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Connection Between Percolation and Lattice Animals

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Abstract

An n-state Potts lattice gas Hamiltonian is constructed whose partition function is shown to reproduce in the limit n→0 the generating function for the statistics of either lattice animals or percolating clusters for appropriate choices of potentials. This model treats an ensemble of *single* clusters terminated by weighted perimeter bonds rather than clusters distributed uniformly throughout the lattice. The model is studied within mean-field theory as well as via the ε expansion. In general, cluster statistics are described by the lattice animal's fixed point. The percolation fixed point appears as a multicritical point in a space of potentials not obviously related to that of the usual one-state Potts model.

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Connection between percolation and lattice animals

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An n-state Potts lattice gas Hamiltonian is constructed whose partition function is shown to reproduce in the limit $n \rightarrow 0$ the generating function for the statistics of either lattice animals or percolating clusters for appropriate choices of potentials. This model treats an ensemble of *single* clusters terminated by weighted perimeter bonds rather than clusters distributed uniformly throughout the lattice. The model is studied within mean-field theory as well as via the ϵ expansion. In general, cluster statistics are described by the lattice animal's fixed point. The percolation fixed point appears as a multicritical point in a space of potentials not obviously related to that of the usual one-state Potts model.

I. INTRODUCTION

In the problem of bond percolation,[1,2] bonds on a lattice of N sites and $\frac{1}{2}zN$ bonds are occupied with probability p and vacant with probability $1-p$. For any given value of p, there are many different arrangements of the $\frac{1}{2}pNz$ occupied bonds, constituting an ensemble of states of the lattice. Clusters are formed of adjacent sites connected by occupied bonds. The statistics of percolating clusters can be described in terms of $B(n,p)$, the average number of clusters per site containing n sites. There are two ways of calculating this quantity. In the first, one constructs all possible states of the lattice for each p, counts all clusters with n sites for each state, divides by N, and averages over the ensemble of possible states. To calculate $B(n,p)$ in this way one uses information about the whole lattice. In the second but equivalent approach,[3,4] one determines the number $A(n_b, n_p, n)$ of distinct clusters (sometimes called animals[5]) having n_b bonds, n_p perimeter bonds (unoccupied bonds adjacent to occupied bonds), and n sites. Then

$$B(n,p) = \sum_{n_p, n_b} A(n_b, n_p, n)\, p^{n_b} q^{n_p} . \qquad (1.1)$$

In this formulation, one only needs information about conformations of a single cluster. It is isolated from other clusters by unoccupied perimeter bonds. Note that in this formulation it is natural to relax the constraints $q = 1-p$ and assume $p, q > 0$ to obtain generating functions for A.

The free energy of the one-state Potts model in a field yields directly the generating function for $B(n,p)$ and is interpreted most directly in terms of the first counting procedure.[6,7,8] The problem of finding $A(n_b, n_p, n)$ is very similar to the problem of finding the number of configurations of dilute branched polymers with n monomers.[9] A field-theoretic formulation of the latter problem in terms of an nm-component field ψ, was introduced in Ref. 9. In that paper, the dilute limit was obtained by allowing m, the polymer fugacity, to go to zero. In this paper, we will introduce an n-state Potts lattice gas model which in the limit $n \rightarrow 0$ will yield directly the generating function $A(n_b, n_p, n)$. Thus the Potts lattice gas in the $n \rightarrow 0$ limit reproduces the information contained in the s-state Potts model in the $s \rightarrow 1$ limit.

