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Kripke Models for the Second-Order Lambda-Calculus

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Preliminary Version

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Abstract. We define a new class of Kripke structures for the second-order λ-calculus, and investigate the soundness and completeness of some proof systems for proving inequalities (rewrite rules) or equations. The Kripke structures under consideration are equipped with preorders that correspond to an abstract form of reduction, and they are not necessarily extensional. A novelty of our approach is that we define these structures directly as functors $A: \mathcal{W} \to \mathbf{Preor}$ equipped with certain natural transformations corresponding to application and abstraction (where $\mathcal{W}$ is a preorder, the set of worlds, and $\mathbf{Preor}$ is the category of preorders). We make use of an explicit construction of the exponential of functors in the Cartesian-closed category $\mathbf{Preor}^{\mathcal{W}}$, and we also define a kind of exponential $\prod_{\Phi}(A^s)_{s\in T}$ to take care of type abstraction. We obtain soundness and completeness theorems that generalize some results of Mitchell and Moggi to the second-order λ-calculus, and to sets of inequalities (rewrite rules).

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1 Introduction

In order to have a deeper and hopefully more intuitive understanding of various typed λ-calculi and their logical properties, it is useful to define and study classes of models for these calculi. Typically, given some typed λ-calculus, we are interested in reduction or conversion properties of this calculus, and the crucial properties of reduction and conversion are axiomatized by a proof system for deriving equations or rewrite rules (for example, β-conversion). Models will be useful only if they are sound with respect to the given proof system, in the sense that provable equations (or rewrite rules) must be valid. Then, models can be helpful for showing that a certain equation $M \doteq N$ is not derivable from a given set E of equations: it is sufficient to exhibit a model in which all equations in E are valid and in which $M \doteq N$ is falsified. Conversely, we can better calibrate the strength of a proof system if we can prove a completeness theorem. For example, we say that we have *strong completeness* if we can show that for any set E of equations and any equation $M \doteq N$, if $M \doteq N$ is valid in every model of the equations in E, then $M \doteq N$ is provable from E. Then, we know that if $M \doteq N$ is not a consequence of E, then there is a model of E that falsifies $M \doteq N$. One can also consider refinements of strong completeness theorems where completeness is shown for classes of models with certain required properties.

For the simply-typed λ-calculus, models inspired by Henkin models [7] were defined by Friedman [2], who proved a strong completeness theorem, as well as another interesting completeness theorem. Plotkin [13] and Statman [15], [16], also proved some refinements of the strong completeness theorem for the simply-typed λ-calculus.

So far, we have assumed that the models under considerations have nonempty carriers for all types. However, in computer science applications, the assumption that carriers are nonempty may be unreasonable, because too restrictive. This fact was first observed by Goguen and Meseguer [5] in the framework of many-sorted algebras, and later on, by Meyer, Mitchell, Moggi, and Statman [10], for the second-order λ-calculus. Unfortunately, the usual proof systems that are complete for models with nonempty carriers, are not complete in the more general situation where carriers may be empty, and even though complete proof systems can be designed, they are quite complicated, since they involve reasoning by cases (see [10]). Furthermore, the existence of the minimal model of a set of equations is lost: there exist sets of equations that are not the theory of any single model.

Mitchell and Moggi [12] observed that after all, proof systems for typed λ-calculi are intuitionistic (in most cases), and that the semantics in terms of Henkin-like models with possibly empty carriers is just too classical in nature, in the sense that arguments where we assume that a carrier is either empty or nonempty, may be used freely. Thus, Mitchell and Moggi suggested to consider intuitionistic semantics such as Kripke-style semantics. Indeed, a Kripke-style semantics forces an intuitionistic interpretation of the connectives, and extended completeness holds again for the usual proof system, regardless of the fact that carriers may be empty. Also, in the Kripke semantics, for any set E of equations, there is a Kripke model $\mathcal{A}$ such that, an equation $M \doteq N$ is valid in $\mathcal{A}$ iff $M \doteq N$ is provable from E. Besides having the virtue that these desirable completeness properties are regained in the Kripke semantics, from a categorical point of view, Kripke models are essentially equivalent to arbitrary CCC's, as sketched in Mitchell and Moggi [12]. However, this relationship will not be considered in the present paper.

In this paper, we define a new class of Kripke structures for the second-order λ-calculus, and investigate the soundness and completeness of some proof systems for proving inequalities (rewrite rules) or equations. Actually, we consider a more general class of structures. The Kripke structures considered in this paper are equipped with preorders that correspond to an abstract form of reduction, and they are not necessarily extensional. This approach allows us to consider models of sets of rewrite rules, as well as sets of equations. We obtain soundness and completeness theorems that generalize some results of Mitchell and Moggi [12] to the second-order λ-calculus, and to sets of inequalities (rewrite rules).

Although we were not expecting to use any category theory in this paper, we realized that this was almost unvoidable in order to come up with the "right" concepts. In particular, we don't believe that we would have come up with the right notion of dependent product for interpreting typed λ-abstraction, if we had not known that categories of presheaves are Cartesian-closed. Thus, we found it convenient to define these structures directly as functors $A: \mathcal{W} \rightarrow \mathtt{Preor}$ equipped with certain natural transformations corresponding to application and abstraction (where $\mathcal{W}$ is a preorder, the set of worlds, and $\mathtt{Preor}$ is the category of preorders). We make use of an explicit construction of the exponential of functors in the Cartesian-closed category $\mathtt{Preor}^{\mathcal{W}}$, and we also define a kind of exponential $\prod_{\Phi}(A^s)_{s\in\mathcal{T}}$ to take care of type abstraction. However, our use of categorical concepts is minimal, and we do not appeal to any fancy machinery.

In order to understand what motivated our definition of a Kripke structure for the second-order λ-calculus, it is useful to review the usual definition of an applicative structure for the simply-typed λ-calculus (for example, as presented in Gunter [6]). For simplicity, we are restricting our attention to arrow types. Let $\mathcal{T}$ be the set of simple types built up from some base types using the constructor $\rightarrow$. Given a signature Σ of function symbols, where each symbol in Σ is assigned some type in $\mathcal{T}$, an *applicative structure* $\mathcal{A}$ is defined as a triple

$$\langle (A^\sigma)_{\sigma\in\mathcal{T}},\ (\mathsf{app}^{\sigma,\tau})_{\sigma,\tau\in\mathcal{T}},\ \mathit{Const}\rangle,$$

where

$(A^\sigma)_{\sigma\in\mathcal{T}}$ is a family of nonempty sets called *carriers*,

$(\mathsf{app}^{\sigma,\tau})_{\sigma,\tau\in\mathcal{T}}$ is a family of *application operators*, where each $\mathsf{app}^{\sigma,\tau}$ is a total function $\mathsf{app}^{\sigma,\tau}: A^{\sigma\rightarrow\tau} \times A^\sigma \rightarrow A^\tau$;

and *Const* is a function assigning a member of A^σ to every symbol in Σ of type σ.

The meaning of simply-typed λ-terms is usually defined using the notion of an *environment*, or *valuation*. A valuation is a function $\rho: \mathcal{X} \rightarrow \bigcup(A^\sigma)_{\sigma\in\mathcal{T}}$, where $\mathcal{X}$ is the set of term variables. Although when nonempty carriers are considered (which is the case right now), it is not really necessary to consider judgements for interpreting λ-terms, since we are going to consider more general applicative structures, we define the semantics of terms using judgements. Recall that a judgement is an expression of the form $\Gamma \triangleright M:\sigma$, where Γ, called a context, is a set of variable declarations of the form $x_1:\sigma_1,\ldots,x_n:\sigma_n$, where the x_i are pairwise distinct and the σ_i are types, M is a simply-typed λ-term, and σ is a type. There is a standard proof system that allows to type-check terms. A term M type-checks with type σ in the context Γ (where Γ contains an assignment of types to all the variables in M) iff the judgement $\Gamma \triangleright M:\sigma$ is derivable in this proof system. Given a context Γ, we say that a valuation ρ *satisfies* Γ iff $\rho(x) \in A^\sigma$ for every $x:\sigma \in \Gamma$ (in other words, ρ respects the typing of the variables declared in Γ). Then given a context Γ and a valuation ρ satisfying Γ, the meaning $[\![\Gamma \triangleright M:\sigma]\!]\rho$ of a judgement $\Gamma \triangleright M:\sigma$ is defined by induction on the derivation of $\Gamma \triangleright M:\sigma$, according to the following clauses:

$[\![\Gamma \triangleright x\colon \sigma]\!]\rho = \rho(x)$, if x is a variable;

$[\![\Gamma \triangleright c\colon \sigma]\!]\rho = \mathit{Const}(c)$, if c is a constant;

$[\![\Gamma \triangleright MN\colon \tau]\!]\rho = \mathbf{app}^{\sigma,\tau}([\![\Gamma \triangleright M\colon (\sigma \to \tau)]\!]\rho, [\![\Gamma \triangleright N\colon \sigma]\!]\rho)$,

$[\![\Gamma \triangleright \lambda x\colon \sigma. M\colon (\sigma \to \tau)]\!]\rho = f$, where f is the unique element of $A^{\sigma\to\tau}$ such that $\mathbf{app}^{\sigma,\tau}(f, a) = [\![\Gamma, x\colon \sigma \triangleright M\colon \tau]\!]\rho[x\colon= a]$, for every $a \in A^{\sigma}$.

Note that in order for the element $f \in A^{\sigma\to\tau}$ to be uniquely defined in the last clause, we need to make certain additional assumptions. First, we assume that we are considering *extensional* applicative structures, which means that for all $f, g \in A^{\sigma\to\tau}$, if $\mathbf{app}(f, a) = \mathbf{app}(g, a)$ for all $a \in A^{\sigma}$, then $f = g$. This condition garantees the uniqueness of f if it exists. The second condition is more technical, and asserts that each A^{σ} contains enough elements so that there is an element $f \in A^{\sigma\to\tau}$ such that $\mathbf{app}^{\sigma,\tau}(f, a) = [\![\Gamma, x\colon \sigma \triangleright M\colon \tau]\!]\rho[x\colon= a]$, for every $a \in A^{\sigma}$.

Note that each operator $\mathbf{app}^{\sigma,\tau}\colon A^{\sigma\to\tau} \times A^{\sigma} \to A^{\tau}$ induces a function $\mathtt{fun}^{\sigma,\tau}\colon A^{\sigma\to\tau} \to [A^{\sigma} \Rightarrow A^{\tau}]$, where $[A^{\sigma} \Rightarrow A^{\tau}]$ denotes the set of functions from A^{σ} to A^{τ}, defined such that

$$\mathtt{fun}^{\sigma,\tau}(f)(a) = \mathbf{app}^{\sigma,\tau}(f, a),$$

for all $f \in A^{\sigma\to\tau}$, and all $a \in A^{\sigma}$. Then, extensionality is equivalent to the fact that each $\mathtt{fun}^{\sigma,\tau}$ is injective. Note that $\mathtt{fun}^{\sigma,\tau}\colon A^{\sigma\to\tau} \to [A^{\sigma} \Rightarrow A^{\tau}]$ is the "curried" version of $\mathbf{app}^{\sigma,\tau}\colon A^{\sigma\to\tau} \times A^{\sigma} \to A^{\tau}$, and it exists because the category of sets is Cartesian-closed. For the category of sets, the fact that $[A^{\sigma} \Rightarrow A^{\tau}]$ is an exponential object is a triviality, but for more general categories, as this will be the case when we define Kripke structures (categories of presheaves), the existence of exponentials is no longer a trivial fact (but not a difficult one).

The clause defining $[\![\Gamma \triangleright \lambda x\colon \sigma. M\colon (\sigma \to \tau)]\!]\rho$ suggests that a partial map $\mathtt{abst}^{\sigma,\tau}\colon [A^{\sigma} \Rightarrow A^{\tau}] \to A^{\sigma\to\tau}$, "abstracting" a function $\varphi \in [A^{\sigma} \Rightarrow A^{\tau}]$ into an element $\mathtt{abst}^{\sigma,\tau}(\varphi) \in A^{\sigma\to\tau}$, can be defined. For example, the function φ defined such that $\varphi(a) = [\![\Gamma, x\colon \sigma \triangleright M\colon \tau]\!]\rho[x\colon= a]$ would be mapped to $[\![\Gamma \triangleright \lambda x\colon \sigma. M\colon (\sigma \to \tau)]\!]\rho$. In order for the resulting structure to be a model of β-reduction, we just have to require that $\mathtt{fun}^{\sigma,\tau}$ and $\mathtt{abst}^{\sigma,\tau}$ satisfy the axiom

$$\mathtt{fun}^{\sigma,\tau}(\mathtt{abst}^{\sigma,\tau}(\varphi)) = \varphi,$$

whenever $\varphi \in [A^{\sigma} \Rightarrow A^{\tau}]$ is in the domain of $\mathtt{abst}^{\sigma,\tau}$. But now, observe that if pairs of operators $\mathtt{fun}^{\sigma,\tau}, \mathtt{abst}^{\sigma,\tau}$ satisfying the above axiom are defined, the injectivity of $\mathtt{fun}^{\sigma,\tau}$ is superfluous for defining $[\![\Gamma \triangleright \lambda x\colon \sigma. M\colon (\sigma \to \tau)]\!]\rho$.

Thus, by defining a more general kind of applicative structure using the operators $\mathtt{fun}^{\sigma,\tau}$ and $\mathtt{abst}^{\sigma,\tau}$, we can still give meanings to λ-terms, even when these structures are nonextensional. In particular, our approach is an alternative to the method where one considers applicative structures with meaning functions, as for example in Mitchell [11]. In particular, the term structure together with the meaning function defined using substitution can be seen to be an applicative structure according to our definition. In fact, this approach allows us to go further. We can assume that each carrier A^{σ} is equipped with a preorder $\preceq^{\sigma}$, and rather than considering the equality

$$\mathtt{fun}^{\sigma,\tau}(\mathtt{abst}^{\sigma,\tau}(\varphi)) = \varphi,$$

we can consider inequalities

$$\mathtt{fun}^{\sigma,\tau}(\mathtt{abst}^{\sigma,\tau}(\varphi)) \succeq \varphi.$$

This way, we can deal with intentional (nonapplicative) structures that model reduction rather than conversion. We learned from Gordon Plotkin that models of β-reduction (or $\beta\eta$-reduction) have been considered before, in particular by Girard [4], Jacobs, Margaria, and Zacchi [8], and Plotkin [14]. However, except for Girard who studies qualitative domains for system F, the other authors consider models of the untyped λ-calculus. We now show how to construct Kripke structures along the ideas sketched above. First, we review Mitchell and Moggi's definition [12]. The main new ingredient is that we have a preordered set $\langle \mathcal{W}, \preceq \rangle$, intuitively, a set of worlds. Then, a *Kripke applicative structure* is defined as a tuple

$$\langle \mathcal{W}, \preceq, (A^{\sigma}_{w})_{\sigma\in\mathcal{T},w\in\mathcal{W}}, (\mathsf{app}^{\sigma,\tau}_{w})_{\sigma,\tau\in\mathcal{T},w\in\mathcal{W}}, (i^{\sigma}_{w_1,w_2})_{\sigma\in\mathcal{T},w_1,w_2\in\mathcal{W}} \rangle,$$

where,

$\mathcal{W}$ is a set of worlds preordered by $\preceq$,

$(A^{\sigma}_{w})_{\sigma\in\mathcal{T},w\in\mathcal{W}}$ is a family of (possibly empty) sets called *carriers*,

$(\mathsf{app}^{\sigma,\tau}_{w})_{\sigma,\tau\in\mathcal{T},w\in\mathcal{W}}$ is a family of *application operators*, where each $\mathsf{app}^{\sigma,\tau}_{w}$ is a total function $\mathsf{app}^{\sigma,\tau}_{w}: A^{\sigma\to\tau}_{w} \times A^{\sigma}_{w} \to A^{\tau}_{w}$;

$i^{\sigma}_{w_1,w_2}: A^{\sigma}_{w_1} \to A^{\sigma}_{w_2}$ is a *transition function*, whenever $w_1 \preceq w_2$.

Furthermore, certain conditions hold, making each A^{σ} into a functor from $\mathcal{W}$ to $\mathtt{Sets}$, and each $\mathsf{app}^{\sigma,\tau}$ into a natural transformation between the functors $A^{\sigma\to\tau} \times A^{\sigma}$ and A^{τ}. For example, we have

$$i^{\tau}_{w_1,w_2}(\mathsf{app}^{\sigma,\tau}_{w_1}(f, a)) = \mathsf{app}^{\sigma,\tau}_{w_2}(i^{\sigma\to\tau}_{w_1,w_2}(f), i^{\sigma}_{w_1,w_2}(a)),$$

for all $f \in A^{\sigma\to\tau}_{w_1}$ and all $a \in A^{\sigma}_{w_1}$.[1]

If we want to adapt this definition to give a more general definition in terms of the operators $\mathtt{fun}^{\sigma,\tau}$ and $\mathtt{abst}^{\sigma,\tau}$, we need to define $\mathtt{fun}^{\sigma,\tau}$ as the "curried" version of the natural transformation $\mathsf{app}^{\sigma,\tau}$ between the functors $A^{\sigma\to\tau} \times A^{\sigma}$ and A^{τ}. This is where we use a bit of category theory. Each A^{σ} can be viewed as a functor $A^{\sigma}: \mathcal{W} \to \mathtt{Sets}$ from the preorder $\mathcal{W}$ viewed as a category, and the category of sets, and these functors together with the natural transformations between them form a category, a *presheaf category*, which is known to be Cartesian-closed (in fact, a topos, see Mac Lane and Moerdijk [9]). Furthermore, it is possible to give an explicit construction of the exponential $[A^{\sigma} \Rightarrow A^{\tau}]$ (see definition 3.5) between two functors A^{σ} and A^{τ}, and to define $\mathtt{fun}$ as $\mathtt{curry}(\mathsf{app})$. Then, it is easy to define a Kripke applicative structure in terms of the natural transformations $\mathtt{fun}^{\sigma,\tau}$ and $\mathtt{abst}^{\sigma,\tau}$.

In order to deal with second-order types, first, we need to provide an interpretation of the type variables. Thus, as in Breazu-Tannen and Coquand [1], we assume that we have an *algebra of types* T, which consists of a quadruple

$$\langle T, \to, [T \Rightarrow T], \forall \rangle,$$

where T is a nonempty set of types, $\to: T \times T \to T$ is a binary operations on T, $[T \Rightarrow T]$ is a nonempty set of functions from T to T, and $\forall$ is a function $\forall: [T \Rightarrow T] \to T$.

[1] Constants can be handled too, but for simplicity, they are dropped.

Intuitively, given a valuation $\theta: \mathcal{V} \rightarrow \mathcal{T}$ (where $\mathcal{V}$ is the set of type variables), a type $\sigma \in \mathcal{T}$ will be interpreted as an element $[\![\sigma]\!]\theta$ of T. Then, a *second-order applicative structure* is defined as a tuple

$$\langle T, (A^s)_{s \in T}, (\mathbf{app}^{s,t})_{s,t \in T}, (\mathtt{tapp}^{\Phi})_{\Phi \in [T \rightarrow T]} \rangle,$$

where

T is an algebra of types;

$(A^s)_{s \in T}$ is a family of nonempty sets called *carriers*,

$(\mathbf{app}^{s,t})_{s,t \in T}$ is a family of *application operators*, where each $\mathbf{app}^{s,t}$ is a total function $\mathbf{app}^{s,t}: A^{s \rightarrow t} \times A^s \rightarrow A^t$;

$(\mathtt{tapp}^{\Phi})_{\Phi \in [T \rightarrow T]}$ is a family of *type-application operators*, where each $\mathtt{tapp}^{\Phi}$ is a total function $\mathtt{tapp}^{\Phi}: A^{\forall(\Phi)} \times T \rightarrow \coprod(A^{\Phi(s)})_{s \in T}$, such that $\mathtt{tapp}^{\Phi}(f, t) \in A^{\Phi(t)}$, for every $f \in A^{\forall(\Phi)}$, and every $t \in T$.

In order to define second-order applicative structures using operators like **fun** and **abst**, we need to define the curried version $\mathtt{tfun}^{\Phi}$ of $\mathtt{tapp}^{\Phi}: A^{\forall(\Phi)} \times T \rightarrow \coprod(A^{\Phi(s)})_{s \in T}$. For this, we define a kind of *dependent product* $\prod_{\Phi}(A^s)_{s \in T}$ (see definition 3.8). Then, we have families of operators $\mathtt{tfun}^{\Phi}: A^{\forall(\Phi)} \rightarrow \prod_{\Phi}(A^s)_{s \in T}$, and $\mathtt{tabst}^{\Phi}: \prod_{\Phi}(A^s)_{s \in T} \rightarrow A^{\forall(\Phi)}$, for every $\Phi \in [T \Rightarrow T]$.

Now, if we want to adapt the above definition to define Kripke applicative structures, we have to view $A^{\forall(\Phi)} \times T$ and $\coprod(A^{\Phi(s)})_{s \in T}$ as functors, and $\mathtt{tapp}^{\Phi}: A^{\forall(\Phi)} \times T \rightarrow \coprod(A^{\Phi(s)})_{s \in T}$ as a natural transformation between them. Then, we need to define some form of exponential of T and $\coprod(A^{\Phi(s)})_{s \in T}$. Such an exponential can indeed be constructed as a functor $\prod_{\Phi}(A^s)_{s \in T}$ defined in terms of the dependent products $\prod_{\Phi}(A^s_w)_{s \in T}$ (see definition 3.8). We also need to show that the functor $\prod_{\Phi}(A^s)_{s \in T}$ satisfies a universal property analogous to the property satisfied by the functor $[A^s \Rightarrow A^t]$. For this, we define the set $\mathtt{Nat}_{\Phi}(H \times T, \coprod(A^{\Phi(s)})_{s \in T})$ as the set of natural transformations $\eta: H \times T \rightarrow \coprod(A^{\Phi(s)})_{s \in T}$, such that, $\eta_u(a, t) \in A^{\Phi(t)}_u$, for every $a \in H_u$ and every $t \in T$ (see definition 3.9). Then, we can prove a lemma (lemma 3.11) that shows that $\prod_{\Phi}(A^s)_{s \in T}$ is indeed a certain kind of exponential. Thus, at the level of presheaf categories, we have the usual maps **curry** and **uncurry** that set up a (natural) bijection between $\mathtt{Nat}(H \times F, G)$ and $\mathtt{Nat}(H, [F \Rightarrow G])$, but also some maps $\mathtt{curry}_{\Phi}$ and $\mathtt{uncurry}_{\Phi}$ that set up a (natural) bijection between the sets of natural transformations $\mathtt{Nat}_{\Phi}(H \times T, \coprod(A^{\Phi(s)})_{s \in T})$ and $\mathtt{Nat}(H, \prod_{\Phi}(A^s)_{s \in T})$.

Armed with the definition of the functors $[A^s \Rightarrow A^t]$ and $\prod_{\Phi}(A^s)_{s \in T}$, and the natural transformations **fun**, **abst**, **tfun**, and **tabst**, we can define Kripke applicative structures (see definition 4.1). In fact, the definition also applies to the product and sum types, and to carriers A^s_w equipped with preorders. This way, we can define models of sets of rewrite rules, as well as models of sets of equations.

The paper is organized as follows. Section 2 is a review of the syntax of the second-order typed λ-calculus $\lambda^{\rightarrow,\times,+,\forall^2}$. Section 3 contains a review of some elementary notions of category theory. An explicit construction of the exponential of functors $F, G: \mathcal{W} \rightarrow \mathbf{Preor}$, where $\mathcal{W}$ is a preorder, and **Preor** is the category of preorders, is given. The dependent product $\prod_{\Phi}(A^s)_{s \in T}$ is also defined. Kripke pre-applicative structures are defined in section 4. In section 5, we show how to interpet second-order λ-terms using Kripke applicative structures. A number of proof systems for proving inequalities (rewrite rules) and equations are defined in section 6. Satisfaction and validity (in a

Kripke structure) is also defined. Some soundness and completeness results are proved in section 7. The results of section 7 are adapted to equations in section 8. Section 9 contains the conclusion and some suggestions for further research.

2 Syntax of the Second-Order Typed λ-Calculus $\lambda^{\to,\times,+,\forall^2}$

In this section, we review quickly the syntax of the second-order typed λ-calculus $\lambda^{\to,\times,+,\forall^2}$. This includes a definition of the second-order types under consideration, of raw terms, or the type-checking rules for judgements, and of the reduction rules. For more details (on the subsystem $\lambda^{\to,\forall^2}$), the reader should consult Breazu-Tannen and Coquand [1].

Let $\mathcal{T}$ denote the set of second-order types. This set comprises type variables X, type constants k, and compound types $(\sigma \to \tau)$, $(\sigma \times \tau)$, $(\sigma + \tau)$, and $\forall X.\, \sigma$. It is assumed that we have a set TC of *type constants* (also called *base types of kind* $\star$). We have a countably infinite set $\mathcal{V}$ of type variables (denoted as upper case letters X, Y, Z), and a countably infinite set $\mathcal{X}$ of term variables (denoted as lower case letters x, y, z). We denote the set of free type variables occurring in a type σ as $FTV(\sigma)$. We use the notation $\star$ for the kind of types. Since we are only considering second-order quantification over predicate symbols (of kind $\star$) of arity 0, this is superfluous. However, it will occasionally be useful to consider contexts Γ in which type variables are explicitly present, since this makes the type-checking rules more uniform in the case of λ-abstraction and typed λ-abstraction. Thus, officially, a context Γ is a set $\{x_1{:}\,\sigma_1, \ldots, x_n{:}\,\sigma_n\}$, where $x_1, \ldots, x_n$ are term variables, and $\sigma_1, \ldots, \sigma_n$ are types. We let $dom(\Gamma) = \{x_1, \ldots, x_n\}$. As usual, we assume that the variables x_j are pairwise distinct. We also assume that $x \notin dom(\Gamma)$ in a context $\Gamma, x{:}\,\sigma$. Informally, we will also consider contexts $\{X_1{:}\,\star, \ldots, X_m{:}\,\star, x_1{:}\,\sigma_1, \ldots, x_n{:}\,\sigma_n\}$, where $X_1, \ldots, X_m$ are type variables, and $x_1, \ldots, x_n$ are term variables, with the two sets $\{X_1, \ldots, X_m\}$ and $\{x_1, \ldots, x_n\}$ disjoint, the variables X_i pairwise distinct, and the variables x_j pairwise distinct. We assume that $X \notin dom(\Gamma)$ in a context $\Gamma, X{:}\,\star$. For the sake of brevity, rather than writing typed λ-abstraction as $\lambda X{:}\,\star.\, M$, it will be written as $\lambda X.\, M$.

It is assumed that we have a set *Const* of constants, together with a function *Type*: *Const* $\to \mathcal{T}$, such that every constant c is assigned a *closed* type *Type*(c) in $\mathcal{T}$. The set TC of type constants, together with the set *Const* of constants, and the function *Type*, constitute a *signature* Σ. Let us review the definition of raw terms.

Definition 2.1 The set of *raw terms* is defined inductively as follows: every variable $x \in \mathcal{X}$ is a raw term, every constant $c \in Const$ is a raw terms, and if M, N are raw terms and σ, τ are types, then (MN), $(M\tau)$, $\lambda x{:}\,\sigma.\, M$, $\lambda X.\, M$, $\pi_1(M)$, $\pi_2(M)$, $\langle M,\ N\rangle$, $\mathtt{inl}(M)$, $\mathtt{inr}(M)$, and $[M,\ N]$, are raw terms.

We let $FV(M)$ denote the set of free term-variables in M. Raw terms may contain free variables and may not type-check (for example, (xx)). In order to define which raw terms type-check, we consider expressions of the form $\Gamma \triangleright M{:}\,\sigma$, called *judgements*, where Γ is a context in which all the free term variables in M are declared. A term M type-checks with type σ in the context Γ iff the judgement $\Gamma \triangleright M{:}\,\sigma$ is provable using axioms and rules summarized in the following definition.

Definition 2.2 The judgements of the polymorphic typed λ-calculus $\lambda^{\to,\times,+,\forall^2}$ are defined by the following rules.

$$\Gamma \triangleright x{:}\,\sigma, \quad \text{when } x{:}\,\sigma \in \Gamma,$$

$$\Gamma \triangleright c\colon \mathit{Type}(c), \quad \text{when } c \text{ is a constant},$$

$$\frac{\Gamma, x\colon\sigma \triangleright M\colon\tau}{\Gamma \triangleright (\lambda x\colon\sigma.\, M)\colon(\sigma \rightarrow \tau)} \quad (\mathit{abstraction})$$

$$\frac{\Gamma \triangleright M\colon(\sigma \rightarrow \tau) \quad \Gamma \triangleright N\colon\sigma}{\Gamma \triangleright (MN)\colon\tau} \quad (\mathit{application})$$

$$\frac{\Gamma \triangleright M\colon\sigma \quad \Gamma \triangleright N\colon\tau}{\Gamma \triangleright \langle M,\ N\rangle\colon\sigma \times \tau} \quad (\mathit{pairing})$$

$$\frac{\Gamma \triangleright M\colon\sigma \times \tau}{\Gamma \triangleright \pi_1(M)\colon\sigma} \quad (\mathit{projection}) \qquad \frac{\Gamma \triangleright M\colon\sigma \times \tau}{\Gamma \triangleright \pi_2(M)\colon\tau} \quad (\mathit{projection})$$

$$\frac{\Gamma \triangleright M\colon\sigma}{\Gamma \triangleright \mathbf{inl}(M)\colon\sigma + \tau} \quad (\mathit{injection}) \qquad \frac{\Gamma \triangleright M\colon\tau}{\Gamma \triangleright \mathbf{inr}(M)\colon\sigma + \tau} \quad (\mathit{injection})$$

$$\frac{\Gamma \triangleright M\colon(\sigma \rightarrow \delta) \quad \Gamma \triangleright N\colon(\tau \rightarrow \delta)}{\Gamma \triangleright [M,\ N]\colon(\sigma + \tau) \rightarrow \delta} \quad (\mathit{co\text{-}pairing})$$

$$\frac{\Gamma, X\colon\star \triangleright M\colon\sigma}{\Gamma \triangleright (\lambda X.\, M)\colon\forall X.\, \sigma} \quad (\forall\text{-}\mathit{intro})$$

provided that $X \notin \bigcup_{x:\tau\in\Gamma} FTV(\tau)$;

$$\frac{\Gamma \triangleright M\colon\forall X.\, \sigma}{\Gamma \triangleright (M\tau)\colon\sigma[\tau/X]} \quad (\forall\text{-}\mathit{elim})$$

The reason why we do not officially consider that a context contains type variables, is that in the rule ($\forall$-*elim*), the type τ could contain type variables not declared in Γ, and it would be necessary to have a weakening rule to add new type variables to a context (or some other mechanism to add new type variables to a context). As long as we do not deal with dependent types, this technical annoyance is most simply circumvented by assuming that type variables are not included in contexts.

