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José-Antonio N. Caraza
University of Pennsylvania

Xiaoping Yun
University of Pennsylvania

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José-Antonio N. Caraza
Xiaoping Yun

Department of Computer and Information Science
School of Engineering and Applied Science
University of Pennsylvania
Philadelphia, PA 19104-6389

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José-Antonio N. Caraza and Xiaoping Yun *

University of Pennsylvania
GRASP Lab - Room 301C
3401 Walnut St.
Philadelphia, PA 19104/6228

Abstract

This paper studies force-closure grasps of rigid objects by using two palms. The two palms are instrumented with tactile sensors capable of detecting the presence of contacts, and are assumed to be respectively installed on two robotic manipulators capable of motion and force control. Established in this paper is an existence condition under which the two palms form a force-closure grasp. The salient feature of this condition is that it does not require the information on the shape of the object and the contact locations. A configuration of the two palms in contact with the object satisfying this condition is called a force-closure grasp configuration (FCGC). Further, an algorithm is developed to check the condition for FCGC in terms of the position and orientation of the palms.

1 Introduction

Grasping in robotics is a study of kinematics, dynamics, and control of mechanical hands (or grippers) for manipulating objects. Due to the fact that motions of the fingers of a hand are generally slow while manipulating an object, the major effort in the study of grasping has been focused on static analysis, e. g., [1, 2, 3, 4, 5], with a few exceptions dealing with dynamics [6, 7, 8, 9]. A challenging issue in grasping is the determination of the stability of a grasp, that is, determining force equilibrium given all the information such as contact points, surface normals of the contact points, and friction coefficients. The work dealing with this issue includes [2, 5, 7, 10, 11, 12], among others.

Modeling of contacts is a primary concern in the study of the grasp stability. Various models have been considered, including point contact, line contact, surface contact, and soft contact [1, 13, 14]. In terms of the occurrence of contact points on hands/grippers,

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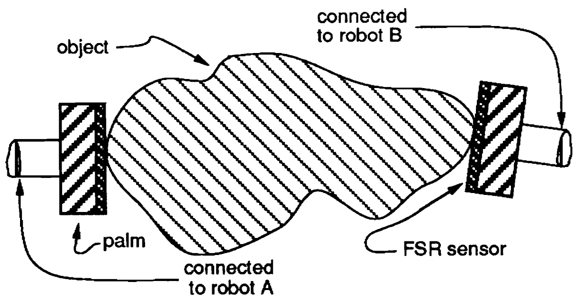

Figure 1: Two Handed Grasping – Palm Pushing

there are finger-tip grasps analyzed in most of the works, and enveloping grasps (or whole finger grasps) treated in a few works, e. g., [15].

In dealing with grasping stability, most of the existing approaches assume the availability of the complete contact information, including the contact locations, surface normals of the object at the contact points (or equivalently the shape of the object), and friction coefficients at the contacts. The information assumed in these approaches may not be available at the required accuracy, or not available at all, in an unstructed environment. To remedy the problem, an alternative approach is to investigate the grasping issues based on partial information. Towards this direction, Nguyen [16] proposed an algorithm to find the independent regions of contacts on the polygonal and polyhedral objects. As long as the contacts on the object occur in the regions, a force-closure grasp is possible (however, the contact locations on the gripper are assumed to be known). Faverjon and Ponce have extended the Nguyen's algorithm to piecewise smooth curved objects [17].

In this paper, the requirement on the available information is further relaxed. Using two palms to grasp a rigid object, this paper establishes an existence condition for force-closure grasp, which does not require any information about the shape of the object and the contact locations on the object. Furthermore, the contact locations on each palm are not required either, as long as the contacts occur at the interior of the palms. Nevertheless, the coefficients of friction are needed in this study. A typical scenario of the two palms grasping an irregularly-shaped object is illustrated in Figure 1. The palms considered in this study are flat surface in shape. Attached to the surface of each palm is an Interlink Force Sensing Resistor (FSR[1]) sensor which is used to detect the presence of contacts with objects. By using flat surface palms, the common contact normals can always be obtained from the orientation of the palms. In seeking conditions for force-closure grasps, the approach taken in this study is to establish useful sufficient conditions, rather than necessary and sufficient conditions, since the latter requires the complete knowledge of the contact parameters.

The motivation of using two palms in this study, compared with a single hand with multiple fingers, is that a hand is limited to grasp relatively small objects, and objects with special features such as handles. A task as simple as picking up a relatively large carton can not be achieved by a single arm/hand. However, such a task can be easily performed by

[1]FSR is a trademark of Interlink Electronics, P.O. Box 40760, Santa Barbara, CA 93103.

using two palms to push from two opposite sides. Owing to this motivation, the treatment on force-closure grasps in this paper is targeted to relatively large objects.

2 Background

Grasping an object is the process of exerting contact forces on it with the purpose of gaining control over its movement. The whole grasping process involves: (1) approaching the object, (2) detecting contacts, (3) evaluating the capability of the contact configuration to grasp the object, (4) determining the grasping forces required to manipulate it, and (5) applying the required contact forces. The focus of this work is on step (3).

This section provides some definitions and results from previous works which are relevant to this study. The precise problem of this study is also defined

A palm can be considered as a very primitive type of hand. Therefore the definitions that follow are applicable to grasps using two palms. However, in order to generalize the concepts explained, the term hand is used instead.

A set of contacts between the hands and the object are established in order to constrain the movement of the object. To be able to move the grasped object in any direction, the hands must be able to apply arbitrary forces and moments to it.

Definition 2.1 *A* force-closure grasp *is a set of contacts able to apply arbitrary force and moment to the object through the set of contacts.*

A force-closure grasp completely constrains the movement of the grasped object, therefore there is no direction in which the object can move freely.

Define matrix $\mathbf{G}$ as the matrix whose columns are the contact wrenches. Denote by $\mathbf{t}^S$ the spatial transpose of the twist $\mathbf{t}$. From the concept of virtual work of a wrench $\mathbf{W}_i$ against a twist $\mathbf{t}$, if the system of linear inequalities $\mathbf{G}^T\mathbf{t}^S \geq \mathbf{0}$ has no other solution than the null vector, a force-closure grasp has been obtained. Otherwise, there is a direction along the solution twist $\mathbf{t} \neq \mathbf{0}$ in which motion is not opposed by any force.

The following theorem states the requirements for vector-closure in an n-dimensional space [18]:

Theorem 2.2 (Goldman and Tucker) *In an n-dimensional vector space, a set of vectors $\mathbf{V}$ is vector closure if and only if $\mathbf{V}$ has at least $n+1$ vectors $(\mathbf{v}_1, \ldots, \mathbf{v}_{n+1})$ such that*

1. *n of the $n+1$ vectors are linearly independent.*
2. *A strictly positive combination of the $n+1$ vectors is the zero vector.*

$$\sum_{i=1}^{n+1} \alpha_i \mathbf{v}_i = \mathbf{0} \tag{1}$$

At each contact with the object, a palm can only apply unidirectional force (it can only push, but cannot pull the object) further, if the contact is a point contact, the palm can not apply torque to the object.

Theorem 2.3 (Nguyen) *In two-dimensional (2-D) space, the necessary and sufficient conditions for force closure with point contacts is that there exist four forces such that:*

1. *Three of the four forces have lines of action that do not intersect at a common point or at infinity,*
2. *Let $\mathbf{u}_1, \ldots, \mathbf{u}_4$ be the force directions of wrenches $\mathbf{W}_1, \ldots, \mathbf{W}_4$. Let $\mathbf{p}_{ij}$ be the intersection point of the nonparallel lines of action of $\mathbf{W}_i$ and $\mathbf{W}_j$. There exists α, β, γ and δ all greater than zero, such that:*

$$\mathbf{p}_{34} - \mathbf{p}_{12} = \pm(\alpha \mathbf{u}_1 + \beta \mathbf{u}_2) = \mp(\gamma \mathbf{u}_3 + \delta \mathbf{u}_4) \tag{2}$$

These conditions are the same as those expressed in the Goldman-Tucker theorem stated before. They were derived by Nguyen [16] by applying the force-closure requirement: no $\mathbf{t} \neq \mathbf{0}$ solution to the system $\mathbf{G}^T \mathbf{t}^S \geq \mathbf{0}$. The first condition corresponds to no homogeneous solution, and the second condition implies no particular solution to the system.

For two point contacts with friction, Theorem 2.3 leads to the following corollary [16].

Corollary 2.4 *Two point contacts with friction at $\mathbf{p}_1$ and $\mathbf{p}_2$ form a planar force-closure grasp if and only if the segment $\overline{\mathbf{p}_1\mathbf{p}_2}$ points strictly out of the friction cone at $\mathbf{p}_1$ and strictly into the friction cone at $\mathbf{p}_2$.*

Two point contacts form a planar force-closure grasp if and only if the line segment joining the two point contacts is inside the cone of friction of each of the point contacts.