We study the Potts lattice gas model both in mean-field theory and also with the aid of the ϵ expansion. The mean-field theory reduces exactly to that obtained using the one-state Potts model. The structure of the ϵ expansion recursion relations is identical to that of dilute branched polymers discussed in Ref. 9. In particular, for most functional relations between p and q, there is deviation from mean-field behavior for $d < 8$ described by the $(8-\epsilon)$-dimensional "animals" fixed point. For special choices of potentials (in particular for $q = 1-p$), the animals fixed point is not reached. In this case, mean-field theory remains valid down to six dimensions, and one seeks a fixed point in $6-\epsilon$ dimensions controlled by third-order potentials in the field theory. In the treatment of the percolation problem based on the one-state Potts model, there is a single third-order potential leading to a single $(6-\epsilon)$-dimensional fixed point.[10] In the present treatment based on the n-state lattice gas, there are three third-order potentials and three fixed points. The most stable fixed point describes the statistics of branched polymers in θ solvents and was analyzed in Ref. 9. The next most stable fixed point describes percolation though its relation to the Potts-model fixed point is not obvious. Thus percolation emerges as a multicritical point that is unstable with respect to both a second-order potential leading to the animals (or dilute branched polymers) fixed point and also a third-order potential leading to the θ-solvent branched-polymer fixed point.

II. FORMULATION OF A FIELD THEORY FOR CLUSTER STATISTICS

We now construct a generating function, Z, which counts various cluster properties as follows. Associate with each site $\vec{x}$ or a lattice of N sites an n-state variable $\nu(\vec{x}) = 1, 2, \ldots, n$ and an occupation variable $p(\vec{x})$ such that $p(\vec{x}) = 1$ if the site $\vec{x}$ is occupied and $p(\vec{x}) = 0$ if the site is unoccupied. Now we define the "partition function" to be

$$Z = \mathrm{Tr} \prod_{\langle \vec{x}, \vec{x}' \rangle} \left([1 + v p(\vec{x}) p(\vec{x}') \delta_{\nu(\vec{x}), \nu(\vec{x}')}] \{\exp[-K_2 p(\vec{x}) p(\vec{x}') - K_1 p(\vec{x}) q(\vec{x}') - K_1 q(\vec{x}) p(\vec{x}')]\} \right) \times \prod_{\vec{x}} [n^{-1} q(\vec{x}) + e^{-H} p(\vec{x})] , \tag{2.1}$$

where $q(\vec{x}) = 1 - p(\vec{x})$ is the indicator for vacant sites, $\langle \vec{x}, \vec{x}' \rangle$ indicates that the product is taken over pairs of nearest-neighboring sites, δ is the Kronecker delta, and the trace is over all values of $\{p(\vec{x})\}$ and $\{\nu(\vec{x})\}$. The factor of n^{-1} in the last term of Eq. (2.1) ensures that the trace over $\nu(\vec{x})$ for unoccupied sites yields unity.

One can associate Z with a sum over graphs, $\mathcal{G}$, by associating the factor $p(\vec{x}) e^{-H}$ with occupied sites and the factor $p(\vec{x}) p(\vec{x}') \delta_{\nu(\vec{x}), \nu(\vec{x}')}$ with occupied bonds. The most general graph is constructed by selecting a set of occupied bonds and a set of occupied sites. The set of occupied sites must include all sites interesecting occupied bonds, but can also include arbitrary numbers of "isolated" sites, i.e., sites to which no occupied bonds are connected. An example of a graph with its associated counting variables is given in Fig. 1. One can express Z as

$$Z = \sum_{\mathcal{G}} n^{N_c(\mathcal{G})} (v e^{-K_2})^{N_b(\mathcal{G})} e^{-K_2 N_{pi}(\mathcal{G})} \times e^{-K_1 N_{pe}(\mathcal{G})} e^{-H N_s(\mathcal{G})} , \tag{2.2}$$

where $N_c(\mathcal{G})$, $N_b(\mathcal{G})$, $N_{pi}(\mathcal{G})$, $N_{pe}(\mathcal{G})$, and $N_s(\mathcal{G})$ are, respectively, the number of clusters, bonds, internal perimeter bonds, external perimeter bonds, and sites in the graph $\mathcal{G}$. A cluster is a group of occupied sites connected by occupied bonds. A single isolated occupied site is considered to be a (one-site) cluster. Adjacent occupied sites need not be in the same cluster. Internal perimeter bonds are unoccupied bonds which connect two occupied sites whether or not the two sites are in the same cluster. Exterior perimeter bonds are unoccupied bonds which connect an unoccupied site to an occupied site.