Instead of using the construct **case** P **of** $\mathbf{inl}(x\colon\sigma) \Rightarrow M \mid \mathbf{inr}(y\colon\tau) \Rightarrow N$, we found it more convenient and simpler to use the slightly more general construct $[M,\ N]$, where M is of type $\sigma \rightarrow \delta$ and N is of type $\tau \rightarrow \delta$, even when M and N are not λ-abstractions. This will be especially advantageous for the semantic treatment to follow. Then, we can define the conditional construct **case** P **of** $\mathbf{inl}(x\colon\sigma) \Rightarrow M \mid \mathbf{inr}(y\colon\tau) \Rightarrow N$, where P is of type $\sigma + \tau$, as $[\lambda x\colon\sigma.\, M,\ \lambda y\colon\tau.\, N]P$.

Definition 2.3 The reduction rules of the system $\lambda^{\rightarrow,\times,+,\forall}$ are listed below:

$$\begin{aligned}
(\lambda x\colon\sigma.\, M)N &\longrightarrow M[N/x],\\
\pi_1(\langle M, N\rangle) &\longrightarrow M,\\
\pi_2(\langle M, N\rangle) &\longrightarrow N,\\
[M,\ N]\mathbf{inl}(P) &\longrightarrow MP,\\
[M,\ N]\mathbf{inr}(P) &\longrightarrow NP,\\
(\lambda X.\, M)\tau &\longrightarrow M[\tau/X].
\end{aligned}$$

The reduction relation defined by the rules of definition 2.3 is denoted as $\longrightarrow_\beta$ (even though there are reductions other that β-reduction). From now on, when we refer to a λ-term, we mean a λ-term that type-checks. In order to define Kripke models for $\lambda^{\rightarrow,\times,+,\forall^2}$, we need to review a few concepts from category theory.

3 Exponentials and Dependent Products in the Category $\mathtt{Preor}^{\mathcal{W}}$

In this section, we define an algebra of polymorphic types, and review some elementary notions of category theory. We give an explicit construction of the exponential of functors $F, G: \mathcal{W} \rightarrow \mathtt{Preor}$, where $\mathcal{W}$ is a preorder, and **Preor** is the category of preorders. We also define the dependent product $\prod_\Phi(A^s)_{s\in T}$, and show that this functor is a certain kind of exponential, if the right set of natural transformations is considered.

Definition 3.1 An *algebra of (polymorphic) types* is a tuple

$$\langle T, \rightarrow, \times, +, [T \Rightarrow T], \forall \rangle,$$

where T is a nonempty set of *types*, $\rightarrow, \times, +: T \times T \rightarrow T$ are binary operations on T, $[T \Rightarrow T]$ is a nonempty set of functions from T to T, and $\forall$ is a function $\forall: [T \Rightarrow T] \rightarrow T$.

Intuitively, given a valuation $\theta: \mathcal{V} \rightarrow T$, a type $\sigma \in T$ will be interpreted as an element $[\![\sigma]\!]\theta$ of T.

We need to define two categories of preorders.

Definition 3.2 The category **Preor** is the category whose objects are preordered sets $\langle W, \preceq\rangle$, and whose arrows $f: W_1 \rightarrow W_2$ are monotonic functions (with respect to $\preceq_1$ and $\preceq_2$). The category $\mathtt{Preor}_p$ is the category whose objects are preordered sets $\langle W, \preceq\rangle$, and whose arrows $f: W_1 \rightarrow W_2$ are monotonic partial functions (with respect to $\preceq_1$ and $\preceq_2$).

It is obvious that **Preor** and $\mathtt{Preor}_p$ are categories. Given a monotonic function $f: W_1 \rightarrow W_2$, where W_1 and W_2 are preorders, we say that f is *isotone* iff $f(w_1) \preceq f(w_2)$ implies that $w_1 \preceq w_2$, for all $w_1, w_2 \in W_1$.

Any preordered set $\langle \mathcal{W}, \preceq\rangle$ can be viewed as the category whose objects are the elements of $\mathcal{W}$, and such that there is a single arrow denoted $w_1 \rightarrow w_2$ from w_1 to w_2 iff $w_1 \preceq w_2$. We will be interested in functors $F: \mathcal{W} \rightarrow \mathtt{Preor}$. Such a functor assigns a preorder $F(w)$ to every $w \in \mathcal{W}$, and an arrow $F(w_1 \rightarrow w_2): F(w_1) \rightarrow F(w_2)$ to every pair such that $w_1 \preceq w_2$. The preorder $F(w)$ is also denoted as $\langle F_w, \preceq_w^F\rangle$, and the arrow $F(w_1 \rightarrow w_2)$ is a monotonic function denoted as $i^F_{w_1,w_2}: F_{w_1} \rightarrow F_{w_2}$. The fact that F is a functor means that $i^F_{w,w} = \mathtt{id}$, and that $i^F_{w_1,w_3} = i^F_{w_2,w_3} \circ i^F_{w_1,w_2}$, whenever $w_1 \preceq w_2 \preceq w_3$.

Recall that a natural transformation $\eta: F \rightarrow G$ between two functors $F, G: \mathcal{W} \rightarrow \mathtt{Preor}$ is a family $\eta = (\eta_w)_{w\in\mathcal{W}}$, where $\eta_w: F_w \rightarrow G_w$ is an arrow in **Preor**, and such that the following naturality conditions hold whenever $w_1 \preceq w_2$:

$$\eta_{w_2} \circ i^F_{w_1,w_2} = i^G_{w_1,w_2} \circ \eta_{w_1}.$$

Definition 3.3 The set of natural transformations between two functors $F, G: \mathcal{W} \to \mathbf{Preor}$ is denoted as $\mathbf{Nat}(F, G)$. The set of natural transformations between two functors $F, G: \mathcal{W} \to \mathbf{Preor}_p$ is denoted as $\mathbf{Nat}_p(F, G)$. Functors $F: \mathcal{W} \to \mathbf{Preor}$ and natural transformations between them form a category (of *presheaves*), denoted as $\mathbf{Preor}^{\mathcal{W}}$. Similarly, we have the category $\mathbf{Preor}_p^{\mathcal{W}}$.

The categories $\mathbf{Preor}^{\mathcal{W}}$ (and $\mathbf{Preor}_p^{\mathcal{W}}$) are Cartesian-closed (in fact, they are topos, see Mac Lane and Moerdijk [9]), and we will be interested in an explicit description of the exponentials.

Given an indexed family of sets $(A_i)_{i\in I}$, we let $\prod(A_i)_{i\in I}$ be the product of the family $(A_i)_{i\in I}$, and $\coprod(A_i)_{i\in I}$ be the coproduct (or disjoint sum) of the family $(A_i)_{i\in I}$. The disjoint sum $\coprod(A_i)_{i\in I}$ is the set $\bigcup\{\langle a,\ i\rangle \mid a \in A_i\}_{i\in I}$. If the sets A_i are preorders, then $\prod(A_i)_{i\in I}$ is a preorder under the product preorder, where $(a_i)_{i\in I} \preceq (b_i)_{i\in I}$ iff $a_i \preceq_i b_i$ for all $i \in I$, and $\coprod(A_i)_{i\in I}$ is a preorder under the (disjoint) sum preorder, where $\langle a,\ i\rangle \preceq \langle b,\ i\rangle$ iff $i = j$ and $a \preceq_i b$. When $I = \{1, 2\}$, we also denote $\prod(A_i)_{i\in I}$ as $A_1 \times A_2$, and $\coprod(A_i)_{i\in I}$ as $A_1 + A_2$.

Definition 3.4 Given a family of functors $(F_i)_{i\in I}$, where $F_i: \mathcal{W} \to \mathbf{Preor}$, we define the functors $\prod(F_i)_{i\in I}: \mathcal{W} \to \mathbf{Preor}$ and $\coprod(F_i)_{i\in I}: \mathcal{W} \to \mathbf{Preor}$ as follows. In order to abbreviate the notation, let $P_I = \prod(F_i)_{i\in I}$, and $S_I = \coprod(F_i)_{i\in I}$. Then

(i) For every $w \in \mathcal{W}$, $P_I(w) = \prod(F_i(w))_{i\in I}$, and arrows are defined in the following way: $i^{P_I}_{w_1,w_2}: P_I(w_1) \to P_I(w_2)$ is the I-indexed family $\prod(i^{F_i}_{w_1,w_2})_{i\in I}$, where $w_1 \preceq w_2$.

(ii) For every $w \in \mathcal{W}$, $S_I(w) = \coprod(F_i(w))_{i\in I}$, and arrows are defined in the following way: $i^{S_I}_{w_1,w_2}: S_I(w_1) \to S_I(w_2)$ is the I-indexed family $\coprod(i^{F_i}_{w_1,w_2})_{i\in I}$, where $w_1 \preceq w_2$.

It is immediately verified that $\prod(F_i)_{i\in I}$ and $\coprod(F_i)_{i\in I}$ are functors $\prod(F_i)_{i\in I}: \mathcal{W} \to \mathbf{Preor}$ and $\coprod(F_i)_{i\in I}: \mathcal{W} \to \mathbf{Preor}$. Thus, the category of functors $F: \mathcal{W} \to \mathbf{Preor}$ has products and coproducts. It also has a terminal object, the constant functor from $\mathcal{W}$ to the one object preorder (and an initial object). We will now define a notion of exponential, showing that the category of functors $F: \mathcal{W} \to \mathbf{Preor}$ (with natural transformations between them) is Cartesian-closed. This can be shown using the Yoneda lemma (see Mac Lane and Moerdijk [9]), but we will give an explicit construction.

Definition 3.5 Given a preorder $\langle \mathcal{W}, \preceq\rangle$ and two functors $F: \mathcal{W} \to \mathbf{Preor}$ and $G: \mathcal{W} \to \mathbf{Preor}$, we define the functor $[F \Rightarrow G]$ as follows: For any $u \in \mathcal{W}$, $[F \Rightarrow G]_u$ is the set of families $\varphi = (\varphi_w)_{w\succeq u}$, where each φ_w is an arrow $\varphi_w: F_w \to G_w$ (in the category $\mathbf{Preor}$), such that the following naturality conditions hold whenever $w_2 \succeq w_1 \succeq w$:

$$\varphi_{w_2} \circ i^F_{w_1,w_2} = i^G_{w_1,w_2} \circ \varphi_{w_1}.$$

The preorder on $[F \Rightarrow G]_u$ is defined as follows: Given two families $\varphi = (\varphi_w)_{w\succeq u}$ and $\psi = (\psi_w)_{w\succeq u}$, $\varphi \preceq_u \psi$ iff $\varphi_w \preceq_w \psi_w$ for all $w \succeq u$.[2] Whenever $w_1 \preceq w_2$, we define $i^{F\Rightarrow G}_{w_1,w_2}: [F \Rightarrow G]_{w_1} \to [F \Rightarrow G]_{w_2}$ as follows:

For every family $\varphi = (\varphi_w)_{w\succeq w_1}$ in $[F \Rightarrow G]_{w_1}$ (where $\varphi_w: F_w \to G_w$),

$$i^{F\Rightarrow G}_{w_1,w_2}((\varphi_w)_{w\succeq w_1}) = (\varphi_w)_{w\succeq w_2}.$$

Thus, $i^{F\Rightarrow G}_{w_1,w_2}$ is the *restriction function* that restricts every family $(\varphi_w)_{w\succeq w_1}$ in $[F \Rightarrow G]_{w_1}$ to the subfamily $(\varphi_w)_{w\succeq w_2}$ in $[F \Rightarrow G]_{w_2}$, where $w_1 \preceq w_2$.

[2] Given two functions $f, g: F_w \to G_w$, $f \preceq_w g$ iff $f(a) \preceq^G_w g(a)$ for all $a \in F_w$.

It is clear that $[F \Rightarrow G]$ is a functor $[F \Rightarrow G]: \mathcal{W} \to \mathbf{Preor}$. In fact, $[F \Rightarrow G]$ is an exponential in the category of functors $F: \mathcal{W} \to \mathbf{Preor}$, and this makes this category Cartesian-closed. To make this precise, we have to define the evaluation map $\mathtt{eval}: [F \Rightarrow G] \times F \to G$.

Definition 3.6 Given a preorder $\langle \mathcal{W}, \preceq \rangle$ and two functors $F: \mathcal{W} \to \mathbf{Preor}$ and $G: \mathcal{W} \to \mathbf{Preor}$, we define the *evaluation map* $\mathtt{eval}^{F,G}: [F \Rightarrow G] \times F \to G$ as follows:

For every $u \in \mathcal{W}$, for every family $\varphi = (\varphi_w)_{w \succeq u}$ in $[F \Rightarrow G]_u$ (where $\varphi_w: F_w \to G_w$), for every $a \in F_u$,

$$\mathtt{eval}_u^{F,G}((\varphi_w)_{w \succeq u}, a) = \varphi_u(a).$$

Given any functors $F, G, H: \mathcal{W} \to \mathbf{Preor}$, for any natural transformation $\eta: H \times F \to G$, we define the natural transformation $\mathtt{curry}(\eta): H \to [F \Rightarrow G]$ as follows:

For every $u \in \mathcal{W}$, $\mathtt{curry}(\eta)_u: H_u \to [F \Rightarrow G]_u$ is the arrow (in the category $\mathbf{Preor}$) such that, for every $a \in H_u$,

$$\mathtt{curry}(\eta)_u(a) = \{\mathtt{curry}(\eta_w)(i_{u,w}^H(a)): F_w \to G_w \mid w \succeq u\},$$

where $\mathtt{curry}(\eta_w)(i_{u,w}^H(a)): F_w \to G_w$, is the arrow (in the category $\mathbf{Preor}$), such that, for every $b \in F_w$, $\mathtt{curry}(\eta_w)(i_{u,w}^H(a))(b) = \eta_w(i_{u,w}^H(a), b)$.

Lemma 3.7 *Given any two functors $F, G: \mathcal{W} \to \mathbf{Preor}$, $\mathtt{eval}^{F,G}: [F \Rightarrow G] \times F \to G$ is a natural transformation. Furthermore, Given any functors $F, G, H: \mathcal{W} \to \mathbf{Preor}$, for any natural transformation $\eta: H \times F \to G$, $\mathtt{curry}(\eta): H \to [F \Rightarrow G]$ (as in definition 3.6) is the unique natural transformation such that*

$$\eta = \mathtt{eval}^{F,G} \circ (\mathtt{curry}(\eta) \times \mathtt{id}_F).$$

If $\theta: H \to [F \Rightarrow G]$ is a natural transformation, then $\theta = \mathtt{curry}(\mathtt{eval}^{F,G} \circ (\theta \times \mathtt{id}_F))$.

Proof. It is easily verified that $\mathtt{eval}^{F,G}: [F \Rightarrow G] \times F \to G$ and $\mathtt{curry}(\eta): H \to [F \Rightarrow G]$ are indeed natural transformations. It can also be checked that for any $\eta: H \times F \to G$, the natural transformation $\mathtt{curry}(\eta): H \to [F \Rightarrow G]$ is the unique natural transformation such that

$$\eta = \mathtt{eval}^{F,G} \circ (\mathtt{curry}(\eta) \times \mathtt{id}_F).$$

Finally, letting $\eta = \mathtt{eval}^{F,G} \circ (\theta \times \mathtt{id}_F)$, since θ satisfies the property $\eta = \mathtt{eval}^{F,G} \circ (\theta \times \mathtt{id}_F)$, by uniqueness of $\mathtt{curry}(\eta)$, we have $\theta = \mathtt{curry}(\mathtt{eval}^{F,G} \circ (\theta \times \mathtt{id}_F))$. $\square$

Thus, the category of all functors $F: \mathcal{W} \to \mathbf{Preor}$ is Cartesian-closed. Given a natural transformation $\theta: H \to [F \Rightarrow G]$, if we define the natural transformation $\mathtt{uncurry}$ such that $\mathtt{uncurry}(\theta) = \mathtt{eval}^{F,G} \circ (\theta \times \mathtt{id}_F)$, then we have immediately that

$$\mathtt{uncurry} \circ \mathtt{curry} = \mathtt{id} \quad \text{and} \quad \mathtt{curry} \circ \mathtt{uncurry} = \mathtt{id},$$

which shows that $\mathtt{curry}$ and $\mathtt{uncurry}$ set up a (natural) bijection between $\mathbf{Nat}(H \times F,\ G)$ and $\mathbf{Nat}(H,\ [F \Rightarrow G])$.

We view T as the constant functor $T: \mathcal{W} \to \mathbf{Preor}$ such that $T_w = T$ for every $w \in \mathcal{W}$, the preorder on T being the identity relation. Before defining a Kripke pre-applicative structure, we need to define the notion of a dependent product. The construction is quite similar to that of definition 3.5.

Definition 3.8 Given an algebra of types T, and a T-indexed family of preorders $\langle A^s, \preceq^s\rangle$, for every function $\Phi \in [T \Rightarrow T]$, the *dependent product* $\prod_\Phi (A^s)_{s\in T}$ is the cartesian product $\prod(A^{\Phi(t)})_{t\in T}$, which is also described explicitly as the set of functions in $(\coprod(A^{\Phi(s)})_{s\in T})^T$ defined as follows:

$$\prod_\Phi (A^s)_{s\in T} = \{f : T \to \coprod(A^{\Phi(s)})_{s\in T} \mid f(t) \in A^{\Phi(t)}, \text{ for all } t \in T\}.$$

The set $\prod_\Phi (A^s)_{s\in T}$ is given the preorder $\preceq^\Phi$ defined such that, $f \preceq^\Phi g$ iff $f(t) \preceq^{\Phi(t)} g(t)$, for every $t \in T$.

Given a preorder $\langle \mathcal{W}, \preceq\rangle$, an algebra of types T, and a family of functors $A^s : \mathcal{W} \to \mathbf{Preor}$ (where $s \in T$), for every $\Phi \in [T \Rightarrow T]$, we define the functor $\prod_\Phi (A^s)_{s\in T} : \mathcal{W} \to \mathbf{Preor}$ as follows: for any $u \in \mathcal{W}$, $\prod_\Phi (A^s_u)_{s\in T}$ is the set of families $\varphi = (\varphi_w)_{w\succeq u}$, where $\varphi_w \in \prod_\Phi (A^s_w)_{s\in T}$, such that the following naturality conditions hold whenever $w_2 \succeq w_1 \succeq w$:

$$\varphi_{w_2}(t) = i^{\Phi(t)}_{w_1,w_2}(\varphi_{w_1}(t)),$$

for every $t \in T$. The preorder on $\prod_\Phi (A^s_u)_{s\in T}$ is defined as follows: Given two families $\varphi = (\varphi_w)_{w\succeq u}$ and $\psi = (\psi_w)_{w\succeq u}$, $\varphi \preceq_u \psi$ iff $\varphi_w \preceq^\Phi_w \psi_w$ for all $w \succeq u$. Whenever $w_1 \preceq w_2$, we define $i^{\Pi_\Phi}_{w_1,w_2} : \prod_\Phi (A^s_{w_1})_{s\in T} \to \prod_\Phi (A^s_{w_2})_{s\in T}$ as follows:

For every family $\varphi = (\varphi_w)_{w\succeq w_1}$ in $\prod_\Phi (A^s_{w_1})_{s\in T}$ (where $\varphi_w : T \to \coprod(A^{\Phi(s)}_w)_{s\in T}$),

$$i^{\Pi_\Phi}_{w_1,w_2}((\varphi_w)_{w\succeq w_1}) = (\varphi_w)_{w\succeq w_2}.$$

Thus, $i^{\Pi_\Phi}_{w_1,w_2}$ is the *restriction function* that restricts every family $(\varphi_w)_{w\succeq w_1}$ in $\prod_\Phi (A^s_{w_1})_{s\in T}$ to the subfamily $(\varphi_w)_{w\succeq w_2}$ in $\prod_\Phi (A^s_{w_2})_{s\in T}$, where $w_1 \preceq w_2$.

It is clear that $\prod_\Phi (A^s)_{s\in T}$ is a functor $\prod_\Phi (A^s)_{s\in T} : \mathcal{W} \to \mathbf{Preor}$. The functor $\prod_\Phi (A^s)_{s\in T}$ is universal in a certain sense that makes it a kind of exponential with respect to certain natural tranformations. This universality is made precise in what follows.

Definition 3.9 Given any functor $H : \mathcal{W} \to \mathbf{Preor}$ and any family of functors $A^s : \mathcal{W} \to \mathbf{Preor}$ (where $s \in T$), we define the set of natural transformation $\mathtt{Nat}_\Phi(H \times T, \coprod(A^{\Phi(s)})_{s\in T})$ as the set of natural transformations $\eta : H \times T \to \coprod(A^{\Phi(s)})_{s\in T}$, such that, $\eta_u(a, t) \in A^{\Phi(t)}_u$, for every $a \in H_u$ and every $t \in T$.

Definition 3.10 Given a preorder $\langle \mathcal{W}, \preceq\rangle$, and a family of functors $A^s : \mathcal{W} \to \mathbf{Preor}$ (where $s \in T$), we define the *polymorphic evaluation map* $\mathtt{eval}^A_\Phi : (\prod_\Phi (A^s_u)_{s\in T}) \times T \to \coprod(A^{\Phi(s)}_u)_{s\in T}$ as follows:

For every $u \in \mathcal{W}$, for every family $\varphi = (\varphi_w)_{w\succeq u}$ in $\prod_\Phi (A^s_u)_{s\in T}$ (where $\varphi_w : T \to \coprod(A^{\Phi(s)}_w)_{s\in T}$), for every $t \in T$,

$$\mathtt{eval}^A_{\Phi,u}((\varphi_w)_{w\succeq u}, t) = \varphi_u(t).$$

Given any functor $H : \mathcal{W} \to \mathbf{Preor}$ and any family of functors $A^s : \mathcal{W} \to \mathbf{Preor}$ (where $s \in T$), for any natural transformation $\eta \in \mathtt{Nat}_\Phi(H \times T, \coprod(A^{\Phi(s)})_{s\in T})$, we define the natural transformation $\mathtt{curry}_\Phi(\eta) : H \to \prod_\Phi (A^s)_{s\in T}$ as follows:

For every $u \in \mathcal{W}$, $\mathtt{curry}_\Phi(\eta)_u : H_u \to \prod_\Phi (A^s_u)_{s\in T}$ is the arrow (in the category $\mathtt{Preor}$), such that, for every $a \in H_u$,

$$\mathtt{curry}_\Phi(\eta)_u(a) = \{\mathtt{curry}_\Phi(\eta_w)(i^H_{u,w}(a)) : T \to \coprod (A^{\Phi(s)}_w)_{s\in T}, \mid w \succeq u\},$$

where $\mathtt{curry}_\Phi(\eta_w)(i^H_{u,w}(a)) : T \to \coprod (A^{\Phi(s)}_w)_{s\in T}$ is the arrow (in $\prod_\Phi (A^s_w)_{s\in T}$) such that, for every $t \in T$, $\mathtt{curry}_\Phi(\eta_w)(i^H_{u,w}(a))(t) = \eta_w(i^H_{u,w}(a), t)$.

Lemma 3.11 *Given an algebra of types T and family of functors $A^s : \mathcal{W} \to \mathtt{Preor}$ (where $s \in T$), $\mathtt{eval}^A_\Phi : (\prod_\Phi (A^s)_{s\in T}) \times T \to \coprod (A^{\Phi(s)}_w)_{s\in T}$ is a natural transformation. Furthermore, Given any functor $H : \mathcal{W} \to \mathtt{Preor}$ and any family of functors $A^s : \mathcal{W} \to \mathtt{Preor}$ (where $s \in T$), for any natural transformation $\eta \in \mathtt{Nat}_\Phi(H \times T, \coprod (A^{\Phi(s)})_{s\in T})$, $\mathtt{curry}_\Phi(\eta) : H \to \prod_\Phi (A^s)_{s\in T}$ (as in definition 3.10) is the unique natural transformation such that*

$$\eta = \mathtt{eval}^A_\Phi \circ (\mathtt{curry}_\Phi(\eta) \times \mathtt{id}_T).$$

If $\theta \in \mathtt{Nat}_\Phi(H \times T, \coprod (A^{\Phi(s)})_{s\in T})$, then $\theta = \mathtt{curry}_\Phi(\mathtt{eval}^A_\Phi(\theta \times \mathtt{id}_T))$.

Proof. The calculations are straightforward. □

Thus, given a natural transformation $\theta \in \mathtt{Nat}_\Phi(H \times T, \coprod (A^{\Phi(s)})_{s\in T})$, if we define the natural transformation $\mathtt{uncurry}_\Phi$ such that $\mathtt{uncurry}_\Phi(\theta) = \mathtt{eval}^A_\Phi \circ (\theta \times \mathtt{id}_T)$, then we have immediately that

$$\mathtt{uncurry}_\Phi \circ \mathtt{curry}_\Phi = \mathtt{id} \quad \text{and} \quad \mathtt{curry}_\Phi \circ \mathtt{uncurry}_\Phi = \mathtt{id},$$

which shows that $\mathtt{curry}_\Phi$ and $\mathtt{uncurry}_\Phi$ set up a (natural) bijection between the sets of natural transformations $\mathtt{Nat}_\Phi(H \times T, \coprod (A^{\Phi(s)})_{s\in T})$ and $\mathtt{Nat}(H, \prod_\Phi (A^s)_{s\in T})$.

4 Kripke Pre-Applicative Structures

In this section, we define Kripke pre-applicative structures, as suggested in the introduction. The basic version (see definition 4.1) is intentional (i.e. nonextensional). We also consider a version with η-like rules, and an extensional version. An important example of a Kripke pre-applicative structure is given in definition 4.4. Definition 4.8 contains an example also satisfying the η-like rules. We conclude this section with a characterization of extensionality, showing the equivalence between our definition of extentionality and Mitchell and Moggi's definition [12], in the case of first-order applicative structures.

Definition 4.1 Given a preorder $\langle \mathcal{W}, \preceq \rangle$ viewed as a category, and T an algebra of types, a *Kripke pre-applicative β-structure* is a structure

$$\mathcal{A} = \langle A, \mathtt{fun}, \mathtt{abst}, \mathtt{tfun}, \mathtt{tabst}, \Pi, \langle -,- \rangle, \mathtt{inl}, \mathtt{inr}, [-,\ -] \rangle,$$

where

$A = (A^s)_{s\in T}$, a family of functors $A^s : \mathcal{W} \to \mathtt{Preor}$ (recall that for every $w \in \mathcal{W}$, we write $A^s(w)$ as A^s_w);

$\mathtt{fun}^{s,t} : A^{s\to t} \to [A^s \Rightarrow A^t]$, a family of natural transformations in $\mathtt{Nat}(A^{s\to t}, [A^s \Rightarrow A^t])$;

$\mathtt{abst}^{s,t}: [A^s \Rightarrow A^t] \to A^{s\to t}$, a family of natural transformations in $\mathtt{Nat}_p([A^s \Rightarrow A^t],\ A^{s\to t})$;
$\mathtt{tfun}^{\Phi}: A^{\forall(\Phi)} \to \prod_{\Phi}(A^s)_{s\in T}$, a family of natural transformations in $\mathtt{Nat}(A^{\forall(\Phi)},\ \prod_{\Phi}(A^s)_{s\in T})$, for every $\Phi \in [T \Rightarrow T]$;
$\mathtt{tabst}^{\Phi}: \prod_{\Phi}(A^s)_{s\in T} \to A^{\forall(\Phi)}$, a family of natural transformations in $\mathtt{Nat}_p(\prod_{\Phi}(A^s)_{s\in T},\ A^{\forall(\Phi)})$, for every $\Phi \in [T \Rightarrow T]$;
$\Pi^{s,t}: A^{s\times t} \to A^s \times A^t$, a family of natural transformations in $\mathtt{Nat}(A^{s\times t},\ A^s \times A^t)$,
$\langle -, - \rangle^{s,t}: A^s \times A^t \to A^{s\times t}$, a family of natural transformations in $\mathtt{Nat}_p(A^s \times A^t,\ A^{s\times t})$;
$[-,\ -]^{s,t,d}: A^{s\to d} \times A^{t\to d} \to A^{(s+t)\to d}$, a family of natural transformations in $\mathtt{Nat}_p(A^{s\to d} \times A^{t\to d},\ A^{(s+t)\to d})$;
$\mathtt{inl}^{s,t}: A^s \to A^{s+t}$, a family of natural transformations in $\mathtt{Nat}(A^s,\ A^{s+t})$;
$\mathtt{inr}^{s,t}: A^t \to A^{s+t}$, a family of natural transformations in $\mathtt{Nat}(A^t,\ A^{s+t})$.

For every $u \in \mathcal{W}$, we define $\mathtt{cinl}_u: A_u^{(s+t)\to d} \to [A^s \Rightarrow A^d]_u$ and $\mathtt{cinr}_u: A_u^{(s+t)\to d} \to [A^t \Rightarrow A^d]_u$ as follows: For every $h \in A_u^{(s+t)\to d}$, for every $w \succeq u$,

$$(\mathtt{cinl}_u(h))_w(a) = \mathtt{eval}_w(\mathtt{fun}_w(i_{u,w}^{(s+t)\to d}(h)),\ \mathtt{inl}_w(a)),$$

for every $a \in A_w^s$, and

$$(\mathtt{cinr}_u(h))_w(b) = \mathtt{eval}_w(\mathtt{fun}_w(i_{u,w}^{(s+t)\to d}(h)),\ \mathtt{inr}_w(b)),$$

for every $b \in A_w^t$.

Furthermore, the following conditions are satisfied for every $w \in \mathcal{W}$:

(1) For all $s, t \in T$, if $A_w^s \neq \emptyset$ and $A_w^t \neq \emptyset$, then $A_w^{s\to t} \neq \emptyset$, and $\mathtt{fun}_w^{s,t}(\mathtt{abst}_w^{s,t}(\varphi)) \succeq_w \varphi$, whenever $\mathtt{abst}_w^{s,t}(\varphi)$ is defined, for $\varphi \in [A^s \Rightarrow A^t]_w$;

(2) If $A_w^{\Phi(t)} \neq \emptyset$ for every $t \in T$, then $A_w^{\forall(\Phi)} \neq \emptyset$, and $\mathtt{tfun}_w^{\Phi}(\mathtt{tabst}_w^{\Phi}(\varphi)) \succeq_w \varphi$, whenever $\mathtt{tabst}_w^{\Phi}(\varphi)$ is defined, for $\varphi \in \prod_{\Phi}(A_w^s)_{s\in T}$;

(3) For all $s, t \in T$, if $A_w^s \neq \emptyset$ and $A_w^t \neq \emptyset$, then $A_w^{s\times t} \neq \emptyset$, and $\Pi_w^{s,t}(\langle a,\ b\rangle) \succeq_w (a, b)$, for all $a \in A_w^s$, $b \in A_w^t$, whenever $\langle a,\ b\rangle$ is defined;

(4) For all $s, t \in T$, if $A_w^s \neq \emptyset$ and $A_w^t \neq \emptyset$, then $A_w^{s+t} \neq \emptyset$, and $\mathtt{cinl}_w([f,\ g]) \succeq_w \mathtt{fun}_w(f)$, and $\mathtt{cinr}_w([f,\ g]) \succeq_w \mathtt{fun}_w(g)$, whenever $[f,\ g]$ is defined, for $f \in A_w^{s\to d}$ and $g \in A_w^{t\to d}$.

