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Renormalization-Group Approach to the Critical Behavior of Random-Spin Models

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Abstract
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Renormalization-Group Approach to the Critical Behavior of Random-Spin Models*

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A renormalization-group technique is used to study the critical behavior of spin models in which each interaction has a small independent random width about its average value. The cluster approximation of Niemeyer and Van Leeuwen indicates that the two-dimensional Ising model has the same critical behavior as the homogeneous system. The ϵ expansion for n-component continuous spins shows that this behavior holds to first order in ϵ for $n>4$. For $n<4$, there is a new stable fixed point with $2\nu = 1+[3n/16(n-1)]\epsilon$.

The critical behavior of randomly diluted magnetic systems has been the object of some interest for many years.[1-3] Until recently, except for special models,[2] phase transitions in such systems could only be studied via series expansions.[1,2] This technique is not entirely satisfactory since it does not indicate whether the phase transition remains sharp with possibly different exponents or whether it washes out upon randomization of the interactions. A simple heuristic argument[4] indicates that the former behavior occurs when the specific heat exponent α is negative and the latter when α is positive. However, the validity of this argument is uncertain, and in any event, it gives no prediction for two-dimensional Ising models where $\alpha = 0$.

In this paper, we will discuss the application of the renormalization group,[5] which has been used so successfully in the calculation of critical exponents in pure systems, to phase transition in random systems. We introduce additional variables describing the randomization of the potential. These variables can be either irrelevant or relevant[6] in the vicinity of the pure-system fixed point. In the former case, we argue that the transition will be sharp with expoments of the pure system. In the latter case, either the system goes to a new stable fixed point, indicating a sharp phase transition with new critical exponents, or the renormalized randomization variables become infinite and the transition is probably smeared. We have applied two versions of the renormalization group to the random problem: the Niemeyer–Van Leeuwen (NL) cluster expansion for the two-dimensional Ising model, and the Wilson-Fisher ϵ expansion for n-component continuous spins with lattice dimensionality $d = 4 - \epsilon$. In the first case, we find for the small clusters we treat here that the randomization variables are irrelevant. In the second case, we find the same to be true for $n>4$ to first order in ϵ. For $n<4$, we find a new stable fixed point (i.e., a sharp transition) in which fluctuations in the local transition temperature have a nonvanishing value. Since space is limited, we will present in detail the calculations for the two-dimensional Ising model and only outline the results from the ϵ expansion. A detailed presentation of the ϵ expansion will follow shortly.

We consider Ising models described by the Hamiltonian

$$-\beta\mathcal{H} = \sum_{\langle ij\rangle} J_{ij} s_i s_j . \tag{1}$$

Here $\beta = 1/kT$, the sum is over nearest neighboring pairs of sites, $s_i = \pm 1$, and each J is an independent random variable governed by a probabil-

ity distribution $\rho(J)$ having the properties

$$\int \rho(J)\, dJ = 1, \tag{2a}$$

$$\int \rho(J)(J - J_0)^n\, dJ \leq (\zeta J_0)^n, \tag{2b}$$

with $\zeta \ll 1$. Thus, $\rho(J)$ has a narrow width ζJ_0 about its mean value J_0. The free energy is then calculated as the average over all the ρ's of the free energy as a function of all the J's. This is the so-called "quenched-bond" dilution problem. Since the main issue is whether randomness alters the critical behavior, we will carry out the calculation only to lowest order in the width ζJ_0.

The recursion relations may be written as

$$\mathcal{H}^{(N+1)}(\{s_N\}) = \ln(\mathrm{Tr}_{\{\xi\}_N} \exp[\mathcal{H}^{(N)}(\{s\}_{N-1})]) \tag{3a}$$

$$= \ln(\mathrm{Tr}_{\{\xi\}_N} \exp[\mathcal{H}^{(N)}(\{s\}_N, \{\xi\}_N)]). \tag{3b}$$