In the limit $n \to 0$ one has

$$Z = 1 - n\mathcal{F} + O(n^2) , \tag{2.3}$$

where

$$-\mathcal{F} = N \sum_{C} A(N_b, N_{pi}, N_{pe}, N_s) (v e^{-K_2})^{N_b} \times e^{-K_2 N_{pi}} e^{-K_1 N_{pe}} e^{-H N_s} , \tag{2.4}$$

where $A(N_b, N_{pi}, N_{pe}, N_s)$ is the number of clusters per site having N_b bonds, N_{pi} internal perimeter bonds, N_{pe} external perimeter bonds, and N_s sites. In Eq. (2.4) the sum is over all clusters C, i.e., over all graphs consisting of a single cluster. If z denotes the coordination number of the lattice we note the relation

$$z N_s = 2N_b + 2N_{pi} + N_{pe} . \tag{2.5}$$

We now write $\mathcal{F}$ in terms of the partition function

$$\mathcal{F} = -\lim_{n \to 0} \frac{1}{n} \ln \mathrm{Tr} e^{-\mathcal{H}} \tag{2.6a}$$

$$\equiv -\lim_{n \to 0} \frac{1}{n} \ln Z^{(n)} , \tag{2.6b}$$

where

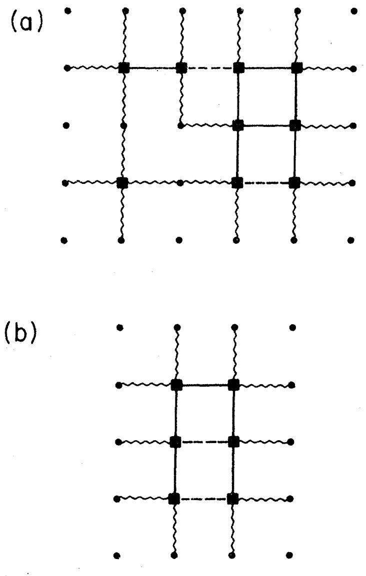

FIG. 1. (a) A graph, $\mathcal{G}$, on a square lattice with $N_c(\mathcal{G}) = 3$, $N_b(\mathcal{G}) = 7$, $N_{pi}(\mathcal{G}) = 2$, $N_{pe}(\mathcal{G}) = 18$, and $N_s(\mathcal{G}) = 9$. Occupied sites are indicated by dark squares, occupied bonds by full lines, interior perimeter bonds by dashed lines, and exterior perimeter bonds by wiggly lines. (b) A single cluster, C, on a square lattice ($z = 4$) with $N_b(C) = 5$, $N_{pi}(C) = 2$, $N_{pe}(C) = 10$, and $N_s(C) = 6$ with the same legend as in (a). One can verify Eq. (2.5).

$$-\mathcal{H}=\sum_{\langle\vec{x},\vec{x}'\rangle}[Kp(\vec{x})p(\vec{x}')\delta_{\nu(\vec{x}),\nu(\vec{x}')}-(K_2-2K_1)p(\vec{x})p(\vec{x}')]+\sum_{\vec{x}}\{[p(\vec{x})-1]\ln n-(H+zK_1)p(\vec{x})\}, \tag{2.7}$$

where

$$K=\ln(1+v). \tag{2.8}$$

According to Eq. (2.6) we may regard $\mathcal{F}$ as the free energy corresponding to $\mathcal{H}$. One sees that $\mathcal{H}$ depends on three independent linear combinations of the original parameters K, K_1, K_2, and H, in conformity with the constraint of Eq. (2.5). It is interesting to note that this Hamiltonian describes a lattice gas Potts model which has received considerable attention lately.[11,12] It differs from the usual models, however, in that the chemical potential for site vacancy becomes infinite as $\ln n^{-1}$ in the limit $n\rightarrow 0$.

