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Series Analysis of Randomly Diluted Nonlinear Networks With Negative Nonlinearity Exponent

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Abstract
The behavior of randomly diluted networks of nonlinear resistors, for each of which the voltage-current relationship is $|V|=r|I|^{\alpha}$, where α is negative, is studied using low-concentration series expansions on d-dimensional hypercubic lattices. The average nonlinear resistance $\langle R \rangle$ between a pair of points on the same cluster, a distance r apart, scales as $r^{\zeta(\alpha)/\nu}$, where ν is the correlation-length exponent for percolation, and we have estimated $\zeta(\alpha)$ in the range $-1 \leq \alpha \leq 0$ for $1 \leq d \leq 6$. $\zeta(\alpha)$ is discontinuous at $\alpha=0$ but, for $\alpha<0$, $\zeta(\alpha)$ is shown to vary continuously from ζ_{max}, which describes the scaling of the maximal self-avoiding-walk length (for $\alpha \to 0-$), to ζ_{BB}, which describes the scaling of the backbone (at $\alpha=-1$). As α becomes large and negative, the loops play a more important role, and our series results are less conclusive.

Disciplines
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Comments
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Series analysis of randomly diluted nonlinear networks with negative nonlinearity exponent

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The behavior of randomly diluted networks of nonlinear resistors, for each of which the voltage-current relationship is $|V|=r|I|^{\alpha}$, where α is negative, is studied using low-concentration series expansions on d-dimensional hypercubic lattices. The average nonlinear resistance $\langle R\rangle$ between a pair of points on the same cluster, a distance r apart, scales as $r^{\zeta(\alpha)/\nu}$, where ν is the correlation-length exponent for percolation, and we have estimated $\zeta(\alpha)$ in the range $-1\leq\alpha\leq 0$ for $1\leq d\leq 6$. $\zeta(\alpha)$ is discontinuous at $\alpha=0$ but, for $\alpha<0$, $\zeta(\alpha)$ is shown to vary continuously from $\zeta_{\max}$, which describes the scaling of the maximal self-avoiding-walk length (for $\alpha\to 0-$), to $\zeta_{\rm BB}$, which describes the scaling of the backbone (at $\alpha=-1$). As α becomes large and negative, the loops play a more important role, and our series results are less conclusive.

Kenkel and Straley[1] proposed a model of networks in which bonds on percolation clusters consist of nonlinear resistors, each of which obeys the generalized Ohm's law

$$\Delta V=r|I|^{\alpha}\operatorname{sgn}I\ , \tag{1}$$

where ΔV is the voltage drop across the resistor, I is current flowing through it, r is the nonlinear resistance, and α is the exponent characterizing the nonlinearity. Interest in this model centers on the region of concentration p for $p\sim p_c$, where p_c is the critical concentration for percolation.

Blumenfeld and Aharony[2] showed that the nonlinear resistance $R_\alpha(L)$ between two terminals on the same cluster, a distance L apart, reduces to some geometrical characteristics of the cluster for specific values of α. For $\alpha\to 0+$ the nonlinear resistance describes the length of the minimal path between the two terminals, while for $\alpha\to\infty$ the resistance reduces to the number of singly connected bonds between the terminals. For $\alpha=1$ the nonlinear resistance reduces trivially to the linear one. Consequently, the exponent $\tilde{\zeta}(\alpha)$, which describes how the nonlinear resistance $R_\alpha(L)\sim L^{\tilde{\zeta}(\alpha)}$ scales with the distance L for $L\ll\xi$, where ξ is the percolation correlation length, should reduce in the above limits to the exponents that describe the scaling of the corresponding geometrical quantities. These results were confirmed by series expansions[3] and the ϵ expansion.[4]

Recently, we proposed[5] that the above results could be extended to negative α, where again the resistance for some particular values of α corresponds to other geometrical characteristics of the network. In particular, for $\alpha\to 0-$ the resistance reduces to the maximal self-avoiding path between the terminals; for $\alpha=-1$ the resistance describes the number of backbone bonds (the number of bonds that carry current), while for $\alpha\to-\infty$ the resistance scales with an exponent $z|\alpha|$, where z describes the scaling of the maximal "cutting surface" of the backbone between the terminals, i.e., the largest number of bonds, $N_{\max}$, which one can cut in order to break the backbone into two pieces, each connected to one terminal, $N_{\max}\sim L^z$. Based on available values for $\zeta(\alpha)\equiv\tilde{\zeta}(\alpha)\nu$, where ν is the exponent that describes the scaling of the correlation length, at $\alpha=1,0^+$, and -1, we constructed[5] an approximant function for $\zeta(\alpha)$, which is reproduced (for $d=2$) in Fig. 1.

In this work we carry on the series expansion described in Ref. 3 (later referred to as I) to negative α, in order to obtain estimates of $\zeta(\alpha)$ for all d and α.