After the Nth stage of renormalization a Hamiltonian, $\mathcal{H}^{(N)}$, is obtained which depends on the site $\{s\}_{N-1}$. These are classified into cell variables $\{s\}_N$ and internal variables $\{\xi\}_N$, which are traced over to yield a new site Hamiltonian, $\mathcal{H}^{(N+1)}$, according to Eq. (3). Following NL we consider a two-dimensional triangular lattice and with a triangular cell, A, we associate a cell variable $s_A = \mathrm{sgn}[\sum_i s_i]$, where the sum is over the three sites ($i = 1, 2, 3$) in A. The internal variables may be taken to be $\xi_{A1} = s_1 s_2$ and $\xi_{A2} = s_1 s_3$. In general $\mathcal{H}^{(N)}$ will be of the form

$$\mathcal{H}^{(N)} = \sum_\alpha J_\alpha s_\alpha, \tag{4}$$

where

$$s_\alpha = \prod_{i \in \alpha} s_i,$$

where the sum over α in Eq. (4) is over all subsets of sites. Then Eq. (3) is of the form

$$J_\alpha' = \varphi_\alpha(\{J\}), \tag{5}$$

where the prime indicates a renormalized value. For the random system the recursion relations involve, not the J's, but rather their probability distribution, $P(\{J\})$. Thus, for the random system we write

$$P'(\{J'\}) = \int P(\{J\}) \prod_\alpha \delta(J_\alpha' - \varphi_\alpha(\{J\}))\, d\vec{J}, \tag{6}$$

where $d\vec{J} \equiv \prod dJ_\alpha$. One can easily verify that Eq. (6) does indeed correspond to the quenched bond problem where the free energy, and not the partition function, is configurationally averaged. We wish to study Eq. (6) in the vicinity of the fixed point of the homogeneous system. It is clear that $P(\{J\}) = \prod_\alpha \delta(J_\alpha - J_\alpha^*)$, where J_α^* is the fixed-point value of J_α for the homogeneous system, is a fixed point of Eq. (6). We now ask whether this fixed point is stable with respect to a small width in $P(\{J\})$. To do this we replace the renormalization equation for the functional P by the recursion relations for the cumulants of P. These recursion relations are generated by multiplying Eq. (6) by the desired powers of J_α' and evaluating the right-hand side using a Taylor series expansion about the average value of $\{J\}$. We define

$$\langle J_\alpha J_\beta \ldots J_\rho \rangle = \int P(\{J\}) J_\alpha J_\beta \ldots J_\rho\, d\vec{J}, \tag{7a}$$

$$\langle J_\alpha' J_\beta' \ldots J_\rho' \rangle = \int P'(\{J'\}) J_\alpha' J_\beta' \ldots J_\rho'\, d\vec{J}'. \tag{7b}$$

In terms of moments Eq. (6) is

$$\langle J_\alpha' J_\beta' \ldots J_\rho' \rangle = \psi + \tfrac{1}{2} \sum_{\mu\nu} \langle \delta J_\mu \delta J_\nu \rangle \frac{d^2\psi}{dJ_\mu dJ_\nu} + \tfrac{1}{6} \sum_{\mu\nu\sigma} \langle \delta J_\mu \delta J_\nu \delta J_\sigma \rangle \frac{d^3\psi}{dJ_\mu dJ_\nu dJ_\sigma} + \ldots \Big|_{\{J\} = \{\langle J \rangle\}}, \tag{8}$$

where $\psi = \varphi_\alpha(\{J\}) \varphi_\beta(\{J\}) \ldots \varphi_\rho(\{J\})$, and $\delta J_\mu = J_\mu - \langle J_\mu \rangle$.

To determine the flow near the critical point[6] we linearize these relations for $\{J\}$ near $\{J^*\}$. Furthermore, for narrow width it suffices to consider only terms of order $\langle \delta J_\mu \delta J_\nu \rangle$. Thus we obtain the lin-

earized recursion relations

$$\langle J_\alpha' \rangle - J_\alpha^* = \sum_\beta \frac{\partial \varphi_\alpha(\{J\})}{\partial J_\beta} (\langle J_\beta \rangle - J_\beta^*) + \tfrac{1}{2} \sum_{\beta\gamma} \frac{\partial^2 \varphi_\alpha(\{J\})}{\partial J_\beta \partial J_\gamma} \langle \delta J_\beta \delta J_\gamma \rangle, \tag{9a}$$