We develop a field theory as follows. We introduce a complete orthonormal set of n unit vectors $\vec{a}_l$ for $l=0,1,\ldots,(n-1)$: $\vec{a}_l=(a_l^1,a_l^2,\ldots,a_l^n)$, and we set $\vec{a}_0=n^{-1/2}(1,1,\ldots,1)$. The $\vec{a}$'s obey

$$\sum_l a_l^\nu a_l^{\nu'}=\delta_{\nu,\nu'}, \tag{2.9a}$$

$$\sum_\nu a_l^\nu a_{l'}^\nu=\delta_{l,l'}. \tag{2.9b}$$

We write

$$S_l(\vec{x})=p(\vec{x})a_l^{\nu(\vec{x})} \tag{2.10}$$

in which case $\mathcal{H}$ can be written as

$$\mathcal{H}=-\tfrac{1}{2}\sum_{\vec{x},\vec{x}',l}K_l(\vec{x},\vec{x}')S_l(\vec{x})S_l(\vec{x}')+n^{1/2}H'\sum_{\vec{x}}S_0(\vec{x})+\ln n\sum_{\vec{x}}[p(\vec{x})-1], \tag{2.11}$$

where

$$H'=H+zK_1, \tag{2.12a}$$

$$K_l(\vec{x},\vec{x}')=[K-n(K_2-2K_1)\delta_{l,0}]\gamma_{\vec{x},\vec{x}'}, \tag{2.12b}$$

where $\gamma_{\vec{x},\vec{x}'}$ is zero unless $\vec{x}$ and $\vec{x}'$ are nearest-neighboring sites, in which case it is unity.

We now use the Hubbard[13] transformation to write $Z^{(n)}$ to leading order in n as

$$Z^{(n)}=\int\Big(\prod_{l,\vec{x}}d\psi_l(\vec{x})\Big)\exp\Big(-\tfrac{1}{2}\sum_{\vec{x},\vec{x}',l}K_l(\vec{x},\vec{x}')\psi_l(\vec{x})\psi_l(\vec{x}')-n^{1/2}H'\sum_{\vec{x}}\psi_0(\vec{x})-\frac{1}{2}\frac{nN}{zK}(H')^2-\sum_{\vec{x}}\mathcal{S}(\vec{x})\Big) \tag{2.13a}$$

$$\equiv\int\Big(\prod_{l,\vec{x}}d\psi_l(\mathrm{x})\Big)e^{-F(\{\psi\})}, \tag{2.13b}$$

where

$$e^{-\mathcal{S}(\vec{x})}=\operatorname*{Tr}_{\nu(\vec{x}),p(\vec{x})}\exp\Big(-q(\vec{x})\ln n+\sum_{l,\vec{x}_1}K_l(\vec{x},\vec{x}_1)\psi_l(\vec{x}_1)S_l(\vec{x})\Big). \tag{2.14}$$

Explicitly we may write

$$F(\{\psi\})=\tfrac{1}{2}\sum_{\vec{x},\vec{x}',l}K_l(\vec{x},\vec{x}')\psi_l(\vec{x})\psi_l(\vec{x}')+\sum_{\vec{x}}\ln\Big[1+\sum_\nu\exp\Big(\sum_{l,\vec{x}'}K_l(\vec{x},\vec{x}')\psi_l(\vec{x}')a_l^\nu\Big)\Big]+n^{1/2}H'\sum_{\vec{x}}\psi_0(\vec{x})+\frac{1}{2}\frac{nN}{zK}(H')^2. \tag{2.15}$$

We have thus constructed a field-theoretic expression for $Z^{(n)}$ in terms of a free-energy functional F.

