10),

$$\tilde{V} = e^{tP} e^{q^I C_I + \xi^I B_I} \ . \tag{6.6.6}$$

(Recall that the unbroken generators, in this case the internal rotation generators, are not included in the coset, but the unbroken translations are). The Maurer–Cartan form (6.2.12) is nearly the same as in the point particle case (6.5.7), except that some of the components now carry an extra internal index

$$\omega_P = \mathrm{d}t \ , \qquad \omega_C^I = \mathrm{d}q^I - \xi^I \mathrm{d}t \ , \qquad \omega_B^I = \mathrm{d}\xi^I \ . \tag{6.6.7}$$

Similarly, the inverse Higgs constraint is now given by

$$\xi^I = \dot{q}^I \ . \tag{6.6.8}$$

As before, the only invariant form left for constructing actions is $\ddot{q}^I dt$, so all coset constructible actions contain at least two derivatives per field.

To construct the Wess–Zumino terms, we again create 2-forms by wedging together the 1-forms (6.6.7), but now we must be sure that the forms are $\mathfrak{so}(N)$-invariant so that they are well defined on the coset. This means that the $\mathfrak{so}(N)$ indices in (6.6.7) must be contracted using $\mathfrak{so}(N)$ invariant tensors, and the only such tensors are δ_{IJ} and $\epsilon_{I_1 \cdots I_N}$. These forms will therefore be identified with the relative Lie algebra cohomology

$$H^2\left(\mathfrak{Gal}(0+1,N), \mathfrak{so}(N)\right) \ . \tag{6.6.9}$$

We construct the kinetic terms of the fields by considering

$$\omega_1^{\mathrm{wz}} = \delta_{IJ}\omega_B^I \wedge \omega_C^J = \delta_{IJ}\mathrm{d}\xi^I \wedge \left(\mathrm{d}q^J - \xi^J \mathrm{d}t\right) \ , \tag{6.6.10}$$

which can be written as the exterior derivative of a 1-form,

$$\omega_1^{\mathrm{wz}} = \mathrm{d}\beta_1^{\mathrm{wz}} \ , \qquad \beta_1^{\mathrm{wz}} = \delta_{IJ}\left(\xi^I \mathrm{d}q^J - \frac{1}{2}\xi^I\xi^J\mathrm{d}t\right) \ . \tag{6.6.11}$$

Pulling back to the surface space-time manifold M, defined by $q^I = q^I(t)$, $\xi^I = \xi^I(t)$, and then integrating, we have

$$S_1^{\mathrm{wz}} = \int_M \beta_1^{\mathrm{wz}} = \int \mathrm{d}t \ \delta_{IJ}\left(\xi^I \dot{q}^J - \frac{1}{2}\xi^I\xi^J\right) \ , \tag{6.6.12}$$

and then imposing the inverse Higgs constraint $\xi^I = \dot{q}^I$ (or equivalently, integrating out ξ^I), we recover

$$S_1^{\mathrm{wz}} = \int_M \beta_1^{\mathrm{wz}} = \int \mathrm{d}t \ \frac{1}{2}\delta_{IJ}\dot{q}^I\dot{q}^J \ . \tag{6.6.13}$$

For $N \geq 2$ there is no longer a tadpole term, since the Lagrangian must be invariant under an $\mathrm{SO}(N)$ rotation of the fields q^I. There are also no more non-trivial Wess–Zumino terms beyond the kinetic term (once the inverse Higgs constraints are imposed), with one exception: for $N = 2$ a novel Lagrangian appears involving the ϵ_{IJ} tensor,

$$\omega^{\mathrm{wz}} = \epsilon_{IJ}\omega_B^I \wedge \omega_B^J = \epsilon_{IJ}\mathrm{d}\xi^I \wedge \mathrm{d}\xi^J = \mathrm{d}\beta_2^{\mathrm{wz}}, \qquad \beta_2^{\mathrm{wz}} = \epsilon_{IJ}\xi^I\mathrm{d}\xi^J \ , \tag{6.6.14}$$

$$S_2 = \int_M \beta_2^{\mathrm{wz}} = \int \mathrm{d}t \ \epsilon_{IJ}\xi^I\dot{\xi}^J \ , \tag{6.6.15}$$

which upon imposing the inverse Higgs constraint becomes

$$S_2 = \int_M \beta_2^{\mathrm{wz}} = \int \mathrm{d}t \ \epsilon_{IJ}\dot{q}^I\ddot{q}^J \ . \tag{6.6.16}$$

Note that this is an example in which the imposition of the inverse Higgs constraint is not equivalent to integrating out redundant fields from the action.

Thus, in the bi-galileon case there is the extra Lagrangian $\mathcal{L} = \epsilon_{IJ}\dot{q}^J\dddot{q}^J$ which has third-order equations of motion, unlike the other galileons, so the relation between second order equations of motion and Wess–Zumino terms is not a perfectly tight one, though it holds in all other cases. Even so, this term still describes fewer degrees of freedom (there are two fields each with third order equations, indicating six phase space degrees of freedom, or three real degrees of freedom) than the non-galileon terms, though it describes more than the kinetic term.

6.7 Galileons

We now perform the coset construction for galileons in four dimensions. This is the situation of greatest physical interest. We will consider the case which is inspired by a co-dimension 1 braneworld model, in which the galileons are related to the brane-bending mode into the bulk.

The galileons non-linearly realize the shift symmetries

$$\delta_C\pi = 1 \ , \qquad \delta_{B^\mu}\pi = x^\mu \ , \tag{6.7.1}$$

and we have the non-trivial commutators

$$[P_\mu, B_\nu] = \eta_{\mu\nu}C \ , \qquad [J_{\rho\sigma}, B_\nu] = \eta_{\rho\nu}B_\sigma - \eta_{\sigma\nu}B_\rho \ , \tag{6.7.2}$$

which, along with the commutators of Poincaré transformations, fill out the galileon algebra $\mathfrak{Gal}(3+1,1)$. The $4d$ galileons non-linearly realize the symmetry breaking pattern

$$\mathfrak{Gal}(3+1,1) \longrightarrow \mathfrak{iso}(3,1) \ , \tag{6.7.3}$$

and the coset is parametrized by (6.2.10)

$$\tilde{V} = e^{x\cdot P}e^{\pi C+\xi\cdot B} \ . \tag{6.7.4}$$

Note that the linearly realized generators consists of only the Lorentz transformations, so we are working with the coset

$$\mathrm{Gal}(3+1,1)/\mathrm{SO}(3,1) \ . \tag{6.7.5}$$

The coefficients of the components of the Maurer–Cartan form (6.2.12) are

$$\omega_P^\mu = \mathrm{d}x^\mu \ , \qquad \omega_C = \mathrm{d}\pi + \xi_\mu \mathrm{d}x^\mu \ , \qquad \omega_B^\mu = \mathrm{d}\xi^\mu \ , \qquad \omega_J^{\mu\nu} = 0 \ . \tag{6.7.6}$$

As is the norm when breaking space-time symmetries, there are fewer Goldstone modes than naive counting would lead us to believe. We have broken generators, V_μ and C, but we only have a single Goldstone mode π, and this can be seen from the presence of an inverse Higgs constraint—the commutator $[P_\mu, B_\nu] = \eta_{\mu\nu} C$ tells us that we may eliminate the ξ_μ field in favor of π by setting $\omega_C = 0$, which leads to the relation

$$\xi_\mu = -\partial_\mu \pi \ . \tag{6.7.7}$$

This allows us to write the components of the Maurer–Cartan form as

$$\omega_P^\mu = \mathrm{d}x^\mu \ , \qquad \omega_B^\mu = -\mathrm{d}x^\nu \partial_\nu \partial^\mu \pi \ . \tag{6.7.8}$$

Since we can only build Lagrangians by using these ingredients (along with the higher covariant derivatives on π, which in this case are the same as ordinary derivatives), the field π will only ever appear with at least 2 derivatives per field. Thus we can never obtain the galileons from this construction, since the galileon terms (1.37) all have fewer than two derivatives per field (the galileon Lagrangians with n π's have $2n-2$ derivatives).

The fact that they cannot be built by the coset construction is suggestive of the fact that the 4D galileons are Wess–Zumino terms in the same sense as the free particle kinetic term—they are 4-form potentials for non-trivial 5-co-cycles in Lie algebra cohomology. The construction proceeds similarly to the $1d$ case.

We work on the coset space, the space in which π and ξ^μ are considered as new coordinates in addition to the x^μ directions of space-time. The Lagrangian will be obtained by integrating a Wess–Zumino form on the subspace where $\pi = \pi(x)$ and $\xi^\mu = \xi^\mu(x)$. The symmetries on the coset space are generated by the vector fields

$$C = \partial_\pi \ , \qquad B_\mu = \partial_{\xi^\mu} - x_\mu \partial_\pi \ , \qquad P_\mu = \partial_\mu \ . \tag{6.7.9}$$

The components of the Maurer–Cartan form (6.7.6), where we treat π and ξ^μ as independent coordinates, are the (left) invariant 1-forms on the coset space parametrized by $\{\pi, \xi^\mu, x^\mu\}$, so that we have $\pounds_X \omega = 0$ where X is any of the vector fields (6.7.9) and ω is any of the forms (6.7.6).

To construct the Wess–Zumino terms, we create invariant 5-forms by wedging together the 1-forms (6.7.6). However, we must ensure that the forms are invariant under the Lorentz transformations $\mathfrak{so}(3,1)$ so that they are well defined on the coset. This means that the Lorentz indices in (6.7.6) must be contracted using Lorentz invariant tensors, and the only such tensors are $\eta_{\mu\nu}$ and $\epsilon_{\mu\nu\rho\sigma}$. From the cohomology perspective, this means that the galileon terms are members of the relative Lie algebra cohomology group

$$H^5\left(\mathfrak{Gal}(3+1,1), \mathfrak{so}(3,1)\right) \ . \tag{6.7.10}$$

Start by considering the invariant 5-form

$$\omega_1^{\mathrm{wz}} = \epsilon_{\mu\nu\rho\sigma}\ \omega_C \wedge \omega_P^\mu \wedge \omega_P^\nu \wedge \omega_P^\rho \wedge \omega_P^\sigma = \epsilon_{\mu\nu\rho\sigma} \mathrm{d}\pi \wedge \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \mathrm{d}x^\rho \wedge \mathrm{d}x^\sigma \ , \tag{6.7.11}$$

which can be written as the exterior derivative of a 4-form,

$$\omega_1^{\mathrm{wz}} = \mathrm{d}\beta_1^{\mathrm{wz}}, \qquad \beta_1^{\mathrm{wz}} = \epsilon_{\mu\nu\rho\sigma} \pi \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \mathrm{d}x^\rho \wedge \mathrm{d}x^\sigma \ . \tag{6.7.12}$$

Pulling back to the space-time manifold M, defined by $\pi = \pi(x)$, $\xi = \xi(x)$, and then integrating,

$$S_1^{\mathrm{wz}} = \int_M \beta_1^{\mathrm{wz}} = \int_M \pi \epsilon_{\mu\nu\rho\sigma} \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \mathrm{d}x^\rho \wedge \mathrm{d}x^\sigma \sim \int \mathrm{d}^4x\ \pi \ , \tag{6.7.13}$$

we recover the tadpole term, which is the first galileon. Just as in the free particle case, the tadpole term appears as a 4-form which shifts by a total derivative under the symmetries and whose exterior derivative is a strictly invariant 5-form.