We say that a Kripke pre-applicative β-structure is an *applicative β-structure* iff in conditions (1)-(4), $\succeq_w$ is replaced by the identity relation $=_w$.

We think of $\mathcal{W}$ as a set of *worlds*. When $\mathcal{A}$ is a Kripke applicative β-structure, then, in definition 4.1, conditions (1)-(4) amount to

(1) $\mathtt{fun}_w^{s,t} \circ \mathtt{abst}_w^{s,t} = \mathtt{id}_w$ on the domain of definition of $\mathtt{abst}_w$;

(2) $\mathtt{tfun}_w^{\Phi} \circ \mathtt{tabst}_w^{\Phi} = \mathtt{id}_w$ on the domain of definition of $\mathtt{tabst}_w$;

(3) $\Pi_w^{s,t} \circ \langle -,\ - \rangle_w^{s,t} = \mathtt{id}_w$ on the domain of definition of $\langle -,\ - \rangle_w$; and

(4) $\langle \mathtt{cinl}_w,\ \mathtt{cinr}_w \rangle \circ [-,\ -] = \mathtt{fun}_w^{s,d} \times \mathtt{fun}_w^{t,d}$ on the domain of definition of $[-,\ -]$.

In view of (1), from (4), we get

$\langle \mathtt{cinl}_w, \mathtt{cinr}_w \rangle \circ ([-, -]_w \circ (\mathtt{abst}_w^{s,d} \times \mathtt{abst}_w^{t,d})) = \mathtt{id}_w$ on the domain of definition of $[-, -]_w \circ (\mathtt{abst}_w^{s,d} \times \mathtt{abst}_w^{t,d})$.

In this case, $\mathtt{abst}_w$ is injective and $\mathtt{fun}_w$ is surjective on the domain of definition of $\mathtt{abst}_w$ (and left inverse to $\mathtt{abst}_w$), $\mathtt{tabst}_w$ is injective and $\mathtt{tfun}_w$ is surjective on the domain of definition of $\mathtt{tabst}_w$ (and left inverse to $\mathtt{tabst}_w$), $\langle -, - \rangle_w$ is injective and Π_w is surjective on the domain of definition of $\langle -, - \rangle_w$ (and left inverse to $\langle -, - \rangle_w$), $[-, -]_w \circ (\mathtt{abst}_w^{s,d} \times \mathtt{abst}_w^{t,d})$ is injective on its domain of definition, and $\langle \mathtt{cinl}_w, \mathtt{cinr}_w \rangle$ is surjective on this domain (and left inverse to $[-, -]_w \circ (\mathtt{abst}_w^{s,d} \times \mathtt{abst}_w^{t,d})$).

When we use a Kripke pre-applicative β-structure to interpret λ-terms, we assume that $\langle -, - \rangle$ and $[-, -]$ are total, and that the domains of abst and tabst are sufficently large, but we have not elucidated this last condition yet.

Using lemma 3.7, given $\mathtt{fun}^{s,t}: A^{s \to t} \to [A^s \Rightarrow A^t]$, we can define a natural transformation $\mathtt{app}^{s,t}: A^{s \to t} \times A^s \to A^t$, by

$$\mathtt{app}^{s,t} = \mathtt{eval}^{A^s, A^t} \circ (\mathtt{fun}^{s,t} \times \mathtt{id}_{A^s}).$$

Since $\theta = \mathtt{curry}(\mathtt{eval}^{F,G} \circ (\theta \times \mathtt{id}_F))$, from lemma 3.7, we also have

$$\mathtt{fun}^{s,t} = \mathtt{curry}(\mathtt{app}^{s,t}).$$

Thus, $\mathtt{app}^{s,t}$ and $\mathtt{fun}^{s,t}$ correspond to each other in the isomorphism between $\mathtt{Nat}(A^{s \to t} \times A^s, A^t)$ and $\mathtt{Nat}(A^{s \to t}, [A^s \Rightarrow A^t])$ set up by curry, uncurry. Thus, we could have used $\mathtt{app}^{s,t}$ instead of $\mathtt{fun}^{s,t}$ in definition 4.1. More explicitly, $\mathtt{app}_w^{s,t}(f, a)$ is defined such that, for every $f \in A_w^{s \to t}$ and every $a \in A_w^s$,

$$\mathtt{app}_w^{s,t}(f, a) = \mathtt{eval}^{A^s, A^t}(\mathtt{fun}_w^{s,t}(f), a).$$

Then, the functions $\mathtt{cinl}_u$ and $\mathtt{cinr}_u$ of definition 4.1 can be expressed in terms of app as follows: For every $h \in A_u^{(s+t) \to d}$,

$$(\mathtt{cinl}_u(h))_w(a) = \mathtt{app}_w(i_{u,w}^{(s+t) \to d}(h), \mathtt{inl}_w(a)),$$

for every $a \in A_w^s$, and

$$(\mathtt{cinr}_u(h))_w(b) = \mathtt{app}_w(i_{u,w}^{(s+t) \to d}(h), \mathtt{inr}_w(b)),$$

for every $b \in A_w^t$.

Using lemma 3.11, given $\mathtt{tfun}^{s,t}: A^{\forall(\Phi)} \to \prod_\Phi (A^s)_{s \in T}$, we can define a natural transformation $\mathtt{tapp}^\Phi: A^{\forall(\Phi)} \times T \to \coprod (A^{\Phi(s)})_{s \in T}$, by

$$\mathtt{tapp}^\Phi = \mathtt{eval}_\Phi^A \circ (\mathtt{tfun}^\Phi \times \mathtt{id}_T).$$

Since $\theta = \mathtt{curry}_\Phi(\mathtt{eval}_\Phi^A \circ (\theta \times \mathtt{id}_T))$, from lemma 3.11, we also have

$$\mathtt{tfun}^\Phi = \mathtt{curry}_\Phi(\mathtt{tapp}^\Phi).$$

Thus, $\mathtt{tapp}^{\Phi}$ and $\mathtt{tfun}^{\Phi}$ correspond to each other in the isomorphism between the sets of natural transformations $\mathtt{Nat}_{\Phi}(A^{\forall(\Phi)} \times T, \coprod(A^{\Phi(s)})_{s\in T})$ and $\mathtt{Nat}(A^{\forall(\Phi)}, \prod_{\Phi}(A^{s})_{s\in T})$ set up by $\mathtt{curry}_{\Phi}$, $\mathtt{uncurry}_{\Phi}$. Thus, we could have used $\mathtt{tapp}^{\Phi}$ instead of $\mathtt{tfun}^{\Phi}$ in definition 4.1. More explicitly, $\mathtt{tapp}^{\Phi}_{w}(f,t)$ is defined such that, for every $f \in A^{\forall(\Phi)}_{w}$ and every $t \in T$,

$$\mathtt{tapp}^{\Phi}_{w}(f,t) = \mathtt{eval}^{A}_{\Phi}(\mathtt{tfun}^{\Phi}_{w}(f), t).$$

The projection operators Π_w induce projections $\pi^{s,t}_{1,w}\colon A^{s\times t}_{w} \to A^{s}_{w}$ and $\pi^{s,t}_{2,w}\colon A^{s\times t}_{w} \to A^{t}_{w}$, such that for every $a \in A^{s\times t}_{w}$, if $\Pi^{s,t}_{w}(a) = (a_1, a_2)$, then

$$\pi^{s,t}_{1,w}(a) = a_1 \quad \text{and} \quad \pi^{s,t}_{2,w}(a) = a_2.$$

Let us now unravel the naturality conditions.

Definition 4.2 The following conditions hold whenever $w_1 \preceq w_2$.

(1) $\mathtt{fun}^{s,t}\colon A^{s\to t} \to [A^{s} \Rightarrow A^{t}]$. The naturality conditions are

$$\mathtt{fun}_{w_2} \circ i^{s\to t}_{w_1,w_2} = i^{s\Rightarrow t}_{w_1,w_2} \circ \mathtt{fun}_{w_1}.$$

These can be rewritten as follows: for any $g \in A^{s\to t}_{w_1}$, if $\mathtt{fun}_{w_1}(g) = (\varphi_w)_{w\succeq w_1}$, then

$$\mathtt{fun}_{w_2}(i^{s\to t}_{w_1,w_2}(g)) = (\varphi_w)_{w\succeq w_2}.$$

In terms of the operators app (recall that $\mathtt{app} = \mathtt{eval}^{A^{s},A^{t}} \circ (\mathtt{fun} \times \mathtt{id}_{A^{s}})$), the condition is written as

$$\mathtt{app}_{w_2}(i^{s\to t}_{w_1,w_2}(g), i^{s}_{w_1,w_2}(b)) = i^{t}_{w_1,w_2}(\mathtt{app}_{w_1}(g,b)),$$

for every $g \in A^{s\to t}_{w_1}$, and every $b \in A^{s}_{w_1}$.

(2) $\mathtt{abst}^{s,t}\colon [A^{s} \Rightarrow A^{t}] \to A^{s\to t}$. The naturality conditions are

$$\mathtt{abst}_{w_2} \circ i^{s\Rightarrow t}_{w_1,w_2} = i^{s\to t}_{w_1,w_2} \circ \mathtt{abst}_{w_1}.$$

These can be rewritten as follows:

$$\mathtt{abst}_{w_2}((\varphi_w)_{w\succeq w_2}) = i^{s\to t}_{w_1,w_2}(\mathtt{abst}_{w_1}((\varphi_w)_{w\succeq w_1})),$$

for every $\varphi = (\varphi_w)_{w\succeq w_1} \in [A^{s} \Rightarrow A^{t}]_{w_1}$.

(3) $\mathtt{tfun}^{\Phi}\colon A^{\forall(\Phi)} \to \prod_{\Phi}(A^{s})_{s\in T}$. The naturality conditions are

$$\mathtt{tfun}_{w_2} \circ i^{\forall(\Phi)}_{w_1,w_2} = i^{\Pi_{\Phi}}_{w_1,w_2} \circ \mathtt{tfun}_{w_1}.$$

These can be rewritten as follows: for any $g \in A^{\forall(\Phi)}_{w_1}$, if $\mathtt{tfun}_{w_1}(g) = (\varphi_w)_{w\succeq w_1}$, then

$$\mathtt{tfun}_{w_2}(i^{\forall(\Phi)}_{w_1,w_2}(g)) = (\varphi_w)_{w\succeq w_2}.$$

In terms of the operators tapp (recall that $\mathtt{tapp}^{\Phi} = \mathtt{eval}^{A}_{\Phi} \circ (\mathtt{tfun}^{\Phi} \times \mathtt{id}_{T})$), the condition is written as

$$\mathtt{tapp}_{w_2}(i^{\forall(\Phi)}_{w_1,w_2}(g), t) = i^{\Phi(t)}_{w_1,w_2}(\mathtt{tapp}_{w_1}(g,t)),$$

for every $g \in A_{w_1}^{\forall(\Phi)}$, and every $t \in T$.

(4) $\mathtt{tabst}^{\Phi}\colon \prod_{\Phi}(A^s)_{s\in T} \to A^{\forall(\Phi)}$. The naturality conditions are

$$\mathtt{tabst}_{w_2} \circ i_{w_1,w_2}^{\Pi_\Phi} = i_{w_1,w_2}^{\forall(\Phi)} \circ \mathtt{tabst}_{w_1}.$$

These can be rewritten as follows:

$$\mathtt{tabst}_{w_2}((\varphi_w)_{w\succeq w_2}) = i_{w_1,w_2}^{\forall(\Phi)}(\mathtt{tabst}_{w_1}((\varphi_w)_{w\succeq w_1})),$$

for every $\varphi = (\varphi_w)_{w\succeq w_1} \in \prod_{\Phi}(A_{w_1}^s)_{s\in T}$.

(5) $\Pi^{s,t}\colon A^{s\times t} \to A^s \times A^t$. The naturality conditions are

$$\Pi_{w_2} \circ i_{w_1,w_2}^{s\times t} = (i_{w_1,w_2}^s \times i_{w_1,w_2}^t) \circ \Pi_{w_1}.$$

These can be rewritten as

$$\Pi_{w_2}(i_{w_1,w_2}^{s\times t}(b)) = (i_{w_1,w_2}^s(\pi_{1,w_1}(b)),\ i_{w_1,w_2}^t(\pi_{2,w_1}(b)),$$

for all $b \in A_{w_1}^{s\times t}$.

(6) $\langle -,-\rangle^{s,t}\colon A^s \times A^t \to A^{s\times t}$. The naturality conditions are

$$\langle -,\ -\rangle_{w_2} \circ (i_{w_1,w_2}^s \times i_{w_1,w_2}^t) = i_{w_1,w_2}^{s\times t} \circ \langle -,\ -\rangle_{w_1}.$$

These can be rewritten as

$$\langle i_{w_1,w_2}^s(b_1),\ i_{w_1,w_2}^t(b_2)\rangle_{w_2} = i_{w_1,w_2}^{s\times t}(\langle b_1,\ b_2\rangle_{w_1}),$$

for all $b_1 \in A_{w_1}^s$ and all $b_2 \in A_{w_1}^t$.

(7) $\mathtt{inl}^{s,t}\colon A^s \to A^{s+t}$ and $\mathtt{inr}^{s,t}\colon A^t \to A^{s+t}$. The naturality conditions are

$$\mathtt{inl}_{w_2} \circ i_{w_1,w_2}^s = i_{w_1,w_2}^{s+t} \circ \mathtt{inl}_{w_1} \quad \text{and} \quad \mathtt{inr}_{w_2} \circ i_{w_1,w_2}^t = i_{w_1,w_2}^{s+t} \circ \mathtt{inr}_{w_1}.$$

These can be rewritten as

$$\mathtt{inl}_{w_2}(i_{w_1,w_2}^s(a)) = i_{w_1,w_2}^{s+t}(\mathtt{inl}_{w_1}(a)) \quad \text{and} \quad \mathtt{inr}_{w_2}(i_{w_1,w_2}^t(b)) = i_{w_1,w_2}^{s+t}(\mathtt{inr}_{w_1}(b)),$$

where in the first case, $a \in A_{w_1}^s$, and in the second case, $b \in A_{w_1}^t$.

(8) $[-,\ -]^{s,t,d}\colon A^{s\to d} \times A^{t\to d} \to A^{(s+t)\to d}$. The naturality conditions are

$$[-,\ -]_{w_2} \circ (i_{w_1,w_2}^{s\to d} \times i_{w_1,w_2}^{t\to d}) = i_{w_1,w_2}^{(s+t)\to d} \circ [-,\ -]_{w_1}.$$

These can be rewritten as

$$[i_{w_1,w_2}^{s\to d}(f),\ i_{w_1,w_2}^{t\to d}(g)]_{w_2} = i_{w_1,w_2}^{(s+t)\to d}([f,\ g]_{w_1}),$$

where $f \in A_{w_1}^{s\to d}$ and $g \in A_{w_1}^{t\to d}$.

Let us give an (important) example of a Kripke pre-applicative structure. First, we review the notion of a substitution.

Definition 4.3 A substitution φ is a function $\varphi: \mathcal{V} \cup \mathcal{X} \to \mathcal{T} \cup \mathit{Terms}$, such that $\varphi(X) \in \mathcal{T}$ if $X \in \mathcal{V}$, $\varphi(x) \in \mathit{Terms}$ if $x \in \mathcal{X}$, and $\varphi(x) \neq x$ only for finitely many variables. We let $\mathit{dom}(\varphi) = \{x \in \mathcal{V} \cup \mathcal{X} \mid \varphi(x) \neq x\}$. We say that φ is a *type-substitution* if $\mathit{dom}(\varphi) \subseteq \mathcal{V}$. Given two contexts Γ and Δ, we say that φ *satisfies* Γ *at* Δ, denoted as $\Delta \Vdash \Gamma[\varphi]$, iff $\Delta \triangleright \varphi(x): \sigma[\varphi]$, for every $x: \sigma \in \Gamma$ (Compare with definition 5.4: φ is a valuation, the type-substitution part of φ being a type valuation).

Definition 4.4 Let $\langle \mathcal{W}, \preceq \rangle$ be the poset of all type assignments $\Gamma = x_1: \sigma_1, \ldots, x_n: \sigma_n$ ordered by inclusion, T be the free algebra of second-order types, and let A^σ_Γ be the set of all provable typing judgements $\Gamma \triangleright M: \sigma$. For $[T \Rightarrow T]$, we can take the set of all functions Φ of the form $\tau \mapsto \sigma[\tau/X]$, where $\sigma, \tau \in T$ are any types, and X is any fixed variable that does not occur in Γ. Then, $\forall(\Phi) = \forall X. \sigma$.[3] The map $i^\sigma_{\Gamma_1,\Gamma_2}: A^\sigma_{\Gamma_1} \to A^\sigma_{\Gamma_2}$ is the function such that $i^\sigma_{\Gamma_1,\Gamma_2}(\Gamma_1 \triangleright M: \sigma) = \Gamma_2 \triangleright M: \sigma$.

We let Π, $\langle -, - \rangle$, **inl**, **inr**, and $[-, -]$, be the obvious. For example, $\langle \Gamma \triangleright M_1: \sigma,\ \Gamma \triangleright M_2: \tau \rangle = \Gamma \triangleright \langle M_1,\ M_2 \rangle: \sigma \times \tau$. Define $\Gamma \triangleright N: \sigma \preceq \Gamma \triangleright M: \sigma$ iff $M \xrightarrow{*}_\beta N$. Finally, we need to define **fun**, **abst**, **tfun**, and **tabst**.

We define $\mathbf{fun}_\Gamma(\Gamma \triangleright M: \sigma \to \tau)$ as the family of functions $([\Gamma \triangleright M: \sigma \to \tau]_\Delta)_{\Gamma \subseteq \Delta}$, where the function $[\Gamma \triangleright M: \sigma \to \tau]_\Delta$ is from A^σ_Δ to A^τ_Δ, such that

$$[\Gamma \triangleright M: \sigma \to \tau]_\Delta(\Delta \triangleright N: \sigma) = \Delta \triangleright MN: \tau,$$

for every $\Delta \triangleright N: \sigma \in A^\sigma_\Delta$.

We define $\mathbf{tfun}_\Gamma(\Gamma \triangleright M: \forall X. \sigma)$ as the family of functions $([\Gamma \triangleright M: \forall X. \sigma]_\Delta)_{\Gamma \subseteq \Delta}$, where the function $[\Gamma \triangleright M: \forall X. \sigma]_\Delta$ is from T to $\coprod(A^\sigma_\Delta)_{\sigma \in T}$, such that

$$[\Gamma \triangleright M: \forall X. \sigma]_\Delta(\tau) = \Delta \triangleright M\tau: \sigma[\tau/X],$$

for every $\tau \in T$. In this case, the Φ in $\mathbf{tfun}^\Phi_\Gamma$ is the function from T to T induced by σ, such that $\Phi(\tau) = \sigma[\tau/X]$ for every $\tau \in T$.

For every (type and term)-substitution φ, every judgement $\Gamma, x: \sigma \triangleright M: \tau$, and every context Δ such that $\Delta \Vdash (\Gamma, x: \sigma)[\varphi]$, consider the family of functions $(\varphi[\Gamma, x: \sigma \triangleright M: \tau]_{\Delta'})_{\Delta \subseteq \Delta'}$, where the function $\varphi[\Gamma, x: \sigma \triangleright M: \tau]_{\Delta'}$ is from $A^{\sigma[\varphi]}_{\Delta'}$ to $A^{\tau[\varphi]}_{\Delta'}$, defined such that,

$$\varphi[\Gamma, x: \sigma \triangleright M: \tau]_{\Delta'}(\Delta' \triangleright N: \sigma[\varphi]) = \Delta' \triangleright M[\varphi[x := N]]: \tau[\varphi],$$

for every $\Delta' \triangleright N: \sigma[\varphi] \in A^{\sigma[\varphi]}_{\Delta'}$. Given any such family $(\varphi[\Gamma, x: \sigma \triangleright M: \tau]_{\Delta'})_{\Delta \subseteq \Delta'}$, we let

$$\mathbf{abst}_\Delta((\varphi[\Gamma, x: \sigma \triangleright M: \tau]_{\Delta'})_{\Delta \subseteq \Delta'}) = \Delta \triangleright (\lambda x: \sigma. M)[\varphi]: \sigma[\varphi] \to \tau[\varphi].$$

For every (type and term)-substitution φ, every judgement $\Gamma, X: \star \triangleright M: \sigma$, and every context Δ such that $\Delta \Vdash (\Gamma, X: \star)[\varphi]$, consider the family of functions $(\varphi[\Gamma, X: \star \triangleright M: \sigma]_{\Delta'})_{\Delta \subseteq \Delta'}$, where the function $\varphi[\Gamma, X: \star \triangleright M: \sigma]_{\Delta'}$ is from T to $\coprod(A^\sigma_{\Delta'})_{\sigma \in T}$, defined such that,

$$\varphi[\Gamma, X: \star \triangleright M: \sigma]_{\Delta'}(\tau) = \Delta' \triangleright M[\varphi[X := \tau]]: \sigma[\varphi[X := \tau]],$$

[3] The choice of X is irrelevant as long as X does not occur in Γ, since X is bound in $\forall X. \sigma$.

for every $\tau \in T$.

Given any such family $(\varphi[\Gamma, X{:}\star \triangleright M{:}\sigma]_{\Delta'})_{\Delta\subseteq\Delta'}$, we let

$$\mathtt{tabst}_\Delta((\varphi[\Gamma, X{:}\star \triangleright M{:}\sigma]_{\Delta'})_{\Delta\subseteq\Delta'}) = \Delta \triangleright (\lambda X.\, M)[\varphi]{:}\, \forall X.\, \sigma[\varphi].$$

The Kripke pre-applicative β-structure just defined is denoted as $\mathcal{LT}_\beta$.

It is clear that $(\varphi[\Gamma, x{:}\sigma \triangleright M{:}\tau]_{\Delta'})_{\Delta\subseteq\Delta'}$ is in $[A^{\sigma[\varphi]} \Rightarrow A^{\tau[\varphi]}]_\Delta$. Let us verify that

$$\mathtt{fun}_\Delta(\mathtt{abst}_\Delta((\varphi[\Gamma, x{:}\sigma \triangleright M{:}\tau]_{\Delta'})_{\Delta\subseteq\Delta'})) \succeq (\varphi[\Gamma, x{:}\sigma \triangleright M{:}\tau]_{\Delta'})_{\Delta\subseteq\Delta'}.$$

Since

$$\mathtt{fun}_\Delta(\mathtt{abst}_\Delta((\varphi[\Gamma, x{:}\sigma \triangleright M{:}\tau]_{\Delta'})_{\Delta\subseteq\Delta'})) = \mathtt{fun}_\Delta(\Delta \triangleright (\lambda x{:}\sigma.\, M)[\varphi]{:}\,\sigma[\varphi] \to \tau[\varphi]),$$
$$\mathtt{fun}_\Delta(\Delta \triangleright (\lambda x{:}\sigma.\, M)[\varphi]{:}\,\sigma[\varphi] \to \tau[\varphi]) = ([\Delta \triangleright (\lambda x{:}\sigma.\, M)[\varphi]{:}\,\sigma[\varphi] \to \tau[\varphi]]_{\Delta'})_{\Delta\subseteq\Delta'},$$
$$[\Delta \triangleright (\lambda x{:}\sigma.\, M)[\varphi]{:}\,\sigma[\varphi] \to \tau[\varphi]]_{\Delta'}(\Delta' \triangleright N{:}\sigma[\varphi]) = \Delta' \triangleright ((\lambda x{:}\sigma.\, M)[\varphi])N{:}\,\tau[\varphi],$$
$$\varphi[\Gamma, x{:}\sigma \triangleright M{:}\tau]_{\Delta'}(\Delta' \triangleright N{:}\sigma[\varphi]) = \Delta' \triangleright M[\varphi[x{:}= N]]{:}\,\tau[\varphi],$$

and

$$((\lambda x{:}\sigma.\, M)[\varphi])N \longrightarrow_\beta M[\varphi[x{:}= N]],$$

the inequality holds. Indeed, $(\lambda x{:}\sigma.\, M)[\varphi]$ is α-equivalent to $(\lambda y{:}\sigma.\, M[y/x])[\varphi]$ for any variable y such that $y \notin dom(\varphi)$ and $y \notin \varphi(z)$ for every $z \in dom(\varphi)$, and for such a y, $(\lambda y{:}\sigma.\, M[y/x])[\varphi] = (\lambda y{:}\sigma[\varphi].\, M[y/x][\varphi])$. Then, for this choice of y,

$$(\lambda y{:}\sigma[\varphi].\, M[y/x][\varphi])N \longrightarrow_\beta M[y/x][\varphi][N/y] = M[\varphi[x{:}= N]].$$

Regarding the definition of **tabst**, letting Φ be the function from T to T induced by σ, such that $\Phi(\tau) = \sigma[\tau/X]$ for every $\tau \in T$, it is clear that $(\varphi[\Gamma, X{:}\star \triangleright M{:}\sigma]_{\Delta'})_{\Delta\subseteq\Delta'}$ is in $\prod_\Phi(A^s_\Delta)_{s\in T}$. Let us now verify that

$$\mathtt{tfun}_\Delta(\mathtt{tabst}_\Delta((\varphi[\Gamma, X{:}\star \triangleright M{:}\sigma]_{\Delta'})_{\Delta\subseteq\Delta'})) \succeq (\varphi[\Gamma, X{:}\star \triangleright M{:}\sigma]_{\Delta'})_{\Delta\subseteq\Delta'}.$$

Since

$$\mathtt{tfun}_\Delta(\mathtt{tabst}_\Delta((\varphi[\Gamma, X{:}\star \triangleright M{:}\sigma]_{\Delta'})_{\Delta\subseteq\Delta'})) = \mathtt{tfun}_\Delta(\Delta \triangleright (\lambda X.\, M)[\varphi]{:}\, \forall X.\, \sigma[\varphi]),$$
$$\mathtt{tfun}_\Delta(\Delta \triangleright (\lambda X.\, M)[\varphi]{:}\, \forall X.\, \sigma[\varphi]) = ([\Delta \triangleright (\lambda X.\, M)[\varphi]{:}\, \forall X.\, \sigma[\varphi]]_{\Delta'})_{\Delta\subseteq\Delta'},$$
$$[\Delta \triangleright (\lambda X.\, M)[\varphi]{:}\, \forall X.\, \sigma[\varphi]]_{\Delta'}(\tau) = \Delta' \triangleright ((\lambda X.\, M)[\varphi])\tau{:}\, \sigma[\varphi][\tau/X],$$
$$\varphi[\Gamma, X{:}\star \triangleright M{:}\sigma]_{\Delta'}(\tau) = \Delta' \triangleright M[\varphi[X{:}= \tau]]{:}\, \sigma[\varphi[X{:}= \tau]],$$
$$\sigma[\varphi][\tau/X] = \sigma[\varphi[X{:}= \tau]],$$

(by a suitable α-renaming on X), and

$$((\lambda X.\, M)[\varphi])\tau \longrightarrow_\beta M[\varphi[X{:}= \tau]],$$

the inequality holds (the details of the verification using α-renaming are similar to the previous case).

The other conditions of definition 4.1 are easily verified.

We now define extensional Kripke pre-applicative β-structures and Kripke pre-applicative $\beta\eta$-structures.

Definition 4.5 A Kripke pre-applicative β-structure $\langle \mathcal{W}, T, \mathcal{A}\rangle$ is *extensional* iff $\mathtt{fun}_w$, $\mathtt{tfun}_w$, Π_w, and $\langle \mathtt{cinl}_w, \mathtt{cinr}_w\rangle$, are isotone, and the following conditions hold for every $w \in \mathcal{W}$:

(1) $ran(\mathtt{fun}_w) \subseteq dom(\mathtt{abst}_w)$;

(2) $ran(\mathtt{tfun}_w) \subseteq dom(\mathtt{tabst}_w)$;

(3) $ran(\Pi_w) \subseteq dom(\langle -, -\rangle_w)$;

(4) $ran(\langle \mathtt{cinl}_w^{s,t,d}, \mathtt{cinr}_w^{s,t,d}\rangle) \subseteq dom([-, -]_w \circ (\mathtt{abst}_w^{s,d} \times \mathtt{abst}_w^{t,d}))$.

When $\mathcal{A}$ is an applicative Kripke β-structure, conditions (1)-(4) hold, and the functions $\mathtt{fun}_w$, $\mathtt{tfun}_w$, Π_w, and $\langle \mathtt{cinl}_w, \mathtt{cinr}_w\rangle$, are injective, we say that we have an *extensional Kripke applicative β-structure*.

When $\mathcal{A}$ is a Kripke extensional pre-applicative β-structure, by condition (1), $\mathtt{abst}_w(\mathtt{fun}_w(f))$ is defined for any $f \in A_w^{s\to t}$. Observe that by condition (1) of definition 4.1, we have $\mathtt{fun}_w(f) \preceq \mathtt{fun}_w(\mathtt{abst}_w(\mathtt{fun}_w(f)))$, and since $\mathtt{fun}_w$ is isotone, this implies that

(1) $\mathtt{abst}_w(\mathtt{fun}_w(f)) \succeq_w f$, for all $f \in A_w^{s\to t}$.

Similarly, we can prove that

(2) $\mathtt{tabst}_w(\mathtt{tfun}_w(f)) \succeq_w f$, for all $f \in A_w^{\forall(\Phi)}$;

(3) $\langle \pi_1(a), \pi_2(a)\rangle_w \succeq_w a$, for all $a \in A_w^{s\times t}$; and

(4) $[\mathtt{abst}_w(\mathtt{cinl}_w(h)), \mathtt{abst}_w(\mathtt{cinr}_w(h))]_w \succeq_w h$, for all $h \in A_w^{(s+t)\to d}$.

We will call the above inequalities the *η-like rules*.

In many cases, a Kripke pre-applicative β-structure that satisfies the η-like rules is not extensional. This motivates the next definition.

Definition 4.6 A Kripke pre-applicative β-structure $\langle \mathcal{W}, T, \mathcal{A}\rangle$ is a *$\beta\eta$-structure* if the following conditions hold for every $w \in \mathcal{W}$:

(1) $ran(\mathtt{fun}_w) \subseteq dom(\mathtt{abst}_w)$, and $\mathtt{abst}_w(\mathtt{fun}_w(f)) \succeq_w f$, for all $f \in A_w^{s\to t}$;

(2) $ran(\mathtt{tfun}_w) \subseteq dom(\mathtt{tabst}_w)$, and $\mathtt{tabst}_w(\mathtt{tfun}_w(f)) \succeq_w f$, for all $f \in A_w^{\forall(\Phi)}$;

(3) $ran(\Pi_w) \subseteq dom(\langle -, -\rangle_w)$, and $\langle \pi_1(a), \pi_2(a)\rangle_w \succeq_w a$, for all $a \in A_w^{s\times t}$; and

(4) $ran(\langle \mathtt{cinl}_w^{s,t,d}, \mathtt{cinr}_w^{s,t,d}\rangle) \subseteq dom([-, -]_w \circ (\mathtt{abst}_w^{s,d} \times \mathtt{abst}_w^{t,d}))$, and $[\mathtt{abst}_w(\mathtt{cinl}_w(h)), \mathtt{abst}_w(\mathtt{cinr}_w(h))]_w \succeq_w h$, for all $h \in A_w^{(s+t)\to d}$.