III. MEAN-FIELD THEORY

We generate mean-field theory by minimizing F with respect to a spatially uniform $\psi_l(\vec{x})$. It is convenient to write

$$\psi_l(\vec{x})=-n^{1/2}\Psi\delta_{l,0}. \tag{3.1}$$

With this choice of sign, Ψ is positive for the ordered phase of the percolation model. The mean-field free energy per site, $F_{\mathrm{MF}}\equiv\lim_{n\rightarrow 0}(nN)^{-1}F^{(n)}$, is then given by

$$F_{\mathrm{MF}}=\tfrac{1}{2}zK\Big(\Psi-\frac{H'}{zK}\Big)^2-e^{-zK\Psi}. \tag{3.2}$$

This result is identical to that obtained previously[8] for the very similar model phrased in terms of a Potts Hamiltonian. The mean-field value of Ψ must minimize F_{MF}, so we require

$$zK\Psi-H'+zKe^{-zK\Psi}=0. \tag{3.3}$$

The critical point occurs when $\chi^{-1}=0$, i.e., when

$$\frac{\partial^2F_{\mathrm{MF}}}{\partial\Psi^2}\equiv\chi^{-1}=zK(1-zKe^{-zK\Psi})=0 \tag{3.4}$$

simultaneously with the constraint of Eq. (3.3). The critical surface is thus defined by

$$1 - zK + \ln zK = H + z(K_1 - K) \equiv \bar{H} . \tag{3.5}$$

Thus the critical surface occurs for $\bar{H} < 0$, except at $zK = 1$, when $\bar{H} = 0$.

Near the critical point we can determine the behavior of χ by expanding in powers of $\Delta K = K_c - K$ and $\Delta\Psi = \Psi - \Psi_c$, where K_c and Ψ_c are the values of K and Ψ at the transition. Thus

$$\chi^{-1} \sim z\Delta K(1 - zK_c\Psi_c) + (zK_c)^3\Delta\Psi . \tag{3.6}$$

If we solve the constraint equation near the critical point, we find that

$$\Delta\Psi = \left[2\Delta K\left(1 - zK_c \frac{dK_1}{dK}\right) + O(\Delta K^2)\right]^{1/2} \tag{3.7}$$

so that

$$\chi \sim (\Delta K)^{-\gamma} , \tag{3.8}$$

where $\gamma = \frac{1}{2}$ except when the critical point is approached along a path for which

$$zK_c \frac{dK_1}{dK} = 1 . \tag{3.9}$$

Setting

$$e^{-K_1} = q , \tag{3.10a}$$

$$e^{K} = (p+q)/q , \tag{3.10b}$$

we see that the critical path for which $\gamma = 1$ is defined by

$$1 - \frac{q}{p+q}\left(1 + \frac{dp}{dq}\right) = \left(z \ln \frac{p+q}{q}\right)^{-1} \tag{3.11}$$

which is satisfied for percolation, for which

$$p + q = 1 \tag{3.12a}$$

$$p_c = 1 - \exp(-z^{-1}) . \tag{3.12b}$$

More generally, one can have $p + q \neq 1$ but still retain percolationlike behavior in mean-field theory, providing Eq. (3.11) holds true along the path approaching the critical point.

IV. ϵ EXPANSION

We now study the effects of fluctuations. We therefore write

$$\psi_l(\vec{x}) = -n^{1/2}\Psi\delta_{l,0} + \varphi_l(\vec{x}) \tag{4.1}$$

and require

$$\langle\varphi_l(\vec{x})\rangle = 0 , \tag{4.2}$$

where $\langle\;\rangle$ denotes an average with respect to the weight function $\exp[-F(\Psi)]$. Using the substitution, Eq. (4.1), we obtain F as an expansion in powers of φ, $F = \sum_m F_m$, where F_m contains all terms in F of order φ^m. F_0 is the mean-field value of F. Also

$$F_1 = n^{1/2}(\Gamma_1 - \bar{H}) \sum_{\vec{x}} \varphi_0(\vec{x}) , \tag{4.3}$$