Next consider

$$\omega_2^{\rm wz} = \epsilon_{\mu\nu\rho\sigma}\ \omega_C \wedge \omega_B^\mu \wedge \omega_P^\nu \wedge \omega_P^\rho \wedge \omega_P^\sigma = \epsilon_{\mu\nu\rho\sigma} \left(\mathrm{d}\pi + \xi_\lambda \mathrm{d}x^\lambda\right) \wedge \mathrm{d}\xi^\mu \wedge \mathrm{d}x^\nu \wedge \mathrm{d}x^\rho \wedge \mathrm{d}x^\sigma\ , \tag{6.7.14}$$

which can be written as the exterior derivative of a 4-form,[23]

$$\omega_2^{\rm wz} = \mathrm{d}\beta_2^{\rm wz}, \qquad \beta_2^{\rm wz} = \epsilon_{\mu\nu\rho\sigma} \left(\pi \mathrm{d}\xi^\mu - \frac{1}{8}\xi^2 \mathrm{d}x^\mu\right) \wedge \mathrm{d}x^\nu \wedge \mathrm{d}x^\rho \wedge \mathrm{d}x^\sigma\ . \tag{6.7.16}$$

Pulling back to the space-time manifold M and integrating, we obtain

$$S_2^{\rm wz} = \int_M \beta_2^{\rm wz} = 3! \int \mathrm{d}^4x\ \left(\pi \partial_\mu \xi^\mu - \frac{1}{2}\xi^2\right)\ . \tag{6.7.17}$$

Imposing the Higgs constraint $\xi_\mu = -\partial_\mu \pi$ (or equivalently, integrating out ξ^μ), we recover the kinetic term, which is the second galileon,

$$S_2^{\rm wz} \sim \int \mathrm{d}^4x\ (\partial\pi)^2\ . \tag{6.7.18}$$

The construction of $\mathcal{L}_3$ is similar. We consider

$$\omega_3^{\rm wz} = \epsilon_{\mu\nu\rho\sigma}\ \omega_C \wedge \omega_B^\mu \wedge \omega_B^\nu \wedge \omega_P^\rho \wedge \omega_P^\sigma = \epsilon_{\mu\nu\rho\sigma} \left(\mathrm{d}\pi + \xi_\lambda \mathrm{d}x^\lambda\right) \wedge \mathrm{d}\xi^\mu \wedge \mathrm{d}\xi^\nu \wedge \mathrm{d}x^\rho \wedge \mathrm{d}x^\sigma\ , \tag{6.7.19}$$

[23]In showing this, it is helpful to use the identity

$$\begin{aligned} &\epsilon_{\mu\nu\rho\sigma}\xi_\lambda \mathrm{d}\xi^\mu \wedge \mathrm{d}x^\lambda \wedge \mathrm{d}x^\nu \wedge \mathrm{d}x^\rho \wedge \mathrm{d}x^\sigma \\ &= \frac{1}{4}\epsilon_{\mu\nu\rho\sigma}\xi_\lambda \mathrm{d}\xi^\lambda \wedge \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \mathrm{d}x^\rho \wedge \mathrm{d}x^\sigma = 3!\xi_\mu d\xi^\mu \wedge \mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3\ . \end{aligned} \tag{6.7.15}$$

which can be written as the exterior derivative of a 4-form[24]

$$\omega_3^{\rm wz} = {\rm d}\beta_3^{\rm wz}, \qquad \beta_3^{\rm wz} = \epsilon_{\mu\nu\rho\sigma}\left(\pi {\rm d}\xi^\mu \wedge {\rm d}\xi^\nu \wedge {\rm d}x^\rho \wedge {\rm d}x^\sigma - \frac{1}{3}\xi^2 {\rm d}\xi^\mu \wedge {\rm d}x^\nu \wedge {\rm d}x^\rho \wedge {\rm d}x^\sigma\right) . \tag{6.7.21}$$

Pulling back to the space-time manifold M and integrating yields

$$S_3^{\rm wz} = \int_M \beta_3^{\rm wz} = \int_M {\rm d}^4x \Big[-2\pi \left[(\partial_\mu\xi^\mu)^2 - \partial_\mu\xi^\nu\partial_\nu\xi^\mu\right] + 2\xi_\alpha\xi^\alpha\partial_\mu\xi^\mu\Big] . \tag{6.7.22}$$

Imposing the Higgs constraint $\xi_\mu = -\partial_\mu\pi$, and performing a 4D integration by parts, we recover the cubic galileon,

$$S_3^{\rm wz} \sim \int_M {\rm d}^4x \, \Box\pi(\partial\pi)^2 . \tag{6.7.23}$$

The pattern in now clear. The expressions for $\mathcal{L}_4$ and $\mathcal{L}_5$ will be given by the forms

$$\begin{aligned} \omega_4^{\rm wz} &= \epsilon_{\mu\nu\rho\sigma}\omega_C \wedge \omega_B^\mu \wedge \omega_B^\nu \wedge \omega_B^\rho \wedge \omega_P^\sigma , \\ \omega_5^{\rm wz} &= \epsilon_{\mu\nu\rho\sigma}\omega_C \wedge \omega_B^\mu \wedge \omega_B^\nu \wedge \omega_B^\rho \wedge \omega_B^\sigma , \end{aligned} \tag{6.7.24}$$

respectively. From the cohomology perspective, the galileon terms are members of the relative Lie algebra cohomology group $H^5\left(\mathfrak{Gal}(3+1,1), \mathfrak{so}(3,1)\right)$.

6.7.1 d dimensional galileons

This procedure is easily generalized to d space-time dimensions, in which case the breaking pattern is

$$\mathfrak{Gal}((d-1)+1,1) \to \mathfrak{iso}(d-1,1), \tag{6.7.25}$$

[24]In showing this, it is helpful to use the identity

$$-\frac{2}{3}\epsilon_{\mu\nu\rho\sigma}\xi_\lambda {\rm d}\xi^\lambda \wedge {\rm d}\xi^\mu \wedge {\rm d}x^\nu \wedge {\rm d}x^\rho \wedge {\rm d}x^\sigma = \epsilon_{\mu\nu\rho\sigma}\xi_\lambda {\rm d}\xi^\mu \wedge {\rm d}\xi^\nu \wedge {\rm d}x^\lambda \wedge {\rm d}x^\rho \wedge {\rm d}x^\sigma = -4\xi_\lambda {\rm d}\xi^\lambda \wedge {\rm d}\xi_\mu \wedge (*_4 {\rm d}x^\mu), \tag{6.7.20}$$

where $*_4$ is the Hodge star on the space of x^μ's.

and the coset is

$$\mathrm{Gal}((d-1)+1,1)/\mathrm{SO}(d-1,1). \tag{6.7.26}$$

The n-th single field galileon term descends from the $(d+1)$-form

$$\begin{aligned}\omega_n^{\mathrm{wz}} &= \epsilon_{\mu_1\cdots\mu_d}\omega_C \wedge \omega_B^{\mu_1} \wedge \cdots \wedge \omega_B^{\mu_{n-1}} \wedge \omega_P^{\mu_n} \wedge \cdots \wedge \omega_P^{\mu_d}\ , \\ &= \epsilon_{\mu_1\cdots\mu_d}\left(\mathrm{d}\pi + \xi_\lambda \mathrm{d}x^\lambda\right) \wedge \mathrm{d}\xi^{\mu_1} \wedge \cdots \wedge \mathrm{d}\xi^{\mu_{n-1}} \wedge \mathrm{d}x^{\mu_n} \wedge \cdots \wedge \mathrm{d}x^{\mu_d},\end{aligned} \tag{6.7.27}$$

where the basis 1-forms are the d-dimensional versions of (6.7.6). This is the total derivative of the non-invariant Wess–Zumino d-form[25] (in the following expressions $n \geq 2$, the tadpole is easily treated as before),

$$\begin{aligned}\omega_n^{\mathrm{wz}} &= \mathrm{d}\beta_n^{\mathrm{wz}}, \\ \beta_n^{\mathrm{wz}} &= \epsilon_{\mu_1\cdots\mu_d}\bigg(\pi \mathrm{d}\xi^{\mu_1} \wedge \cdots \wedge \mathrm{d}\xi^{\mu_{n-1}} \wedge \mathrm{d}x^{\mu_n} \wedge \cdots \wedge \mathrm{d}x^{\mu_d} \\ &\quad - \frac{(n-1)}{2(d-n+2)}\xi^2 \mathrm{d}\xi^{\mu_1} \wedge \cdots \wedge \mathrm{d}\xi^{\mu_{n-2}} \wedge \mathrm{d}x^{\mu_{n-1}} \wedge \cdots \wedge \mathrm{d}x^{\mu_d}\bigg).\end{aligned} \tag{6.7.29}$$

Pulling back to the space-time manifold M and integrating yields

$$\begin{aligned}S_n^{\mathrm{wz}} = \int_M \beta_n^{\mathrm{wz}} = \int_M \mathrm{d}^d x\, &(d-n+1)!(n-1)!\pi \delta^{[\nu_1}_{\mu_1} \cdots \delta^{\nu_{n-1}]}_{\mu_{n-1}} \partial_{\nu_1}\xi^{\mu_1} \cdots \partial_{\nu_{n-1}}\xi^{\mu_{n-1}} \\ &- \frac{n-1}{2}(d-n+1)!(n-2)!\xi^2 \delta^{[\nu_1}_{\mu_1} \cdots \delta^{\nu_{n-2}]}_{\mu_{n-2}} \partial_{\nu_1}\xi^{\mu_1} \cdots \partial_{\nu_{n-2}}\xi^{\mu_{n-2}}.\end{aligned} \tag{6.7.30}$$

[25] We use the identity,

$$\begin{aligned}&\frac{1}{(d-n+2)!}\xi_\lambda d\xi^\lambda \wedge \mathrm{d}\xi^{\mu_1} \wedge \cdots \wedge \mathrm{d}\xi^{\mu_{n-2}} \wedge \mathrm{d}x^{\mu_{n-1}} \wedge \cdots \wedge \mathrm{d}x^{\mu_d}\epsilon_{\mu_1\cdots\mu_d} \\ &= -\frac{1}{(n-1)(d-n+1)!}\xi_\lambda dx^\lambda \wedge \mathrm{d}\xi^{\mu_1} \wedge \cdots \wedge \mathrm{d}\xi^{\mu_{n-1}} \wedge \mathrm{d}x^{\mu_n} \wedge \cdots \wedge \mathrm{d}x^{\mu_d}\epsilon_{\mu_1\cdots\mu_d} \\ &= \xi_\lambda d\xi^\lambda \wedge \mathrm{d}\xi_{\mu_1} \wedge \cdots \wedge \mathrm{d}\xi_{\mu_{n-2}} \wedge *_d\left(\mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_{n-2}}\right),\end{aligned} \tag{6.7.28}$$

where $*_d$ is the Hodge star on the space of x^μ's.

Imposing the Higgs constraint $\xi_\mu = -\partial_\mu \pi$, and integrating the last term by parts, we recover the general galileon (1.37),

$$S_n^{\rm wz} \sim \int_M \mathrm{d}^d x \;\; \pi \delta^{[\nu_1}_{\mu_1} \cdots \delta^{\nu_{n-1}]}_{\mu_{n-1}} \partial_{\nu_1} \partial^{\mu_1} \pi \cdots \partial_{\nu_{n-1}} \partial^{\mu_{n-1}} \pi. \tag{6.7.31}$$

The d dimensional galileon terms are members of the relative Lie algebra cohomology group

$$H^{d+1} \left(\mathfrak{Gal}((d-1)+1,1), \mathfrak{so}(d-1,1) \right) \ . \tag{6.7.32}$$

6.8 Multi-galileons

It is straightforward to extend the analysis to the multi-galileon case. The action and commutation relations are those of the algebra $\mathfrak{Gal}(3+1,N)$ described in Section 6.4, and the galileons realize the symmetry breaking pattern

$$\mathfrak{Gal}(3+1,N) \longrightarrow \mathfrak{iso}(3,1) \oplus \mathfrak{so}(N) \ . \tag{6.8.1}$$

The coset is parametrized by (6.2.10)

$$\tilde{V} = e^{x \cdot P} e^{\pi^I \cdot C_I + \xi^I \cdot B_I} \ , \tag{6.8.2}$$

and the linearly realized subgroup consists of the Lorentz transformations and the $\mathfrak{so}(N)$ rotations, so we are working with the coset

$$\mathrm{Gal}(3+1,N) / \left(\mathrm{SO}(3,1) \times \mathrm{SO}(N) \right) \ . \tag{6.8.3}$$

The coefficients of the components of the Maurer–Cartan form (6.2.12) are

$$\omega_P^\mu = \mathrm{d}x^\mu \, , \quad \omega_C^I = \mathrm{d}\pi^I + \xi^I{}_\mu \mathrm{d}x^\mu \, , \quad \omega_B^{I\mu} = \mathrm{d}\xi^{I\mu} \, , \quad \omega_J^{\mu\nu} = \omega_J^{IJ} = 0 \, , \tag{6.8.4}$$

and the inverse Higgs constraint is

$$\xi^I{}_\mu = -\partial_\mu \pi^I \ , \tag{6.8.5}$$

so that we again find that we cannot construct any terms with fewer than two derivatives per π^I.

To construct the Wess–Zumino terms, we create invariant 5-forms by wedging together the 1-forms (6.8.4), making sure that the forms are invariant under both the Lorentz transformations $\mathfrak{so}(3,1)$ and the internal $\mathfrak{so}(N)$ transformations so that they are well defined on the coset. The two possible 5-forms that lead to non-trivial Lagrangians for $N \geq 2$ are

$$\begin{aligned} \omega_2^{\text{wz}} &= \delta_{IJ}\epsilon_{\mu\nu\rho\sigma}\ \omega_C^I \wedge \omega_B^{J\mu} \wedge \omega_P^\nu \wedge \omega_P^\rho \wedge \omega_P^\sigma\ , \\ \omega_4^{\text{wz}} &= \delta_{IJ}\delta_{KL}\epsilon_{\mu\nu\rho\sigma}\ \omega_C^I \wedge \omega_B^{J\mu} \wedge \omega_B^{K\nu} \wedge \omega_B^{L\rho} \wedge \omega_P^\sigma\ , \end{aligned} \tag{6.8.6}$$

leading to the kinetic term, and the quartic term studied in [73].