Based on the fact that (1) each soft finger contact can generate four independent wrenches; (2) two soft finger contacts are able to generate six independent wrenches; and (3) if two soft finger contacts are inside of each other's friction cone, a strictly positive combination of all eight wrenches applied to the object add to zero; Nguyen [16] proved that:

Theorem 2.5 *A grasp with at least two distinct soft finger contacts is force closure if it is in equilibrium, with contact forces pointing strictly into the friction cones at the respective points of contact.*

The problem of this study is precisely characterized in the following two definitions.

Definition 2.6 *A grasp configuration of two hands is the relative position and orientation of the two hands in contact with the object to be grasped.*

Definition 2.7 *Two hands are in a force-closure grasp configuration (FCGC) if they can apply arbitrary force and moment to the object at the grasp configuration.*

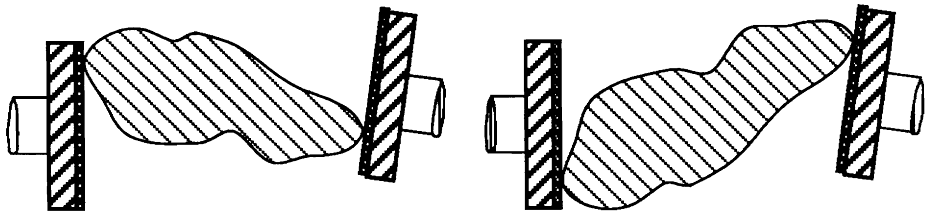

Figure 2: Example of Two Palms in an Identical Grasp Configuration

The definition above is a natural extension of force-closure grasps defined in Definition 2.1. A FCGC is a grasp configuration able to apply to the grasped object a resultant wrench with any line of action.

The objective of this paper is to establish a condition for FCGCs, and to develop an algorithm verifying this condition.

It is noted that a grasp configuration is independent of contact locations. Therefore, the two grasps illustrated in Figure 2 correspond to the same grasp configuration. Also, from the definition of grasp configuration, the condition obtained for FCGC is independent of the shape of the object to be grasped.

3 Friction Cones

The idea of *friction cone* of a point is associated with the range of directions that contact forces can have when exerted at that point. Because of the constant proportionality between the normal force applied to an object and its maximum applicable friction force, no contact force can be applied with a direction outside the friction cone of the contact point. If a friction force bigger than the maximum applicable for a given normal force is intended, sliding will occur and contact at that point will be lost.

For the study of grasps in this paper, a set approach to the concept of friction cones is introduced in this section. Instead of focusing on the directions of the permissible contact forces, the friction cone of a point will be associated to all the points in the space touched by vectors representing exertable contact forces.

3.1 2-D Friction Cones

3.1.1 Friction Cone of a Point

Assuming Coulomb friction model, the maximum tangential force exertable through a point contact is proportional to the normal force, with the proportionality coefficient μ being the friction coefficient. All the forces (or wrenches) exertable through the contact point form a cone (or a cone and its reflected image) which is the friction cone of the contact point. Due to the extensive use of wrenches, the friction cone of a contact point in this discussion includes the cone itself and its reflected image.

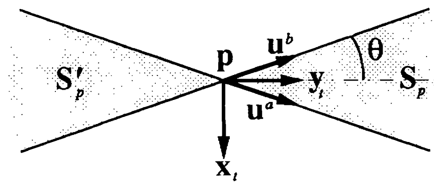

Figure 3: Friction Cone of a Point

In the point-set representation used here, the friction cone is represented by two convex cones including all the points through which lines along the contact forces lie. This representation does not convey information about the direction of the contact force; this must be specified separately. Contact forces are line vectors having magnitude, direction and moment.

Definition 3.1 *The* friction cone of a contact point *is the set of points belonging to lines of action of forces exertable through the contact point.*

Since the palms are flat surfaces in shape, in 2-D space a palm is modeled by a straight line segment, denoted by $\mathbf{P}_i$ with the subscript i designated to indicate the specific palm of concern.

To represent the position and orientation of a palm in 2-D space, a world coordinate frame $(\mathbf{x}_W, \mathbf{y}_W)$ with origin at a fixed point, and a palm (or tool) coordinate frame $(\mathbf{x}_t, \mathbf{y}_t)$ located at the center of the palm, $\mathbf{c}$, with $\mathbf{y}_t$ being normal to the palm and pointing towards the object, are defined.

Let $\mathbf{p}$ be a point on the palm and μ its friction coefficient. Denote the set of contact forces exertable at point $\mathbf{p}$ as $\boldsymbol{\Gamma}_p$. Let cordinate frame $(\mathbf{x}_t, \mathbf{y}_t)$ be the reference frame.

A contact force $\mathbf{f}_p = f_p\mathbf{u}_p$ of magnitude $f_p > 0$ and unit vector $\mathbf{u}_p$, exerted at point $\mathbf{p}$, can be expressed as $\mathbf{f}_p = f_p[u_{px}, u_{py}]^T$.

The set $\boldsymbol{\Gamma}_p$ is given by

$$\boldsymbol{\Gamma}_p = \left\{ \mathbf{f}_p = f_p\mathbf{u}_p \;\middle|\; \mathbf{u}_p = [u_{px}, u_{py}]^T, f_p > 0, 0 \le \frac{|u_{px}|}{u_{py}} < \mu \right\} \tag{3}$$

According to Definition 3.1, the friction cone $\boldsymbol{\mathcal{V}}_p$ of point $\mathbf{p}$ is

$$\boldsymbol{\mathcal{V}}_p = \left\{ \mathbf{s} \in \Re^2 \;\mid \mathbf{p} \times \mathbf{u}_p = \mathbf{s} \times \mathbf{u}_p, f_p\mathbf{u}_p \in \boldsymbol{\Gamma}_p \right\} \tag{4}$$

That is, the friction cone of point $\mathbf{p}$ comprises all the lines of action of wrenches exertable at $\mathbf{p}$. For a point $\mathbf{p}$, shown in Figure 3, its friction cone is

$$\boldsymbol{\mathcal{V}}_p = \mathbf{S}_p \cup \mathbf{S}'_p \tag{5}$$

$\mathbf{S}_p$ and $\mathbf{S}'_p$ are the two convex cones:

$$\mathbf{S}_p = \{\mathbf{s} = \mathbf{p} + \psi_1\mathbf{u}^a + \psi_2\mathbf{u}^b \mid \psi_1 \ge 0, \psi_2 \ge 0\}, \tag{6a}$$

$$\mathbf{S}'_p = \{\mathbf{s} = \mathbf{p} - \psi_1 \mathbf{u}^a - \psi_2 \mathbf{u}^b \mid \psi_1 \geq 0, \psi_2 \geq 0\}, \tag{6b}$$

$\mathbf{S}'_p$ is said to be the image of $\mathbf{S}_p$. Unit vectors $\mathbf{u}^a$ and $\mathbf{u}^b$ obtained after rotating vector $\mathbf{y}_t$ by an angle of $-\theta$ and θ respectively, with $\theta = \arctan\mu$, denote the extreme directions of forces $\mathbf{f}_p \in \boldsymbol{\Gamma}_p$.

Remark 3.2 *Each point* $\mathbf{s} \in \boldsymbol{\mathcal{V}}_p$ *is associated with a directed line* $\boldsymbol{\mathcal{U}}_{s:p}$ *passing through it.* $\boldsymbol{\mathcal{U}}_{s:p}$ *is the line of action of the force* $\mathbf{f}_p$ *exerted through the contact at point* $\mathbf{p}$.

Using Plücker coordinates, $\boldsymbol{\mathcal{U}}_{s:p}$ is expressed as

$$\boldsymbol{\mathcal{U}}_{s:p} = \left\{ \mathbf{U} = [\mathbf{u}_p^T,\ (\mathbf{s} \times \mathbf{u}_p)^T]^T \;\middle|\; \begin{array}{l} \mathbf{p} \times \mathbf{u}_p = \mathbf{s} \times \mathbf{u}_p, \\ f_p \mathbf{u}_p \in \boldsymbol{\Gamma}_p, \mathbf{s} \in \boldsymbol{\mathcal{V}}_p \end{array} \right\} \tag{7}$$

3.1.2 Friction Cone of a Palm

Based on the concept of friction cone of a point, we can define the friction cone of a palm.

