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Abstract
It is shown that, unlike the case of the distribution of currents in a random resistor network, all negative moments of the harmonic measure in diffusion-limited aggregates exhibit power-law scaling with sample size.

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Scaling of the negative moments of the harmonic measure in diffusion-limited aggregates

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It is shown that, unlike the case of the distribution of currents in a random resistor network, all negative moments of the harmonic measure in diffusion-limited aggregates exhibit power-law scaling with sample size.

Recently it has been established[1] that the multifractal picture[2-4] does *not* completely describe the scaling behavior of the current distribution in a randomly diluted resistor network at the critical concentration for percolation. In that case, one studies the moments of the current distribution $M_q(L) \equiv \sum_b' [i_b^{2q}]_{\rm av}$, where i_b is the current in bond b when a unit current is inserted at $\mathbf{x}$ and removed at $\mathbf{x}'$, and $[\;]_{\rm av}$ denotes an average over all configurations of occupied (i.e., unit resistance) bonds which occur with probability p, and unoccupied (i.e., infinite resistance) bonds that occur with probability $1-p$. Also, L denotes the size of the sample with $|\mathbf{x}-\mathbf{x}'| \sim L$, and $\sum_b'$ indicates omission in the sum over bonds of terms with $i_b=0$ (this restriction is necessary if $q<0$). For non-negative q, $M_q(L) \sim L^{\psi_q/\nu}$, where the ψ_q's are noise exponents[5] equivalent to those previously introduced by Rammal and co-workers[6] and de Arcangelis, Redner, and Coniglio.[4] For q less than some critical value q_c (with $q_c<0$), it was shown[1] that $\ln M_q(L)$ grew with increasing L at least as fast as

$$A(q)L^y , \tag{1}$$

where y was an exponent greater than one. Thus, for $q<q_c$ $(q>q_c)$, the current moments exhibit exponential (power-law) scaling with size L.

The purpose of this Brief Report is to point out that for the case of the growth probability for diffusion-limited aggregates[7] (DLA) there is no similar range of negative moments of the growth probability having non-power-law scaling. To see this, it is useful to recall[1] that the exponential behavior of $M_q(L)$ as in Eq. (1) in the resistor network comes from the existence of ladders shown in Fig. 1. A ladder of l rungs occurs with probability $P(l)$, with

$$\ln P(l) \sim l \ln p = -A(p)l , \tag{2a}$$

and the minimal current (which dominates the negative moments) decreases with l as

$$\ln(i_{\rm min}^2) = -Bl , \tag{2b}$$

where B is given in Ref. 1. The contribution of such a current to M_{-q} is then of order $\exp\{[Bq-A(p)]l\}$, which is dominant if $Bq>A(p)$, as happens for large enough q. Since $l \sim L^y$, one obtains Eq. (1).

For DLA, one introduces the growth probability $P(S_i,C)$ for boundary sites S_i of a cluster C as follows.[7] One considers the growth of a cluster C on a hypercubic lattice starting from a seed C_1 consisting of a single occupied site at the origin. The growth process consists of adding a particle by allowing it to diffuse from a release point randomly chosen on the inner surface of an infinitely large sphere. When the particle first reaches a site S neighboring to the existing cluster, the particle sticks to and increments the size of the existing cluster. Diffusion, in this case, is modeled by a random walk on the hypercubic lattice. At any stage, one can define a probability $P(S_i;C)$ that a particle be added to any one of the available growth sites, S_i of the cluster C. Note that we only consider sites S_i which have a nonzero probability P for growth. In this way, we generate the probability measure for growth, from which we can obtain the normalized growth probability $W(C_i \rightarrow C_{i+1})$ that a cluster C_{i+1} be formed by adding a site to cluster C_i. We can use this probability of growth to define precisely the probability weight $F_N(C)$ to be assigned to any cluster C of sites in the ensemble of all DLA clusters having N sites. To do this note that normally a cluster C of N sites can be grown from the initial seed