When $\mathcal{A}$ is an applicative Kripke β-structure and in conditions (1)-(4), $\succeq_w$ is replaced by $=_w$, we say that we have a *Kripke applicative $\beta\eta$-structure*.

From the remark before definition 4.6, an extensional Kripke pre-applicative β-structure is a $\beta\eta$-structure. When $\mathcal{A}$ is a Kripke applicative $\beta\eta$-structure, conditions (1)-(4) of definition 4.6 amount to:

(1) $\mathtt{abst}_w^{s,t} \circ \mathtt{fun}_w^{s,t} = \mathtt{id}_w$;

(2) $\mathtt{tabst}_w^\Phi \circ \mathtt{tfun}_w^\Phi = \mathtt{id}_w$;

(3) $\langle -,\ - \rangle_w^{s,t} \circ \Pi_w^{s,t} = \mathtt{id}_w$; and

(4) $([-,\ -]_w \circ (\mathtt{abst}_w^{s,d} \times \mathtt{abst}_w^{t,d})) \circ \langle \mathtt{cinl}_w^{s,t,d},\ \mathtt{cinr}_w^{s,t,d} \rangle = \mathtt{id}_w$.

This implies that $\mathtt{fun}_w$, $\mathtt{tfun}_w$, Π_w, and $\langle \mathtt{cinl}_w,\ \mathtt{cinr}_w \rangle$, are injective. Thus, a Kripke applicative $\beta\eta$-structure is extensional. In this case, (together with conditions (1)-(4) of definition 4.1 in the case of a Kripke applicative β-structure), we have $dom(\mathtt{abst}_w) = \mathtt{fun}_w(A_w^{s\to t})$, $\mathtt{fun}_w$ is a bijection between $A_w^{s\to t}$ and a subset of $[A^s \Rightarrow A^t]_w$ (with inverse $\mathtt{abst}_w$), $dom(\mathtt{tabst}_w) = \mathtt{tfun}_w(A_w^{\forall(\Phi)})$, $\mathtt{tfun}_w$ is a bijection between $A_w^{\forall(\Phi)}$ and a subset of $\prod_\Phi (A_w^s)_{s\in T}$ (with inverse $\mathtt{tabst}_w$), Π_w is a bijection between $A_w^{s\times t}$ and a subset of $A_w^s \times A_w^t$ (with inverse $\langle -,\ - \rangle_w$), and $\langle \mathtt{cinl}_w^{s,t,d},\ \mathtt{cinr}_w^{s,t,d} \rangle$ is a bijection between $A_w^{(s+t)\to d}$ and a subset of $[A^s \Rightarrow A^d]_w \times [A^t \Rightarrow A^d]_w$ (with inverse $[-,\ -]_w \circ (\mathtt{abst}_w^{s,d} \times \mathtt{abst}_w^{t,d})$).

We now show how the structure $\mathcal{LT}_\beta$ of definition 4.4 can be made into a pre-applicative $\beta\eta$-structure. First, we recall the η-like rules.

Definition 4.7 The set of *η-like reduction rules* is defined as follows.

$$\begin{aligned}
\lambda x{:}\,\sigma.\,(Mx) &\longrightarrow M, && \text{if } x \notin FV(M),\\
\lambda X.\,(MX) &\longrightarrow M, && \text{if } X \notin FTV(M),\\
\langle \pi_1(M),\ \pi_2(M) \rangle &\longrightarrow M, &&\\
[\lambda x{:}\,\sigma.\,(M\,\mathtt{inl}(x)),\ \lambda y{:}\,\tau.\,(M\,\mathtt{inr}(y))] &\longrightarrow M. &&
\end{aligned}$$

We will denote the reduction relation defined by the union of the rules of definition 2.3 and of definition 4.7 as $\longrightarrow_{\beta\eta}$ (even though there are reductions other that β-reduction and η-reduction).

Definition 4.8 We define a Kripke pre-applicative structure as in definition 4.4, except that $\Gamma \triangleright M{:}\,\sigma \preceq \Gamma \triangleright N{:}\,\sigma$ iff $N \xrightarrow{*}_{\beta\eta} M$, and that $\mathtt{abst}$ and $\mathtt{tabst}$ have a larger domain of definition. First, recall the definition of families of functions used in defining $\mathtt{fun}$ and $\mathtt{tfun}$.

$\mathtt{fun}_\Gamma(\Gamma \triangleright M{:}\,\sigma \to \tau)$ is defined as the family of functions $([\Gamma \triangleright M{:}\,\sigma \to \tau]_\Delta)_{\Gamma\subseteq\Delta}$, where the function $[\Gamma \triangleright M{:}\,\sigma \to \tau]_\Delta$ is from A_Δ^σ to A_Δ^τ, such that

$$[\Gamma \triangleright M{:}\,\sigma \to \tau]_\Delta(\Delta \triangleright N{:}\,\sigma) = \Delta \triangleright MN{:}\,\tau,$$

for every $\Delta \triangleright N{:}\,\sigma \in A_\Delta^\sigma$.

$\mathtt{tfun}_\Gamma(\Gamma \triangleright M{:}\,\forall X.\,\sigma)$ is defined as the family of functions $([\Gamma \triangleright M{:}\,\forall X.\,\sigma]_\Delta)_{\Gamma\subseteq\Delta}$, where the function $[\Gamma \triangleright M{:}\,\forall X.\,\sigma]_\Delta$ is from T to $\coprod (A_\Delta^\sigma)_{\sigma\in T}$, such that

$$[\Gamma \triangleright M{:}\,\forall X.\,\sigma]_\Delta(\tau) = \Delta \triangleright M\tau{:}\,\sigma[\tau/X],$$

for every $\tau \in T$. In this case, the Φ in $\mathtt{tfun}_\Gamma^\Phi$ is the function from T to T induced by σ, such that $\Phi(\tau) = \sigma[\tau/X]$ for every $\tau \in T$.

Then, we define

$$\mathtt{abst}_\Gamma(([\Gamma \triangleright M{:}\,\sigma \to \tau]_\Delta)_{\Gamma\subseteq\Delta}) = \Gamma \triangleright \lambda x{:}\,\sigma.\,(Mx){:}\,\sigma \to \tau,$$

where $x \notin FV(M)$, and

$$\mathbf{tabstr}_\Gamma(([\Gamma \rhd M{:}\forall X.\,\sigma]_\Delta)_{\Gamma\subseteq\Delta}) = \Gamma \rhd \lambda X.\,(MX){:}\forall X.\,\sigma,$$

where $X \notin FTV(M)$. The structure just defined is denoted as $\mathcal{LT}_{\beta\eta}$.

We need to check that $\mathcal{LT}_{\beta\eta}$ is a Kripke pre-applicative $\beta\eta$-structure. Let us first verify that

$$\mathbf{fun}_\Gamma(\mathbf{abstr}_\Gamma(([\Gamma \rhd M{:}\sigma \to \tau]_\Delta)_{\Gamma\subseteq\Delta})) \succeq ([\Gamma \rhd M{:}\sigma \to \tau]_\Delta)_{\Gamma\subseteq\Delta}.$$

Since

$$\mathbf{fun}_\Gamma(\mathbf{abstr}_\Gamma(([\Gamma \rhd M{:}\sigma \to \tau]_\Delta)_{\Gamma\subseteq\Delta})) = \mathbf{fun}_\Gamma(\Gamma \rhd \lambda x{:}\sigma.\,(Mx){:}\sigma \to \tau),$$

$$\mathbf{fun}_\Gamma(\Gamma \rhd \lambda x{:}\sigma.\,(Mx){:}\sigma \to \tau) = ([\Gamma \rhd \lambda x{:}\sigma.\,(Mx){:}\sigma \to \tau]_\Delta)_{\Gamma\subseteq\Delta},$$

$$[\Gamma \rhd \lambda x{:}\sigma.\,(Mx){:}\sigma \to \tau]_\Delta(\Delta \rhd N{:}\sigma) = \Delta \rhd (\lambda x{:}\sigma.\,(Mx))N{:}\tau,$$

$$[\Gamma \rhd M{:}\sigma \to \tau]_\Delta(\Delta \rhd N{:}\sigma) = \Delta \rhd MN{:}\tau,$$

and

$$(\lambda x{:}\sigma.\,(Mx))N \longrightarrow_\beta MN,$$

since $x \notin FV(M)$, the inequality holds.

Let us now verify that

$$\mathbf{tfun}_\Gamma(\mathbf{tabstr}_\Gamma(([\Gamma \rhd M{:}\forall X.\,\sigma]_\Delta)_{\Gamma\subseteq\Delta})) \succeq ([\Gamma \rhd M{:}\forall X.\,\sigma]_\Delta)_{\Gamma\subseteq\Delta}.$$

Since

$$\mathbf{tfun}_\Gamma(\mathbf{tabstr}_\Gamma(([\Gamma \rhd M{:}\forall X.\,\sigma]_\Delta)_{\Gamma\subseteq\Delta})) = \mathbf{tfun}_\Gamma(\Gamma \rhd \lambda X.\,(MX){:}\forall X.\,\sigma),$$

$$\mathbf{tfun}_\Gamma(\Gamma \rhd \lambda X.\,(MX){:}\forall X.\,\sigma) = ([\Gamma \rhd \lambda X.\,(MX){:}\forall X.\,\sigma]_\Delta)_{\Gamma\subseteq\Delta},$$

$$[\Gamma \rhd \lambda X.\,(MX){:}\forall X.\,\sigma]_\Delta(\tau) = \Delta \rhd (\lambda X.\,(MX))\tau{:}\sigma[\tau/X],$$

$$[\Gamma \rhd M{:}\forall X.\,\sigma]_\Delta(\tau) = \Delta \rhd M\tau{:}\sigma[\tau/X],$$

and

$$(\lambda X.\,(MX))\tau \longrightarrow_\beta M\tau,$$

since $X \notin FTV(M)$, the inequality holds.

We also need to verify the conditions of definition 4.6.

We have $\mathbf{abstr}_\Gamma(\mathbf{fun}_\Gamma(\Gamma \rhd M{:}\sigma \to \tau)) = \mathbf{abstr}_\Gamma(([\Gamma \rhd M{:}\sigma \to \tau]_\Delta)_{\Gamma\subseteq\Delta})$, and since

$$\mathbf{abstr}_\Gamma(([\Gamma \rhd M{:}\sigma \to \tau]_\Delta)_{\Gamma\subseteq\Delta}) = \Gamma \rhd \lambda x{:}\sigma.\,(Mx){:}\sigma \to \tau,$$

where $x \notin FV(M)$, and by the η-like rule, $\lambda x{:}\sigma.\,(Mx) \longrightarrow_{\beta\eta} M$, we have

$$\mathbf{abstr}_\Gamma(\mathbf{fun}_\Gamma(\Gamma \rhd M{:}\sigma \to \tau)) \succeq \Gamma \rhd M{:}\sigma \to \tau.$$

Similarly, we have $\mathbf{tabstr}_\Gamma(\mathbf{tfun}_\Gamma(\Gamma \rhd M{:}\forall X.\,\sigma)) = \mathbf{tabstr}_\Gamma(([\Gamma \rhd M{:}\forall X.\,\sigma]_\Delta)_{\Gamma\subseteq\Delta})$, and since

$$\mathbf{tabstr}_\Gamma(([\Gamma \rhd M{:}\forall X.\,\sigma]_\Delta)_{\Gamma\subseteq\Delta}) = \Gamma \rhd \lambda X.\,(MX){:}\forall X.\,\sigma,$$

where $X \notin FTV(M)$, and by the η-like rule, $\lambda X.(MX) \longrightarrow_{\beta\eta} M$, we have

$$\mathtt{tabst}_\Gamma(\mathtt{tfun}_\Gamma(\Gamma \triangleright M{:}\forall X.\sigma)) \succeq \Gamma \triangleright M{:}\forall X.\sigma.$$

The other conditions of definition 4.6, are immediately verified. We now give a convenient characterization of the isotonicity of $\mathtt{fun}_u$ and $\mathtt{tfun}_u$. This lemma shows the equivalence between our definition of extentionality and Mitchell and Moggi's definition [12], in the case of first-order applicative structures.

Lemma 4.9 *Given a Kripke pre-applicative β-structure $\mathcal{A}$, then the following properties hold for every $u \in \mathcal{W}$: (1) $\mathtt{fun}_u$ is isotone iff for every $f, g \in A_u^{s\to t}$, if $\mathtt{app}_v(i_{u,v}^{s\to t}(f), b) \preceq \mathtt{app}_v(i_{u,v}^{s\to t}(g), b)$ for all $b \in A_v^s$ and all $v \succeq u$, then $f \preceq g$.*

(2) $\mathtt{tfun}_u$ is isotone iff for every $f, g \in A_u^{\forall(\Phi)}$, if $\mathtt{tapp}_v(i_{u,v}^{\forall(\Phi)}(f), t) \preceq \mathtt{tapp}_v(i_{u,v}^{\forall(\Phi)}(g), t)$ for all $t \in T$ and all $v \succeq u$, then $f \preceq g$.

Proof. (1) First, assume that $\mathtt{fun}_u$ is isotone. Recall that the naturality condition for $\mathtt{fun}$ is

$$\mathtt{fun}_{w_2}(i_{w_1,w_2}^{s\to t}(g)) = (\varphi_w)_{w\succeq w_2},$$

for any $g \in A_{w_1}^{s\to t}$, if $\mathtt{fun}_{w_1}(g) = (\varphi_w)_{w\succeq w_1}$. Let $\mathtt{fun}_u(f) = (\varphi_w)_{w\succeq u}$ and $\mathtt{fun}_u(g) = (\psi_w)_{w\succeq u}$. If $\mathtt{app}_v(i_{u,v}^{s\to t}(f), b) \preceq \mathtt{app}_v(i_{u,v}^{s\to t}(g), b)$ for all $b \in A_v^s$ and all $v \succeq u$, since $\mathtt{app}$ is defined from $\mathtt{fun}$ as $\mathtt{app} = \mathtt{eval}^{A^s,A^t} \circ (\mathtt{fun} \times \mathtt{id}_{A^s})$, and $\mathtt{eval}_u^{A^s,A^t}((\varphi_w)_{w\succeq u}, a) = \varphi_u(a)$, we have

$$\begin{aligned}\mathtt{app}_v(i_{u,v}^{s\to t}(f), b) &= \mathtt{eval}_v^{A^s,A^t}(\mathtt{fun}_v(i_{u,v}^{s\to t}(f)), b)\\ &= \mathtt{eval}_v^{A^s,A^t}((\varphi_w)_{w\succeq v}, b)\\ &= \varphi_v(b).\end{aligned}$$

Similarly, we get

$$\mathtt{app}_v(i_{u,v}^{s\to t}(g), b) = \psi_v(b).$$

Thus, the hypothesis implies that $\varphi_v(b) \preceq \psi_v(b)$ for all $b \in A_v^s$, and thus $\varphi_v \preceq \psi_v$. Since this holds for all $v \succeq u$, we have $(\varphi_v)_{v\succeq u} \preceq (\psi_v)_{v\succeq u}$, that is, $\mathtt{fun}_u(f) \preceq \mathtt{fun}_v(f)$, and since $\mathtt{fun}_u$ is isotone, we have $f \preceq g$.

Now, assume that $f \preceq g$ whenever $\mathtt{app}_v(i_{u,v}^{s\to t}(f), b) \preceq \mathtt{app}_v(i_{u,v}^{s\to t}(g), b)$ for all $b \in A_v^s$ and all $v \succeq u$. Again, let $\mathtt{fun}_u(f) = (\varphi_w)_{w\succeq u}$ and $\mathtt{fun}_u(g) = (\psi_w)_{w\succeq u}$, and assume that $\mathtt{fun}_u(f) \preceq \mathtt{fun}_u(g)$. Then, we have $(\varphi_v)_{v\succeq u} \preceq (\psi_v)_{v\succeq u}$, that is, $\varphi_v \preceq \psi_v$ for every $v \succeq u$. By the calculations above, we have

$$\mathtt{app}_v(i_{u,v}^{s\to t}(f), b) = \varphi_v(b) \quad \text{and} \quad \mathtt{app}_v(i_{u,v}^{s\to t}(g), b) = \psi_v(b),$$

and so, we have $\mathtt{app}_v(i_{u,v}^{s\to t}(f), b) \preceq \mathtt{app}_v(i_{u,v}^{s\to t}(g), b)$ for all $b \in A_v^s$ and all $v \succeq u$. Then, $f \preceq g$.

(2) First, assume that $\mathtt{tfun}_u$ is isotone. Recall that the naturality condition for $\mathtt{tfun}$ is

$$\mathtt{tfun}_{w_2}(i_{w_1,w_2}^{\forall(\Phi)}(g)) = (\varphi_w)_{w\succeq w_2},$$

for any $g \in A_{w_1}^{\forall(\Phi)}$, if $\mathtt{tfun}_{w_1}(g) = (\varphi_w)_{w \succeq w_1}$. Let $\mathtt{tfun}_u(f) = (\varphi_w)_{w \succeq u}$ and $\mathtt{tfun}_u(g) = (\psi_w)_{w \succeq u}$. If $\mathtt{tapp}_v(i_{u,v}^{\forall(\Phi)}(f), t) \preceq \mathtt{tapp}_v(i_{u,v}^{\forall(\Phi)}(g), t)$ for all $t \in T$ and all $v \succeq u$, since $\mathtt{tapp}$ is defined from $\mathtt{tfun}$ as $\mathtt{tapp} = \mathtt{eval}_{\Phi}^{A} \circ (\mathtt{tfun} \times \mathtt{id}_T)$, and $\mathtt{eval}_{\Phi,u}^{A}((\varphi_w)_{w \succeq u}, t) = \varphi_u(t)$, we have

$$\begin{aligned}\mathtt{tapp}_v(i_{u,v}^{\forall(\Phi)}(f), t) &= \mathtt{eval}_{\Phi,v}(\mathtt{tfun}_v(i_{u,v}^{\forall(\Phi)}(f)), t) \\ &= \mathtt{eval}_{\Phi,v}((\varphi_w)_{w \succeq v}, t) \\ &= \varphi_v(t).\end{aligned}$$

Similarly, we get

$$\mathtt{tapp}_v(i_{u,v}^{\forall(\Phi)}(g), t) = \psi_v(t).$$

Thus, the hypothesis implies that $\varphi_v(t) \preceq \psi_v(t)$ for all $t \in T$, and thus $\varphi_v \preceq \psi_v$. Since this holds for all $v \succeq u$, we have $(\varphi_v)_{v \succeq u} \preceq (\psi_v)_{v \succeq u}$, that is, $\mathtt{tfun}_u(f) \preceq \mathtt{tfun}_v(f)$, and since $\mathtt{tfun}_u$ is isotone, we have $f \preceq g$.

Now, assume that $f \preceq g$ whenever $\mathtt{tapp}_v(i_{u,v}^{\forall(\Phi)}(f), t) \preceq \mathtt{tapp}_v(i_{u,v}^{\forall(\Phi)}(g), t)$ for all $t \in T$ and all $v \succeq u$. Again, let $\mathtt{tfun}_u(f) = (\varphi_w)_{w \succeq u}$ and $\mathtt{tfun}_u(g) = (\psi_w)_{w \succeq u}$, and assume that $\mathtt{tfun}_u(f) \preceq \mathtt{tfun}_u(g)$. Then, we have $(\varphi_v)_{v \succeq u} \preceq (\psi_v)_{v \succeq u}$, that is, $\varphi_v \preceq \psi_v$ for every $v \succeq u$. By the calculations above, we have

$$\mathtt{tapp}_v(i_{u,v}^{\forall(\Phi)}(f), t) = \varphi_v(t) \quad \text{and} \quad \mathtt{tapp}_v(i_{u,v}^{\forall(\Phi)}(g), t) = \psi_v(t),$$

and so, we have $\mathtt{tapp}_v(i_{u,v}^{\forall(\Phi)}(f), t) \preceq \mathtt{tapp}_v(i_{u,v}^{\forall(\Phi)}(g), t)$ for all $t \in T$ and all $v \succeq u$. Then, $f \preceq g$. □

For the sake of brevity, we will abbreviate Kripke pre-applicative (β or $\beta\eta$)-structures as Kripke pre-applicative structures. We now show how to interpret λ-terms in a Kripke pre-applicative structure. For this, we will use valuations.

5 Interpreting λ-Terms in Kripke Pre-Applicative Structures

In this section, we show how to interpet second-order λ-terms using Kripke applicative structures. Then, we prove several basic lemmas that will be needed in section 7, in particular, lemma 5.10 (and lemma 5.11), the "substitution lemma", which is crucial in proving the soundness of β-reduction and typed β-reduction.

Definition 5.1 Given an algebra of polymorphic types T, it is assumed that we have a function $TI\colon TC \to T$ assigning an element $TI(k) \in T$ to every type constant $k \in TC$. A *type valuation* is a function $\theta\colon \mathcal{V} \to T$. Given a type valuation θ, every type $\sigma \in \mathcal{T}$ is interpreted as an element $[\![\sigma]\!]\theta$ of T as follows:

$$\begin{aligned}[\![X]\!]\theta &= \theta(X), \quad \text{where } X \text{ is a type variable,} \\ [\![k]\!]\theta &= TI(k), \quad \text{where } k \text{ is a type constant,} \\ [\![\sigma \to \tau]\!]\theta &= [\![\sigma]\!]\theta \to [\![\tau]\!]\theta, \\ [\![\sigma \times \tau]\!]\theta &= [\![\sigma]\!]\theta \times [\![\tau]\!]\theta, \\ [\![\sigma + \tau]\!]\theta &= [\![\sigma]\!]\theta + [\![\tau]\!]\theta, \\ [\![\forall X.\, \sigma]\!]\theta &= \forall(\Lambda t \in T.\, [\![\sigma]\!]\theta[X := t]).\end{aligned}$$

In the above definition, $\Lambda t \in T.\, [\![\sigma]\!]\theta[X{:=}\, t]$ denotes the function Φ from T to T such that $\Phi(t) = [\![\sigma]\!]\theta[X{:=}\, t]$ for every $t \in T$. We say that T is a *type interpretation* iff $\Phi \in [T \to T]$ for every type σ and every valuation θ.

In other words, T is a type interpretation iff $[\![\sigma]\!]\theta$ is well-defined for every valuation θ. The following lemmas will be needed later.

Lemma 5.2 *For every type $\sigma \in \mathcal{T}$, and every pair of type valuations θ_1 and θ_2, if $\theta_1(X) = \theta_2(X)$, for all $X \in FTV(\sigma)$, then $[\![\sigma]\!]\theta_1 = [\![\sigma]\!]\theta_2$.*

Proof. A straightforward induction on σ. □

Lemma 5.3 *Given a type interpretation T, for all $\sigma, \tau \in \mathcal{T}$, for every type valuation θ, we have*

$$[\![\sigma[\tau/X]]\!]\theta = [\![\sigma]\!]\theta[X{:=}\, [\![\tau]\!]\theta].$$

Proof. The proof is by induction on σ. The case where $\sigma = X$ is trivial, since then $X[\tau/X] = \tau$, and

$$[\![X]\!]\theta[X{:=}\, [\![\tau]\!]\theta] = \theta[X{:=}\, [\![\tau]\!]\theta](X) = [\![\tau]\!]\theta.$$

The induction steps are straightforward, and we only treat the case where $\sigma = \forall Y.\, \sigma_1$. In this case,

$$[\![(\forall Y.\, \sigma_1)[\tau/X]]\!]\theta = \forall(\Lambda t \in T.\, [\![\sigma_1[\tau/X]]\!]\theta[Y{:=}\, t]),$$

(where the bound variable Y is renamed in a suitable fashion if necessary), and where $\Lambda t \in T.\, [\![\sigma_1[\tau/X]]\!]\theta[Y{:=}\, t]$ denotes the function Φ from T to T such that $\Phi(t) = [\![\sigma_1[\tau/X]]\!]\theta[Y{:=}\, t]$ for every $t \in T$. By the induction hypothesis, we have

$$\Phi(t) = [\![\sigma_1[\tau/X]]\!]\theta[Y{:=}\, t] = [\![\sigma_1]\!]\theta[X{:=}\, [\![\tau]\!]\theta,\, Y{:=}\, t].$$

Then, since

$$[\![\forall Y.\, \sigma_1]\!]\theta[X{:=}\, [\![\tau]\!]\theta] = \forall(\Lambda t \in T.\, [\![\sigma_1]\!]\theta[X{:=}\, [\![\tau]\!]\theta,\, Y{:=}\, t]),$$

we have

$$[\![(\forall Y.\, \sigma_1)[\tau/X]]\!]\theta = [\![\forall Y.\, \sigma_1]\!]\theta[X{:=}\, [\![\tau]\!]\theta].$$

□

Definition 5.4 Given a type interpretation T, given a Kripke pre-applicative structure $\mathcal{A}$, a *valuation* is a pair $\rho = \langle \theta,\, \eta \rangle$, where $\theta{:}\, \mathcal{V} \to T$ is a type valuation, and $\eta{:}\, \mathcal{X} \times \mathcal{W} \to \bigcup(A^t_w)_{t\in T,\, w\in\mathcal{W}}$ is a partial function called an *environment* satisfying the following condition:

For every $x \in \mathcal{X}$, whenever $w_1 \preceq w_2$, if $\eta(x, w_1)$ is defined and $\eta(x, w_1) \in A^t_{w_1}$ (where $t \in T$) then $\eta(x, w_2)$ is defined and

$$\eta(x, w_2) = i^t_{w_1,w_2}(\eta(x, w_1)).$$

We denote $\eta(x, u)$ as $\eta_u(x)$. Given a valuation $\rho = \langle \theta,\, \eta \rangle$, for any $s \in T$ and $a \in A^s_u$ we let $\rho[X{:=}\, s,\, x{:=}\, a] = \langle \theta[X{:=}\, s],\, \eta[x{:=}\, a] \rangle$ be the valuation, such that, $\theta[X{:=}\, s](Y) = \theta(Y)$ for every $Y \neq X$ and $\theta[X{:=}\, s](X) = s$, and $\eta_w[x{:=}\, a](y) = \eta_w(y)$ for all $w \in \mathcal{W}$ and all $y \neq x$, and

$$\eta_w[x{:=}\, a](x) = i^s_{u,w}(a), \qquad \text{for all } w \succeq u,$$

and undefined otherwise.

A *global element* of A^s is a function $a\colon \mathcal{W} \to \bigcup(A^s_w)_{w\in\mathcal{W}}$, such that, $a_u \in A^s_u$ and $a_v = i^s_{u,v}(a_u)$ whenever $v \succeq u$.

Given a context Γ, we say that $w \in \mathcal{W}$ *satisfies* Γ *at* ρ, written as $w \Vdash \Gamma[\rho]$ (where $\rho = \langle\theta,\ \eta\rangle$) iff

$$\eta_w(x) \in A^{[\sigma]\theta}_w \qquad \text{for every } x\colon\sigma \in \Gamma.$$

Given a valuation $\rho = \langle\theta,\ \eta\rangle$, we often denote θ as $[\rho]$ (or ρ_t), and η as ρ or (ρ_x).

Note that if $w_1 \preceq w_2$, by the definition of a valuation $\rho = \langle\theta,\ \eta\rangle$ (the condition $\eta(x, w_2) = i^t_{w_1,w_2}(\eta(x, w_1)))$, if $w_1 \Vdash \Gamma[\rho]$, then $w_2 \Vdash \Gamma[\rho]$. Also, conditions (1)-(4) of definition 4.1 imply that the following conditions hold:

For all $w \in \mathcal{W}$, for all types $\sigma, \tau \in \mathcal{T}$, if $A^{[\sigma]\theta}_w \neq \emptyset$ and $A^{[\tau]\theta}_w \neq \emptyset$, then $A^{[\sigma\to\tau]\theta}_w \neq \emptyset$, $A^{[\sigma\times\tau]\theta}_w \neq \emptyset$, $A^{[\sigma+\tau]\theta}_w \neq \emptyset$, and if $A^{[\sigma[\tau/X]]\theta}_w \neq \emptyset$ for every $\tau \in \mathcal{T}$, then $A^{[\forall X.\sigma]\theta}_w \neq \emptyset$.

We are now ready to interpret λ-terms.

Definition 5.5 Given a type interpretation T and a Kripke pre-applicative structure $\mathcal{A}$, let $\mathcal{A}I\colon Const \to \mathcal{A}$ be a function assigning a global element $\mathcal{A}I(c)$ of $A^{TI(Type(c))}$ to every constant $c \in Const$. For every valuation $\rho = \langle\theta,\ \eta\rangle$, every context Γ, and every world $u \in \mathcal{W}$, if $u \Vdash \Gamma[\rho]$, we define the *interpretation* (or *meaning*) $\mathcal{A}[\![\Gamma \triangleright M\colon\sigma]\!]\rho u$ of a judgement $\Gamma \triangleright M\colon\sigma$, inductively as follows:

$\mathcal{A}[\![\Gamma \triangleright x\colon\sigma]\!]\rho u = \eta_u(x)$

$\mathcal{A}[\![\Gamma \triangleright c\colon Type(c)]\!]\rho u = \mathcal{A}I(c)_u$

$\mathcal{A}[\![\Gamma \triangleright MN\colon\tau]\!]\rho u = \mathtt{app}^{[\sigma]\theta,\,[\tau]\theta}_u(\mathcal{A}[\![\Gamma \triangleright M\colon\sigma\to\tau]\!]\rho u,\ \mathcal{A}[\![\Gamma \triangleright N\colon\sigma]\!]\rho u)$

$\mathcal{A}[\![\Gamma \triangleright \lambda x\colon\sigma.\, M\colon\sigma\to\tau]\!]\rho u = \mathtt{abst}^{[\sigma]\theta,\,[\tau]\theta}_u(\varphi)$,

where $\varphi = (\varphi_w)_{w\succeq u}$ is the family of functions defined such that,
$\varphi_w(a) = \mathcal{A}[\![\Gamma, x\colon\sigma \triangleright M\colon\tau]\!]\rho[x\colon= a]w$, for every $a \in A^{[\sigma]\theta}_w$

$\mathcal{A}[\![\Gamma \triangleright M\tau\colon\sigma[\tau/X]]\!]\rho u = \mathtt{tapp}^{\Phi}_u(\mathcal{A}[\![\Gamma \triangleright M\colon\forall X.\,\sigma]\!]\rho u,\ [\![\tau]\!]\theta)$,

where Φ is the function such that $\Phi(s) = [\![\sigma]\!]\theta[X\colon= s]$ for every $s \in T$

$\mathcal{A}[\![\Gamma \triangleright \lambda X.\, M\colon\forall X.\,\sigma]\!]\rho u = \mathtt{tabst}^{\Phi}_u(\varphi)$,

where $\varphi = (\varphi_w)_{w\succeq u}$ is the family of functions defined such that,
$\varphi_w(s) = \mathcal{A}[\![\Gamma, X\colon\star \triangleright M\colon\sigma]\!]\rho[X\colon= s]w$, for every $s \in T$, and where Φ is the function such that $\Phi(s) = [\![\sigma]\!]\theta[X\colon= s]$ for every $s \in T$

$\mathcal{A}[\![\Gamma \triangleright \pi_1(M)\colon\sigma]\!]\rho u = \pi_1(\mathcal{A}[\![\Gamma \triangleright M\colon\sigma\times\tau]\!]\rho u)$

$\mathcal{A}[\![\Gamma \triangleright \pi_2(M)\colon\tau]\!]\rho u = \pi_2(\mathcal{A}[\![\Gamma \triangleright M\colon\sigma\times\tau]\!]\rho u)$

$\mathcal{A}[\![\Gamma \triangleright \langle M_1,\ M_2\rangle\colon\sigma\times\tau]\!]\rho u = \langle\mathcal{A}[\![\Gamma \triangleright M_1\colon\sigma]\!]\rho u,\ \mathcal{A}[\![\Gamma \triangleright M_2\colon\tau]\!]\rho u\rangle$

$\mathcal{A}[\![\Gamma \triangleright \mathtt{inl}(M)\colon\sigma+\tau]\!]\rho u = \mathtt{inl}(\mathcal{A}[\![\Gamma \triangleright M\colon\sigma]\!]\rho u)$

$\mathcal{A}[\![\Gamma \rhd \mathbf{inr}(M): \sigma + \tau]\!]\rho u = \mathbf{inr}(\mathcal{A}[\![\Gamma \rhd M: \tau]\!]\rho u)$

$\mathcal{A}[\![\Gamma \rhd [M,\ N]: (\sigma + \tau) \to \delta]\!]\rho u = [\mathcal{A}[\![\Gamma \rhd M: (\sigma \to \delta)]\!]\rho u,\ \mathcal{A}[\![\Gamma \rhd N: (\tau \to \delta)]\!]\rho u].$

We are assuming that $\langle -, - \rangle$ and $[-,\ -]$ are total, and that the domains of **abst** and **tabst** are sufficently large for the above definitions to be well-defined for all ρ, $\Gamma \rhd M: \sigma$, and $u \in \mathcal{W}$. In this case, we say that $\mathcal{A}$ is a *Kripke pre-interpretation.*

In the special case where $\mathcal{W} = \{0\}$ consists of a single world, and $\mathcal{A}$ is an extentional applicative structure, it is not difficult to show that definition 5.5 is equivalent to Breazu-Tannen and Coquand's definition of a *polymorphic λ-interpretation*, or *pli* (see [1]).