Definition 3.3 *The* friction cone of a palm *is the intersection of friction cones of all the points on the palm.*

Denoting the friction cone of a palm by $\boldsymbol{\mathcal{V}}$, according to Definition 3.3,

$$\boldsymbol{\mathcal{V}} = \bigcap_{\mathbf{p} \in \mathbf{P}} \boldsymbol{\mathcal{V}}_p = \{\mathbf{S}_\zeta, \mathbf{S}'_{\zeta'}\} \tag{8}$$

where $\mathbf{S}_\zeta$ and $\mathbf{S}'_{\zeta'}$ are as shown in Figure 4.

The two vertex points of the friction cone of a palm, labeled $\boldsymbol{\zeta}$ and $\boldsymbol{\zeta}'$ in Figure 4, are called focal points. Their coordinates are

$$\boldsymbol{\zeta} = \mathbf{c} + b\mathbf{y}_t \tag{9a}$$

and

$$\boldsymbol{\zeta}' = \mathbf{c} - b\mathbf{y}_t, \tag{9b}$$

with $b = a/\mu$, and a being half of the palm's length.

Geometrically, the friction cone of a palm is the friction cone of its center point $\mathbf{c}$ shifted away from the palm by a distance $b = a/\mu$ i.e., $\mathbf{S}_\zeta$, is $\mathbf{S}_c$ shifted by a distance b along $\mathbf{y}_t$ direction, and $\mathbf{S}'_{\zeta'}$ is $\mathbf{S}'_c$ shifted by a distance b along $-\mathbf{y}_t$ direction.

Remark 3.4 *Each point* $\mathbf{s} \in \boldsymbol{\mathcal{V}}$ *is associated with a set of directed lines,* $\boldsymbol{\mathcal{U}}_s$ *connecting it with all* $\mathbf{p} \in \mathbf{P}$. $\boldsymbol{\mathcal{U}}_s$ *corresponds to all possible lines of action of contact forces —exerted by the palm— passing through* $\mathbf{s}$. *Each line has the same direction as its corresponding contact force.* $\boldsymbol{\mathcal{U}}_s$ *is expressed by*

$$\boldsymbol{\mathcal{U}}_s = \bigcup_{\mathbf{p} \in \mathbf{P}} \boldsymbol{\mathcal{U}}_{s:p} \tag{10}$$

The size of the palms represents the level of resolution with which the contact location can be specified.

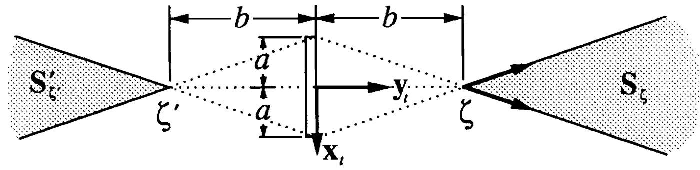

Figure 4: Friction Cone of a Palm

3.1.3 Multiple Contacts on a Palm

Since the palm is modeled as a straight line segment, and no contact is assumed at the edge of the palm, the normal directions of the points on the palm (contact normals) are all parallel. Both the object and the palms are considered rigid, therefore, the direction of the force exerted by a palm on one contact point will be parallel to the direction of all other contact forces exerted by the same palm on the object. If there is more than one contact point on the palm, the equivalent to the total force exerted by the palm on the object, can be placed at some point $\mathbf{p}$. The force $\mathbf{f}$, placed at point $\mathbf{p}$ equivalent to contact forces applied at m different points in the palm has the properties:

$$\mathbf{f} = \mathbf{f}_1 + \mathbf{f}_2 + \cdots + \mathbf{f}_m \tag{11}$$

$$\mathbf{p} \times \mathbf{f} = \mathbf{p}_1 \times \mathbf{f}_1 + \mathbf{p}_2 \times \mathbf{f}_2 + \cdots + \mathbf{p}_m \times \mathbf{f}_m \tag{12}$$

All contact forces are parallel and with the same direction. Let us denote by f_i the magnitude of $\mathbf{f}_i$, and $\mathbf{u}$ its unit vector, hence

$$\mathbf{f} = f_1\mathbf{u} + f_2\mathbf{u} + \cdots + f_m\mathbf{u} = \sum_{i=1}^{m} f_i\mathbf{u} \tag{13}$$

$$\mathbf{p} \times \mathbf{f} = \mathbf{p}_1 \times f_1\mathbf{u} + \mathbf{p}_2 \times f_2\mathbf{u} + \cdots + \mathbf{p}_m \times f_m\mathbf{u} = \sum_{i=1}^{m} \mathbf{p}_i \times f_i\mathbf{u} \tag{14}$$

also

$$\mathbf{p} \times \mathbf{f} = \mathbf{p} \times \sum_{i=1}^{m} f_i\mathbf{u} \tag{15}$$

All points $\mathbf{p}_i$ belong to the same line segment along axis $\mathbf{x}_t$. Expressing $\mathbf{p}_i$ in terms of point $\mathbf{p}$,

$$\mathbf{p}_i = \mathbf{p} + (p^t_{ix} - p^t_x)\mathbf{x}_t = \mathbf{p} + \nu_i\mathbf{x}_t \tag{16}$$

where p^t_{ix}, and p^t_x represent the ordinate value of the points $\mathbf{p}^t_i$, and $\mathbf{p}^t$ respectively in the palm coordinate frame t, $\nu_i = p^t_{ix} - p^t_x$ represents the directed distance between the points $\mathbf{p}$ and $\mathbf{p}_i$. It follows that

$$\mathbf{p}_i \times f_i\mathbf{u} = \mathbf{p} \times f_i\mathbf{u} + f_i\nu_i(\mathbf{x}_t \times \mathbf{u})$$

$$\begin{aligned}\sum_{i=1}^{m} \mathbf{p}_i \times f_i \mathbf{u} &= \sum_{i=1}^{m} \mathbf{p} \times f_i \mathbf{u} + f_i \nu_i (\mathbf{x}_t \times \mathbf{u}) \\ &= \mathbf{p} \times \mathbf{f} + \left(\sum_{i=1}^{m} f_i \nu_i \right) \mathbf{x}_t \times \mathbf{u} \end{aligned} \tag{17}$$

which means that point $\mathbf{p}$ is such that

$$\sum_{i=1}^{m} f_i \nu_i = \sum_{i=1}^{m} f_i (p_{ix}^t - p_x^t) = 0 \tag{18}$$

Since the abscissa value for all points of the palm in coordinate frame t is equal to zero,

$$p_x^t = \frac{\sum_{i=1}^{m} f_i p_{ix}^t}{\sum_{i=1}^{m} f_i} \tag{19}$$

$$\mathbf{p}^t = [p_x^t, 0]^T \tag{20}$$

i.e., the coordinates of $\mathbf{p}^t$ are obtained through a weighted sum of the position vectors of each contact point.

Then, the equivalent contact force will always be placed at a point $\mathbf{p}$ between the two extreme points of a palm contacting the object. This force will also have a direction inside the friction cone of point $\mathbf{p}$, hence, all properties derived here for only one contact point are valid for the case of multiple contact points on a palm.

3.2 3-D Friction Cones

In this subsection, the definitions stated for the 2-D case are extended to the 3-D case.

To represent the position and orientation of the palm in 3-D space, a world coordinate frame $(\mathbf{x}_W, \mathbf{y}_W, \mathbf{z}_W)$ with origin at a fixed point, and a palm (or tool) coordinate frame $(\mathbf{x}_t, \mathbf{y}_t, \mathbf{z}_t)$ located at the center of the palm, $\mathbf{c}$, with $\mathbf{z}_t$ being normal to the palm and pointing towards the object, are defined.

3.2.1 Friction Cone of a Point

Similarly to the 2-D case, let $\mathbf{p}$ be a point on the palm and μ its friction coefficient. Denote the set of contact forces exertable at point $\mathbf{p}$ as $\boldsymbol{\Gamma}_p$. Cordinate frame $(\mathbf{x}_t, \mathbf{y}_t, \mathbf{z}_t)$ will be the reference frame. A contact force $\mathbf{f}_p = f_p \mathbf{u}_p$ of magnitude $f_p > 0$ and unit vector $\mathbf{u}_p$, exerted at point $\mathbf{p}$, can be expressed as $\mathbf{f}_p = f_p [u_{px}, u_{py}, u_{pz}]^T$.