From the cohomology perspective, the multi-galileon terms are members of the relative Lie algebra cohomology group

$$H^5\left(\mathfrak{Gal}(3+1,N), \mathfrak{so}(3,1) \oplus \mathfrak{so}(N)\right)\ . \tag{6.8.7}$$

Generalizing to d-dimensions, there are $d/2$ possible terms for d even, and $(d+1)/2$ possible terms for d odd. The Wess–Zumino $(d+1)$-forms are

$$\omega_{2n}^{\text{wz}} = \delta_{I_1J_1}\cdots\delta_{I_nJ_n}\epsilon_{\mu_1\cdots\mu_d}\ \omega_C^{I_1} \wedge \omega_B^{J_1\mu_1} \wedge \cdots \wedge \omega_B^{I_n\mu_{2n-2}} \wedge \omega_B^{J_n\mu_{2n-1}} \wedge \omega_P^{\mu_{2n}} \wedge \cdots \wedge \omega_P^{\mu_d}\ , \tag{6.8.8}$$

which lead to the Lagrangian (1.73). They are members of the relative Lie algebra cohomology group

$$H^{d+1}\left(\mathfrak{Gal}((d-1)+1,N), \mathfrak{so}(d-1,1) \oplus \mathfrak{so}(N)\right)\ . \tag{6.8.9}$$

Note that using $\epsilon_{I_1\cdots I_N}$ to contract indices gives nothing new, leading only to Lagrangians which are total derivatives (with the exception of $d=1$, $N=1$ in (6.6.14)).

6.9 Speculations on Quantum Properties of Galileons

The fact that galileons arise due to local algebraic properties is somewhat tantalizing—it is well-known that there is a non-renormalization theorem for galileons; they are not renormalized to any loop order in perturbation theory [73, 82]. It may be possible that this non-renormalization is tied to the algebraic properties of the galileon terms. A possibly instructive example is that of anomalies—whose existence is similarly forecast by algebraic properties á la BRST—which also have a non-renormalization theorem, although of a slightly different type (anomalies are not renormalized past 1-loop). This raises the possibility that the non-renormalization of galileons may be understood in terms of some deeper topological or algebraic context based upon their construction as Wess–Zumino terms, but unlike the Wess–Zumino–Witten term of the chiral Lagrangian (which are not renormalized due to a quantization condition on their coefficients), there does not appear to be an obvious global topological condition requiring the coefficients of the galileon terms to be quantized.

6.10 Counting the galileons

While the construction of the four dimensional single field galileons makes it hard to imagine any other possible galileon invariant Lagrangians (and it has been shown by other methods that there aren't any [91]), it is good to have a formal check that we have indeed found every possible Wess–Zumino term. After all, every Lagrangian that is compatible with the symmetries of the theory should be included when constructing an effective field theory, and so proper bookkeeping and accounting of terms is a worthwhile endeavor.

In order to verify that we have found all possible Wess–Zumino terms, we want to compute the relative Lie algebra cohomology $H^5\left(\mathfrak{Gal}(3+1,1), \mathfrak{so}(3,1)\right)$. Noting that (6.7.6) is a basis for left-invariant forms, we determine the action of the exterior derivative, d, on these forms

$$\mathrm{d}\omega_P^\mu = 0\ , \qquad \mathrm{d}\omega_B^\mu = 0\ , \qquad \mathrm{d}\omega_C = \eta_{\mu\nu}\omega_B^\mu \wedge \omega_P^\nu\ . \tag{6.10.1}$$

To meet the requirement of SO(3, 1) invariance, all Greek indices must be contracted with $\eta_{\mu\nu}$ or $\epsilon_{\mu\nu\rho\sigma}$. Then, the SO(3, 1) invariant 5-co-cycles can be explicitly constructed and are given by

$$\begin{aligned}
\omega_1 &= \epsilon_{\mu\nu\rho\sigma}\omega_C \wedge \omega_P^\mu \wedge \omega_P^\nu \wedge \omega_P^\rho \wedge \omega_P^\sigma \ , \\
\omega_2 &= \epsilon_{\mu\nu\rho\sigma}\omega_C \wedge \omega_P^\mu \wedge \omega_P^\nu \wedge \omega_P^\rho \wedge \omega_B^\sigma \ , \\
\omega_3 &= \epsilon_{\mu\nu\rho\sigma}\omega_C \wedge \omega_P^\mu \wedge \omega_P^\nu \wedge \omega_B^\rho \wedge \omega_B^\sigma \ , \\
\omega_4 &= \epsilon_{\mu\nu\rho\sigma}\omega_C \wedge \omega_P^\mu \wedge \omega_B^\nu \wedge \omega_B^\rho \wedge \omega_B^\sigma \ , \\
\omega_5 &= \epsilon_{\mu\nu\rho\sigma}\omega_C \wedge \omega_B^\mu \wedge \omega_B^\nu \wedge \omega_B^\rho \wedge \omega_B^\sigma \ .
\end{aligned} \tag{6.10.2}$$

It is clear that each of these forms are closed, $\mathrm{d}\omega = 0$. Furthermore, due to the presence of a factor of ω_C in each form, none of these are expressible as the exterior derivative of a 4-form. In order to not vanish there must have been exactly one factor of ω_C in the 4-co-chain, but such a form is not Lorentz invariant; therefore all of the 5-co-cycles in (6.10.2) are non-trivial elements of $H^5(\mathfrak{gal}(1+3,1), \mathfrak{so}(3,1))$.

This provides a formal check that there only exist the five galileon Lagrangians and we have not missed any other Wess–Zumino terms in our construction. Similar remarks apply to all other dimensions and co-dimensions.

6.11 Conformal galileons

The conformal galileon is a higher derivative theory of a single scalar field, with second order equations of motion, and which non-linearly realizes the conformal group. The relevant Lagrangians are those of (3.149) (with the replacement $\hat{\pi} \to -\pi$) and were first constructed in Sec. 3.1 of [91],

$$\begin{aligned}
\mathcal{L}_1 &= -\frac{1}{4}e^{4\pi} \ , \\
\mathcal{L}_2 &= -\frac{1}{2}e^{2\pi}(\partial\pi)^2 \ ,
\end{aligned}$$

$$
\begin{aligned}
\mathcal{L}_3 &= \frac{1}{2}(\partial\pi)^2\Box\pi + \frac{1}{4}(\partial\pi)^4 \ , \\
\mathcal{L}_4 &= -\frac{1}{2}e^{-2\pi}(\partial\pi)^2\left([\Pi]^2 - [\Pi^2] + \frac{2}{5}(-(\partial\pi)^2\Box\pi + [\pi^3]) + \frac{3}{10}(\partial\pi)^4\right) \ , \\
\mathcal{L}_5 &= -\frac{1}{2}e^{-4\pi}(\partial\pi)^2\Big[- [\Pi]^3 + 3[\Pi][\Pi^2] - 2[\Pi^3] + 3(\partial\pi)^2([\Pi]^2 - [\Pi^2]) \\
&\quad + \frac{30}{7}(\partial\pi)^2(-(\partial\pi)^2[\Pi] + [\pi^3]) - \frac{3}{28}(\partial\pi)^6\Big] \ .
\end{aligned} \tag{6.11.1}
$$

where we've used the bracket shorthand for traces as explained in the conventions in Appendix A. Indices are raised and lowered with $\eta_{\mu\nu}$.

The conformal galileons linearly realize Poincaré symmetry,

$$
\begin{aligned}
\delta_{P_\mu}\pi &= -\partial_\mu\pi \ , \\
\delta_{J_{\mu\nu}}\pi &= (x_\mu\partial_\nu - x_\nu\partial_\mu)\pi \ ,
\end{aligned} \tag{6.11.2}
$$

while the conformal symmetry is non-linearly realized

$$
\begin{aligned}
\delta_D\pi &= -1 - x^\mu\partial_\mu\pi \ , \\
\delta_{K_\mu}\pi &= -2x_\mu - (2x_\mu x^\nu\partial_\nu - x^2\partial_\mu)\pi \ .
\end{aligned} \tag{6.11.3}
$$

Taken together, the transformations satisfy the commutators of the conformal algebra $\mathfrak{so}(4,2)$,

$$
\begin{aligned}
&[P_\mu, D] = P_\mu \ , && [D, K_\mu] = K_\mu \ , \\
&[J_{\mu\nu}, K_\sigma] = \eta_{\mu\sigma}K_\nu - \eta_{\nu\sigma}K_\mu \ , && [J_{\mu\nu}, P_\sigma] = \eta_{\mu\sigma}P_\nu - \eta_{\nu\sigma}P_\mu \ , \\
&[K_\mu, P_\nu] = 2J_{\mu\nu} - 2\eta_{\mu\nu}D \ , && [J_{\mu\nu}, J_{\rho\sigma}] = \eta_{\mu\rho}J_{\nu\sigma} - \eta_{\nu\rho}J_{\mu\sigma} + \eta_{\nu\sigma}J_{\mu\rho} - \eta_{\mu\sigma}J_{\nu\rho} \ .
\end{aligned} \tag{6.11.4}
$$

The conformal galileons may be interpreted as the Goldstone field associated with the symmetry breaking pattern

$$
\mathfrak{so}(4,2) \longrightarrow \mathfrak{iso}(3,1) \ . \tag{6.11.5}
$$

As we shall see, it is possible to obtain the conformal galileon terms via the coset construction, with the exception of the term quartic in derivatives, $\mathcal{L}_3$, which appears as a Wess–Zumino term.

The coset space is

$$\mathrm{SO}(4,2)/\mathrm{SO}(3,1), \tag{6.11.6}$$

which we parametrize as[26]

$$\tilde{V} = e^{x\cdot P} e^{\pi D} e^{\xi\cdot K}. \tag{6.11.7}$$

Calculating the Maurer–Cartan form (6.2.12),

$$\omega = \tilde{V}^{-1}\mathrm{d}\tilde{V} = \omega_P^\mu P_\mu + \omega_D D + \omega_K^\mu K_\mu + \frac{1}{2}\omega_J^{\mu\nu} J_{\mu\nu} \ , \tag{6.11.8}$$

the components are found to be [11, 72, 85, 112]

$$\begin{aligned}
\omega_P^\mu &= e^\pi \mathrm{d}x^\mu, \\
\omega_D &= \mathrm{d}\pi + 2e^\pi \xi_\mu \mathrm{d}x^\mu, \\
\omega_K^\mu &= \mathrm{d}\xi^\mu + \xi^\mu \mathrm{d}\pi + e^\pi \left(2\xi^\mu \xi_\nu \mathrm{d}x^\nu - \xi^2 \mathrm{d}x^\mu\right), \\
\omega_J^{\mu\nu} &= -4e^\pi \left(\xi^\mu \mathrm{d}x^\nu - \xi^\nu \mathrm{d}x^\mu\right) \ .
\end{aligned} \tag{6.11.9}$$

where indices have been raised and lowered with $\eta_{\mu\nu}$.

Due to the commutator $[K_\mu, P_\nu] = 2J_{\mu\nu} - 2\eta_{\mu\nu}D$, there is an inverse Higgs constraint, $\omega_D = 0$ yielding the relation

$$\xi_\mu = -\frac{1}{2}e^{-\pi}\partial_\mu \pi \ . \tag{6.11.10}$$

Plugging back into the Maurer–Cartan form, we have

[26] This differs slightly from our general expression (6.2.10) since we write a product of exponentials for the broken generators rather than the exponential of a sum. This just amounts to a different choice of parametrization for the coset.

$$\begin{aligned}&\omega_P^\mu = e^{\pi}\mathrm{d}x^\mu,\\ &\omega_K^\mu = e^{-\pi}\left(\tfrac{1}{2}\partial_\nu\pi\partial^\mu\pi\mathrm{d}x^\nu - \tfrac{1}{2}\partial_\nu\partial^\mu\pi\mathrm{d}x^\nu - \tfrac{1}{4}(\partial\pi)^2 dx^\nu\right),\\ &\omega_J^{\mu\nu} = 2\left(\partial^\mu\pi\mathrm{d}x^\nu - \partial^\nu\pi\mathrm{d}x^\mu\right)\ .\end{aligned} \tag{6.11.11}$$

The vielbein (6.2.14) can be extracted from ω_P,

$$e_\nu{}^\alpha = e^\pi\delta_\nu^\alpha, \tag{6.11.12}$$

giving the invariant metric

$$g_{\mu\nu} = e_\mu{}^\alpha e_\nu{}^\beta\eta_{\alpha\beta} = e^{2\pi}\eta_{\mu\nu}. \tag{6.11.13}$$

The invariant measure (6.2.16) is

$$\sqrt{-g} = e^{4\pi}. \tag{6.11.14}$$

The derivative (6.2.18) associated to ξ^β (here another Lorentz index β plays the role of the index a in Section 6.2.2) is given by the expression $(\omega_K)_\mu{}^\beta = e_\mu{}^\alpha\mathcal{D}_\alpha\xi^\beta$. By contracting with the vielbein, we can instead work with the object $\mathcal{D}_\mu\xi_\nu \equiv e_\mu{}^\alpha\mathcal{D}_\alpha\xi^\beta e_\nu{}^\gamma\eta_{\beta\gamma} = (\omega_K)_\mu{}^\beta e_\nu{}^\alpha\eta_{\beta\alpha}$,

$$\mathcal{D}_\mu\xi_\nu = \frac{1}{2}\partial_\nu\pi\partial_\mu\pi - \frac{1}{2}\partial_\nu\partial_\mu\pi - \frac{1}{4}(\partial\pi)^2\eta_{\mu\nu}\ . \tag{6.11.15}$$

We construct invariant Lagrangians by using $D_\mu\xi_\nu$, contracting up indices with the metric (6.11.13) and multiplying by the measure (6.2.16).