In order to be sure that in definition 5.5, $\mathcal{A}[\![\Gamma \rhd M: \sigma]\!]\rho u$ is a well defined element of $A_u^{[\![\sigma]\!]\theta}$, we need to verify that $(\varphi_w)_{w \succeq u} \in [A^{[\![\sigma]\!]\theta} \Rightarrow A^{[\![\tau]\!]\theta}]_u$ in the case of λ-abstraction, and that $(\varphi_w)_{w \succeq u} \in \prod_\Phi (A_u^s)_{s \in T}$, in the case of typed λ-abstraction. For this, we show the following lemma.

Lemma 5.6 *Given a type interpretation T and a Kripke pre-applicative (β or $\beta\eta$)-structure $\mathcal{A}$, for every valuation $\rho = \langle \theta,\ \eta \rangle$, every context Γ, and every world $u \in \mathcal{W}$, if $u \Vdash \Gamma[\rho]$, then for every judgement $\Gamma \rhd M: \sigma$, whenever $v \succeq u$,*

$$\mathcal{A}[\![\Gamma \rhd M: \sigma]\!]\rho v = i_{u,v}^{[\![\sigma]\!]\theta}(\mathcal{A}[\![\Gamma \rhd M: \sigma]\!]\rho u).$$

Proof. We proceed by induction on typing derivations. Except for the cases of λ-abstraction and typed λ-abstraction, the induction is straightforward and uses the naturality conditions of definition 4.2. Let us consider the case of λ-abstraction. We need to show that the family of functions $\varphi = (\varphi_w)_{w \succeq u}$ defined such that,

$$\varphi_w(a) = \mathcal{A}[\![\Gamma, x: \sigma \rhd M: \tau]\!]\rho[x := a]w,$$

for every $a \in A_w^{[\![\sigma]\!]\theta}$, satisfies the naturality condition

$$\varphi_v(i_{u,v}^{[\![\sigma]\!]\theta}(a)) = i_{u,v}^{[\![\tau]\!]\theta}(\varphi_u(a)),$$

for every $a \in A_u^{[\![\sigma]\!]\theta}$, whenever $v \succeq u$. Thus, we need to show that

$$\mathcal{A}[\![\Gamma, x: \sigma \rhd M: \tau]\!]\rho[x := i_{u,v}^{[\![\sigma]\!]\theta}(a)]v = i_{u,v}^{[\![\tau]\!]\theta}(\mathcal{A}[\![\Gamma, x: \sigma \rhd M: \tau]\!]\rho[x := a]u).$$

By the induction hypothesis applied to $\rho[X := a]$ and $\Gamma, x: \sigma \rhd M: \tau$, which is legitimate, since $u \Vdash \Gamma[\rho]$ implies that $u \Vdash (\Gamma, x: \sigma)[\rho[X := a]]$, since $a \in A_u^{[\![\sigma]\!]\theta}$, we have

$$\mathcal{A}[\![\Gamma, x: \sigma \rhd M: \tau]\!]\rho[x := a]v = i_{u,v}^{[\![\tau]\!]\theta}(\mathcal{A}[\![\Gamma, x: \sigma \rhd M: \tau]\!]\rho[x := a]u).$$

However, by definition 5.4, $\eta_v[x := a](x) = i_{u,v}^{[\![\sigma]\!]\theta}(a)$ and thus,

$$\mathcal{A}[\![\Gamma, x: \sigma \rhd M: \tau]\!]\rho[x := i_{u,v}^{[\![\sigma]\!]\theta}(a)]v = \mathcal{A}[\![\Gamma, x: \sigma \rhd M: \tau]\!]\rho[x := a]v,$$

and thus, we have

$$\mathcal{A}[\![\Gamma, x: \sigma \rhd M: \tau]\!]\rho[x := i_{u,v}^{[\![\sigma]\!]\theta}(a)]v = i_{u,v}^{[\![\tau]\!]\theta}(\mathcal{A}[\![\Gamma, x: \sigma \rhd M: \tau]\!]\rho[x := a]u).$$

Thus, we know that $\mathcal{A}[\![\Gamma \rhd \lambda x{:}\,\sigma.\, M{:}\,\sigma \to \tau]\!]\rho u$ is well defined, and we have

$$\mathcal{A}[\![\Gamma \rhd \lambda x{:}\,\sigma.\, M{:}\,\sigma \to \tau]\!]\rho u = \mathbf{abst}_u^{[\![\sigma]\!]\theta,\,[\![\tau]\!]\theta}((\varphi_w)_{w \succeq u}),$$

and

$$\mathcal{A}[\![\Gamma \rhd \lambda x{:}\,\sigma.\, M{:}\,\sigma \to \tau]\!]\rho v = \mathbf{abst}_v^{[\![\sigma]\!]\theta,\,[\![\tau]\!]\theta}((\varphi_w)_{w \succeq v}).$$

Recalling that the naturality condition (2) of definition 4.2 is

$$\mathbf{abst}_{w_2}((\varphi_w)_{w \succeq w_2}) = i_{w_1,w_2}^{s \to t}(\mathbf{abst}_{w_1}((\varphi_w)_{w \succeq w_1})),$$

we have

$$\mathcal{A}[\![\Gamma \rhd \lambda x{:}\,\sigma.\, M{:}\,\sigma \to \tau]\!]\rho v = i_{u,v}^{[\![\sigma \to \tau]\!]\theta}(\mathcal{A}[\![\Gamma \rhd \lambda x{:}\,\sigma.\, M{:}\,\sigma \to \tau]\!]\rho u).$$

Let us now consider the case of typed λ-abstraction. We need to show that the family of functions $\varphi = (\varphi_w)_{w \succeq u}$ defined such that,

$$\varphi_w(s) = \mathcal{A}[\![\Gamma, X{:}\star \rhd M{:}\,\sigma]\!]\rho[X{:}= s]w,$$

for every $s \in T$, satisfies the naturality condition

$$\varphi_v(s) = i_{u,v}^{\Phi(s)}(\varphi_u(s)),$$

for every $s \in T$, whenever $v \succeq u$, where Φ is the function such that $\Phi(s) = [\![\sigma]\!]\theta[X{:}= s]$ for every $s \in T$. Thus, we need to show that

$$\mathcal{A}[\![\Gamma, X{:}\star \rhd M{:}\,\sigma]\!]\rho[X{:}= s]v = i_{u,v}^{[\![\sigma]\!]\theta[X:=s]}(\mathcal{A}[\![\Gamma, X{:}\star \rhd M{:}\,\sigma]\!]\rho[X{:}= s]u).$$

However, this follows directly from the induction hypothesis applied to $\rho[X{:}= s]$ and $\Gamma, X{:}\star \rhd M{:}\,\sigma$, which is legitimate, since $u \Vdash \Gamma[\rho]$ obviously implies that $u \Vdash (\Gamma, X{:}\star)[\rho[X{:}= s]]$.

Thus, we know that $\mathcal{A}[\![\Gamma \rhd \lambda X.\, M{:}\forall X.\, \sigma]\!]\rho u$ is well defined, and we have

$$\mathcal{A}[\![\Gamma \rhd \lambda X.\, M{:}\forall X.\, \sigma]\!]\rho u = \mathbf{tabst}_u^{\Phi}((\varphi_w)_{w \succeq u}),$$

and

$$\mathcal{A}[\![\Gamma \rhd \lambda X.\, M{:}\forall X.\, \sigma]\!]\rho v = \mathbf{tabst}_v^{\Phi}((\varphi_w)_{w \succeq v}),$$

where Φ is the function defined above. Recalling that the naturality condition (4) of definition 4.2 is

$$\mathbf{tabst}_{w_2}((\varphi_w)_{w \succeq w_2}) = i_{w_1,w_2}^{\forall(\Phi)}(\mathbf{tabst}_{w_1}((\varphi_w)_{w \succeq w_1})),$$

since by definition 5.1, $\forall(\Phi) = [\![\forall X.\, \sigma]\!]\theta$, we have

$$\mathcal{A}[\![\Gamma \rhd \lambda X.\, M{:}\forall X.\, \sigma]\!]\rho v = i_{u,v}^{[\![\forall X.\sigma]\!]\theta}(\mathcal{A}[\![\Gamma \rhd \lambda X.\, M{:}\forall X.\, \sigma]\!]\rho u).$$

□

Consider the pre-applicative structure $\mathcal{LT}_\beta$ of definition 4.4. Note that, according to definition 5.4, a valuation is a pair $\rho = \langle \theta,\ \eta \rangle$, where θ is an infinite type substitution, and η is a partial function $\eta{:}\, \mathcal{X} \times \mathcal{W} \to \bigcup(A_w^t)_{t \in T,\ w \in \mathcal{W}}$. Thus, recalling that worlds are contexts, $\eta_\Delta(x) = \Gamma \rhd M{:}\,\sigma$ for some judgement $\Gamma \rhd M{:}\,\sigma$, when defined. Furthermore, the condition for ρ to satisfy a context

Γ at a world Δ, is $\eta_\Delta(x) \in A_\Delta^{\theta(\sigma)}$, that is, $\eta_\Delta(x) = \Delta \triangleright M_x : \theta(\sigma)$, for some M_x, for every $x : \sigma \in \Gamma$. Thus, if $\rho = \langle \theta, \eta \rangle$ satisfies a context Γ at Δ, the valuation ρ defines a substitution φ such that $\varphi(X) = \theta(X)$ for every $X \in \bigcup(FTV(\sigma))_{x:\sigma \in \Gamma}$, and $\varphi(x) = M_x$ for every $x \in dom(\Gamma)$ (where $\eta_\Delta(x) = \Delta \triangleright M_x : \theta(\sigma)$), and we have $\Delta \Vdash \Gamma[\varphi]$, as in definition 4.3. Then, we have the following useful property.

Lemma 5.7 *For the pre-applicative structure $\mathcal{LT}_\beta$ of definition 4.4, for every pair of contexts Γ and Δ, for every valuation $\rho = \langle \theta, \eta \rangle$, if $\Delta \Vdash \Gamma[\rho]$, then for every judgement $\Gamma \triangleright M : \sigma$, we have*

$$\mathcal{LT}_\beta[\![\Gamma \triangleright M : \sigma]\!]\rho\Delta = \Delta \triangleright M[\varphi] : \sigma[\varphi],$$

and $\Delta \Vdash \Gamma[\varphi]$, where φ is the substitution defined by the restriction of ρ_Δ to Γ, as explained just before stating this lemma. The same result holds for the $\beta\eta$-structure $\mathcal{LT}_{\beta\eta}$ of definition 4.8.

Proof. A straighforward induction on the derivation of $\Gamma \triangleright M : \sigma$. □

The following lemmas will be needed later.

Lemma 5.8 *Given a type interpretation T and a Kripke pre-applicative (β or $\beta\eta$)-structure $\mathcal{A}$, for every pair of contexts Γ_1 and Γ_2, for every world $u \in \mathcal{W}$, for every pair of valuations $\rho_1 = \langle \theta_1, \eta_1 \rangle$ and $\rho_2 = \langle \theta_2, \eta_2 \rangle$, for every pair of judgements $\Gamma_1 \triangleright M : \sigma$ and $\Gamma_2 \triangleright M : \sigma$, if $u \Vdash \Gamma_1[\rho_1]$ and $u \Vdash \Gamma_2[\rho_2]$, $\Gamma_1(x) = \Gamma_2(x)$, for all $x \in FV(M)$, $\theta_1(X) = \theta_2(X)$, for all $X \in \bigcup(FTV(\tau))_{x:\tau \in \Gamma} \cup FTV(M)$, and $\eta_1(x) = \eta_2(x)$, for all $x \in FV(M)$, then*

$$\mathcal{A}[\![\Gamma_1 \triangleright M : \sigma]\!]\rho_1 u = \mathcal{A}[\![\Gamma_2 \triangleright M : \sigma]\!]\rho_2 u.$$

Proof. A straightforward induction on typing derivations (and using lemma 5.2). □

Lemma 5.9 *Given a type interpretation T and a Kripke pre-applicative (β or $\beta\eta$)-structure $\mathcal{A}$, for every context Γ, for every world $u \in \mathcal{W}$, for every pair of valuations $\rho_1 = \langle \theta_1, \eta_1 \rangle$ and $\rho_2 = \langle \theta_2, \eta_2 \rangle$, for every judgement $\Gamma \triangleright M : \sigma$, if $u \Vdash \Gamma[\rho_1]$ and $u \Vdash \Gamma[\rho_2]$, $\theta_1(X) = \theta_2(X)$, for all $X \in \bigcup(FTV(\tau))_{x:\tau \in \Gamma} \cup FTV(M)$, and $\eta_1(x) \preceq \eta_2(x)$, for all $x \in FV(M)$, then*

$$\mathcal{A}[\![\Gamma \triangleright M : \sigma]\!]\rho_1 u \preceq \mathcal{A}[\![\Gamma \triangleright M : \sigma]\!]\rho_2 u.$$

Proof. A straightforward induction on typing derivations. □

The following "substitution lemma" is needed to establish the soundness of Kripke interpretations with respect to β-reduction and typed β-reduction.

Lemma 5.10 *Given a type interpretation T and a Kripke pre-applicative (β or $\beta\eta$)-structure $\mathcal{A}$, for every context Γ, for every world $u \in \mathcal{W}$, for every valuation $\rho = \langle \theta, \eta \rangle$, the following properties hold: (1) for every judgements $\Gamma, x : \sigma \triangleright M : \tau$ and $\Gamma \triangleright N : \sigma$, if $u \Vdash \Gamma[\rho]$, then*

$$\mathcal{A}[\![\Gamma \triangleright M[N/x] : \tau]\!]\rho u = \mathcal{A}[\![\Gamma, x : \sigma \triangleright M : \tau]\!]\rho[x := \mathcal{A}[\![\Gamma \triangleright N : \sigma]\!]\rho u]u.$$

(2) for every judgement $\Gamma, X : \star \triangleright M : \sigma$ and every $\tau \in \mathcal{T}$, if $u \Vdash \Gamma[\rho]$, then

$$\mathcal{A}[\![\Gamma \triangleright M[\tau/X] : \sigma[\tau/X]]\!]\rho u = \mathcal{A}[\![\Gamma, X : \star \triangleright M : \sigma]\!]\rho[X := [\![\tau]\!]\theta]u.$$

Proof. We proceed by induction on typing derivations.

(1) When $M = x$, we have $x[N/x] = N$, and by definition 5.5,

$$\mathcal{A}[\![\Gamma, x{:}\,\sigma \rhd M{:}\,\tau]\!]\rho[x{:}= \mathcal{A}[\![\Gamma \rhd N{:}\,\sigma]\!]\rho u]u = \eta_u[x{:}= \mathcal{A}[\![\Gamma \rhd N{:}\,\sigma]\!]\rho u](x) = \mathcal{A}[\![\Gamma \rhd N{:}\,\sigma]\!]\rho u.$$

The induction steps are straightforward, except for λ-abstraction and typed λ-abstraction.

(1a) Consider the judgements $\Gamma, x{:}\,\sigma \rhd \lambda y{:}\,\delta.\, M_1{:}\,(\delta \to \tau)$ and $\Gamma \rhd N{:}\,\sigma$, and assume that $u \Vdash \Gamma[\rho]$. Recall that

$$\mathcal{A}[\![\Gamma \rhd \lambda y{:}\,\delta.\,(M_1[N/x]){:}\,(\delta \to \tau)]\!]\rho u = \mathbf{abst}_u^{[\![\delta]\!]\theta,\,[\![\tau]\!]\theta}(\varphi),$$

where $\varphi = (\varphi_w)_{w \succeq u}$ is the family of functions defined such that,

$$\varphi_w(a) = \mathcal{A}[\![\Gamma, y{:}\,\delta \rhd M_1[N/x]{:}\,\tau]\!]\rho[y{:}= a]w,$$

for every $a \in A_w^{[\![\delta]\!]\theta}$. Since $u \Vdash \Gamma[\rho]$ implies $w \Vdash \Gamma[\rho]$ when $u \preceq w$, and $a \in A_w^{[\![\delta]\!]\theta}$, we have $w \Vdash (\Gamma, y{:}\,\delta)[\rho]$ for every $w \succeq u$. Thus, we can apply the induction hypothesis to $(\Gamma, y{:}\,\delta)$, $w \in \mathcal{W}$, $\rho = \langle\theta,\ \eta[y{:}= a]\rangle$, and the judgements $\Gamma, x{:}\,\sigma, y{:}\,\delta \rhd M_1{:}\,\tau$, and $\Gamma, y{:}\,\delta \rhd N{:}\,\sigma$, and we have

$$\mathcal{A}[\![\Gamma, y{:}\,\delta \rhd M_1[N/x]{:}\,\tau]\!]\rho[y{:}= a]w = \mathcal{A}[\![\Gamma, x{:}\,\sigma, y{:}\,\delta \rhd M_1{:}\,\tau]\!]\rho[x{:}= \mathcal{A}[\![\Gamma, y{:}\,\delta \rhd N{:}\,\sigma]\!]\rho[y{:}= a]w,\ y{:}= a]w.$$

By lemma 5.8, since $y \notin dom(\Gamma)$, we have

$$\mathcal{A}[\![\Gamma, y{:}\,\delta \rhd N{:}\,\sigma]\!]\rho[y{:}= a]w = \mathcal{A}[\![\Gamma \rhd N{:}\,\sigma]\!]\rho w,$$

and so, we have

$$\mathcal{A}[\![\Gamma, y{:}\,\delta \rhd M_1[N/x]{:}\,\tau]\!]\rho[y{:}= a]w = \mathcal{A}[\![\Gamma, x{:}\,\sigma, y{:}\,\delta \rhd M_1{:}\,\tau]\!]\rho[x{:}= \mathcal{A}[\![\Gamma \rhd N{:}\,\sigma]\!]\rho w,\ y{:}= a]w,$$

that is,

$$\varphi_w(a) = \mathcal{A}[\![\Gamma, x{:}\,\sigma, y{:}\,\delta \rhd M_1{:}\,\tau]\!]\rho[x{:}= \mathcal{A}[\![\Gamma \rhd N{:}\,\sigma]\!]\rho w,\ y{:}= a]w.$$

However, we also have

$$\mathcal{A}[\![\Gamma, x{:}\,\sigma \rhd \lambda y{:}\,\delta.\, M_1{:}\,(\delta \to \tau)]\!]\rho[x{:}= \mathcal{A}[\![\Gamma \rhd N{:}\,\sigma]\!]\rho u]u = \mathbf{abst}_u^{[\![\delta]\!]\theta,\,[\![\tau]\!]\theta}(\psi),$$

where $\psi = (\psi_w)_{w \succeq u}$ is the family of functions defined such that,

$$\psi_w(a) = \mathcal{A}[\![\Gamma, x{:}\,\sigma, y{:}\,\delta \rhd M_1{:}\,\tau]\!]\rho[x{:}= \mathcal{A}[\![\Gamma \rhd N{:}\,\sigma]\!]\rho u,\ y{:}= a]w,$$

for every $a \in A_w^{[\![\delta]\!]\theta}$. However, letting $b = \mathcal{A}[\![\Gamma \rhd N{:}\,\sigma]\!]\rho u$, by definition 5.4, for any valuation ρ, we have $\rho_w[x{:}= b](x) = i^s_{u,w}(b)$ for all $w \succeq u$, and since by lemma 5.6,

$$\mathcal{A}[\![\Gamma \rhd N{:}\,\sigma]\!]\rho w = i_{u,w}^{[\![\sigma]\!]\theta}(\mathcal{A}[\![\Gamma \rhd N{:}\,\sigma]\!]\rho u),$$

we have

$$\psi_w(a) = \mathcal{A}[\![\Gamma, x{:}\,\sigma, y{:}\,\delta \rhd M_1{:}\,\tau]\!]\rho[x{:}= \mathcal{A}[\![\Gamma \rhd N{:}\,\sigma]\!]\rho w,\ y{:}= a]w,$$

for every $a \in A_w^{[\![\delta]\!]\theta}$. Thus, $\varphi_w(a) = \psi_w(a)$, for every $a \in A_w^{[\![\delta]\!]\theta}$ and all $w \succeq u$, that is, $\varphi = \psi$, and thus

$$\mathcal{A}[\![\Gamma \rhd \lambda y{:}\,\delta.\,(M_1[N/x]){:}\,(\delta \to \tau)]\!]\rho u = \mathcal{A}[\![\Gamma, x{:}\,\sigma \rhd \lambda y{:}\,\delta.\, M_1{:}\,(\delta \to \tau)]\!]\rho[x{:}= \mathcal{A}[\![\Gamma \rhd N{:}\,\sigma]\!]\rho u]u.$$

(1b) Consider the judgements $\Gamma, x:\sigma \rhd \lambda Y.\, M_1 : \forall Y.\, \sigma_1$ and $\Gamma \rhd N:\sigma$, and assume that $u \Vdash \Gamma[\rho]$. Recall that

$$\mathcal{A}[\![\Gamma \rhd \lambda Y.\,(M_1[N/x]) : \forall Y.\, \sigma_1]\!]\rho u = \mathtt{tabst}_u^{\Phi}(\varphi),$$

where $\varphi = (\varphi_w)_{w \succeq u}$ is the family of functions defined such that,

$$\varphi_w(s) = \mathcal{A}[\![\Gamma, Y:\star \rhd M_1[N/x] : \sigma_1]\!]\rho[Y := s]w,$$

for every $s \in T$, and where Φ is the function such that $\Phi(s) = [\![\sigma_1]\!]\theta[Y := s]$ for every $s \in T$. Since $u \Vdash \Gamma[\rho]$ implies $w \Vdash \Gamma[\rho]$ when $u \preceq w$, and $s \in T$, we have $w \Vdash (\Gamma, Y:\star)[\rho]$ for every $w \succeq u$. Thus, we can apply the induction hypothesis to $(\Gamma, Y:\star)$, $w \in \mathcal{W}$, $\rho = \langle \theta[Y := s],\ \eta \rangle$, the judgements $\Gamma, x:\sigma, Y:\star \rhd M_1:\sigma_1$, and $\Gamma \rhd N:\sigma$, and we have

$$\mathcal{A}[\![\Gamma, Y:\star \rhd M_1[N/x] : \sigma_1]\!]\rho[Y := s]w = \\ \mathcal{A}[\![\Gamma, x:\sigma, Y:\star \rhd M_1 : \sigma_1]\!]\rho[x := \mathcal{A}[\![\Gamma, Y:\star \rhd N:\sigma]\!]\rho[Y := s]w,\ Y := s]w.$$

By lemma 5.8, since $Y \notin dom(\Gamma)$, we have

$$\mathcal{A}[\![\Gamma, Y:\star \rhd N:\sigma]\!]\rho[Y := s]w = \mathcal{A}[\![\Gamma \rhd N:\sigma]\!]\rho w,$$

and we get

$$\varphi_w(s) = \mathcal{A}[\![\Gamma, x:\sigma, Y:\star \rhd M_1:\sigma_1]\!]\rho[x := \mathcal{A}[\![\Gamma \rhd N:\sigma]\!]\rho w,\ Y := s]w.$$

However, we also have

$$\mathcal{A}[\![\Gamma, x:\sigma \rhd \lambda Y.\, M_1 : \forall Y.\, \sigma_1]\!]\rho[x := \mathcal{A}[\![\Gamma \rhd N:\sigma]\!]\rho u]u = \mathtt{tabst}_u^{\Phi}(\psi),$$

where $\psi = (\psi_w)_{w \succeq u}$ is the family of functions defined such that,

$$\psi_w(s) = \mathcal{A}[\![\Gamma, x:\sigma, Y:\star \rhd M_1:\sigma_1]\!]\rho[x := \mathcal{A}[\![\Gamma \rhd N:\sigma]\!]\rho u,\ Y := s]w,$$

for every $s \in T$, and where Φ is the function such that $\Phi(s) = [\![\sigma_1]\!]\theta[Y := s]$ for every $s \in T$. As in case (1a), by lemma 5.6, we get

$$\psi_w(s) = \mathcal{A}[\![\Gamma, x:\sigma, Y:\star \rhd M_1:\sigma_1]\!]\rho[x := \mathcal{A}[\![\Gamma \rhd N:\sigma]\!]\rho w,\ Y := s]w,$$

for every $s \in T$. Then, as in (1a), we have $\varphi = \psi$, and thus

$$\mathcal{A}[\![\Gamma \rhd \lambda Y.\,(M_1[N/x]) : \forall Y.\, \sigma_1]\!]\rho u = \mathcal{A}[\![\Gamma, x:\sigma \rhd \lambda Y.\, M_1 : \forall Y.\, \sigma_1]\!]\rho[x := \mathcal{A}[\![\Gamma \rhd N:\sigma]\!]\rho u]u.$$

(2) The only cases worth examining are λ-abstraction and typed λ-abstraction.

(2a) Consider the judgement $\Gamma, X:\star \rhd \lambda Y.\, M_1 : \forall Y.\, \sigma$, and assume that $u \Vdash \Gamma[\rho]$. Recall that

$$\mathcal{A}[\![\Gamma \rhd \lambda Y.\,(M_1[\tau/X]) : \forall Y.\,(\sigma[\tau/X])]\!]\rho u = \mathtt{tabst}_u^{\Phi}(\varphi),$$

where $\varphi = (\varphi_w)_{w \succeq u}$ is the family of functions defined such that,

$$\varphi_w(s) = \mathcal{A}[\![\Gamma, Y:\star \rhd M_1[\tau/X] : \sigma[\tau/X]]\!]\rho[Y := s]w,$$

for every $s \in T$, and where Φ is the function such that $\Phi(s) = [\![\sigma]\!]\theta[Y := s]$ for every $s \in T$. Since $u \Vdash \Gamma[\rho]$ implies $w \Vdash \Gamma[\rho]$ when $u \preceq w$, and $s \in T$, we have $w \Vdash (\Gamma, Y: \star)[\rho]$ for every $w \succeq u$. Thus, we can apply the induction hypothesis to $(\Gamma, Y: \star)$, $w \in \mathcal{W}$, $\rho = \langle \theta[Y := s],\ \eta \rangle$, the judgement $\Gamma, X: \star, Y: \star \triangleright M_1 : \sigma$, and $s \in T$, and we have

$$\mathcal{A}[\![\Gamma, Y: \star \triangleright M_1[\tau/X] : \sigma[\tau/X]]\!]\rho[Y := s]w = \mathcal{A}[\![\Gamma, X: \star, Y: \star \triangleright M_1 : \sigma]\!]\rho[X := [\![\tau]\!]\theta[Y := s],\ Y := s]w.$$

By lemma 5.2, since $Y \notin dom(\Gamma)$, we have

$$[\![\tau]\!]\theta[Y := s] = [\![\tau]\!]\theta,$$

and so, we have

$$\mathcal{A}[\![\Gamma, Y: \star \triangleright M_1[\tau/X] : \sigma[\tau/X]]\!]\rho[Y := s]w = \mathcal{A}[\![\Gamma, X: \star, Y: \star \triangleright M_1 : \sigma]\!]\rho[X := [\![\tau]\!]\theta,\ Y := s]w,$$

that is,

$$\varphi_w(s) = \mathcal{A}[\![\Gamma, X: \star, Y: \star \triangleright M_1 : \sigma]\!]\rho[X := [\![\tau]\!]\theta,\ Y := s]w.$$

However, we also have

$$\mathcal{A}[\![\Gamma, X: \star \triangleright \lambda Y.\, M_1 : \forall Y.\, \sigma]\!]\rho[X := [\![\tau]\!]\theta]u = \mathtt{tabst}_u^{\Phi}(\psi),$$

where $\psi = (\psi_w)_{w \succeq u}$ is the family of functions defined such that,

$$\psi_w(s) = \mathcal{A}[\![\Gamma, X: \star, Y: \star \triangleright M_1 : \sigma]\!]\rho[X := [\![\tau]\!]\theta,\ Y := s]w,$$

for every $s \in T$, and where Φ is the function such that $\Phi(s) = [\![\sigma]\!]\theta[Y := s]$ for every $s \in T$. Thus, $\varphi_w(s) = \psi_w(s)$, for every $s \in T$ and all $w \succeq u$, that is, $\varphi = \psi$, and thus

$$\mathcal{A}[\![\Gamma \triangleright \lambda Y.\, (M_1[\tau/X]) : \forall Y.\, (\sigma[\tau/X])]\!]\rho u = \mathcal{A}[\![\Gamma, X: \star \triangleright \lambda Y.\, M_1 : \forall Y.\, \sigma]\!]\rho[X := [\![\tau]\!]\theta]u.$$

(2b) Consider the judgement $\Gamma, X: \star \triangleright \lambda y : \delta.\, M_1 : (\delta \to \gamma)$, and assume that $u \Vdash \Gamma[\rho]$. Recall that

$$\mathcal{A}[\![\Gamma \triangleright \lambda y : \delta[\tau/X].\, (M_1[\tau/X]) : (\delta \to \gamma)[\tau/X]]\!]\rho u = \mathtt{abst}_u^{[\![\delta[\tau/X]]\!]\theta,\, [\![\gamma[\tau/X]]\!]\theta}(\varphi),$$

where $\varphi = (\varphi_w)_{w \succeq u}$ is the family of functions defined such that,

$$\varphi_w(a) = \mathcal{A}[\![\Gamma, y : \delta[\tau/X] \triangleright M_1[\tau/X] : \gamma[\tau/X]]\!]\rho[y := a]w,$$

for every $a \in A_w^{[\![\delta[\tau/X]]\!]\theta}$. Since $u \Vdash \Gamma[\rho]$ implies $w \Vdash \Gamma[\rho]$ when $u \preceq w$, and $a \in A_w^{[\![\delta[\tau/X]]\!]\theta}$, we have $w \Vdash (\Gamma, y : \delta[\tau/X])[\rho]$ for every $w \succeq u$. Thus, we can apply the induction hypothesis to $(\Gamma, y : \delta[\tau/X])$, $w \in \mathcal{W}$, $\rho = \langle \theta,\ \eta[y := a] \rangle$, the judgement $\Gamma, X: \star, y : \delta \triangleright M_1 : \gamma$, and $\tau \in \mathcal{T}$, and we have

$$\mathcal{A}[\![\Gamma, y : \delta[\tau/X] \triangleright M_1[\tau/X] : \gamma[\tau/X]]\!]\rho[y := a]w = \mathcal{A}[\![\Gamma, X: \star, y : \delta \triangleright M_1 : \gamma]\!]\rho[X := [\![\tau]\!]\theta,\ y := a]w,$$

and so, we have

$$\varphi_w(a) = \mathcal{A}[\![\Gamma, X: \star, y : \delta \triangleright M_1 : \gamma]\!]\rho[X := [\![\tau]\!]\theta,\ y := a]w.$$

However, we also have

$$\mathcal{A}[\![\Gamma, X\colon\star \rhd \lambda y\colon\delta.\, M_1\colon(\delta\to\gamma)]\!]\rho[X\colon= [\![\tau]\!]\theta]u = \mathbf{abst}_u^{[\![\delta]\!]\theta[X:=[\![\tau]\!]\theta],\,[\![\gamma]\!]\theta[X:=[\![\tau]\!]\theta]}(\psi),$$

where $\psi = (\psi_w)_{w\succeq u}$ is the family of functions defined such that,

$$\psi_w(a) = \mathcal{A}[\![\Gamma, X\colon\star, y\colon\delta \rhd M_1\colon\gamma]\!]\rho[X\colon= [\![\tau]\!]\theta,\ y\colon= a]w,$$

for every $a \in A_w^{[\![\delta]\!]\theta[X:=[\![\tau]\!]\theta]}$. By lemma 5.3, we have

$$[\![\delta[\tau/X]]\!]\theta = [\![\delta]\!]\theta[X\colon= [\![\tau]\!]\theta] \quad\text{and}\quad [\![\gamma[\tau/X]]\!]\theta = [\![\gamma]\!]\theta[X\colon= [\![\tau]\!]\theta],$$

and so we have $\varphi_w(a) = \psi_w(a)$, for every $a \in A_w^{[\![\delta]\!]\theta[X:=[\![\tau]\!]\theta]}$ and all $w \succeq u$, that is, $\varphi = \psi$. We also have

$$\mathbf{abst}_u^{[\![\delta]\!]\theta[X:=[\![\tau]\!]\theta],\,[\![\gamma]\!]\theta[X:=[\![\tau]\!]\theta]}(\psi) = \mathbf{abst}_u^{[\![\delta[\tau/X]]\!]\theta,\,[\![\gamma[\tau/X]]\!]\theta}(\varphi),$$

and thus,

$$\mathcal{A}[\![\Gamma \rhd \lambda y\colon\delta[\tau/X].\,(M_1[\tau/X])\colon(\delta\to\gamma)[\tau/X]]\!]\rho u = \mathcal{A}[\![\Gamma, X\colon\star \rhd \lambda y\colon\delta.\, M_1\colon(\delta\to\gamma)]\!]\rho[X\colon= [\![\tau]\!]\theta]u.$$

□

Actually, the following generalization of lemma 5.10 will also be needed.