The set $\boldsymbol{\Gamma}_p$ is given by

$$\boldsymbol{\Gamma}_p = \left\{ \mathbf{f}_p = f_p \mathbf{u}_p \;\middle|\; \mathbf{u}_p = [u_{px}, u_{py}, u_{pz}]^T, f_p > 0, 0 \leq \frac{\left|\sqrt{u_{px}^2 + u_{py}^2}\right|}{u_{pz}} < \mu \right\} \tag{21}$$

The friction cone $\boldsymbol{\mathcal{V}}_p$ of point $\mathbf{p}$ can be defined on the basis of its geometry as

$$\boldsymbol{\mathcal{V}}_p = \mathbf{S}_p \cup \mathbf{S}'_p \tag{22}$$

where $\mathbf{S}_p$ and $\mathbf{S}'_p$ are the two convex cones:

$$\mathbf{S}_p = \{\mathbf{s} = \mathbf{p} + \psi\mathbf{u} \mid \psi \geq 0, \| \mathbf{u} \| = 1, \mathbf{u} \cdot \mathbf{z}_t > 0, (\mathbf{u} \cdot \mathbf{z}_t)^2 > 1 + \mu^2\}, \tag{23a}$$

$$\mathbf{S}'_p = \{\mathbf{s} = \mathbf{p} - \psi\mathbf{u} \mid \psi \geq 0, \| \mathbf{u} \| = 1, \mathbf{u} \cdot \mathbf{z}_t > 0, (\mathbf{u} \cdot \mathbf{z}_t)^2 > 1 + \mu^2\} \tag{23b}$$

Each point $\mathbf{s} \in \boldsymbol{\mathcal{V}}_p$ is associated with a directed line $\boldsymbol{\mathcal{U}}_{s;p}$ defined as in 7, but now using the new definitions of $\boldsymbol{\Gamma}_p$, $\mathbf{S}_p$, and $\mathbf{S}'_p$.

3.2.2 Friction Cone of a Palm

The friction cone of a palm in 3-D space can be defined as the intersection of the friction cones of all points contained in that palm. It represents the set of all points belonging to lines of action of any force exertable on an object by the palm when the palm and the object are in contact.

Two points: $\boldsymbol{\zeta}$ at the front, and $\boldsymbol{\zeta}'$ at the back of a palm, both inside the friction cone of the palm, and closest to the palm contact surface are called *focal points* of the palm. From the focal points, the friction cone of the palm is projected.

For the purposes of this study, flat palms with circular shapes are considered. The characteristics of the 3-D friction cone of a circular palm, with diameter $2a$, can be mathematically stated in basically the same way as expressed in Section 3.1.2. The only exception being the substitution of $\mathbf{z}_t$ instead of $\mathbf{y}_t$ as the vector normal to the palm. Therefore, Equations 9a, and 9b must be read as

$$\boldsymbol{\zeta} = \mathbf{c} + b\mathbf{z}_t \tag{24a}$$

and

$$\boldsymbol{\zeta}' = \mathbf{c} - b\mathbf{z}_t \tag{24b}$$

Similarly to the analysis given in section 3.1.3, for the 3-D case, when multiple contacts occur on a palm, their effect can be expressed as the effect of a single contact in the same palm placed at an equivalent point.

4 FCGC Condition

This section establishes a condition for FCGC. The following theorem is for two palms in 2-D space.

Theorem 4.1 *In 2-D space, a grasp configuration of two palms is a FCGC if*

- *For an outside-in grasp,* $\mathbf{P}_j \subset \mathbf{S}_{\zeta i}, i, j = 1, 2, i \neq j$.
- *For an inside-out grasp,* $\mathbf{P}_j \subset \mathbf{S}'_{\zeta' i}, i, j = 1, 2, i \neq j$.

Proof

We first prove it for outside-in grasps.

Let $\mathbf{p}_1 \in \mathbf{P}_1$ and $\mathbf{p}_2 \in \mathbf{P}_2$ be two contact points on the two palms respectively. All that is needed is to show that $\mathbf{p}_1$ is in the friction cone of $\mathbf{p}_2$, and vice versa.

Since $\mathbf{S}_{\zeta 2}$ is the intersection of friction cones of all points in $\mathbf{P}_2$, the condition $\mathbf{P}_1 \subset \mathbf{S}_{\zeta 2}$ implies that $\mathbf{P}_1$, and thus $\mathbf{p}_1$, is in the friction cone of $\mathbf{p}_2$.

Similarly, we can show that $\mathbf{p}_2$ is in the friction cone of $\mathbf{p}_1$. Therefore, from Corollary 2.4, the grasp configuration is a FCGC.

The case of inside-out grasps can be proved similarly. ■

Extending the results to the 3-D case requires to consider soft finger (compliant) contacts:

Theorem 4.2 *In 3-D space, a grasp configuration of two palms is a FCGC if it satisfies the condition of Theorem 4.1, and the contacts between the palms and the object are soft finger contacts.*

Proof

By using Theorem 2.5, the proof is similar to that of Theorem 4.1. ■

Geometrically, the requirement for a FCGC is that all the points on each palm are defined in the friction cone of the other palm.

5 Position and Orientation of Two Palms in a FCGC

In this section, an algorithm to test the FCGC condition is developed in terms of the relative position and orientation of the two palms.

5.1 The 2-D Case

Denote the frame coordinates of palm i as $(\mathbf{x}_i, \mathbf{y}_i)$ with $\mathbf{y}_i$ normal to palm i and pointing towards the object. Assume both palms having the same length $2a$, and the same friction coefficient μ. Use palm 1 coordinates as the reference frame. Denote by β the angle, from palm 1 to palm 2. The coordinates of the center of palm 2 will be given by $\mathbf{c}_2 = [x, y]^T$.

In this subsection, relationships between the extreme values of x, and β for a FCGC will be found in terms of y. The strategy used here to determine the actual bounds of the variables will be to take y as a constant value, then specify (1) the variation of the extreme values of β as a function of x, and (2) the variation of the extreme values of x as a function of β. Therefore, for each different value of y sets of extreme values of x and β will be found.

Assuming $a_1 = a_2 = a$, and $\mu_1 = \mu_2 = \mu$, the permissible values for x, y, and β, in a FCGC, satisfy the following inequalities

$$y \geq 2b \tag{25a}$$

$$x < \mu y - a \tag{25b}$$

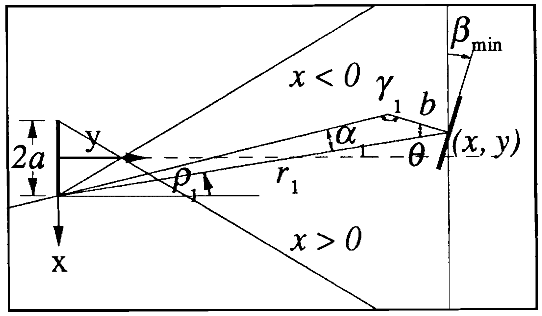

Figure 5: Minimum Value of β in a FCGC

$$|\beta| < 2\theta \tag{25c}$$

5.1.1 Permissible Range of Values for β

In a FCGC the position of $\mathbf{c}_2$ is constrained by the friction cone of palm 1, and the range of values for β is determined according to the position of $\mathbf{c}_2$. The extreme permissible values of the angle β will be called β_{min}, and β_{max}, for the minimum and maximum values of β respectively.

The angles and distances shown in Figure 5 are: $\rho_1 = \text{atan2}(a-x, y)$, $r_1 = \sqrt{(a-x)^2 + y^2}$, and $\gamma_1 = \rho_1 - \beta_{\text{min}}$.

Using sine's law

$$\sin \alpha_1 = \frac{b}{r_1} \sin \theta \tag{26}$$

Note that $|\alpha_1| < \frac{\pi}{2}$. It follows that,

$$\alpha_1 = \arcsin\left(\frac{b}{r_1}\sin\theta\right) \tag{27}$$

$$\beta_{\text{min}} = \alpha_1 - \theta + \rho_1 \tag{28}$$

A similar approach yields the value of β_{max}

$$\beta_{\text{max}} = \alpha_2 - \theta + \rho_2 \tag{29}$$

where

$$\rho_2 = \text{atan2}(-a + x, y) \tag{30}$$

$$\alpha_2 = \arcsin(b \sin\theta / \sqrt{(a+x)^2 + y^2}) \tag{31}$$

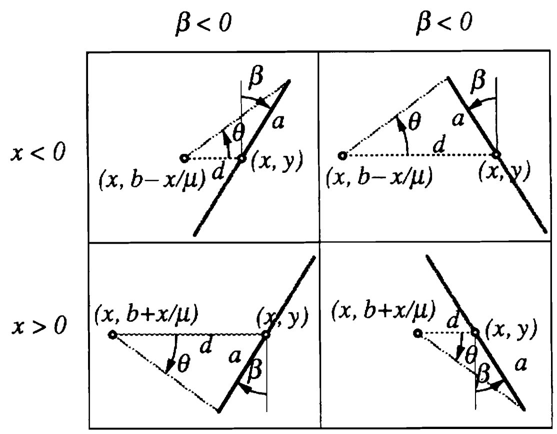

Figure 6: Relationship Between x and β at their Extreme Values in a FCGC

5.1.2 Extreme Values for x for Permissible Values of β in a FCGC.

Figure 6 shows palm 2 inside the friction cone of palm 1, having one of its edges touching the border of the same friction cone. Using Figure 6, a relationship between the extreme values of x as a function of β can be obtained.