Another method is used in [91][27]. The conformal galileons are constructed by forming diffeomorphism scalars of the conformal metric $g_{\mu\nu} = e^{2\pi}\eta_{\mu\nu}$. This method is in fact completely equivalent to the coset construction, because we have for the Ricci tensor

$$R_{\mu\nu}(g) = 2\partial_\mu\pi\partial_\nu\pi - 2\partial_\mu\partial_\nu\pi - \Box\pi\eta_{\mu\nu} - 2(\partial\pi)^2\eta_{\mu\nu}\ , \tag{6.11.16}$$

[27] There is also a method called tractor calculus, which is designed for constructing realizations of conformal symmetry [10, 12, 53, 68, 69, 104, 105].

which can be expressed in terms of the covariant derivative (6.11.15),

$$R_{\mu\nu}(g) = 4\mathcal{D}_\mu \xi_\nu + 2\mathcal{D}_\rho \xi^\rho g_{\mu\nu} \ . \tag{6.11.17}$$

The Ricci scalar for the conformal metric is $R[g] = 12\mathcal{D}_\rho \xi^\rho$, and the Riemann tensor gives nothing beyond the Ricci tensor because the Weyl tensor vanishes for the conformally flat metric (6.11.13). Furthermore, higher covariant derivatives $\mathcal{D}$ in the coset are equivalent to higher covariant derivatives $\nabla(g)$ with respect to the metric (6.11.13). We therefore see that the invariant actions constructible by the coset method correspond to all possible diffeomorphism scalars constructed from the metric $g_{\mu\nu} = e^{2\pi}\eta_{\mu\nu}$, its curvature tensors and its covariant derivative.

The zero derivative term in (6.11.1) comes from the volume element

$$\mathcal{L}_1 \sim \sqrt{-g} = e^{4\pi} \ , \tag{6.11.18}$$

while the kinetic term comes from the Ricci curvature, after an integration by parts

$$\mathcal{L}_2 \sim \sqrt{-g}R = 6e^{2\pi}(\partial\pi)^2 \ . \tag{6.11.19}$$

The terms $\mathcal{L}_4$ and $\mathcal{L}_5$ are constructed from particular curvature invariants of order R^3 and R^4, respectively [91].

The term $\mathcal{L}_3$, however, presents a problem. It should be constructible from curvature invariants of order R^2, but all three curvature invariants which are quadratic in the Ricci curvature give the same contribution after integration by parts (and in fact, only two could have been independent since the Gauss-Bonnet combination $R^2 - 4R^{\mu\nu}R_{\mu\nu} + R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}$ is a total derivative) [91].

$$\left.\begin{matrix} \sqrt{-g}R^2 \\ \sqrt{-g}R^{\mu\nu}R_{\mu\nu} \\ \sqrt{-g}R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} \end{matrix}\right\} \propto (\Box\pi)^2 + (\partial\pi)^4 + 2\Box\pi(\partial\pi)^2 \ , \tag{6.11.20}$$

which is not of the form $\mathcal{L}_3$ and gives rise to higher order equations of motion due to the $(\Box\pi)^2$ term. It would thus appear that it is impossible to create the conformal galileon $\mathcal{L}_3$ by the coset method.

However, one can create a linearly independent invariant Lagrangian by using a trick, as is done in [91]. We go to d dimensions,[28] and consider the following combination of curvature invariants,

$$\begin{aligned}&\frac{\sqrt{-g}}{(d-4)}\left(\frac{R_{\mu\nu}^2}{(d-1)}-\frac{R^2}{(d-1)^2}\right)\\&=e^{(d-4)\pi}\left((\Box\pi)^2+\frac{(d-2)(3d-4)}{2(d-1)}\Box\pi(\partial\pi)^2+\frac{(d-2)^3}{2(d-1)}(\partial\pi)^4\right)\ .\end{aligned} \tag{6.11.21}$$

This combination is finite in the limit $d\to 4$ and leads to the Lagrangian

$$\mathcal{L}\sim\frac{3}{4}(\Box\pi)^2+(\partial\pi)^4+2\Box\pi(\partial\pi)^2\ . \tag{6.11.22}$$

This combination is linearly independent of (6.11.20), and can be used to subtract off the offending $(\Box\pi)^2$ term giving the cubic galileon

$$\mathcal{L}_3\sim(\partial\pi)^4+2\Box\pi(\partial\pi)^2\ . \tag{6.11.23}$$

The fact that we must do this dimensional continuation to construct $\mathcal{L}_3$ is a harbinger of the fact that this is a Wess–Zumino term. The fact that Wess–Zumino terms are not captured by the coset construction appears here through the fact that we have to move away from four dimensions. In fact, it is easy to show that $\mathcal{L}_3$ changes by a total derivative under the non-linear symmetries while the remaining Lagrangians are strictly invariant (modulo the total derivative associated with changing the field coordinates), so we expect the necessity of a Wess–Zumino type construction for $\mathcal{L}_3$.

Starting with the conformal algebra (6.11.4), we wish to compute the relative Lie algebra cohomology

$$H^5(\mathfrak{so}(4,2),\mathfrak{so}(3,1)), \tag{6.11.24}$$

[28]The d-dimensional metric is $e^{2\pi}$ times the d-dimensional Minkowski metric.

in order to catalog the possible Wess–Zumino terms. Recall from Section 6.3 that the basis forms which are dual to the Lie algebra vectors are written with upper indices and the forms which annihilate the vector subspace spanned by $\mathfrak{so}(3,1)$ are $\{D, K^\mu, P^\mu\}$. These are used to create n-co-chains for computing the relative Lie algebra cohomology. The co-boundary operator δ acts on the basis forms as

$$\begin{aligned} \delta D &= 2\eta_{\mu\nu} K^\mu \wedge P^\nu \ , \\ \delta P^\mu &= D \wedge P^\mu + 2P^\beta \wedge J^{\alpha\mu}\eta_{\alpha\beta} \ , \\ \delta K^\mu &= -D \wedge K^\mu + 2K^\beta \wedge J^{\alpha\mu}\eta_{\alpha\beta} \ . \end{aligned} \tag{6.11.25}$$

We can construct the following six $\mathfrak{so}(3,1)$ invariant 5-co-chains

$$\begin{aligned} \omega_1 &= \epsilon_{\mu\nu\rho\sigma} D \wedge P^\mu \wedge P^\nu \wedge P^\rho \wedge P^\sigma, \\ \omega_2 &= \epsilon_{\mu\nu\rho\sigma} D \wedge P^\mu \wedge P^\nu \wedge P^\rho \wedge K^\sigma, \\ \omega_3 &= \epsilon_{\mu\nu\rho\sigma} D \wedge P^\mu \wedge P^\nu \wedge K^\rho \wedge K^\sigma, \\ \omega_4 &= \epsilon_{\mu\nu\rho\sigma} D \wedge P^\mu \wedge K^\nu \wedge K^\rho \wedge K^\sigma, \\ \omega_5 &= \epsilon_{\mu\nu\rho\sigma} D \wedge K^\mu \wedge K^\nu \wedge K^\rho \wedge K^\sigma, \\ \omega_6 &= \eta_{\mu\nu}\eta_{\rho\sigma} D \wedge P^\mu \wedge K^\nu \wedge P^\rho \wedge K^\sigma \ . \end{aligned} \tag{6.11.26}$$

The co-chains ω_1 to ω_5 are closed ($\delta\omega = 0$), and we therefore have five possible non-trivial co-cycles. However, four of these turn out to be co-boundaries

$$\begin{aligned} \omega_1 &= \frac{1}{4}\delta\Big[\epsilon_{\mu\nu\rho\sigma} P^\mu \wedge P^\nu \wedge P^\rho \wedge P^\sigma\Big], \\ \omega_2 &= \frac{1}{2}\delta\Big[\epsilon_{\mu\nu\rho\sigma} D \wedge P^\mu \wedge P^\nu \wedge P^\rho \wedge K^\sigma\Big], \\ \omega_4 &= -\frac{1}{2}\delta\Big[\epsilon_{\mu\nu\rho\sigma} D \wedge P^\mu \wedge K^\nu \wedge K^\rho \wedge K^\sigma\Big], \\ \omega_5 &= -\frac{1}{4}\delta\Big[\epsilon_{\mu\nu\rho\sigma} D \wedge K^\mu \wedge K^\nu \wedge K^\rho \wedge K^\sigma\Big] \ . \end{aligned} \tag{6.11.27}$$

However, it turns out that ω_3 is a non-trivial co-cycle. The only possible $\mathfrak{so}(3,1)$ invariant potential for ω_3 would be of the form $\alpha_3 \sim \epsilon_{\mu\nu\rho\sigma} P^\mu \wedge P^\nu \wedge K^\rho \wedge K^\sigma$ but, due to the sign difference between δP^μ and δK^μ, the co-boundary operator annihilates this form, $\delta\alpha_3 = 0$. Therefore, there is a single non-trivial element of $H^5(\mathfrak{so}(4,2), \mathfrak{so}(3,1))$ and correspondingly, a single Wess–Zumino term.

The 5-form corresponding to the non-trivial co-cycle ω_3 is given by

$$\omega_3^{\text{wz}} = \epsilon_{\mu\nu\rho\sigma}\, \omega_D \wedge \omega_P^\mu \wedge \omega_P^\nu \wedge \omega_K^\rho \wedge \omega_K^\sigma \tag{6.11.28}$$

$$\begin{aligned} &= \epsilon_{\mu\nu\rho\sigma}\Big[e^{4\pi}\left(\xi^4 \mathrm{d}\pi \wedge \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \mathrm{d}x^\rho \wedge \mathrm{d}x^\sigma - 4\xi^2\xi_\lambda \mathrm{d}x^\lambda \wedge \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \mathrm{d}x^\rho \wedge \mathrm{d}x^\sigma\right) \\ &\quad + e^{3\pi}\left(-2\xi^2 \mathrm{d}\pi \wedge \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \mathrm{d}\xi^\rho \wedge \mathrm{d}x^\sigma + 2\xi_\lambda \mathrm{d}x^\lambda \wedge \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \mathrm{d}\xi^\rho \wedge \mathrm{d}\xi^\sigma\right) \\ &\quad + e^{2\pi} \mathrm{d}\pi \wedge \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \mathrm{d}\xi^\rho \wedge \mathrm{d}\xi^\sigma\Big] , \end{aligned} \tag{6.11.29}$$

and can be written as a total derivative,

$$\begin{aligned} \omega_3^{\text{wz}} &= \mathrm{d}\beta_3^{\text{wz}}, \\ \beta_3^{\text{wz}} &= \epsilon_{\mu\nu\rho\sigma}\Big[\frac{e^{4\pi}}{4}\xi^4 \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \mathrm{d}x^\rho \wedge \mathrm{d}x^\sigma - \frac{e^{3\pi}}{3}\xi^2 \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \mathrm{d}\xi^\rho \wedge \mathrm{d}x^\sigma \\ &\quad + \frac{e^{2\pi}}{2} \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \mathrm{d}\xi^\rho \wedge \mathrm{d}\xi^\sigma\Big]. \end{aligned} \tag{6.11.30}$$

Pulling back and imposing the inverse Higgs constraint (6.11.10), the final result is

$$S_3^{\text{wz}} = \int_M \beta_3^{\text{wz}} = -\frac{1}{2}\int \mathrm{d}^4x \left[\frac{1}{2}\Box\pi(\partial\pi)^2 + \frac{1}{4}(\partial\pi)^4\right] , \tag{6.11.31}$$

which reproduces $\mathcal{L}_3$.