Lemma 5.11 *Given a type interpretation T and a Kripke pre-applicative (β or $\beta\eta$)-structure $\mathcal{A}$, for every pair of contexts Γ, Δ, for every world $u \in \mathcal{W}$, for every valuation $\rho_1 = \langle\theta,\ \eta\rangle$, the following property holds: for every judgement $\Gamma \rhd M\colon\sigma$, for every substitution φ, if $\Delta \Vdash \Gamma[\varphi]$ and $u \Vdash \Delta[\rho_1]$, then*

$$\mathcal{A}[\![\Delta \rhd M[\varphi]\colon\sigma[\varphi]]\!]\rho_1 u = \mathcal{A}[\![\Gamma \rhd M\colon\sigma]\!]\rho_2 u,$$

where, if $\Gamma = \{X_1\colon\star, \ldots, X_m\colon\star, x_1\colon\sigma_1, \ldots, x_n\colon\sigma_n\}$, for $1 \le i \le m$, we let $s_i = [\![\varphi(X_i)]\!]\theta$, and for $1 \le j \le n$, we let $a_j = \mathcal{A}[\![\Delta \rhd \varphi(x_j)\colon\sigma_j[\varphi]]\!]\rho_1 u$, then

$$\rho_2 = \rho_1[X_1\colon= s_1, \ldots, X_m\colon= s_m, x_1\colon= a_1, \ldots, x_n\colon= a_n].$$

Proof. It is very similar to that of lemma 5.10, but the notation becomes quite formidable. □

We will now consider inequalities on Kripke pre-applicative structures and equations on Kripke applicative structures, and prove some soundness and completeness theorems.

6 Proving Inequalities (Rewrite rules) in $\lambda^{\to,\times,+,\forall^2}$

In this section, we define a number of proof systems for proving inequalities (rewrite rules) and equations. We also define satisfaction and validity (in a Kripke structure). There are three variations of satisfaction and validity, depending whether we consider Kripke applicative β-structures, Kripke applicative $\beta\eta$-structures, or extensional Kripke applicative β-structures.

Inequalities and equations are only defined between terms M and N such that $\Gamma \rhd M\colon\sigma$ and $\Gamma \rhd N\colon\sigma$ for some common Γ and σ. An inequality is denoted as $\Gamma \rhd M \preceq N\colon\sigma$, and an equation as $\Gamma \rhd M \doteq N\colon\sigma$, and provability is defined as follows.

Definition 6.1 The axioms and inference rules of the *inequational β-theory* of $\lambda^{\rightarrow,\times,+,\forall^2}$ are defined below.

Axioms:

$$\Gamma \triangleright M \preceq M : \sigma \quad (\textit{reflexivity})$$

$$\Gamma \triangleright M[N/x] \preceq (\lambda x : \sigma.\, M)N : \tau \quad (\beta)$$

$$\Gamma \triangleright M[\tau/X] \preceq (\lambda X.\, M)\tau : \sigma[\tau/X] \quad (\textit{type-}\beta)$$

$$\Gamma \triangleright M \preceq \pi_1(\langle M,\ N\rangle) : \sigma \quad (\pi_1)$$

$$\Gamma \triangleright N \preceq \pi_2(\langle M,\ N\rangle) : \tau \quad (\pi_2)$$

$$\Gamma \triangleright MP \preceq [M,\ N]\mathtt{inl}(P) : \delta \quad (\mathtt{inl})$$

$$\Gamma \triangleright NP \preceq [M,\ N]\mathtt{inr}(P) : \delta \quad (\mathtt{inr})$$

Inference Rules:

$$\frac{\Gamma \triangleright M_1 \preceq M_2 : \sigma}{\Delta \triangleright M_1 \preceq M_2 : \sigma} \quad (\textit{addvar})$$

where $\Gamma \subseteq \Delta$

$$\frac{\Gamma \triangleright M_1 \preceq M_2 : \sigma \quad \Gamma \triangleright M_2 \preceq M_3 : \sigma}{\Gamma \triangleright M_1 \preceq M_3 : \sigma} \quad (\textit{transitivity})$$

$$\frac{\Gamma \triangleright M_1 \preceq M_2 : (\sigma \rightarrow \tau) \quad \Gamma \triangleright N_1 \preceq N_2 : \sigma}{\Gamma \triangleright (M_1 N_1) \preceq (M_2 N_2) : \tau} \quad (\rightarrow\textit{-congruence})$$

$$\frac{\Gamma, x : \sigma \triangleright M_1 \preceq M_2 : \tau}{\Gamma \triangleright \lambda x : \sigma.\, M_1 \preceq \lambda x : \sigma.\, M_2 : (\sigma \rightarrow \tau)} \quad (\xi)$$

$$\frac{\Gamma \triangleright M_1 \preceq M_2 : \forall X.\, \sigma}{\Gamma \triangleright (M_1 \tau) \preceq (M_2 \tau) : \sigma[\tau/X]} \quad (\forall\textit{-congruence})$$

$$\frac{\Gamma, X : \star \triangleright M_1 \preceq M_2 : \sigma}{\Gamma \triangleright \lambda X.\, M_1 \preceq \lambda X.\, M_2 : \forall X.\, \sigma} \quad (\textit{type-}\xi)$$

$$\frac{\Gamma \triangleright M_1 \preceq M_2 : \sigma \quad \Gamma \triangleright N_1 \preceq N_2 : \tau}{\Gamma \triangleright \langle M_1,\ N_1\rangle \preceq \langle M_2,\ N_2\rangle : \sigma \times \tau} \quad (\times\textit{-congruence})$$

$$\frac{\Gamma \triangleright M_1 \preceq M_2 : \sigma \times \tau}{\Gamma \triangleright \pi_1(M_1) \preceq \pi_1(M_2) : \sigma} \quad (\pi_1\textit{-congruence})$$

$$\frac{\Gamma \triangleright M_1 \preceq M_2 : \sigma \times \tau}{\Gamma \triangleright \pi_2(M_1) \preceq \pi_2(M_2) : \tau} \quad (\pi_2\textit{-congruence})$$

$$\frac{\Gamma \triangleright M_1 \preceq M_2 : (\sigma \rightarrow \delta) \quad \Gamma \triangleright N_1 \preceq N_2 : (\tau \rightarrow \delta)}{\Gamma \triangleright [M_1,\ N_1] \preceq [M_2,\ N_2] : (\sigma + \tau) \rightarrow \delta} \quad (\mathtt{copair}\textit{-congruence})$$

$$\frac{\Gamma \triangleright M_1 \preceq M_2 : \sigma}{\Gamma \triangleright \mathtt{inl}(M_1) \preceq \mathtt{inl}(M_2) : \sigma + \tau} \quad (\mathtt{inl}\textit{-congruence})$$

$$\frac{\Gamma \triangleright M_1 \preceq M_2 : \tau}{\Gamma \triangleright \mathtt{inr}(M_1) \preceq \mathtt{inr}(M_2) : \sigma + \tau} \quad (\mathtt{inr}\text{-}congruence)$$

$$\frac{\Gamma \triangleright M_1 \preceq M_2 : \sigma}{\Delta \triangleright M_1[\varphi] \preceq M_2[\varphi] : \sigma[\varphi]} \quad (substitution)$$

for every substitution φ such that $\Delta \Vdash \Gamma[\varphi]$.

The notation $\vdash_\beta \Gamma \triangleright M \preceq N : \sigma$ means that the inequality $\Gamma \triangleright M \preceq N : \sigma$ is provable from the above axioms and inference rules.

The *inequational $\beta\eta$-theory* of the system $\lambda^{\rightarrow,\times,+,\forall^2}$ is obtained by adding the following *η-like rules* to the axioms and inference rules of the β-theory:

$$\Gamma \triangleright M \preceq \lambda x : \sigma . (Mx) : (\sigma \rightarrow \tau) \quad (\eta)$$

where $x \notin FV(M)$;

$$\Gamma \triangleright M \preceq \lambda X . (MX) : \forall X . \sigma \quad (type\text{-}\eta)$$

where $X \notin FTV(M)$;

$$\Gamma \triangleright M \preceq \langle \pi_1(M), \pi_2(M) \rangle : \sigma \times \tau \quad (\mathtt{pair})$$

$$\Gamma \triangleright M \preceq [\lambda x : \sigma . (M\,\mathtt{inl}(x)), \lambda y : \tau . (M\,\mathtt{inr}(y))] : (\sigma + \tau) \rightarrow \delta \quad (\mathtt{copair})$$

The notation $\vdash_{\beta\eta} \Gamma \triangleright M \preceq N : \sigma$ means that the inequality $\Gamma \triangleright M \preceq N : \sigma$ is provable from all the axioms and the inference rules of the $\beta\eta$-theory, including the η-like rules.

The *extensional inequational $\beta\eta$-theory* of the system $\lambda^{\rightarrow,\times,+,\forall^2}$ is obtained by adding the following inference rules (*extensionality rules*) to the axioms and inference rules of the β-theory of $\lambda^{\rightarrow,\times,+,\forall^2}$:

$$\frac{\Gamma, x : \sigma \triangleright M_1 x \preceq M_2 x : \tau}{\Gamma \triangleright M_1 \preceq M_2 : (\sigma \rightarrow \tau)} \quad (\mathtt{fun}\text{-}extentionality)$$

where $x \notin FV(M_1) \cup FV(M_2)$;

$$\frac{\Gamma, X : \star \triangleright M_1 X \preceq M_2 X : \sigma}{\Gamma \triangleright M_1 \preceq M_2 : \forall X . \sigma} \quad (\mathtt{tfun}\text{-}extentionality)$$

where $X \notin FTV(M_1) \cup FTV(M_2)$;

$$\frac{\Gamma \triangleright \pi_1(M_1) \preceq \pi_1(M_2) : \sigma \quad \Gamma \triangleright \pi_2(M_1) \preceq \pi_2(M_2) : \tau}{\Gamma \triangleright M_1 \preceq M_2 : (\sigma \times \tau)} \quad (\Pi\text{-}extentionality)$$

$$\frac{\Gamma, x : \sigma \triangleright M_1 \mathtt{inl}(x) \preceq M_2 \mathtt{inl}(x) : \delta \quad \Gamma, y : \tau \triangleright M_1 \mathtt{inr}(y) \preceq M_2 \mathtt{inr}(y) : \delta}{\Gamma \triangleright M_1 \preceq M_2 : (\sigma + \tau) \rightarrow \delta} \quad (\mathtt{inl}, \mathtt{inr}\text{-}extentionality)$$

where $x, y \notin FV(M_1) \cup FV(M_2)$.

The notation $\vdash_{ex\beta\eta} \Gamma \triangleright M \preceq N : \sigma$ means that the inequality $\Gamma \triangleright M \preceq N : \sigma$ is provable from all the axioms and the inference rules of the extensional β-theory, including the extensionality rules.

By rule (*addvar*), if $\vdash_\beta \Gamma \triangleright M \preceq N : \sigma$, then $\vdash_\beta \Delta \triangleright M \preceq N : \sigma$, for any Δ such that $\Gamma \subseteq \Delta$, and similarly for $\vdash_{\beta\eta}$ and $\vdash_{ex\beta\eta}$. Actually, this rule is only needed when we consider deductions from nonempty sets of inequalities other than the axioms. Otherwise, due to the form of the axioms, by induction on the structure of proofs, it is easily shown that rule (*addvar*) is a derived rule.

The following lemma shows the relationship between the η-like rules and the extensionality rules. Given an inequality $\Gamma \triangleright M \preceq N : \sigma$, its *converse* is the inequality $\Gamma \triangleright N \preceq M : \sigma$.

Lemma 6.2 *In the $ex\beta\eta$-theory, the η-like rules are provable from the extensionality rules. If we add the converse of each η-like rule to the $\beta\eta$-theory, then the extensionality rules are provable.*

Proof. First, we prove that in the $ex\beta\eta$-theory, the η-like rules are provable from the extensionality rules.

If $x \notin FV(M)$, observe that

$$\Gamma, x : \sigma \triangleright Mx \preceq (\lambda x : \sigma . (Mx))x : \tau$$

is a consequence of axiom (β), since $(Mx)[x/x] = Mx$. Thus, by the first extensionality rule, we have

$$\Gamma \triangleright M \preceq \lambda x : \sigma . (Mx) : (\sigma \to \tau)$$

where $x \notin FV(M)$. We prove in a similar fashion that

$$\Gamma \triangleright M \preceq \lambda X . (MX) : \forall X . \sigma .$$

where $X \notin FTV(M)$. Proving

$$\Gamma \triangleright M \preceq \langle \pi_1(M),\ \pi_2(M) \rangle : \sigma \times \tau$$

is easy, and we prove that

$$\Gamma \triangleright M \preceq [\lambda x : \sigma . (M\,\mathtt{inl}(x)),\ \lambda y : \tau . (M\,\mathtt{inr}(y))] : (\sigma + \tau) \to \delta .$$

Assume that $x \notin FV(M)$ and $y \notin FV(M)$. Then, by axioms (β), ($\mathtt{inl}$), and ($\mathtt{inr}$), we have

$$M\,\mathtt{inl}(x) \preceq (\lambda x : \sigma . (M\,\mathtt{inl}(x)))x \preceq [\lambda x : \sigma . (M\,\mathtt{inl}(x)),\ \lambda y : \tau . (M\,\mathtt{inr}(y))]\mathtt{inl}(x),$$

and

$$M\,\mathtt{inr}(y) \preceq (\lambda y : \tau . (M\,\mathtt{inr}(y)))y \preceq [\lambda x : \sigma . (M\,\mathtt{inl}(x)),\ \lambda y : \tau . (M\,\mathtt{inr}(y))]\mathtt{inr}(y).$$

We conclude using the last extensionality rule.

Conversely, we prove that from the η-like rules and their converse, we obtain the extensionality rules. We consider the first rule, the others being similar.

Assume that $\vdash_{\beta\eta} \Gamma, x : \sigma \triangleright M_1 x \preceq M_2 x : \tau$, where $x \notin FV(M_1) \cup FV(M_2)$. Then, by ($\xi$), we get

$$\vdash_{\beta\eta} \Gamma \triangleright \lambda x : \sigma . (M_1 x) \preceq \lambda x : \sigma . (M_2 x) : (\sigma \to \tau).$$

Since $x \notin FV(M_1) \cup FV(M_2)$, using ($\eta$), we get

$$\vdash_{\beta\eta} \Gamma \triangleright M_1 \preceq \lambda x : \sigma . (M_1 x) : (\sigma \to \tau),$$

and using the converse of (η), we get

$$\vdash_{\beta\eta} \Gamma \triangleright \lambda x{:}\,\sigma.\,(M_2 x) \preceq M_2{:}\,(\sigma \to \tau),$$

and by transitivity (twice), we have

$$\vdash_{\beta\eta} \Gamma \triangleright M_1 \preceq M_2{:}\,(\sigma \to \tau).$$

□

The following lemma shows the relationship between the (ξ)- rule, the (*substitution*)-rule, and the converse of the (β)-axioms. If $\Gamma = \{X_1{:}\star, \ldots, X_m{:}\star, x_1{:}\sigma_1, \ldots, x_n{:}\sigma_n\}$, given an inequality $\Gamma \triangleright M \preceq N{:}\,\sigma$, we let

$$\triangleright\, \lambda X_1 \ldots \lambda X_m.\, \lambda x_1{:}\,\sigma_1 \ldots \lambda x_n{:}\,\sigma_n.\, M \preceq \lambda X_1 \ldots \lambda X_m.\, \lambda x_1{:}\,\sigma_1 \ldots \lambda x_n{:}\,\sigma_n.\, N{:}\,\delta,$$

where $\delta = \forall X_1 \ldots \forall X_m.\,(\sigma_1 \to (\ldots(\sigma_n \to \sigma)\ldots))$, be the *closure* of $\Gamma \triangleright M \preceq N{:}\,\sigma$, and we denote it as $\triangleright\, \lambda\overrightarrow{X}.\,\lambda\overrightarrow{x}{:}\,\overrightarrow{\sigma}.\,M \preceq \lambda\overrightarrow{X}.\,\lambda\overrightarrow{x}{:}\,\overrightarrow{\sigma}.\,N{:}\forall\overrightarrow{X}.\,\overrightarrow{\sigma}$.

Lemma 6.3 *If we add the converse of the β-rule and the converse of the (type-β)-rule to the β-theory, then the following properties hold: (1) the substitution-rule is provable; (2) an inequality $\Gamma \triangleright M \preceq N{:}\,\sigma$ is β-provable iff its closure is β-provable.*

Proof. (1) Let φ be a substitution such that $\Delta \Vdash \Gamma[\varphi]$, and assume that $\vdash_\beta \Gamma \triangleright M_1 \preceq M_2{:}\,\sigma$. By applications of the (ξ)-rule and the (*type-ξ*)-rule, we get

$$\vdash_\beta \triangleright\, \lambda\overrightarrow{X}.\,\lambda\overrightarrow{x}{:}\,\overrightarrow{\sigma}.\,M_1 \preceq \lambda\overrightarrow{X}.\,\lambda\overrightarrow{x}{:}\,\overrightarrow{\sigma}.\,M_2{:}\forall\overrightarrow{X}.\,\overrightarrow{\sigma}.$$

Thus, by a previous remark, we also have

$$\vdash_\beta \Delta \triangleright\, \lambda\overrightarrow{X}.\,\lambda\overrightarrow{x}{:}\,\overrightarrow{\sigma}.\,M_1 \preceq \lambda\overrightarrow{X}.\,\lambda\overrightarrow{x}{:}\,\overrightarrow{\sigma}.\,M_2{:}\forall\overrightarrow{X}.\,\overrightarrow{\sigma}.$$

We can make sure that $X_i \notin \bigcup(FVT(\varphi(X_k)))_{1\le k\le m}$, and that $x_j \notin \bigcup(FV(\varphi(x_l)))_{1\le l\le n}$, using α-conversion, and since $\Delta \Vdash \Gamma[\varphi]$, by applications of the (*congruence*)-rules for $\to$ and $\forall$, we get

$$\vdash_\beta \Delta \triangleright ((\lambda\overrightarrow{X}.\,\lambda\overrightarrow{x}{:}\,\overrightarrow{\sigma}.\,M_1)\overrightarrow{\varphi(X_i)})\overrightarrow{\varphi(x_j)} \preceq ((\lambda\overrightarrow{X}.\,\lambda\overrightarrow{x}{:}\,\overrightarrow{\sigma}.\,M_2)\overrightarrow{\varphi(X_i)})\overrightarrow{\varphi(x_j)}{:}\,\sigma[\varphi].$$

Then, by applications of the (β, *type-β*)-rules and their converse, and using transitivity, we get

$$\vdash_\beta \Delta \triangleright M_1[\varphi] \preceq M_2[\varphi]{:}\,\sigma[\varphi].$$

(2) By applications of the (ξ)-rule and the (*type-ξ*)-rule, if $\vdash_\beta \Gamma \triangleright M_1 \preceq M_2{:}\,\sigma$, then

$$\vdash_\beta \triangleright\, \lambda\overrightarrow{X}.\,\lambda\overrightarrow{x}{:}\,\overrightarrow{\sigma}.\,M_1 \preceq \lambda\overrightarrow{X}.\,\lambda\overrightarrow{x}{:}\,\overrightarrow{\sigma}.\,M_2{:}\forall\overrightarrow{X}.\,\overrightarrow{\sigma}.$$

Conversely, if

$$\vdash_\beta \triangleright\, \lambda\overrightarrow{X}.\,\lambda\overrightarrow{x}{:}\,\overrightarrow{\sigma}.\,M_1 \preceq \lambda\overrightarrow{X}.\,\lambda\overrightarrow{x}{:}\,\overrightarrow{\sigma}.\,M_2{:}\forall\overrightarrow{X}.\,\overrightarrow{\sigma},$$

then

$$\vdash_\beta \Gamma \triangleright \lambda \overrightarrow{X}.\, \lambda \overrightarrow{x}{:}\, \overrightarrow{\sigma}.\, M_1 \preceq \lambda \overrightarrow{X}.\, \lambda \overrightarrow{x}{:}\, \overrightarrow{\sigma}.\, M_2{:}\forall \overrightarrow{X}.\, \overrightarrow{\sigma},$$

and by choosing φ to be the identity substitution on Γ, by the previous argument, we have

$$\vdash_\beta \Gamma \triangleright M_1 \preceq M_2{:}\, \sigma,$$

since $M_1[\varphi] = M_1$, $N_1[\varphi] = N_1$, and $\sigma[\varphi] = \sigma$. □

We now define provability from a set of inequalities, and the the notions of satisfaction and validity. Let $\mathcal{E}$ be a set of inequalities (of the form $\Gamma \triangleright M \preceq N{:}\,\sigma$).

Definition 6.4 An inequality $\Gamma \triangleright M \preceq N{:}\,\sigma$ is *β-provable from a set $\mathcal{E}$ of inequalities*, denoted as $\mathcal{E} \vdash_\beta \Gamma \triangleright M \preceq N{:}\,\sigma$, iff $\Gamma \triangleright M \preceq N{:}\,\sigma$ is β-provable from the system obtained by adding all inequalities in $\mathcal{E}$ to the axioms of the system of definition 6.1. Note that when $\mathcal{E} = \emptyset$, this notion coincides with $\vdash_\beta \Gamma \triangleright M \preceq N{:}\,\sigma$. An inequality $\Gamma \triangleright M \preceq N{:}\,\sigma$ is *$\beta\eta$-provable from a set $\mathcal{E}$ of inequalities*, denoted as $\mathcal{E} \vdash_{\beta\eta} \Gamma \triangleright M \preceq N{:}\,\sigma$, iff $\Gamma \triangleright M \preceq N{:}\,\sigma$ is $\beta\eta$-provable from the system obtained by adding all inequalities in $\mathcal{E}$ to the axioms of the $\beta\eta$-system of definition 6.1, and $\mathcal{E} \vdash_{ex\beta\eta} \Gamma \triangleright M \preceq N{:}\,\sigma$ is defined similarly for the $ex\beta\eta$-theory. Note that when $\mathcal{E} = \emptyset$, these notions coincides with $\vdash_{\beta\eta} \Gamma \triangleright M \preceq N{:}\,\sigma$ and $\vdash_{ex\beta\eta} \Gamma \triangleright M \preceq N{:}\,\sigma$, respectively.

We also define the notion of satisfaction and validity.

Definition 6.5 Given a type interpretation T and a Kripke pre-applicative β-structure $\mathcal{A}$, for every Γ, for every world $u \in \mathcal{W}$, for every valuation $\rho = \langle \theta,\ \eta \rangle$, we have the following definitions:

(1) For every inequality $\Gamma \triangleright M \preceq N{:}\,\sigma$, we say that $\Gamma \triangleright M \preceq N{:}\,\sigma$ *holds at u and ρ in $\mathcal{A}$*, denoted as $\mathcal{A},\ u \Vdash_\beta (\Gamma \triangleright M \preceq N{:}\,\sigma)[\rho]$, iff whenever $u \Vdash \Gamma[\rho]$, then

$$\mathcal{A}[\![\Gamma \triangleright M{:}\,\sigma]\!]\rho u \preceq \mathcal{A}[\![\Gamma \triangleright N{:}\,\sigma]\!]\rho u.$$

(2) *$\mathcal{A}$ satisfies* $\Gamma \triangleright M \preceq N{:}\,\sigma$, denoted as $\mathcal{A} \Vdash_\beta \Gamma \triangleright M \preceq N{:}\,\sigma$, iff $\mathcal{A},\ u \Vdash_\beta (\Gamma \triangleright M \preceq N{:}\,\sigma)[\rho]$ for every world u and every valuation ρ for $\mathcal{A}$.

(3) Given a set $\mathcal{E}$ of inequalities, *$\mathcal{A}$ satisfies $\mathcal{E}$*, denoted as $\mathcal{A} \Vdash_\beta \mathcal{E}$, iff $\mathcal{A} \Vdash_\beta \Gamma \triangleright M \preceq N{:}\,\sigma$ for all $\Gamma \triangleright M \preceq N{:}\,\sigma \in \mathcal{E}$; We say that $\Gamma \triangleright M \preceq N{:}\,\sigma$ *is a semantic consequence of $\mathcal{E}$*, denoted as $\mathcal{E} \Vdash_\beta \Gamma \triangleright M \preceq N{:}\,\sigma$, iff $\mathcal{A} \Vdash_\beta \Gamma \triangleright M \preceq N{:}\,\sigma$ whenever $\mathcal{A} \Vdash_\beta \mathcal{E}$, for every Kripke β-structure $\mathcal{A}$.

(4) We say that $\Gamma \triangleright M \preceq N{:}\,\sigma$ *is valid*, denoted as $\Vdash_\beta \Gamma \triangleright M \preceq N{:}\,\sigma$, iff $\mathcal{A} \Vdash_\beta \Gamma \triangleright M \preceq N{:}\,\sigma$ for every $\mathcal{A}$.

The above notions are defined in a similar fashion for Kripke $\beta\eta$-structures and extensional Kripke $\beta\eta$-structures, in which case we use $\Vdash_{\beta\eta}$ and $\Vdash_{ex\beta\eta}$ instead of $\Vdash_\beta$.

7 Soundness and Completeness Results for Rewrite rules

In this section, we prove some soundness and completeness results. Soundness is shown in lemma 7.1. Lemma 7.2 shows the existence of a Kripke model associated with a set of inequalities (rewrite rules). Extended compeletenes is shown in theorem 7.3. We also consider completeness with respect to Kripke structures with nonempty carriers. By adding the rule (*nonempty*), we obtain completeness (theorem 7.7).

First, we show a soundness lemma.

Lemma 7.1 *For any set $\mathcal{E}$ of inequalities, for every inequality $\Gamma \triangleright M \preceq N{:}\,\sigma$, the following properties hold: (1) if $\mathcal{E} \vdash_\beta \Gamma \triangleright M \preceq N{:}\,\sigma$, then $\mathcal{E} \Vdash_\beta \Gamma \triangleright M \preceq N{:}\,\sigma$; (2) if $\mathcal{E} \vdash_{\beta\eta} \Gamma \triangleright M \preceq N{:}\,\sigma$, then $\mathcal{E} \Vdash_{\beta\eta} \Gamma \triangleright M \preceq N{:}\,\sigma$; (3) if $\mathcal{E} \vdash_{ex\beta\eta} \Gamma \triangleright M \preceq N{:}\,\sigma$, then $\mathcal{E} \Vdash_{ex\beta\eta} \Gamma \triangleright M \preceq N{:}\,\sigma$.*

Proof. (1) We proceed by induction on the structure of the proof $\mathcal{E} \vdash_\beta \Gamma \triangleright M \preceq N{:}\,\sigma$.