The distance d shown in Figure 6 can be obtained using sine's law. Its value is given by:

$$d = a\left[\frac{\cos\beta}{\mu} + \sin|\beta|\right] \tag{32}$$

Therefore, the maximum absolute value attainable by x at a given β and y satisfy the following equation

$$|x|_{\max} = \mu[y - b - d] \tag{33}$$

Note that the sign of x at its maximum absolute value for given β and y, has a sign opposite to that of β.

These results allow us to test if for a given y, the pair of values x and β correspond to a FCGC.

5.1.3 Extreme Values for x and β in a FCGC

The maximum values for the magnitude of both, x and β, can be obtained from the two triangles that the palms and friction cones form, as illustrated in Figure 7. The internal angles of both triangles are $-\theta + \pi/2$, $2\theta - |\beta|$, and $|\beta| - \theta + \pi/2$, one of the sides of each triangle is $2a$, therefore the other two sides of each other's triangle are equal. Name v and w to the unknown length of the sides of the triangles, and r_1 and r_2 to the distance between the extreme points of the palms, as illustrated in Figure 7. Note that,

$$v + w = r_1 = r_2 \tag{34}$$

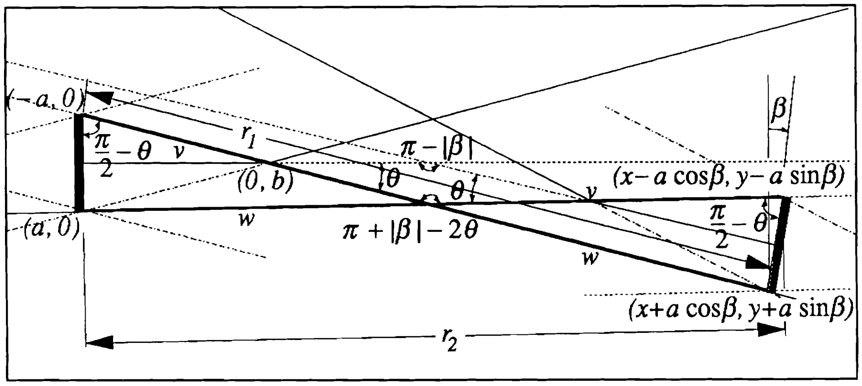

Figure 7: Relationship Between the Maximum Values of x and β

There is symmetry in the positive and negative extreme values of x with respect to β. Therefore, the following relationship can be found for x and β

$$|x| \leq |x|_{\max} = y\frac{\sin|\beta|}{1+\cos\beta} \tag{35}$$

5.1.4 Features of the Extreme Values for β

From the analysis performed in Section 5.1.1, it can be noted that

$$\lim_{y\to\infty} \beta_{\max}|_{x=0} = \lim_{y\to\infty} -\beta_{\min}|_{x=0} = \theta \tag{36}$$

For a given value of y, let

$$|\beta|_{\max} = \begin{cases} \beta_{\max} & \text{if } x < 0, \\ -\beta_{\min} & \text{otherwise.} \end{cases} \tag{37}$$

An implication of Equations 33 and 35 is that, when $y \to \infty$, the extreme values of β at extreme values of x are such that

$$\lim_{y\to\infty} \sin|\beta|_{\max} = \frac{2\mu}{1+\mu^2} = \sin 2\theta \tag{38}$$

Therefore,

$$\lim_{y\to\infty} |\beta|_{\max} = 2\theta \tag{39}$$

Another interesting feature of the relationship between the variables x, y and β is that, for $y \geq b\left(2+\frac{\mu^2+2}{\sqrt{1+\mu^2}}\right)$, if $|x| = a(1+\cos\theta)$ then $|\beta|_{\max} = \theta$, i.e., for an $|x| = a(1+\cos\theta)$, having a FCGC requires that $y \geq b\left(2+\frac{\mu^2+2}{\sqrt{1+\mu^2}}\right)$, and for those values of x and y, the value of $|\beta|_{\max} = \theta$.

Figure 8 shows curves with the maximum permissible values of x and β corresponding to different values on the y axis, in a FCGC. The values of β are given in radians, and the values of x, y and a are in any length units. For the example shown in the figure, $a = 1$ and $\mu = 1$. The narrower boundaries for x and β correspond to the smaller values of y.

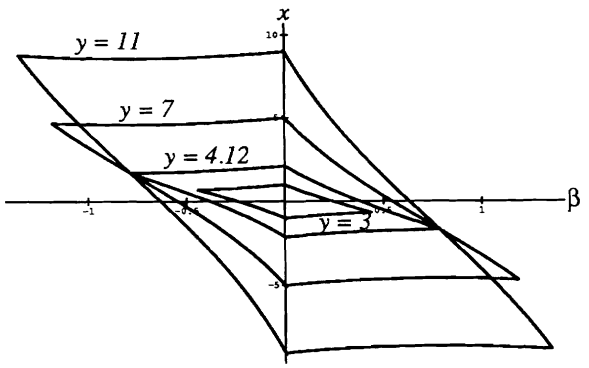

Figure 8: Boundary values for x and β corresponding to different values of y

5.1.5 Algorithm

Summarizing the results obtained, to verify if a grasp configuration is a FCGC, the procedure applied can be described as follows:

1. Choose one palm as palm 1.
2. Obtain values x, y and β.
3. Verify if Inequalities 25 are satisfied. If not, exit with failure.
4. Verify if $\beta_{\min} \leq \beta \leq \beta_{\max}$ where $\beta_{\min}$, and $\beta_{\max}$, are given by Equations 28, and 29, respectively. If not, exit with failure.
5. Verify if $|x|$ is smaller than or equal to the value obtained on Equation 33. If not, exit with failure; otherwise, the grasp configuration is a FCGC.

5.2 The 3-D Case

As stated in Section 4, to get a FCGC using two palms, the two palms must be in the friction cone of each other. Here, an equivalent condition will be obtained in terms of relative position and orientation of the two palms.

Figure 9 shows palm 2 being inside the friction cone of palm 1. The coordinate frame of palm 1 is $(\mathbf{x}_1, \mathbf{y}_1, \mathbf{z}_1)$, in this coordinate frame, the center of palm 2, $\mathbf{c}_2$, is expressed as $[X, Y, Z]^T$. In the figure, $\boldsymbol{\zeta}_1$ is the focal point of palm 1, r represents the distance from $\mathbf{c}_2$, to the z_1 axis.

Define plane G as the plane passing through point $\mathbf{c}_2$ and z_1 axis. The orientation of G varies according to the values of X and Y (abscissa and ordinate values of the center of palm 2). As it can be seen in Figure 9, due to the axial symmetry of the friction cone of a palm, if the characteristics of a grasp configuration are described having as a reference plane G, all different positions of palm 2 bearing the same relative position and orientation with respect to G, are equivalent for the purpose of its force-closure analysis. Hence, the

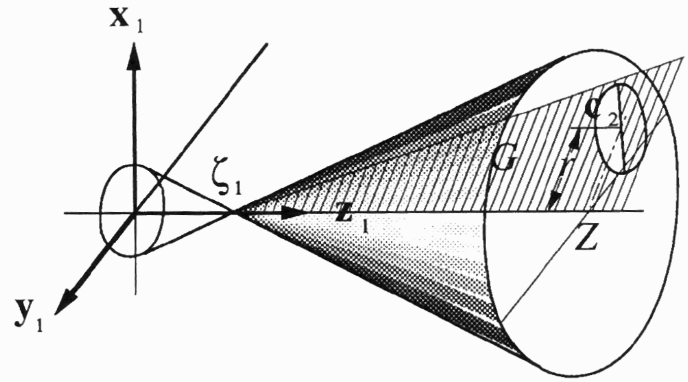

Figure 9: Palm 2 Inside the 3-D Friction Cone of Palm 1

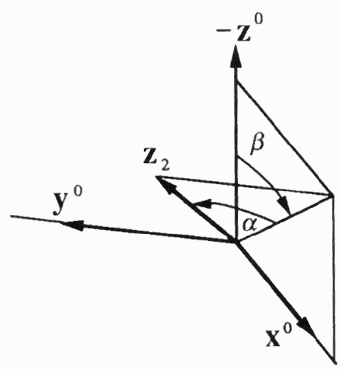

Figure 10: Unit vector in Spherical Coordinates

conclusions drawn for one case are valid for all other grasp configurations having palm 2 with the same relative position and orientation with respect to G.