The extension to d space-time dimensions proceeds without too much trouble. When d is even, there is a single Wess–Zumino galileon, the middle one $\mathcal{L}_{\frac{d}{2}+1}$. The others are all coset constructible. As an example, in $d = 2$ the kinetic term $\mathcal{L}_2$ is a Wess–Zumino term. It is impossible to construct with the coset method, since the only possible curvature term which could give it, R, is a total derivative in two dimensions. When d is odd, there is no Wess–Zumino term, and all the conformal galileons are coset constructible.

It is worth noting that the 4-dimensional Wess–Zumino term

$$\mathcal{L}_3 \sim (\partial\pi)^4 + 2\Box\pi(\partial\pi)^2 \ , \tag{6.11.32}$$

has been of some interest recently in connection with the a-theorem in four dimensions [78, 79]. This term for the 4 dimensional conformal group plays a similar role to that of the more well-known 2 dimensional Wess–Zumino term in the trace anomaly. The extension to d dimensions reflects the fact that there is no anomaly for odd d, and in even d it is associated with terms of order $d/2$ in the curvature.

6.12 DBI galileons

Finally, we demonstrate that coset methods can reproduce the DBI galileons of Sec.3.3.1, a 4D scalar field theory which non-linearly realized 5D Poincaré symmetries. Once again, the DBI lagrangians are (3.9)

$$\begin{aligned}
&\mathcal{L}_1 = \pi, \\
&\mathcal{L}_2 = -\sqrt{1+(\partial\pi)^2} \ , \\
&\mathcal{L}_3 = -[\Pi] + \gamma^2 \left[\pi^3\right] \ , \\
&\mathcal{L}_4 = -\gamma\left([\Pi]^2 - \left[\Pi^2\right]\right) - 2\gamma^3\left(\left[\pi^4\right] - [\Pi]\left[\pi^3\right]\right) \ , \\
&\mathcal{L}_5 = -\gamma^2\left([\Pi]^3 + 2\left[\Pi^3\right] - 3[\Pi]\left[\Pi^2\right]\right) - \gamma^4\left(6[\Pi]\left[\pi^4\right] - 6\left[\pi^5\right] - 3\left([\Pi]^2 - \left[\Pi^2\right]\right)\left[\pi^3\right]\right) \ ,
\end{aligned} \tag{6.12.1}$$

where

$$\gamma \equiv \frac{1}{\sqrt{1+(\partial\pi)}}. \tag{6.12.2}$$

As we have seen, the last four DBI galileons are obtained from Lovelock invariants of the induced brane metric for M_4 in M_5 and the boundary terms associated to 5D Lovelock invariants. However, the first term, the tadpole, is not constructed from local terms on the brane, but rather as the five-dimensional volume bounded by the brane (as discussed in [66]), and is the only Wess–Zumino term, as we will see.

The DBI galileons realize spontaneous breaking of the 5D Poincaré algebra to its 4D Poincaré subalgebra,

$$\mathfrak{iso}(4,1) \longrightarrow \mathfrak{iso}(3,1). \tag{6.12.3}$$

The broken transformations are translations and rotations into the fifth direction [39, 66]

$$\delta_{P_5}\pi = 1 \ , \qquad\qquad \delta_{J_{\mu 5}}\pi = x_\mu + \pi\partial_\mu\pi \ . \tag{6.12.4}$$

The 5D Poincaré algebra has the commutation relations

$$\begin{aligned} &[J_{MP}, P_Q] = \eta_{MQ}P_N - \eta_{NQ}P_M \\ &[J_{MN}, J_{PQ}] = \eta_{MP}J_{NQ} - \eta_{NP}J_{MQ} + \eta_{NQ}J_{MP} - \eta_{MQ}J_{NP} \ , \end{aligned} \tag{6.12.5}$$

where $\eta_{AB} = \mathrm{diag}\,(-1,1,1,1,1)$. The preserved subalgebra is the Poincaré subalgebra generated by $(J_{\mu\nu}, P_\rho)$, where Greek indices run from 0 to 3, acting as in (6.11.2).

The broken generators are P_5 and $J_{\mu 5}$, and the coset space is

$$\mathrm{ISO}(4,2)/\mathrm{SO}(3,1), \tag{6.12.6}$$

parametrized by[29]

$$\tilde{V} = e^{x\cdot P}e^{\pi P_5}e^{\xi^\alpha J_{\alpha 5}} \ . \tag{6.12.7}$$

From this, we can compute the Maurer–Cartan form (6.2.12)

$$\omega = \tilde{V}^{-1}\mathrm{d}\tilde{V} = \omega_P^\alpha P_\alpha + \omega_{P_5}P_5 + \omega_J^\alpha J_{\alpha 5} + \frac{1}{2}\omega_J^{\alpha\beta}J_{\alpha\beta} \ , \tag{6.12.8}$$

[29] As in the conformal galileon example (6.11.7), this differs slightly from our general expression (6.2.10), which just amounts to a different choice of parametrization of the coset.

where the needed components are

$$\omega_P^\alpha = \mathrm{d}x^\alpha - \frac{\frac{1}{2}\psi^\alpha\psi_\nu}{1+\frac{\psi^2}{4}}\mathrm{d}x^\nu + \frac{\psi^\alpha}{1+\frac{\psi^2}{4}}\mathrm{d}\pi \ , \tag{6.12.9}$$

$$\omega_{P_5} = \frac{1-\frac{\psi^2}{4}}{1+\frac{\psi^2}{4}}\mathrm{d}\pi - \frac{\psi_\mu}{1+\frac{\psi^2}{4}}\mathrm{d}x^\mu \ , \tag{6.12.10}$$

$$\omega_J^\alpha = \frac{\mathrm{d}\psi^\alpha}{1+\frac{\psi^2}{4}} \ . \tag{6.12.11}$$

Here, inspired by [11], we have made the field redefinition

$$\psi_\mu \equiv \xi_\mu \frac{\tanh\sqrt{\frac{-\xi^2}{4}}}{\sqrt{\frac{-\xi^2}{4}}} \ , \tag{6.12.12}$$

to make the field ψ appear quadratically, which simplifies the expressions. We will not consider the coupling of π to matter fields, so the explicit form of $\omega_J^{\mu\nu}$ will not be important.

There is an inverse Higgs constraint, since the commutator of $J_{\mu 5}$ with the unbroken translations

$$[P_\mu, J_{\nu 5}] = -\eta_{\mu\nu}P_5 \ , \tag{6.12.13}$$

is proportional to the other unbroken generator P_5, so the ψ_μ field is unphysical and may be eliminated in favor of the π by setting $\omega_{P_5} = 0$, leading to the following relationship between the π and ψ_μ fields

$$\psi_\mu = \frac{2\partial_\mu\pi}{1+\sqrt{1+(\partial\pi)^2}} \ . \tag{6.12.14}$$

The choice of sign for the square root just leads to an overall sign in front of the Lagrangian, and we will choose the + branch. Using this, we may simplify slightly the expressions for the Maurer–Cartan forms

$$\omega_P^\alpha = \left(\delta_\mu^\alpha + \frac{\frac{1}{2}\psi_\mu\psi^\alpha}{1-\frac{\psi^2}{4}}\right)\mathrm{d}x^\mu \ , \tag{6.12.15}$$

$$\omega_J^\alpha = \frac{\mathrm{d}\psi^\alpha}{1+\frac{\psi^2}{4}}. \tag{6.12.16}$$

The vielbein (6.2.14) and inverse vielbein can be extracted from ω_P,

$$e_\mu{}^\alpha = \delta_\mu^\alpha + \frac{\frac{1}{2}\psi_\mu\psi^\alpha}{1-\frac{\psi^2}{4}}, \quad e^\mu{}_\alpha = \delta_\alpha^\mu - \frac{\frac{1}{2}\psi_\alpha\psi^\mu}{1+\frac{\psi^2}{4}} \ . \tag{6.12.17}$$

Here we see explicitly that the coset construction is exactly equivalent to the brane construction of [39] and Sec.3.3.1 by noting that the induced metric associated to the vielbein (6.12.15) is

$$g_{\mu\nu} = \eta_{\alpha\beta} e_\mu{}^\alpha e_\nu{}^\beta = \eta_{\mu\nu} + \partial_\mu\pi\partial_\nu\pi \ . \tag{6.12.18}$$

Similarly, the covariant derivative (6.2.18) of ξ, written with spacetime rather than Lorentz indices, is precisely the extrinsic curvature

$$\mathcal{D}_\mu\xi_\nu \equiv e_\mu{}^\alpha \left(\omega_J\right)_\nu{}^\beta \eta_{\alpha\beta} = \gamma\partial_\mu\partial_\nu\pi = -K_{\mu\nu} \ . \tag{6.12.19}$$

From the Gauss–Codazzi relation for a flat bulk

$$R_{\mu\nu\rho\sigma} - K_{\mu\rho}K_{\nu\sigma} + K_{\nu\rho}K_{\mu\sigma} = 0 \ , \tag{6.12.20}$$

we see that the coset procedure gives us all the ingredients we need for constructing the generic brane action $\mathcal{L}(\bar{g}_{\mu\nu}, K_{\mu\nu}, R_{\mu\nu\rho\sigma})$ of (2.8) and hence we can construct the DBI galileon terms $\mathcal{L}_2$ through $\mathcal{L}_5$ of Sec.3.3.1.[30]

[30] The DBI terms, save the tadpole $\mathcal{L}_1$, can also be constructed just by wedging the Maurer–Cartan components together as

$$\begin{aligned}
\mathcal{L}_2 &= -\frac{1}{4!}\epsilon_{\mu\nu\rho\sigma}\ \omega_P^\mu \wedge \omega_P^\nu \wedge \omega_P^\rho \wedge \omega_P^\sigma, \\
\mathcal{L}_3 &= \frac{1}{3!}\epsilon_{\mu\nu\rho\sigma}\ \omega_J^\mu \wedge \omega_P^\nu \wedge \omega_P^\rho \wedge \omega_P^\sigma, \\
\mathcal{L}_4 &= -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\omega_J^\mu \wedge \omega_J^\nu \wedge \omega_P^\rho \wedge \omega_P^\sigma, \\
\mathcal{L}_5 &= \epsilon_{\mu\nu\rho\sigma}\ \omega_J^\mu \wedge \omega_J^\nu \wedge \omega_J^\rho \wedge \omega_P^\sigma \ ,
\end{aligned}$$

and then integrating over the spacetime.

Note that—just as in the brane construction—we have failed to construct the tadpole term, $\mathcal{L}_1 = \pi$, from the coset methods in four dimensions. However, it is possible to construct this tadpole as a Wess–Zumino term by considering the 5-form

$$\omega_1^{\mathrm{wz}} = \epsilon_{\mu\nu\rho\sigma}\omega_{P_5} \wedge \omega_P^\mu \wedge \omega_P^\nu \wedge \omega_P^\rho \wedge \omega_P^\sigma \ . \tag{6.12.21}$$

A fairly straightforward calculation reveals that this 5-form is exact,

$$\begin{aligned} \omega_1^{\mathrm{wz}} &= \mathrm{d}\beta_1^{\mathrm{wz}}, \\ \beta_1^{\mathrm{wz}} &= \pi\epsilon_{\mu\nu\rho\sigma}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \mathrm{d}x^\rho \wedge \mathrm{d}x^\sigma \ . \end{aligned} \tag{6.12.22}$$

The action given by integrating this 4-form is then

$$S_1 = \int_M \beta_1^{\mathrm{wz}} = \int \mathrm{d}^4x \ \pi \ , \tag{6.12.23}$$

which is the action corresponding to the tadpole Lagrangian $\mathcal{L}_1$. Therefore we see that the tadpole term is a Wess–Zumino term for spontaneously broken Poincaré invariance, in contrast to the other DBI galileon terms.

The DBI galileons are obtainable from the coset construction and so are not Wess–Zumino terms (except for the tadpole term). Taking a small-field limit gives the ordinary galileon terms, indicating that the procedure of contracting the algebra can change which terms are Wess–Zumino. For concreteness, here we derived the DBI galileons in four dimensions, but similar remarks apply in all dimensions: none of the DBI galileons will be Wess–Zumino except for the tadpole.

The case of higher co-dimensions is more subtle (the DBI galileons for higher co-dimension are discussed in [73]), but the extension should not be too difficult. The coset construction used here is not new—there are many examples of authors deriving low-energy effective actions for membranes using non-linear realization techniques, for example [29, 64, 117]—but to our knowledge the construction of the full set of DBI galileons from this perspective has not appeared elsewhere in the literature.

Based on the expectation that the brane constructions used in [39, 65, 66, 73] are equivalent to the coset construction, we can surmise that the DBI-like galileons living on (A)dS and flat spaces and realizing higher dimensional (A)dS and Poincaré symmetries, catalogued in [65, 66] (before taking any small field limits), have the same Wess–Zumino properties as the original DBI galileons studied in this section, that is, the tadpole is Wess–Zumino and the higher order galileons are not.