Axiom (β). Assume that $u \Vdash \Gamma[\rho]$. Recall from definition 4.1 that **app** is defined from **fun** as $\mathbf{app} = \mathbf{eval}^{A^s,A^t} \circ (\mathbf{fun} \times \mathbf{id}_{A^s})$, and from definition 3.6, that $\mathbf{eval}_u^{A^s,A^t}((\varphi_w)_{w \succeq u}, a) = \varphi_u(a)$, for any $\varphi = (\varphi_w)_{w \succeq u} \in [A^s \Rightarrow A^t]_u$ and any $a \in A^s_u$. Also, recall from condition (1) of definition 4.1 that we have, $\mathbf{fun}_u(\mathbf{abst}_u(\varphi)) \succeq_u \varphi$, for every $\varphi \in [A^s \Rightarrow A^t]_u$. Thus, we have

$$\mathbf{app}_u(\mathbf{abst}_u(\varphi), a) = \mathbf{eval}_u(\mathbf{fun}_u(\mathbf{abst}_u(\varphi)), a) \succeq_u \mathbf{eval}_u(\varphi, a) = \varphi_u(a),$$

that is, $\mathbf{app}_u(\mathbf{abst}_u(\varphi), a) \succeq_u \varphi_u(a)$. From definition 5.5, we have

$$\mathcal{A}[\![\Gamma \triangleright MN{:}\,\tau]\!]\rho u = \mathbf{app}_u^{[\![\sigma]\!]\theta,\,[\![\tau]\!]\theta}(\mathcal{A}[\![\Gamma \triangleright M{:}\,\sigma \to \tau]\!]\rho u,\ \mathcal{A}[\![\Gamma \triangleright N{:}\,\sigma]\!]\rho u),$$

and

$$\mathcal{A}[\![\Gamma \triangleright \lambda x{:}\,\sigma.\, M{:}\,\sigma \to \tau]\!]\rho u = \mathbf{abst}_u^{[\![\sigma]\!]\theta,\,[\![\tau]\!]\theta}(\varphi),$$

where $\varphi = (\varphi_w)_{w \succeq u}$ is the family of functions defined such that,

$$\varphi_w(a) = \mathcal{A}[\![\Gamma, x{:}\,\sigma \triangleright M{:}\,\tau]\!]\rho[x{:}= a]w,$$

for every $a \in A_w^{[\![\sigma]\!]\theta}$. Then,

$$\mathcal{A}[\![\Gamma \triangleright (\lambda x{:}\,\sigma.\, M)N{:}\,\tau]\!]\rho u = \mathbf{app}_u^{[\![\sigma]\!]\theta,\,[\![\tau]\!]\theta}(\mathcal{A}[\![\Gamma \triangleright \lambda x{:}\,\sigma.\, M{:}\,\sigma \to \tau]\!]\rho u,\ \mathcal{A}[\![\Gamma \triangleright N{:}\,\sigma]\!]\rho u),$$

and letting $a = \mathcal{A}[\![\Gamma \triangleright N{:}\,\sigma]\!]\rho u$, by the definition of $\mathcal{A}[\![\Gamma \triangleright \lambda x{:}\,\sigma.\, M{:}\,\sigma \to \tau]\!]\rho u$ and the fact that $\mathbf{app}_u(\mathbf{abst}_u(\varphi), a) \succeq_u \varphi_u(a)$, we have

$$\mathcal{A}[\![\Gamma \triangleright (\lambda x{:}\,\sigma.\, M)N{:}\,\tau]\!]\rho u \succeq \mathcal{A}[\![\Gamma, x{:}\,\sigma \triangleright M{:}\,\tau]\!]\rho[x{:}= a]u,$$

with $a = \mathcal{A}[\![\Gamma \triangleright N{:}\,\sigma]\!]\rho u$. However, by lemma 5.10, we have

$$\mathcal{A}[\![\Gamma \triangleright M[N/x]{:}\,\tau]\!]\rho u = \mathcal{A}[\![\Gamma, x{:}\,\sigma \triangleright M{:}\,\tau]\!]\rho[x{:}= \mathcal{A}[\![\Gamma \triangleright N{:}\,\sigma]\!]\rho u]u,$$

and thus,

$$\mathcal{A}[\![\Gamma \triangleright (\lambda x{:}\,\sigma.\, M)N{:}\,\tau]\!]\rho u \succeq \mathcal{A}[\![\Gamma \triangleright M[N/x]{:}\,\tau]\!]\rho u.$$

Axiom (*type-β*). Recall from definition 4.1 that $\mathtt{tapp}$ is defined from $\mathtt{tfun}$ as $\mathtt{tapp} = \mathtt{eval}^A_\Phi \circ (\mathtt{tfun} \times \mathtt{id}_T)$, and from definition 3.10, that $\mathtt{eval}^A_{\Phi,u}((\varphi_w)_{w \succeq u}, s) = \varphi_u(s)$, for any $\varphi = (\varphi_w)_{w \succeq u} \in \prod_\Phi (A^s_u)_{s \in T}$ and any $s \in T$. Also, recall from condition (2) of definition 4.1 that we have, $\mathtt{tfun}_u(\mathtt{tabst}_u(\varphi)) \succeq_u \varphi$, for every $\varphi \in \prod_\Phi (A^s_u)_{s \in T}$. Thus, we have

$$\mathtt{tapp}_u(\mathtt{tabst}_u(\varphi), s) = \mathtt{eval}_{\Phi,u}(\mathtt{tfun}_u(\mathtt{tabst}_u(\varphi)), s) \succeq_u \mathtt{eval}_{\Phi,u}(\varphi, s) = \varphi_u(s),$$

that is, $\mathtt{tapp}_u(\mathtt{tabst}_u(\varphi), s) \succeq_u \varphi_u(s)$. From definition 5.5, we have

$$\mathcal{A}[\![\Gamma \triangleright M\tau : \sigma[\tau/X]]\!]\rho u = \mathtt{tapp}^\Phi_u(\mathcal{A}[\![\Gamma \triangleright M : \forall X.\, \sigma]\!]\rho u,\ [\![\tau]\!]\theta),$$

and

$$\mathcal{A}[\![\Gamma \triangleright \lambda X.\, M : \forall X.\, \sigma]\!]\rho u = \mathtt{tabst}^\Phi_u(\varphi),$$

where $\varphi = (\varphi_w)_{w \succeq u}$ is the family of functions defined such that,

$$\varphi_w(s) = \mathcal{A}[\![\Gamma, X : \star \triangleright M : \sigma]\!]\rho[X := s]w,$$

for every $s \in T$, and where Φ is the function such that $\Phi(s) = [\![\sigma]\!]\theta[X := s]$ for every $s \in T$. Then,

$$\mathcal{A}[\![\Gamma \triangleright (\lambda X.\, M)\tau : \sigma[\tau/X]]\!]\rho u = \mathtt{tapp}^\Phi_u(\mathcal{A}[\![\Gamma \triangleright \lambda X.\, M : \forall X.\, \sigma]\!]\rho u,\ [\![\tau]\!]\theta),$$

and letting $s = [\![\tau]\!]\theta$, by the definition of $\mathcal{A}[\![\Gamma \triangleright \lambda X.\, M : \forall X.\, \sigma]\!]\rho u$ and since $\mathtt{tapp}_u(\mathtt{tabst}_u(\varphi), s) \succeq_u \varphi_u(s)$, we have

$$\mathcal{A}[\![\Gamma \triangleright (\lambda X.\, M)\tau : \sigma[\tau/X]]\!]\rho u \succeq \mathcal{A}[\![\Gamma, X : \star \triangleright M : \sigma]\!]\rho[X := s]u,$$

where $s = [\![\tau]\!]\theta$. However, by lemma 5.10, we have

$$\mathcal{A}[\![\Gamma \triangleright M[\tau/X] : \sigma[\tau/X]]\!]\rho u = \mathcal{A}[\![\Gamma, X : \star \triangleright M : \sigma]\!]\rho[X := [\![\tau]\!]\theta]u,$$

and thus

$$\mathcal{A}[\![\Gamma \triangleright (\lambda X.\, M)\tau : \sigma[\tau/X]]\!]\rho u \succeq \mathcal{A}[\![\Gamma \triangleright M[\tau/X] : \sigma[\tau/X]]\!]\rho u.$$

The other axioms are treated easily, and so are the inference rules. As an illustration, we treat the rule (ξ) and the (*substitution*) rule.

Rule (ξ). Assume that $\vdash_\beta \Gamma, x : \sigma \triangleright M_1 \preceq M_2 : \tau$. By the rule ($\xi$), we have

$$\vdash_\beta \Gamma \triangleright \lambda x : \sigma.\, M_1 \preceq \lambda x : \sigma.\, M_2 : (\sigma \to \tau).$$

By the induction hypothesis, we have

$$\mathcal{A} \Vdash_\beta \Gamma, x : \sigma \triangleright M_1 \preceq M_2 : \tau,$$

which means that

$$\mathcal{A},\, w \Vdash_\beta (\Gamma, x : \sigma \triangleright M_1 \preceq M_2 : \tau)[\rho_1]$$

for every world $w \in \mathcal{W}$ and every valuation ρ_1. We need to show that

$$\mathcal{A} \Vdash_\beta \Gamma \triangleright \lambda x : \sigma.\, M_1 \preceq \lambda x : \sigma.\, M_2 : (\sigma \to \tau)$$

for every Kripke pre-applicative β-structure $\mathcal{A}$.

Let $\mathcal{A}$ be any Kripke pre-applicative β-structure, $u \in \mathcal{W}$ any world, and ρ_2 any valuation, and assume that $u \Vdash \Gamma[\rho_2]$. By definition 5.5, we have

$$\mathcal{A}[\![\Gamma \triangleright \lambda x{:}\,\sigma.\, M_1{:}\,\sigma \to \tau]\!]\rho_2 u = \mathbf{abst}_u^{[\sigma]\theta,\,[\tau]\theta}(\varphi),$$

where $\varphi = (\varphi_w)_{w \succeq u}$ is the family of functions defined such that,

$$\varphi_w(a) = \mathcal{A}[\![\Gamma, x{:}\,\sigma \triangleright M_1{:}\,\tau]\!]\rho_2[x{:}= a]w,$$

for every $a \in A_w^{[\sigma]\theta}$, and similarly

$$\mathcal{A}[\![\Gamma \triangleright \lambda x{:}\,\sigma.\, M_2{:}\,\sigma \to \tau]\!]\rho_2 u = \mathbf{abst}_u^{[\sigma]\theta,\,[\tau]\theta}(\psi),$$

where

$$\psi_w(a) = \mathcal{A}[\![\Gamma, x{:}\,\sigma \triangleright M_2{:}\,\tau]\!]\rho_2[x{:}= a]w,$$

for every $a \in A_w^{[\sigma]\theta}$. Since $u \Vdash \Gamma[\rho_2]$, for every $a \in A_w^{[\sigma]\theta}$, and every $w \succeq u$, we have $w \Vdash (\Gamma, x{:}\,\sigma)[\rho_2[x{:}= a]]$, and since

$$\mathcal{A},\, w \Vdash_\beta (\Gamma, x{:}\,\sigma \triangleright M_1 \preceq M_2{:}\,\tau)[\rho_1],$$

for every $w \in \mathcal{W}$ and every valuation ρ_1, by definition 6.5, we have

$$\mathcal{A}[\![\Gamma, x{:}\,\sigma \triangleright M_1{:}\,\tau]\!]\rho_2[x{:}= a]w \preceq \mathcal{A}[\![\Gamma, x{:}\,\sigma \triangleright M_2{:}\,\tau]\!]\rho[x{:}= a]w.$$

Since this holds for every $a \in A_w^{[\sigma]\theta}$, we have $\varphi_w \preceq \psi_w$ for all $w \succeq u$, and thus $\varphi \preceq \psi$, that is,

$$\mathcal{A}[\![\Gamma \triangleright \lambda x{:}\,\sigma.\, M_1{:}\,\sigma \to \tau]\!]\rho_2 u \preceq \mathcal{A}[\![\Gamma \triangleright \lambda x{:}\,\sigma.\, M_2{:}\,\sigma \to \tau]\!]\rho_2 u.$$

This shows that

$$\mathcal{A},\, u \Vdash_\beta (\Gamma \triangleright \lambda x{:}\,\sigma.\, M_1 \preceq \lambda x{:}\,\sigma.\, M_2{:}\,(\sigma \to \tau))[\rho_2],$$

for every $u \in \mathcal{W}$ and every valuation ρ_2. Thus, we just showed that

$$\mathcal{A} \Vdash_\beta \Gamma \triangleright \lambda x{:}\,\sigma.\, M_1 \preceq \lambda x{:}\,\sigma.\, M_2{:}\,(\sigma \to \tau)$$

for every Kripke pre-applicative β-structure $\mathcal{A}$, as desired.

Rule (*substitution*). Let φ be a substitution such that $\Delta \Vdash \Gamma[\varphi]$, and assume that $\vdash_\beta \Gamma \triangleright M_1 \preceq M_2{:}\,\sigma$. By the induction hypothesis,

$$\mathcal{A} \Vdash_\beta \Gamma \triangleright M_1 \preceq M_2{:}\,\sigma,$$

for every Kripke pre-applicative β-structure $\mathcal{A}$. We need to prove that

$$\mathcal{A} \Vdash_\beta \Delta \triangleright M_1[\varphi] \preceq M_2[\varphi]{:}\,\sigma[\varphi],$$

for every Kripke pre-applicative β-structure $\mathcal{A}$.

Let $\Gamma = \{X_1{:}\,\star, \ldots, X_m{:}\,\star, x_1{:}\,\sigma_1, \ldots, x_n{:}\,\sigma_n\}$, and let $\mathcal{A}$ be any Kripke pre-applicative β-structure, $u \in \mathcal{W}$ any world, ρ_1 any valuation, and assume that $u \Vdash \Delta[\rho_1]$. By lemma 5.11, we have

$$\mathcal{A}[\![\Delta \triangleright M[\varphi]{:}\,\sigma[\varphi]]\!]\rho_1 u = \mathcal{A}[\![\Gamma \triangleright M{:}\,\sigma]\!]\rho_2 u,$$

where $s_i = [\![\varphi(X_i)]\!]\theta$, for $1 \leq i \leq m$, $a_j = \mathcal{A}[\![\Delta \triangleright \varphi(x_j) \colon \sigma_j[\varphi]]\!]\rho_1 u$, for $1 \leq j \leq n$, and

$$\rho_2 = \rho_1[X_1 := s_1, \ldots, X_m := s_m, x_1 := a_1, \ldots, x_n := a_n].$$

Note that $u \Vdash \Gamma[\rho_2]$, and since we assumed that $\mathcal{A} \Vdash_\beta \Gamma \triangleright M_1 \preceq M_2 \colon \sigma$ holds, we have $\mathcal{A}, u \Vdash_\beta (\Gamma \triangleright M_1 \preceq M_2 \colon \sigma)[\rho_2]$, which means that

$$\mathcal{A}[\![\Gamma \triangleright M_1 \colon \sigma]\!]\rho_2 u \preceq \mathcal{A}[\![\Gamma \triangleright M_1 \colon \sigma]\!]\rho_2 u,$$

which, in view of previous identities, is equivalent to

$$\mathcal{A}[\![\Delta \triangleright M_1[\varphi] \colon \sigma[\varphi]]\!]\rho_1 u \preceq \mathcal{A}[\![\Delta \triangleright M_2[\varphi] \colon \sigma[\varphi]]\!]\rho_1 u,$$

that is

$$\mathcal{A}, u \Vdash_\beta (\Delta \triangleright M_1[\varphi] \preceq M_2[\varphi] \colon \sigma[\varphi])[\rho_1].$$

The above holds for all $u \in \mathcal{W}$ and all ρ_1, and thus

$$\mathcal{A} \Vdash_\beta \Delta \triangleright M_1[\varphi] \preceq M_2[\varphi] \colon \sigma[\varphi],$$

for every Kripke pre-applicative β-structure $\mathcal{A}$, as desired.

(2) We proceed by induction on the structure of the proof $\mathcal{E} \vdash_{\beta\eta} \Gamma \triangleright M \preceq N \colon \sigma$. The only new cases are the η-like axioms.

Axiom (η). Assume that $u \Vdash \Gamma[\rho]$. As in the case of axiom (β), we have $\mathbf{app} = \mathbf{eval}^{A^s, A^t} \circ (\mathbf{fun} \times \mathbf{id}_{A^s})$ and $\mathbf{eval}_u^{A^s, A^t}((\varphi_w)_{w \succeq u}, a) = \varphi_u(a)$, for any $\varphi = (\varphi_w)_{w \succeq u} \in [A^s \Rightarrow A^t]_u$ and any $a \in A_u^s$. For any $f \in A_u^{s \to t}$, if $\mathbf{fun}_u(f) = (\varphi_w)_{w \succeq u}$, for every $a \in A_w^s$, we have

$$\mathbf{app}_w(i_{u,w}^{s \to t}(f),\, a) = \mathbf{eval}_w(\mathbf{fun}_w(i_{u,w}^{s \to t}(f)),\, a) = \mathbf{eval}_w((\varphi_{w'})_{w' \succeq w},\, a) = \varphi_w(a).$$

This shows that $(\mathbf{fun}_u(f))_w = \varphi_w$ is the function such that $(\mathbf{fun}_u(f))_w(a) = \mathbf{app}_w(i_{u,w}^{s \to t}(f),\, a)$, for every $a \in A_w^s$. By definition 5.5, we have

$$\mathcal{A}[\![\Gamma \triangleright \lambda x \colon \sigma.\, (Mx) \colon \sigma \to \tau]\!]\rho u = \mathbf{abst}_u^{[\![\sigma]\!]\theta,\, [\![\tau]\!]\theta}(\varphi),$$

where $\varphi = (\varphi_w)_{w \succeq u}$ is the family of functions defined such that,

$$\varphi_w(a) = \mathcal{A}[\![\Gamma, x \colon \sigma \triangleright Mx \colon \tau]\!]\rho[x := a]w,$$

for every $a \in A_w^{[\![\sigma]\!]\theta}$. Again, by definition 5.5, we have

$$\mathcal{A}[\![\Gamma, x \colon \sigma \triangleright Mx \colon \tau]\!]\rho w = \mathbf{app}_w^{[\![\sigma]\!]\theta,\, [\![\tau]\!]\theta}(\mathcal{A}[\![\Gamma, x \colon \sigma \triangleright M \colon \sigma \to \tau]\!]\rho w,\; \mathcal{A}[\![\Gamma, x \colon \sigma \triangleright x \colon \sigma]\!]\rho w),$$

and since $\mathcal{A}[\![\Gamma, x \colon \sigma \triangleright x \colon \sigma]\!]\rho[x := a]w = a$, we have

$$\mathcal{A}[\![\Gamma, x \colon \sigma \triangleright Mx \colon \tau]\!]\rho[x := a]w = \mathbf{app}(\mathcal{A}[\![\Gamma, x \colon \sigma \triangleright M \colon \sigma \to \tau]\!]\rho[x := a]w,\; a).$$

Since $x \notin FV(M)$, by lemma 5.8, we have $\mathcal{A}[\![\Gamma, x \colon \sigma \triangleright M \colon \sigma \to \tau]\!]\rho[x := a]w = \mathcal{A}[\![\Gamma \triangleright M \colon \sigma \to \tau]\!]\rho w$, and so

$$\mathcal{A}[\![\Gamma, x \colon \sigma \triangleright Mx \colon \tau]\!]\rho[x := a]w = \mathbf{app}(\mathcal{A}[\![\Gamma \triangleright M \colon \sigma \to \tau]\!]\rho w,\; a),$$

for every $a \in A_w^{[\sigma]\theta}$. Since $(\mathbf{fun}_u(f))_w(a) = \mathbf{app}_w(i_{u,w}^{s\to t}(f), a)$, for every $a \in A_w^s$, letting

$$f = \mathcal{A}[\![\Gamma \triangleright M{:}\, \sigma \to \tau]\!]\rho u,$$

since by lemma 5.6, $\mathcal{A}[\![\Gamma \triangleright M{:}\, \sigma \to \tau]\!]\rho w = i_{u,w}^{[\sigma\to\tau]\theta}(\mathcal{A}[\![\Gamma \triangleright M{:}\, \sigma \to \tau]\!]\rho u)$, the above shows that $\mathbf{fun}(\mathcal{A}[\![\Gamma \triangleright M{:}\, \sigma \to \tau]\!]\rho u)$ is the family of functions $\varphi = (\varphi_w)_{w \succeq u}$ defined such that,

$$\varphi_w(a) = \mathcal{A}[\![\Gamma, x{:}\, \sigma \triangleright Mx{:}\, \tau]\!]\rho[x{:}= a]w,$$

for every $a \in A_w^{[\sigma]\theta}$. However, by condition (1) of definition 4.6, we have, $\mathbf{abst}_u(\mathbf{fun}_u(f)) \succeq_u f$, for every $f \in A_u^{s\to t}$, and since

$$\mathbf{abst}_u(\mathbf{fun}_u(\mathcal{A}[\![\Gamma \triangleright M{:}\, \sigma \to \tau]\!]\rho u)) = \mathbf{abst}_u(\varphi) = \mathcal{A}[\![\Gamma \triangleright \lambda x{:}\, \sigma.\, (Mx){:}\, \sigma \to \tau]\!]\rho u,$$

we have

$$\mathcal{A}[\![\Gamma \triangleright M{:}\, \sigma \to \tau]\!]\rho u \preceq \mathcal{A}[\![\Gamma \triangleright \lambda x{:}\, \sigma.\, (Mx){:}\, \sigma \to \tau]\!]\rho u,$$

which shows that $\mathcal{A},\ u \Vdash_{\beta\eta} (\Gamma \triangleright M \preceq \lambda x{:}\, \sigma.\, (Mx){:}\, \sigma \to \tau)[\rho]$, as desired.

The other η-like rules are treated in a similar fashion.

(3) We only have to consider the extensional rules. Consider the rule

$$\frac{\Gamma, x{:}\, \sigma \triangleright M_1 x \preceq M_2 x{:}\, \tau}{\Gamma \triangleright M_1 \preceq M_2{:}\, (\sigma \to \tau)} \quad (\mathbf{fun}\text{-}\mathit{extentionality})$$

where $x \notin FV(M_1) \cup FV(M_2)$. By the induction hypothesis, we have

$$\mathcal{A} \Vdash_{ex\beta\eta} \Gamma, x{:}\, \sigma \triangleright M_1 x \preceq M_2 x{:}\, \tau.$$

Thus, for every extensional Kripke pre-applicative $\beta\eta$-structure $\mathcal{A}$, every $w \in \mathcal{W}$, and every valuation ρ_1, if $w \Vdash (\Gamma, x{:}\, \sigma)[\rho_1]$, then

$$\mathcal{A}[\![\Gamma, x{:}\, \sigma \triangleright M_1 x{:}\, \tau]\!]\rho_1 w \preceq \mathcal{A}[\![\Gamma, x{:}\, \sigma \triangleright M_2 x{:}\, \tau]\!]\rho_1 w.$$

Consider any $u \in \mathcal{W}$ and any valuation ρ such that $u \Vdash \Gamma[\rho]$. The proof for the soundness of the axiom (η) showed that $\mathbf{fun}(\mathcal{A}[\![\Gamma \triangleright M{:}\, \sigma \to \tau]\!]\rho u)$ is the family of functions $\varphi = (\varphi_w)_{w \succeq u}$ defined such that,

$$\varphi_w(a) = \mathcal{A}[\![\Gamma, x{:}\, \sigma \triangleright Mx{:}\, \tau]\!]\rho[x{:}= a]w,$$

for every $a \in A_w^{[\sigma]\theta}$. Thus, letting $\rho_1 = \rho[x{:}= a]$, for any $w \succeq u$, we have $w \Vdash (\Gamma, x{:}\, \sigma)[\rho_1]$, and so

$$\mathcal{A}[\![\Gamma, x{:}\, \sigma \triangleright M_1 x{:}\, \tau]\!]\rho[x{:}= a]w \preceq \mathcal{A}[\![\Gamma, x{:}\, \sigma \triangleright M_2 x{:}\, \tau]\!]\rho[x{:}= a]w,$$

which shows that

$$\mathbf{fun}(\mathcal{A}[\![\Gamma \triangleright M_1{:}\, \sigma \to \tau]\!]\rho u) \preceq \mathbf{fun}(\mathcal{A}[\![\Gamma \triangleright M_2{:}\, \sigma \to \tau]\!]\rho u).$$

Since $\mathcal{A}$ is extensional, $\mathbf{fun}$ is isotone, and then

$$\mathcal{A}[\![\Gamma \triangleright M_1{:}\, \sigma \to \tau]\!]\rho u \preceq \mathcal{A}[\![\Gamma \triangleright M_2{:}\, \sigma \to \tau]\!]\rho u,$$

which shows that $\mathcal{A},\ u \Vdash_{ex\beta\eta} (\Gamma \triangleright M_1 \preceq M_2{:}\, \sigma \to \tau)[\rho]$, for every $u \in \mathcal{W}$ and every ρ, as desired.

The proofs for the other extentionality rules are similar. □

Next, we turn to completeness results.

Lemma 7.2 *For any set $\mathcal{E}$ of inequalities, the following properties hold: (1) There is a Kripke pre-applicative β-structure $\mathcal{A}$, such that for every inequality $\Gamma \triangleright M \preceq N{:}\,\sigma$, $\mathcal{E} \vdash_\beta \Gamma \triangleright M \preceq N{:}\,\sigma$ iff $\mathcal{A} \Vdash_\beta \Gamma \triangleright M \preceq N{:}\,\sigma$; (2) There is a Kripke pre-applicative $\beta\eta$-structure $\mathcal{A}$, such that for every inequality $\Gamma \triangleright M \preceq N{:}\,\sigma$, $\mathcal{E} \vdash_{\beta\eta} \Gamma \triangleright M \preceq N{:}\,\sigma$ iff $\mathcal{A} \Vdash_{\beta\eta} \Gamma \triangleright M \preceq N{:}\,\sigma$; (3) There is an extensional Kripke pre-applicative β-structure $\mathcal{A}$, such that for every inequality $\Gamma \triangleright M \preceq N{:}\,\sigma$, $\mathcal{E} \vdash_{ex\beta\eta} \Gamma \triangleright M \preceq N{:}\,\sigma$ iff $\mathcal{A} \Vdash_{ex\beta\eta} \Gamma \triangleright M \preceq N{:}\,\sigma$.*

Proof. (1) We modify the construction of definition 4.4. Rather that defining A^σ_Γ as the set of all provable typing judgements $\Gamma \triangleright M{:}\,\sigma$, we define A^σ_Γ as the set of equivalence classes $[\Gamma \triangleright M{:}\,\sigma]$ of the equivalence relation $\equiv_\mathcal{E}$ induced by the precongruence $\preceq_\mathcal{E}$ defined such that

$$\Gamma \triangleright M{:}\,\sigma \preceq_\mathcal{E} \Gamma \triangleright N{:}\,\sigma \quad \text{iff} \quad \mathcal{E} \vdash_\beta \Gamma \triangleright M \preceq N{:}\,\sigma,$$

with $\Gamma \triangleright M{:}\,\sigma \equiv_\mathcal{E} \Gamma \triangleright N{:}\,\sigma$ iff $\Gamma \triangleright M{:}\,\sigma \preceq_\mathcal{E} \Gamma \triangleright N{:}\,\sigma$ and $\Gamma \triangleright N{:}\,\sigma \preceq_\mathcal{E} \Gamma \triangleright M{:}\,\sigma$.

The congruence rules of definition 6.1 ensure that **fun**, **abst**, **tfun**, **tabst**, Π, $\langle -,- \rangle$, **inl**, **inr**, and $[-,\ -]$, are well-defined. Rule (*addvar*) is used to show that if $[\Gamma \triangleright M{:}\,\sigma] \in A^\sigma_\Gamma$, when $\Gamma \triangleright M \preceq N{:}\,\sigma \in \mathcal{E}$, then $[\Delta \triangleright M{:}\,\sigma] \in A^\sigma_\Delta$, for any Δ such that $\Gamma \subseteq \Delta$.

Recall that, according to definition 5.4, a valuation is a pair $\rho = \langle \theta,\ \eta \rangle$, where θ is an infinite type substitution, and η is a partial function $\eta{:}\,\mathcal{X} \times \mathcal{W} \to \bigcup(A^t_w)_{t\in T,\ w\in\mathcal{W}}$. Thus, recalling that worlds are contexts, $\eta_\Delta(x) = [\Gamma \triangleright M{:}\,\sigma]$ for some judgement $\Gamma \triangleright M{:}\,\sigma$, when defined. Furthermore, the condition for ρ to satisfy a context Γ at a world Δ (since worlds are contexts), is $\eta_\Delta(x) \in A^{\theta(\sigma)}_\Delta$, that is, $\eta_\Delta(x) = [\Delta \triangleright M_x{:}\,\theta(\sigma)]$, for some M_x, for every $x{:}\,\sigma \in \Gamma$. Thus, if $\rho = \langle \theta,\ \eta \rangle$ satisfies a context Γ at Δ, the valuation ρ defines a substitution φ such that $\varphi(X) = \theta(X)$ for every $X \in \bigcup(FTV(\tau))_{x:\tau\in\Gamma}$, and $\varphi(x) = M_x$ for every $x \in dom(\Gamma)$ (where $\eta_\Delta(x) = [\Delta \triangleright M_x{:}\,\theta(\sigma)]$), and we have $\Delta \Vdash \Gamma[\varphi]$, as defined just before definition 4.4. Note that such a substitution φ depends on the selection of representatives chosen from the classes $[\Delta \triangleright M_x{:}\,\theta(\sigma)]$, but as we will see, this does not matter. Then, the following property can be shown by induction on the derivation of typing judgements.

Claim: For the pre-applicative structure $\mathcal{A}$ just defined, for every pair of contexts Γ and Δ, for every valuation $\rho = \langle \theta,\ \eta \rangle$, if $\Delta \Vdash \Gamma[\rho]$, then for every judgement $\Gamma \triangleright M{:}\,\sigma$, we have $\Delta \Vdash \Gamma[\varphi]$ and

$$\mathcal{A}[\![\Gamma \triangleright M{:}\,\sigma]\!]\rho\Delta = [\Delta \triangleright M[\varphi]{:}\,\sigma[\varphi]],$$

where φ is the substitution defined by the restriction of ρ_Δ to Γ, as explained above.

One also verifies easily that if φ_1 and φ_2 are two substitutions constructed by selecting representatives chosen from the classes $[\Delta \triangleright M_x{:}\,\theta(\sigma)]$, as explained above, then

$$[\Delta \triangleright M[\varphi_1]{:}\,\sigma[\varphi_1]] = [\Delta \triangleright M[\varphi_2]{:}\,\sigma[\varphi_2]].$$

It remains to show that $\mathcal{E} \vdash_\beta \Gamma \triangleright M \preceq N{:}\,\sigma$ iff $\mathcal{A} \Vdash_\beta \Gamma \triangleright M \preceq N{:}\,\sigma$.