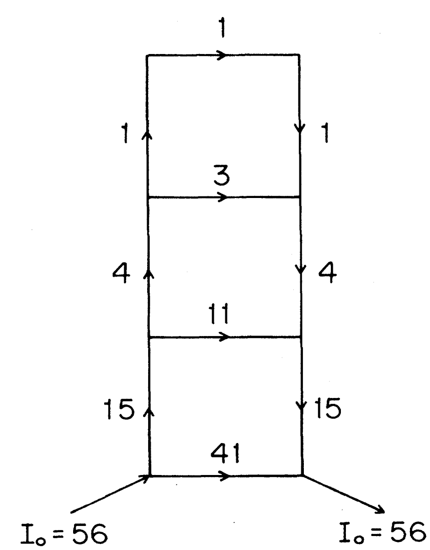

FIG. 1. Ladders which dominate the negative current moments. Each bond connecting nodes has unit resistance. When a unit current is inserted and removed at the bottom of the ladder, the current through the top-most rung is of order $\exp(-bl)$, where l is the number of rungs in the ladder and b is a constant (Ref. 1). Here we give the currents when the input current is normalized so that $i_{\rm min}=1$.

by many different growth sequences $\Gamma(C) \equiv C_1, C_2, C_3, \ldots, C_{N-1}, C$. Then

$$F_N(C) = \sum_{\Gamma(C)} \prod_{i=1}^{N-1} W(C_i \rightarrow C_{i+1}) , \tag{3}$$

where the sum is over all growth sequences producing the desired cluster C of N sites. Note that $F_N(C)$ is normalized in that when summed over all N-site clusters, it gives unity. We now define moments of the growth probability of DLA clusters of N sites by

$$M_q(N) = \left[\sum_S P(S;C)^q \right]_{\mathrm{av}} = \sum_C F_N(C) \sum_S P(S;C)^q , \tag{4}$$

where the sum over S is over all possible growth sites for a fixed cluster C, and the average is over the ensemble of DLA clusters of N sites with the probability weighting according to Eq. (3). For the non-negative moments, $M_q(N)$ with $q \geq 0$, one has $M_q(N) \sim N^{x(q)}$, where $x(q)$ is related to $D(q)$ used in Refs. 8 and 9. The obvious question now arises: Are there special configurations, analogous to the ladders in the resistor network, which can cause the negative moments to increase exponentially rapidly with increasing N?

Note that $P(S;C)$ becomes small when the site S is heavily screened by large branches from the diffusing particle. Then $P(S;C)$ is small by virtue of the small probability that the diffusing particle finds its way to S between the branches. An exceptionally small probability $P(S;C)$ occurs for the "tube" configuration shown in Fig. 2. Note that the probability for a particle to diffuse from the open end to the closed end of the tube (without sticking to the side) is of order $\exp(-aL)$, where a is a constant which can be obtained exactly, if desired, and L is the length of the tube. Assuming the probability that the diffusing particle reaches the opening of the tube to be of order N^{-1}, we find that the contribution to $M_{-q}(N)$ from the tube is of order $N^{-1} \exp(qa'N) F_N(C)$, where $a' = aL/N$ is of order unity. Here $F_N(C)$ is the probability weight of the tube according to Eq. (3). If $F_N(C)$ were of order $\exp(-bN)$, then the situation would be as for the random resistor network: For $q > q_c = b/a'$, $M_{-q}(N)$ would grow exponentially with N. Now we estimate $F_N(C)$ for the tube of Fig. 2, assuming the initial seed site to be at the bottom of the tube. Since the tube is simply a linear structure of N sites, it can be formed from at most 2^N growth sequences. We estimate that when there are N_g potential growth sites, the chance of adding a particle at one end is of order $1/N_g$. This estimate neglects the fact that the most exposed sites at the end of the tube are the ones where growth is most likely.[7,9] A rigorous estimate can be obtained by considering the harmonic measure, which is the continuum version of the growth probability. The harmonic measure is obtained by solving for the electric field distribution external to the cluster when the cluster is treated as a conductor carrying charge Q embedded in an infinite vacuum. The harmonic measure associated with any point on the surface of the cluster is equal to the electric field (which is normal to the surface) at that point if the charge is chosen in suitable units. For the present discussion we consider a straight linear cluster. In view of the cutoff implied by the lattice constant we consider a cylinder of unit radius and length L, whose ends are capped by hemispheres of unit radius. Using an electrostatic argument[10] one can establish that the ratio of the electric field near the end of the cylinder to that near the center of the cylinder remains finite in the limit $L \rightarrow \infty$. This result no doubt also holds for a bent chain or tube. This argument implies that the growth probability has an upper bound of order $(N_g)^{-1} \sim cN^{-1}$, where c is a constant. Thus for a tube of N sites we have

$$P(N) \leq 2^N \prod_{j=1}^{N} (c/j) = (2c)^N / N! , \tag{5}$$

and a similar lower bound with a different (smaller) value of c. Thus, we set $P(N) \sim \alpha^N / N!$, where α is a constant, and the contribution to $M_{-q}(N)$ from these tubes is of order $\delta M_{-q}(N) \sim \alpha^N \exp(qa'N)/N!$, which decreases faster than any finite power of N as N becomes large. Thus, this anomalous contribution to $M_{-q}(N)$ is completely negligible compared to the "normal" power-law contribution described by the multifractal approach.