Chapter 7

Summary of Part III

In Part III we demonstrated that galileons are naturally viewed as Goldstone modes borne from the breaking of spacetime symmetries. In particular, the ghost free galileon interactions are special in that they arise as Wess–Zumino terms for the appropriate spacetime SSB pattern. Their existence is linked to the existence of non-trivial co-cycles in relative Lie algebra cohomology. The galileon terms are the d-form potentials for the $(d+1)$-form non-trivial co-cycles. The existence of the galileons is due to the local algebraic properties of the relevant groups.

We also used the techniques of non-linear realizations to address multi-galileon theories, showing that they too are Wess–Zumino terms. Finally, we considered the DBI galileons, showing that they are not Wess–Zumino terms (except for the tadpole term), and we considered the conformal galileons, showing that only the middle conformal galileon is a Wess–Zumino term.

Part IV

Conclusions

Chapter 8

Conclusion

Galileon field theories are of great interest, both due to their natural appearance in well motivated theories of modified gravity and in their own right, as they have fascinating field theoretic properties. First discovered in the context of the DGP [51] higher dimensional braneworld model, they were later incorporated into other modifications of gravity [91] and found to appear in the first consistent theory of massive gravity [36, 37].

On the modified gravity side, galileon interactions are special because they provide the unique natural realization of the Vainshtein screening mechanism for scalar fields. The mechanism utilizes higher derivative interactions in order to generate non-linear regimes which effectively shut off the scalar fifth force at distances close to massive sources. In the context of massive gravity, this mechanism is essential in order to resolve the vDVZ discontinuity in which the $m \to 0$ limit of typical massive gravity theories does not reproduce general relativity.

On the field theory side, galileons are quite interesting as they represent non-trivial derivative interactions for which the equations of motion remain second order in time derivative (thereby evading ghostly modes) and which are technically distinct from other generic actions which also obey the galileon symmetry $\pi \to \pi + c + b_\mu x^\mu$. Further, the galileons enjoy a non-renormalization theorem which ensures that galileon interactions are not renormalized to any order in loops.

Befitting their braneworld origins, there is an elegant geometric interpretation of galileons in which they describe the bending mode of a probe brane living in a higher dimensional space. This approach was first elucidated in [39] and the purpose of Part II of this thesis was to extend and generalize the geometric construction. In Sec. 2.2 we gave the entirely general construction for describing probe brane theories and discuss how the galileon-like symmetries originate from isometries of the bulk. After identifying the unique actions which lead to second order equations of motion, we explicitly built the appropriate theories for maximally symmetric bulk spaces in Chapter 3. These are the natural generalizations of galileons. They are higher derivative theories of scalar fields that retain second order equations of motion and also exhibit a host of non-linear shift symmetries. They also exhibit novel features such as the presence of masses and potentials whose values are fixed by symmetries. We explored whether one class of these generalized lagrangians is free of the pathology demonstrated by the original galileons in which perturbations on top of non-trivial $\pi(x)$ configurations propagate superluminally, but found they too are afflicted. Finally, in Chapter 4 we derived the appropriate generalization of galileon theories which live on FRW spacetimes.

While the specific models generated by these geometric methods are interesting, the overall conceptual viewpoint they provide is important, too. These procedure starts by specifying an arbitrary bulk spacetime with a given set of isometries and ends up generating stable scalar field theories which can be nearly arbitrarily complicated and yet realize every symmetry of the bulk space. This perspective inspired the work of Part III in which the galileons were viewed as Goldstone modes arising from spontaneous symmetry breaking.

The non-linear realization of bulk isometries via the shift symmetries of the scalar field and the resulting derivative interaction make the association between galileons and spontaneous symmetry breaking tempting and in Part III we make the identification explicit.

The viewpoint is that the mere presence of the brane spontaneously breaks the spacetime symmetries previously enjoyed by the bulk space, and hence there ought to be corresponding Goldstone modes which describe the motion of the brane within the bulk space. Using canonical techniques of SSB, we demonstrated that the construction of galileon field theories can be performed entirely within the standard framework. In particular, within the coset construction, the galileon interactions, which have fewer derivatives than their generic, symmetric counterparts, are found to be technically special. They are the analogue of the Wess-Zumino-Witten term from the chiral lagrangian of pion physics and require a higher dimensional construction. More than anything, the methods presented provide a new way of constructing and thinking about galileon field theories and their generalizations.

Galileons and their generalizations have turned up everywhere from theories of modified gravity to proofs of the a-theorem in conformal field theory [78, 79] and studies of halo biasing [8]. Studying them in their own right, we not only gain a greater understanding of an interesting class of non-trivial field theories, but also a better understanding of these other physical scenarios in which they appear. This thesis has provided a host of technical methods and viewpoints for galileon field theories and represents significant progress in understanding their underlying nature. Certainly, these field theories are not without their flaws, as can be seen from their superluminal modes and the apparent inability to UV complete them in a standard Wilsonian way [1], but this potentially makes them more interesting as the pursuit for resolutions of these apparent shortcomings is likely to be illuminating itself.

Appendices

Appendix A

Conventions

We use the mostly plus metric signature convention. The 3-brane worldvolume coordinates are x^μ, $\mu = 0, 1, 2, 3$, bulk coordinates are X^A, $A = 0, 1, 2, 3, 5$. Occasionally we use 6-dimensional cartesian coordinates $Y^{\mathcal{A}}$, $\mathcal{A} = 0, 1, 2, 3, 4, 5$, for constructing five dimensional AdS_5 and dS_5 as embeddings. Tensors are symmetrized and anti-symmetrized with unit weight, i.e $T_{(\mu\nu)} = \frac{1}{2}(T_{\mu\nu} + T_{\nu\mu})$, $T_{[\mu\nu]} = \frac{1}{2}(T_{\mu\nu} - T_{\nu\mu})$. Curvature tensors are defined by $[\nabla_\mu, \nabla_\nu] V^\rho = R^\rho{}_{\sigma\mu\nu} V^\sigma$ and $R_{\mu\nu} = R^\rho{}_{\mu\rho\nu}$, $R = R^\mu{}_\mu$. The flat space epsilon tensor is defined so that $\epsilon_{01\cdots d} = +1$. n-dimensional Minkowski space, de Sitter space and Anti-de Sitter space are abbreviated as M_n, dS_n and AdS_n, respectively. The Planck mass is $M_{\rm pl} \equiv 1/\sqrt{8\pi G_N}$. Throughout the thesis, $\cong$ indicates equivalence up to total derivatives.

When writing actions for a scalar field π in curved space with metric $g_{\mu\nu}$ and covariant derivative ∇_μ, we use the notation Π for the matrix of second derivatives $\Pi_{\mu\nu} \equiv \nabla_\mu \nabla_\nu \pi$. For traces of powers of Π we write $[\Pi^n] \equiv Tr(\Pi^n)$, e.g. $[\Pi] = \nabla_\mu \nabla^\mu \pi$, $[\Pi^2] = \nabla_\mu \nabla_\nu \pi \nabla^\mu \nabla^\nu \pi$, where all indices are raised with respect to $g^{\mu\nu}$. We also define the contractions of powers of Π with $\nabla\pi$ using the notation $[\pi^n] \equiv \nabla\pi \cdot \Pi^{n-2} \cdot \nabla\pi$, e.g. $[\pi^2] = \nabla_\mu \pi \nabla^\mu \pi$, $[\pi^3] = \nabla_\mu \pi \nabla^\mu \nabla^\nu \pi \nabla_\nu \pi$, where again all indices are raised with $\eta_{\mu\nu}$.

Appendix B

Some useful expressions

Here we collect some expressions useful in the calculation leading to (3.6).

First some transformations in which we set $\bar{g}_{\mu\nu} = \tilde{g}_{\mu\nu} + \partial_\mu\pi\partial_\nu\pi$. Define

$$\gamma = \frac{1}{\sqrt{1+\tilde{g}^{\mu\nu}\partial_\mu\pi\partial_\nu\pi}}, \quad \tilde{\Pi}_{\mu\nu} = \tilde{\nabla}_\mu\tilde{\nabla}_\nu\pi \ . \tag{B.0.1}$$

Brackets with tildes denote a trace with respect to $\tilde{g}^{\mu\nu}$, e.g. $[\tilde{\Pi}] = \tilde{g}^{\mu\nu}\tilde{\nabla}_\mu\tilde{\nabla}_\nu\pi$, $[\tilde{\Pi}^2] = \tilde{g}^{\alpha\mu}\tilde{g}^{\beta\nu}\tilde{\nabla}_\mu\tilde{\nabla}_\nu\pi\tilde{\nabla}_\alpha\tilde{\nabla}_\beta\pi$, etc. and $\left[\tilde{\pi}^2\right] = \tilde{g}^{\mu\nu}\tilde{\nabla}_\mu\pi\tilde{\nabla}_\nu\pi$, $\left[\tilde{\pi}^3\right] = \tilde{g}^{\alpha\mu}\tilde{g}^{\beta\nu}\tilde{\nabla}_\alpha\pi\tilde{\nabla}_\mu\tilde{\nabla}_\nu\pi\tilde{\nabla}_\beta\pi$, etc.

We have,

$$\bar{\Gamma}^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu} + \gamma^2\tilde{\Pi}_{\mu\nu}\tilde{\nabla}^\lambda\pi \ , \tag{B.0.2}$$

$$\begin{aligned}
\bar{R}^\alpha{}_{\beta\mu\nu} &= \tilde{R}^\alpha{}_{\beta\mu\nu} - \gamma^2\tilde{R}_{\gamma\beta\mu\nu}\tilde{\nabla}^\gamma\pi\tilde{\nabla}^\alpha\pi + 2\gamma^2\left(\tilde{\Pi}^{\ \alpha}_{[\mu}\tilde{\Pi}_{\nu]\beta} - \gamma^2\tilde{\Pi}_{\gamma[\mu}\tilde{\Pi}_{\nu]\beta}\tilde{\nabla}^\alpha\pi\tilde{\nabla}^\gamma\pi\right) \ , &\text{(B.0.3)}\\
\bar{R}_{\mu\nu} &= \tilde{R}_{\mu\nu} - \gamma^2\tilde{R}_{\alpha\mu\beta\nu}\tilde{\nabla}^\alpha\pi\tilde{\nabla}^\beta\pi \\
&\quad + \gamma^2\left[\left([\tilde{\Pi}] - \gamma^2[\tilde{\pi}^3]\right)\tilde{\Pi}_{\mu\nu} - \tilde{\Pi}^2_{\mu\nu} + \gamma^2\tilde{\Pi}_{\mu\alpha}\tilde{\Pi}_{\nu\beta}\tilde{\nabla}^\alpha\pi\tilde{\nabla}^\beta\pi\right] \ , &\text{(B.0.4)}\\
\bar{R} &= \tilde{R} - 2\gamma^2\tilde{R}_{\mu\nu}\tilde{\nabla}^\mu\pi\tilde{\nabla}^\nu\pi + \gamma^2\left([\tilde{\Pi}]^2 - [\tilde{\Pi}^2]\right) + 2\gamma^4\left([\tilde{\pi}^4] - [\tilde{\pi}^3][\tilde{\Pi}]\right) \ . &\text{(B.0.5)}
\end{aligned}$$

For performing the conformal transformation, $\tilde{g}_{\mu\nu} = f^2 g_{\mu\nu}$, we use

$$\tilde{\Gamma}^\rho_{\mu\nu} = \Gamma^\rho_{\mu\nu} + f^{-1}\left(\delta^\rho_\mu \partial_\nu f + \delta^\rho_\nu \partial_\mu f - g_{\mu\nu} g^{\rho\sigma}\partial_\sigma f\right) \ , \tag{B.0.6}$$

$$\begin{aligned}
\tilde{R}^\rho{}_{\sigma\mu\nu} &= R^\rho{}_{\sigma\mu\nu} + 2\left(-\frac{f''}{f} + 2\frac{f'^2}{f^2}\right)\delta^\rho_{[\mu}\nabla_{\nu]}\pi\nabla_\sigma\pi - 2\frac{f'}{f}\delta^\rho_{[\mu}\nabla_{\nu]}\nabla_\sigma\pi \\
&\quad + 2\left(\frac{f''}{f} - 2\frac{f'^2}{f^2}\right) g_{\sigma[\mu}\nabla_{\nu]}\pi\nabla^\rho\pi + 2\frac{f'}{f} g_{\sigma[\mu}\nabla_{\nu]}\nabla^\rho\pi + 2\frac{f'^2}{f^2} g_{\sigma[\mu}\delta^\rho_{\nu]}(\nabla\pi)^2 \ , \qquad \text{(B.0.7)} \\
\tilde{R}_{\mu\nu} &= R_{\mu\nu} + 2\left(2\frac{f'^2}{f^2} - \frac{f''}{f}\right)\nabla_\mu\pi\nabla_\nu\pi - 2\frac{f'}{f}\Pi_{\mu\nu} - g_{\mu\nu}\left(\frac{f'}{f}[\Pi] + \left(\frac{f''}{f} + \frac{f'^2}{f^2}\right)[\pi^2]\right) \ , \\
\tilde{R} &= \frac{1}{f^2}R - \frac{6}{f^3}\left(f''[\pi^2] + f'[\Pi]\right) \ . \qquad \text{(B.0.8)}
\end{aligned}$$