Therefore, the analysis that follows has as a reference the coordinate frame $(\mathbf{x}^0, \mathbf{y}^0, \mathbf{z}^0)$ which is obtained through a rotation of the coordinate frame of palm 1: $(\mathbf{x}_1, \mathbf{y}_1, \mathbf{z}_1)$ along its z axis by an angle ψ defined as:

$$\psi = \text{atan2}(Y, X) \tag{40}$$

The frame coordinates $(\mathbf{x}^0, \mathbf{y}^0, \mathbf{z}^0)$ are such that the center of palm 2 lies in the x_1^0–z_1^0 plane, with $\mathbf{z}_1^0$ normal to palm 1 and pointing inwards the object.

Name palm 1 to the palm with the smallest friction cone. Vector $\mathbf{z}_2$ is defined as $\mathbf{z}_2 = [z_{2x}, z_{2y}, z_{2z}]^T$.

Using spherical coordinates, angles α and β are the two required parameters to specify the unit vector $\mathbf{z}_2$. Figure 10 illustrates the meaning of these parameters in the context of this study.

The center of palm 2 is expressed, in the reference frame coordinates, as:

$$\mathbf{c}_2 = [r, 0, Z]^T \tag{41}$$

where

$$r = \sqrt{X^2 + Y^2} \tag{42}$$

The angle β is given by

$$\beta = \text{atan2}(-z_{2x}, -z_{2z}) \tag{43}$$

and α is given by

$$\alpha = \text{asin}(z_{2y}) \tag{44}$$

i.e., β is a rotation of palm 2 with an axis of rotation parallel to $\mathbf{y}_1$, and α is a rotation of palm 2 with an axis of rotation along the intersection of plane x_1–z_1 and the palm's plane.

In this section, relationships between the extreme values of r, β, and α, for a FCGC, will be found in terms of Z. The strategy used here to determine the actual bounds of the variables will be to take Z as a constant value, then specify (1) the variation of the extreme values of α and β as a function of r, and (2) the variation of the extreme values of r as a function of α and β. Therefore, for each different value of Z sets of extreme values of r, α and β will be found.

5.2.1 Extreme Values for α and β

The extreme values of the angles α and β can be obtained for each value of r at a constant Z as described in this section.

As it can be seen from Figure 9, in a FCGC the position of $\mathbf{c}_2$ is constrained by the friction cone of palm 1. Having palm 2 in the friction cone of palm 1, the range of values of α and β are bound according to the position of $\mathbf{c}_2$.

Assuming $a_1 = a_2 = a$, and $\mu_1 = \mu_2 = \mu$, the permissible values for r, Z, α and β, in a FCGC, satisfy the following inequalities

$$Z \geq 2b \tag{45a}$$

$$r < \mu Z - a \tag{45b}$$

$$|\beta| < 2\theta \tag{45c}$$

$$|\alpha| < 2\theta \tag{45d}$$

In Figure 11 it can be noted that for extreme values of α and β at a given position of $\mathbf{c}_2$, if a cross section of the friction cone of palm 2 is drawn passing through the axis of the cone and through the tangency point between palm 1 and the cone, that cross section also goes through the center of palm 1 (origin of $(\mathbf{x}^0, \mathbf{y}^0, \mathbf{z}^0)$ coordinate axis). The reason for this is that the tangent to a circle at some point is always normal to the radius of the circle touching that point.

Figure 12 will be used to obtain the extreme values for the angles α and β caracterizing the orientation of palm 2 as a function of its position with respect to palm 1, in the reference frame coordinates. The tangent point between palm 1 and the friction cone of palm 2, is expressed as point $\mathbf{e}$ with coordinates $(x_1, y_1, 0)$. The origin of the coordinate frame is given

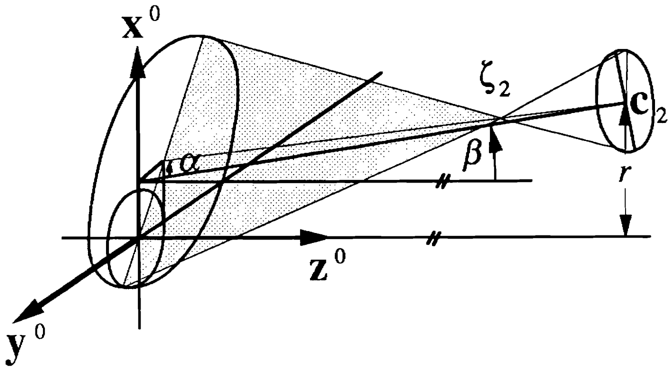

Figure 11: A Cross Section of the Friction Cone of Palm 2 at the Tangency Between Palm 1 and the Cone

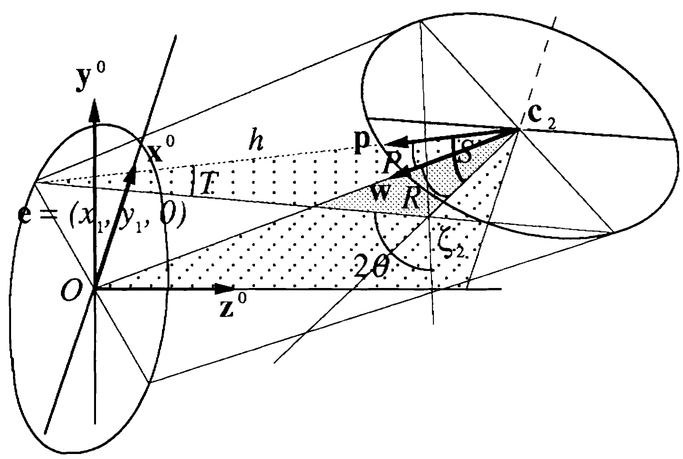

Figure 12: Geometrical Relationships for Extreme Values of α and β at a Given Position of $\mathbf{c}_2$

by O. The triangle formed by points $\mathbf{e}$, $\mathbf{c}_2$, and $\boldsymbol{\zeta}_2$ lies on a plane that also passes through O. The relationship between angles and distances in Figure 12 are as follows.

$$h = \sqrt{(r - x_1)^2 + y_1^2 + Z^2} \tag{46}$$

Using sine's law

$$\sin T = \frac{a}{h\sqrt{1 + \mu^2}} \tag{47}$$

Then, we have that since

$$0 < \theta < \pi/2 \tag{48}$$

the value of S can be obtained as follows:

$$T > 0, S > 0 \Rightarrow S = \theta - T \tag{49}$$

In Figure 12, defining the unit vectors

$$\mathbf{p} = \frac{\mathbf{e} - \mathbf{c}}{h} \tag{50}$$

and

$$\mathbf{w} = \frac{-\mathbf{c}_2}{\| \mathbf{c}_2 \|} = \frac{-\mathbf{c}_2}{\sqrt{r^2 + Z^2}} \tag{51}$$

and naming the angle between $\mathbf{p}$ and $\mathbf{w}$ as P, we have that

$$\cos P = \mathbf{p} \cdot \mathbf{w} \tag{52}$$

The angle R is, therefore, given by

$$R = S - P \tag{53}$$

note that the angles P, R, and S are all non-negative. Define frame A, a new set of coordinates $(\mathbf{x}^A, \mathbf{y}^A, \mathbf{z}^A)$ such that

$$\mathbf{z}^A = \mathbf{w} \tag{54}$$

$$\mathbf{x}^A = \frac{\mathbf{e} \times \mathbf{z}^A}{\| \mathbf{e} \times \mathbf{z}^A \|} \tag{55}$$

and

$$\mathbf{y}^A = \mathbf{z}^A \times \mathbf{x}^A \tag{56}$$

i.e., the unit vector $\mathbf{z}^A$ has the same direction as vector $-\mathbf{c}_2$, and points $\mathbf{e}$, $\mathbf{c}_2$, $\boldsymbol{\zeta}_2$, and O all lie on the y–z plane of frame A.

Finally, define coordinate frame B, a coordinate frame obtained after rotating coordinate frame A about $\mathbf{x}^A$ by an angle R.

The basis vector $\mathbf{z}^B$, of coordinate frame B, is the normal vector to palm 2 when the orientation of palm 2 is such that its friction cone touches the edge of palm 1 (at point $\mathbf{e}$), hence, $\mathbf{z}^B$ yields an extreme vector for FCGC.

From $\mathbf{z}^B$, it is possible to obtain the corresponding extreme values of α and β, through Equations 43, and 44.