The percolation susceptibility is defined by

$$\chi_p=\left[\sum_j \nu_{ij}\right]_{\rm av}\ , \tag{2}$$

where ν_{ij} is 1 if the two sites i and j belong to the same cluster and zero otherwise, and $[\ ]_{\rm av}$ denotes an average over all configurations of occupied and unoccupied bonds. The nonlinear resistive susceptibility is defined by[3,6]

$$\chi_R(\alpha)=\left[\sum_j R_{ij}(\alpha)\nu_{ij}\right]_{\rm av}\ , \tag{3}$$

where $R_{ij}(\alpha)$ is the nonlinear resistance between i and j. In Eq. (3) we interpret $R_{ij}(\alpha)\nu_{ij}$ to be zero when $\nu_{ij}=0$ and $R_{ij}(\alpha)=\infty$.

Section II of I describes in detail the construction of the

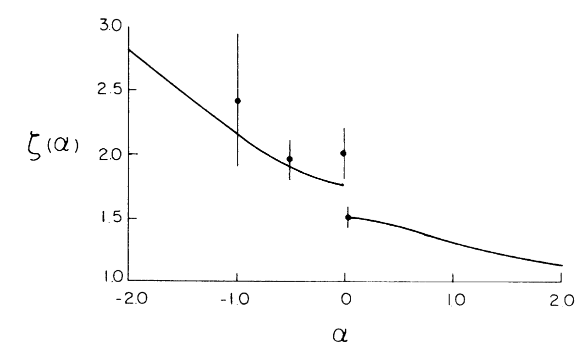

FIG. 1. Approximant function for $\zeta(\alpha)$ at $d=2$ (from Ref. 5) (solid line), and series results (solid circles).

series and we will not repeat it here. However, for negative α there is an additional complexity. As described in Ref. 5, one may find a number of solutions to the nonlinear Kirchhoff's equations, each of which corresponds to a given allowed assignment of directions of the currents on the bonds. Each solution corresponds to a local extremum of the power

$$P=\frac{1}{\alpha+1}\sum_b r_b\,|I_b|^{\alpha+1}=\frac{1}{\alpha+1}\sum_b r_b^{-\alpha^{-1}}|\Delta V_b|^{1+\alpha^{-1}}\ , \tag{4}$$

where the sum is over occupied bonds b. (Note that this definition is slightly different from that of the power in Ref. 5). Allowed regions Ω in voltage space correspond to choosing current directions such that ΔV is irrotational and has no internal sources or sinks. Each region Ω has a boundary B on which one or more of the ΔV_b's vanish, so that for $-1<\alpha<0$ P is infinite on B. Thus, for this range of α each region Ω has a local minimum of P corresponding to a solution to Kirchhoff's equations, $\partial P/\partial V_i=0$. Since there is *a priori* no reason to choose any one of these solutions as the "physical" one, we constructed the series both for the solution that gives the minimal power and for the one that gives the maximal power. We found no difference (within our error bars) in the estimates for the exponents for these two, although their amplitudes do differ. In this way one constructs a series in p and d,

$$\chi_R(\alpha)=\sum_{k,l}A(\alpha,k,l)p^k d^l\ . \tag{5}$$

For $\alpha=-1$ all the solutions coincide and we recovered the series for the backbone.[7] For $\alpha\rightarrow 0-$, the series for the nonlinear resistance, which give the maximal power, reduce to the series for the length of the maximal self-avoiding walk between the two terminals on the cluster, which we constructed independently. In Table I, we give the coefficients $A(\alpha,k,l)$ of the series for the nonlinear resistance susceptibility for $\alpha=-\frac{1}{2}$, based on the solution which minimizes the power P.

In order to analyze the series, we divided the coefficients of each series (for any α), by those of the series for the percolation susceptibility, χ_p, term by term. One can show (see, e.g., Ref. 8) that if two series diverge at the same critical point, the series that results from dividing one by the other, term by term, diverges at $p=1$ no matter what the value of the critical point was, with an exponent that is equal to the difference between the two exponents plus one. Since in our case we believe that the series diverge at p_c, and the difference between the exponents is $\zeta(\alpha)$, the constructed series is expected to diverge at $p=1$, with an exponent $\zeta(\alpha)+1$. In this way, we obtain an estimate for $\zeta(\alpha)$ which is not biased by the value of p_c nor by the value of γ_p, the exponent that describes the divergence of χ_p. The resulting series were analyzed by the nonhomogeneous differential Padé method.[9] The results of the analysis, however, are less conclusive than those for positive α. In the range $-1\le\alpha\le 0$ one can still obtain reasonable results, and those for $\alpha=-1,\ -\frac{1}{2},\ 0^-$, and for $2\le d\le 5$ are listed in Table II and shown (with the error bars) for $d=2$ in Fig. 1. For $d=1$ and at dimensions $d\ge 6$, one has $\zeta(\alpha)\equiv 1$ exactly. We also list the estimates obtained in I for $\zeta(0^+)$. The values of $\zeta(-\frac{1}{2})$ are very close to those of $\zeta(0^-)$, and this agrees with the prediction of Ref. 5 that the derivative of $\zeta(\alpha)$ vanishes as $\alpha\rightarrow 0-$. Also, as predicted in Ref. 5, there is a finite difference between $\zeta(0^+)$ and $\zeta(0^-)$ which corresponds to the difference between the exponents that describe the scaling with separation of the maximal and minimal self-