where

$$\Gamma_1 = zK(\Psi - 1 - e^{-zK\Psi}) \tag{4.4}$$

and

$$\begin{aligned} F_2 = {} & \tfrac{1}{2} \sum_{\vec{x},\vec{x}',l} (K\gamma_{\vec{x},\vec{x}'} - K^2 e^{-zK\Psi}\gamma^2_{\vec{x},\vec{x}'})\varphi_l(\vec{x})\varphi_l(\vec{x}') \\ & + \tfrac{1}{2} \sum_{\vec{x},\vec{x}'} [(2K_1 - K_2)\gamma_{\vec{x},\vec{x}'} + K^2 e^{-2zK\Psi}\gamma^2_{\vec{x},\vec{x}'} \\ & \qquad - 2K(K_2 - 2K_1)e^{-zK\Psi}\gamma^2_{\vec{x},\vec{x}'}]\,\varphi_0(\vec{x})\varphi_0(\vec{x}') \end{aligned} \tag{4.5}$$

and

$$\begin{aligned} F_3 = z^{-3} & \sum_{\vec{x}_1,\ldots,\vec{x}_4} \gamma_{\vec{x}_1,\vec{x}_4}\gamma_{\vec{x}_2,\vec{x}_4}\gamma_{\vec{x}_3,\vec{x}_4} \\ & \times \sum_{l_1,l_2,l_3} \varphi_{l_1}(\vec{x}_1)\varphi_{l_2}(\vec{x}_2)\varphi_{l_3}(\vec{x}_3)(-\tfrac{1}{6}w_0\lambda_{l_1,l_2,l_3} + n^{1/2}u_0\delta_{l_1,0}\delta_{l_2,l_3} + \tfrac{1}{6}n^{3/2}v_0\delta_{l_1,0}\delta_{l_2,0}\delta_{l_3,0}) , \end{aligned} \tag{4.6}$$

where γ^k is the kth power of the matrix γ, and the coefficients in F_3 are given by

$$z^{-3}w_0 = K^3 e^{-zK\Psi} \tag{4.7a}$$

$$z^{-3}u_0 = \tfrac{1}{2}K^3 e^{-2zK\Psi} + \tfrac{1}{2}K^2(K_2 - 2K_1)e^{-zK\Psi} \tag{4.7b}$$

$$z^{-3}v_0 = \tfrac{1}{3}K^3 e^{-3zK\Psi} - \tfrac{1}{2}K(K_2 - 2K_1)^2 e^{-zK\Psi} - K^2(K_2 - 2K_1)e^{-2zK\Psi} \tag{4.7c}$$

and

$$\lambda_{l,m,n} = \sum_{\nu} a_l^{\nu} a_m^{\nu} a_n^{\nu} . \tag{4.8}$$

We are thus led to consider a continuum field theory with a free-energy functional of the form

$$\begin{aligned} \mathcal{H} = \int d\vec{x}\Big(& \tfrac{1}{2}r\sum_l \varphi_l(\vec{x})^2 + \tfrac{1}{2}nT\varphi_0(\vec{x})^2 + \tfrac{1}{2}\sum_l |\nabla\varphi_l(\vec{x})|^2 + n^{1/2}u_0\sum_l \varphi_l(\vec{x})^2\varphi_0(\vec{x}) \\ & - \tfrac{1}{6}w_0 \sum_{l_1,l_2,l_3} \lambda_{l_1,l_2,l_3}\varphi_{l_1}(\vec{x})\varphi_{l_2}(\vec{x})\varphi_{l_3}(\vec{x}) + \tfrac{1}{6}n^{3/2}v_0\varphi_0(\vec{x})^3\Big) . \end{aligned} \tag{4.9}$$

Terms of higher order in φ or $\nabla\varphi$ are irrelevant to our analysis. This model is identical to that studied by Lubensky and Isaacson[9] and we will use the recursion relations they give for the potentials appearing in Eq. (4.9).