The transformation of the matrix of derivatives is

$$\tilde{\Pi}_{\mu\nu} = \Pi_{\mu\nu} - 2\frac{f'}{f}\nabla_\mu\pi\nabla_\nu\pi + g_{\mu\nu}\frac{f'}{f}[\pi^2] \ , \tag{B.0.9}$$

and, finally, some useful relations for the contractions are

$$[\tilde{\Pi}] = \frac{1}{f^2}[\Pi] + 2\frac{f'}{f^3}[\pi^2] \ , \tag{B.0.10}$$

$$[\tilde{\Pi^2}] = \frac{1}{f^4}[\Pi^2] + 2\frac{f'}{f^5}\left([\Pi][\pi^2] - 2[\pi^3]\right) + 4\frac{f'^2}{f^6}[\pi^2]^2 \ , \tag{B.0.11}$$

$$[\tilde{\pi^2}] = \frac{1}{f^2}[\pi^2] \ , \tag{B.0.12}$$

$$[\tilde{\pi^3}] = \frac{1}{f^4}[\pi^3] - \frac{f'}{f^5}[\pi^2]^2 \ , \tag{B.0.13}$$

$$[\tilde{\pi^4}] = \frac{1}{f^6}[\pi^4] - 2\frac{f'}{f^7}[\pi^3][\pi^2] + \frac{f'^2}{f^8}[\pi^2]^3 \ . \tag{B.0.14}$$

Appendix C

Lagrangians for Gaussian Normal Foliations of Flat Spaces

In this appendix, we present the general expressions for the $\mathcal{L}_i$'s for probe branes whose bulk metric is both flat (so that $R_{ABCD} = 0$) and written in Gaussian normal form,

$$G_{AB}\mathrm{d}X^A\mathrm{d}X^B = f_{\mu\nu}(x^\sigma, \rho)\mathrm{d}x^\mu\mathrm{d}x^\nu + \mathrm{d}\rho^2 \ . \tag{C.0.1}$$

In all cases, we use the definition $\tilde{\gamma} = 1/\sqrt{1+(\partial\pi)^2}$ to replace $(\partial\pi)^2$ in favor of $\tilde{\gamma}$ (recall that indices on the derivatives are raised with $f^{\mu\nu}$). In addition, we employ a shorthand notation. We define $\Pi_{\mu\nu} = \nabla_\mu\nabla_\nu\pi$, where the covariant derivative ∇_μ is calculated from $f_{\mu\nu}$ at fixed π. $f'_{\mu\nu}$ denotes the derivative of $f_{\mu\nu}(x,\pi)$ with respect to π.

For this appendix, we also use a slightly different shorthand for contractions than we do elsewhere in this thesis. This is needed as there is more ambiguity in the ordering of contractions here than there is in the case of Sec.2.2.1.2 and we wish to differentiate the two from each other. We use angular brackets $\langle \ldots \rangle$ to denote traces of the enclosed product as matrices, with all contractions performed using $f^{\mu\nu}$. For example, we have

$$\langle f' \rangle = f^{\mu\nu}\partial_\pi f_{\mu\nu},$$

$$
\begin{aligned}
\langle \Pi f' \rangle &= \Pi_{\mu\nu} f^{\nu\lambda} \left(\partial_\pi f_{\lambda\sigma}\right) f^{\sigma\mu}, \\
\langle \Pi^3 \rangle &= \Pi_{\mu\nu} f^{\nu\lambda} \Pi_{\lambda\sigma} f^{\sigma\rho} \Pi_{\rho\kappa} f^{\kappa\mu} \ .
\end{aligned} \tag{C.0.2}
$$

In addition, when π appears within a angled bracket, it does so only at both ends, and denotes contraction with $\nabla_\mu \pi$, for example,

$$
\begin{aligned}
\langle \pi f' \pi \rangle &= \nabla_\mu \pi \; f^{\mu\nu} \left(\partial_\pi f_{\nu\lambda}\right) f^{\lambda\sigma} \nabla_\sigma \pi, \\
\langle \pi \Pi f' \pi \rangle &= \nabla_\mu \pi \; f^{\mu\nu} \Pi_{\nu\lambda} f^{\lambda\sigma} \left(\partial_\pi f_{\sigma\rho}\right) f^{\rho\kappa} \nabla_\kappa \pi \ .
\end{aligned} \tag{C.0.3}
$$

Employing this notation, the lagrangians (2.38) are calculated to be (no integrations by parts have been made in obtaining these expressions)

$$
\begin{aligned}
\mathcal{L}_1 &= \int^{\pi(x)} d\pi' \sqrt{-\det f_{\mu\nu}(x, \pi')}, \\
\mathcal{L}_2 &= -\sqrt{-f}\frac{1}{\tilde{\gamma}}, \\
\mathcal{L}_3 &= \sqrt{-f}\Big[-\langle \Pi \rangle + \frac{1}{2}\langle f' \rangle + \tilde{\gamma}^2 \left(\langle \pi \Pi \pi \rangle + \frac{1}{2}\langle \pi f' \pi \rangle \right) \Big], \\
\mathcal{L}_4 &= \sqrt{-f}\Big[-\frac{1}{2}\langle \pi f' \pi \rangle^2 \tilde{\gamma}^3 - \langle f' \rangle \langle \pi \Pi \pi \rangle \tilde{\gamma}^3 - 2\langle \pi \Pi^2 \pi \rangle \tilde{\gamma}^3 + 2\langle \pi \Pi \pi \rangle \langle \Pi \rangle \tilde{\gamma}^3 \\
&\quad - \frac{1}{2}\langle f' \rangle \langle \pi f' \pi \rangle \tilde{\gamma}^3 + \langle \Pi \rangle \langle \pi f' \pi \rangle \tilde{\gamma}^3 - \frac{\langle f' \rangle^2 \tilde{\gamma}}{4} - \langle \Pi \rangle^2 \tilde{\gamma} + \frac{\langle f'^2 \rangle \tilde{\gamma}}{4} \\
&\quad - \langle \Pi f' \rangle \tilde{\gamma} + \langle f' \rangle \langle \Pi \rangle \tilde{\gamma} + \langle \Pi^2 \rangle \tilde{\gamma} + \frac{\langle \pi f'^2 \pi \rangle \tilde{\gamma}}{2} \Big],
\end{aligned} \tag{C.0.4}
$$

$$
\begin{aligned}
\mathcal{L}_5 = \sqrt{-f}\Big[& 3\langle \pi \Pi \pi \rangle \langle \Pi \rangle^2 \tilde{\gamma}^4 + \frac{3}{4}\langle f' \rangle \langle \pi f' \pi \rangle^2 \tilde{\gamma}^4 - \frac{3}{2}\langle \Pi \rangle \langle \pi f' \pi \rangle^2 \tilde{\gamma}^4 + \frac{3}{4}\langle f' \rangle^2 \langle \pi \Pi \pi \rangle \tilde{\gamma}^4 \\
& - \frac{3}{4}\langle f'^2 \rangle \langle \pi \Pi \pi \rangle \tilde{\gamma}^4 + 3\langle \Pi f' \rangle \langle \pi \Pi \pi \rangle \tilde{\gamma}^4 + 6\langle \pi \Pi^3 \pi \rangle \tilde{\gamma}^4 + 3\langle f' \rangle \langle \pi \Pi^2 \pi \rangle \tilde{\gamma}^4 \\
& - 3\langle f' \rangle \langle \pi \Pi \pi \rangle \langle \Pi \rangle \tilde{\gamma}^4 - 6\langle \pi \Pi^2 \pi \rangle \langle \Pi \rangle \tilde{\gamma}^4 - 3\langle \pi \Pi \pi \rangle \langle \Pi^2 \rangle \tilde{\gamma}^4 + \frac{3}{8}\langle f' \rangle^2 \langle \pi f' \pi \rangle \tilde{\gamma}^4 \\
& + \frac{3}{2}\langle \Pi \rangle^2 \langle \pi f' \pi \rangle \tilde{\gamma}^4 - \frac{3}{8}\langle f'^2 \rangle \langle \pi f' \pi \rangle \tilde{\gamma}^4 + \frac{3}{2}\langle \Pi f' \rangle \langle \pi f' \pi \rangle \tilde{\gamma}^4
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{3}{2}\langle f'\rangle\langle\Pi\rangle\langle\pi f'\pi\rangle\tilde{\gamma}^4-\frac{3}{2}\langle\Pi^2\rangle\langle\pi f'\pi\rangle\tilde{\gamma}^4-\frac{3}{2}\langle\pi\Pi\pi\rangle\langle\pi f'^2\pi\rangle\tilde{\gamma}^4\\
&-\frac{3}{4}\langle\pi f'\pi\rangle\langle\pi f'^2\pi\rangle\tilde{\gamma}^4-3\langle\pi\Pi f'\Pi\pi\rangle\tilde{\gamma}^4+3\langle\pi f'\pi\rangle\langle\pi\Pi f'\pi\rangle\tilde{\gamma}^4\\
&+\frac{\langle f'\rangle^3\tilde{\gamma}^2}{8}-\langle\Pi\rangle^3\tilde{\gamma}^2+\frac{3}{2}\langle f'\rangle\langle\Pi\rangle^2\tilde{\gamma}^2-\frac{3}{8}\langle f'\rangle\langle f'^2\rangle\tilde{\gamma}^2+\frac{\langle f'^3\rangle\tilde{\gamma}^2}{4}\\
&+\frac{3}{2}\langle f'\rangle\langle\Pi f'\rangle\tilde{\gamma}^2-\frac{3\langle\Pi f'^2\rangle\tilde{\gamma}^2}{2}-\frac{3\langle\Pi f'\Pi f'\pi\rangle\tilde{\gamma}^2}{2}-\frac{3}{4}\langle f'\rangle^2\langle\Pi\rangle\tilde{\gamma}^2\\
&+\frac{3}{4}\langle f'^2\rangle\langle\Pi\rangle\tilde{\gamma}^2-3\langle\Pi f'\rangle\langle\Pi\rangle\tilde{\gamma}^2-2\langle\Pi^3\rangle\tilde{\gamma}^2-\frac{3}{2}\langle f'\rangle\langle\Pi^2\rangle\tilde{\gamma}^2\\
&+3\langle\Pi\rangle\langle\Pi^2\rangle\tilde{\gamma}^2+3\langle\pi f'\pi\rangle\tilde{\gamma}^2-\frac{3}{4}\langle f'\rangle\langle\pi f'^2\pi\rangle\tilde{\gamma}^2+\frac{3}{2}\langle\Pi\rangle\langle\pi f'^2\pi\rangle\tilde{\gamma}^2+\frac{3\langle\pi f'^3\pi\rangle\tilde{\gamma}^2}{4}\bigg]\ .
\end{aligned}
\tag{C.0.5}
$$

Appendix D

Explicit expression for $\mathcal{L}_3$

Here we present the full expression for $\mathcal{L}_3$ in the FRW case. No integrations by parts have been made.