The orthogonal matrix that represents the rotation from the base frame (palm 1 frame), to coordinate frame A, is given by

$$\mathbf{R}_A^0 = \left[\mathbf{x}^A, \mathbf{y}^A, \mathbf{z}^A\right] \tag{57}$$

The matrix representing the rotation from coordinate frame A to coordinate frame B is

$$\mathbf{R}_B^A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos R & -\sin R \\ 0 & \sin R & \cos R \end{bmatrix} \tag{58}$$

In frame B coordinates, vector $\mathbf{z}^B$ is expressed as $\mathbf{z}^B = [0, 0, 1]^T$.Therefore, the representation of the vector $\mathbf{z}^B$ in global frame coordinates is given by:

$$(\mathbf{z}^B)^0 = \mathbf{R}_A^0 \mathbf{R}_B^A \mathbf{z}^B = \left[\mathbf{x}^A, \mathbf{y}^A, \mathbf{z}^A\right] \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos R & -\sin R \\ 0 & \sin R & \cos R \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{59}$$

The extreme values of α and β: α_{extr} and β_{extr}, respectively, are implicit in the evaluation of $(\mathbf{z}^B)^0$; i.e., α_{extr} and β_{extr}, for given values of x and Z, are such that:

$$\begin{bmatrix} -\sin\beta_{\text{extr}} \cos\alpha_{\text{extr}} \\ \sin\alpha_{\text{extr}} \\ -\cos\beta_{\text{extr}} \cos\alpha_{\text{extr}} \end{bmatrix} = \begin{bmatrix} -(\sin R)\mathbf{y}^A + (\cos R)\mathbf{z}^A \end{bmatrix} \tag{60}$$

5.2.2 Extreme Values r for Given Values of Z, α and β in a FCGC

Define a new coordinate frame, frame H with origin at $O^H = \zeta_1$, and basis vectors $(\mathbf{x}^H, \mathbf{y}^H, \mathbf{z}^H)$ such that:

$$\mathbf{z}^H = \mathbf{z}^0 \tag{61}$$

$$\mathbf{y}^H = \begin{cases} \mathbf{y}_1 & \text{if } \mathbf{z}_2 \parallel \mathbf{z}_1 \\ \frac{\mathbf{z}_2 \times \mathbf{z}_1}{\|\mathbf{z}_2 \times \mathbf{z}_1\|} & \text{otherwise} \end{cases} \tag{62}$$

$$\mathbf{x}^H = \frac{\mathbf{y}^H \times \mathbf{z}^H}{\| \mathbf{y}^H \times \mathbf{z}^H \|} \tag{63}$$

By expanding the respective values of the vectors, it can be concluded that the orientation of coordinate frame H is obtained by rotating the base frame about its $\mathbf{z}$ axis by an angle $\rho = \text{atan2}(-\sin\alpha, \sin\beta\cos\alpha)$.

The matrix describing the translation from base frame to H frame is given by

$$\mathbf{T}_H^0 = \begin{bmatrix} & \mathbf{R}_H^0 & & 0 \\ & & & 0 \\ & & & b \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\sin\beta\cos\alpha}{\sqrt{\sin^2\beta\cos^2\alpha+\sin^2\alpha}} & \frac{\sin\alpha}{\sqrt{\sin^2\beta\cos^2\alpha+\sin^2\alpha}} & 0 & 0 \\ \frac{-\sin\alpha}{\sqrt{\sin^2\beta\cos^2\alpha+\sin^2\alpha}} & \frac{\sin\beta\cos\alpha}{\sqrt{\sin^2\beta\cos^2\alpha+\sin^2\alpha}} & 0 & 0 \\ 0 & 0 & 1 & b \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{64}$$

Resolving vector $\mathbf{z}_2$ in the H coordinate frame:

$$\mathbf{z}_2^H = (\mathbf{R}_H^0)^{-1}\mathbf{z}_2 = \begin{bmatrix} -\sqrt{\sin^2\beta\cos^2\alpha + \sin^2\alpha} \\ 0 \\ -\cos\beta\cos\alpha \end{bmatrix} \tag{65}$$

Similarly, the center of palm 2 can be resolved in the H coordinate frame as:

$$\begin{bmatrix} \mathbf{c}_2^H \\ 1 \end{bmatrix} = (\mathbf{T}_H^0)^{-1}\begin{bmatrix} \mathbf{c}_2 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{r\sin\beta\cos\alpha}{\sqrt{\sin^2\beta\cos^2\alpha+\sin^2\alpha}} \\ \frac{r\sin\alpha}{\sqrt{\sin^2\beta\cos^2\alpha+\sin^2\alpha}} \\ Z-b \\ 1 \end{bmatrix} \tag{66}$$

The friction cone of palm 1 is drawn from the focal point $\boldsymbol{\zeta}_1$, with an axis of symmetry along the vector $\mathbf{z}^H$. The equation representing the external points of this friction cone in H frame coordinates is

$$x^2 + y^2 - \mu^2 z^2 = 0 \tag{67}$$

The condition for a FCGC is that palm 2 must be entirely included in that cone. To determine if this is true, the equation of the extreme points of the friction cone intersecting the plane on which the palm j is placed, will be found.

In H frame coordinates, the plane containing palm 2 is given by:

$$D = -z_{2z}z - z_{2x}x = z\cos\beta\cos\alpha + x\sqrt{\sin^2\beta\cos^2\alpha + \sin^2\alpha} = Az + Bx \tag{68}$$

where D is a constant. The center of the palm is in that plane, therefore the value of D is given by

$$D = (Z-b)\cos\beta\cos\alpha + r\sin\beta\cos\alpha \tag{69}$$

Another coordinate frame, frame I, will be defined such that its plane x–y is parallel to the plane given by Equation 69, and its y axis is the same as the y axis of frame H. Coordinate frame I is obtained by rotating frame H about its $\mathbf{y}$ axis by an angle ϕ such that

$$\phi = \operatorname{atan2}(-z_{2x}^H, -z_{2z}^H) \tag{70}$$

The matrix describing the rotation from H frame to I frame is given by

$$\mathbf{R}_I^H = \begin{bmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{bmatrix} = \begin{bmatrix} \cos\beta\cos\alpha & 0 & \sqrt{\sin^2\beta\cos^2\alpha + \sin^2\alpha} \\ 0 & 1 & 0 \\ -\sqrt{\sin^2\beta\cos^2\alpha + \sin^2\alpha} & 0 & \cos\beta\cos\alpha \end{bmatrix} \tag{71}$$

For a given vector $\mathbf{z}_2$, and a given Z, it is possible to find the maximum value that r (shift from the z^0 axis between the center of the two palms) can have in a FCGC.

The composite translation from the global frame to I frame is given by

$$\mathbf{T}_I^0 = \mathbf{T}_H^0 \mathbf{T}_I^H = \begin{bmatrix} \frac{\sin\beta\cos\alpha\cos\phi}{\sin\phi} & \frac{\sin\alpha}{\sin\phi} & \sin\beta\cos\alpha & 0 \\ -\frac{\sin\alpha\cos\phi}{\sin\phi} & \frac{\sin\beta\cos\alpha}{\sin\phi} & -\sin\alpha & 0 \\ -\sin\phi & 0 & \cos\phi & b \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{72}$$

By using matrix $\mathbf{T}_I^0$, it is possible to find the maximum value of r for given Z, β and α (the vector $\mathbf{z}_2$ is expressed in terms of β and α). The center of palm 2 can be expressed in I frame coordinates as:

$$\begin{bmatrix} \mathbf{c}_2^I \\ 1 \end{bmatrix} = (\mathbf{T}_I^0)^{-1} \begin{bmatrix} r \\ 0 \\ Z \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{r\sin\beta\cos\alpha\cos\phi}{\sin\phi} - (Z-b)\sin\phi \\ \frac{r\sin\alpha}{\sin\phi} \\ r\sin\beta\cos\alpha + (Z-b)\cos\phi \\ 1 \end{bmatrix} = \begin{bmatrix} c_{2x}^I \\ c_{2y}^I \\ c_{2z}^I \\ 1 \end{bmatrix} \tag{73}$$

The new set of coordinates can be used to describe the intersection of the cone and the plane expressed in H frame coordinates, respectively, by Equations 67 and 69. The equations of the cone and the plane are represented in terms of these new coordinates respectively as:

$$(x\cos\phi + z\sin\phi)^2 + y^2 - \mu_i^2(-x\sin\phi + z\cos\phi)^2 = 0 \tag{74}$$

and

$$z = D \tag{75}$$

substituting the value of z in the equation of the cone yields

$$y^2 + (1 - \sin^2\phi(1+\mu^2))x^2 + 2D\sin\phi\cos\phi(1+\mu^2)x - D^2(\mu^2 - \sin^2\phi(1+\mu^2)) = 0$$

$$q = y^2 + Mx^2 + Nx - P = 0 \tag{76}$$

When $M = 1 - \sin^2\phi(1+\mu_i^2) > 0$, the curve represented by the equation is an ellipse (the particular case for which $M = 1$, i.e., when $\phi = 0$, is when the curve is a circle). When $M = 0$, then $\sin\phi = \cos\theta_i$, and the curve is a parabola. Finally, when $M < 0$ the curve is a hyperbola.