To prove that $\mathcal{A} \Vdash_\beta \Gamma \triangleright M \preceq N{:}\,\sigma$ implies $\mathcal{E} \vdash_\beta \Gamma \triangleright M \preceq N{:}\,\sigma$, we choose a particular valuation $\rho = \langle \theta,\ \eta \rangle$ as follows: θ is the identity, and η is defined such that, for every Γ and Δ such that $\Gamma \subseteq \Delta$, for every $x \in \mathcal{X}$,

$$\eta_\Gamma(x) = \begin{cases} [\Delta \triangleright x{:}\,\sigma] & \text{if } x{:}\,\sigma \in \Gamma, \\ \textit{undefined} & \text{otherwise.} \end{cases}$$

Then, the substitution φ associated with ρ is the identity, and by the above claim, we have

$$\mathcal{A}[\![\Gamma \triangleright M\colon \sigma]\!]\rho\Delta = [\Delta \triangleright M\colon \sigma],$$

and

$$\mathcal{A}[\![\Gamma \triangleright N\colon \sigma]\!]\rho\Delta = [\Delta \triangleright N\colon \sigma].$$

If $\mathcal{A} \Vdash_\beta \Gamma \triangleright M \preceq N\colon \sigma$, since by definition of ρ, $\Gamma \Vdash \Gamma[\rho]$, we have

$$[\Delta \triangleright M\colon \sigma] \preceq_\mathcal{E} [\Delta \triangleright N\colon \sigma],$$

and by the definition of $\preceq_\mathcal{E}$, we have $\mathcal{E} \vdash_\beta \Gamma \triangleright M \preceq N\colon \sigma$.

Assume that $\mathcal{E} \vdash_\beta \Gamma \triangleright M \preceq N\colon \sigma$. Consider any Δ and any ρ such that $\Delta \Vdash \Gamma[\rho]$. Then, by the claim, we have $\Delta \Vdash \Gamma[\varphi]$,

$$\mathcal{A}[\![\Gamma \triangleright M\colon \sigma]\!]\rho\Delta = [\Delta \triangleright M[\varphi]\colon \sigma[\varphi]],$$

and

$$\mathcal{A}[\![\Gamma \triangleright N\colon \sigma]\!]\rho\Delta = [\Delta \triangleright N[\varphi]\colon \sigma[\varphi]],$$

where φ is the substitution defined by the restriction of ρ_Δ to Γ, as explained earlier. Since we have $\Delta \Vdash \Gamma[\varphi]$, by the (*substitution*) rule, we get

$$\mathcal{E} \vdash_\beta \Delta \triangleright M[\varphi] \preceq N[\varphi]\colon \sigma[\varphi],$$

which, by the definition of $\preceq_\mathcal{E}$, means that

$$[\Delta \triangleright M[\varphi]\colon \sigma[\varphi]] \preceq_\mathcal{E} [\Delta \triangleright N[\varphi]\colon \sigma[\varphi]],$$

that is, $\mathcal{A}[\![\Gamma \triangleright M\colon \sigma]\!]\rho\Delta \preceq_\mathcal{E} \mathcal{A}[\![\Gamma \triangleright N\colon \sigma]\!]\rho\Delta$, which shows that $\mathcal{A}, \Delta \Vdash_\beta (\Gamma \triangleright M \preceq N\colon \sigma)[\rho]$. Since this holds for all Δ and ρ, we have $\mathcal{A} \Vdash_\beta \Gamma \triangleright M \preceq N\colon \sigma$.

(2) The proof is similar to that of (1), except that we define $\preceq_\mathcal{E}$ such that

$$\Gamma \triangleright M\colon \sigma \preceq_\mathcal{E} \Gamma \triangleright N\colon \sigma \quad \text{iff} \quad \mathcal{E} \vdash_{\beta\eta} \Gamma \triangleright M \preceq N\colon \sigma.$$

The argument showing that the resulting Kripke pre-applicative structure is a $\beta\eta$-structure is identical to the argument given just after definition 4.8.

(3) The proof is similar to that of (2), except that we define $\preceq_\mathcal{E}$ such that

$$\Gamma \triangleright M\colon \sigma \preceq_\mathcal{E} \Gamma \triangleright N\colon \sigma \quad \text{iff} \quad \mathcal{E} \vdash_{ex\beta\eta} \Gamma \triangleright M \preceq N\colon \sigma.$$

We also need to verify that the resulting Kripke pre-applicative structure is extensional, that is, that the functions **fun**, **tfun**, Π, and $\langle$`cinl`, `cinr`$\rangle$, are isotone.

Assume that

$$\mathbf{fun}_\Gamma([\Gamma \triangleright M_1\colon \sigma \to \tau]) \preceq_\mathcal{E} \mathbf{fun}_\Gamma([\Gamma \triangleright M_2\colon \sigma \to \tau]).$$

Since $\mathbf{fun}_\Gamma([\Gamma \triangleright M_1\colon \sigma \to \tau])$ is the family of functions $([\Gamma \triangleright M_1\colon \sigma \to \tau]_\Delta)_{\Gamma\subseteq\Delta}$, such that

$$[\Gamma \triangleright M_1\colon \sigma \to \tau]_\Delta([\Delta \triangleright N_1\colon \sigma]) = [\Delta \triangleright M_1 N_1\colon \tau],$$

for every $[\Delta \triangleright N_1 : \sigma] \in A^\sigma_\Delta$, and similarly for $\mathbf{fun}_\Gamma([\Gamma \triangleright M_2 : \sigma \to \tau])$, letting $\Delta = \Gamma, x : \sigma$, where $x \notin FV(M_1) \cup FV(M_2)$, we have

$$[\Gamma \triangleright M_1 : \sigma \to \tau]_{\Gamma, x:\sigma} \preceq_{\mathcal{E}} [\Gamma \triangleright M_2 : \sigma \to \tau]_{\Gamma, x:\sigma},$$

and thus, in particular,

$$[\Gamma, x : \sigma \triangleright M_1 x : \tau] \preceq_{\mathcal{E}} [\Gamma, x : \sigma \triangleright M_2 x : \tau].$$

This means that $\mathcal{E} \vdash_{ex\beta\eta} \Gamma, x : \sigma \triangleright M_1 x \preceq M_2 x : \tau$, and since $x \notin FV(M_1) \cup FV(M_2)$, by the first extensionality rule, we get $\mathcal{E} \vdash_{ex\beta\eta} \Gamma \triangleright M_1 \preceq M_2 : \sigma \to \tau$. Then, $[\Gamma \triangleright M_1 : \sigma \to \tau] \preceq_{\mathcal{E}} [\Gamma \triangleright M_2 : \sigma \to \tau]$, showing that **fun** is isotone. The proofs for the other cases are similar. □

As a corollary of lemma 7.2 and lemma 7.1, we obtain the following soundness and completeness theorem.

Theorem 7.3 *For any set $\mathcal{E}$ of inequalities, the following properties hold: (1) $\mathcal{E} \Vdash_\beta \Gamma \triangleright M \preceq N : \sigma$ iff $\mathcal{E} \vdash_\beta \Gamma \triangleright M \preceq N : \sigma$; (2) $\mathcal{E} \Vdash_{\beta\eta} \Gamma \triangleright M \preceq N : \sigma$ iff $\mathcal{E} \vdash_{\beta\eta} \Gamma \triangleright M \preceq N : \sigma$; and (3) $\mathcal{E} \Vdash_{ex\beta\eta} \Gamma \triangleright M \preceq N : \sigma$ iff $\mathcal{E} \vdash_{ex\beta\eta} \Gamma \triangleright M \preceq N : \sigma$.*

Proof. (1) The direction ($\Leftarrow$) is just lemma 7.1. Conversely, $\mathcal{E} \Vdash_\beta \Gamma \triangleright M \preceq N : \sigma$ means that $\mathcal{A} \Vdash_\beta \Gamma \triangleright M \preceq N : \sigma$ whenever $\mathcal{A} \Vdash_\beta \mathcal{E}$, for every Kripke β-structure $\mathcal{A}$. By lemma 7.2, for any $\mathcal{E}$, there is some Kripke pre-applicative structure $\mathcal{A}$ such that for every inequality $\Gamma \triangleright M \preceq N : \sigma$, $\mathcal{E} \vdash_\beta \Gamma \triangleright M \preceq N : \sigma$ iff $\mathcal{A} \Vdash_\beta \Gamma \triangleright M \preceq N : \sigma$. Then, in particular, we have $\mathcal{A} \Vdash_\beta \Gamma_1 \triangleright M_1 \preceq N_1 : \sigma_1$ for every $\Gamma_1 \triangleright M_1 \preceq N_1 : \sigma_1 \in \mathcal{E}$, which implies that $\mathcal{A} \Vdash_\beta \mathcal{E}$. Then, we have $\mathcal{A} \Vdash_\beta \Gamma \triangleright M \preceq N : \sigma$, which implies that $\mathcal{E} \vdash_\beta \Gamma \triangleright M \preceq N : \sigma$, by the definition of $\mathcal{A}$. Cases (2) and (3) are similar. □

Another interesting corollary of lemma 7.2 which shows the correspondence between provability and inhabitation, is the following lemma, which generalizes a result of Mitchell and Moggi [12].

Lemma 7.4 *Given a signature Σ and a set $\mathcal{E}$ of inequalities over Σ, there is a Kripke pre-applicative β-structure $\mathcal{A}$ such that $\mathcal{A} \Vdash_\beta \mathcal{E}$ and the following property holds: A^σ_w is nonempty for every $w \in \mathcal{W}$ iff the type σ, when viewed as a second-order proposition, is intuitionistically provable from the types of constants in Σ. The same result holds for a $\beta\eta$-structure when $\mathcal{A} \Vdash_{\beta\eta} \mathcal{E}$, and for an $ex\beta\eta$-structure when $\mathcal{A} \Vdash_{ex\beta\eta} \mathcal{E}$*

The special case where we consider soundness and completeness with respect to Kripke structures where $A^s_w \neq \emptyset$ for all $s \in T$ and all worlds $w \in \mathcal{W}$, is of particular interest. First, observe that the proof system of definition 6.1 is incomplete in this case. Consider the set of inequalities

$$\mathcal{E} = \{x : \sigma \triangleright T \preceq f(x) : \tau,\ x : \sigma \triangleright f(x) \preceq F : \tau\},$$

where $f : (\sigma \to \tau)$, $T, F : \tau$, and $\sigma \neq \tau$. Clearly, we can prove $x : \sigma \triangleright T \preceq F : \tau$ from $\mathcal{E}$. However, in Kripke structures with nonempty carriers, $\triangleright T \preceq F : \tau$ is valid, whereas we have no way of proving it. However, if we had a constant $c : \sigma$, then by the (*substitution*)-rule, we would be able to prove $\triangleright T \preceq F : \tau$.

The above discussion suggests adding a new rule to the system of definition 6.1.

Definition 7.5 The rule (*nonempty*) is defined as follows.

$$\frac{\Gamma, x\colon\sigma \triangleright M_1 \preceq M_2\colon\sigma}{\Gamma \triangleright M_1 \preceq M_2\colon\sigma} \quad (\textit{nonempty})$$

provided that $x \notin FV(M_1) \cup FV(M_2)$.

The notation $\mathcal{E} \vdash_{\beta+} \Gamma \triangleright M \preceq N\colon\sigma$ means that $\Gamma \triangleright M \preceq N\colon\sigma$ is provable from $\mathcal{E}$ using the axioms and rules of the inequational β-theory of definition 6.1, plus the rule (*nonempty*), and similarly for $\mathcal{E} \vdash_{\beta\eta+} \Gamma \triangleright M \preceq N\colon\sigma$ and $\mathcal{E} \vdash_{ex\beta\eta+} \Gamma \triangleright M \preceq N\colon\sigma$. The notation $\mathcal{E} \Vdash_{\beta+} \Gamma \triangleright M \preceq N\colon\sigma$ means that $\mathcal{E} \Vdash_{\beta} \Gamma \triangleright M \preceq N\colon\sigma$ in all Kripke pre-applicative β-structures with all carriers nonempty, and similarly for $\mathcal{E} \Vdash_{\beta\eta+} \Gamma \triangleright M \preceq N\colon\sigma$ and $\mathcal{E} \Vdash_{ex\beta\eta+} \Gamma \triangleright M \preceq N\colon\sigma$.

It is easily verified that the rule (*nonempty*) is sound with respect to Kripke structures with nonempty carriers. Completeness also holds. Unfortunately, lemma 7.2 does not immediately yield this result, because some of the carriers of the Kripke structure used in the proof of that lemma may be empty. There is an easy way around, which consists in adding new constants, as we now explain.

Let us expand our signature Σ by adding new constants c_σ such that $Type(c_\sigma) = \sigma$, for every closed type $\sigma \in \mathcal{T}$. If the original signature is Σ, the new signature is denoted as Σ_c. Then we have the following lemma.

Lemma 7.6 *Given any set $\mathcal{E}$ of inequalities and any inequality $\Gamma \triangleright M \preceq N\colon\sigma$ over the original signature Σ, if $\mathcal{E} \vdash_\beta \Gamma \triangleright M \preceq N\colon\sigma$ using any terms over the expanded signature Σ_c, then there is some Δ such that $dom(\Delta) \cap dom(\Gamma) = \emptyset$, and $\mathcal{E} \vdash_\beta \Gamma \cup \Delta \triangleright M \preceq N\colon\sigma$, using only terms over the original signature Σ. The same result holds for $\vdash_{\beta\eta}$ and $\vdash_{ex\beta\eta}$.*

Proof. We proceed by induction on the structure of proofs. The only interesting cases are the axioms and the (*substitution*) rule. The idea is the following: whenever a term N containing new constants is used, we replace every new constant c_σ in N by a new variable x, and we add $x\colon\sigma$ to the context. This way, every term N involving new constants is replaced by a term N' over the original signature Σ. For example, if we are dealing with the axiom $\Gamma \triangleright M[N/x] \preceq (\lambda x\colon\sigma.\, M)N\colon\tau$, letting Γ' be the declaration of all the new variables needed to eliminate new constants from M and N, we obtain the new axiom $\Gamma \cup \Gamma' \triangleright M'[N'/x] \preceq (\lambda x\colon\sigma.\, M')N'\colon\tau$. In the case of the (*substitution*) rule,

$$\frac{\Gamma \triangleright M_1 \preceq M_2\colon\sigma}{\Delta \triangleright M_1[\varphi] \preceq M_2[\varphi]\colon\sigma[\varphi]} \quad (\textit{substitution})$$

where φ is a substitution such that $\Delta \Vdash \Gamma[\varphi]$, let Δ' be the set of declarations needed to convert every term $\varphi(x)$ to a term $\varphi'(x)$ over the signature Σ, for every $x \in dom(\Gamma)$. Then, it is immediate that $\Delta \cup \Delta' \Vdash \Gamma[\varphi']$, and we have $\vdash_\beta \Delta \cup \Delta' \triangleright M_1[\varphi'] \preceq M_2[\varphi']\colon\sigma[\varphi']$.

Since a proof is finite, and we have infinitely many variables, we can always use fresh variables that do not clash with the variables occurring in the original proof. □

We can now prove the following soundness and completeness theorem.

Theorem 7.7 *For any set $\mathcal{E}$ of inequalities, the following properties hold: (1) $\mathcal{E} \Vdash_{\beta+} \Gamma \triangleright M \preceq N\colon\sigma$ iff $\mathcal{E} \vdash_{\beta+} \Gamma \triangleright M \preceq N\colon\sigma$; (2) $\mathcal{E} \Vdash_{\beta\eta+} \Gamma \triangleright M \preceq N\colon\sigma$ iff $\mathcal{E} \vdash_{\beta\eta+} \Gamma \triangleright M \preceq N\colon\sigma$; and (3) $\mathcal{E} \Vdash_{ex\beta\eta+} \Gamma \triangleright M \preceq N\colon\sigma$ iff $\mathcal{E} \vdash_{ex\beta\eta+} \Gamma \triangleright M \preceq N\colon\sigma$.*

Proof. (1) We go back to the proof of lemma 7.2. Given the set $\mathcal{E}$ over the signature Σ, we define the structure $\mathcal{A}$, but this time, over the expanded signature Σ_c. Thus, A^{σ}_{Γ} is the set of equivalence classes $[\Gamma \triangleright M\colon\sigma]$ of the equivalence relation $\equiv_{\mathcal{E}}$ induced by the precongruence $\preceq_{\mathcal{E}}$, where the terms M are over the expanded signature Σ_c. For every type σ, if $FVT(\sigma) = \{X_1, \ldots, X_m\}$, letting $\widehat{\sigma} = \forall X_1 \ldots \forall X_m.\, \sigma$ be the closure of σ, there is a new constant $c_{\widehat{\sigma}}$ such that $Type(c_{\widehat{\sigma}}) = \widehat{\sigma}$, and so, we have $X_1\colon\star, \ldots, X_m\colon\star \triangleright c_{\widehat{\sigma}} X_1 \ldots X_m\colon\sigma$, which shows that every carrier is nonempty. The rest of the proof is unchanged. Thus, we have constructed a Kripke structure with nonempty carriers such that, $\mathcal{E} \vdash_{\beta} \Gamma \triangleright M \preceq N\colon\sigma$ using any terms over the expanded signature Σ_c iff $\mathcal{A} \Vdash_{\beta+} \Gamma \triangleright M \preceq N\colon\sigma$. Using the reasoning of theorem 7.3, if $\mathcal{E} \Vdash_{\beta+} \Gamma \triangleright M \preceq N\colon\sigma$, then $\mathcal{E} \vdash_{\beta} \Gamma \triangleright M \preceq N\colon\sigma$, using any terms over the expanded signature Σ_c.

Now, given any set $\mathcal{E}$ of inequalities and any inequality $\Gamma \triangleright M \preceq N\colon\sigma$ over the original signature Σ, we observe that if $\mathcal{E} \vdash_{\beta} \Gamma \triangleright M \preceq N\colon\sigma$ using any terms over the expanded signature Σ_c, then $\mathcal{E} \vdash_{\beta+} \Gamma \triangleright M \preceq N\colon\sigma$. Indeed, by lemma 7.6, we have $\mathcal{E} \vdash_{\beta} \Gamma \cup \Delta \triangleright M \preceq N\colon\sigma$, using only terms over the original signature Σ, for some Δ such that $dom(\Delta) \cap dom(\Gamma) = \emptyset$, and we eliminate all variables in Δ using the rule (*nonempty*). This shows the compeleteness part of (1). The soundness part is trivial. The proof for (2) and (3) is similar. □

We now consider equations.

8 Proving Equations

In this section, we adapt the results of section 7 to equations. Some simplifications take place.

Formally, an equation $\Gamma \triangleright M \doteq N\colon\sigma$ is equivalent to the pair of inequalities $\Gamma \triangleright M \preceq N\colon\sigma$ and $\Gamma \triangleright N \preceq M\colon\sigma$, which amounts to adding the (*symmetry*) rule

$$\frac{\Gamma \triangleright M_1 \preceq M_2\colon\sigma}{\Gamma \triangleright M_2 \preceq M_1\colon\sigma} \quad (symmetry)$$

to the rules of the system of definition 6.1.

In view of lemma 6.2 and lemma 6.3, the (*substitution*) rule becomes redundant, and the $\beta\eta$-theory is equivalent to the $ex\beta\eta$-theory. Some of the other congruence rules also become redundant, for example for π_1, π_2, **inl**, **inr**, $\langle -,\ - \rangle$ and $[-,\ -]$. For example, from $\vdash_{\beta} \Gamma \triangleright \lambda x\colon\sigma.\, \lambda y\colon\tau.\, \langle x,\ y\rangle \doteq \lambda x\colon\sigma.\, \lambda y\colon\tau.\, \langle x,\ y\rangle\colon \sigma \times \tau$, $\vdash_{\beta} \Gamma \triangleright M_1 \doteq N_1\colon\sigma$, and $\vdash_{\beta} \Gamma \triangleright M_2 \doteq N_2\colon\tau$, we can show that $\vdash_{\beta} \Gamma \triangleright \langle M_1,\ N_1\rangle \doteq \langle M_2,\ N_2\rangle\colon \sigma \times \tau$, using ($\rightarrow$-*congruence*) and ($\beta$). The resulting simplified equational proof system is given next.

Definition 8.1 The axioms and inference rules of the *equational β-theory* of $\lambda^{\rightarrow,\times,+,\forall^2}$ are defined below.

Axioms:

$$\Gamma \triangleright M \doteq M\colon\sigma \quad (reflexivity)$$

$$\Gamma \triangleright M[N/x] \doteq (\lambda x\colon \sigma.\, M)N\colon \tau \quad (\beta)$$

$$\Gamma \triangleright M[\tau/X] \doteq (\lambda X.\, M)\tau\colon \sigma[\tau/X] \quad (\textit{type-}\beta)$$

$$\Gamma \triangleright M \doteq \pi_1(\langle M,\ N\rangle)\colon \sigma \quad (\pi_1)$$

$$\Gamma \triangleright N \doteq \pi_2(\langle M,\ N\rangle)\colon \tau \quad (\pi_2)$$

$$\Gamma \triangleright MP \doteq [M,\ N]\mathtt{inl}(P)\colon \delta \quad (\mathtt{inl})$$

$$\Gamma \triangleright NP \doteq [M,\ N]\mathtt{inr}(P)\colon \delta \quad (\mathtt{inr})$$

Inference Rules:

$$\frac{\Gamma \triangleright M_1 \doteq M_2\colon \sigma}{\Delta \triangleright M_1 \doteq M_2\colon \sigma} \quad (\textit{addvar})$$

where $\Gamma \subseteq \Delta$

$$\frac{\Gamma \triangleright M_1 \doteq M_2\colon \sigma}{\Gamma \triangleright M_2 \doteq M_1\colon \sigma} \quad (\textit{symmetry})$$

$$\frac{\Gamma \triangleright M_1 \doteq M_2\colon \sigma \quad \Gamma \triangleright M_2 \doteq M_3\colon \sigma}{\Gamma \triangleright M_1 \doteq M_3\colon \sigma} \quad (\textit{transitivity})$$

$$\frac{\Gamma \triangleright M_1 \doteq M_2\colon (\sigma \to \tau) \quad \Gamma \triangleright N_1 \doteq N_2\colon \sigma}{\Gamma \triangleright (M_1 N_1) \doteq (M_2 N_2)\colon \tau} \quad (\to\textit{-congruence})$$

$$\frac{\Gamma, x\colon \sigma \triangleright M_1 \doteq M_2\colon \tau}{\Gamma \triangleright \lambda x\colon \sigma.\, M_1 \doteq \lambda x\colon \sigma.\, M_2\colon (\sigma \to \tau)} \quad (\xi)$$

$$\frac{\Gamma \triangleright M_1 \doteq M_2\colon \forall X.\, \sigma}{\Gamma \triangleright (M_1 \tau) \doteq (M_2 \tau)\colon \sigma[\tau/X]} \quad (\forall\textit{-congruence})$$

$$\frac{\Gamma, X\colon \star \triangleright M_1 \doteq M_2\colon \sigma}{\Gamma \triangleright \lambda X.\, M_1 \doteq \lambda X.\, M_2\colon \forall X.\, \sigma} \quad (\textit{type-}\xi)$$

The notation $\vdash_\beta \Gamma \triangleright M \doteq N\colon \sigma$ means that the equation $\Gamma \triangleright M \doteq N\colon \sigma$ is provable from the above axioms and inference rules.

The equational extensional $\beta\eta$-theory of the system $\lambda^{\to,\times,+,\forall^2}$ is obtained by adding the following *η-like rules* to the axioms and inference rules of the β-theory:

$$\Gamma \triangleright M \doteq \lambda x\colon \sigma.\, (Mx)\colon (\sigma \to \tau) \quad (\eta)$$

where $x \notin FV(M)$;

$$\Gamma \triangleright M \doteq \lambda X.\, (MX)\colon \forall X.\, \sigma \quad (\textit{type-}\eta)$$

where $X \notin FTV(M)$;

$$\Gamma \triangleright M \doteq \langle \pi_1(M),\ \pi_2(M)\rangle\colon \sigma \times \tau \quad (\mathtt{pair})$$

$$\Gamma \triangleright M \doteq [\lambda x\colon \sigma.\, (M\,\mathtt{inl}(x)),\ \lambda y\colon \tau.\, (M\,\mathtt{inr}(y))]\colon (\sigma + \tau) \to \delta \quad (\mathtt{copair})$$

The notation $\vdash_{\beta\eta} \Gamma \triangleright M \doteq N\colon \sigma$ means that the equation $\Gamma \triangleright M \doteq N\colon \sigma$ is provable from all the axioms and the inference rules of the $\beta\eta$-theory, including the η-like rules.

Definition 6.4 can be restated for equations rather than inequalities, using the proof system of definition 8.1. Similarly, definition 6.5 can be restated for equations, but (1) has to be redefined in terms of $=$, instead of $\preceq$:

(1) For every equation $\Gamma \triangleright M \doteq N\!:\sigma$, we say that $\Gamma \triangleright M \doteq N\!:\sigma$ *holds at* u *and* ρ *in* $\mathcal{A}$, denoted as $\mathcal{A},\, u \Vdash_\beta (\Gamma \triangleright M \doteq N\!:\sigma)[\rho]$, iff whenever $u \Vdash \Gamma[\rho]$, then

$$\mathcal{A}[\![\Gamma \triangleright M\!:\sigma]\!]\rho u = \mathcal{A}[\![\Gamma \triangleright N\!:\sigma]\!]\rho u.$$

We have the following soundness and completeness theorem.

Theorem 8.2 *For any set $\mathcal{E}$ of equations, the following properties hold: (1) $\mathcal{E} \Vdash_\beta \Gamma \triangleright M \doteq N\!:\sigma$ iff $\mathcal{E} \vdash_\beta \Gamma \triangleright M \doteq N\!:\sigma$; (2) $\mathcal{E} \Vdash_{\beta\eta} \Gamma \triangleright M \doteq N\!:\sigma$ iff $\mathcal{E} \vdash_{\beta\eta} \Gamma \triangleright M \doteq N\!:\sigma$.*

Proof. (1) We consider the set $\mathcal{E}'$ of inequalities obtained from $\mathcal{E}$ by adding the converse of every axiom and the converse of every equation in $\mathcal{E}$. It easily verified that $\mathcal{E} \vdash_\beta \Gamma \triangleright M \doteq N\!:\sigma$ iff $\mathcal{E}' \vdash_\beta \Gamma \triangleright M \preceq N\!:\sigma$ and $\mathcal{E}' \vdash_\beta \Gamma \triangleright N \preceq M\!:\sigma$. Then, we apply theorem 7.7. The proof for (2) is similar. □

The equational version of rule (*nonempty*) is shown below:

$$\frac{\Gamma, x\!:\sigma \triangleright M_1 \doteq M_2\!:\sigma}{\Gamma \triangleright M_1 \doteq M_2\!:\sigma} \quad (\textit{nonempty})$$

provided that $x \notin FV(M_1) \cup FV(M_2)$. Then, we also have the following equational version of theorem 7.7.

Theorem 8.3 *For any set $\mathcal{E}$ of equations, the following properties hold: (1) $\mathcal{E} \Vdash_{\beta+} \Gamma \triangleright M \doteq N\!:\sigma$ iff $\mathcal{E} \vdash_{\beta+} \Gamma \triangleright M \doteq N\!:\sigma$; (2) $\mathcal{E} \Vdash_{\beta\eta+} \Gamma \triangleright M \doteq N\!:\sigma$ iff $\mathcal{E} \vdash_{\beta\eta+} \Gamma \triangleright M \doteq N\!:\sigma$.*

Proof. Immediate from theorem 7.7, in view of the proof of theorem 8.2. □

9 Conclusion and Suggestions for Further Research

A new class of Kripke structures for the second-order λ-calculus was defined, and the soundness and completeness of some proof systems for proving inequalities (rewrite rules) or equations was investigated. The Kripke structures considered in this paper form a more general class of structures than the applicative structures introduced by Mitchell and Moggi, since they are equipped with preorders that correspond to an abstract form of reduction, and they are not necessarily extensional. This approach allows us to consider models of sets of rewrite rules, as well as sets of equations. We obtained soundness and completeness theorems that generalize some results of Mitchell and Moggi to the second-order λ-calculus, and to sets of inequalities (rewrite rules).

Since this paper is already quite long, we have not considered Kripke second-order logical relations and their applications, which have been considered by Mitchell and Moggi [12] in the first-order case. We are confident that some of the basic results will go through, for example the construction of quotient structures, but the well-known problem of finding useful ways of constructing second-order logical relations remains. We also believe that it would be worth investigating whether Breazu-Tannen and Coquand's extensional collapse construction ([1]) can be adapated to our class of Kripke structures. It would also be interesting to see if the definition of HRO_2 and HEO_2 models can be recast in our formalism (Girard [3]). We believe that this is possible. Finally, it would be interesting to see if the structures of this paper can be extended to richer type theories, such as generalized type systems (in particular, the theory of constructions).

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References

[1] V. Breazu-Tannen and T. Coquand. Extensional models for polymorphism. *Theoretical Computer Science*, 59:85–114, 1988.

[2] H. Friedman. Equality between functionals. In R. Parikh, editor, *Logic Colloquium*, volume 453 of *Lecture Notes in Math.*, pages 22–37. Springer-Verlag, 1975.

[3] Jean-Yves Girard. *Interprétation fonctionnelle et élimination des coupures de l'arithmétique d'ordre supérieur.* PhD thesis, Université de Paris VII, June 1972. Thèse de Doctorat d'Etat.

[4] Jean-Yves Girard. The system F of variable types, fifteen years later. *Theoretical Computer Science*, 45(2):159–192, 1986.

[5] J. Goguen and J. Meseguer. Completeness of many-sorted equational logic. *SIGPLAN Notices*, 417:9–17, 1982.

[6] C. A. Gunter. *Semantics of Programming Languages.* Foundations of Computing. MIT Press, 1992.

[7] L. Henkin. Completeness in the theory of types. *J. Symbolic Logic*, 15(2):81–91, 1950.

[8] B. Jacobs, I. Margaria, and M. Zacchi. Filter models with polymorphic types. *Theoretical Computer Science*, 95(1):143–158, 1992.

[9] S. MacLane and I. Moerdijk. *Sheaves in Geometry and Logic.* Springer Verlag, New York, 1992.

[10] A.R. Meyer, J.C. Mitchell, E. Moggi, and R. Statman. Empty types in polymorphic lambda calculus. In *Proc. 14th ACM Symp. on Principles of Programming Languages*, pages 253–262. ACM, 1987.

[11] J. C. Mitchell. A type-inference approach to reduction properties and semantics of polymorphic expressions. In *ACM Conference on LISP and Functional Programming*, pages 308–319. ACM, 1986. Reprinted in *Logical Foundations of Functional Programming*, G. Huet, Ed., Addison Wesley, 1990, 195-212.

[12] J.C. Mitchell and E Moggi. Kripke-style models for typed lambda calculus. *Annals of Pure and Applied Logic*, 51:99–124, 1991.

[13] G.D. Plotkin. Lambda definability in the full type hierarchy. In J. P. Seldin and J. R. Hindley, editors, *To H. B. Curry: Essays on Combinatory Logic, Lambda Calculus and Formalism*, pages 363–373, London, 1980. Academic Press.

[14] G.D. Plotkin. A semantics for static type inference. *Theoretical Computer Science*, 1993. To appear.

[15] R. Statman. Completeness, invariance, and λ-definability. *J. Symbolic Logic*, 47(1):17–26, 1982.

[16] R. Statman. Equality between functionals, revisited. In Harrington, Morley, Scedrov, and Simpson, editors, *Harvey Friedman's Research on the Foundations of Mathematics*, pages 331–338. North-Holland, 1985.