$$
\begin{aligned}
\mathcal{L}_3 = \Big\{ & \dot{a}^2\ddot{a}a^5 + 3\dot{a}^4a^4 - 2\pi\dot{a}\ddot{a}^2a^5 - 14\pi\dot{a}^3\ddot{a}a^4 - 12\pi\dot{a}^5a^3 - 3\dot{\pi}a^4\dot{a}^4 + (\nabla^2\pi)a^3\dot{a}^3 \\
& - \ddot{\pi}a^5\dot{a}^3 + 18\pi^2a^2\dot{a}^6 + 46\pi^2a^3\ddot{a}\dot{a}^4 + 19\pi^2a^4\ddot{a}^2\dot{a}^2 + \pi^2a^5\ddot{a}^3 + \pi\dot{\pi}\dot{a}\ddot{a}^2a^5 \\
& - \pi\dot{\pi}\dot{a}^2a^{(3)}a^5 + 6\pi\dot{\pi}\dot{a}^3\ddot{a}a^4 + 12\pi\dot{\pi}\dot{a}^5a^3 - 2\dot{\pi}^2\dot{a}^2\ddot{a}a^5 + \pi\ddot{\pi}\dot{a}^2\ddot{a}a^5 - 3\dot{\pi}^2\dot{a}^4a^4 \\
& + 5\pi\ddot{\pi}\dot{a}^4a^4 + (\nabla\pi)^2\dot{a}^2\ddot{a}a^3 - 3(\nabla^2\pi)\pi\dot{a}^2\ddot{a}a^3 + 4(\nabla\pi)^2\dot{a}^4a^2 - 3(\nabla^2\pi)\pi\dot{a}^4a^2 \\
& - 12\pi^3a\dot{a}^7 - 64\pi^3a^2\ddot{a}\dot{a}^5 - 56\pi^3a^3\ddot{a}^2\dot{a}^3 - 8\pi^3a^4\ddot{a}^3\dot{a} - 18\pi^2\dot{\pi}a^2\dot{a}^6 \\
& - 24\pi^2\dot{\pi}a^3\ddot{a}\dot{a}^4 + 5\pi^2\dot{\pi}a^4a^{(3)}\dot{a}^3 - 8\pi^2\dot{\pi}a^4\ddot{a}^2\dot{a}^2 + 12\pi\dot{\pi}^2a^3\dot{a}^5 - 10\pi^2\ddot{\pi}a^3\dot{a}^5 \\
& + 3(\nabla^2\pi)\pi^2a\dot{a}^5 - 8(\nabla\pi)^2\pi a\dot{a}^5 + 13\pi\dot{\pi}^2a^4\ddot{a}\dot{a}^3 - 5\pi^2\ddot{\pi}a^4\ddot{a}\dot{a}^3 \\
& + 9(\nabla^2\pi)\pi^2a^2\ddot{a}\dot{a}^3 - 15(\nabla\pi)^2\pi a^2\ddot{a}\dot{a}^3 + 3(\nabla^2\pi)\pi^2a^3\ddot{a}^2\dot{a} - 2(\nabla\pi)^2\pi a^3\ddot{a}^2\dot{a} \\
& + 3\dot{\pi}^3a^4\dot{a}^4 - 4(\nabla\pi)^2\dot{\pi}a^2\dot{a}^4 - (\nabla^2\pi)\dot{\pi}^2a^3\dot{a}^3 + 2\nabla\dot{\pi}\cdot\nabla\pi\dot{\pi}a^3\dot{a}^3 \\
& - (\nabla\pi)^2\ddot{\pi}a^3\dot{a}^3 + (\nabla\pi)^2(\nabla^2\pi)a\dot{a}^3 - \delta^{ij}\delta^{kl}\partial_i\pi\partial_j\partial_k\pi\partial_l\pi a\dot{a}^3 + 3\pi^4\dot{a}^8 \\
& + 41\pi^4a\ddot{a}\dot{a}^6 + 74\pi^4a^2\ddot{a}^2\dot{a}^4 + 22\pi^4a^3\ddot{a}^3\dot{a}^2 + 12\pi^3\dot{\pi}a\dot{a}^7 + 36\pi^3\dot{\pi}a^2\ddot{a}\dot{a}^5
\end{aligned}
$$

$$
\begin{aligned}
&- 10\pi^3\dot\pi a^3 a^{(3)}\dot a^4 + 22\pi^3\dot\pi a^3\ddot a^2\dot a^3 - (\nabla^2\pi)\pi^3\dot a^6 + 4(\nabla\pi)^2\pi^2\dot a^6 \\
&- 18\pi^2\dot\pi^2 a^2\dot a^6 + 10\pi^3\ddot\pi a^2\dot a^6 - 32\pi^2\dot\pi^2 a^3\ddot a\dot a^4 + 10\pi^3\ddot\pi a^3\ddot a\dot a^4 \\
&- 9(\nabla^2\pi)\pi^3 a\ddot a\dot a^4 + 27(\nabla\pi)^2\pi^2 a\ddot a\dot a^4 - 9(\nabla^2\pi)\pi^3 a^2\ddot a^2\dot a^2 \\
&+ 18(\nabla\pi)^2\pi^2 a^2\ddot a^2\dot a^2 - (\nabla^2\pi)\pi^3 a^3\ddot a^3 + (\nabla\pi)^2\pi^2 a^3\ddot a^3 - 12\pi\dot\pi^3 a^3\dot a^5 \\
&+ 8(\nabla\pi)^2\pi\dot\pi a\dot a^5 + 8(\nabla\pi)^2\pi\dot\pi a^2\ddot a\dot a^3 - (\nabla\pi)^2\pi\dot\pi a^3 a^{(3)}\dot a^2 + (\nabla\pi)^2\pi\dot\pi a^3\ddot a^2\dot a \\
&+ 3(\nabla^2\pi)\pi\dot\pi^2 a^2\dot a^4 - 6\nabla\dot\pi\cdot\nabla\pi\pi\dot\pi a^2\dot a^4 + 3(\nabla\pi)^2\pi\ddot\pi a^2\dot a^4 - (\nabla\pi)^2(\nabla^2\pi)\pi\dot a^4 \\
&+ \pi\delta^{ij}\delta^{kl}\partial_i\pi\partial_j\partial_k\pi\partial_l\pi\dot a^4 + (\nabla^2\pi)\pi\dot\pi^2 a^3\ddot a\dot a^2 - 2\nabla\dot\pi\cdot\nabla\pi\pi\dot\pi a^3\ddot a\dot a^2 \\
&+ (\nabla\pi)^2\pi\ddot\pi a^3\ddot a\dot a^2 - 3(\nabla\pi)^2(\nabla^2\pi)\pi a\ddot a\dot a^2 + 3\pi\delta^{ij}\delta^{kl}\partial_i\pi\partial_j\partial_k\pi\partial_l\pi a\ddot a\dot a^2 \\
&- 10\pi^5\ddot a\dot a^7 - 46\pi^5 a\ddot a^2\dot a^5 - 28\pi^5 a^2\ddot a^3\dot a^3 - 3\pi^4\dot\pi\dot a^8 - 24\pi^4\dot\pi a\ddot a\dot a^6 \\
&+ 10\pi^4\dot\pi a^2 a^{(3)}\dot a^5 - 28\pi^4\dot\pi a^2\ddot a^2\dot a^4 + 12\pi^3\dot\pi^2 a\dot a^7 - 5\pi^4\ddot\pi a\dot a^7 + 3(\nabla^2\pi)\pi^4\ddot a\dot a^5 \\
&- 13(\nabla\pi)^2\pi^3\ddot a\dot a^5 + 38\pi^3\dot\pi^2 a^2\ddot a\dot a^5 - 10\pi^4\ddot\pi a^2\ddot a\dot a^5 + 9(\nabla^2\pi)\pi^4 a\ddot a^2\dot a^3 \\
&- 30(\nabla\pi)^2\pi^3 a\ddot a^2\dot a^3 + 3(\nabla^2\pi)\pi^4 a^2\ddot a^3\dot a - 7(\nabla\pi)^2\pi^3 a^2\ddot a^3\dot a + 18\pi^2\dot\pi^3 a^2\dot a^6 \\
&- 4(\nabla\pi)^2\pi^2\dot\pi\dot a^6 - 16(\nabla\pi)^2\pi^2\dot\pi a\ddot a\dot a^4 + 3(\nabla\pi)^2\pi^2\dot\pi a^2 a^{(3)}\dot a^3 - 7(\nabla\pi)^2\pi^2\dot\pi a^2\ddot a^2\dot a^2 \\
&- 3(\nabla^2\pi)\pi^2\dot\pi^2 a\dot a^5 + 6\nabla\dot\pi\cdot\nabla\pi\pi^2\dot\pi a\dot a^5 - 3(\nabla\pi)^2\pi^2\ddot\pi a\dot a^5 + 3(\nabla\pi)^2(\nabla^2\pi)\pi^2\ddot a\dot a^3 \\
&- 3(\nabla^2\pi)\pi^2\dot\pi^2 a^2\ddot a\dot a^3 + 6\nabla\dot\pi\cdot\nabla\pi\pi^2\dot\pi a^2\ddot a\dot a^3 - 3(\nabla\pi)^2\pi^2\ddot\pi a^2\ddot a\dot a^3 \\
&- 3\pi^2\delta^{ij}\delta^{kl}\partial_i\pi\partial_j\partial_k\pi\partial_l\pi\ddot a\dot a^3 + 3(\nabla\pi)^2(\nabla^2\pi)\pi^2 a\ddot a^2\dot a \\
&- 3\pi^2\delta^{ij}\delta^{kl}\partial_i\pi\partial_j\partial_k\pi\partial_l\pi a\ddot a^2\dot a + 17a\dot a^4\ddot a^3\pi^6 + 11\dot a^6\ddot a^2\pi^6 + 6\pi^5\dot\pi\ddot a\dot a^7 - 5\pi^5\dot\pi a a^{(3)}\dot a^6 \\
&+ 17\pi^5\dot\pi a\ddot a^2\dot a^5 - 3\pi^4\dot\pi^2\dot a^8 + \pi^5\ddot\pi\dot a^8 - 22\pi^4\dot\pi^2 a\ddot a\dot a^6 + 5\pi^5\ddot\pi a\ddot a\dot a^6 - 3(\nabla^2\pi)\pi^5\ddot a^2\dot a^4 \\
&+ 14(\nabla\pi)^2\pi^4\ddot a^2\dot a^4 - 3(\nabla^2\pi)\pi^5 a\ddot a^3\dot a^2 + 11(\nabla\pi)^2\pi^4 a\ddot a^3\dot a^2 - 12\pi^3\dot\pi^3 a\dot a^7 \\
&+ 8(\nabla\pi)^2\pi^3\dot\pi\ddot a\dot a^5 - 3(\nabla\pi)^2\pi^3\dot\pi a a^{(3)}\dot a^4 + 11(\nabla\pi)^2\pi^3\dot\pi a\ddot a^2\dot a^3 + (\nabla^2\pi)\pi^3\dot\pi^2\dot a^6 \\
&- 2\nabla\dot\pi\cdot\nabla\pi\pi^3\dot\pi\dot a^6 + (\nabla\pi)^2\pi^3\ddot\pi\dot a^6 + 3(\nabla^2\pi)\pi^3\dot\pi^2 a\ddot a\dot a^4 - 6\nabla\dot\pi\cdot\nabla\pi\pi^3\dot\pi a\ddot a\dot a^4 \\
&+ 3(\nabla\pi)^2\pi^3\ddot\pi a\ddot a\dot a^4 - 3(\nabla\pi)^2(\nabla^2\pi)\pi^3\ddot a^2\dot a^2 + 3\pi^3\delta^{ij}\delta^{kl}\partial_i\pi\partial_j\partial_k\pi\partial_l\pi\ddot a^2\dot a^2 \\
&- (\nabla\pi)^2(\nabla^2\pi)\pi^3 a\ddot a^3 + \pi^3\delta^{ij}\delta^{kl}\partial_i\pi\partial_j\partial_k\pi\partial_l\pi a\ddot a^3 - 4\pi^7\dot a^5\ddot a^3 + \pi^6\dot\pi\dot a^7 a^{(3)} - 4\pi^6\dot\pi\dot a^6\ddot a^2
\end{aligned}
$$

$$
\begin{aligned}
&+5\pi^5\dot{\pi}^2\ddot{a}\dot{a}^7 - \pi^6\dddot{\pi}\ddot{a}\dot{a}^7 + (\nabla^2\pi)\pi^6\ddot{a}^3\dot{a}^3 - 5(\nabla\pi)^2\pi^5\ddot{a}^3\dot{a}^3 + 3\pi^4\dot{\pi}^3\dot{a}^8 \\
&+(\nabla\pi)^2\pi^4\dot{\pi}a^{(3)}\dot{a}^5 - 5(\nabla\pi)^2\pi^4\dot{\pi}\ddot{a}^2\dot{a}^4 - (\nabla^2\pi)\pi^4\dot{\pi}^2\ddot{a}\dot{a}^5 + 2\nabla\dot{\pi}\cdot\nabla\pi\pi^4\dot{\pi}\ddot{a}\dot{a}^5 \\
&-(\nabla\pi)^2\pi^4\ddot{\pi}\ddot{a}\dot{a}^5 + (\nabla\pi)^2(\nabla^2\pi)\pi^4\ddot{a}^3\dot{a} - \pi^4\delta^{ij}\delta^{kl}\partial_i\pi\partial_j\partial_k\pi\partial_l\pi\ddot{a}^3\dot{a}\Big\}/ \\
&\Big\{\dot{a}^3(a-\dot{a}\pi)^2\dot{\pi}^2 - \dot{a}(\dot{a}-\ddot{a}\pi)^2\Big((a-\dot{a}\pi)^2 + (\vec{\nabla}\pi)^2\Big)\Big\},
\end{aligned}
\tag{D.0.1}
$$

where $(\vec{\nabla}\pi)^2 = \delta^{ij}\partial_i\pi\partial_j\pi$, $\vec{\nabla}^2\pi = \delta^{ij}\partial_i\partial_j\pi$ and $a^{(n)}$ is the n-th time derivative of the scale factor.

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