The points (x, y) for which $q < 0$ are interior points of the curve, they represent points on the plane inside the friction cone of palm 1, the points (x, y) for which $q > 0$ are exterior points to the curve, and they are points outside the friction cone of palm 1. If the center of palm 2 is inside of the friction cone of palm 1, there is a possibility that the whole palm is entirely included in it. This condition can be verified by evaluating q at the coordinates $(x = c_{2x}^I, y = c_{2y}^I)$, which are coordinates of the center point of palm j along the axes x and y.

The minimum distance between point $\mathbf{c}_2$, and the extreme points of the friction cone of palm 1 along the plane of palm 2, will then be determined. The problem can be stated as:

$$\min(x - c_{2x}^I)^2 + (y - c_{2y}^I)^2 \tag{77}$$

$$\text{subject to:} \qquad q = 0 \tag{78}$$

If the minimum distance between those extreme points and point $\mathbf{c}_2$ is greater than or equal to a, then palm 2 is completely included in the friction cone of palm 1, otherwise the grasp configuration is not a FCGC.

Applying Lagrange multipliers, the solution to the optimization problem can be found.

The strategy used here to determine the boundary value for r in a FCGC will be to set fixed values of α, β, and Z, and increment the value of r until the maximum r is reached in a FCGC. Initially, values of r and Z will be chosen in such a way that the friction cone of palm 1 includes palm 2.

Figure 13 shows incomplete overlapped boundaries of permissible values r, α, and β for different values of Z, except for $Z = 3.12$ in which the whole boundary is shown. The parameters a, and μ, are equal to 1 for the case shown in Figure 13. The complete bounds are symmetrical with respect to the plane $\beta - r$. To better describe their features, the faces corresponding to negative values for α have been omited except for the top of each boundary surface, and lines emphasizing the intersection between plane $\beta - r$ and the boundary surfaces have been drawn on top of the surfaces corresponding to the values shown. For $\alpha = 0$, the values of r and β for a given Z, are respectively the same as the values of x and $-\beta$ for a given y in the 2-D case (Figure 8).

5.2.3 Algorithm

An interpretation of the previous subsections can be given as follows. In Subsection 5.2.1, finding the extreme values of α and β at given r and Z, is the same as finding the extreme orientation, given the position, of palm 2 relative to palm 1, for which palm 1 is in the friction cone of palm 2. In Subsection 5.2.2, finding the extreme values for r for given values of Z, α and β is the same as finding the extreme position, given the orientation, of palm 2 relative to palm 1, for which palm 2 is in the friction cone of palm 1. Therefore, the problem of verifying if a grasp configuration is a FCGC, has been decomposed into two parts.

However, the results obtained in Subsection 5.2.1 cannot be directly applied, since the coordinates of the position vector $\mathbf{e}$ were assumed instead of derived, and from it the values for $\alpha_{\mathbf{extr}}$ and $\beta_{\mathbf{extr}}$ were obtained.

To make these results useful, note that, if $Z > 2b$, i.e., if $\mathbf{w} \neq \mathbf{z}_2$, where $\mathbf{w}$ is as given by Equation 51:

$$\mathbf{e} = a\frac{\mathbf{w} - (\sqrt{1 + \tan^2 R})\mathbf{z}_2}{\| \mathbf{w} - (\sqrt{1 + \tan^2 R})\mathbf{z}_2 \|} \tag{79}$$

angle R (shown in Figure 12) can be obtained from:

$$\cos R = \mathbf{w} \cdot \mathbf{z}_2 \tag{80}$$

$\mathbf{z}_2$ is the unit normal to palm 2; if $Z = 2b$, the only solution is that α, β, and r be all zero.

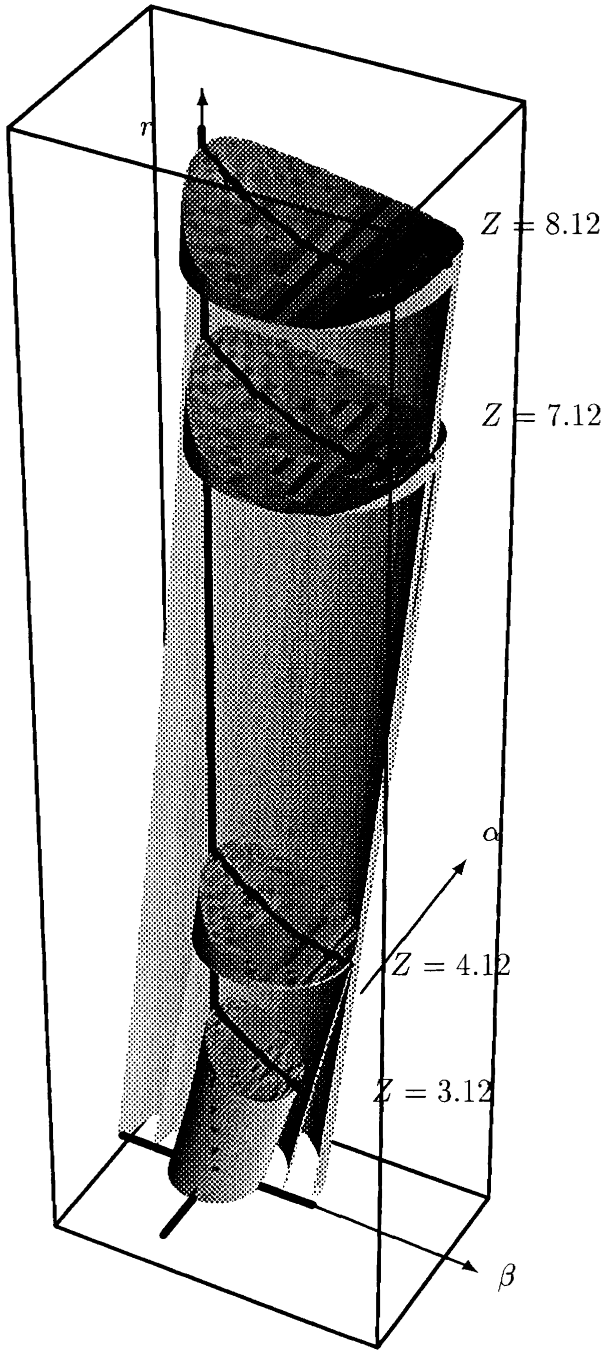

Figure 13: Boundary values for α, β and r corresponding to different values of Z

From this values of $\mathbf{e}$, and R, values for α_{extr} and β_{extr} can be obtained. If their absolute values are bigger than the absolute values of α and β, respectively obtained from Equations 43 and 44, palm 1 is in the friction cone of palm 2.

Therefore, to verify if a grasp configuration is a FCGC, the procedure applied can be described as follows:

1. Choose one of the palms as palm 1.

2. Obtain values Z, r, α and β, as expressed in Equations 41, 42, 43, and 44.

3. Verify if Inequalities 45 are satisfied. If not, exit with failure.

4. Verify if for given r, and Z, the values of α and β are such that, by using position vector $\mathbf{e}$ and angle R, obtained in Equations 79 and 80, the respective values of α and β have absolute values of magnitude smaller than or equal to the absolute values of the angles α_{extr} and β_{extr} satisfying Equation 60. If not, exit with failure.

5. Verify if r is such that the solution to the minimization problem of Expression 77, under the condition given by Equation 78, is greater than or equal to a, the radius of the plate. If not, exit with failure; otherwise, the grasp configuration is a FCGC.

6 Conclusions

We established a condition and a computational algorithm for force-closure grasps by using two flat-surface palms. The result is particularly useful for determining force-closure grasp of relatively large objects. The condition may not be satisfied when grasping a small object though a force closure clearly exists. It is noted that the study is based on static analysis. Dynamic properties such as mass are not considered. This implies that the result is applicable to lightweight objects.

The simplicity of the FCGC condition is made possible by taking advantage of the flat-surface palms and their intended applications (grasping large objects). On one hand, the palm is "viewed closely" as a flat surface to provide the contact normal. On the other hand, since the object is relatively larger than the palm, the palm is "viewed remotely" as an equivalent point. Thus the FCGC condition for the two palms is similar to that of the two-point contacts. The practical assumptions and the simplicity of the resulting condition make it promising to implement it in real time. An experimental setup, using two PUMA 250 robots, and based on the theoretical result of this paper, is currently being developed in the GRASP Laboratory, University of Pennsylvania.

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