TABLE I. Series coefficients for χ_R as in Eq. (5) for $\alpha=-\frac{1}{2}$. $\chi_R(\frac{1}{2})=dp+(4d^2-2d)p^2+(12d^3-12d^2+3d)p^3+(32d^4-48d^3+12.828d^2+7.172d)p^4+\sum_{lm}A(\frac{1}{2},l,m)p^l d^m$.

l,m	$A(\frac{1}{2},l,m)$	l,m	$A(\frac{1}{2},l,m)$	l,m	$A(\frac{1}{2},l,m)$
5,5	80	5,4	−160	5,3	55.314
5,2	79.373	5,1	−49.686	6,6	192
6,5	−480	6,4	237.625	6,3	205.518
6,2	−22.836	6,1	−126.307	7,7	448
7,6	−1344	7,5	858.088	7,4	559.637
7,3	798.010	7,2	−2686.202	7,1	1373.467
8,8	1024	8,7	−3584	8,6	2918.624
8,5	1207.713	8,4	1548.465	8,3	1899.363
8,2	−9571.211	8,1	8363.771	9,9	2304
9,8	−9216	9,7	9266.144	9,6	2011.409
9,5	1692.313	9,4	25934.034	9,3	−145633.951
9,2	190601.658	9,1	−76950.607	10,10	5120
10,9	−23040	10,8	27822.079	10,7	1220.992
10,6	−2005.823	10,5	73309.420	10,4	−183113.648
10,3	−454888.320	10,2	1319107.895	10,1	−763522.594
11,11	11264	11,10	56320	11,9	79855.740
11,8	−9325.247	11,7	−19507.448	11,6	186423.664
11,5	494997.289	11,4	−7058955.097	11,3	17576754.921
11,2	−17049095.324	11,1	5843918.501		

TABLE II. Estimates of $\zeta(\alpha)$.

d	$\alpha=-1$	$\alpha=-\frac{1}{2}$	$\alpha=0^-$	$\alpha=0^+$
2	2.4 ± 0.5	1.95 ± 0.15	2.0 ± 0.2	1.51 ± 0.08
3	1.7 ± 0.2	1.4 ± 0.3	1.43 ± 0.06	1.15 ± 0.03
4	1.37 ± 0.18	1.17 ± 0.05	1.17 ± 0.02	1.08 ± 0.02
5	1.15 ± 0.15	1.1 ± 0.2	1.1 ± 0.1	1.03 ± 0.01
5[a]	1.14	1.12	1.11	1.04

[a]Numerical evaluations of ϵ expansion (Ref. 4) extended to negative α.

avoiding walks between the terminals, ζ_{max} and ζ_{min}, respectively. The value of ζ_{min}, obtained in I, agrees very well with the results of several workers (see Table I in I). On the other hand, we are aware of only one other work[10] which estimated ζ_{max} (by real-space renormalization group), and gave the value $\zeta_{max}=1.835$ in two dimensions, which is slightly smaller than our estimate, but still within the error range. Our results can also be compared with the continuation to negative α of the ϵ-expansion results,[4] where $\epsilon=6-d$. For instance, we find $\zeta_{max}\equiv\zeta(0^-)=1+(3\epsilon/28)$, $\zeta_{BB}\equiv\zeta(-1)=1+\epsilon/7$, and $\zeta(-\frac{1}{2})=1+(\epsilon/14)\times[1+2^{-1/2}\ln(1+2^{1/2})]$, each of which is given in Table II for $d=5$.

As α decreases towards $-\infty$, the series are dominated by the smaller currents, which run through the bonds in the loops. Our series, which involve clusters with up to 11 bonds, are not sensitive enough to the small currents. Therefore, as α decreases the analysis of the series becomes harder and the estimates obtained for the exponents are less trustworthy, especially for two dimensions, where the loops play the most significant role. One can see this already for $\alpha=-1$, where our results for ζ_{BB}, which agree with other estimates from series expansions,[7] are significantly larger than the value 2.16 ± 0.03 obtained from Monte Carlo simulations[11] in two-dimensions. Thus we could not check the prediction of Ref. 5 on the behavior of the exponent $\zeta(\alpha)$ as $\alpha\rightarrow-\infty$.

To conclude, we constructed and analyzed series for the nonlinear resistive susceptibility for negative α and confirmed the prediction of Ref. 5 that there will be a discontinuity in $\zeta(\alpha)$ at $\alpha=0$. (This discontinuity does not appear in structures with high symmetry, like those studied in Ref. 12.) We also confirmed the predictions that the resistance reduces to the length of the maximal self-avoiding path and the number of backbone bonds for $\alpha\rightarrow0-$ and $\alpha=-1$, respectively, and our results also agree well with the prediction that $d\zeta/d\alpha=0$ at $\alpha\rightarrow0+$. For α less than -1 our series did not lead to any conclusive results.

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