$$\langle \delta J_\alpha' \delta J_\beta' \rangle = \sum_{\gamma\delta} \frac{\partial \varphi_\alpha(\{J\})}{\partial J_\gamma} \frac{\partial \varphi_\beta(\{J\})}{\partial J_\delta} \langle \delta J_\gamma \delta J_\delta \rangle, \tag{9b}$$

where all the derivatives are evaluated at $\{J\} = \{J^*\}$. These equations are of the form

$$x_i' = \sum_j M_{ij} x_j. \tag{10}$$

Equations (9a) and (9b) are the linearized recursion relations valid very near the fixed point. Nonlinear corrections have been studied but with inconclusive results. These corrections would bear on the possible existence of other fixed points accessible from large initial values of the $\langle \delta J_\alpha \delta J_\beta \rangle$. Our aim is to see whether these quantities increase or decrease as the renormalization is repeated. In the latter case the system evolves towards the pure-system fixed point and the critical indices are those of the pure system. In the former case the transition is qualitatively modified by even an infinitesimal amount of randomness.

We now use the cluster approximation suggested by NL to treat Eq. (9). The additional complication introduced by randomness is that in general all $\langle \delta J_\alpha \delta J_\beta \rangle$'s will be generated by repeated use of Eq. (9). This, of course, would also happen for J_α if the infinite lattice were used. In the spirit of the cluster approximation we will only allow $\langle \delta J_\alpha \delta J_\beta \rangle$ to be nonzero when α and β are sufficiently near to one another. As NL have shown, J_α becomes rapidly less important as either the separation between sites in α or the number of sites in α becomes large. It is hard to see why $\langle \delta J_\alpha \delta J_\beta \rangle$ should be long ranged if J_α itself is not.

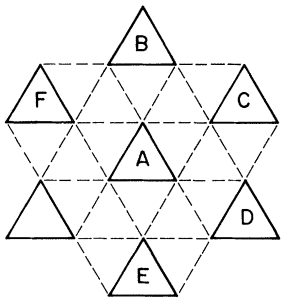

FIG. 1. Section of the triangular lattice divided into cells A, B, C, ..., each consisting of three sites. The full lines represent intracell interactions; the dashed lines, intercell interactions.

For the two-cell cluster approximation we retain only the correlations $\langle \delta J_\alpha \delta J_\beta \rangle$ if J_α and J_β are two spin interactions involving a common site. Thus in Eq. (10) there are five variables, $x_1 = J - J^*$, $x_2 = \langle \delta J_{AB} \delta J_{AB} \rangle$, $x_3 = \langle \delta J_{AB} \delta J_{AC} \rangle$, $x_4 = \langle \delta J_{AB} \delta J_{AD} \rangle$, and $x_5 = \langle \delta J_{AB} \delta J_{AE} \rangle$, and the matrix $\underline{M}$ of Eq. (10) has the numerical value given in Table I. (The labeling of the J's is shown in Fig. 1.) We know that if $P(J)$ is initially a δ function, it will remain so after renormalization. Therefore, one eigenvalue of $\underline{M}$ must be η_T, where η_T is the eigenvalue for the homogeneous system.[7] The left eigenvectors of $\underline{M}$ define new coordinates, u_i. The variable u_T corresponding to η_T is a "relevant" variable since $\eta_T \approx 1.54 > 1$, and $u_T = 0$ defines criticality. Since the next largest eigenvalue, $\eta_\sigma = 0.764$, is less than 1 the other u_i's involving only $\langle \delta J_\alpha \delta J_\beta \rangle$ are "irrelevant" variables and we say that $P(J)$ is therefore an "irrelevant" function. Thus the pure-system fixed point is stable with respect to an infinitesimal amount of randomness in J, and the critical exponents are those of the homogeneous system. For a small initial width we can calculate the shift in J^* (i.e., in T_c) by the condition $u_T = 0$, for which we use the eigenvector corresponding to η_T:

$$u_T = x_1 - 0.546x_2 - 0.055x_3 + 0.605x_4 + 0.321x_5 = 0. \tag{11}$$

This gives $\xi \equiv J^* dJ^* / d\langle \delta J_{AB} \delta J_{AB} \rangle = J^* dJ^* / dx_2 = 0.199$ compared to the exact result[8] $\xi = 0.183$.