One can phrase the above argument in more general terms. It is possible that I may have overlooked some special configurations that would reverse this conclusion. However, this seems unlikely: One can invoke arguments similar to those[11] which rule our superlocalization (in the chemical distance) to see that $-\ln P(S,C)$ has an upper bound which is linear in N. The whole question is then whether there is some way to invoke a structure for which $P(S,C)$ is exponentially small while $\ln F_N(C)$ is of order N (rather than of order $\ln N! \sim N \ln N$ as for the tubes). Since I do not think that this is possible, I suggest that for DLA, $M_{-q}(N)$ scales like a power of N for all finite q.

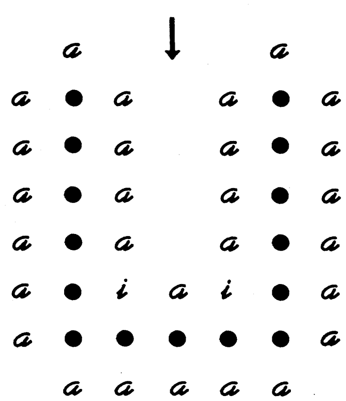

FIG. 2. The tube configuration of a DLA cluster. The filled dots represent occupied sites of the cluster. Sites labeled a are accessible perimeter sites at which the growth probability is nonzero and those labeled i are inaccessible sites. The arrow indicates the direction a diffusing particle would have to assume to reach the bottom of the tube. Inside the tube the particle could only reach the bottom of the tube if, at each step, it does not move transversely and stick to the side of the tube at an a site.

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[1]R. Blumenfeld, Y. Meir, A. Aharony, and A. B. Harris, Phys. Rev. B **35**, 3524 (1987).

[2]H. G. E. Hentschel and I. Procaccia, Physica D **8**, 435 (1983).

[3]T. C. Halsey, M. H. Jensen, L. P. Kadanoff, I. Procaccia, and B. I. Shraiman, Phys. Rev. A **33**, 1141 (1986).

[4]L. de Arcangelis, S. Redner, and A. Coniglio, Phys. Rev. B **31**, 4725 (1985).

[5]Y. Park, A. B. Harris, and T. C. Lubensky, Phys. Rev. B **35**, 5048 (1987).

[6]R. Rammal, C. Tannous, and A.-M. S. Tremblay, Phys. Rev. A **31**, 2662 (1985); R. Rammal, C. Tannous, P. Breton, and A.-M. S. Tremblay, Phys. Rev. Lett. **54**, 1718 (1985).

[7]T. A. Witten and L. M. Sander, Phys. Rev. Lett. **47**, 1400 (1981).

[8]T. C. Halsey, P. Meakin, and I. Procaccia, Phys. Rev. Lett. **56**, 854 (1986).

[9]P. Meakin, A. Coniglio, H. E. Stanley, and T. A. Witten, Phys. Rev. A **34**, 3325 (1986).

[10]I invoke an argument suggested by Professor Michael Cohen. In this argument one considers a semi-infinite cylinder of radius a (whose finite end, capped by a hemisphere, is at the origin) on which the charge per unit length is finite. The field near the origin due to all charges at distances from the end greater than some fixed distance is finite. In this regard, there is no singularity for a cylinder of length L as $L\rightarrow\infty$. Indeed, the finiteness of this field can be used to argue that the surface charge density on the hemispherical ends of a cylinder of length L remains finite in the limit $L\rightarrow\infty$. Thus, we expect the field at the surface of this capped cylinder to be less than $cQ/(\epsilon_0 aL)$ everywhere, where a is taken to be of order the lattice constant, c is some finite constant, and ϵ_0 is the permittivity of vacuum. We agree, that for a needle this quantity does diverge, as stated in Ref 9. However, the limit $a\rightarrow 0$ is not appropriate for the present discussion.

[11]A. B. Harris and A. Aharony, Europhys. Lett. **4**, 12 (1987).