As they showed, there is a fixed point in $8-\epsilon$ dimensions describing the non-mean-field behavior of dilute polymer statistics. Here we study the multicritical fixed point in $6-\epsilon$ dimensions which reproduces the known results[10] for the percolation problem. For this purpose we follow Ref. 9 and set

$$x=K_d w^2\,, \tag{4.10a}$$

$$y=K_d u w\,, \tag{4.10b}$$

$$z=K_d(v w^3)^{1/2}\,, \tag{4.10c}$$

where $K_d^{-1}=2^{d-1}\pi^{d/2}\Gamma(d/2)$ in terms of which the recursion relation are[9]

$$\frac{dx}{dl}=\epsilon x+\tfrac{3}{2}x^2-20xy\,, \tag{4.11a}$$

$$\frac{dy}{dl}=\epsilon y+\tfrac{3}{2}xy-22y^2+\tfrac{1}{2}z^2\,, \tag{4.11b}$$

$$\frac{dz}{dl}=\epsilon z+xz-26yz+56\,\frac{y^3}{z}-6x\,\frac{y^2}{z}\,, \tag{4.11c}$$

and also

$$\eta=\tfrac{1}{6}x-\tfrac{4}{3}y\,. \tag{4.12}$$

These equations have three fixed points which may be classified according to their degree of stability as determined by the signs of the stability exponents. The most stable fixed point, at which all three stability exponents are negative, was analyzed in Ref. 9 and describes the statistics of dilute branched polymers in θ solvents. The next stable fixed point, which was not analyzed in Ref. 9, occurs at

$$x=\tfrac{2}{7}\epsilon\,, \tag{4.13a}$$

$$y=\tfrac{1}{14}\epsilon\,, \tag{4.13b}$$

$$z=\tfrac{1}{7}\epsilon\,, \tag{4.13c}$$

with

$$\eta=-\tfrac{1}{21}\epsilon\,. \tag{4.14}$$

The stability exponents λ_1, λ_2, and λ_3 are the three roots of the equation

$$\lambda^3+17\lambda^2+62\lambda-56=0\,. \tag{4.15}$$

Two roots of this equation are negative and one is positive, indicating stability in two directions and instability in a third. We will comment on this in a moment.

The stability exponents associated with the temperaturelike variables r and T can be found from Eq. (6.51) of Ref. 9 and are

$$\mu_1=\nu^{-1}=2-\tfrac{5}{21}\epsilon\,, \tag{4.16a}$$

$$\mu_2=2-\tfrac{11}{21}\epsilon\,. \tag{4.16b}$$

The interpretation of μ_2 is that it gives the anisotropy crossover exponent φ:

$$\varphi=\mu_2/\mu_1=1-\tfrac{1}{7}\epsilon\,. \tag{4.17}$$

From these results one obtains the known results for the other exponents, e.g.,

$$\beta=1-\tfrac{1}{7}\epsilon\,, \tag{4.18a}$$

$$\gamma=1+\tfrac{1}{7}\epsilon\,. \tag{4.18b}$$

Zia and Wallace[14] have shown that $\varphi=\beta$ holds true to all orders in the ϵ expansion.

The exponents η, ν, γ, β, and φ are identical to those calculated for percolation from the Potts model.[9,14] Since we know that the present model must have a percolation fixed point, the conclusion of this analysis is that the fixed point of Eq. (4.13) is a multicritical point in the space of potentials appearing in the Hamiltonian of Eq. (4.9). In the case of the generalized Potts model[8] we have previously shown that this multicritical point is unstable with respect to anisotropy and Eq. (4.17) is a reflection of this instability which takes one from percolation to dilute polymers. However, in the present context, the fixed point is also unstable with respect to the single direction in the space of the potentials u, v, and w corresponding to the positive root in Eq. (4.15). We assert without proof that the percolation Hamiltonian must correspond to a point in this space on the surface defined by the vanishing of the unstable cubic potential. This condition is clearly required if the flow is to take the initial Hamiltonian to the fixed point which we have shown does indeed describe percolation. Moving off this surface seems to lead to a crossover to the fixed point describing dilute branched polymers in θ solvents, but we do not have any clear physical picture of the consequences of this phenomenon.

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