For the three-cell approximation we retain only correlations $\langle \delta J_\mu' \delta J_\nu' \rangle$ which involve overlapping clusters. We therefore consider the variables x_1–x_5 above and also $x_6 = \langle \delta J_{AB} \delta J_{CL} \rangle$, x_7

TABLE I. Matrix $\underline{M}$ of Eq. (10) for the pair cluster approximation.

1.542	−0.554	0.302	1.108	0.625
0	0.541	0.785	0.491	0.390
0	0.047	0.452	0.367	0.106
0	0.012	0.098	0.121	0.121
0	0.010	0.021	0.123	0.051

$= \langle \delta J_{BC} \delta J_{DE} \rangle$, $x_8 = \langle \delta J_{AB} \delta_{DE} \rangle$, and $x_9 = \langle \delta J_{BF} \delta J_{DE} \rangle$. If φ^{ABC} is the renormalization function obtained using the cluster of cells A, B, and C, and φ^{AB} that using the two-cell cluster A, B, then following NL we set

$$\varphi_{AB} = \varphi_{AB}{}^{ABC} + \varphi_{AB}{}^{ABF} - \varphi_{AB}{}^{AB}. \quad (12)$$

Such a prescription is necessary inasmuch as any interaction can be considered as a part of two different triangular clusters. Apart from η_T which is again the same as for the homogeneous system ($\eta_T = 1.500$), the largest eigenvalue is $\eta_\sigma \simeq 0.758$, again showing $P(J)$ to be "irrelevant." The more widely separated correlations x_6–x_9 have only a minor effect on η_σ: Ignoring them gives $\eta_\sigma = 0.768$. The shift in T_c is found, using the left eigenvector of η_T, to be $\xi = 0.124$. It appears, then, that while our estimates of η_σ are accurate, those of the left eigenvectors are somewhat unreliable and an accurate value of ξ could only be obtained from a rather large cluster. A similar poor result for the critical eigenvector was also obtained by NL. As is well known, eigenvectors are harder to approximate than are eigenvalues.

In the ϵ expansion, recursion relations for potentials in spatially inhomogeneous systems are developed. From these, recursion relations for translationally invariant average potentials and higher cumulants can be obtained as in Eq. (8). We find that, in addition to the quadratic potential r and the quartic potential u, only the zero-wave-number part of the variance, Δ, of the quadratic potential is relevant. The recursion relations for r, u, and Δ are similar to the recursion relations for a spin model with cubic symmetry[9] with Δ playing the role of the hypercubic potential. As in that model, the recursion relations for the random model have four fixed points within the ϵ expansion. In this model, these are the Gaussian fixed points with $u^* = \Delta^* = 0$, the Heisenberg fixed point for the pure system with $u^* = \epsilon[8K_d(n-1)]^{-1}$, $\Delta^* = 0$, an unphysical fixed point with $u^* = 0$, $\Delta^* = -\epsilon[4K_d]^{-1}$, and a randomness-dominated fixed point with $u^* = \epsilon[16K_d(n-1)]^{-1}$ and $\Delta^* = \epsilon[4-n][8K_d(n-1)]^{-1}$, where $K_d{}^{-1} = 2^{d-1}\pi^{d/2}\Gamma(d/2)$. The unphysical fixed point is always stable but inaccessible since Δ must be positive. For $n > 4$, the pure-system fixed point is stable with the Δ exponent given by $\lambda_\Delta = \epsilon(4-n)/(n+8) = 2\alpha$. For $1 < n < 4$, the new fixed point is stable with exponents $\lambda_1 = \frac{1}{4}\epsilon(n-4)/(n-1)$, $\lambda_2 = -\epsilon$, $2\nu = 1 + [3n/16(n-1)]\epsilon$, and $\alpha = \frac{1}{8}\epsilon(n-4)/(n-1)$. Thus, the heuristic argument seems to apply for $n > 4$, possibly because the variable Δ is linearly proportional to the variance in the local transition temperature $\langle \delta T_c(x) \delta T_c(x) \rangle$ at point x. For $1 < n < 4$, there is a sharp transition. However, α has a renormalized value less than zero. The status of the result for n near or equal to 1 is uncertain and is currently under investigation. Preliminary calculations to second order in ϵ indicate that for the pure system fixed point λ_Δ is equal[10] to α/ν to this order in ϵ, again in agreement with the heuristic argument.

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